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*Research Article*

# Fourth-Order Differential Equation with Deviating Argument

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We consider the fourth-order differential equation with middle-term and deviating argument $x^{(4)}(t) + q(t)x^{(2)}(t) + r(t)f(x(\varphi(t))) = 0$, in case when the corresponding second-order equation $h'' + q(t)h = 0$ is oscillatory. Necessary and sufficient conditions for the existence of bounded and unbounded asymptotically linear solutions are given. The roles of the deviating argument and the nonlinearity are explained, too.

## 1. Introduction

The aim of this paper is to investigate the fourth-order nonlinear differential equation with middle-term and deviating argument

$$x^{(4)}(t) + q(t)x^{(2)}(t) + r(t)f\left(x\left(\varphi(t)\right)\right) = 0. \tag{1.1}$$

The following assumptions will be made.

(i) $q$ is a continuously differentiable bounded away from zero function, that is, $q(t) \geq q_0 > 0$ for large $t$ such that

$$\int_0^\infty \left|q'(t)\right| dt < \infty. \tag{1.2}$$

(ii) $r, \varphi$ are continuous functions for $t \geq 0$, $r$ is not identically zero for large $t$, $\varphi(t) \geq 0$, and $\varphi(0) = 0$, $\lim_{t\to\infty}\varphi(t) = \infty$.

(iii) $f$ is a continuous function such that $f(u)u > 0$ for $u \neq 0$.

Observe that (i) implies that there exists a positive constant $Q$ such that $q(t) \leq Q$ and the linear second-order equation

$$h''(t) + q(t)h(t) = 0 \tag{1.3}$$

is oscillatory. Moreover, solutions of (1.3) are bounded together with their derivatives, see for example, [1, Theorem 2].

By a solution of (1.1) we mean a function $x$ defined on $[T_x, \infty)$, $T_x \geq 0$, which is differentiable up to the fourth order and satisfies (1.1) on $[T_x, \infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq T_x$.

A solution $x$ of (1.1) is said to be *asymptotically linear (AL-solution)* if either

$$\lim_{t\to\infty} x(t) = c_x \neq 0, \qquad \lim_{t\to\infty} x'(t) = 0, \tag{1.4}$$

or

$$\lim_{t\to\infty} |x(t)| = \infty, \qquad \lim_{t\to\infty} x'(t) = d_x \neq 0, \tag{1.5}$$

for some constants $c_x, d_x$.

Fourth-order nonlinear differential equations naturally appear in models concerning physical, biological, and chemical phenomena, such as, for instance, problems of elasticity, deformation of structures, or soil settlement, see, for example, [2, 3].

When (1.3) is nonoscillatory and $h$ is its eventually positive solution, it is known that (1.1) can be written as the two-term equation

$$\left(h^2(t)\left(\frac{x''(t)}{h(t)}\right)'\right)' + h(t)r(t)f(x(t)) = 0. \tag{1.6}$$

In this case, the question of oscillation and asymptotics of such class of equations has been investigated with sufficient thoroughness, see, for example, the papers [3–10] or the monographs [11, 12] and references therein.

Nevertheless, as far we known, there are only few results concerning (1.1) when (1.3) is oscillatory. For instance, the equation without deviating argument

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)f(x(t)) = 0 \tag{1.7}$$

has been investigated by Kiguradze in [13] in case $q(t) \equiv 1$ and by the authors in [14, 15] when $q$ satisfies (i). In particular, in [14] the oscillation of (1.1) in the case $n = 3$ is studied. In [15], the existence of positive bounded and unbounded solutions as well as of oscillatory solutions for (1.7) has been considered and the case $n = 4$ has been analyzed in detail. Other results can be found in [16] and references therein, in which the existence and uniqueness of almost periodic solutions for equations of type (1.1) with almost periodic coefficients $q, r$ are studied.

Motivated by [14, 15], here we study the existence of AL-solutions for (1.1). The approach is completely different from the one used in [15], in which an iteration process, jointly with a comparison with the linear equation $y^{(4)} + q(t)y^{(2)} = 0$, is employed. Our tools are based on a topological method, certain integral inequalities, and some auxiliary functions. In particular, for proving the continuity in the Fréchet space $C[t_0, \infty)$ of the fixed point operators here considered, we use a similar argument to that in the Vitali convergence theorem.

Our results extend to the case with deviating argument analogues ones stated in [15] for (1.7) when $n = 4$. We obtain sharper conditions for the existence of unbounded AL-solutions of (1.1), and, in addition, we show that under additional assumptions on $q, r$, these conditions become also necessary for the existence of AL-solutions, in both the bounded and unbounded cases. In the final part, we consider the particular case

$$f(u) = |u|^{\lambda} \operatorname{sgn} u \quad (\lambda > 0) \tag{1.8}$$

and we study the possible coexistence of bounded and unbounded AL-solutions. The role of deviating argument and the one of the growth of the nonlinearity are also discussed and illustrated by some examples.

## 2. Unbounded Solutions

Here we study the existence of unbounded AL-solutions of (1.1). Our first main result is the following.

**Theorem 2.1.** *For any $c$, $0 < c < \infty$, there exists an unbounded solution $x$ of* (1.1) *such that*

$$\lim_{t\to\infty} x'(t) = c, \qquad \lim_{t\to\infty} x^{(i)}(t) = 0, \quad i = 2, 3, \tag{2.1}$$

*provided*

$$\int_0^{\infty} |r(t)| F(\varphi(t))\, dt < \infty, \tag{2.2}$$

*where for $u > 0$*

$$F(u) = \max\left\{ f(v) : |v - u| \le \frac{1}{2}u \right\}. \tag{2.3}$$

*Proof.* Without loss of generality, we prove the existence of solutions of (1.1) satisfying (2.1) for $c = 1$.

Let $u$ and $v$ be two linearly independent solutions of (1.3) with Wronskian $d = 1$. Denote

$$w(s,t) = u(s)v(t) - u(t)v(s), \qquad z(s,t) = \frac{\partial}{\partial t} w(s,t). \tag{2.4}$$

As claimed by the assumptions on $q$, all solutions of (1.3) and their derivatives are bounded. Thus, put

$$M = \sup\{|w(s,t)| + |z(s,t)| : s \geq 0,\ t \geq 0\}, \qquad L = \frac{2(2M+1)}{q_0}. \tag{2.5}$$

Let $\bar{t} \geq t_0$ be such that $\varphi(t) \geq t_0$ for $t \geq \bar{t}$. Define

$$\overline{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \geq \bar{t}, \\ \varphi\left(\bar{t}\right) & \text{if } t_0 \leq t \leq \bar{t}, \end{cases} \tag{2.6}$$

and choose $t_0 \geq 0$ large so that

$$\int_{t_0}^{\infty} |r(s)| F(\overline{\varphi}(s))\, ds \leq \frac{1}{2L}, \qquad \frac{1}{q_0}\int_{t_0}^{\infty} \left|q'(t)\right| dt \leq \frac{1}{2}. \tag{2.7}$$

Denote by $C[t_0,\infty)$ the Fréchet space of all continuous functions on $[t_0,\infty)$, endowed with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$, and consider the set $\Omega \subset C[t_0,\infty)$ given by

$$\Omega = \left\{ x \in C[t_0,\infty) : \frac{t}{2} \leq x(t) \leq \frac{3t}{2} \right\}. \tag{2.8}$$

Let $T > t_0$ and define on $[t_0, T]$ the function

$$g(t) = \gamma''(t) + q(t)\gamma(t), \tag{2.9}$$

where

$$\gamma(t) = -\int_t^T \int_\tau^\infty r(s) f\left(x\left(\overline{\varphi}(s)\right)\right) w(s,\tau)\, ds\, d\tau \tag{2.10}$$

and $x \in \Omega$. Then,

$$\gamma'(t) = \int_t^\infty r(s) f\left(x\left(\overline{\varphi}(s)\right)\right) w(s,t)\, ds, \tag{2.11}$$

$$\begin{gathered} \gamma''(t) = \int_t^\infty r(s) f\left(x\left(\overline{\varphi}(s)\right)\right) z(s,t)\, ds, \\ \gamma'''(t) = -r(t) f\left(x\left(\overline{\varphi}(t)\right)\right) - q(t)\gamma'(t). \end{gathered} \tag{2.12}$$

Moreover, $g(T) = \gamma''(T)$, and it holds for $t \in [t_0, T]$ that

$$g'(t) = \gamma'''(t) + q(t)\gamma'(t) + q'(t)\gamma(t) = -r(t) f\left(x\left(\overline{\varphi}(t)\right)\right) + q'(t)\gamma(t). \tag{2.13}$$

Integrating, we obtain

$$g(t) = g(T) - \int_t^T g'(s)ds = \gamma''(T) + \int_t^T r(s) f\left(x(\overline{\varphi}(s))\right)ds - \int_t^T q'(s)\gamma(s)ds. \tag{2.14}$$

From here and (1.3), we get

$$\gamma(t) = \frac{1}{q(t)}\left(\gamma''(T) - \gamma''(t) + \int_t^T r(s) f\left(x(\overline{\varphi}(s))\right)ds - \int_t^T q'(s)\gamma(s)ds\right). \tag{2.15}$$

Thus,

$$\left|\gamma(t)\right| \leq \frac{1+2M}{q_0}\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds + \frac{1}{q_0}\max_{t\leq s\leq T}\left|\gamma(s)\right|\int_{t_0}^\infty \left|q'(s)\right|ds, \tag{2.16}$$

and so

$$\left(1 - \frac{1}{q_0}\int_{t_0}^\infty \left|q'(s)\right|ds\right)\max_{t\leq s\leq T}\left|\gamma(s)\right| \leq \frac{1+2M}{q_0}\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds, \tag{2.17}$$

or, in view of (2.7),

$$\left|\gamma(t)\right| \leq L\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds. \tag{2.18}$$

Thus, from (2.10), as $T \to \infty$, we get

$$\left|\int_t^\infty \int_\tau^\infty r(s) f\left(x(\overline{\varphi}(s))\right)w(s,\tau)ds\,d\tau\right| \leq L\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds. \tag{2.19}$$

Hence, the operator $\mathcal{T} : \Omega \to \Omega$ given by

$$\mathcal{T}(x)(t) = t - \int_{t_0}^t \int_\sigma^\infty \int_\tau^\infty r(s) f\left(x(\overline{\varphi}(s))\right)w(s,\tau)ds\,d\tau\,d\sigma \tag{2.20}$$

is well defined for any $x \in \Omega$. Moreover, in view of (2.19), we have

$$\left|\mathcal{T}'(x)(t) - 1\right| \leq L\int_t^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds. \tag{2.21}$$

From here, in virtue of (2.7) we get

$$|\mathcal{T}(x)(t) - t| \leq Lt\int_{t_0}^\infty |r(s)|F\left(\overline{\varphi}(s)\right)ds \leq \frac{1}{2}t. \tag{2.22}$$

Hence, $\mathcal{T}(\Omega) \subset \Omega$. From (2.5) and (2.11), we have

$$\left|\mathcal{T}''(x)(t)\right| = \left|\gamma'(t)\right| \leq M\int_t^{\infty}|r(s)|F(\overline{\varphi}(s))ds, \tag{2.23}$$

and so $\lim_{t\to\infty}\mathcal{T}''(x)(t) = 0$. Similarly,

$$\left|\mathcal{T}'''(x)(t)\right| = \left|\gamma''(t)\right| \leq M\int_t^{\infty}|r(s)|F(\overline{\varphi}(s))ds, \tag{2.24}$$

and thus, $\lim_{t\to\infty}\mathcal{T}'''(x)(t) = 0$, too. In addition,

$$\mathcal{T}^{(4)}(x)(t) = \gamma'''(t) = -q(t)\mathcal{T}''(x)(t) - r(t)f(x(\overline{\varphi}(t))). \tag{2.25}$$

Hence, any fixed point of $\mathcal{T}$ is a solution of (1.1) for large $t$.

Let us show that $\mathcal{T}(\Omega)$ is relatively compact, that is, $\mathcal{T}(\Omega)$ consists of functions equibounded and equicontinuous on every compact interval of $[t_0,\infty)$. Because $\mathcal{T}(\Omega) \subset \Omega$, the equiboundedness follows. Moreover, in view of (2.7), $\mathcal{T}'(u)(t)$ is bounded for any $u \in \Omega$, which yields the equicontinuity of the elements in $\mathcal{T}(\Omega)$.

Now we prove the continuity of $\mathcal{T}$ in $\Omega$. Let $\{x_n\}$, $n \in \mathbb{N}$, be a sequence in $\Omega$, which uniformly converges to $\overline{x} \in \Omega$ on every compact interval of $[t_0,\infty)$. Fixing $T > t_0$, in virtue of (2.23), the dominated convergence Lebesgue theorem gives

$$\lim_{n\to\infty}\int_\sigma^T\int_\tau^{\infty} r(s)(f(x_n(\overline{\varphi}(s))))w(s,\tau)ds\,d\tau = \int_\sigma^T\int_\tau^{\infty} r(s)(f(x(\overline{\varphi}(s))))w(s,\tau)ds\,d\tau. \tag{2.26}$$

Moreover,

$$\begin{aligned}
&\left|\int_\sigma^{\infty}\int_\tau^{\infty} r(s)(f(x_n(\overline{\varphi}(s))) - f(\overline{x}(\overline{\varphi}(s))))w(s,\tau)ds\,d\tau\right| \\
&\quad\leq \left|\int_\sigma^{T}\int_\tau^{\infty} r(s)(f(x_n(\overline{\varphi}(s))) - f(\overline{x}(\overline{\varphi}(s))))w(s,\tau)ds\,d\tau\right| \\
&\qquad + \int_T^{\infty}\int_\tau^{\infty}|r(s)|(f(x_n(\overline{\varphi}(s))) + f(\overline{x}(\overline{\varphi}(s))))|w(s,\tau)|ds\,d\tau.
\end{aligned} \tag{2.27}$$

In view of (2.19), we have

$$\int_T^{\infty}\int_\tau^{\infty}|r(s)|(f(x_n(\overline{\varphi}(s))) + f(\overline{x}(\overline{\varphi}(s))))|w(s,\tau)|ds\,d\tau \leq 2M\int_T^{\infty}|r(s)|F(\overline{\varphi}(s))ds. \tag{2.28}$$

Thus, choosing $T$ sufficiently large, we get from (2.27)

$$\lim_{n\to\infty}\int_\sigma^{\infty}\int_\tau^{\infty} r(s)f(x_n(\overline{\varphi}(s)))w(s,\tau)ds\,d\tau = \int_\sigma^{\infty}\int_\tau^{\infty} r(s)f(\overline{x}(\overline{\varphi}(s)))w(s,\tau)ds\,d\tau, \tag{2.29}$$

and so the continuity of $\mathcal{T}$ in $\Omega$ follows. By the Tychonoff fixed point theorem, the operator $\mathcal{T}$ has a fixed point $x$, which is an unbounded solution of (1.1) satisfying (2.1). □

*Remark 2.2.* With minor modifications, Theorem 2.1 gives also the existence of eventually negative unbounded AL-solutions. The details are omitted.

*Remark 2.3.* When $\varphi(t) \equiv t$, Theorem 2.1 is related with Theorem 1 in [15], from which the existence of unbounded AL-solutions of (1.1) can be obtained under stronger assumptions. A comparison between Theorem 1 in [15] and Theorem 2.1 is given in Section 4.

Our next result gives a necessary condition for the existence of unbounded solutions $x$ of (1.1) satisfying for large $t$ and some $\alpha$ and $\beta$

$$0 < \alpha \leq x'(t) \leq \beta. \tag{2.30}$$

**Theorem 2.4.** *Assume either $r(t) \geq 0$ or $r(t) \leq 0$.*

*Equation* (1.1) *does not have eventually positive solutions $x$ satisfying* (2.30) *for large $t$ and some $\alpha$ and $\beta$ provided*

$$\int_0^\infty |r(t)|\overline{F}(\varphi(t))dt = \infty, \tag{2.31}$$

*where for $u > 0$*

$$\overline{F}(u) = \min\left\{ f(v) : \frac{\alpha}{2}u \leq v \leq 2\beta u \right\}. \tag{2.32}$$

*Proof.* Assume $r(t) \geq 0$, and let $x$ be an eventually positive solution of (1.1) satisfying (2.30). Then, there exists $\tau$ such that

$$\frac{\alpha}{2}t \leq x(t) \leq 2t\beta \quad \text{for } t \geq \tau. \tag{2.33}$$

Consequently, in view of (2.31), we have

$$\lim_{t\to\infty} \int_\tau^t r(s)f(x(\varphi(s)))ds = \infty. \tag{2.34}$$

Thus, integrating (1.1), we get

$$\lim_{t\to\infty} \left( x'''(t) + \int_\tau^t q(s)x''(s)ds \right) = -\infty. \tag{2.35}$$

Furthermore,

$$\begin{aligned}\left|\int_{\tau}^{t} q(s)x''(s)ds\right| &= \left|q(t)x'(t) - q(\tau)x'(\tau) - \int_{\tau}^{t} q'(s)x'(s)ds\right| \\ &\leq 2\beta Q + \beta\int_{\tau}^{\infty}\left|q'(s)\right|ds < \infty,\end{aligned} \tag{2.36}$$

where $Q = \sup_{s\geq 0} q(s)$. Hence $\lim_{t\to\infty} x'''(t) = -\infty$, which gives a contradiction with the boundedness of $x'$. Finally, if $r(t) \leq 0$, the argument is similar and the details are left to the reader. □

## 3. Bounded Solutions

In this section we study the existence of bounded AL-solutions of (1.1). The following holds.

**Theorem 3.1.** *If*

$$\int_{0}^{\infty} |r(t)| t\, dt < \infty, \tag{3.1}$$

*then, for any* $c \in \mathbb{R} \setminus \{0\}$, *there exists a solution* $x$ *of* (1.1) *satisfying*

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty} x^{(i)}(t) = 0, \quad i = 1, 2. \tag{3.2}$$

*Proof.* Without loss of generality, we prove the existence of solutions of (1.1) satisfying (3.2) for $c = 1$.

We proceed by a similar way to that in the proof of Theorem 2.1, and we sketch the proof.

Let $M$ be the constant given in (2.5), and let

$$K = \max\left\{ f(u) : \frac{1}{2} \leq u \leq \frac{3}{2} \right\}, \qquad L_1 = \frac{2K(2M+1)}{q_0}. \tag{3.3}$$

Choose $t_0 \geq 0$ large so that

$$\int_{t_0}^{\infty} t|r(t)|dt \leq \frac{1}{2L_1}, \qquad \frac{1}{q_0}\int_{t_0}^{\infty}\left|q'(s)\right|ds \leq \frac{1}{2}, \tag{3.4}$$

and define $\overline{\varphi}$ as in (2.6). Denote by $C[t_0, \infty)$ the Fréchet space of all continuous functions on $[t_0, \infty)$, endowed with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$, and consider the set $\Omega \subset C[t_0, \infty)$ given by

$$\Omega = \left\{ x \in C[t_0, \infty) : \frac{1}{2} \leq x(t) \leq \frac{3}{2} \right\}. \tag{3.5}$$

Let $T > t_0$, and, for any $x \in \Omega$, consider again the function $\gamma$ given in (2.10). Reasoning as in the proof of Theorem 2.1, with minor changes, we obtain

$$\left|\int_t^{\infty}\int_{\tau}^{\infty} r(s)f\left(x(\overline{\varphi}(s))\right)w(s,\tau)ds\,d\tau\right| \leq L_1\int_t^{\infty}|r(s)|ds. \tag{3.6}$$

Hence, in virtue of (3.1), the operator $\mathcal{H} : \Omega \to \Omega$ given by

$$\mathcal{H}(x)(t) = 1 + \int_t^{\infty}\int_{\sigma}^{\infty}\int_{\tau}^{\infty} r(s)f\left(x(\overline{\varphi}(s))\right)w(s,\tau)ds\,d\tau\,d\sigma \tag{3.7}$$

is well defined and $\lim_{t\to\infty}\mathcal{H}(x)(t) = 1$. In view of (3.6), we get

$$\left|\mathcal{H}'(x)(t)\right| \leq L_1\int_t^{\infty}|r(s)|ds. \tag{3.8}$$

A similar estimation holds for $|\mathcal{H}''(x)|$. Thus, $\lim_{t\to\infty}\mathcal{H}^{(i)}(x)(t) = 0,\ i = 1,2$. In view of (3.4), from (3.8), we obtain

$$|\mathcal{H}(x)(t) - 1| \leq L_1\int_t^{\infty} s|r(s)|ds \leq \frac{1}{2}, \tag{3.9}$$

that is, $\mathcal{H}(\Omega) \subset \Omega$. Moreover, a standard calculation gives

$$\mathcal{H}^{(4)}(x)(t) = -q(t)\mathcal{H}^{(2)}(x)(t) - r(t)f\left(x(\overline{\varphi}(s))\right), \tag{3.10}$$

and so any fixed point of $\mathcal{H}$ is, for large $t$, a solution of (1.1). Proceeding by a similar way to that in the proof of Theorem 2.1, we obtain that $\mathcal{H}(\Omega)$ is relatively compact.

Now we prove the continuity of $\mathcal{H}$ in $\Omega$. Let $\{x_n\}$, $n \in \mathbb{N}$, be a sequence in $\Omega$, which uniformly converges to $\overline{x} \in \Omega$ on every compact interval of $[t_0, \infty)$. Since

$$\left|\int_{\tau}^{\infty} r(s)\left(f\left(x_n(\overline{\varphi}(s))\right)\right)w(s,\tau)ds\right| \leq KM\int_{\tau}^{\infty}|r(s)|ds, \tag{3.11}$$

in virtue of (3.1), the dominated convergence Lebesgue theorem gives

$$\lim_{n\to\infty}\int_{\sigma}^{\infty}\int_{\tau}^{\infty} r(s)\left(f\left(x_n(\overline{\varphi}(s))\right)\right)w(s,\tau)ds\,d\tau = \int_{\sigma}^{\infty}\int_{\tau}^{\infty} r(s)\left(f\left(x(\overline{\varphi}(s))\right)\right)w(s,\tau)ds\,d\tau. \tag{3.12}$$

Moreover, fixing $T > t_0$, we have

$$\begin{aligned}
&\left|\int_t^\infty \int_\sigma^\infty \int_\tau^\infty r(s)\left(f\left(x_n(\overline{\varphi}(s))\right) - f\left(\overline{x}(\overline{\varphi}(s))\right)\right)w(s,\tau)ds\,d\tau\,d\sigma\right| \\
&\quad\le \left|\int_t^T \int_\sigma^\infty \int_\tau^\infty r(s)\left(f\left(x_n(\overline{\varphi}(s))\right) - f\left(\overline{x}(\overline{\varphi}(s))\right)\right)w(s,\tau)ds\,d\tau\,d\sigma\right| \\
&\qquad + \int_T^\infty \int_\sigma^\infty \int_\tau^\infty |r(s)|\left(f\left(x_n(\overline{\varphi}(s))\right) + f\left(\overline{x}(\overline{\varphi}(s))\right)\right)|w(s,\tau)|ds\,d\tau\,d\sigma.
\end{aligned} \tag{3.13}$$

In view of (3.9), we have

$$\int_T^\infty \int_\sigma^\infty \int_\tau^\infty |r(s)|\left(f\left(x_n(\overline{\varphi}(s))\right) + f\left(\overline{x}(\overline{\varphi}(s))\right)\right)|w(s,\tau)|ds\,d\tau\,d\sigma \le 2L_1 \int_T^\infty s|r(s)|ds, \tag{3.14}$$

and so, choosing $T$ sufficiently large, from (3.13) we obtain the continuity of $\mathscr{H}$ in $\Omega$. Hence, by the Tychonoff fixed point theorem, the operator $\mathscr{H}$ has a fixed point $x$, which is a bounded solution of (1.1) satisfying (3.2). □

*Remark 3.2.* When $n = 4$, Theorem 3.1 extends to equations with deviating argument of a similar result stated in [15] for (1.7). Observe that our approach used here is completely different from that in [15].

The next result shows that, under additional assumptions, condition (3.1) can be also necessary for the existence of bounded AL-solutions of (1.1).

**Theorem 3.3.** *Assume either*

$$r(t) \ge 0, \qquad q''(t) \ge 0 \quad \text{for large } t \tag{3.15}$$

*or*

$$r(t) \le 0, \qquad q''(t) \le 0 \quad \text{for large } t. \tag{3.16}$$

*If*

$$\int_0^\infty |r(t)|t\,dt = \infty, \tag{3.17}$$

*then* (1.1) *does not have solutions $x$ satisfying*

$$0 < \alpha \le x(t) \le \beta, \tag{3.18}$$

*for large t and some α and β. Consequently, every bounded solution x of* (1.1) *satisfies*

$$\liminf_{t\to\infty}|x(t)| = 0. \tag{3.19}$$

The following lemmas are needed for proving Theorem 3.3.

**Lemma 3.4.** *Assume $q''(t) \geq 0$ for $t \geq T \geq 0$, and let $x$ be a solution of* (1.1) *satisfying* (3.18) *for $t \geq T$. Then, there exist two constants $M_1, M_2$ such that for $t \geq T$*

$$-\int_T^t sq(s)x''(s)ds < tq'(t)x(t) - tq(t)x'(t) + M_1, \tag{3.20}$$

$$\int_T^t (s-T)q'(s)x'(s)ds < M_2. \tag{3.21}$$

*If $q''(t) \leq 0$ for $t \geq T \geq 0$, inequalities* (3.20), (3.21) *hold in the opposite order.*

*Proof.* Suppose $q''(t) \geq 0$ on $[T,\infty)$. We have

$$\int_T^t sq(s)x''(s)ds = tq(t)x'(t) - Tq(T)x'(T) - \int_T^t q(s)x'(s)ds - \int_T^t sq'(s)x'(s)ds. \tag{3.22}$$

Since

$$\begin{gathered}\int_T^t sq'(s)x'(s)ds = tq'(t)x(t) - Tq'(T)x(T) - \int_T^t q'(s)x(s)ds - \int_T^t sq''(s)x(s)ds,\\ \int_T^t q(s)x'(s)ds = q(t)x(t) - q(T)x(T) - \int_T^t q'(s)x(s)ds,\end{gathered} \tag{3.23}$$

from (3.22), we get

$$\begin{aligned}-\int_T^t sq(s)x''(s)ds &= tq'(t)x(t) - tq(t)x'(t) + q(t)x(t)\\ &\quad - 2\int_T^t q'(s)x(s)ds - \int_T^t sq''(s)x(s)ds + K_1,\end{aligned} \tag{3.24}$$

where $K_1$ is a suitable constant. Since $q, x$ are bounded, $q''(t) \geq 0$, in view of (1.1), inequality (3.20) follows.

Moreover, $q'$ is nondecreasing for $t \geq T$. Because $q$ is a positive bounded function, then $q'(t) \leq 0$ on $[T,\infty)$. Thus, inequality (3.21) follows integrating by parts and using (1.1). Finally, if $q''(t) \leq 0$ on $[T,\infty)$, the argument is similar. □

**Lemma 3.5.** *Let $x$ be a solution of* (1.1) *satisfying* (3.18) *for large t. If*

$$\int_0^\infty |r(t)|dt < \infty, \tag{3.25}$$

*then $x''$ is bounded. If, in addition, $r(t) \geq 0$, $q''(t) \geq 0$ for $t \geq T \geq 0$ and* (3.17) *holds, then for large $t$*

$$x'''(t) + q(t)x'(t) < q'(t)x(t). \tag{3.26}$$

*If $r(t) \leq 0$, $q''(t) \leq 0$ for $t \geq T \geq 0$, inequality* (3.26) *holds in the opposite order.*

*Proof.* Since $\lim_{t\to\infty}\varphi(t) = \infty$, there exists $\tau$ such that for $t \geq \tau$

$$0 < \alpha \leq x(\varphi(t)) \leq \beta. \tag{3.27}$$

Without loss of generality, let $\tau = T$. Thus, $\inf_{t\geq T} f(x(\varphi(t))) > 0$.

Let $u$ and $v$ be two linearly independent solutions of (1.3) with Wronskian $d = 1$. By assumptions on $q$, all solutions of (1.3) and their derivatives are bounded. Thus, by the variation constant formula, there exist constants $c_1$ and $c_2$ such that

$$x''(t) = c_1u(t) + c_2v(t) - \int_T^t (u(s)v(t) - u(t)v(s))r(s)f(x(\varphi(s)))ds, \tag{3.28}$$

and, in view of (3.25), $x''$ is bounded.

Let us prove (3.26), and suppose $r(t) \geq 0$, $q''(t) \geq 0$ on $[T,\infty)$. Multiplying (1.1) by $t$ and integrating from $T$ to $t$, we get

$$tx'''(t) - x''(t) + \int_T^t sq(s)x''(s)ds = Tx'''(T) - x''(T) - \int_T^t sr(s)f(x(\varphi(s)))ds, \tag{3.29}$$

or, in view of Lemma 3.4,

$$tx'''(t) \leq x''(t) + tq'(t)x(t) - tq(t)x'(t) - \int_T^t sr(s)f(x(\varphi(s)))ds + K_2, \tag{3.30}$$

where $K_2$ is a suitable constant. Since $x''$ is bounded and

$$\int_T^t sr(s)f(x(\varphi(s)))ds \geq \inf_{t\geq T} f(x(\varphi(t))) \int_T^t sr(s)ds, \tag{3.31}$$

from (3.17) and (3.30), we have

$$\lim_{t\to\infty} t(x'''(t) - q'(t)x(t) + q(t)x'(t)) = -\infty, \tag{3.32}$$

which gives the assertion. The case $r(t) \leq 0$, $q''(t) \leq 0$ on $[T,\infty)$ can be treated in a similar way. □

*Proof of Theorem 3.3.* Suppose $r(t) \geq 0$, $q''(t) \geq 0$ for $t \geq T \geq 0$. Without loss of generality, assume also that (3.27) holds for $t \geq T$. Define

$$v(t) = x''(t) + q(t)x(t), \tag{3.33}$$

$$z(t) = x'''(t) + q(t)x'(t) - \int_T^t q'(s)x'(s)ds. \tag{3.34}$$

Then, $z'(t) = -r(t)f(x(\varphi(t))) \leq 0$ and

$$z(t) = z(T) - \int_T^t r(s)f\big(x(\varphi(s))\big)ds. \tag{3.35}$$

Since $q'(t) \leq 0$ for $t \geq T$, we have

$$v'(t) \leq z(t) + \int_T^t q'(s)x'(s)ds = z(T) - \int_T^t r(s)f\big(x(\varphi(s))\big)ds + \int_T^t q'(s)x'(s)ds. \tag{3.36}$$

*Case I.* Assume

$$\int_0^\infty r(t)dt = \infty. \tag{3.37}$$

Since for $t \geq T$ we have $q''(t) \geq 0$ and, as claimed, $q'(t) \leq 0$, we get

$$\int_T^t q'(s)x'(s)ds = q'(t)x(t) - q'(T)x(T) - \int_T^t q''(s)x(s)ds \leq -q'(T)x(T). \tag{3.38}$$

Thus, from (3.36), we obtain $\lim_{t\to\infty} v'(t) = -\infty$, that is, $v$ is unbounded. Hence, in view of (3.33), we obtain a contradiction with the boundedness of $x$.

*Case II.* Now assume (3.17) and (3.25). In view of Lemma 3.5, without loss of generality, we can suppose that (3.26) holds for $t \geq T$. Then,

$$z(T) = x'''(T) + q(T)x'(T) < q'(T)x(T). \tag{3.39}$$

Hence, $z(T) < 0$. Integrating (3.36), we get

$$v(t) \leq v(T) + z(T)(t-T) - \int_T^t (s-T)r(s)f\big(x(\varphi(s))\big)ds + \int_T^t (s-T)q'(s)x'(s)ds, \tag{3.40}$$

and, in view of Lemma 3.4, we have

$$v(t) \leq v(T) + z(T)(t-T) + M_2. \tag{3.41}$$

Thus, $\lim_{t\to\infty} v(t) = -\infty$, that is, as before, a contradiction. Finally, the case $r(t) \le 0$, $q''(t) \le 0$ for large $t$ follows in a similar way. □

## 4. Applications

Here we present some applications of our results to a particular case of (1.1), namely, the equation

$$x^{(4)}(t) + q(t)x''(t) + r(t)\left|x(\varphi(t))\right|^{\lambda} \operatorname{sgn} x(\varphi(t)) = 0 \quad (\lambda > 0), \tag{4.1}$$

jointly with some suggestions for future research.

### *4.1. Coexistence of Both Types of AL-Solutions*

Applying Theorems 2.1–3.3 to this equation, we obtain the following.

**Corollary 4.1.** (a) *Let $r(t) \neq 0$ for large $t$. Equation* (4.1) *has unbounded AL-solutions if and only if*

$$\int_0^{\infty} |r(t)|\varphi^{\lambda}(t)dt < \infty. \tag{4.2}$$

(b) *Assume either* (3.15) *or* (3.16). *Equation* (4.1) *has bounded AL-solutions if and only if* (3.1) *holds.*

Corollary 4.1 shows also that the deviating argument can produce a different situation concerning the unboundedness of solutions with respect to the corresponding equation without delay, as the following example illustrates.

*Example 4.2.* In view of Corollary 4.1(a), the equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + \frac{1}{(t+1)^2}\left|x\left(\sqrt{t}\right)\right|^{3/2} \operatorname{sgn} x\left(\sqrt{t}\right) = 0, \tag{4.3}$$

where $q$ satisfies (i), has unbounded AL-solutions, while the corresponding ordinary equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + \frac{1}{(t+1)^2}|x(t)|^{3/2} \operatorname{sgn} x(t) = 0, \tag{4.4}$$

in view of Theorem 2.4, does not have unbounded AL-solutions. Moreover, if in addition $q''(t) > 0$ for large $t$, then from Corollary 4.1(b) (4.3) does not have bounded AL-solutions.

The following example shows that the opposite situation to the one described in Example 4.2 can occur.

*Example 4.3.* Consider the equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + \frac{1}{(t+1)^3}x\left(t^2\right) = 0, \tag{4.5}$$

where $q$ satisfies (i). From Theorem 3.1, (4.5) has bounded AL-solutions and the same occurs for the corresponding ordinary equation. Nevertheless, in view of Corollary 4.1(a), (4.5) has no unbounded AL-solutions.

Examples 4.2 and 4.3 illustrate also that the coexistence of both AL-solutions for (4.1) can fail. Sufficient conditions for the coexistence of these solutions immediately follow from Corollary 4.1.

**Corollary 4.4.** *Let $r(t) \neq 0$ for large $t$.*

(a) *Assume for large $t$*

$$\varphi(t) \geq t^{1/\lambda}. \tag{4.6}$$

*If* (4.1) *has unbounded AL-solutions, then* (4.1) *also has AS bounded solutions.*

(b) *Assume for large $t$*

$$\varphi(t) \leq t^{1/\lambda}, \qquad \operatorname{sgn} r(t) = \operatorname{sgn} q''(t). \tag{4.7}$$

*If* (4.1) *has bounded AL-solutions, then* (4.1) *also has unbounded AL-solutions.*

*For the equation without deviating argument*

$$x^{(4)}(t) + q(t)x''(t) + r(t)|x(t)|^{\lambda} \operatorname{sgn} x(t) = 0 \quad (\lambda > 0), \tag{4.8}$$

*from* Corollary 4.4 *we get the following.*

**Corollary 4.5.** *Let $r(t) \neq 0$ for large $t$.*

(a) *Assume $\lambda \geq 1$. If* (4.8) *has unbounded AL-solutions, then* (4.8) *has also bounded AL-solutions.*

(b) *Assume $0 < \lambda \leq 1$ and $\operatorname{sgn} r(t) = \operatorname{sgn} q''(t)$ for large $t$. If* (4.8) *has bounded AL-solutions, then* (4.8) *has also unbounded AL-solutions.*

## *4.2. Comparison with Some Results in [15]*

As claimed, the existence of unbounded AL-solutions for (4.8) follows also from Theorem 1 in [15]. For $n = 4$ this result reads as follows.

**Theorem A.** *If*

$$\int_0^{\infty} |r(t)| t^{\lambda+1} dt < \infty, \tag{4.9}$$

*then there exists a solution* $x$ *of* (4.8) *such that*

$$x^{(i)}(t) = t^{(i)} + \varepsilon_i(t), \quad i = 0, \dots, 3, \tag{4.10}$$

*where* $\varepsilon_i$ *are functions of bounded variation for large* $t$ *and* $\lim_{t\to\infty}\varepsilon_i(t) = 0$.

Therefore, when $\varphi(t) \equiv t$, Theorem 2.1 ensures the existence of unbounded AL-solutions of (4.8) under a weaker condition than (4.9), namely,

$$\int_0^\infty |r(t)| t^\lambda dt < \infty. \tag{4.11}$$

On the other hand, Theorem A gives an asymptotic formula for such solutions.

### *4.3. An Open Problem*

Equation (1.1) can admit also other types of nonoscillatory solutions, as the following examples show.

*Example 4.6.* Consider the equation

$$x^{(4)}(t) + x^{(2)}(t) - \frac{2t^2 + 4t + 26}{(t+1)^{7/2}} |x(t)|^{3/2} \operatorname{sgn} x(t) = 0. \tag{4.12}$$

In virtue of Corollary 4.1(b), (4.12) has no bounded AL-solutions. Nevertheless, this equation admits nonoscillatory bounded solutions because $x(t) = (1+t)^{-1}$ is a solution of (4.12).

*Example 4.7.* Consider the equation

$$x^{(4)}(t) + x^{(2)}(t) + \frac{t^2 + 4t + 10}{(t+2)^4 (\log(t+2))^3} x^3(t) = 0. \tag{4.13}$$

Thus, (3.1) holds, while $\int_0^\infty t^3 r(t) dt = \infty$. Hence, in virtue of Corollary 4.1, (4.13) has bounded AL-solutions, but no unbounded AL-solutions. Nevertheless, this equation admits nonoscillatory unbounded solutions because $x(t) = \log(t+2)$ is a solution of (4.13).

The existence of nonoscillatory solutions $x$ satisfying either $\lim_{t\to\infty} x(t) = 0$ or $\lim |x(t)| = \infty$, $\lim_{t\to\infty} x'(t) = 0$ will be a subject of our next research.

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