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Abstract

We study a simple monetary model in which a central bank faces a boundedly rational private sector and has the goal of stabilizing inflation. The system's dynamics is generated by the interaction of the expectations about inflation of the various agents involved. A modest degree of heterogeneity in such expectations is found to have interesting consequences, in particular when the central bank is uncertain about the relevant behavioral parameters. We find that a simple heuristic based on mean and variance of the distribution of behavioural parameters stabilizes the system for a wide parametric region.

1   Introduction

In recent years the ability of central banks to control inflation by way of inflation targeting policies has generally increased and their credibility has improved accordingly. The issue of whether monetary authorities are also actually able (and therefore should commit themselves to) tackle full employment at the same time is controversial, however (see e.g. Thornton [9]). More recently the worldwide crises has also put forward the issue of the zero lower bound constraint on nominal interest rates reaching effectiveness, with implications on the range of policy instruments which remain viable for monetary policy (see Williams [10]).

In this paper, we study a simple model in which an *inflation targeting monetary policy* is carried out by a Central Bank (CB henceforth), using money supply as its sole instrument. Also we assume bounded rationality for both the monetary authority and the private sector. The model builds on Bischi and Marimon [1] who rank a number of different policies according to the size of the basin of attraction associated with meaningful steady state equilibria. Also, they assume bounded rationality for the private sector, which is in turn modelled as a representative agent but do not especially focus on the CB's forecasts of relevant variables. In contrast, here the CB is assumed to have a reasonable, albeit not fully rational, strategy to forecast the private sector's inflation expectations. Besides, the private sector has heterogeneous agents, even though such behavioral heterogeneity will be rather modest. The criterion of judging policies according to the probability of reaching the target, however, will be maintained. This will make sense in the context of an exercise about parametric uncertainty for the CB. The authorities shall be assumed to have correct information about the sample moments of the distribution of behavioral parameters characterizing the private sector, while ignoring the fine detail of the actual realization of such parameters.

The results obtained with this simple model are as follows: within the representative agent framework we show how local stability depends on the interaction between money demand elasticity to inflation, the inflation target and the adaptive parameters involved. We also document

the relative ranking of this policy with various alternatives, given a criterion focused on the probability of a stable outcome for the dynamics of inflation. With heterogeneous agents we study the effect of the number of different types of agents playing a role within the private sector and find a sort of polarization: the larger such number the less uncertain becomes the issue of whether stability will prevail, with the actual answer depending on a structural parameter. In this context a significant role seems to be played by the dispersion of the private sector's adaptive parameter, which can be taken as a proxy for the amount of behavioral heterogeneity. We also describe a heuristic for the choice of a "good" adaptive parameter for the CB, in terms of the expected long-run outcome.

In the first part of the paper we present the baseline model with a representative agent and rank various policies, using analytical results. Next we proceed to introduce heterogeneous agents and parameter uncertainty and we resort to simulations and numerical techniques.

2 The baseline model

We consider the deterministic monetary model of inflation targeting described by Bischi and Marimon [1]. It is assumed that the government faces the intertemporal budget constraint

$$M_{t+1}^S + B_{t+1}^S = p_t g_t - p_t \tau_t + M_t^S + B_t^S I_t \tag{1}$$

where M^s is the money supply, B^S is the government bonds supply, I is the nominal rate of return on bonds, g are real expenditures, τ a lump-sum tax and p the price level. By defining the real term balance deficit as

$$d_t = g_t - \tau_t + \frac{B_t^S}{p_t} I_t - \frac{B_{t+1}^S}{p_t} \tag{2}$$

and using (1) we have that the money supply evolves according to

$$M_{t+1}^s = M_t^s + d_t p_t \tag{3}$$

i.e. the real term balance deficit equals to the expansion of the money supply.

Let m^d the money demand in real terms. The equilibrium condition on the market of money, $M_{t+1}^s = m_{t+1}^d p_t$, implies that

$$m_{t+1}^d p_t = m_t^d p_{t-1} + d_t p_t. \tag{4}$$

Such condition can be rewritten as

$$\pi_t = \frac{m_t^d}{m_{t+1}^d - d_t} \tag{5}$$

where $\pi_t = \frac{p_t}{p_{t-1}}$ is the gross inflation rate at time t.

The money demand is of the Cagan type, i.e. a function of the expected inflation rate, taking the linear form

$$m_{t+1}^d = b - \pi_{t+1}^e \tag{6}$$

where $b > 0$, and π_{t+1}^e is the private sector forecast at time t of the inflation rate at time $t+1$.[1]

The CB sets the desired level of inflation (the inflation target) π^* (which we assume to be smaller than b to ensure nonnegative demand at the target) and, in order to attain it, determines d_t given the information set at time t. While, in practice, expansions/contractions of the money

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[1]A money demand function of this kind can be founded from a microeconomic point of view through an overlapping generation model of general equilibrium with logarithmic utility function (see [1] for more details).

supply could be the outcome of different policy interventions (e.g. open market operations or interbank rate level), in our model this has no role on the dynamics of the economy.

From (5) it follows that the optimal inflation target policy, conditional on the bank's expectations being correct, is

$$d_t = E_t^c[m_{t+1}^d] - \frac{m_t^d}{\pi^*} \tag{7}$$

where E_t^c is the operator which defines the expectations of the CB. The optimal monetary policy (7) depends on the money demand at time t and on the forecast of the CB about the future money demand of the private sector.[2]

Assume that the money demand function is correctly estimated by the CB. The CB must therefore forecast the private sector's expectation only. So, the expected inflation rate is an intermediate target for the optimal monetary policy. By indicating with π_{t+1}^c the forecast of the CB about private sector's expectations and by substituting (6) and (7) in (5) we have that the inflation rate evolves according to

$$\pi_t = \pi^* \frac{b - \pi_t^e}{\pi^*(\pi_{t+1}^c - \pi_{t+1}^e) + b - \pi_t^e}. \tag{8}$$

Observe that if the CB has perfect foresight the optimal policy immediately drives the system on the target, independently of private sector expectations. In such case, no out-of-equilibrium dynamics is generated.

In what follows, we study the dynamics of the economy in case both the private sector and the CB are boundedly rational. Agents in the private sector have adaptive expectations.

$$\pi_{t+1}^e = (1 - \alpha)\pi_t^e + \alpha\pi_{t-1}. \tag{9}$$

Accordingly we assume that, for the sake of inferring the expectations prevailing in the private sector, the CB uses the same mechanism, so that its device is correctly specified, while forecasting errors are possible due to parameters' uncertainty:

$$\pi_{t+1}^c = (1 - \gamma)\pi_t^c + \gamma\pi_{t-1}. \tag{10}$$

2.1 Local analysis

We now determine stability conditions for the system obtained by substitution of equation (8) for the inflation rate, into (9) and (10). Letting $x_t = \pi_{t-1}^e$ we obtain the following dynamical system

$$\begin{cases} x_{t+1} = \pi_t^e \\ \pi_{t+1}^e = (1 - \alpha)\pi_t^e + \alpha\pi^* \frac{b - x_t}{\pi^*(\pi_t^c - \pi_t^e) + b - x_t} \\ \pi_{t+1}^c = (1 - \gamma)\pi_t^c + \gamma\pi^* \frac{b - x_t}{\pi^*(\pi_t^c - \pi_t^e) + b - x_t} \end{cases} \tag{11}$$

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[2]Under the assumption that the money demand function is correctly estimated by the Central Bank, Eq. (7) can be rewritten as

$$d_t = d^* + \left(\pi^* - E_t^c \pi_{t+1}^e\right) + \frac{1}{\pi^*}\left(\pi_t^e - \pi^*\right)$$

where $d^* = \frac{\pi^* - 1}{\pi^*}(b - \pi^*)$. So the Central Bank would react with an increase of the money supply whether private sector forecasts are expected to be lower than the target or whether past expectations were higher than the target. More details about relations between the policy in eq. (7) and Taylor rules can be found in Bischi and Marimon [1] pg. 152.

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The vector $\bar{\pi}^* = (\pi^*, \pi^*, \pi^*)$ is the unique stationary steady state. The associated Jacobian evaluated in the steady state has an eigenvalue equal to 0. The other eigenvalues come from the following 2×2 submatrix

$$J = \begin{pmatrix} 1 - \alpha + \alpha C & -\alpha C \\ \gamma C & 1 - \gamma - \gamma C \end{pmatrix}$$

where $C = \frac{\pi^{*2}}{b - \pi^*}$. Notice that C can be interpreted as the product of the target π^* and the elasticity to inflation of the money demand at the target, $\frac{\pi^*}{b - \pi^*}$. With π^* reasonably close to 1, C is quite close to such elasticity.

The following proposition summarizes the stability properties of the steady state.

Proposition 1 *The stationary steady state, $\bar{\pi}^* = (\pi^*, \pi^*, \pi^*)$ for the system (11) satisfies the following stability properties:*

1) for $0 < C \leq 1$, the $\bar{\pi}^*$ is locally stable for every choice of (α, γ);

2) for $C > 1$, there are couples (α, γ) for which $\bar{\pi}^*$ is unstable. In particular $\bar{\pi}^*$ can lose stability through a Period-doubling bifurcation with instability for $\alpha < \frac{2\gamma(1+C)-4}{\gamma + 2C - 2}$, or through a Neimark-Hopf bifurcation with instability for $\alpha > \frac{\gamma(1+C)}{\gamma + C - 1}$.

Proof. In the Appendix. ∎

The results in Proposition 1 have a pictorial representation in Figure 1 which shows the bifurcation diagrams of the system for $C < 1$ on the left and $C > 1$ on the right. The parameter γ varies on the horizontal axis, while the vertical axis refers to α. With $\mathbb{R}$ and $\mathbb{C}$, we denote regions in which the eigenvalues of the dynamical system at the steady state are real or complex, respectively. S and U denote the regions in which $\bar{\pi}^*$ is stable (white areas) or unstable (grey areas). The arrows show how the regions expand/contract when C changes. The upshot of Figure 1 is that when $C < 1$ the steady state is locally stable whatever the adaptive parameters. When C grows (beyond 1) the stability region shrinks around the line $\alpha = \gamma$ so that as $C \to +\infty$ the equilibrium is unstable for all couples (α, γ) such that $\alpha \neq \gamma$.

<center>Figure 1 about here</center>

Notice that it would appear that the CB could induce $C < 1$ by choosing an appropriately low π^* therefore ensuring stability automatically. However, this might not be technically feasible, on one hand, e.g. if $b < 2$ (in that case $C < 1$ would require $\pi^* < 1$). On the other hand it might not be desirable, because monetary authorities typically have some form of Phillips-curve trade-off relationship in the back of their minds, even though in our model there is no explicit dual mandate (regarding the output gap and inflation) for the CB. In view of this observation we concentrate on the case $C > 1$ in the rest of the paper.

We now compare the stability properties of (11) with those obtained by Bischi and Marimon [1] who study the same model under different policy rules. They rank such policies with regard to the stability properties of the system, preferring a monetary rule over another if the steady state of the inflation dynamics is stable for a larger set of α values under the first rule then under the second one. The policies they consider are:

F - a constant money growth rule *à la* Friedman

$$d_t^F = d^* = \frac{\pi^* - 1}{\pi^*} m^d (\pi^*)$$

O - an optimal-on-equilibrium money rule conditional on past real balances;

$$d_t^O = m^d (\pi^*) - \frac{m_{t-1}^d}{\pi^*}$$

<center>4</center>

I - a rule depending on past real balances and on the assumption of the private sector behavioral inertia;

$$d_t^I = \frac{\pi^* - 1}{\pi^*} m_{t-1}^d.$$

The policy we study here is

AD - a rule depending on past real balances which is optimal provided that the private sector adaptive behaviour is correctly estimated

$$d_t^{AD} = \pi_{t+1}^c - \frac{m_t^d}{\pi^*} \quad \text{with} \quad \pi_{t+1}^c = (1-\gamma)\,\pi_t^c + \gamma\pi_{t-1}$$

Proposition 1 shows that for (AD), if the CB has adaptive expectations and the value of α is known, it is always possible to find an adaptive parameter γ which locally stabilizes the system. More interesting is the case with α unknown where the system could be stable or unstable depending on γ. We study this case in the following proposition where we compare the four policies.

Proposition 2 *Consider the inflation dynamics defined in (5), (6), (7) and (9) under the four different policies d^F, d^O, d^I and d^{AD}. Let $\bar{\pi}^*$ the target steady state of the system. Let $A^i \subset (0,1)$ the set of α values for which the steady state of the system under the policy i is locally stable. Let $\succ$ be the order relation on the set of policies such that $d^i \succ d^j$ if and only if $A^j \subset A^i$. Then for all $\pi^* > 1$ we have*

$$AD \succ F \succ I \succ O \quad if \quad C < \frac{\pi^{*2}}{\pi^{*2} - \pi^*}$$
$$AD \succ I \succ F \succ O \quad if \quad C > \frac{\pi^{*2}}{\pi^{*2} - \pi^*}$$

Proof. In the Appendix. ■

Figure 2 about here

Figure 2 illustrates the result stated in Proposition 2. The curves show, for each value of C, the maximum value of α which guarantees the local stability of the system, which is achieved, for each policy, for parameter values that lie south-west of the corresponding curve. The inertial behaviour of a CB trying to mimic an adaptive private sector has positive consequences on the stability of the target steady state. This fact is somewhat reminiscent of a stream of literature (e.g. Clarida et al. [3]) showing that the behaviour of central banks is better explained by introducing a partial adjustment mechanism into the optimal policy.

Proposition 2 is obtained by setting $\gamma = \hat{\gamma}$ in the (AD) policy such that the interval of stabilizing α is of the type $[0,\hat{\alpha})$ with $\hat{\alpha}$ the largest possible value, to ease the comparison. Focusing on the (AD) policy it is possible to obtain a related interesting result if we drop the constraint that the interval is of the type $[0,\hat{\alpha})$. The problem we have in mind is as follows. Suppose that the CB knows that α is the realization of a random draw from a given probability distribution f. Then an optimal strategy for the CB would be that of choosing the value of γ maximizing the probability of ending up with a stable system as shown in the following example.

Example 3 *Suppose that $\alpha \sim f$. Let A_γ^{AD} the set of α values for which the steady state $\bar{\pi}^*$ of the system (11) is locally stable. It is easy to see that the set A_γ^{AD} is an interval. We want to find the value of γ such that the probability that the sampled α belongs to A_γ^{AD} is maximized.*

When $C < 1 + \sqrt{2}$ the two curves of Period-Doubling and Neimark-Hopf bifurcations do not overlap: in this case, by choosing any $\gamma \in \left(\frac{C-1}{C}, \frac{2}{C+1}\right)$ the system is locally stable for any α. On the contrary, when $C > 1 + \sqrt{2}$ there is no γ value which would work for all α and the optimal γ value is the solution of the problem

$$\max_{\gamma \in \left(\frac{2}{C+1}, \frac{C-1}{C}\right)} \int_{\frac{2\gamma(1+C)-4}{\gamma+2C-2}}^{\frac{\gamma(1+C)}{\gamma+C-1}} f(x)\, dx$$

In the case of uniform distribution $f = U_{(0,1)}$ we obtain the solution $\gamma = \frac{2}{C+1}$. For such value, $A_\gamma^{AD} = \left[0, \frac{2C+2}{C^2+1}\right)$.

In the next Section we will address the problem of choosing an optimal adaptive parameter for the CB in the heterogeneous agents case.

3 Heterogeneous agents

In this section we remove the assumption of a representative agent and allow instead for n different agents, each having a different adaptive parameter, α_i. This assumption of heterogeneous adaptive behaviour is consistent with previous experimental evidence (see e.g. Colucci and Valori [4], and [7]). The model modifies in a rather straightforward way. Assume the n agents come up with a specific inflation expectation π_t^i with demand for money $m_t^d = \phi_i \left(b - \pi_t^i\right)$ where the weight ϕ_i reflects the relative importance of agent i in the economy. The aggregate money demand thus becomes

$$
\begin{aligned}
m_t^d &= b - \sum_{i=1}^n \phi_i \pi_t^i \\
&= b - \pi_t^e
\end{aligned}
$$

where $\pi_t^e = \sum_{i=1}^n \phi_i \pi_t^i$ is now the (weighted) mean forecast for time-t inflation. The dynamical system now becomes $(2n+1)$-dimensional:

$$
\begin{cases}
x_{t+1}^i = \pi_t^i \\
\pi_{t+1}^i = (1-\alpha)\,\pi_t^i + \alpha \pi^* \frac{b - \sum_{i=1}^n \phi_i x_t^i}{\pi^*(\pi_t^c - \pi_t^e) + b - \sum_{i=1}^n \phi_i x_t^i} \quad i = 1, \dots, n \\
\pi_{t+1}^c = (1-\gamma)\,\pi_t^c + \gamma \pi^* \frac{b - \sum_{i=1}^n \phi_i x_t^i}{\pi^*(\pi_t^c - \pi_t^e) + b - \sum_{i=1}^n \phi_i x_t^i}
\end{cases}
\tag{12}
$$

A $(2n+1)$-dimensional vector with each component equal to π^* is the unique stationary steady state. The associated Jacobian evaluated in the steady state has an eigenvalue equal to 0 with multiplicity n, while the remaining eigenvalues come from the following $n+1 \times n+1$ matrix

$$
J_{het} = \begin{pmatrix}
(1-\alpha_1) + \alpha_1 \phi_1 C & \alpha_1 \phi_2 C & \cdots & \alpha_1 \phi_n C & -\alpha_1 C \\
\alpha_2 \phi_1 C & (1-\alpha_2) + \alpha_2 \phi_2 C & \cdots & \alpha_2 \phi_n C & -\alpha_2 C \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha_n \phi_1 C & \alpha_n \phi_2 C & \cdots & (1-\alpha_n) + \alpha_n \phi_n C & -\alpha_n C \\
\gamma \phi_1 C & \gamma \phi_2 C & \cdots & \gamma \phi_n C & (1-\gamma) - \gamma C
\end{pmatrix}.
\tag{13}
$$

It has been shown elsewhere (see Colucci and Valori [6]) that in spite of its simplicity such modest form of behavioral heterogeneity may have relevant dynamic consequences, and the same

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applies here. Indeed, we know from Proposition 1, that with a representative agent such that $\alpha = 0.5$ and a parameter $\gamma = 0.5$ there is local stability (there is a single eigenvalue equal to 0.5 with multiplicity equal 2). However a couple of agents having for example $\alpha_1 = 0.2, \alpha_2 = 0.8$ (so that the average α is again 0.5) and $\gamma = 0.5$ will generate an unstable system (for large enough C). Heterogeneous agents in the private sector therefore matter. It turns out that the outcome in terms of the steady state's stability depends in a complicated way on the overall distribution of the behavioral parameters of the private sector *and* the CB, beside the structural parameter C.

Reflecting the broad range of possible behaviors we have already seen in the simpler case with a representative agents, the local dynamics generated by the system (12) around its steady state can assume various forms, depending on the parameters. In particular, local stability can be lost either through a Period-doubling or a Neimark-Hopf bifurcation. While it proved intractable (to us!) to draw analytical conclusions matching those seen above, in the next subsection we resort to numerical evidence to shed light on a number of interesting features characterizing this situation of heterogeneity.[3]

3.1 Uncertainty about behavioral parameters.

Assume the CB is uncertain about the behavioral parameters used by the agents in the private sector. Clearly, we might as well conjecture uncertainty about other relevant parameters. However here we suppose the CB is more reliable in estimating relevant structural parameters than it is in dealing with the private sector's expectations.[4] We then look for a good choice of its own gain parameter γ for the CB given such uncertainty. A similar exercise has been carried out e.g. in Karozumi [8] in the context of a DSGE model.

Let the parameters α_i be unknown to the monetary authority. Instead suppose such parameters are drawn (independently) from a known probability distribution with support on the unit interval. Whilst the effect of a larger value of C is unexpectedly similar to that we have seen in the representative agent case examined above, it is less obvious to figure out the effect of a larger n, i.e. a larger number of different types of agents.

Our first set of simulation entails drawing samples of adaptive parameters from the uniform distribution over [0,1], while γ is set to a constant value (0.5). This is done taking increasing values of n and increasing values of C. For each n and C we drew 10000 independent samples and recorded whether or not the system's steady state was locally stable (by looking at the implied spectral radius of the J_{het} matrix).[5] This generates a measure of the empirical probability of stability as a function of C for various values of n. An interesting phenomenon we observed is depicted in Figure 3.

Figure 3 about here

As n increases there emerges a threshold value for C such that below the threshold there is almost certainly convergence whereas past the threshold almost certainly instability will prevail. This feature, can also be found in other related but different models with heterogeneous agents (see Colucci and Valori [5] and [6]), and is a kind of polarization. The result represented in

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[3] Colucci and Valori [6] study the effect of introducing heterogeneity in a population of adaptive agents within a cobweb model and derive analytic conditions for stability in the more general case of n agents. In the model we study here, the interaction between the CB and the private sector changes substantially the nature of the Jacobian matrix and such results are no more valid.

[4] We will address the issue of uncertainty about the parameter b (or equivalently C) which determines the money demand schedule with the second simulation set.

[5] The simulations are all implemented with equal weights for each agent, i.e. $\phi_i = \frac{1}{n}$ for simplicity.

Figure 3 does not depend either on the probability distribution of α_is or on the particular choice of γ: in fact it can also be observed if γ itself is the realization of a random variable (with the same distribution as the α_is). For different values of (fixed) γ polarization still emerges, but the value of C discriminating between the two asymptotic outcomes (0 or 1) changes.

This probability polarization is relevant for the policy maker. Suppose that, in the context of an economy with a large number of heterogeneous agents, the CB copes with the problem of choosing its adaptive parameter γ having the stabilization of inflation around the target as a primary objective. In this case the CB would choose, if it exists, a value of γ such that the related threshold value is larger than the C. This choice, at least on a probabilistic basis, would grant stability to the system.

Assume now that C is unknown (e.g. when the value of b is uncertain) then a good question to answer is: "Which is the value of γ for which the threshold value of C is maximized?". One may suppose that a good strategy is that of choosing a small value of γ (adaptive expectations with small adaptive parameters are usually stabilizing); alternatively $\gamma = E[\alpha_i]$ could also make sense (coherently with the fact that in the representative agent case $\gamma = \alpha$ is always a good choice). In fact the right answer is a third one as we discuss below.

The problem we want to address can be summarized as follows. Let $p_{n,f,\gamma}(C)$ be the set of functions - depending on the population numerosity n, on the α_is probability distribution and on the CB adaptive parameter γ - which defines the probability of convergence for each value of C. This is the kind of function we have estimated in the first set of simulations (indeed figure 3 shows the graphs of $p_{n,f,\gamma}(C)$ with $\gamma = 0.5$, f the uniform distribution and $n = 2, 5, 15, 50, 100, 250$). If we suppose that each couple of successive functions has a unique intersection point C_n (i.e. a point such that $p_{n+1,f,\gamma}(C_n) - p_{n,f,\gamma}(C_n) = 0$) and that the sequence $\{C_n\}$ converges to $\hat{C}$, then we are looking for the value of γ which maximizes $\hat{C}$ given f. To obtain an approximate value of this limit we have estimated the root of $p_{50,f,\gamma}(C_n) - p_{15,f,\gamma}(C_n)$ using Newton's method and evaluating successive points of such difference using the algorithm of the first simulation set. The exercise has been repeated drawing samples of α_is from several Beta distributions.[6] The three graphs in Figure 4 show the result obtained for three different Beta distributions (parameters are: top $(1, 1)$, bottom left $(0.7, 0.3)$, bottom right $(0.7, 2)$). In all cases there is evidence of a strong sensitivity to the choice of γ. The optimal value of γ is always larger than the distribution mean (distributions mean is: top 0.5, bottom left 0.7, bottom right 0.26), so neither the distribution mean nor a generic "small" value are a good choice for γ. To a close-up view it emerges that a key role is played by the distribution variance (distributions variance is: top 0.083, bottom left 0.11, bottom right 0.052); indeed the peak of the plots can be found for a value of γ approximately equal to $E[f] + Var[f]$. In the end, being the α_is randomly drawn from a given probability distribution f, the value of γ maximizing the size of the set of C values for which the system's probability of convergence shows polarization toward 1 is $\gamma = E[f] + Var[f]$. More easily said: in the case of heterogeneity of expectations and parameter uncertainty with regard to money demand functions the best strategy for the CB is to set its adaptive parameter equal to the sum of mean and variance of the behavioral parameters' distribution.

As we will see now, the value of $\gamma = E[f] + Var[f]$ seems to be a robust optimal choice within this model.

Figure 4 about here

To conclude this section, we want to study reinforce the previous findings to answer the question of what is a smart choice for γ, from the point of view of a CB that cares for stabilizing

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[6] The parameter A and B of the Beta distributions have been chosen in $\{0.3, 0.7, 1, 2\}$ in all possible combinations.

inflation towards a target. We show here that the rather simple heuristic $\gamma = E[f] + Var[f]$ shows a strong capacity of stabilizing the system in presence of uncertainty about the private sector's adaptive parameters, α_i. Notice that the informational requirements on the CB's part implied by this are not very heavy. We tested such simple strategy against a wealth of other fixed choices and against similar competitors using the sample mean and variance: such numerical exercises involve a wide choice of distributions within the Beta family (both symmetric, including the special case of the uniform distribution, and asymmetric) for the α parameters, as well as several different values for the structural parameter C. For each set of parameters we ran 10000 runs. The parameter γ is set in various different ways: constant and variable (depending on the sampled α_is). The simulations covered therefore a grid described in the following table:

A	B	n
$\{0.5, 1, 2, 4\}$	$\{0.5, 1, 2, 4\}$	100
C	Fixed γ	Variable γ
$\{5, 10, 15, 20\}$	$\{0.1, 0.2, \ldots, 0.9, E[f], E[f] + var[f]\}$	$\{\bar{\alpha}, \bar{\alpha} + Var(\alpha_1, \ldots, \alpha_n)\}$

where A and B are the parameters that define the relevant Beta distribution used to draw the α_is, the fixed γ are just constant values for gamma evenly distributed on the unit interval or depending on the distribution mean and variance, and the variable γ are obtained for each sample of α_is as their sample mean or as their sample mean augmented by their sample variance. The use of variable values for γ implies that the CB knows the whole set of behavioral parameters or, at least, that it is capable to estimate it: clearly this is a very strong assumption. Nonetheless, as we supposed that $\tilde{\gamma} = \bar{\alpha} + Var(\alpha_1, \ldots, \alpha_n)$ would have been the best choice, we have decided to consider it among the alternatives so as to have a benchmark to compare the other possible choices. A summary of the results is contained in Table \ref.

Table 5 about here

The result of this simulation set confirm what we expected: in order to obtain a stable system, $\gamma = \tilde{\gamma}$ gives the best result in all cases but one (cell with grey background). On the the other hand, the interesting fact is that the choice $\gamma = E[f] + var[f]$ is also an excellent one. Results obtained using this heuristics are, with few exceptions (again cells with grey background), the second best, and when it is not the case they are very near the best outcome (never more than 4% absolute difference).

Conclusion

We developed a simple model in which a central bank and a boundedly rational private sector interact as the monetary authority pursuits a target for the level of inflation. The expectations about inflation of the various agents involved effectively drive the dynamics of the system. Introducing heterogeneity in such expectations is shown to have a significant impact, in particular when the central bank is uncertain about the relevant behavioral parameters. A heuristic for the central bank, based on the knowledge mean and variance of the probability distribution of the adaptive parameters used within the private sector, is shown to be able to stabilize the system for a rather large parametric region. This kind of analysis could benefit from further inquiry in various directions, for example modelling explicitly the goals of the authorities in terms of controlling the output gap and unemployment and allowing for more interesting expectations mechanisms in the private sector, such as fitness-based endogenous switching among different rules.

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Appendix

Proof of Proposition 1.

At first we establish the nature (real or complex) of eigenvalues. From condition $\text{tr}^2\left(J_{(1,1)}\right) - 4\det\left(J_{(1,1)}\right) < 0$, which ensures that eigenvalues are complex, we have:

$$\alpha^2\left(1-C\right)^2 - \alpha\left[2\gamma\left(1+C^2\right)\right] + \gamma^2\left(1+C\right)^2 < 0. \tag{14}$$

The second degree polynomial has the two roots $\alpha_{1,2} = \left[\gamma, \gamma\left(\frac{1+C}{1-C}\right)^2\right]$. The inequality is satisfied, given $\alpha, \gamma \in (0,1)$, in the following cases:

1.1 $\gamma\left(\frac{1+C}{1-C}\right)^2 < \alpha < \gamma$ if $C < 0$,

1.2 $\gamma < \alpha < \gamma\left(\frac{1+C}{1-C}\right)^2$ if $C > 0$,

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while for $C = 0$ complex eigenvalues are not allowed.

To expose the geometric locus of the Neimark-Hopf bifurcation, we need the solution of

$$\det\left(J_{(1,1)}\right) = 1 - \gamma\left(1 + C\right) - \alpha\left(1 - \gamma - C\right) > 1. \tag{15}$$

We obtain (for $\gamma \neq 1 - C$):

1.3 $\alpha > \frac{\gamma(1+C)}{\gamma+C-1}$ se $\gamma > 1 - C$

1.4 $\alpha < \frac{\gamma(1+C)}{\gamma+C-1}$ se $\gamma < 1 - C$.

The case $\gamma = 1 - C$ gives $C^2 > 1$ and the solution $|C| > 1$ is of no interest as $\gamma \in (0, 1)$. The Neimark-Hopf bifucation curve correspond to an hyperbola with asymptotes in $1 - C$ (vertical) e $1 + C$ (horizontal). The two regions specified in 1.3 e 1.4 belong to the feasible region of parameters when $C > 1$ and $C < -1$ respectively.

From $\operatorname{tr}\left(J_{(1,1)}\right) > \det\left(J_{(1,1)}\right) + 1$, we have $0 > \alpha\gamma$ which is never satisfied in the region we are interested in. So, the Saddle-Node bifurcation could never occur.

To find the geometric locus in which a Period-doubling bifurcation occurs we have to find out when the inequality $\operatorname{tr}\left(J_{(1,1)}\right) < -\det\left(J_{(1,1)}\right) - 1$ is satisfied, we have:

$$2 - \alpha\left(1 - C\right) - \gamma\left(1 + C\right) < -1 + \gamma\left(1 + C\right) + \alpha\left(1 - \gamma - C\right) - 1. \tag{16}$$

Whenever $\gamma \neq 2 - 2C$, we have two cases:

1.5 $\alpha < \frac{2\gamma(1+C)-4}{\gamma+2C-2}$ if $\gamma > 2 - 2C$

1.6 $\alpha > \frac{2\gamma(1+C)-4}{\gamma+2C-2}$ if $\gamma < 2 - 2C$.

When $\gamma = 2 - 2C$ there are no solutions.

The Period-doubling bifurcation curve is given by an hyperbola with asymptotes in $2 - 2C$ (vertical) e $2 + 2C$ (horizontal). As in the case of the Neimark-Hopf, conditions 1.5 e 1.6 can be satisfied, given the restictions on parameter values, for $C > 1$ and $C < -1$ respectively. ■

Proof of Proposition 2. Bischi and Marimon [1] show that:

- $A^F = \left\{\alpha \in [0,1] : \alpha < \frac{\pi}{C} \wedge C > \frac{\pi^{*2}}{\pi^{*2} - \pi^*}\right\}$
- $A^O = [0,1]$ if $C < 1$, ∅ if $C \geq 1$
- $A^I = \left\{\alpha \in [0,1] : \alpha < \frac{1}{C}\right\}$

As regards the policy AD, we have to decide how to set the value of the parameter γ as the set A^{AD} depends from it. In order to facilitate the comparison we set $\gamma = \hat{\gamma}$ such that the interval of "stable" α is of the type $[0, \hat{\alpha})$ with $\hat{\alpha}$ the greatest possible value. It is immediate to check that these values are $\hat{\gamma} = \frac{2}{1+C}$ and:

- $A^{AD} = \left\{\alpha \in [0,1] : \alpha < \hat{\alpha} = \frac{2(1+C)}{1+C^2}\right\}$.
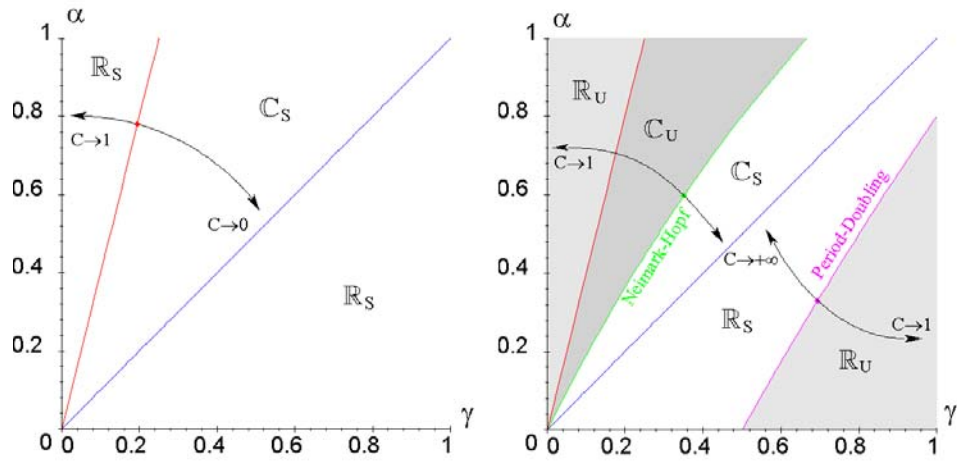
The result follows immediately. ■
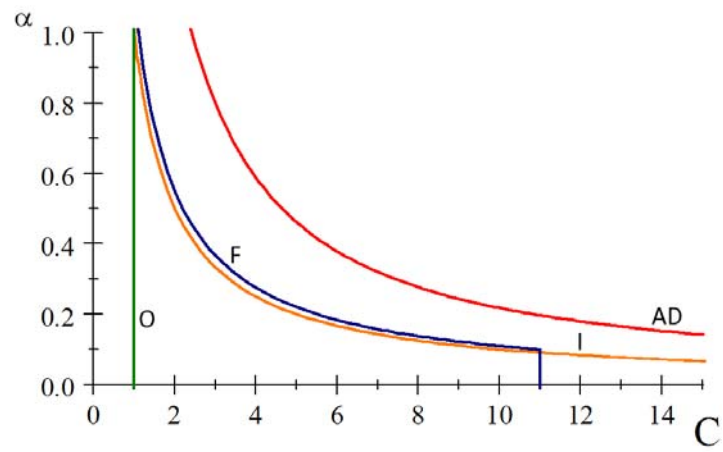
Figures and tables

Figure 1: Bifurcation diagrams.

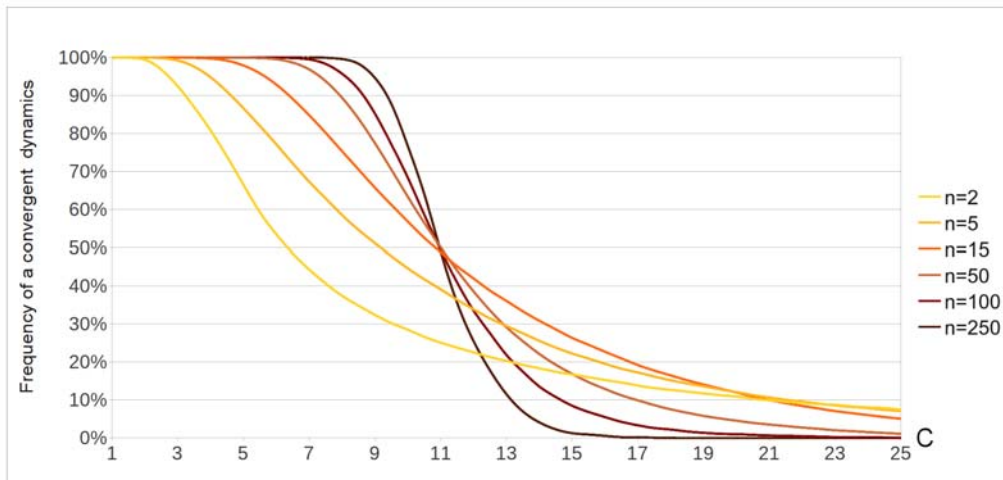Figure 2: Policy ranking $(\pi^* = 1.1)$

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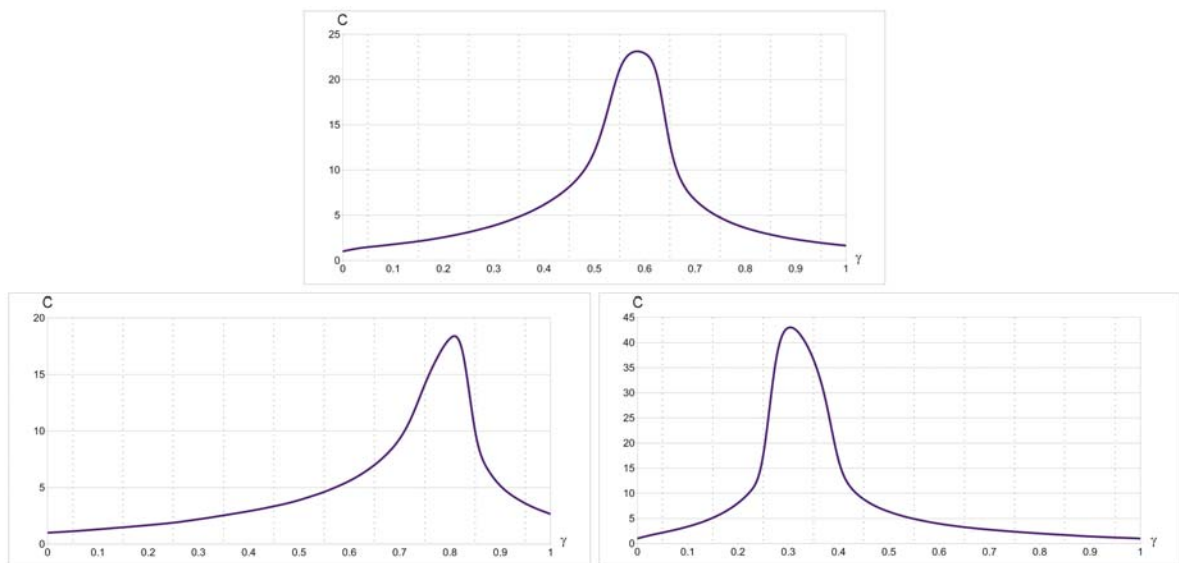Figure 3: Polarization of probabilities of convergence as n increases.

Figure 4: Highest value of C granting polarization of probability towards 1 as a function of γ.

A	B	C	Best fixed 0.1-0.9	Sample mean	Sample mean+var	Dist. mean	Dist. Mean+var
0.5	0.5	5	1	1	1	1	1
0.5	1	5	1	1	1	1	1
0.5	2	5	1	0	1	1	1
0.5	4	5	1	1	1	1	1
1	0.5	5	1	1	1	1	1
1	1	5	1	1	1	1	1
1	2	5	1	1	1	1	1
1	4	5	1	1	1	1	1
2	0.5	5	1	1	1	1	1
2	1	5	1	1	1	1	1
2	2	5	1	1	1	1	1
2	4	5	1	1	1	1	1
4	0.5	5	1	1	1	1	1
4	1	5	1	1	1	1	1
4	2	5	1	1	1	1	1
4	4	5	1	1	1	1	1
0.5	0.5	10	0.85	0	1	0.06	0.92
0.5	1	10	0.94	0.49	1	0.49	0.99
0.5	2	10	1	1	1	0.93	1
0.5	4	10	1	1	1	1	1
1	0.5	10	0.96	0.69	1	0.62	0.97
1	1	10	0.98	0.89	1	0.68	1
1	2	10	1	1	1	0.94	1
1	4	10	1	1	1	1	1
2	0.5	10	1	1	1	1	1
2	1	10	1	1	1	1	1
2	2	10	1	1	1	1	1
2	4	10	1	1	1	1	1
4	0.5	10	1	1	1	1	1
4	1	10	1	1	1	1	1
4	2	10	1	1	1	1	1
4	4	10	1	1	1	1	1
0.5	0.5	15	0.09	0	0.19	0	0.13
0.5	1	15	0.52	0	0.97	0.07	0.67
0.5	2	15	0.9	0.72	1	0.57	0.96
0.5	4	15	0.99	1	1	0.91	0.99
1	0.5	15	0.18	0	1	0.01	0.78
1	1	15	0.77	0	1	0.09	0.82
1	2	15	0.98	0.49	1	0.49	0.97
1	4	15	0.88	1	1	0.87	1
2	0.5	15	0.93	0.96	1	0.93	0.96
2	1	15	0.98	0.77	1	0.65	0.96
2	2	15	0.74	0.96	1	0.74	0.99
2	4	15	0.99	1	1	0.93	1
4	0.5	15	1	1	1	1	1
4	1	15	1	1	1	1	1
4	2	15	1	1	1	1	1
4	4	15	1	1	1	1	1
0.5	0.5	20	0	0	0	0	0
0.5	1	20	0.18	0	0.2	0.01	0.25
0.5	2	20	0.58	0.05	1	0.31	0.83
0.5	4	20	0.67	1	1	0.75	0.96
1	0.5	20	0.07	0	0.5	0	0.18
1	1	20	0.32	0	0.59	0.01	0.33
1	2	20	0.83	0	1	0.2	0.81
1	4	20	0.66	0.96	1	0.66	0.96
2	0.5	20	0.36	0.39	1	0.37	0.87
2	1	20	0.68	0.04	1	0.14	0.83
2	2	20	0.32	0.13	1	0.32	0.9
2	4	20	0.86	0.97	1	0.69	0.98
4	0.5	20	1	1	1	1	0.99
4	1	20	0.99	1	1	0.99	0.97
4	2	20	0.97	1	1	0.9	0.98
4	4	20	0.92	1	1	0.92	1

Figure 5: Probability of stable systems under different choices for γ.