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# A NOTE ON SERRIN'S OVERDETERMINED PROBLEM 

GIULIO CIRAOLO AND ROLANDO MAGNANINI

Abstract. We consider the solution of the torsion problem

$$
-\Delta u=1 \text { in } \Omega, \quad u=0 \text { on } \partial \Omega
$$

Serrin's celebrated symmetry theorem states that, if the normal derivative $u_{\nu}$ is constant on $\partial \Omega$, then $\Omega$ must be a ball. In [6], it has been conjectured that Serrin's theorem may be obtained by stability in the following way: first, for the solution $u$ of the torsion problem prove the estimate

$$
r_{e}-r_{i} \leq C_{t}\left(\max _{\Gamma_{t}} u-\min _{\Gamma_{t}} u\right)
$$

for some constant $C_{t}$ depending on $t$, where $r_{e}$ and $r_{i}$ are the radii of an annulus containing $\partial \Omega$ and $\Gamma_{t}$ is a surface parallel to $\partial \Omega$ at distance $t$ and sufficiently close to $\partial \Omega$; secondly, if in addition $u_{\nu}$ is constant on $\partial \Omega$, show that

$$
\max _{\Gamma_{t}} u-\min _{\Gamma_{t}} u=o\left(C_{t}\right) \quad \text { as } \quad t \rightarrow 0^{+}
$$

The estimate constructed in [6] is not sharp enough to achieve this goal. In this paper, we analyse a simple case study and show that the scheme is successful if the admissible domains $\Omega$ are ellipses.

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^{N}$ and let $u$ be the solution of the torsion problem

$$
\begin{equation*}
-\Delta u=1 \text { in } \Omega, u=0 \text { on } \partial \Omega . \tag{1.1}
\end{equation*}
$$

Serrin's celebrated symmetry theorem [10] states that, if there exists a solution of (1.1) whose (exterior) normal derivative $u_{\nu}$ is constant on $\partial \Omega$, that is such that

$$
\begin{equation*}
u_{\nu}=c \quad \text { on } \partial \Omega, \tag{1.2}
\end{equation*}
$$

then $\Omega$ is a ball and $u$ is radially symmetric.
As is well-known, the proof of Serrin makes use of the method of moving planes (see [10, ?]), a refinement of Alexandrov's reflection principle [2].

The aim of this note is to probe the feasibility of a new proof of Serrin's symmetry theorem based on a comparison with another overdetermined problem for (1.1). In fact, it has been noticed that, under certain sufficient conditions on $\partial \Omega$, if the solution of (1.1) is constant on a surface parallel to $\partial \Omega$, that is, if for some small $t>0$

$$
\begin{equation*}
u=k \text { on } \Gamma_{t}, \text { where } \Gamma_{t}=\{x \in \Omega: \operatorname{dist}(x, \partial \Omega)=t\}, \tag{1.3}
\end{equation*}
$$

[^0]then $\Omega$ must be a ball (see [8, 9, 6] and [11]).
Condition (1.3) was first studied in [8] (see also [9] and [5] for further developments), motivated by an investigation on time-invariant level surfaces of a nonlinear non-degenerate fast diffusion equation (tailored upon the heat equation), and was used to extend to nonlinear equations the symmetry results obtained in [7] for the heat equation. The proof still hinges on the method of moving planes, that can be applied in a much simplified manner, since the overdetermination in (1.3) takes place inside $\Omega$. Under slightly different assumptions and by a different proof - still based on the method of moving planes - a similar result was obtained in [11] independently.

The evident similarity between the two problems arouses a natural question: is condition (1.3) weaker or stronger than (1.2)?

As pointed out in [6], (1.3) seems to be weaker than (1.2), as explained by the following two observations: (i) as (1.3) does not imply (1.2), the latter can be seen as the limit of a sequence of conditions of type (1.3) with $k=k_{n}$ and $t=t_{n}$ and $k_{n}$ and $t_{n}$ vanishing as $n \rightarrow \infty$; (ii) as (1.2) does not imply (1.3) either, if $u$ satisfies (1.1)-(1.2), then the oscillation of $u$ on a surface parallel to the boundary becomes smaller than usual, the closer the surface is to $\partial \Omega$. More precisely, if $u \in C^{1}(\bar{\Omega})$, by a Taylor expansion argument, it is easy to verify that

$$
\begin{equation*}
\max _{\Gamma_{t}} u-\min _{\Gamma_{t}} u=o(t) \text { as } t \rightarrow 0 \tag{1.4}
\end{equation*}
$$

- that becomes a $O\left(t^{2}\right)$ as $t \rightarrow \infty$ when $u \in C^{2}(\bar{\Omega})$.

This remark suggests the possibility that Serrin's symmetry result may be obtained by stability in the following way: first, for the solution $u$ of the torsion problem (1.1) prove the estimate

$$
\begin{equation*}
r_{e}-r_{i} \leq C_{t}\left(\max _{\Gamma_{t}} u-\min _{\Gamma_{t}} u\right) \tag{1.5}
\end{equation*}
$$

for some constant $C_{t}$ depending on $t$, where $r_{e}$ and $r_{i}$ are the radii of an annulus containing $\partial \Omega$; secondly, if in addition $u_{\nu}$ is constant on $\partial \Omega$, show that

$$
\max _{\Gamma_{t}} u-\min _{\Gamma_{t}} u=o\left(C_{t}\right) \text { as } t \rightarrow 0^{+}
$$

In the same spirit of (1.5), based on [1], in [6] we proved an estimate that quantifies the radial symmetry of $\Omega$ in terms of the following quantity:

$$
\begin{equation*}
[u]_{\Gamma_{t}}=\sup _{\substack{z, w \in \Gamma_{t} \\ z \neq w}} \frac{|u(z)-u(w)|}{|z-w|} \tag{1.6}
\end{equation*}
$$

In fact, it was proved that there exist two constants $\varepsilon, C_{t}>0$ such that, if $[u]_{\Gamma_{t}} \leq \varepsilon$, then there are two concentric balls $B_{r_{i}}$ and $B_{r_{e}}$ such that

$$
\begin{equation*}
B_{r_{i}} \subset \Omega \subset B_{r_{e}} \quad \text { and } \quad r_{e}-r_{i} \leq C_{t}[u]_{\Gamma_{t}} \tag{1.7}
\end{equation*}
$$

The constant $C_{t}$ only depends on $t, N$, the regularity of $\partial \Omega$ and the diameter of $\Omega$.

The calculations in [6] imply that $C_{t}$ blows-up exponentially as $t$ tends to 0 , which is too fast for our purposes, since $[u]_{\Gamma_{t}}$ cannot vanish faster than $t^{2}$, when (1.2) holds. The exponential dependence of $C_{t}$ on $t$ is due to the method of proof we employed, which is based on the idea of refining
the method of moving planes from a quantitative point of view. As that method is based on the maximum (or comparison) principle, its quantitative counterpart is based on Harnack's inequality and some quantitative versions of Hopf's boundary lemma. The exponential dependence of the constant involved in Harnack's inequality leads to that of $C_{t}$. Recent (unpublished) calculations, based on more refined versions of Harnack's inequality, show that the growth rate of $C_{t}$ can be improved, but they are still inadequate to achieve our goal. Approaches to stability based on the ideas contained in [3] and [4] do not seem to work for problem (1.1)-(1.3).

In this note, we shall show that our scheme (i)-(ii) is successful, at least if the admissible domains are ellipses: in this case, the deviation from radial symmetry can be exactly computed in terms of the oscillation of $u$ on $\Gamma_{t}$. We obtain (1.5) with $C_{t}=O\left(t^{-1}\right)$ as $t \rightarrow 0^{+}$; thus, formula (1.4) yields the desired symmetry $[1$

## 2. SECtion 2

We begin by defining the three quantities that we shall exactly compute later on. Let $\Gamma$ be a $C^{1}$-regular closed simple plane curve and let $z(s)$, $s \in[0,|\Gamma|)$ be its parameterization by arc-lenght. For a function $u: \Gamma \rightarrow \mathbb{R}$, we will consider the seminorms

$$
\begin{equation*}
|u|_{\Gamma}=\sup _{\substack{0 \leq s, s^{\prime} \leq|\Gamma| \\ s \neq s^{\prime}}} \frac{\left|u(z(s))-u\left(z\left(s^{\prime}\right)\right)\right|}{\min \left(\left|s-s^{\prime}\right|,|\Gamma|-\left|s-s^{\prime}\right|\right)},[u]_{\Gamma}=\sup _{\substack{z, w \in \Gamma \\ z \neq w}} \frac{|u(z)-u(w)|}{|z-w|} \tag{2.1}
\end{equation*}
$$

and the oscillation

$$
\begin{equation*}
\underset{\Gamma}{\operatorname{osc}} u=\max _{\Gamma} u-\min _{\Gamma} u \tag{2.2}
\end{equation*}
$$

We now consider an ellipse

$$
E=\left\{z=(x, y) \in \mathbb{R}^{2}: \frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}<1\right\}
$$

with semi-axes $a$ and $b$ normalized by $a^{-2}+b^{-2}=1$, and let

$$
\begin{equation*}
\Gamma_{t}=\{z \in E: \operatorname{dist}(z, \partial E)=t\} \tag{2.3}
\end{equation*}
$$

be the curve parallel to $\partial E$ at distance $t ; \Gamma_{t}$ is still regular and simple if $t$ is smaller than the minimal radius of curvature of $\partial E$, that is for

$$
\begin{equation*}
0 \leq t<\frac{\min \left(a^{3}, b^{3}\right)}{2 a^{2} b^{2}} \tag{2.4}
\end{equation*}
$$

The solution $u$ of (1.1) is clearly given by

$$
\begin{equation*}
u(x, y)=1-\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}} \tag{2.5}
\end{equation*}
$$

Lemma 2.1. Let $u$ be given by (2.5) and let t satisfy (2.4). Then, we have:
(i) $|u|_{\Gamma_{t}}=|a-b| \frac{a+b}{a^{2} b^{2}} t$;
(ii) $[u]_{\Gamma_{t}}=|u|_{\Gamma_{t}}$;

[^1](iii) $\underset{\Gamma_{t}}{\operatorname{osc}} u=|a-b| \frac{a+b}{a^{2} b^{2}}\left(\frac{2 a b}{a+b}-t\right) t$.

Proof. The standard parametrization of $\partial E$ is

$$
\gamma(\theta)=(a \cos \theta, b \sin \theta), \theta \in[0,2 \pi]
$$

thus,

$$
\Gamma_{t}=\left\{\gamma(\theta)-t J \frac{\gamma^{\prime}(\theta)}{\left|\gamma^{\prime}(\theta)\right|}: \theta \in[0,2 \pi)\right\}
$$

where $J$ is the rotation matrix

$$
\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)
$$

so that the outward unit normal is

$$
\nu(\theta)=J \frac{\gamma^{\prime}(\theta)}{\left|\gamma^{\prime}(\theta)\right|}
$$

(i) The mean value theorem then tells us that

$$
\begin{equation*}
\frac{\left|u(z(s))-u\left(z\left(s^{\prime}\right)\right)\right|}{\min \left(\left|s-s^{\prime}\right|,\left|\Gamma_{t}\right|-\left|s-s^{\prime}\right|\right)}=\left|\left\langle D u(z(\sigma)), z^{\prime}(\sigma)\right\rangle\right| \tag{2.6}
\end{equation*}
$$

for some $\sigma \in\left[0,\left|\Gamma_{t}\right|\right]$. Since $\Gamma_{t}$ is parallel to $\partial E$, we have

$$
z^{\prime}(\sigma)=\frac{\gamma^{\prime}(\theta(\sigma))}{\left|\gamma^{\prime}(\theta(\sigma))\right|}
$$

where $\theta(\sigma)$ is such that

$$
z(\sigma)=\gamma(\theta(\sigma))-t \nu(\theta)
$$

By (2.5), we have that

$$
\left|\left\langle D u(z(\sigma)), z^{\prime}(\sigma)\right\rangle\right|=2\left|\left\langle A z(\sigma), z^{\prime}(\sigma)\right\rangle\right| \quad \text { with } \quad A=\left(\begin{array}{cc}
a^{-2} & 0 \\
0 & b^{-2}
\end{array}\right)
$$

and hence

$$
\left|\left\langle D u(z(\sigma)), z^{\prime}(\sigma)\right\rangle\right|=2\left|\frac{\left\langle A \gamma(\theta), \gamma^{\prime}(\theta)\right\rangle}{\left|\gamma^{\prime}(\theta)\right|}-t \frac{\left\langle A J \gamma^{\prime}(\theta), \gamma^{\prime}(\theta)\right\rangle}{\left|\gamma^{\prime}(\theta)\right|^{2}}\right|
$$

with $\theta=\theta(\sigma)$.
Straightforward computations give:

$$
\begin{aligned}
& \gamma^{\prime}(\theta)=(-a \sin \theta, b \cos \theta), \quad\left|\gamma^{\prime}(\theta)\right|=\sqrt{a^{2} \sin ^{2} \theta+b^{2} \cos ^{2} \theta} \\
& \left\langle A \gamma(\theta), \gamma^{\prime}(\theta)\right\rangle=0, \quad\left\langle A J \gamma^{\prime}(\theta), \gamma^{\prime}(\theta)\right\rangle=\frac{\left|a^{2}-b^{2}\right|}{a b} \sin \theta \cos \theta
\end{aligned}
$$

Therefore,

$$
\left|\left\langle D u(z(\sigma)) \cdot z^{\prime}(\sigma)\right\rangle\right|=\frac{\left|a^{2}-b^{2}\right|}{a b} \frac{2|\tan \theta|}{a^{2} \tan ^{2} \theta+b^{2}} t
$$

this expression achieves its maximum if $|\tan \theta|=b / a$, that gives:

$$
\max _{0 \leq \sigma \leq\left|\Gamma_{t}\right|}\left|\left\langle D u(z(\sigma)), z^{\prime}(\sigma)\right\rangle\right|=\left|a^{-2}-b^{-2}\right| t
$$

¿From (2.6) we conclude.
(ii) By a symmetry argument, we can always assume that $[u]_{\Gamma_{t}}$ is attained for points $z$ and $w$ (that may possibly coincide) in the first quadrant of the cartesian plane.

Now, suppose that the value $[u]_{\Gamma_{t}}$ is attained for two points $z, w \in \Gamma_{t}$ with $z \neq w$. Let $s \rightarrow z(s) \in \Gamma_{t}$ be a parametrization by arclength of $\Gamma_{t}$ such that $z(0)=z$ and let $\omega=z^{\prime}(0)$ be the tangent unit vector to $\Gamma_{t}$ at $z$. The function defined by

$$
f(s)=\frac{u(z(s))-u(w)}{|z(s)-w|}
$$

has a relative maximum at $s=0$ and hence $f^{\prime}(0)=0$; thus,

$$
\frac{\langle D u(z), \omega\rangle}{|z-w|}=\frac{u(z)-u(w)}{|z-w|} \frac{\langle z-w, \omega\rangle}{|z-w|^{2}}
$$

Therefore, since $\langle z-w, \omega\rangle \neq 0$, we have that

$$
[u]_{\Gamma_{t}}=\frac{\langle D u(z), \omega\rangle}{\langle z-w, \omega\rangle}|z-w|,
$$

that gives a contradiction, since the right-hand side increases with $z$ if the angle between $z-w$ and $\omega$ decreases.

As a consequence, we infer that

$$
[u]_{\Gamma_{t}}=\lim _{n \rightarrow \infty} \frac{u\left(z_{n}\right)-u\left(w_{n}\right)}{\left|z_{n}-w_{n}\right|} \text { where } z_{n}, w_{n} \in \Gamma_{t} \text { and }\left|z_{n}-w_{n}\right| \rightarrow 0
$$

Thus, by compactness, we can find a point $z \in \Gamma_{t}$ such that

$$
[u]_{\Gamma_{t}}=\langle D u(z), \omega\rangle
$$

where $\omega$ is the tangent unit vector to $\Gamma_{t}$ at $z$.
It is clear now that $[u]_{\Gamma_{t}}=|u|_{\Gamma_{t}}$.
(iii) If (2.4) holds, the maximum and minimum of $u$ on $\Gamma_{t}$ are attained at the points on $\Gamma_{t}$ whose projections on $\partial E$ respectively maximize and minimize $|D u|$ on $\partial E$. Thus, (iii) follows at once.

In fact, for a point $z=\gamma(\theta)-t \nu(\theta)$ on $\Gamma_{t}$, calculations give that

$$
\begin{aligned}
u(z)= & 1-\langle A \gamma(\theta), \gamma(\theta)\rangle+2 t\langle A \gamma(\theta), \nu(\theta)\rangle-t^{2}\langle A \nu(\theta), \nu(\theta)\rangle= \\
& 2 t\langle A \gamma(\theta), \nu(\theta)\rangle-t^{2}\langle A \nu(\theta), \nu(\theta)\rangle
\end{aligned}
$$

where

$$
\begin{aligned}
& \langle A \gamma(\theta), \nu(\theta)\rangle=\frac{1}{a b} \sqrt{b^{2} \cos ^{2} \theta+a^{2} \sin ^{2} \theta} \\
& \langle A \nu(\theta), \nu(\theta)\rangle=\frac{1}{a^{2} b^{2}} \frac{b^{4} \cos ^{2} \theta+a^{4} \sin ^{2} \theta}{b^{2} \cos ^{2} \theta+a^{2} \sin ^{2} \theta}
\end{aligned}
$$

so that, by the substitution $\xi=\sqrt{b^{2} \cos ^{2} \theta+a^{2} \sin ^{2} \theta}$, we obtain that

$$
u(z)=\frac{2 t}{a b} \xi+\frac{t^{2}}{\xi^{2}}-\left(a^{-2}+b^{-2}\right) t^{2}
$$

Since (2.4) holds, this function is respectively maximal or minimal when $\xi=\min (a, b)$ or $\max (a, b)$.

Therefore, for an ellipse $E$, 6][Theorem 1.1] can be stated as follows, together with two analogues.

Theorem 2.2. Let $u$ be the solution of (1.1) in an ellipse $E$ of semi-axes a and $b$. Let $\Gamma_{t}$ be the curve (2.3) parallel to $\partial E$ at distance $t$ satisfying (2.4).

Then, there are two concentric balls $B_{r_{i}}$ and $B_{r_{e}}$ such that $B_{r_{i}} \subset E \subset B_{r_{e}}$ and

$$
\begin{gathered}
r_{e}-r_{i}=\frac{1}{t} \frac{a^{2} b^{2}}{a+b}|u|_{\Gamma_{t}} ; \quad r_{e}-r_{i}=\frac{1}{t} \frac{a^{2} b^{2}}{a+b}[u]_{\Gamma_{t}} ; \\
r_{e}-r_{i}=\frac{1}{t} \frac{a^{2} b^{2}}{a+b} \underset{\Gamma_{t}}{\operatorname{Osc} u .}
\end{gathered}
$$

Proof. The largest ball contained in $E$ and the smallest ball containing $E$ are centered at the origin and have radii $\min (a, b)$ and $\max (a, b)$, respectively; hence, $r_{e}-r_{i}=|a-b|$ and the desired formulas follow from Lemma 2.1.

Now, we turn to Serrin problem (1.1)-(1.2). The following lemma holds for quite general domains in general dimension .
Lemma 2.3. Let $\Omega \subset \mathbb{R}^{N}$ be a bounded domain with boundary of class $C^{2}$ and let $u \in C^{1}(\bar{\Omega}) \cap C^{2}(\Omega)$ be a solution of (1.1) satisfying (1.2).

Then

$$
\underset{\Gamma_{t}}{\operatorname{osc} u}=o(t) \text { as } t \rightarrow 0^{+} .
$$

If, in addition, $u \in C^{2}(\bar{\Omega})$, then

$$
[u]_{\Gamma_{t}} \text { and }|u|_{\Gamma_{t}}=o(t) \text { as } t \rightarrow 0^{+} .
$$

Proof. Let $z$ and $z^{\prime} \in \Gamma_{t}$ points at which $u$ attains its maximum and minimum, respectively (for notational semplicity, we do not indicate their dependence on $t$ ). If $t$ is sufficiently small, they have unique projections, say $\gamma$ and $\gamma^{\prime}$, on $\partial \Omega$, so that we can write that $z=\gamma-t \nu(\gamma)$ and $z=\gamma^{\prime}-t \nu\left(\gamma^{\prime}\right)$.

Since both $u$ and $u_{\nu}$ are constant on $\partial \Omega$, Taylor's formula gives:

$$
u(z)-u\left(z^{\prime}\right)=\int_{0}^{t}\left[\left\langle D u\left(\gamma^{\prime}-\tau \nu\left(\gamma^{\prime}\right)\right), \nu\left(\gamma^{\prime}\right)\right\rangle-\langle D u(\gamma-\tau \nu(\gamma)), \nu(\gamma)\rangle\right] d \tau
$$

By the (uniform) continuity of the first derivatives of $u$ (and the normals), the right-hand side of the last identity is a $o(t)$ as $t \rightarrow 0^{+}$.

We shall prove the second part of the theorem only for the semi-norm $[u]_{\Gamma_{t}}$, since that for $|u|_{\Gamma_{t}}$ runs similarly.

Let $s$ and $s^{\prime} \in\left[0,\left|\Gamma_{t}\right|\right]$ attain the first supremum in (2.1); we apply (2.6) and obtain that

$$
\frac{\left|u(z(s))-u\left(z\left(s^{\prime}\right)\right)\right|}{\min \left(\left|s-s^{\prime}\right|,\left|\Gamma_{t}\right|-\left|s-s^{\prime}\right|\right)}=\left|\left\langle D u(z(\sigma)), z^{\prime}(\sigma)\right\rangle\right|,
$$

for some $\sigma \in\left[0,\left|\Gamma_{t}\right|\right)$. Let $\gamma \in \partial \Omega$ be the projection of the point $z=z(\sigma)$ on $\partial \Omega$, that is $z=\gamma-t \nu(\gamma)$.

Since $\partial \Omega$ and $\Gamma_{t}$ are parallel, the tangent unit vector $\tau(\gamma)$ to the curve $\sigma \mapsto \gamma(\sigma) \in \partial \Omega$ at $\gamma$ equals the tangent unit vector $\tau(z)$ to the curve $\sigma \mapsto z(\sigma) \in \Gamma_{t}$ at $z$; the same occurs for the corresponding normal unit vectors $\nu(\gamma)$ and $\nu(z)$.

It is clear that $\langle D u(\gamma), \tau(\gamma)\rangle=0$ and, since $u \in C^{2}(\bar{\Omega})$, by differentiating (1.2), we also have that $\left\langle D^{2} u(\gamma) \nu(\gamma), \tau(\gamma)\right\rangle=0$; thus, by Taylor's formula,
we obtain that

$$
\begin{aligned}
\left\langle D u(z(\sigma)), z^{\prime}(\sigma)\right\rangle= & \langle D u(\gamma), \tau(\gamma)\rangle-t\left\langle D^{2} u(\gamma) \nu(\gamma), \tau(\gamma)\right\rangle+R\left(s, s^{\prime}, t\right)= \\
& R\left(s, s^{\prime}, t\right)
\end{aligned}
$$

Since the second derivatives of $u$ are uniformly continuous on $\bar{\Omega}$, we conclude that the remainder term $R\left(s, s^{\prime}, t\right)$ is a $o(t)$ as $t \rightarrow 0^{+}$.

Theorem 2.4. Let $E$ be an ellipse of semi-axes $a$ and $b$ and assume that in $E$ there exists a solution $u$ of (1.1) satisfying (1.2).

Then $a=b$, that is $E$ is a ball and $u$ is radially symmetric.
Proof. Theorem 2.2 and Lemma 2.3 in any case yield that

$$
|a-b|=o(1) \text { as } t \rightarrow 0^{+}
$$

which implies the assertion.

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[^0]:    1991 Mathematics Subject Classification. Primary 35B06, 35J05, 35J61; Secondary 35B35, 35B09.

    Key words and phrases. Serrin's problem, Parallel surfaces, overdetermined problems, method of moving planes, stability.

[^1]:    ${ }^{1}$ Of course, in this very special case, there is a trivial proof of symmetry, but this is not the point.

