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FIGHTING TAX EVASION: A CELLULAR AUTOMATA APPROACH

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Abstract. In this paper we study the dynamics of a population where the individuals can either be contributors (tax payers) or no contributors (tax evaders or cheaters). We introduce a $2D$ cellular automaton on which the probability of transition from one of the above states to the other is the sum of a local effect and a global field. The model also includes the policy that allocates a fraction of the budget to fight tax evasion. This scheme allows to simulate the cases in which inhomogeneous strategies in contrasting tax evasion is applied in a region and the case in which cooperative policies are adopted by neighbor societies.

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1 Introduction

Taxation is the basic instrument adopted by human societies to collect the resources needed to achieve objectives of common interest. But it is well known that there exist individuals that do not want to contribute (at least not in the measure they should) but take advantage of the services provided by the rest. Reducing the size of the group of tax evaders (also referred as to free-riders or cheaters) is one of the most crucial challenges in the industrialized world [4, 17].

There exists a long tradition in the application of Mathematics in the social sciences [11, 8]. However, criminality in its multiple manifestations is being a major issue in applied mathematical research since only very recently (see, for instance, the special double issue of the European Journal of Applied Mathematics (EJAM) devoted to this subject in 2010 [9]). In particular, very fruitful investigations are being performed on the spatial distribution of crimes in urban areas [14, 12, 5], where special attention is being paid to the formation and development of criminal hot spots [15].

In a previous paper, following the tradition of Gary Becker [3], we studied a model aimed at optimizing the strategy to fight cheaters [13], in the sense of finding the fraction of the common wealth W that should be reasonably used to contrast tax evasion, i.e. to reach a compromise between the maximization of W and the minimization of the number of cheaters. Moreover, in [13] we considered the two typical ways of fighting criminality, i.e. repression and education, and discussed some aspects of this multicriteria optimization problem.

In this paper, we address the question of the mutual influence that different policies applied in different areas can have on the final result. For such a purpose, we define a two dimensional classical cellular automaton with probabilistic rules of change [16, 6, 10]. Indeed, we consider the abstract situation in which an individual (i.e. a cell in the 2D cellular automaton) can be either in the state X (law-abiding) or Y (cheater) and that the transition from one state to another is influenced not only by the state of its neighbors but also by a global field.

We start by considering a single isolated society, without contrast to evasion and tune the Cellular Automaton (CA) on the corresponding dynamics governed by a simple Ordinary Differential Equation. Then, we present a simplified model of how the situation can be modified to take into account measures that contrast tax evasion. To implement these measures, some fraction φ of the common wealth has to be used; since the latter depends on the number of tax payers, the dynamics of the CA depends on a "field variable", the wealth W, that itself depends on the global state of the CA at the previous step. Thus, we suggest how to find the optimal value of φ. Finally, we handle the case of several regions mutually interacting. We simulate the case in which different policies are adopted and the effects of cooperative strategies and we draw some conclusions.

2 No law enforcement

We first introduce a CA model that mimics both the transitions from one state to another due to a global field and those influenced by the state of the neighbors. Then, we consider a simple model based on a first order ordinary differential equation (ODE) in which space variations are not taken into account and tune the parameters to compare the two approaches.

2.1 A 2-dimensional cellular automaton

The model that we propose is a 2-D cellular automaton. As it is well known, cellular automata were introduced by John von Neumann (1903-1956) (following an ideas of Stanislav Ulam (1909-1984)) to study global properties from local processes [10]. At present, this technique is extensively used to study the space dependence of the dynamics obtained in ODE models. We assume that the population occupies every cell of a square grid of side n, so that the total dimension of the population is $N = n \times n$. In our case, all the simulations have been performed with $n = 40$.

Each member of the population can be a **tax payer** (X) or a **cheater** (Y). We assume that the probability of changing the state of a given cell depends on the state of its neighbors and on the global properties of the society. Therefore, let us define the probability of changing the current state as sum of these two contributions (see [1], [2]):

$$P^{TOT} = P^{LOC} + P^{GLO} \tag{1}$$

where P^{LOC} and P^{GLO} are the local and global probabilities of changing the state. Obviously, the probability of remaining in the same state is given by $1 - P^{TOT}$.

Let us first define the local probabilities. The local probability that a tax-payer becomes a cheater in the next time step is given by:

$$P^{LOC}_{X\to Y} = l\,\frac{N_{LY}}{N_L}, \qquad 0 < l < 1 \tag{2}$$

where N_{LY} is the number of cheaters in the neighborhood of this cell formed by N_L cells (so, $\frac{N_{LY}}{N_L}$ is the fraction of cheaters in its neighborhood). In the same way we can define the local probability that a cheater becomes a tax-payer as

$$P^{LOC}_{Y\to X} = k\,\frac{N_L - N_{LY}}{N_L}, \qquad 0 < m < 1. \tag{3}$$

Obviously, $N_L - N_{LY}$ corresponds to the number of taxpayers in the neighborhood.

Next, we define the global probability for a tax payer to become a cheater as

$$P^{NONLOC}_{X\to Y} = \tau, \qquad 0 < \tau < 1 \tag{4}$$

and, similarly, the global probability for a cheater to become a tax payer as

$$P^{NONLOC}_{Y\to X} = \alpha, \qquad 0 < \alpha < 1. \tag{5}$$

We use a synchronous or parallel updating rule where all the system sites are updated at the same time step (in contrast to asynchronous or sequential updating where only one randomly selected site is update at each time) [6, 16]. We assume reflection boundary conditions on the sides of the square. The model is then completed by defining the neighborhood of each cell as a square centered in the cell and formed by $m \times m$ cells excluding the cell itself (the so-called Moore neighborhood [10]). In the sequel, we will specify the value of m.

2.2 The ODE model

Let us consider a homogeneous society formed by a constant number N of individuals and let us denote by $Y(t)$ the number of cheaters at time t, so that the number of law abiding individuals at time t will be $X(t) = N - Y(t)$.

In the spirit of classical population dynamics [7], we write an ODE expressing the balance in terms of inflow and outflow. As described before, we consider two kinds of fluxes: one that represents a linear exchange between classes X and Y and another that mimics a sort of contagion term. Thus, we write:

$$\frac{dY}{dt} = \dot{Y} = \tau^* (N - Y) - \alpha^* Y + \frac{l^*}{N} Y (N - Y) - \frac{k^*}{N} (N - Y) Y \tag{6}$$

where the coefficients in the contagion terms are written as $\frac{l^*}{N}$ and $\frac{k^*}{N}$ for normalization purposes and $\tau^*, \alpha^*, l^*, k^* \geq 0$.

If we set

$$y(t) = \frac{Y(t)}{N} \tag{7}$$

and we choose for t a time scale t^* and define:

$$\alpha = \alpha^* t^*; \quad \tau = \tau^* t^*; \quad l = l^* t^*; \quad k = k^* t^*$$

and defining the new parameter:

$$d = l - k \tag{8}$$

we have to solve the following Initial Value Problem:

$$\begin{aligned} \dot{y}(t) &= \tau (1 - y) - \alpha y + d y (1 - y) \\ y(0) &= y_0 \end{aligned} \tag{9}$$

An explicit expression of the solution of the Initial Value Problem (9) for the case $d > 0$ can be written as:

$$y(t) = \frac{1}{2d} \left[\gamma + \sqrt{\gamma^2 + 4 d \tau} \left(tanh \left(\frac{t}{2} \sqrt{\gamma^2 + 4 d \tau} + arctanh \left(\frac{2 d y_0 - \gamma}{\sqrt{\gamma^2 + 4 d \tau}} \right) \right) \right) \right] \tag{10}$$

where $\gamma = d - \tau - \alpha$. When $\gamma^2 + 4 d \tau$ is negative a different formula can be used:

$$y(t) = \frac{1}{2d} \left[\gamma - \sqrt{-\gamma^2 - 4 d \tau} \left(\tan \left(\frac{t}{2} \sqrt{-\gamma^2 - 4 d \tau} - \arctan \left(\frac{2 d y_0 - \gamma}{\sqrt{-\gamma^2 - 4 d \tau}} \right) \right) \right) \right] \tag{11}$$

Its asymptotic value is given by the branch between 0 and 1 of the following expressions:

$$\bar{y} = \frac{d - (\tau + \alpha) \pm \sqrt{(\tau + \alpha)^2 + 2 d (\tau - \alpha) + d^2}}{2d} \tag{12}$$

2.3 Comparing CA and ODE models

Clearly, if we assume that N_L is the entire square or, equivalently, that $m = n$, the CA is a probabilistic approach to the finite difference approximation of equation (9), where the time step is 1 (in the time scale t^*) and t^* is taken small enough to ensure reasonable convergence and $\alpha + \tau + l + k < 1$.

According to the rather slow evolution of the social phenomenon we are considering, we take t^* of the order of one month so that, α and τ are the probabilities of "spontaneous" transition (over one month) from one state to another, while l and k represent the probabilities of changing the state (over the same time period) if all the other cells are in the opposite state. To be specific, we will take α and τ of the order of 1%, while l and k are assumed to be much larger, of the order of 30%.

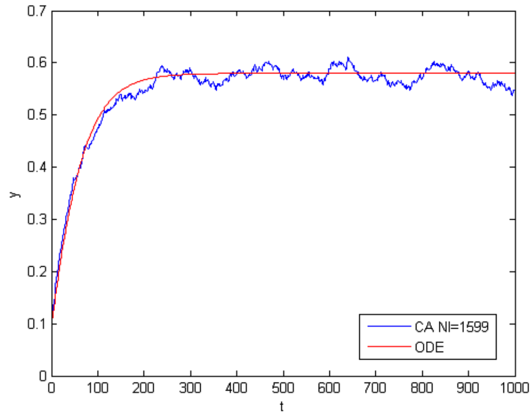


Figure 1: Time evolution of the cheater population y of the ODE and CA with maximum neighborhood ($N_L = 1599$). The parameter setup is $\alpha_0 = 0.01$ $\tau_0 = 0.008$, $l_0 = 0.31$ $k_0 = 0.30$. The initial condition is $y_0 = 0.1$. The CA curve is an average over 10 simulations.

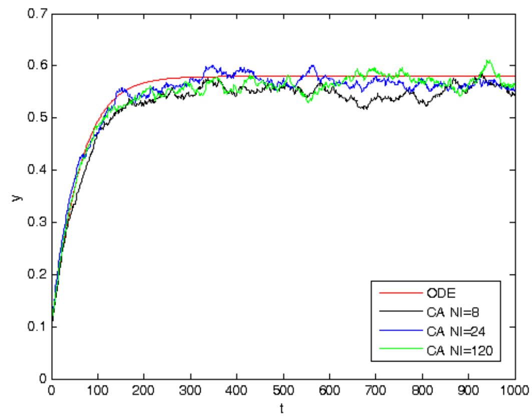


Figure 2: Time evolution of y obtained from the CA model for different values of N_L. Superimposed is the graph of the solution of (9). The initial condition and the parameter setup as in the previous Figure 1. The curves are averages obtained over 10 simulations.

In Figure 1 we display the solution of (9) with $y_0 = 0.1$ and the values of $y_k = \frac{Y_k}{N}$, where Y_k is the total number of cells occupied by cheaters at the kth-step, of a simulation on a CA with $N_L = 1599$. In Figure 2, we also show three other simulations with different N_L (concretely, $m = 3$, $m = 5$ and $m = 11$)

2.4 The evolution of wealth

We can assume that the total rate of fiscal revenue of the society is proportional to the number of tax payers. Of course, this is a simplified picture in which all tax payers contribute with the same amount or that a constant average of contribution can be defined. Moreover, it is assumed that the total rate of expenditure is proportional to the actual value. Thus, the time evolution of the total wealth is given by:

$$\dot{W} = a\,(N - Y) - \theta\,W \tag{13}$$

where a and θ are non negative. Note that the model assumes that cheaters are total evaders. In general, we could define an average rate of contribution $b < a$ and add a term $b\,Y$ on the r.h.s of equation (13).

If the time scale is of the order of one month, it is reasonable to assume that $\theta \approx 0.1$ and if we normalize W by $a\,N$ (the total fiscal revenue referred to one month, if all the citizens are tax payers), we get:

$$\dot{w} = 1 - y - \theta\,w \tag{14}$$

whose asymptotic value is:

$$\bar{w} = \frac{1 - \bar{y}}{\theta} \tag{15}$$

Correspondingly, for the CA model, we define a normalized wealth at each time step k:

$$w_k = w_{k-1} + 1 - y_{k-1} - \theta\,w_{k-1} \tag{16}$$

3 Fighting tax evasion

Next we assume that the coefficients τ, α, k and l may depend on the policy the society is adopting to contrast tax evasion. More specifically, we assume that the policy of the society in controlling tax evasion is characterized in terms of the fraction $\theta\,\varphi$ of the budget devoted to this goal per unit time.

(i) Parameters α and k represent the fraction of cheaters that change their behaviour per unit time spontaneously or by influence of a public opinion. It is then reasonable to assume that they are increasing functions of the amount of resources the society allocates to fight tax evasion. Therefore, we assume

$$\alpha = \alpha_0\,(2 - e^{-p\,\varphi\,w}) \tag{17}$$

and

$$k = k_0\,(2 - e^{-p\,\varphi\,w}) \tag{18}$$

where $\varphi\,w$ accounts for the expenses devoted to prevent tax evasion and $p > 0$ is a sensitivity parameter to be adjusted to experimental data.

(ii) Coefficient τ and l on the contrary, represent the amount of honest contributors that decide per unit time to become tax evaders; thus, it should decrease with the intensity of the contrast to parasitism. We set

$$\tau = \tau_0 \, (1 + e^{-p \, \varphi \, w}) \tag{19}$$

and

$$l = l_0 \, (1 + e^{-p \, \varphi \, w}). \tag{20}$$

According to the definition of φ, the time evolution of w is governed by:

$$\dot{w} = 1 - y - \theta \, (1 + \varphi) \, w \tag{21}$$

and, setting $\theta = 0.1$, it particularizes to:

$$\dot{w} = 1 - y - \frac{1 + \varphi}{10} \, w. \tag{22}$$

Thus, we say that τ_0, α_0, l_0 and k_0 represent the natural or anarchic state of the population, in the sense that its attitude towards taxation is only determined by the ethical principles and the cultural tradition that characterize the society. The parameter p measures the response to the measures (e.g. education and repression) adopted to fight tax evasion. Obviously, assumptions (17), (18), (19) and (20) postulate the simplest case in which there is a single sensitivity parameter. The functional dependence is such that the influence of investments in contrasting tax evasion has a decreasing incremental effect. In Figure 3 we show the dependence on φ of the asymptotic values of w and of y.

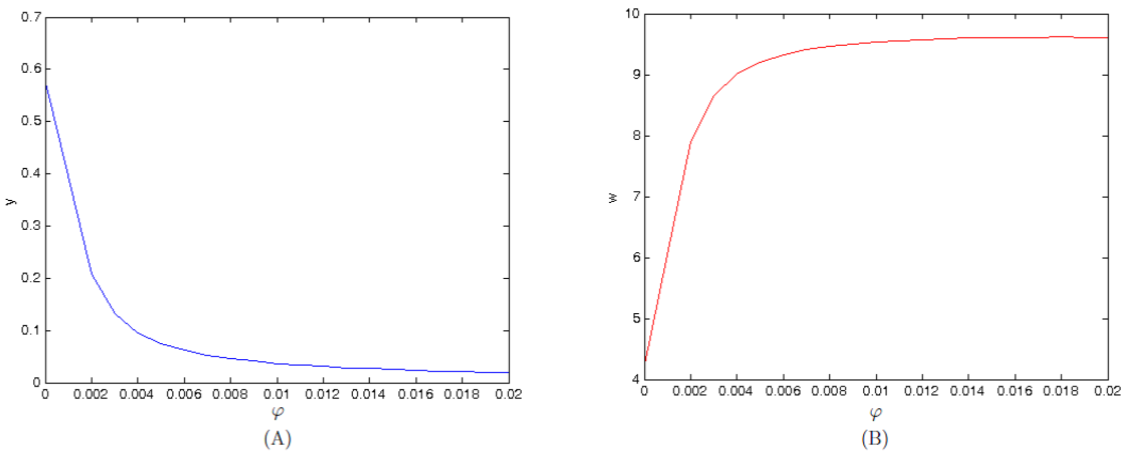


Figure 3: Asymptotic values of y (A) and w (B) obtained from the CA model for different values of φ. The parameter setup is $\alpha_0 = 0.01$ $\tau_0 = 0.008$ $l_0 = 0.31$ $m_0 = 0.30$ and $p = 5$ and $N_L = 120$. The asymptotic value is evaluated as an average over 1000 time steps (after a transient period of 1000 steps) and over 10 simulations.

One could ask which is the optimal value of φ according to this model. We give here a simple criterion: we prescribe a time horizon T (i.e. a number ν of time steps) over which we want to design our policy and give as the initial values the values $\bar{y}_0, \bar{w}_0$ corresponding to the "anarchic" state of the society (i.e. $\varphi = 0$). Then, we compute the time evolution of y and w for increasing values of φ. Denoting by $\bar{y}(\varphi)$ and $\bar{w}(\varphi)$ the values at step ν, the "gain" will be

$G(\varphi) = \bar{w}(\varphi) - \bar{w}_0$. To obtain this gain, the society has spent and amount $E(\phi) = \sum_{j=1}^{n-1} \phi\, w_j$. In Figure 4 we display the difference $G(\varphi) - E(\varphi)$ for increasing values of φ. The simulation shows that there is a value φ_1 of φ for which this difference is maximum and a value, φ_2, beyond which increasing investment in fighting cheaters is no longer convenient (since the difference becomes negative).

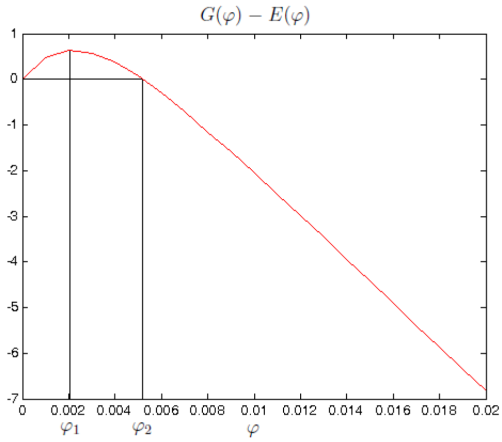


Figure 4: Difference between the gain and the expenditure over a time horizon of five years ($T = 50$). As it can be seen, $\varphi_1 \approx 0.002$ and $\varphi_2 \approx 0.005$. The same parameter setup as before is used: $\alpha_0 = 0.01$ $\tau_0 = 0.008$ $l_0 = 0.31$ $m_0 = 0.30$. Moreover, $p = 5$ and $N_l = 120$

4 Spatial dependence without investment

Of course, the main feature of the CA approach and its main advantage with respect to the ODE consists in taking into account spatial dependence. For instance, confining to the case $\varphi = 0$ for sake of simplicity, consider a CA model with an initial population of cheaters $Y_0 = 120$ (see Figure 5). In one case, the initial distribution of cheaters is randomly distributed (Figures 5A and 5C) and, in the second, they are occupying the three central columns of the grid (Figs. 5B and 5D). Figure 5 depicts the situation in the two cases at times $t = 10$ (Figs. 5A and 5B) and $t = 50$ (Figs. 5C and 5D) (to enhance the effect, we have taken very small global transition probabilities α and τ).

To see how the dimension of N_L influences the propagation of a given behaviour, we have carried out simulations with a slightly different setting of the parameters assuming that contagion is only occurring in one direction, i.e. $k = 0$. We take an initial condition $Y_0 = 120$, concentrated in the central columns of the grid. In Figure 6 we display the total number of cheaters that are found in the first and second columns, third and fourth, etc... at the same time step. The curves represent the situation after 50, 100 and 200 time steps for different neighborhood sizes $N_L = 8$ (Fig. 6A) and $N_L = 24$ (Fig. 6B). In order to avoid the spontaneous generation of criminals so that propagation is only caused by contagion we took $\tau_0 = 0$. In sociological terms, increasing values of m corresponds to increasingly interconnected societies where long-range effects can be observed.

In the same spirit, we consider two different societies (i.e. two different grids $n \times n$) with different parameters, so that their "spontaneous" and independent time evolution brings to different asymptotic levels of tax-evaders. Let us assume now that the two societies are in

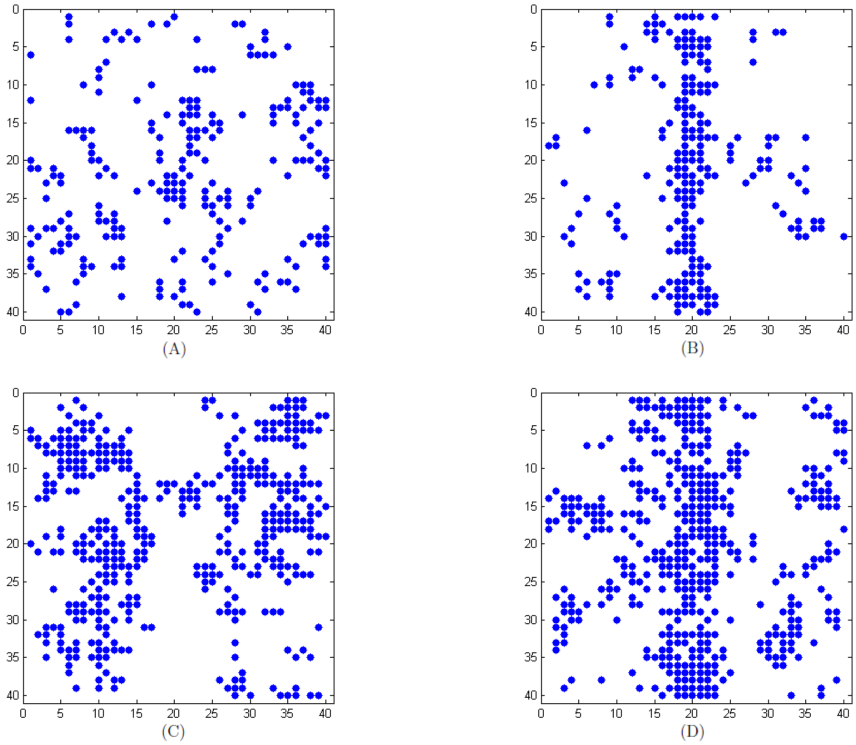


Figure 5: Screenshots of the time evolution of the CA at time $t = 10$ [figures (A) and (B)] and $t = 50$ [figures (C) and (D)] for two initial spatial distributions of the cheaters: random, figures (A) and (C) and, fully distributed in the three central columns of the grid, figures (B) and (D). White cells are occupied by tax payers, whereas dark cells are occupied by cheaters. The initial condition is $Y_0 = 120$. The parameter setup is: $\alpha_0 = 0.005$ $\tau_0 = 0.004$ $l_0 = 0.31$ $m_0 = 0.30$. The neighborhood size is $N_L = 8$.

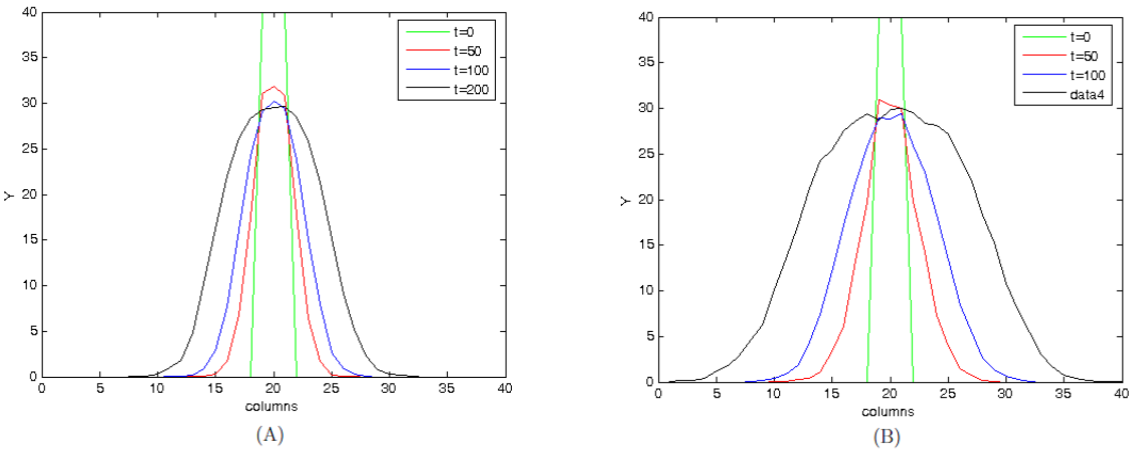
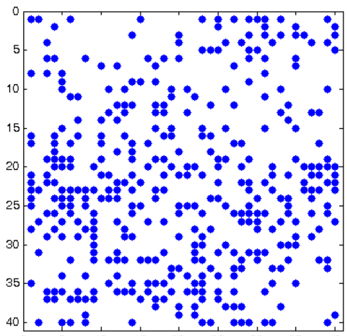


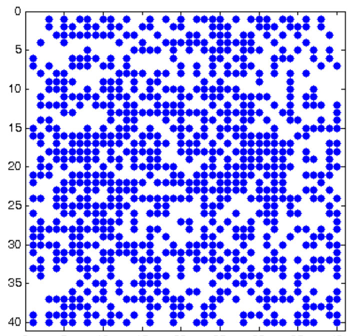
Figure 6: Propagation after $t = 50$, $t = 100$ and $t = 200$ time steps for different neighborhood sizes: $N_l = 8$ (A) and $N_L = 24$ (B). The curves represent the average number of cheaters over 50 simulations in each column of world. Initially, all cheaters are distributed in the three central columns of the grid. The parameter setup is: $\alpha_0 = 0.010$ $\tau_0 = 0$ $l_0 = 0.04$ $k_0 = 0$.

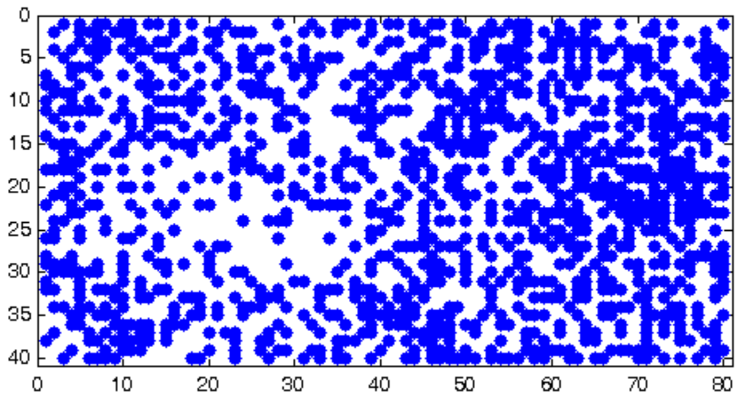
contact through one side of the square, so that they can influence each other. Again, we see that the effects are visible (see Fig. 7) and increasing with m (see Table 1).

	First World	Second World
Isolated	$y_\infty = 0.27$	$y_\infty = 0.60$
In contact $N_L = 120$	$y = 0.39$	$y = 0.54$
In contact $N_L = 24$	$y = 0.31$	$y = 0.55$
In contact $N_L = 8$	$y = 0.27$	$y = 0.56$

Table 1: Density of cheaters y for the case of isolated worlds and when they are in contact for several sizes of the local neighborhood.







(C)

Figure 7: (A) Screenshot at time $t = 100$ of the population of the first society (isolated) characterized by the following parameter setup: $\alpha_0 = 0.01$ $\tau_0 = 0.001$ $l_0 = 0.31$ $k_0 = 0.30$ and $N_l = 120$. (B) Screenshot at time $t = 100$ of the population of the second society (isolated) characterized by the parameter setup: $\alpha_0 = 0.01$ $\tau_0 = 0.009$ $l_0 = 0.31$ $k_0 = 0.30$ and $N_l = 120$. (C) Screenshot at the same time $t = 100$ of the population of the two worlds that share a common boundary (at column 40). In all the simulations the initial condition is $y_0 = 0.1$. As before, white cells and dark cells are occupied by tax payers and tax-evaders, respectively.

5 Non-homogeneous contrast policies

In this section, we briefly investigate the behaviour of the cellular automaton when the budget allocated to combat cheaters is not equally applied. Of course, since spatial inhomogeneities appears, the ODE model is no longer suitable. One possibility would be to replace it by an integro-differential model where the variable y depends on both space and time. This would be the subject of a forthcoming research. Here, we use the CA approach that suits this new situation as well. We assume that the system is divided into different regions that share the same characteristics except the rate of investment φ that is different in each region (although the average rate of investment in the whole world is maintained, e.g. $\varphi = 0.01$). Table 2 depicts the case of one world that is divided into 2, 4 and 16 regions of the same dimension (chessboard-like) in which $\varphi = 0$ and $\varphi = 0.004$, alternatively. We assume that the borders are permeable, in the sense that some agents of one of the regions have neighbors that belong to another region and notice a different policy. In the first three rows we list the asymptotic fraction of cheaters over the whole world, in the case of 2, 4 and 16 regions. The fourth row shows the asymptotic value of y when the policy is homogeneously applied (or rather $\varphi = 0.002$ in all the cell of the world). As it can be seen, an inhomogeneous investment keeping the total investment equal, always yields a larger cheater population.

Case	**Solution**
2 regions	$y = 0.27$
4 regions	$y = 0.25$
16 regions	$y = 0.23$
Homogeneous	$y = 0.20$

Table 2: The equilibrium cheater population y for different policies. The average φ is equal to 0.002 in all cases. The parameter setup is $\alpha_0 = 0.01$ $\tau_0 = 0.008$ $l_0 = 0.31$ $m_0 = 0.30$ $p = 5$ and $N_L = 120$.

6 Advantages of cooperation

As a final application of the model, we consider two societies, separated by a permeable border, and we study the case in which one of the societies decides to devote part of its budget to fight tax-evasion in the other. As before, we consider two adjacent square grids of 40×40 cells each with the following parameter setups:

$$\tau_0^{(1)} = 0.01;\ \alpha_0^{(1)} = 0.05;\ l_0^{(1)} = 0.5;\ k_0^{(1)} = 0.3;\ p = 5;\ N_L = 120 \tag{23}$$

$$\tau_0^{(2)} = 0.01;\ \alpha_0^{(2)} = 0.05;\ l_0^{(2)} = 0.5;\ k_0^{(2)} = 0.3;\ p = 5;\ N_L = 120 \tag{24}$$

We consider the asymptotic behaviour of the two societies when country (1) spends per unit time $\varphi_1 w^{(1)}$ to fight the cheaters in the same country and $\varphi_S w^{(1)}$ in country (2). Note that $\varphi_2 = 0$ (i.e. society (2) does not -or cannot- spend anything in preventing tax evasion from its own budget). Consequently,

$$\begin{array}{rcl}\alpha^{(1)} & = & \alpha_0^{(1)}\,(2 - e^{-p\varphi_1\, w^{(1)}}) \\ \tau^{(1)} & = & \tau_0^{(1)}\,(1 + e^{-p\,\varphi_1\, w^{(1)}}) \\ \alpha^{(2)} & = & \alpha_0^{(2)}\,(2 - e^{-p\varphi_S\, w^{(1)}}) \\ \tau^{(2)} & = & \tau_0^{(2)}\,(1 + e^{-p\,\varphi_S\, w^{(1)}})\end{array} \tag{25}$$

and similarly for parameters l and k. For both societies, the initial densities are $y_0 = 0.10$ homogeneously distributed.

	Setup	Results	
First test	$\varphi_1 = 0.035$ $\varphi_S = 0$ $\varphi_2 = 0$	$y_1 = 0.05$ $y_2 = 0.75$	$w_1 = 9.16$ $w_2 = 2.51$
Second test	$\varphi_1 = 0.025$ $\varphi_S = 0.010$ $\varphi_2 = 0$	$y_1 = 0.04$ $y_2 = 0.32$	$w_1 = 9.26$ $w_2 = 6.75$

Table 3: Results obtained from the simulations of two worlds without (first test) and with (second test) sharing resources. The rest of the parameter values are given in (23) and (24).

Table 3 shows the results of these experiments for $\varphi_S = 0$ and $\varphi_S = 0.01$ but maintaining constant the value $\varphi_1 + \varphi_S$. We remark that in this way the country (1) invests always the same amount for fighting cheaters. As it can be seen, the effect of using the amount $\varphi_S\, w^{(1)}$ for combating tax evasion in society (2) is drastic: the average cheater density reduces from $y_2 \approx 0.75$ to $y_2 \approx 0.32$. But, surprisingly enough, also the cheater density of society (1) is decreased significantly, passing from $y_1 \approx 0.05$ to $y_1 \approx 0.04$. Correspondingly, the total wealth of society (2) is considerably increased, from $w_2 \approx 2.51$ to $w_2 \approx 6.75$; but also w_1 is increased, passing from 9.16 to 9.26.

7 Concluding remarks

In this paper we have investigated the dynamics of a population formed by two types of individuals: contributors (tax payers) and non contributors (tax evaders or cheaters). The contribution of tax payers allows the society to have a positive total fiscal revenue W that influences the behaviour of the society. Indeed, the society can spend a percentage of W to combat tax evasion. As it has been shown in this paper, this extra expenditure can induce relevant changes in the system behaviour both in time and in space.

As a first step we have studied the dynamical system that results assuming spatial homogeneity. The ODE model can be considered as the temporal evolution of spatially averaged values of both the cheater population Y and the total wealth W. To incorporate spatial effects in the model we have introduced a $2D$ cellular automaton where the probability of transition from one state to the other (from cheater to tax payer or viceversa) is the sum of a sort of local contagion term and of a global field. Moreover, the ODE model has been used to term de Cellular Automata.

We have investigated how the CA depends on the parameters and, in particular, we have shown that, when the recruitment and conversion rates depend on the contrast policies, a non-monotonic dynamics appears that allows a minimization of the cheater population with a simultaneous maximization of the total wealth. Therefore, a compromise between these two functions is required and a possible way of choosing an optimal policy is proposed. We have also studied the effects of an inhomogeneous investment to combat tax evasion (section 5) and the case in which two societies with different policies interact through a common border (section 6). Two relevant conclusions can be stated:

(i) A homogeneous (equally distributed) investment provides better strategies for preventing tax evasion and, simultaneously, keeping a large common wealth than an inhomogeneous policy (different in each region).

(ii) To share part of the extra budget with its neighbors (perhaps with larger cheater population) can yield better results not only for the neighbors but also for the own society. In other words, collaboration among neighbor societies is a (sub)optimal strategy for fighting tax evasion.

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