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On a multidimensional model for the codiffusion of isotopes: existence and uniqueness

E. Comparini[a*] and M. Ughi[b]

The paper deals with the existence and uniqueness of classical solutions of the homogeneous Neumann problem for a class of parabolic-hyperbolic system of partial differential equations in n dimensions. The problem arises from a model of the diffusion of N species of radioactive isotopes of the same element.

Keywords: isotopes; diffusion; parabolic-hyperbolic systems.

1. Introduction

In this paper we consider the existence of classical solutions of the following problem:

$$\begin{cases} c_{it} = \operatorname{div}\left(\dfrac{c_i}{c}\nabla c\right) + \displaystyle\sum_{j=1}^{N} \Lambda_{ij} c_j, \quad i = 1, \dots, N, & \text{in } Q_T = \Omega \times (0, T), \\ c = \sum_{j=1}^{N} c_j, & \\ \dfrac{c_i}{c}\nabla c \cdot \mathbf{n} = 0, & \text{in } \Gamma_T = \partial\Omega \times (0, T), \\ c_i(\mathbf{x}, 0) = c_{i0}(\mathbf{x}), \quad i = 1, \dots, N, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded region of $\mathbb{R}^n$, with regular boundary $\partial\Omega$, $\mathbf{n}$ being the outer normal to $\partial\Omega$, and Λ_{ij} are the elements of the constant matrix Λ in the decay law:

$$\dot{c}_i = \sum_{j=1}^{N} \Lambda_{ij} c_j, \quad i = 1, \dots, N \tag{1.2}$$

The problem comes from a model of diffusion of isotopes of the same element, possibly radioactive, in which the flux of the $i-th$ isotope, whose concentration is c_i, depends mainly on the gradient of the total concentration, c, of the element, in a relative percentage $\frac{c_i}{c}$.

The physical motivation of the model is presented in [7], together with a precise study of the one-dimensional case, i.e. $n = 1$. Still for $n = 1$ the qualitative and asymptotic behaviour of the solution is presented in various paper ([10]-[12]), however the method used there is strictly one-dimensional. Here we consider the multidimensional case and we remark at once that in order to have classical solutions we need c_0 to be strictly positive. Therefore we will consider the following assumptions on the data, which are reasonable from a physical point of view:

[a] *Dipartimento di Matematica e Informatica "Ulisse Dini", Università di Firenze, Viale Morgagni 67/A, I-50134 Firenze (Italy), elena.comparini@math.unifi.it*
[b] *Dipartimento di Matematica e Geoscienze, Università di Trieste, V. Valerio 12/b, I-34127 Trieste (Italy), ughi@units.it*
[*] *Correspondence to: Dipartimento di Matematica e Informatica "Ulisse Dini", Università di Firenze, Viale Morgagni 67/A, I-50134 Firenze (Italy). E-mail: elena.comparini@math.unifi.it*

H1) $c_{i0} \in C^{2+\alpha}(\overline{\Omega}),\ \alpha > 0,\ i = 1, ..., N, \quad 0 \le c_{i0} \le k, \qquad c_0 = \sum_{i=1}^{N} c_{i0} \ge k_0 > 0,\ \nabla c_0 \cdot \mathbf{n} = 0$ on $\partial\Omega$,

H2) positivity property of the constant matrix Λ: if $c_{i0} \ge 0$, then $c_i(t) \ge 0, \quad i = 1, ..., N$,

where we have used the notations of [16].
The positivity assumption **H2** is equivalent to assuming that the region $V = \{\mathbf{y} \in \mathbb{R}^N : y_i \ge 0,\ i = 1, ..., N\}$ is invariant for the flux generated by the vector field $\Lambda\mathbf{y}$, that is to the condition $\Lambda\mathbf{y} \cdot \mathbf{n} \ge 0,\ \forall \mathbf{y} \in \partial V$, where $\mathbf{n}$ is the interior normal to ∂V in $\mathbf{y}$. Hence one has to require that all the non-diagonal elements of Λ are non negative (i.e. $\Lambda_{ij} \ge 0\ \forall i \ne j,\ i, j = 1, ..., N$). For a set of isotopes of the same element this assumption is very reasonable from a physical point of view. However one could consider also different linear fields, e.g. in some linear models of population dynamics, for which the positivity assumption holds only up a finite positive time, at which one of the species estinguishes. In this case the results obtained hereafter will hold up to the estinction time.
Prolems somewhat similar to the one in hand had been considered since the papers [6], [15] and [23], see also[2] [3], [4],[6] [19], [20], [21].
Some interesting qualitative properties of the solution of this problem such as localization and asymptotic behaviour have been investigated in [13].
The existence and uniqueness of the complete multidimensional model in which one takes into account also the dependence of the flux of c_i on its gradient, so that the final system is a parabolic one, was proved in [9]. More precisely, in the complete physical model the flux of c_i is given, after a suitable scaling, by $-\epsilon\nabla c_i - \frac{c_i}{c}\nabla c$, with $\epsilon > 0$, so that it is quite reasonable to look at the present model (1.1) as the limit of the complete one as $\epsilon \to 0$ and hence look for weak solutions via the vanishing viscosity method, which in this case would have a precise physical meaning. For a set of stable isotopes, i.e. $\Lambda = 0$, this can be proved by means of the results of [5] (see also [7], Thm.5.2), while in the general case it is an open problem. Let us mention that the numerical simulations for $n = 1$ confirm the convergence in very general assumptions (see [8]).
Let us also mention that for stable isotopes one can prove existence of weak solutions also relaxing the assumption of strict positivity on the total initial concentration c_0. However in this case one has a sort of loss of regularity for the single concentrastion $c_i(\mathbf{x}, t),\ t > 0$, in the sense that there are smooth initial non negative data c_{i0}, with the total c_0 not everywhere positive for which $c_i(\mathbf{x}, t)$ are discontinuous for $t > 0$ (see [7], Prop.5.1, for a one-dimensional example which can be easily generalized to the multidimensional case, see also Remark 2.1 at the end of Sec.2).
In Section 2 we will state the problem and find a priori estimates, in Section 3 we will prove the existence theorem by means of a fixed point argument, and in Section 4 the uniqueness of the solution will be proved.

2. Statement of the problem and a priori estimates

From assumption **H2** on the matrix Λ, we have for the solution $\mathbf{Y}(t, \mathbf{Y}_0)$, $\mathbf{Y} = (y_1, ..., y_N)$, $\mathbf{Y}_0 = (y_{1\,0}, ..., y_{N\,0})$ of the ODE problem

$$\dot{\mathbf{Y}} = \Lambda\mathbf{Y}, \quad \mathbf{Y}(0) = \mathbf{Y}_0, \qquad \mathbf{Y} \in \mathbb{R}^N, \tag{2.1}$$

that if $y_{i0} \ge 0,\ i = 1, ..., N$ and $y_0 = \sum_{i=1}^{N} y_{i0} \ge k_0 > 0$,
then, for any given $T > 0$:

$$y_i(t, \mathbf{Y}_0) \ge 0,\ i = 1, ..., N,\ \sum_{i=1}^{N} y_i(t, \mathbf{Y}_0) \ge \bar{k}_0 > 0,\ 0 \le t \le T. \tag{2.2}$$

k_0, $\bar{k}_0$ positive costants.
Therefore we can define

$$y = \sum_{i=1}^{N} y_i, \qquad R_i = \frac{y_i}{y}, \qquad i = 1, ..., N-1, \quad 0 \le t \le T, \tag{2.3}$$

and the above functions are solutions of the ODE:

$$\begin{cases} \dot{R}_i = P_i(\mathbf{R}), \quad \mathbf{R} = (R_1, ..., R_{N-1}), \quad i = 1, ..., N-1, \\ \dot{y} = y b(\mathbf{R}) = y\left(\beta_0 + \sum_{i=1}^{N-1} \beta_i R_i\right), \\ R_i(0) = \dfrac{y_{i0}}{y_0} = R_{i0}, \quad y(0) = y_0 = \displaystyle\sum_{i=1}^{N} y_{i0}. \end{cases} \tag{2.4}$$

The $P_i(\mathbf{R})$ are polynomial at most of second degree with constant coefficients depending on the element of Λ, Λ_{ij}, and β_0, β_j, $j = 1, ..., N-1$ are also constant depending on Λ_{ij}:

$$P_i = \Lambda_{iN} + \sum_{j=1}^{N-1} (\Lambda_{ij} - \Lambda_{iN}) R_j - R_i \sum_{j=1}^{N-1} \Lambda_{jN} + R_i \left[\sum_{j=1}^{N-1} \left(\sum_{k=1}^{N} (\Lambda_{kN} - \Lambda_{kj}) \right) R_j \right],$$

$$b = \sum_{k=1}^{N} \Lambda_{kN} + \sum_{j=1}^{N-1} R_j \sum_{k=1}^{N} (\Lambda_{kj} - \Lambda_{kN}).$$

Let us remark here that there are physically relevant examples in which $b \equiv \beta_0$ and hence the equation for y is independent of the equations for $\mathbf{R}$

- **Ex. 1** A set of stable isotopes, i.e. $\Lambda = 0$ and $b = \beta_0 = 0$ (e.g. (Cl^{37}, Cl^{35})).
- **Ex. 2** A set of radioactive isotopes that decays out of the element with the same decay coefficient γ, e.g. the couple (U^{235}, U^{238}).
  In this case $\Lambda = -\gamma I$, and $b = -\gamma$, where I is the identity matrix.
- **Ex. 3** A chain of N isotopes such that the i^{th} one decays into the $(i+1)^{th} one$, for $i = 1, ..., N-1$ and the N^{th} one is stable. We have again $\beta_0 = 0$, with the matrix Λ defined by:

$$\begin{cases} \dot{y}_1 = -\gamma_1 y_1 \\ \dot{y}_i = \gamma_{i-1} y_{i-1} - \gamma_i y_i, & i = 2, ..., N-1, \\ \dot{y}_N = \gamma_{N-1} y_{N-1}, \end{cases}$$

  with $\gamma_i > 0$, $i = 1, ..., N-1$.

On the other hand there are examples for which b is not constant, such as the following:

- **Ex. 4** A chain of isotopes similar to the one of Ex.3, but the N^{th} isotope decays out of the element (e.g.the couple (U^{234}, U^{238})) i.e. the sytem is:

$$\begin{cases} \dot{y}_1 = -\gamma_1 y_1 \\ \dot{y}_i = \gamma_{i-1} y_{i-1} - \gamma_i y_i, & i = 2, ..., N, \end{cases}$$

  with $\gamma_i > 0$, $i = 1, ..., N$.

Let us also remark that from (2.2) and the definition of R_i in (2.3) we have, in assumption H1), H2), the following estimate:

$$0 \le R_i \le 1, \; 0 \le \sum_{i=1}^{N-1} R_i \le 1, \quad i = 1, ..., N-1, \; 0 \le t \le T_0. \tag{2.5}$$

Returning to the PDE problem (1.1) and defining $r_i = \dfrac{c_i}{c}$, $i = 1, ..., N-1$, $\mathbf{r} = (r_1, ..., r_{N-1})$, we have that the total concentration c satisfies the strictly parabolic linear problem:

$$\begin{cases} c_t = \Delta c + c b(\mathbf{r}), & \text{in } Q_T, \\ c(\mathbf{x}, 0) = c_0(\mathbf{x}) = \sum_{j=1}^{N} c_{i0} & \text{in } \overline{\Omega}, \\ \nabla c \cdot \mathbf{n} = 0, & \text{in } \Gamma_T. \end{cases} \tag{2.6}$$

Therefore for any bounded b and c_0, say

$$|b| \le B, \; 0 < k_0 \le c_0 \le K_0, \tag{2.7}$$

we have that:

$$0 < \gamma \le k_0 e^{-Bt} \le c(\mathbf{x}, t) \le K_0 e^{Bt}, \quad t \in (0, T). \tag{2.8}$$

Remark that for $\mathbf{r}$ satisfying (2.5) we have

$$B = |\beta_0| + \max_{j=1,...,N} |\beta_j|, \tag{2.9}$$

i.e. B is a constant depending only on $\Lambda_{i,j}$.
Moreover from the classical theory of linear parabolic equation with regular coefficients we have that, if the coefficients, in our case $b(\mathbf{r})$, are $C^{\alpha, \frac{\alpha}{2}}$, and the initial datum is $C^{2+\alpha}(\Omega)$, $\alpha \in (0,1)$, then c is $C^{2+\alpha, 1+\frac{\alpha}{2}}$ and we have the estimate (see [22], Thm.5.3, IV, Sec.5)

$$\begin{cases} ||c||_{2+\alpha,1+\frac{\alpha}{2}} \le \psi\left(||b||_{\alpha,\frac{\alpha}{2}}\right)||c_0||_{2+\alpha} \\ ||b||_{\alpha,\frac{\alpha}{2}} \le |\beta_0| + (N-1)\max_{j=1,\dots,N}|\beta_j|\,||\mathbf{r}||_{\alpha,\frac{\alpha}{2}}, \end{cases} \tag{2.10}$$

where $\psi(\xi)$ is a positive increasing function of ξ, $\psi(0)$ being the constant valid for the heat equation.
As for $\mathbf{r}$, for any smooth positive c, it is solution of the hyperbolic semilinear system:

$$\begin{cases} r_{it} + \nabla r_i \cdot \mathbf{f} = P_i(\mathbf{r}), & i = 1, \dots, N-1, \\ r_i(\mathbf{x}, 0) = \dfrac{c_{i0}(\mathbf{x})}{c_0(\mathbf{x})} = r_{i0}, & i = 1, \dots, N-1, \end{cases} \tag{2.11}$$

with $\mathbf{f} = -\dfrac{\nabla c}{c}$.
Since (2.11) is a very special form of "symmetric hyperbolic" system (see [16], VII, Sec.7.32), one can construct its classical solutions by the method of the characteristics in a standard way. Namely we define the characteristic through $(\mathbf{z}, \tau)$, $\mathbf{z} \in \Omega$, $\tau \ge 0$, $\mathbf{X}(t; \mathbf{z}, \tau)$ as the solution of the ODE:

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X}, t), \qquad \mathbf{X}(\tau; \mathbf{z}, \tau) = \mathbf{z}. \tag{2.12}$$

Since we have homogeneous Neumann boundary conditions, and c is strictly positive in $\overline{Q}_T$, we have from the strong maximum principle for c that all the characteristics starting for $t = 0$ from the interior of Ω remain inside Ω for any time.
Since the evolution in time of $\mathbf{r}$ on any characteristic depends only on $P_i(\mathbf{r})$ and not on c, see (2.11), we can write the solution of (2.11) as

$$\mathbf{r}(\mathbf{x}, t) = \mathbf{R}\left(t; \mathbf{r}_0(\mathbf{X}(0; \mathbf{x}, t))\right). \tag{2.13}$$

From the theory of ODE systems (see e.g. [18], Thm.3.1,V) we have that, for any $c \in C^{2,1}(\overline{Q}_T)$, c positive, satisfying (2.8), there exists a unique $C^{1,1}$ solution of (2.12) for any time.
Moreover one has for the spatial and time derivative of $\mathbf{X}$, that $\mathbf{v}^i = \dfrac{\partial \mathbf{X}}{\partial z_i}$ is solution of the ODE system

$$\begin{cases} \dfrac{d\mathbf{v}^i}{dt} = \mathbf{J}_f \mathbf{v}^i & i = 1, \dots, n, \\ \mathbf{v}^i(\tau) = \mathbf{e}^i, \quad (\mathbf{e}^i)_j = \delta_{ij} & i, j = 1, \dots, n, \end{cases} \tag{2.14}$$

where $\mathbf{J}_f$ is the Jacobian matrix of $\mathbf{f}$ which in the present case is the symmetric matrix whose elements are:

$$\frac{\partial f_i}{\partial x_j} = -\frac{1}{c}\frac{\partial^2 c}{\partial x_i \partial x_j} + \frac{1}{c^2}\frac{\partial c}{\partial x_i}\frac{\partial c}{\partial x_j}. \tag{2.15}$$

Moreover

$$\frac{\partial X_i}{\partial \tau} + \sum_{j=1}^{N} f_j(\mathbf{z}, \tau)\frac{\partial X_i}{\partial z_j} = 0. \tag{2.16}$$

Therefore for $c \in C^{2,1}$ and $c \ge \gamma > 0$ we have

$$\begin{cases} \left|\dfrac{\partial f_i}{\partial x_j}\right| \le \dfrac{1}{\gamma}\max_{\overline{Q}_T}|D_x^2 c| + \dfrac{1}{\gamma^2}\max_{\overline{Q}_T}|D_x c|^2 = a \\ |f_i| \le \dfrac{1}{\gamma}\max_{\overline{Q}_T}|D_x c| = a_1. \end{cases} \tag{2.17}$$

From the Gronwall Lemma applied to (2.14) and (2.16) we have the following:

$$\begin{cases} \left|\dfrac{\partial X_i(t)}{\partial x_j}\right| \le e^{k_2 a|t-\tau|}, & i, j = 1, \dots, n \\ \left|\dfrac{\partial X_i(t)}{\partial \tau}\right| \le n a_1 e^{k_2 a|t-\tau|}, & i = 1, \dots, n, \end{cases} \tag{2.18}$$

where k_2 is a positive constant depending only on n.
Therefore by the explicit expression of $\mathbf{r}$, see (2.13), and the regularity of $\mathbf{R}$ and $\mathbf{r}_0$ we have for the derivative of $\mathbf{r}$ the estimate:

$$\begin{cases} \left|\dfrac{\partial r_i}{\partial x_j}\right| \le k_3 e^{k_2 a t}, & i = 1, \dots, N-1, \ j = 1, \dots, n, \\ \left|\dfrac{\partial r_i)}{\partial t}\right| \le k_3\left(1 + a_1 e^{at}\right) & i = 1, \dots, N-1, \ t > 0, \end{cases} \tag{2.19}$$

where a and a_1 are given in (2.17) and k_3 is a positive constant depending on the data Λ, $||\nabla \mathbf{r}_0||$, n, N.
From (2.19) we get in $\overline{Q}_T$ the estimate:

$$||r_i||_{\alpha,\frac{\alpha}{2}} \leq 1 + k_4 e^{aT} + T^{1-\frac{\alpha}{2}} k_3 \left(1 + a_1 e^{aT}\right), \quad i = 1, ..., N-1, \tag{2.20}$$

where k_4 is again a constant depending on the data.

Remark 2.1

Let us remark that, if b is constant as for stable isotopes or for examples 2 and 3, the problem (2.6) for the total concentration c is decoupled from the problem (2.11) and has a good positive solution for any positive time for $c_0(\mathbf{x}) \geq 0$. Then the hyperbolic semilinear symmetric system (2.11) for $\mathbf{r}$ can be dealt with well known methods, see e.g. [16]). The linear dynamic defined by Λ influences only the behaviour of $\mathbf{r}$ along each characteristic, e.g. for stable isotopes system (2.11) is a set of $N-1$ indipendent linear homogeneous equations so that $\mathbf{r}$ is constant on each characteristic. In a similar way, as in [7] Sect.5, it is then possible to construct examples of the "loss of regularity" we mentioned in Sect.1.
A very simple one is given by the following assumptions:
$\Lambda = 0$, $\Omega = \{\mathbf{x} \in \mathbb{R}^n : -l_i < x_i < l_i, i = 1, ..., n\}$, l_i positive constants, $c_{i0}(\mathbf{x}) = h_i(x_1)c_0(x_1)$,
where $c_0(x_1)$ is a C^∞ function, symmetric with respect to $x_1 = 0$,
$c_0 \equiv 0$ in $[0, \delta]$, $0 < \delta < l_1$, $c_0 > 0$ in (δ, l_1), $\dfrac{\partial c_0}{\partial x_1} = 0$ for $x_1 = \pm l_1$
$h_1(x_1) = H(x_1)$, $h_i(x_1) = \gamma_i H(-x_1)$, $i = 2, ..., N$,
with $H(\xi)$ the Heaviside function and γ_i non negative constants with $\sum_{i=2}^N \gamma_i = 1$.
Then $c(\mathbf{x}, t) = c(x_1, t)$ is symmetric with respect to $x_1 = 0$ and strictly positive in Ω for $t > 0$ and by means of the results of [7] Sec.3 one can show that the solutions are
$c_i(\mathbf{x}, t) = h_i(x_1)c(x_1, t)$, $i = 1, ..., N$.
These functions have a jump across $x_1 = 0$, since $c(0, t) > 0$ for $t > 0$, although the initial data $c_{i0}(\mathbf{x})$ are smooth.

3. Existence of classical solutions

We will prove the following

Theorem 3.1 *In assumptions H1), H2) there exists a classical solution of the coupled problem (2.6)-(2.11)*

Proof. We will use a fixed point argument. Let us define the set $\mathcal{U}$ as

$$\mathcal{U} = \left\{ u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{Q}_T),\ u \geq \gamma > 0,\ \nabla u \cdot \mathbf{n}|_{\Gamma_T} = 0 \right\} \tag{3.1}$$

with the norm $||u||_{2+\alpha,1+\frac{\alpha}{2}}$.
For any $u \in \mathcal{U}$, $\mathbf{r}[u]$ is the solution of:

$$\begin{cases} r_{it} = \nabla r_i \cdot \dfrac{\nabla u}{u} + P_i(\mathbf{r}), \ i = 1, ..., N-1, & \text{in } Q_T, \\ r_i(\mathbf{x}, 0) = r_{i0}(\mathbf{x}), \quad i = 1, ..., N-1, & \text{in } \Omega. \end{cases} \tag{3.2}$$

Then we set $\mathcal{T}u = v$, with v solution of

$$\begin{cases} v_t = \Delta v + v b(\mathbf{r}[u]), & \text{in } Q_T, \\ v(\mathbf{x}, 0) = c_0(\mathbf{x}), & \text{in } \Omega, \\ \nabla v \cdot \mathbf{n} = 0, & \text{in } \Gamma_T. \end{cases} \tag{3.3}$$

From the results of Section 2 we have that $0 \leq r_i \leq 1$, $i = 1, ..., N-1$, and the norm $||r^i||_{\alpha,\frac{\alpha}{2}}$ is estimated as in (2.20) with u instead of c in (2.17). Therefore for any $u \in \mathcal{U}$, with $||u||_{2+\alpha,1+\frac{\alpha}{2}} \leq \rho$, ρ to be fixed later, we have that

$$||v||_{2+\alpha,1+\frac{\alpha}{2}} \leq \psi(k_5 + g(\rho, T))||c_0||_{2+\alpha}, \tag{3.4}$$

where k_5 is a positive constant depending only on the data and $g(\rho, T)$ is an increasing function in ρ and T such that $g(\rho, 0) = 0$ (see (2.20)).
Now we fix ρ as

$$\rho = 2\psi(k_5)\,||c_0||_{2+\alpha}, \tag{3.5}$$

so that ρ depends only on the data. Then we fix $T^* > 0$ such that:

$$\psi(k_5 + g(\rho, T)) \leq 2\psi(k_5), \qquad 0 \leq T \leq T^*. \tag{3.6}$$

Also T^* depends only on the data.
From (3.4)(3.5) we have for any u as above that:

$$||v||_{2+\alpha,1+\frac{\alpha}{2}} \leq 2\psi(k_5)||c_0||_{2+\alpha} = \rho, \qquad 0 \leq T \leq T^*.$$

This means that the operator $\mathcal{T}$ maps $B_\rho \subset \mathcal{U}$ in itself for any $T \leq T^*$.
Since T^* is fixed we can repeat the argument for any time.
Now we have to prove that the operator $\mathcal{T}$ is continuous in the topology indicated in (3.1); to do this we consider two elements of B_ρ, say u and v, and define $w = \mathcal{T}u - \mathcal{T}v$.
Then w is solution of the linear parabolic problem

$$\begin{cases} w_t = \Delta w + b(\mathbf{r}[u])w + h, & \text{in } Q_T, \\ w(\mathbf{x}, 0) = 0, & \text{in } \Omega, \\ \nabla w \cdot \mathbf{n} = 0, & \text{in } \Gamma_T, \end{cases} \tag{3.7}$$

where

$$h = (\mathcal{T}u) \sum_{j=1}^{N-1} \beta_j \left(r_j([u]) - r_j([v]) \right). \tag{3.8}$$

From the explicit form of r_j, see (2.13), we have in $\mathbf{z} \in \Omega$, $\tau > 0$ that:

$$\left(r_j([u]) - r_j([v]) \right)(\mathbf{z}, \tau) = R_j\left(\tau; \mathbf{r}_0(\mathbf{X}(0; \mathbf{z}, \tau)) \right) - R_j\left(\tau; \mathbf{r}_0(\tilde{\mathbf{X}}(0; \mathbf{z}, \tau)) \right), \tag{3.9}$$

where $\mathbf{X}$ and $\tilde{\mathbf{X}}$ are the characteristics from $\mathbf{z}, \tau$ determined respectively by the fields $\mathbf{f} = -\dfrac{\nabla u}{u}$ and $\tilde{\mathbf{f}} = -\dfrac{\nabla v}{v}$ by means of (2.12).
In view of the regularity of $\mathbf{R}$ and $\mathbf{r}_0$, to estimate the norm $||h||_{\alpha,\frac{\alpha}{2}}$ we consider the following:

$$\mathbf{Z}(t; \mathbf{z}, \tau) = \mathbf{X}(t; \mathbf{z}, \tau) - \tilde{\mathbf{X}}(t; \mathbf{z}, \tau), \;\; \mathbf{W}^i(t; \mathbf{z}, \tau) = \frac{\partial \mathbf{Z}}{\partial z_i}, \;\; \mathbf{U}^i(t; \mathbf{z}, \tau) = \frac{\partial \mathbf{Z}_i}{\partial \tau}, \qquad i = 1, \ldots, n. \tag{3.10}$$

We have that $\mathbf{Z}$ is solution of:

$$\begin{cases} \dot{\mathbf{Z}} = \mathbf{J}_{\mathbf{f}}(\mathbf{Z}) + \left[\dfrac{\nabla u}{uv}(u - v) + \dfrac{1}{v}(\nabla u - \nabla v) \right], \\ \mathbf{Z}(\tau; \mathbf{z}, \tau) = \mathbf{0} \end{cases} \tag{3.11}$$

From (2.17) and the Gronwall Lemma applied to the above system (3.11) we have for any u and v in B_ρ:

$$||\mathbf{Z}(0; \mathbf{z}, \tau)|| \leq k_6 \max_{\overline{Q}_T}(|u - v| + ||\nabla(u - v)||), \tag{3.12}$$

where k_6 is a positive constant depending on γ, ρ, T.
Consider now $\mathbf{W}^i$ in (3.10): $\mathbf{W}^i$ is solution of the linear system:

$$\begin{cases} \dot{\mathbf{W}}^i = \mathbf{J}_{\mathbf{f}}\mathbf{W}^i + (\mathbf{J}_{\mathbf{f}} - \mathbf{J}_{\tilde{\mathbf{f}}}) \dfrac{\partial \tilde{\mathbf{X}}}{\partial z_i}, \\ \mathbf{W}^i(\tau; \mathbf{z}, \tau) = \mathbf{0}. \end{cases} \tag{3.13}$$

Proceeding as before we then get

$$||\mathbf{W}^i(0; \mathbf{z}, \tau)|| \leq k_7 \max_{\overline{Q}_T}(|u - v| + ||\nabla(u - v)|| + ||D^2_x(u - v)||), \tag{3.14}$$

with $k_7 = k_7(\rho, T, \gamma)$.
Last for $\mathbf{U}^i$ (3.10) we have from (2.16) that

$$\frac{\partial \mathbf{Z}_i}{\partial \tau} = \sum_{j=1}^{n} \left(\tilde{f}_j(\mathbf{z}, \tau) \frac{\partial \tilde{X}_i}{\partial z_j} - f_j(\mathbf{z}, \tau) \frac{\partial X_i}{\partial z_j} \right).$$

Hence from the previous (3.14) and the definition of $\mathbf{f}, \tilde{\mathbf{f}}$ we have:

$$\left| \frac{\partial \mathbf{Z}_i}{\partial \tau} \right| \leq k_8(\rho, T, \gamma) \max_{\overline{Q}_T} \left(|u - v| + ||\nabla(u - v)|| + ||D^2_x(u - v)|| \right). \tag{3.15}$$

Hence we have for $u, v \in B_\rho \subset \mathcal{U}$ that h defined in (3.8) has a norm

$$||h||_{\alpha,\frac{\alpha}{2}} \leq k_9||u-v||_{2+\alpha,1+\frac{\alpha}{2}}, \tag{3.16}$$

where k_9 depends on ρ, T, γ and the data of the problem through (2.20).
From the already quoted results of [22] for the parabolic problem for w (3.7) we eventually have:

$$||\mathcal{T}u-\mathcal{T}v||_{2+\alpha,1+\frac{\alpha}{2}} \leq k_{10}||u-v||_{2+\alpha,1+\frac{\alpha}{2}}, \tag{3.17}$$

k_{10} depending on the data and on ρ.
Therefore the operator $\mathcal{T}$ is continuous and so we have proved the existence Theorem.

□

Let us remark that in the papers [19], [3] a similar problem is considered, but the equation corresponding to (2.6) is a porous media equation instead of a strictly parabolic one, hence yielding to a weak solution for $\mathbf{r}$.
In a similar way as above one can prove existence of classical solutions for the corresponding Cauchy problem.

4. Uniqueness of the solution

In the following Theorem we will prove that the classical solution of the coupled problem (2.6), (2.11) is unique.

Theorem 4.1 *Assuming hypotheses* **H1**, **H2**, *if there exists a classical solution of (2.6), (2.11), then it is unique* $\forall T > 0$.

Proof. Let us suppose that there exist two distinct solutions of (2.6), (2.11), that we will denote by $(c^I, \mathbf{r}^I)$, $(c^{II}, \mathbf{r}^{II})$.
Define $u = c^I - c^{II}$, $\mathbf{v} = \mathbf{r}^I - \mathbf{r}^{II}$, then we have that u and $\mathbf{v}$ satisfy the following equations:

$$u_t = \Delta u + b(\mathbf{r}^I)u + c^{II}\left(b(\mathbf{r}^I) - b(\mathbf{r}^{II})\right), \tag{4.1}$$

with $b(\mathbf{r}^I - b(\mathbf{r}^{II}) = \sum_{i=1}^{N-1}\beta_i(r_i^I - r_i^{II})$, β_i constant depending on Λ (see (2.6)), and

$$v_{it} = \nabla v_i \cdot \frac{\nabla c^{II}}{c^{II}} + \nabla r_i^I \cdot \left(\frac{\nabla c^I}{c^I} - \frac{\nabla c^{II}}{c^{II}}\right) + P_i(\mathbf{r}^I) - P_i(\mathbf{r}^{II}),\ \ i = 1, ..., N-1. \tag{4.2}$$

Multiplying (4.1) by u and (4.2) by v_i, and integrating on Q_T one obtains:

$$\begin{aligned}\int_\Omega \frac{u^2(\mathbf{x},t)}{2}\,d\mathbf{x} = \int_0^t\int_{\partial\Omega} u\nabla u\cdot\mathbf{n}\,d\mathbf{s}\,d\tau - \int_0^t\int_\Omega ||\nabla u||^2\,d\mathbf{x}\,d\tau + \\ \int_0^t\int_\Omega b(\mathbf{r}^I)u^2\,d\mathbf{x}\,d\tau + \int_0^t\int_\Omega c^{II}[b(\mathbf{r}^I) - b(\mathbf{r}^{II})]u\,d\mathbf{x}\,d\tau,\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\int_\Omega \frac{v_i^2(\mathbf{x},t)}{2}\,d\mathbf{x} = \int_0^t\int_\Omega \frac{\nabla c^{II}}{c^{II}}\cdot\nabla v_i\,v_i\,d\mathbf{x}\,d\tau + \\ \int_0^t\int_\Omega v_i\nabla r_i^I\cdot\left(\frac{\nabla c^I}{c^I} - \frac{\nabla c^{II}}{c^{II}}\right)d\mathbf{x}\,d\tau + \int_0^t\int_\Omega \left(P_i(\mathbf{r}^I) - P_i(\mathbf{r}^{II})\right)v_i\,d\mathbf{x}\,d\tau.\end{aligned} \tag{4.4}$$

where $\mathbf{n}$ denotes the outer normal to $\partial\Omega$.
An estimate on the terms at the right hand side of (4.3) gives

$$\frac{1}{2}\int_\Omega u^2(\mathbf{x},t)\,d\mathbf{x} \leq -\int_0^t\int_\Omega ||\nabla u||^2\,d\mathbf{x}\,d\tau + k\left(\int_0^t\int_\Omega u^2\,d\mathbf{x}\,d\tau + \int_0^t\int_\Omega ||\mathbf{v}||^2\,d\mathbf{x}\,d\tau\right). \tag{4.5}$$

Here and in the following we denote by k any real positive constant depending on the data.
In fact the term $\int_0^t\int_{\partial\Omega} u\nabla u\cdot\mathbf{n}\,d\mathbf{x}\,d\tau$ is null because of the Neumann boundary conditions; moreover, recalling that $0 \leq r_i^j \leq 1$, $i = 1, ..., N-1$, $j = I, II$, we have $|b(\mathbf{r}^I)| < k$ and $|b(\mathbf{r}^I) - b(\mathbf{r}^{II})| \leq \sum_{i=1}^{N-1}|\beta_i|\,|r_i^I - r_i^{II}| < k||\mathbf{v}||$, and c^{II} is a priori bounded.
Let us give an estimate of (4.4). Concerning the first integral at the right hand side of (4.4), denoting by $\mathbf{a}(\mathbf{x},t) = \dfrac{\nabla c^{II}}{c^{II}}$, we have

$$\int_0^t \int_\Omega \mathbf{a}(\mathbf{x},t)\cdot\nabla v_i\, v_i\, d\mathbf{x}\, d\tau = \int_0^t \int_\Omega \operatorname{div}\left(\mathbf{a}\frac{v_i^2}{2}\right)\, d\mathbf{x}\, d\tau - \sum_{j=1}^n \int_0^t \int_\Omega \frac{\partial a_j}{\partial x_j}\frac{v_i^2}{2}\, d\mathbf{x}\, d\tau =$$

$$\int_0^t \int_{\partial\Omega} \frac{v_i^2}{2}\mathbf{a}\cdot\mathbf{n}\, d\mathbf{x}\, d\tau - \sum_{j=1}^n \int_0^t \int_\Omega \frac{\partial a_j}{\partial x_j}\frac{v_i^2}{2}\, d\mathbf{x}\, d\tau \le k\int_0^t \int_\Omega \frac{v_i^2}{2}\, d\mathbf{x}\, d\tau,$$

where we used the Neumann condition on $\partial\Omega$ and we remark that $k \ge \sum_{j=1}^n \left|\frac{\partial a_j}{\partial x_j}\right|$, since $\frac{\partial a_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{c^{II}}\frac{\partial c^{II}}{\partial x_j}\right)$, $c^{II}\in C^2$ in $\overline{Q}_T$ and it is positive.
For the third integral in (4.4), recalling that the P_i are polynomial of second degree with constant coefficients and $0 \le r_i \le 1$, we have

$$\int_0^t \int_\Omega |P_i(\mathbf{r}^I) - P_i(\mathbf{r}^{II})|v_i\, d\mathbf{x}\, d\tau \le \int_0^t \int_\Omega \left|\sum_{j=1}^{N-1}\frac{\partial P_i}{\partial z_i}\right|_{\overline{\mathbf{z}}} |v_j v_i|\, d\mathbf{x}\, d\tau \le k||\mathbf{v}||^2,$$

with suitable $\overline{\mathbf{z}}$.
Let us consider now the coupling term in (4.4), and rewrite it in the form:

$$\int_0^t \int_\Omega v_i\nabla r_i^I\cdot\left(\frac{\nabla c^I}{c^I} - \frac{\nabla c^{II}}{c^{II}}\right)\, d\mathbf{x}\, d\tau = \int_0^t \int_\Omega v_i\nabla r_i^I\cdot\left(-\frac{\nabla c^I}{c^I c^{II}}u\right)\, d\mathbf{x}\, d\tau + \int_0^t \int_\Omega v_i\nabla r_i^I\cdot\frac{\nabla u}{c^{II}}\, d\mathbf{x}\, d\tau = I_1 + I_2.$$

Recalling that c and $\mathbf{r}$ are regular and $c \ge \overline{k}_0 > 0$, we have that

$$I_1 = -\sum_{j=1}^n \int_0^t \int_\Omega \frac{v_i}{c^I c^{II}}\frac{\partial r_i^I}{\partial x_j}\frac{\partial c^I}{\partial x_j}u\, d\mathbf{x}\, d\tau \le k\left[\int_0^t \int_\Omega v_i^2 d\mathbf{x}\, d\tau + \int_0^t \int_\Omega u^2\, d\mathbf{x}\, d\tau\right],$$

$$I_2 = \sum_{j=1}^n \int_0^t \int_\Omega v_i\frac{1}{c^{II}}\frac{\partial r_i^I}{\partial x_j}\frac{\partial u}{\partial x_j}\, d\mathbf{x}\, d\tau \le k\sum_{j=1}^n\left[\int_0^t \int_\Omega v_i^2 d\mathbf{x}\, d\tau + \int_0^t \int_\Omega \left(\frac{\partial u}{\partial x_j}\right)^2\, d\mathbf{x}\, d\tau\right].$$

In conclusion we obtain from (4.4) summing on $i = 1, ..., N-1$

$$\frac{1}{2}\int_\Omega ||\mathbf{v}(\mathbf{x},t)||^2\, d\mathbf{x} \le k\left(\int_0^t \int_\Omega u^2\, d\mathbf{x}\, d\tau + \int_0^t \int_\Omega ||\nabla u||^2\, d\mathbf{x}\, d\tau + \int_0^t \int_\Omega ||\mathbf{v}||^2 d\mathbf{x}\, d\tau\right), \tag{4.6}$$

that can be seen as a Gronwall differential inequality, defining

$$\eta(t) = \int_0^t \int_\Omega ||\mathbf{v}||^2\, d\mathbf{x}\, d\tau, \qquad \psi(t) = k\left(\int_0^t \int_\Omega u^2\, d\mathbf{x}\, d\tau + \int_0^t \int_\Omega ||\nabla u||^2\, d\mathbf{x}\, d\tau\right), \tag{4.7}$$

from which

$$0 \le \eta'(t) \le k\eta(t) + \psi(t), \qquad \eta(0) = 0,$$

and then

$$0 \le \eta(t) \le e^{kt}\int_0^t \psi(\tau)\, d\tau.$$

Remarking that $\psi(t)$ is monotone increasing w.r.t. t, we have

$$0 \le \eta(t) \le te^{kt}\psi(t). \tag{4.8}$$

Using the above estimate in (4.5) we obtain

$$\frac{1}{2}\int_\Omega u^2(\mathbf{x},t)\, d\mathbf{x} \le \left(k + k^2te^{kt}\right)\int_0^t \int_\Omega u^2\, d\mathbf{x}\, d\tau + \left(-1 + k^2te^{kt}\right)\int_0^t \int_\Omega ||\nabla u||^2\, d\mathbf{x}\, d\tau, \tag{4.9}$$

then it is possible to choose $T^* > 0$ such that

$$k^2T^*e^{kT^*} < 1,$$

in order to have $\forall t < T^*$

$$\int_\Omega u^2(\mathbf{x},t)\, d\mathbf{x} \le k\int_0^t \int_\Omega u^2\, d\mathbf{x}d\tau.$$

Recalling that $u(\mathbf{x}, 0) = 0$ we obtain finally from Gronwall lemma

$$\int_\Omega u^2(\mathbf{x}, t)\, d\mathbf{x} \equiv 0, \qquad \forall t < T^*,$$

from which, being $u \in C^2\left(\overline{Q}_T\right)$, we have

$$u \equiv 0, \quad \nabla u \equiv 0, \qquad \forall t < T^*.$$

Going back to (4.8) we obtain moreover that

$$\mathbf{v} \equiv 0, \qquad \forall t < T^*.$$

Let us remark that this proof can be repeated for the same problem with initial data assigned for $t = T^*$, being T^* a fixed value, so that we obtain

$$u \equiv 0, \quad \mathbf{v} \equiv 0, \qquad \forall t < T,$$

with arbitrary $T > 0$.

□

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