

## Multi-mode bosonic Gaussian channels

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**Abstract.** A complete analysis of multi-mode bosonic Gaussian channels (BGCs) is proposed. We clarify the structure of unitary dilations of general Gaussian channels involving any number of bosonic modes and present a normal form. The maximum number of auxiliary modes that is needed is identified, including all rank deficient cases, and the specific role of additive classical noise is highlighted. By using this analysis, we derive a canonical matrix form of the noisy evolution of  $n$ -mode BGCs and of their weak complementary counterparts, based on a recent generalization of the normal mode decomposition for non-symmetric or locality constrained situations. This allows us to simplify the weak-degradability classification. Moreover, we investigate the structure of some singular multi-mode channels, like the additive classical noise channel that can be used to decompose a noisy channel in terms of a less noisy one in order to find new sets of maps with zero quantum capacity. Finally, the two-mode case is analyzed in detail. By exploiting the composition rules of two-mode maps and the fact that anti-degradable channels cannot be used to transfer quantum information, we identify sets of two-mode bosonic channels with zero capacity.

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Bosonic Gaussian channels (BGCs) are ubiquitous in physics. They arise whenever a harmonic system interacts linearly with a number of bosonic modes which are inaccessible in principle or in practice [1]–[7]. They provide realistic noise models for a variety of quantum optical and solid-state systems when treated as open quantum systems, including models for wave guides and quantum condensates. They play a fundamental role in characterizing the efficiency of a variety of tasks in continuous-variables quantum information processing [8], including quantum communication [9] and cryptography [10]. Most importantly, communication channels such as optical fibers can, to a good approximation, be described by Gaussian quantum channels.

Not very surprisingly in the light of the central status of such quantum channels, a lot of effort has been recently devoted to studying their properties (see [4] for a review), based on a long tradition of work on Gaussian channels [2, 3, 6]. Specifically, from a quantum information perspective, a key question is whether or not a channel allows for the reliable transmission of classical or quantum information [3, 4], [11]–[18]. Significant progress has been made in this respect in recent years, although for some important cases, like the thermal noise channel modelling a realistic fiber with offset noise, the quantum capacity is still not yet known. In this context, the degradability properties represent a powerful tool to simplify the quantum capacity issue of such Gaussian channels. Indeed, in [16, 17] it has been shown that

with some (important) exceptions, Gaussian channels which operate on a single bosonic mode (i.e. one-mode Gaussian channels) can be classified as weakly degradable or anti-degradable. This paved the way for the solution of the quantum capacity [19] for a large class of these maps [15].

Here, first we propose a general construction of unitary dilations of multi-mode quantum channels, including all rank-deficient cases. We characterize the minimal noise maps involving only true quantum noise. Then, by using a generalized normal mode decomposition recently introduced in [20], we generalize the results of [16, 17] concerning Gaussian weak complementary channels to the multi-mode case giving a simple weak-degradability/anti-degradability condition for such channels. The paper ends with a detailed analysis of the two-mode case. This is important since any  $n$ -mode channel can always be reduced to single- and two-mode components [20]. We detail the degradability analysis and investigate a useful decomposition of a channel with the additive classical noise map that allows us to find new sets of channels with zero quantum capacity.

## 1. Multi-mode BGCs

Gaussian channels arise from linear dynamics of an open bosonic system interacting with a Gaussian environment via quadratic Hamiltonians. Loosely speaking, they can be characterized as completely positive trace-preserving (CPT) maps that transform Gaussian states into Gaussian states [21, 22].

### 1.1. Notation and preliminaries

Consider a system composed of  $n$ -bosonic modes having canonical coordinates  $\hat{Q}_1, \hat{P}_1, \dots, \hat{Q}_n, \hat{P}_n$ . The canonical commutation relations of the canonical coordinates,  $[\hat{R}_j, \hat{R}_{j'}] = i(\sigma_{2n})_{j,j'}$ , where  $\hat{R} := (\hat{Q}_1, \dots, \hat{Q}_n; \hat{P}_1, \dots, \hat{P}_n)$ , are grasped by the  $2n \times 2n$  commutation matrix

$$\sigma_{2n} = \begin{bmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{bmatrix}, \quad (1)$$

when this order of canonical coordinates is chosen, (here  $\mathbb{1}_n$  is the  $n \times n$  identity matrix) [3, 4, 21]. Even though different reordering of the elements of  $\hat{R}$  will not affect the definitions that follow, we find it useful to assume a specific form for  $\sigma_{2n}$ . One defines the group of real symplectic matrices  $Sp(2n, \mathbb{R})$  as the set of  $2n \times 2n$  real matrices  $S$  which satisfy the condition

$$S\sigma_{2n}S^T = \sigma_{2n}. \quad (2)$$

Since  $\text{Det}[\sigma_{2n}] = 1$  and  $\sigma_{2n}^{-1} = -\sigma_{2n}$ , any symplectic matrix  $S$  has  $\text{Det}[S] = 1$  and it is invertible with  $S^{-1} \in Sp(2n, \mathbb{R})$ . Similarly, one has  $S^T \in Sp(2n, \mathbb{R})$ . Symplectic matrices play a key role in the characterization of bosonic systems. Indeed, define the Weyl (displacement) operators as  $\hat{V}(z) = \hat{V}^\dagger(-z) := \exp[i\hat{R}z]$  with  $z := (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)^T$  being a column vector of  $\mathbb{R}^{2n}$ . Then, it is possible to show [1] that for any  $S \in Sp(2n, \mathbb{R})$  there exists a canonical unitary transformation  $\hat{U}$ , which maps the canonical observables of the system into a linear combination of the operators  $\hat{R}_j$ , satisfying the condition

$$\hat{U}^\dagger \hat{V}(z) \hat{U} = \hat{V}(Sz) \quad (3)$$

for all  $z$ . This is often referred to as the metaplectic representation. Conversely, one can show that any unitary  $\hat{U}$  which transforms  $\hat{V}(z)$  as in equation (3) must correspond to an  $S \in Sp(2n, \mathbb{R})$ .

Weyl operators allow one to rewrite the canonical commutation relations as

$$\hat{V}(z)\hat{V}(z') = \exp\left[-\frac{i}{2}z^T\sigma_{2n}z'\right]\hat{V}(z+z') \quad (4)$$

and permit a complete description of the system in terms of (characteristic) complex functions. Specifically, any trace-class operator  $\hat{\Theta}$  (in particular, any density operator) can be expressed as

$$\hat{\Theta} = \int \frac{d^{2n}z}{(2\pi)^n} \phi(\hat{\Theta}; z) \hat{V}(-z), \quad (5)$$

where  $d^{2n}z := dx_1 \dots dx_n dy_1 \dots dy_n$  and  $\phi(\hat{\Theta}; z)$  is the characteristic function associated with the operator  $\hat{\Theta}$  defined by

$$\phi(\hat{\Theta}; z) := \text{Tr}\left[\hat{\Theta} \hat{V}(z)\right]. \quad (6)$$

Within this framework, a density operator  $\hat{\rho}$  of the  $n$  modes is said to represent a *Gaussian state* if its characteristic function  $\phi(\hat{\rho}; z)$  has a Gaussian form, i.e.

$$\phi(\hat{\rho}; z) = \exp\left[-\frac{1}{4}z^T\gamma z + im^T z\right], \quad (7)$$

with  $m$  being a real vector of mean values  $m_j := \text{Tr}[\hat{\rho} \hat{R}_j]$ , and the  $2n \times 2n$  real symmetric matrix  $\gamma$  being the *covariance matrix* [1, 4, 7] of  $\hat{\rho}$ . For generic density operators  $\hat{\rho}$  (not only the Gaussian ones), the latter is defined as the variance of the canonical coordinates  $\hat{R}$ , i.e.

$$\gamma_{j,j'} := \text{Tr}\left[\rho\{(R_j - m_j), (R_{j'} - m_{j'})\}\right], \quad (8)$$

with  $\{\cdot, \cdot\}$  being the anti-commutator, and it is bound to satisfy the uncertainty relations

$$\gamma \geq i\sigma_{2n} \quad (9)$$

with  $\sigma_{2n}$  being the commutation matrix (1). Up to an arbitrary vector  $m$ , the uncertainty inequality presented above uniquely characterizes the set of Gaussian states, i.e. any  $\gamma$  satisfying (9) defines a Gaussian state. Let us first notice that if  $\gamma$  satisfies (9) then it must be (strictly) positive definite  $\gamma > 0$ , and have  $\text{Det}[\gamma] \geq 1$ . From Williamson's theorem [23], it follows that there exists a symplectic  $S \in Sp(2n, \mathbb{R})$  such that

$$\gamma = S \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} S^T, \quad (10)$$

where  $D := \text{diag}(d_1, \dots, d_n)$  is a diagonal matrix formed by the *symplectic eigenvalues*  $d_j \geq 1$  of  $\gamma$ . For  $S = \mathbb{1}_{2n}$ , equation (10) gives the covariance matrix associated with thermal bosonic states. This also shows that any covariance matrix  $\gamma$  satisfying (9) can be written as

$$\gamma = SS^T + \Delta, \quad (11)$$

with  $\Delta \geq 0^6$ . The extremal solutions of equation (11), i.e.  $\gamma = SS^T$ , are *minimal uncertainty solutions* and correspond to the *pure Gaussian states* of  $n$  modes (e.g. multi-mode squeezed

<sup>6</sup> This is indeed the matrix

$$\Delta := S \begin{bmatrix} D - \mathbb{1}_n & 0 \\ 0 & D - \mathbb{1}_n \end{bmatrix} S^T \quad (12)$$

with  $D$  as in equation (10), which is positive since  $D \geq \mathbb{1}_n$ .

vacuum states). They are uniquely determined by the condition  $\text{Det}[\gamma] = 1$  and satisfy the condition [18]

$$\gamma = -\sigma_{2n}(\gamma^{-1})\sigma_{2n}. \quad (13)$$

### 1.2. BGCs

In the Schrödinger picture, evolution is described by applying the transformation to the states (i.e. the density operators)  $\hat{\rho} \mapsto \Phi(\hat{\rho})$ . In the Heisenberg picture, the transformation is applied to the observables of the system, while leaving the states unchanged,  $\hat{\Theta} \mapsto \Phi_H(\hat{\Theta})$ . The two pictures are related through the identity  $\text{Tr}[\Phi(\hat{\rho})\hat{\Theta}] = \text{Tr}[\hat{\rho}\Phi_H(\hat{\Theta})]$ , which holds for all  $\hat{\rho}$  and  $\hat{\Theta}$ . The map  $\Phi_H$  is called the *dual* of  $\Phi$ .

Due to the representation (5) and (6) any CPT transformation on the  $n$  modes can be characterized by its action on the Weyl operators of the system in the Heisenberg picture (e.g. see [17]). In particular, a BGC is defined as a map which, for all  $z$ , operates on  $V(z)$  according to [3]

$$\hat{V}(z) \mapsto \Phi_H(\hat{V}(z)) := \hat{V}(Xz) \exp\left[-\frac{1}{4}z^T Yz + iv^T z\right], \quad (14)$$

with  $v$  being some fixed real vector of  $\mathbb{R}^{2n}$ , and with  $Y, X \in \mathbb{R}^{2n \times 2n}$  being some fixed real  $2n \times 2n$  matrices satisfying the complete positivity condition

$$Y \geq i\Sigma \quad \text{with} \quad \Sigma := \sigma_{2n} - X^T \sigma_{2n} X. \quad (15)$$

In the context of BGCs, the above inequality is the quantum channel counterpart of the uncertainty relation (9). Indeed up to a vector  $v$ , equation (15) uniquely determines the set of BGCs and bounds  $Y$  to be positive-semidefinite,  $Y \geq 0$ . However, differently from (9), in this case strict positivity is not a necessary prerequisite for  $Y$ . A completely positive map defined by equations (14) and (15) will be referred to as BGC. As mentioned before, such a map is a model for a wide class of physical situations, ranging from communication channels such as optical fibers, to open quantum systems, and to dynamics in harmonic lattice systems. Whenever one has only partial access to the dynamics of a system that can be well described by a time evolution governed by a Hamiltonian that is a quadratic polynomial in the canonical coordinates, one will arrive at a model described by equations (14) and (15)<sup>7</sup>.

An important subset of the BGCs is given by set of *Gaussian unitary* transformations which have  $Y = 0$ ,  $X \in Sp(2n, \mathbb{R})$  and  $v$  arbitrary. They include the canonical transformations of equation (3) (characterized by  $v = 0$ ), and the displacement transformations (characterized by having  $X = \mathbb{1}_{2n}$  and  $v$  arbitrary). The latter simply adds a phase to the Weyl operators and corresponds to unitary transformations of the form  $\Phi_H(\hat{V}(z)) := \hat{V}(-v)\hat{V}(z)\hat{V}(v) = \hat{V}(z) \exp[iv^T z]$ .

In the Schrödinger picture, the BGC transformation (14) induces a mapping of the characteristic functions of the form

$$\phi(\hat{\rho}; z) \mapsto \phi(\Phi(\hat{\rho}); z) := \phi(\hat{\rho}; Xz) \exp\left[-\frac{1}{4}z^T Yz + iv^T z\right], \quad (16)$$

<sup>7</sup> This set does not contain ideal Gaussian measurements, like optical homodyning [24].

which in turn yields the following transformation of the mean and the covariance matrix:

$$\begin{aligned} m &\longmapsto X^T m + v, \\ \gamma &\longmapsto X^T \gamma X + Y. \end{aligned} \quad (17)$$

Clearly, BGCs always map Gaussian input states into Gaussian output states.

For purposes of assessing quantum or classical information capacities, output entropies, or studying degradability or anti-degradability of a channel [14]–[17], the full knowledge of the channel is not required: transforming the input or the output with any unitary operation (say, Gaussian unitaries) will not alter any of these quantities. It is then convenient to take advantage of this freedom to simplify the description of the BGCs. To do so, we first notice that the set of Gaussian maps is closed under composition. Consider then  $\Phi'$  and  $\Phi''$  two BGCs described respectively by the elements  $X', Y', v'$  and  $X'', Y'', v''$ . The composition  $\Phi'' \circ \Phi'$  where, in Schrödinger representation, we first operate with  $\Phi'$  and then with  $\Phi''$ , is still a BGC and it is characterized by the parameters

$$\begin{aligned} v &= (X'')^T v' + v'', \\ X &= X' X'', \\ Y &= (X'')^T Y' X'' + Y''. \end{aligned} \quad (18)$$

Exploiting these composition rules it is then easy to verify that the vector  $v$  can always be compensated by properly displacing either the input state or the output state (or both) of the channel. For instance by taking  $X'' = \mathbb{1}_{2n}$ ,  $Y'' = 0$  and  $v'' = -v'$ , equation (18) shows that  $\Phi'$  is unitarily equivalent to the Gaussian channel  $\Phi$ , which has  $v = 0$  and  $X = X'$ ,  $Y = Y'$ . Therefore, without loss of generality, in the following, we will focus on BGCs having  $v = 0$ .

More generally consider the case where we cascade a generic BGC  $\Phi'$  described by matrices  $X'$  and  $Y'$  as in equation (15) with a couple of canonical unitary transformation  $\hat{U}_1$  and  $\hat{U}_2$  described by the symplectic matrices  $S_1$  and  $S_2$ , respectively. The resulting BGC  $\Phi$  is then described by the matrices

$$X = S_1(X')S_2, \quad Y = S_2^T(Y')S_2. \quad (19)$$

For single mode ( $n = 1$ ), this procedure induces a simplified canonical form [13, 14, 17] which, up to a Gaussian unitarily equivalence, allows one to focus only on transformations characterized by  $X$  and  $Y$  which, apart from some special cases, are proportional to the identity. In this paper, we will generalize some of these results to an arbitrary number of modes  $n$ . To achieve this goal, in the following section, we first present an explicit dilation representation in which the mapping (14) is described as a (canonical) unitary coupling between the  $n$  modes of the system and some extra *environmental* modes which are initially prepared into a Gaussian state. Then, we will introduce the notion of a minimal noise channel, showing a useful decomposition rule.

## 2. Unitary dilation theorem

In this section, we introduce a general construction of unitary dilations of multi-mode quantum channels. Specifically, we show that a CPT channel acting on  $n$  modes is a BGC if and only if it can be realized by invoking  $\ell \leq 2n$  additional (environmental) modes  $E$  through the expression

$$\Phi(\hat{\rho}) = \text{Tr}_E \left[ \hat{U}(\hat{\rho} \otimes \hat{\rho}_E) \hat{U}^\dagger \right], \quad (20)$$

where  $\hat{\rho}$  is the input  $n$ -mode state of the system,  $\hat{\rho}_E$  is a Gaussian state of an environment,  $\hat{U}$  is a canonical unitary transformation which couples the system with the environment and  $\text{Tr}_E$  denotes the partial trace over  $E$ . In the case in which  $\hat{\rho}_E$  is pure, equation (20) corresponds to a Stinespring dilation [25] of the channel  $\Phi$ , otherwise it is a physical representation analogous to those employed in [16, 17] for the single-mode case.

### 2.1. General dilations

In this subsection, we will construct Gaussian dilations, including a discussion of all rank-deficient cases, and later focus on dilations involving the minimal number of modes. To proceed, we will first establish some conventions and notation. To start with, we write the commutation matrix of our  $n + \ell$  modes in the block structure

$$\sigma := \sigma_{2n} \oplus \sigma_{2\ell}^E = \left[ \begin{array}{c|c} \sigma_{2n} & 0 \\ \hline 0 & \sigma_{2\ell}^E \end{array} \right] \begin{array}{l} \} 2n, \\ \} 2\ell, \end{array} \quad (21)$$

where  $\sigma_{2n}$  and  $\sigma_{2\ell}^E$  are  $2n \times 2n$  and  $2\ell \times 2\ell$  commutation matrices associated with the system and environmental modes, respectively. For  $\sigma_{2n}$ , we assume the structure as defined in equation (1). For  $\sigma_{2\ell}^E$ , in contrast, we do not make any assumption at this point, leaving open the possibility of defining it later on<sup>8</sup>. Accordingly, the canonical unitary transformation  $\hat{U}$  of equation (20) will be uniquely determined by a  $2(n + \ell) \times 2(n + \ell)$  real matrix  $S \in Sp(2(n + \ell), \mathbb{R})$  of block form

$$S := \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}, \quad (22)$$

which satisfies the condition

$$S\sigma S^T = \sigma, \quad \iff \begin{cases} s_1 \sigma_{2n} s_1^T + s_2 \sigma_{2\ell}^E s_2^T = \sigma_{2n}, \\ s_1 \sigma_{2n} s_3^T + s_2 \sigma_{2\ell}^E s_4^T = 0, \\ s_3 \sigma_{2n} s_3^T + s_4 \sigma_{2\ell}^E s_4^T = \sigma_{2\ell}^E. \end{cases} \quad (23)$$

In the above expressions,  $s_1$  and  $s_4$  are  $2n \times 2n$  and  $2\ell \times 2\ell$  real square matrices, whereas  $s_2$  and  $s_3^T$  are  $2n \times 2\ell$  real rectangular matrices. Introducing then the covariance matrices  $\gamma \geq i\sigma_{2n}$  and  $\gamma_E \geq i\sigma_{2\ell}^E$  of the states  $\hat{\rho}$  and  $\hat{\rho}_E$ , the identity (20) can be written as

$$S \begin{bmatrix} \gamma & 0 \\ 0 & \gamma_E \end{bmatrix} S^T \Big|_{2n} = s_1 \gamma s_1^T + s_2 \gamma_E s_2^T = X^T \gamma X + Y, \quad (24)$$

where  $|_{2n}$  denotes the upper principle submatrix of degree  $2n$ , and where  $X, Y \in \mathbb{R}^{2n \times 2n}$  satisfying the condition (15) are the matrices associated with the channel  $\Phi$ . In writing equation (24), we use the fact that due to the definition (21) the covariance matrix of the composite

<sup>8</sup> With this choice the canonical commutation relations of the  $n + \ell$  mode read as  $[\hat{R}_j, \hat{R}_{j'}] = i\sigma_{j,j'}$  where  $\hat{R} := (\hat{Q}_1, \dots, \hat{Q}_n; \hat{P}_1, \dots, \hat{P}_n; \hat{r}_1, \dots, \hat{r}_{2\ell})$  with  $\hat{Q}_j$  and  $\hat{P}_j$  being the canonical coordinates of the  $j$ th system mode and with and  $\hat{r}_1, \dots, \hat{r}_{2\ell}$  being some ordering of the canonical coordinates  $\hat{Q}_1^E, \hat{P}_1^E; \dots; \hat{Q}_\ell^E, \hat{P}_\ell^E$  of the environmental modes. For instance, taking  $\sigma_{2\ell}^E = \sigma_{2\ell}$  corresponds to having  $\hat{R} := (\hat{Q}_1, \dots, \hat{Q}_n; \hat{P}_1, \dots, \hat{P}_n; \hat{Q}_1^E, \dots, \hat{Q}_\ell^E; \hat{P}_1^E, \dots, \hat{P}_\ell^E)$ .



state  $\hat{\rho} \otimes \hat{\rho}_E$  can be expressed as  $\gamma \oplus \gamma_E$ . With these definitions, the first part of the unitary dilation property (20) can be written as follows:

**Proposition 1 (Unitary dilations of Gaussian channels).** *Let  $\gamma_E$  be the covariance matrix of a Gaussian state of  $\ell$  modes and let  $S \in Sp(2(n + \ell), \mathbb{R})$  be a symplectic transformation. Then there exists a symmetric  $2n \times 2n$ -matrix  $Y \geq 0$  and a  $2n \times 2n$ -matrix  $X$  satisfying the condition (15), such that equation (24) holds for all  $\gamma$ .*

**Proof.** The proof is straightforward: we write  $S$  in the block form (22) and take  $X = s_1^T$  and  $Y = s_2 \gamma_E s_2^T$ . Since  $\gamma_E$  is a covariance matrix of  $\ell$  modes,  $\gamma_E - i\sigma_{2\ell} \geq 0$  and therefore  $s_2(\gamma_E - i\sigma_{2\ell})s_2^T \geq 0$ . This leads to equation (15) through the identity the symplectic condition  $s_1 \sigma_{2n} s_1^T + s_2 \sigma_{2\ell} s_2^T = \sigma_{2n}$  which follows by comparing the upper principle submatrices of degree  $n$  of both terms of equation (23).  $\square$

This proves that any CPT map obtained by coupling the  $n$  modes with a Gaussian state of  $\ell$  environmental bosonic modes through a Gaussian unitary  $\hat{U}$  is a BGC. The converse property is more demanding. In order to present it we find it useful to state first the following:

**Lemma 1 (Extensions of symplectic forms).** *Let, for some skew symmetric  $\sigma_{2\ell}^E$ ,  $s_1$  and  $s_2$  be  $2n \times 2n$  and  $2n \times 2\ell$  real matrices forming a symplectic system, i.e.  $s_1 \sigma_{2n} s_1^T + s_2 \sigma_{2\ell}^E s_2^T = \sigma_{2n}$ . Then, we can always find real matrices  $s_3$  and  $s_4$  such that  $S$  of equation (22) is symplectic with respect to the commutation matrix (21).*

**Proof.** Since the rows of  $S$  form a symplectic basis, given  $s_1$  and  $s_2$  (an incomplete symplectic basis), it is always possible to find  $s_3$  and  $s_4$  as above. The proof easily follows from a skew-symmetric version of the Gram–Schmidt process to construct a symplectic basis [26]. For a special subset of BGCs, in section 2.5 we will present an explicit expression for  $S$  based on a simplified (canonical) representation of the  $X$  matrix that defines  $\Phi$ . See also appendix A.  $\square$

Due to the above result, the possibility of realizing unitary dilation equation (20) for a generic BGC described by the matrices  $X$  and  $Y \geq i\Sigma = i(\sigma_{2n} - X^T \sigma_{2n} X)$ , can be proven by simply taking  $s_1 = X^T$  and finding some  $2n \times 2\ell$  real matrix  $s_2$  and an  $\ell$ -mode covariance matrix  $\gamma_E \geq i\sigma_{2\ell}^E$  that solve the equations

$$s_2 \sigma_{2\ell}^E s_2^T = \sigma_{2n} - s_1 \sigma_{2n} s_1^T = \Sigma, \quad (25)$$

$$s_2 \gamma_E s_2^T = Y. \quad (26)$$

With this choice in fact equation (24) is trivially satisfied for all  $\gamma$ , while  $s_1$  and  $s_2$  can be completed to a symplectic matrix  $S \in Sp(2(n + \ell), \mathbb{R})$ . The unitary dilation property (20) can hence be restated as follows:

**Theorem 1 (Unitary dilations of Gaussian channels: converse implication).** *For any real  $2n \times 2n$ -matrices  $X$  and  $Y$  satisfying the condition (15), there exists  $\ell$  smaller than or equal to  $2n$ ,  $S \in Sp(2(n + \ell), \mathbb{R})$ , and a covariance matrix  $\gamma_E$  of  $\ell$  modes, such that equation (24) is satisfied.*



**Proof.** As already noted the whole problem can be solved by assuming  $s_1 = X^T$  and finding  $s_2$  and  $\gamma_E$  that satisfy equations (25) and (26). We start by observing that the  $2n \times 2n$  matrix  $\Sigma$  defined in equation (15) is skew-symmetric, i.e.  $\Sigma = -\Sigma^T$ . Moreover according to equation (15) its support must be contained in the support of  $Y$ , i.e.  $\text{Supp}[\Sigma] \subseteq \text{Supp}[Y]$ . Consequently, given  $k := \text{rank}[Y]$  and  $r := \text{rank}[\Sigma]$  as the ranks of  $Y$  and  $\Sigma$ , respectively, one has that  $k \geq r$ . We can hence identify three different regimes:

- (i)  $k = 2n$  and  $r = 2n$ , i.e. both  $Y$  and  $\Sigma$  are full rank and hence invertible. Loosely speaking, this means that all the noise components in the channel are basically quantum (although may involve classical noise as well).
- (ii)  $k = 2n$  and  $r < 2n$ , i.e.  $Y$  is full rank and hence invertible, whereas  $\Sigma$  is singular. This means that some of the noise components can be purely classical, but still nondegenerate.
- (iii)  $2n > k \geq r$ , i.e. both  $Y$  and  $\Sigma$  are singular. There are noise components with zero variance.

Even though (i) and (ii) admit similar solutions, it is instructive to analyze them separately. In the former case, in fact, there is a simple direct way of constructing a physical dilation of the channel with  $\ell = n$  environmental modes.

(i) Since  $\Sigma$  is skew-symmetric and invertible there exists an  $O \in O(2n, \mathbb{R})$  orthogonal such that

$$O\Sigma O^T = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}, \quad (27)$$

where  $\mu = \text{diag}(\mu_1, \dots, \mu_n)$  and  $\mu_i > 0$  for all  $i = 1, \dots, n$  (see p 107 in reference [27]). Hence,  $K := M^{-1/2}O$  with  $M := \mu \oplus \mu$  satisfies

$$K\Sigma K^T = \sigma_{2n}. \quad (28)$$

Taking then  $s_2 := K^{-1}$  we get<sup>9</sup>

$$s_2 \sigma_{2n} s_2^T = K^{-1} \sigma_{2n} K^{-T} = \Sigma, \quad (29)$$

which corresponds to equation (25) for  $\ell = n$ . Since  $s_1 = X^T$ , lemma 1 guarantees that this is sufficient to prove the existence of  $S$ . The condition (24) finally follows by taking  $\gamma_E = KYK^T$  which is strictly positive (indeed  $K$  is invertible and  $Y > 0$  because it has full rank) and which satisfies the uncertainty relation (9), i.e.

$$Y \geq i\Sigma \implies \gamma_E = KYK^T \geq iK\Sigma K^T = i\sigma_{2n}. \quad (30)$$

This shows that the channel admits a unitary dilation of the form as specified in equation (20) with  $\ell = n$  environmental modes with commutation matrix,  $\sigma_{2n}^E = \sigma_{2n}$ —see discussion after equation (21). Such a solution, however, will involve a pure state  $\hat{\rho}_E$  only if  $\text{Det}[\gamma_E] = 1$ , i.e.

$$\text{Det}[Y]\text{Det}[K]^2 = 1 \iff \text{Det}[Y] = \text{Det}[\Sigma]. \quad (31)$$

When  $\text{Det}[\gamma_E] > 1$ , i.e.  $\text{Det}[Y] > \text{Det}[\Sigma]$ , we can still construct a pure dilation by simply adding further  $n$  modes which purify the state associated with the covariance matrix  $\gamma_E$  and by extending the unitary operator  $\hat{U}$  associated with  $S$  as the identity operator on them.

<sup>9</sup> From now on, the symbol  $A^{-T}$  will be used to indicate the transpose of the inverse of the matrix  $A$ , i.e.  $A^{-T} := (A^{-1})^T = (A^T)^{-1}$ .

For details, see the discussion of case (ii) given below. This corresponds to constructing a unitary dilation (20) with the pure state  $\hat{\rho}_E$  being defined on  $\ell = 2n$  modes.

(ii) In this case,  $Y$  is still invertible, whereas  $\Sigma$  is not. Differently from the approach we adopted in solving case (i), we here derive directly a Stinespring unitary dilation, i.e. we construct a solution with a pure  $\gamma_E$  that involves  $\ell = 2n$  environmental modes. In the next section, however, we will show that, dropping the purity requirement, one can construct unitary dilation that involves  $\hat{\rho}_E$  with only  $\ell = 2n - r/2$  modes.

To find  $s_2$  and  $\gamma_E$  which solve equations (25) and (26), it is useful to first transform  $Y$  into a simpler form by a congruent transformation, i.e.

$$CYC^T = \mathbb{1}_{2n} \quad (32)$$

with  $C \in Gl(2n, \mathbb{R})$  being not singular, e.g.  $C := Y^{-1/2}$ . From equation (15), it then follows that

$$\mathbb{1}_{2n} \geq i\Sigma' \quad (33)$$

with  $\Sigma' := Y^{-1/2} \Sigma Y^{-1/2}$  being skew-symmetric (i.e.  $\Sigma' = -(\Sigma')^T$ ) and singular with  $\text{rank}[\Sigma'] = \text{rank}[\Sigma] = r$  [27]. We then observe that introducing

$$s_2 = Y^{1/2} s'_2, \quad (34)$$

the conditions (25) and (26) can be written as

$$s'_2 \sigma_{2\ell}^E (s'_2)^T = \Sigma', \quad (35)$$

$$s'_2 \gamma_E (s'_2)^T = \mathbb{1}_{2n}. \quad (36)$$

Finding  $s'_2$  and  $\gamma_E$  which satisfy these expressions will provide us also a solution of equations (25) and (26).

As in the case of equation (27), there exists an orthogonal matrix  $O \in O(2n, \mathbb{R})$  which transforms the skew-symmetric matrix  $\Sigma'$  in a simplified block form. In this case, however, since  $\Sigma'$  is singular, we find [27]

$$O\Sigma'O^T = \left[ \begin{array}{c|c|c} 0 & \begin{array}{c|c} \mu & 0 \\ 0 & 0 \end{array} & \\ \hline \begin{array}{c|c} -\mu & 0 \\ 0 & 0 \end{array} & & 0 \\ \hline & & \end{array} \right] \begin{array}{l} \} r/2, \\ \} n - r/2, \\ \} r/2, \\ \} n - r/2, \end{array} \quad (37)$$

where now  $\mu = \text{diag}(\mu_1, \dots, \mu_{r/2})$  is the  $r/2 \times r/2$  diagonal matrix formed by the strictly positive eigenvalues of  $|\Sigma'|$  which satisfy the conditions  $1 \geq \mu_j > 0$ , this being equivalent to

$$\mathbb{1}_{r/2} \geq \mu, \quad (38)$$

as a consequence of inequality (33). Define then  $K := M^{-1/2} O$  with

$$M = \left[ \begin{array}{c|c|c} \begin{array}{c|c} \mu & 0 \\ 0 & \mathbb{1}_{n-r/2} \end{array} & & 0 \\ \hline & & \begin{array}{c|c} \mu & 0 \\ 0 & \mathbb{1}_{n-r/2} \end{array} \\ \hline & & \end{array} \right] \begin{array}{l} \} r/2, \\ \} n - r/2, \\ \} r/2, \\ \} n - r/2. \end{array} \quad (39)$$

It satisfies the identity

$$K\Sigma'K^T = \left[ \begin{array}{c|c|c} 0 & \begin{array}{c|c} \mathbb{1}_{r/2} & 0 \\ 0 & 0 \end{array} & \\ \hline \begin{array}{c|c} -\mathbb{1}_{r/2} & 0 \\ 0 & 0 \end{array} & & 0 \\ \hline & & \end{array} \right] \begin{array}{l} \} r/2, \\ \} n - r/2, \\ \} r/2, \\ \} n - r/2. \end{array} \quad (40)$$

To show that equations (35) and (36) admit a solution we take  $\ell = 2n$  and write  $\sigma_{4n}^E = \sigma_{2n} \oplus \sigma_{2n} = \sigma_{4n}$  with  $\sigma_{2n}$  as in equation (1). With these definitions the  $2n \times 4n$  rectangular matrix  $s'_2$  can be chosen to have the block structure

$$s'_2 = [K^{-1} \mid O^T A] \quad (41)$$

with  $A$  being the following  $2n \times 2n$  symmetric matrix

$$A = A^T = \left[ \begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & \mathbb{1}_{n-r/2} \\ \hline 0 & 0 & & \\ \hline 0 & \mathbb{1}_{n-r/2} & & 0 \end{array} \right] \begin{array}{l} \} r/2, \\ \} n-r/2, \\ \} r/2, \\ \} n-r/2. \end{array} \quad (42)$$

By direct substitution one can easily verify that equation (35) is indeed satisfied, see appendix B for details. Inserting equation (41) into (36) yields now the following equation:

$$\alpha + A\delta^T + \delta A^T + A\beta A^T = M^{-1} \quad (43)$$

for the  $4n \times 4n$  covariance matrix

$$\gamma_E = \begin{bmatrix} \alpha & \delta \\ \delta^T & \beta \end{bmatrix}, \quad (44)$$

see appendix C for details. A solution can be easily derived by taking

$$\alpha = \beta = \left[ \begin{array}{cc|cc} \mu^{-1} & 0 & & \\ 0 & \xi \mathbb{1}_{n-r/2} & & \\ \hline & & 0 & \\ \hline & & \mu^{-1} & 0 \\ & & 0 & \xi \mathbb{1}_{n-r/2} \end{array} \right] \begin{array}{l} \} r/2, \\ \} n-r/2, \\ \} r/2, \\ \} n-r/2 \end{array} \quad (45)$$

with  $\xi = 5/4$  and

$$\delta = \left[ \begin{array}{cc|cc} & & f(\mu^{-1}) & 0 \\ & & 0 & f(\xi \mathbb{1}_{n-r/2}) \\ \hline f(\mu^{-1}) & 0 & & \\ \hline 0 & f(\xi \mathbb{1}_{n-r/2}) & & 0 \end{array} \right] \begin{array}{l} \} r/2, \\ \} n-r/2, \\ \} r/2, \\ \} n-r/2, \end{array} \quad (46)$$

with  $f(\theta) := -(\theta^2 - \mathbb{1})^{1/2}$ . For all diagonal matrices  $\mu$  compatible with the constraint (38) the resulting  $\gamma_E$  satisfies the uncertainty relation  $\gamma_E \geq i\sigma_{4n}$ . Moreover, since it has  $\text{Det}[\gamma_E] = 1$ , this is also a minimal uncertainty state, i.e. a pure Gaussian state of  $2n$  modes. It is worth stressing that for  $r = 2n$ , i.e. when also the rank of  $\Sigma$  is maximum, the above solution provides an alternative derivation of the unitary dilation discussed in part (i) of the theorem. In this case, the covariance matrix  $\gamma_E$  has block elements

$$\alpha = \beta = \begin{bmatrix} \mu^{-1} & 0 \\ 0 & \mu^{-1} \end{bmatrix} \begin{array}{l} \} n, \\ \} n, \end{array} \quad \delta = \begin{bmatrix} 0 & f(\mu^{-1}) \\ f(\mu^{-1}) & 0 \end{bmatrix} \begin{array}{l} \} n, \\ \} n, \end{array} \quad (47)$$

where  $\mu$  is now a strictly positive  $n \times n$  matrix, whereas equations (34) and (41) yield

$$s_2 := Y^{1/2} O^T \left[ \begin{array}{cc|cc} \mu^{1/2} & 0 & 0 & 0 \\ 0 & \mu^{1/2} & 0 & 0 \end{array} \right] \begin{array}{l} \} n, \\ \} n. \end{array} \quad (48)$$

(iii) Here, both  $Y$  and  $\Sigma$  are singular. This case is very similar to case (ii). Here, the dilation can be constructed by introducing a strictly positive matrix  $\bar{Y} > 0$  which satisfies the condition

$$\Pi \bar{Y} \Pi = Y \quad (49)$$

with  $\Pi$  being the projector onto the support of  $Y$ . Such a  $\bar{Y}$  always exists ( $\bar{Y} = Y + (\mathbb{1} - \Pi)$ ). By construction, it satisfies the inequality  $\bar{Y} \geq Y \geq i\Sigma$ . According to section 2.2,  $\bar{Y}$  and  $X$  define thus a BGC. Moreover, since  $\bar{Y}$  is strictly positive, it has full rank. Therefore, we can use part (ii) of the proof to find a  $2n \times 2\ell$  matrix  $\bar{s}_2$  and  $\bar{\gamma}_E \geq i\sigma_{2\ell}$  which satisfy the conditions (25) and (26), i.e.

$$\bar{s}_2 \sigma_{2\ell}^E \bar{s}_2^T = \Sigma, \quad (50)$$

$$\bar{s}_2 \bar{\gamma}_E \bar{s}_2^T = \bar{Y}. \quad (51)$$

A unitary dilation for the channel  $Y$ ,  $X$  is then obtained by choosing  $\gamma_E = \bar{\gamma}_E$  and  $s_2 = \Pi \bar{s}_2$ . In fact from equation (51), we get

$$s_2 \gamma_E s_2^T = \Pi \bar{s}_2 \bar{\gamma}_E \bar{s}_2^T \Pi = \Pi \bar{Y} \Pi = Y, \quad (52)$$

whereas from equation (50)

$$s_2 \sigma_{2\ell}^E s_2^T = \Pi \bar{s}_2 \sigma_{2\ell}^E \bar{s}_2^T \Pi = \Pi \Sigma \Pi = \Sigma, \quad (53)$$

where we have used the fact that  $\text{Supp}[\Sigma] \subseteq \text{Supp}[Y]$ .  $\square$

In proving the second part of the unitary dilations theorem, we provided explicit expressions for the environmental state  $\hat{\rho}_E$  of equation (20). Specifically, such a state is given by the pure  $2n$ -modes Gaussian state  $\hat{\rho}_E$  characterized by the covariance matrix  $\gamma_E$  of elements (45) and (46). A trivial observation is that this can always be replaced by the  $2n$ -modes vacuum state  $|\emptyset\rangle\langle\emptyset|$  having the covariance matrix  $\gamma_E^{(0)} = \mathbb{1}_{2n}$ . This is a consequence of the obvious property that according to equation (11) all pure Gaussian states are equivalent to  $|\emptyset\rangle\langle\emptyset|$  up to a Gaussian unitary transformation. On the level of covariance matrices, Gaussian unitaries correspond to symplectic transformations. For a remark on unitarily equivalent dilations, see also appendix D. Hence, by means of a congruence with an appropriate symplectic transformation, we immediately arrive at the following corollary:

**Corollary 1 (Gaussian channels with pure Gaussian dilations).** *Any  $n$ -mode Gaussian channel  $\Phi$  admits a Gaussian unitary dilation (20) with  $\hat{\rho}_E = |\emptyset\rangle\langle\emptyset|$  being the vacuum state on  $2n$  modes.*

## 2.2. Reducing the number of environmental modes

An interesting question is the characterization of the minimal number of environmental modes  $\ell$  that need to be involved in the unitary dilation (20). From theorem 1, we know that this number is certainly smaller than or equal to twice the number  $n$  of modes on which the BGC is operating: we have in fact explicitly constructed one such representation that involves  $\ell = 2n$  modes in a minimal uncertainty, i.e. pure Gaussian state. We also know, however, that there are situations<sup>10</sup> in which  $\ell$  can be reduced to just  $n$ : this happens for instance for BGCs  $\Phi$  with  $\text{rank}[Y] = \text{rank}[\Sigma] = 2n$ , i.e. case (i) of theorem 1. In this case, one can represent the channel  $\Phi$  in terms of a Gaussian unitary coupling with  $\ell = n$  environmental modes which are prepared into a Gaussian state with covariance matrix

$$\gamma_E = KYK^T, \quad (54)$$

<sup>10</sup> Not mentioning the trivial case of Gaussian unitary transformation, which does not require any environmental mode to construct a unitary dilation.

—see equation (30). In general, this will not be of Stinespring form (not be a pure unitary dilation) since  $\gamma_E$  is not a minimal uncertainty covariance matrix. In fact, for  $n = 1$  this corresponds to the physical representation of  $\Phi$  of [17]. However, if  $Y$  and  $X$  satisfy the condition (31), our analysis provides a unitary dilation involving merely  $\ell = n$  modes in a pure Gaussian state.

We can then formulate a necessary and sufficient condition for the channels  $\Phi$  of class (i) which can be described in terms of  $n$ -environmental modes prepared into a pure state. It is given by

$$Y = \Sigma Y^{-1} \Sigma^T, \quad (55)$$

which follows by imposing the minimal uncertainty condition (13) on the  $n$ -mode covariance matrix (54) and by using (28). Similarly, one can verify that given a pure  $n$ -modes Gaussian state  $\hat{\rho}_E$  and an  $S \in Sp(4n, \mathbb{R})$  (22) with an invertible subblock  $s_2$ , then the corresponding BGC satisfies condition (55). The above result can be strengthened by looking at the solutions for channels of class (ii) of which the channels of class (i) are a proper subset.

To achieve this goal, let us first note that with the choice we made on  $\sigma_{2\ell}^E = \sigma_{4n}$ , the two matrices  $\alpha$  and  $\beta$  of equation (45) are  $2n \times 2n$  covariance matrices for two sets of independent  $n$  bosonic modes satisfying the uncertainty relations (9) with respect to the form  $\sigma_{2n}$ . In turn, the matrices  $\delta$  and  $\delta^T$  of equation (46) represent cross-correlation terms among such sets. After all, the entire covariance matrix  $\gamma_E$  corresponds to a pure Gaussian state.

The key point is now the observation that in equation (43), the matrix  $A$  couples only with those rows and columns of the matrices  $\delta$  and  $\beta$ , which contain elements  $\xi \mathbb{1}_{n-r/2}$  or  $f(\xi \mathbb{1}_{n-r/2})$ : as far as  $A$  is concerned, one could indeed replace the element  $\mu^{-1}$  and  $f(\mu^{-1})$  of such matrices with zeros. The only reason we keep these elements the way they are in equations (45) and (46), is to render  $\gamma_E$  the covariance matrix of a minimal uncertainty state. In other words, the elements of  $\delta$  and  $\beta$  proportional to  $\mu^{-1}$  or  $f(\mu^{-1})$  are only introduced to purify the corresponding elements of the submatrix  $\alpha$ , which is in itself hence a covariance matrix of a mixed Gaussian state.

Suppose then that  $\mu$  of equation (38) has (say) the first  $r'/2$  eigenvalues equal to 1, i.e.  $\mu_1 = \mu_2 = \dots = \mu_{r'/2} = 1$ , while for  $j \in \{r'/2 + 1, \dots, r/2\}$  we have that  $\mu_j \in (0, 1)$ . In this case, the corresponding sub-matrix of  $\alpha$  associated with those elements represents a pure Gaussian state, specifically the vacuum state. Accordingly, there is no need to add further modes to purify them. Taking this into account, one can hence reduce the number of environmental modes  $\ell_{\text{pure}}$  that allows one to represent  $\Phi$  as in equation (20) in terms of a *pure state*  $\hat{\rho}_E$  from  $2n$  to

$$\ell_{\text{pure}} = n + (n - r'/2) = 2n - r'/2, \quad (56)$$

i.e. we need the  $n$  modes of  $\alpha$  plus  $n - r'/2$  additional modes of  $\beta$  to purify those of  $\alpha$  which are not in a pure state already. An easy way to characterize the parameter  $r'$  is to observe that, according to equation (37), it corresponds to the number of eigenvalues having modulus 1 of the matrix of  $O\Sigma'O^T$ , i.e.

$$\begin{aligned} r' &= 2n - \text{rank}[\mathbb{1}_{2n} - O\Sigma'(\Sigma')^T O^T] = 2n - \text{rank}[\mathbb{1}_{2n} - \Sigma'(\Sigma')^T] \\ &= 2n - \text{rank}[Y - \Sigma Y^{-1} \Sigma^T]. \end{aligned} \quad (57)$$

The explicit expressions for corresponding values of  $\gamma_E$  and  $s_2$  are given in appendix C.1. Here, we notice that for  $r' = r = 2n$ , we get  $\ell_{\text{pure}} = n$ . This should correspond to the channels (55) of class (i) for which one can construct a unitary dilation with pure input states. Indeed, according to equation (57), when  $r' = 2n$  the matrix  $Y - \Sigma Y^{-1} \Sigma^T$  must be zero, leading to the identity (55).

Taking into account that  $r' \leq r = \text{rank}[\Sigma]$ , a further reduction in the number of modes  $\ell$  can be obtained by dropping the requirement of  $\gamma_E$  being a minimal uncertainty covariance matrix. Indeed, an alternative unitary representation (20) of  $\Phi$  can be constructed with only

$$\ell = n + (n - r/2) = 2n - r/2 \quad (58)$$

environmental modes (see appendix C.2 for the explicit solution).

The whole analysis can be finally generalized to the BGCs of class (iii), corresponding to channels that have non-invertible matrices  $Y$ . We have seen in fact that, in this case, the state  $\hat{\rho}_E$  which provides us the unitary dilation of theorem 1 is constructed by replacing  $Y$  with the strictly positive operator  $\bar{Y}$  of equation (49). Therefore for these channels  $\ell_{\text{pure}}$  of equation (56) are defined by equation (57) with  $Y$  replaced by  $\bar{Y}$ , i.e.

$$r' = 2n - \text{rank}[\bar{Y} - \Sigma \bar{Y}^{-1} \Sigma^T]. \quad (59)$$

Taking  $\bar{Y} := Y + (\mathbb{1}_{2n} - \Pi)$  with  $\Pi$  being the projector on  $\text{Supp}[Y]$  this gives,

$$\begin{aligned} r' &= 2n - \text{rank}[\bar{Y} - \Sigma Y^{\ominus 1} \Sigma^T] = 2n - \text{rank}[Y - \Sigma Y^{\ominus 1} \Sigma^T] - \text{rank}[\mathbb{1}_{2n} - \Pi] \\ &= k - \text{rank}[Y - \Sigma Y^{\ominus 1} \Sigma^T], \end{aligned} \quad (60)$$

where  $k = \text{rank}[Y] = \text{rank}[\Pi]$ , where  $Y^{\ominus 1} := \Pi \bar{Y}^{-1} \Pi$  denotes the Moore–Penrose inverse [27] of  $Y$ , and where we have used the fact that  $\text{Supp}[\Sigma] \subseteq \text{Supp}[Y]$ . Remembering then that for channels of class (ii)  $k = 2n$  and  $Y^{\ominus 1} = Y^{-1}$  these results can be summarized as follows:

**Theorem 2 (Dilations of BGCs involving fewer additional modes).** *Given  $\Phi$  a BGC described by matrices  $X$  and  $Y$  satisfying the conditions (15) and characterized by the quantities*

$$r = \text{rank}[\Sigma], \quad r' = \text{rank}[Y] - \text{rank}[Y - \Sigma Y^{\ominus 1} \Sigma^T]. \quad (61)$$

*Then it is possible to construct a unitary dilation (20) of Stinespring form (i.e. involving a pure Gaussian state  $\hat{\rho}_E$ ) with at most  $\ell_{\text{pure}} = 2n - r'/2$  environmental modes. It is also always possible to construct a unitary dilation (20) using  $\ell = 2n - r/2$  environmental modes which are prepared in a Gaussian, but not necessarily pure state.*

It is worth stressing that, for channels of class (ii) and (iii), theorem 2 only provides upper bounds for the minimal values of  $\ell$  and  $\ell_{\text{pure}}$ . Only in the generic case (i) do these bounds coincide with the real minima.

### 2.3. Minimal noise channels

In a very analogous fashion to the extremal covariance matrices corresponding to pure Gaussian states, one can introduce the concept of a minimal noise channel. In this section, we review the concept of such minimal noise channels [18] and provide criteria to characterize them. Given  $X, Y \in \mathbb{R}^{2n \times 2n}$  satisfying the inequality (15), any other  $Y' = Y + \Delta Y$  with  $\Delta Y \geq 0$  will also satisfy this condition, i.e.

$$Y' \geq Y \geq i(\sigma_{2n} - X^T \sigma_{2n} X). \quad (62)$$

Furthermore, due to the composition rules (18), the BGC  $\Phi'$  associated with the matrices  $X$  and  $Y'$  can be described as the composition

$$\Phi' = \Psi \circ \Phi, \quad (63)$$

between the channel  $\Phi$  associated with the matrices  $X$  and  $Y$ , and the channel  $\Psi$  described by the matrices  $X = \mathbb{1}_n$  and  $Y = \Delta Y$ . The latter belongs to a special case of BGC that includes the so-called *additive classical noise channels* [3, 4, 17]—see section 2.4 for details.

For any  $X \in \mathbb{R}^{2n \times 2n}$ , one can then ask how much *noise*  $Y$  it is necessary to add in order to obtain a map satisfying the condition (15). This gives rise to the notion of *minimal noise* [18], as the extremal solutions  $Y$  of equation (15) for a given  $X$ . The corresponding *minimal noise channels* are the natural analogue of the Gaussian pure state and allow one to represent any other BGC as in equation (63) with a proper choice of the additive classical noise map  $\Psi$ .

Let us start by considering the case of a generic channel  $\Phi'$  of class (i) described by matrices  $X$  and  $Y'$ . According to theorem 1, it admits unitary dilation with  $\ell = n$  modes described by some covariance matrix  $\gamma'_E$  satisfying the condition

$$Y' = s_2 \gamma'_E s_2^T, \quad (64)$$

for some proper  $2n \times 2n$  real matrix  $s_2$ . According to equation (11)  $\gamma_E$  can be written as

$$\gamma'_E = \gamma_E + \Delta, \quad (65)$$

with  $\Delta \geq 0$  and  $\gamma_E$  the minimal uncertainty state. Therefore, writing  $Y = s_2 \gamma_E s_2^T$  and  $\Delta Y = s_2 \Delta s_2^T$ , we can express  $\Phi'$  as in (63), where now  $\Phi$  is the BGC associated with the minimal noise environmental state  $\gamma_E$ . Most importantly since the decomposition (65) is optimal for  $\gamma'_E$ , the channel  $\Phi$  is an extremal solution of equation (15). We stress that by construction  $\Phi$  is still a channel of class (i): in fact, it has the same  $\Sigma$  as  $\Phi'$ , whereas  $Y$  is still strictly positive since  $\gamma_E > 0$  and  $s_2$  is invertible—see equation (64). We can then use the results of section 2.2 to claim that  $\Phi$  must satisfy the equality (55). This leads us to establish three equivalent necessary and sufficient conditions for minimal noise channels of class (i):

$$(m_1) \quad Y = \Sigma Y^{-1} \Sigma^T, \quad (66)$$

$$(m_2) \quad \text{Det}[Y] = \text{Det}[\Sigma], \quad (67)$$

$$(m_3) \quad r = r' \quad (68)$$

with  $r$  and  $r'$  as in equation (61). Since for class (i) we have that  $r = 2n$ , the minimal noise condition  $m_3$  simply requires the eigenvalues of the matrix  $\mu$  of equation (37) to be equal to unity. Similarly, minimal noise channels in cases (ii) and (iii) can be characterized.

**Theorem 3 (Minimal noise condition).** *A Gaussian bosonic channel characterized by the matrices  $Y$  and  $X \in \mathbb{R}^{2n \times 2n}$  is a minimal noise channel if and only if*

$$Y = \Sigma Y^{\ominus 1} \Sigma^T, \quad (69)$$

where, as throughout this work,  $\Sigma = \sigma_{2n} - X^T \sigma_{2n} X$ .

**Proof.** The complete positivity condition (15) of a generic BGC is a positive semidefinite constraint for the symplectic form  $\Sigma$ , corresponding to the constraint  $\gamma - i\sigma_{2n} \geq 0$  in the case of covariance matrices of states of  $n$  modes. In general,  $r = \text{rank}[\Sigma]$  is not maximal, i.e. not equal to  $2n$ . When identifying the minimal solutions of the inequality (15), without loss of generality we can look for the minimal solutions of

$$Y' - i\Sigma' \geq 0, \quad (70)$$



where here

$$\Sigma' = \left[ \begin{array}{c|c|c} 0 & \mu & \\ \hline -\mu & 0 & \\ \hline & & 0 \end{array} \right] \quad (71)$$

with  $\mu > 0$  being diagonal of rank  $r/2$  (here  $Y' = OYO^T$  and  $\Sigma' = O\Sigma O^T$  with  $O \in O(2n, \mathbb{R})$  orthogonal). The minimal solutions of inequality (70) are then given by  $Y' = SS^T \oplus 0$ , where  $S$  is an  $r \times r$  matrix satisfying

$$S \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix} S^T = \begin{bmatrix} 0 & \mu \\ -\mu & 0 \end{bmatrix}, \quad (72)$$

so a symplectic matrix with respect to the modified symplectic form, so an element of  $\{M \in Gl(r, \mathbb{R}) : M = (\mu^{1/2} \oplus \mu^{1/2})S(\mu^{-1/2} \oplus \mu^{-1/2}), S \in Sp(r, \mathbb{R})\}$ . From this, it follows that the minimal solutions of (70) are exactly given by the solutions of  $Y' = \Sigma'(Y')^{\ominus 1}(\Sigma')^T$ , from which the statement of the theorem follows.  $\square$

#### 2.4. Additive classical noise channel

In this subsection, we focus on the maps  $\Psi$  which enter in the decomposition (63). They are characterized by having  $X = \mathbb{1}_{2n}$  and  $Y \geq 0$ . Note that with this choice the condition (15) is trivially satisfied. This is the classical noise channel that has frequently been considered in the literature (for a review, e.g., see [4]). For completeness of the presentation, we briefly discuss this class of multi-mode BGCs.

If the matrix  $Y$  is strictly positive, the channel  $\Psi$  is the multi-mode generalization of the single mode additive classical noise channel [3, 4, 17]. In the language of [17], these are the maps which have a canonical form  $B_2$ . Indeed, one can show that these maps are the (Gaussian) unitary equivalent to a collection of  $n$  single mode additive classical noise maps. To see this, let us apply symplectic transformations ( $S_1$  and  $S_2$ ) before and after the channel  $\Psi$ . Following equation (19) this leads to  $\{\mathbb{1}_n, Y\} \mapsto \{S_1 S_2, S_2^T Y S_2\}$ . Now, since  $Y > 0$ , according to *Williamson's theorem* [23], we can find a  $S_2 \in Sp(2n, \mathbb{R})$  such that  $S_2^T Y S_2$  is diagonal  $\text{diag}(\lambda_1, \dots, \lambda_n, \lambda_1, \dots, \lambda_n)$  with  $\lambda_i > 0$ . We can then take  $S_1 = S_2^{-1}$  to have  $S_1 S_2 = \mathbb{1}_{2n}$ . For  $Y \geq 0$  but not  $Y > 0$ , the maps  $\Psi$  that enter the decomposition equation (63), however, include also channels which are not unitarily equivalent to a collection of  $B_2$  maps. An explicit example of this situation is constructed in appendix E.

#### 2.5. Canonical form for generic channels

Analogously to [13, 14, 17], any BGC  $\Phi$  described by the transformation equation (17) can be simplified through unitarily equivalence by applying unitary canonical transformations before and after the action of the channel which induces transformations of the form (19). Specifically, given a  $n$ -mode Gaussian channel  $\Phi$  described by matrix  $X$  and  $Y$  we can transform it into a new  $n$ -mode Gaussian channel  $\Phi_c$  described by the matrices

$$X_c = S_1 X S_2, \quad Y_c = S_2^T Y S_2, \quad (73)$$

with  $S_{1,2} \in Sp(2n, \mathbb{R})$ . As already discussed in the introductory sections, from an information theoretical perspective  $\Phi$  and  $\Phi_c$  are equivalent in the sense that, for instance, their unconstrained quantum capacities coincide. We can then simplify the analysis of the  $n$ -mode

Gaussian channels by properly choosing  $S_1$  and  $S_2$  to induce a parametrization of the interaction part (i.e.  $X$ ) of the evolution. The resulting canonical form follows from the generalization of the Williamson theorem [23] presented in [20]. According to this result, for every non-singular matrix  $X \in Gl(2n, \mathbb{R})$ , there exist matrices  $S_{1,2} \in Sp(2n, \mathbb{R})$  such that

$$X_c = S_1 X S_2 = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & J^T \end{bmatrix}, \quad (74)$$

where  $J^T$  is an  $n \times n$  block-diagonal matrix in the real Jordan form [27]. This can be developed a little further by constructing a canonical decomposition for the symplectic matrix  $S$  associated with the unitary dilation (20) of the channel.

For the sake of simplicity in the following, we will focus on the case of generic quantum channels  $\Phi$  which have non-singular  $X \in Gl(2n, \mathbb{R})$  and belong to class (i) of theorem 1 (i.e. which have  $r = \text{rank}[\Sigma] = 2n$ ). Under these conditions,  $X$  must admit a canonical decomposition of the form (74) in which all the eigenvalues of  $J$  are different from 1. In fact one has

$$\Sigma = \sigma_{2n} - X^T \sigma_{2n} X = S_2^{-T} [\sigma_{2n} - X_c^T \sigma_{2n} X_c] S_2^{-1} = S_2^{-T} \Sigma_c S_2^{-1}, \quad (75)$$

with  $\Sigma_c$  being the skew-symmetric matrix associated with the channel  $\Phi_c$ , i.e.

$$\Sigma_c := \begin{bmatrix} 0 & \mathbb{1}_n - J \\ J^T - \mathbb{1}_n & 0 \end{bmatrix}. \quad (76)$$

Since  $\text{rank}[\Sigma_c] = \text{rank}[\Sigma] = 2n$ , it follows that  $J$  cannot have eigenvalues equal to 1. Similarly, it is not difficult to see that if  $X$  has a canonical form (74) with all the eigenvalues of  $J$  being different from 1, then  $\Phi$  and  $\Phi_c$  are of class (i). However, a special case in which  $X = \mathbb{1}_{2n}$  is investigated in detail in appendix E.

Consider then a unitary dilation (20) of the channel  $\Phi_c$  constructed with a not necessarily pure Gaussian state  $\hat{\rho}_E$  of  $\ell = n$  environmental modes. According to the above considerations, such a dilation always exists. Let  $S \in Sp(4n, \mathbb{R})$  be the  $4n \times 4n$  real symplectic transformation (22) associated with the corresponding unitary  $\hat{U}$ . Assuming  $s_1 = X_c^T$ , an explicit expression for this dilation can be obtained by writing

$$s_4 = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & J' \end{bmatrix}, \quad s_j = \begin{bmatrix} F_j & 0 \\ 0 & G_j \end{bmatrix}, \quad (77)$$

where, for  $j = 2, 3$ ,  $F_j, G_j$  are  $n \times n$  real matrices. Imposing equation (23), one obtains the following relations:

$$\begin{aligned} J^T + F_2 G_2^T &= \mathbb{1}_n, & J'^T + F_3 G_3^T &= \mathbb{1}_n, \\ G_3^T + F_2 J'^T &= 0, & G_2^T + F_3 J^T &= 0, \end{aligned} \quad (78)$$

whose solution gives an  $S \in Sp(4n, \mathbb{R})$  of the form

$$S = \left[ \begin{array}{cc|cc} \mathbb{1}_n & 0 & (\mathbb{1}_n - J^T)G^{-T} & 0 \\ 0 & J & 0 & G \\ \hline -G^T J^{-T} & 0 & \mathbb{1}_n & 0 \\ 0 & G^{-1}J(J - \mathbb{1}_n) & 0 & G^{-1}JG \end{array} \right], \quad (79)$$

with  $G$  being an arbitrary matrix  $G \in Gl(n, \mathbb{R})$ . As a consequence of this fact, and because the eigenvalues of  $J$  are assumed to be different from 1,  $s_2, s_3$  and  $s_4$  are also non-singular. This is

important because it shows that in choosing  $S$  as in the canonical form (79) we are not restricting generality: the value of  $s_2$  can always be absorbed into the definition of the covariance matrix  $\gamma_E$  of  $\hat{\rho}_E$  by writing

$$\gamma_E = s_2^{-1} Y_c s_2^{-T}, \quad (80)$$

(see also appendix D). Taking this into account, we can conclude that equation (79) provides an explicit demonstration of lemma 1 for channels of class (i) with non-singular  $X$ .

Since  $\Phi_c$  is fully determined by  $X_c$  and  $Y_c$ , the above expressions show that the action of  $\Phi_c$  on the input state does not depend on the choice of  $G$ . As a matter of fact, the latter can be seen as a Gaussian unitary operation  $\hat{U}_G$  characterized by the  $n$ -mode symplectic transformation  $Sp(2n, \mathbb{R})$ ,

$$\Delta_G = \left[ \begin{array}{c|c} G^T & 0 \\ \hline 0 & G^{-1} \end{array} \right], \quad (81)$$

applied to the final state of the environment after the interaction with the input, i.e.  $\tilde{\Phi}_G = \hat{U}_G \tilde{\Phi} \hat{U}_G^T$ , where  $\tilde{\Phi}$  is the *weak complementary map* for  $G = \mathbb{1}_n$ , and  $\tilde{\Phi}_G$  is the weak complementary map in presence of  $G \neq \mathbb{1}_n$ —see the next section for details. Since the relevant properties of a channel (e.g. weak degradability [16, 17]), do not depend on local unitary transformations to the input/output states, without loss of generality, we can consider  $G = -J$  and the canonical form for  $S \in Sp(4n, \mathbb{R})$  assumes the following simple expression:

$$S = \left[ \begin{array}{cc|cc} \mathbb{1}_n & 0 & \mathbb{1}_n - J^{-T} & 0 \\ 0 & J & 0 & -J \\ \hline \mathbb{1}_n & 0 & \mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n - J & 0 & J \end{array} \right]. \quad (82)$$

The possibility of constructing different, but unitarily equivalent, canonical forms for  $S$  is discussed in appendix D.

### 3. Weak degradability

Among other properties, the unitary dilations introduced in section 2 are useful to define *complementary* or *weak complementary* channels of a given BGC  $\Phi$ . These are defined as the CPT map  $\tilde{\Phi}$  which describes the evolution of the environment under the influence of the physical operation describing the channel [16, 17], i.e.

$$\tilde{\Phi}(\hat{\rho}) := \text{Tr}_S \left[ \hat{U}(\hat{\rho} \otimes \hat{\rho}_E) \hat{U}^\dagger \right], \quad (83)$$

where  $\hat{\rho}$ ,  $\hat{\rho}_E$  and  $\hat{U}$  are defined as in equation (20), but the partial trace is now taken over the system modes.

Specifically, if the state  $\hat{\rho}_E$  we employed in constructing the unitary dilation of equation (20) is pure, then the map  $\tilde{\Phi}$  is said to be the *complementary* of  $\Phi$  and, up to partial isometry, it is unique [28]–[31]. Otherwise it is called *weak complementary* [16, 17]. Since in equation (20) the state  $\hat{\rho}_E$  is Gaussian and  $\hat{U}$  is a unitary Gaussian transformation, one can verify that  $\tilde{\Phi}$  is also BGC<sup>11</sup>. Expressing the Gaussian unitary transformation  $\hat{U}$  in terms of its

<sup>11</sup> In general, however, it will not map the  $n$ -input modes into  $n$ -output modes. Instead it will transform them into  $\ell$  modes, with  $\ell$  being the number of modes assumed in the unitary dilation (20).

symplectic matrix  $S$  of equation (22) the action of  $\tilde{\Phi}$  is fully characterized by the following mapping of the covariance matrices  $\gamma$  of  $\hat{\rho}$ , i.e.

$$\tilde{\Phi} : \gamma \longmapsto s_3 \gamma s_3^T + s_4 \gamma_E s_4^T, \quad (84)$$

which is counterpart of the transformations (16) and (24) that characterize  $\Phi$ . The channel  $\tilde{\Phi}$  is then described by the matrices  $\tilde{X} = s_3^T$  and  $\tilde{Y} = s_4 \gamma_E s_4^T$  which, according to the symplectic properties (23), satisfy the condition

$$\tilde{Y} \geq i \tilde{\Sigma} \quad \text{with} \quad \tilde{\Sigma} := \sigma_{2\ell}^E - \tilde{X}^T \sigma_{2n} \tilde{X}. \quad (85)$$

The relations between  $\Phi$  and its weak complementary  $\tilde{\Phi}$  contain useful information about the channel  $\Phi$  itself. In particular, we say that the channel  $\Phi$  is *weakly degradable* (WD), whereas  $\tilde{\Phi}$  is *anti-degradable* (AD), if there exists a CPT map  $\mathcal{T}$  which, for all inputs  $\hat{\rho}$ , allows one to recover  $\tilde{\Phi}(\hat{\rho})$  by acting on the output state  $\Phi(\hat{\rho})$ , i.e.

$$\mathcal{T} \circ \Phi = \tilde{\Phi}. \quad (86)$$

Similarly, one says that  $\Phi$  is AD and  $\tilde{\Phi}$  is WD if there exists a CPT map  $\tilde{\mathcal{T}}$  such that

$$\tilde{\mathcal{T}} \circ \tilde{\Phi} = \Phi. \quad (87)$$

Weak degradability [16, 17] is a property of quantum channels  $\Phi$  generalizing the *degradability property* introduced in [28]. The relevance of weak-degradability analysis stems from the fact that it allows one to simplify the quantum capacity scenario. Indeed, it is known that AD channels have zero quantum capacity [16, 17], whereas WD channels with  $\hat{\rho}_E$  pure are degradable and thus admit a single letter expression for this quantity [28]. A complete weak-degradability analysis of single-mode BGCs has been provided in [16, 17]. Here, we generalize some of these results to  $n > 1$ .

### 3.1. A criterion for weak degradability

In this section, we review a general criterion for degradability of BGCs which was introduced in [15], adapting it to include also weak degradability. Before entering the details of our derivation, however, it is worth noting that generic multi-mode Gaussian channels are neither WD nor AD. Consider in fact a WD single-mode Gaussian channel  $\Phi$  having no zero quantum capacity  $Q > 0$  (e.g. a beam-splitter channel with transmissivity  $> 1/2$ ). Define then the two-mode channel  $\Phi \otimes \tilde{\Phi}$  with  $\tilde{\Phi}$  being its weak complementary defined in [16, 17]. This is Gaussian since both  $\Phi$  and  $\tilde{\Phi}$  are Gaussian. The claim is that  $\Phi \otimes \tilde{\Phi}$  is neither WD nor AD. Indeed, its weak complementary can be identified with the map  $\tilde{\Phi} \otimes \Phi$ . Consequently, since  $\Phi \otimes \tilde{\Phi}$  and  $\tilde{\Phi} \otimes \Phi$  differ by a permutation, they must have the same quantum capacity  $Q'$ . Therefore, if one of the two is WD then *both* of them must also be AD. In this case,  $Q'$  should be zero which is clearly not possible given that  $Q' \geq Q$ . In fact, one can use  $\Phi \otimes \tilde{\Phi}$  to reliably transfer quantum information by encoding it into the inputs of  $\Phi$ . In this respect, the possibility of classifying (almost) all single-mode Gaussian maps in terms of weak degradability turns to be rather a remarkable property. We now turn to investigating the weak degradability properties of multi-mode BGCs deriving a criterion that will be applied in section 4.1 for studying in detail the two-mode channel case.

Consider an  $n$ -mode BGC  $\Phi$  characterized by the unitary dilation (20) and its weak complementary  $\tilde{\Phi}$  (83). Let  $\{X, Y\}$  and  $\{\tilde{X}, \tilde{Y}\}$  be the matrices which define such channels. For the sake of simplicity, we will assume  $X$  and  $\tilde{X}$  to be non-singular, with  $X, \tilde{X} \in GL(2n, \mathbb{R})$ .

Examples of such maps are for instance the channels of class (i) with  $X$  non-singular described in section 2.5. Adopting in fact the canonical form (82) for  $S$  we have that

$$X = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & J^T \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} \mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n - J^T \end{bmatrix} \quad (88)$$

with all the eigenvalues of  $J$  being different from 1.

Suppose now that  $\Phi$  is weakly degradable with  $\mathcal{T}$  being the connecting CPT map which satisfies the weak degradability condition (86). As in [16, 17], we will focus on the case in which  $\mathcal{T}$  is BGC and described by matrices  $\{X_{\mathcal{T}}, Y_{\mathcal{T}}\}$ . Under these hypotheses the identity (86) can be simplified by using the composition rules for BGCs given in equation (18). Accordingly, one must have

$$X_{\mathcal{T}} = X^{-1} \tilde{X}, \quad Y_{\mathcal{T}} = \tilde{Y} - X_{\mathcal{T}}^T Y X_{\mathcal{T}}. \quad (89)$$

These definitions must be compatible with the requirement that  $\mathcal{T}$  should be a CPT map which transforms the  $n$ -system modes into the  $\ell$ -environmental modes, i.e.

$$Y_{\mathcal{T}} \geq i(\sigma_{2\ell}^E - X_{\mathcal{T}}^T \sigma_{2n} X_{\mathcal{T}}). \quad (90)$$

Combining the expressions above, one finds the following weak-degradability condition for  $n$ -mode BGCs [15], i.e.

$$\tilde{Y} - \tilde{X}^T X^{-T} (Y + i\sigma_{2n}) X^{-1} \tilde{X} + i\sigma_{2\ell}^E \geq 0. \quad (91)$$

In order to obtain the anti-degradability condition (87), it is sufficient to swap  $\{X, Y\}$  with  $\{\tilde{X}, \tilde{Y}\}$  and the system commutation matrix  $\sigma_{2n}$  with  $\sigma_{2\ell}^E$ , in equation (91), i.e.

$$Y - X^T \tilde{X}^{-T} (\tilde{Y} + i\sigma_{2\ell}^E) \tilde{X}^{-1} X + i\sigma_{2n} \geq 0. \quad (92)$$

Equations (91) and (92) are strictly related. Indeed since

$$Y - X^T \tilde{X}^{-T} (\tilde{Y} + i\sigma_{2\ell}^E) \tilde{X}^{-1} X + i\sigma_{2n} = -X^T \tilde{X}^{-T} \left( \tilde{Y} - \tilde{X}^T X^{-T} (Y + i\sigma_{2n}) X^{-1} \tilde{X} + i\sigma_{2\ell}^E \right) \tilde{X}^{-1} X, \quad (93)$$

equation (92) corresponds to reverse the sign of the inequality (91), i.e.

$$\tilde{Y} - \tilde{X}^T X^{-T} (Y + i\sigma_{2n}) X^{-1} \tilde{X} + i\sigma_{2\ell}^E \leq 0. \quad (94)$$

Hence to determine if  $\Phi$  is a WD or AD channel, it is then sufficient to study the positivity of the Hermitian matrix

$$W := \tilde{Y} - \tilde{X}^T X^{-T} (Y + i\sigma_{2n}) X^{-1} \tilde{X} + i\sigma_{2\ell}^E. \quad (95)$$

In the case in which  $\ell = n$ , this can be simplified by recalling that a Hermitian  $2n \times 2n$  matrix  $W$  partitioned as

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2^\dagger & W_3 \end{bmatrix} \quad (96)$$

with  $W_i$  being  $n \times n$  matrices, is semi-positive definite if and only if

$$W_1 \geq 0 \quad \text{and} \quad W_3 - W_2^\dagger W_1^{-1} W_2 \geq 0, \quad (97)$$

the right-hand side being the Schur complement of  $W$  (see, e.g., p 472 in [27]). Using this result and the canonical form (82), equation (91) can be written as in equation (97) with

$$\begin{aligned} W_1 &= (\mathbb{1}_n - J^{-T})^{-1} Y_1 (\mathbb{1}_n - J^{-1})^{-1} - Y_1, \\ W_2 &= i(J^{-T} - 2\mathbb{1}_n) - Y_2 (J^{-T} - \mathbb{1}_n) - (\mathbb{1}_n - J^{-T})^{-1} Y_2, \\ W_3 &= Y_3 - (J^{-1} - \mathbb{1}_n) Y_3 (J^{-T} - \mathbb{1}_n) \end{aligned} \quad (98)$$

and

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}. \quad (99)$$

For the anti-degradability condition (92) simply replace  $[\geq]$  with  $[\leq]$  in equation (97).

#### 4. Two-mode BGCs

Here, we consider a particular case of the  $n$ -mode BGC analysis above, namely, the case of  $n = 2$ . This is by no means such a special case as one might at first be tempted to think since any  $n$ -mode channel can always be reduced to single-mode and two-mode parts [20]. For two-mode channels, the interaction part and the noise term of a generic two-mode BGC,  $X$  and  $Y$ , respectively, are  $4 \times 4$  real matrices. Particularly, we will focus on two-mode channels  $\Phi$  which have non-singular  $X$  and belong to class (i) of theorem 1 (i.e. which have  $r = \text{rank}[\Sigma] = 4$ ), like in section 2.5. These maps can be grasped in terms of a unitary dilation of the form (82) coupling the two system bosonic modes with two additional (environmental) modes, where  $J$  is a  $2 \times 2$  real Jordan block. In order to characterize this large class of two-mode BGCs, one has to examine only three possible forms of  $J$ :

- Class A: this corresponds to taking a diagonalizable Jordan block, that is,

$$J := J_0 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad (100)$$

where  $a$  and  $b$  are real nonzero numbers. This represents the trivial case of a two-mode BGC, whose interaction term does not couple the two modes. Actually, we call it class  $A_1$  if  $a \neq b$  and class  $A_2$  otherwise.

- Class B: this is to take  $J$  as a non-diagonalizable matrix with a nonzero real eigenvalue  $a$  with double algebraic multiplicity (but with geometric multiplicity equal to one), i.e.

$$J := J_1 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}. \quad (101)$$

In this case, the Jordan block is called defective [27]. Here, a noisy interaction between the bosonic system and the environment, coupling the two system modes, is switched on.

- Class C: here the real Jordan block  $J$  has complex eigenvalues, i.e.

$$J := J_2 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad (102)$$

with  $b \neq 0$ ; the eigenvalues of  $J$  are  $a \pm ib$ . Again, the two system modes are coupled by a noisy interaction with the environment through the presence of the element  $b$ .

In order to explicit the form of  $Y = s_2 \gamma_E s_2^T$ , with  $s_2$  being defined as in equation (82), we consider a generic two-mode covariance matrix in the so-called standard form [32] for the environmental initial covariance matrix  $\gamma_E$ , i.e.

$$\gamma_E = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix}, \quad (103)$$

where

$$\Gamma_{1,2} := \begin{bmatrix} x & z_{-,+} \\ z_{-,+} & y \end{bmatrix}, \quad (104)$$

and  $x, y, z_{+,-}$  are real numbers satisfying  $x + y \geq 0$ ,  $xy - z_-^2 \geq 1$  and  $x^2y^2 - y^2 - x^2 + (z_-z_+ - 1)^2 - xy(z_-^2 + z_+^2) \geq 0$  because of the uncertainty principle. More generally, one can apply a generic two-mode (symplectic) squeezing operator  $V(\epsilon)$  to the environmental input state, i.e.

$$\gamma'_E = V(\epsilon)\gamma_E V(\epsilon)^T, \quad (105)$$

where

$$V(\epsilon) = \begin{bmatrix} R^{-T} & 0 \\ 0 & R \end{bmatrix}, \quad R = \begin{bmatrix} c + hs & -qs \\ -qs & c - hs \end{bmatrix}, \quad (106)$$

and  $c = \cosh(2r)$ ,  $s = \sinh(2r)$ ,  $h = \cos(2\phi)$ ,  $q = \sin(2\phi)$  and  $\epsilon = re^{2i\phi}$  being the squeezing parameter [32]. Finally, it is interesting to study how the canonical forms of two-mode BGCs compose under the product. A simple calculation shows that the following rules apply

$\circ$	$A$	$B$	$C$
$A$	$A$	$A_1/B$	$A_1/B/C$
$B$	$A_1/B$	$A_2/B$	$A_1/B/C$
$C$	$A_1/B/C$	$A_1/B/C$	$A/C$

In this table, for instance, the element on row 1 and column 1 represents the class (i.e.  $A$ ) associated with the composition of two channels of the same class  $A$ . Note that the canonical form of the products with a ‘coupled’ channel (i.e. with  $B$  or  $C$ ) is often not uniquely defined. For instance, composing two class  $B$  channels characterized by the matrices

$$(J_1)_i = \begin{bmatrix} a_i & 1 \\ 0 & a_i \end{bmatrix} \quad (107)$$

with  $i = 1, 2$ , will give us either a class  $A_2$  channel (if  $a_1 + a_2 = 0$ ) or a class  $B$  channel (if  $a_1 + a_2 \neq 0$ ). Composition rules analogous to those reported above have been analyzed in detail for the one-mode case in [17]. In the following, we will study the weak-degradability properties of these three classes of two-mode Gaussian channels.

#### 4.1. Weak-degradability properties

The weak-degradability conditions in equation (97) become

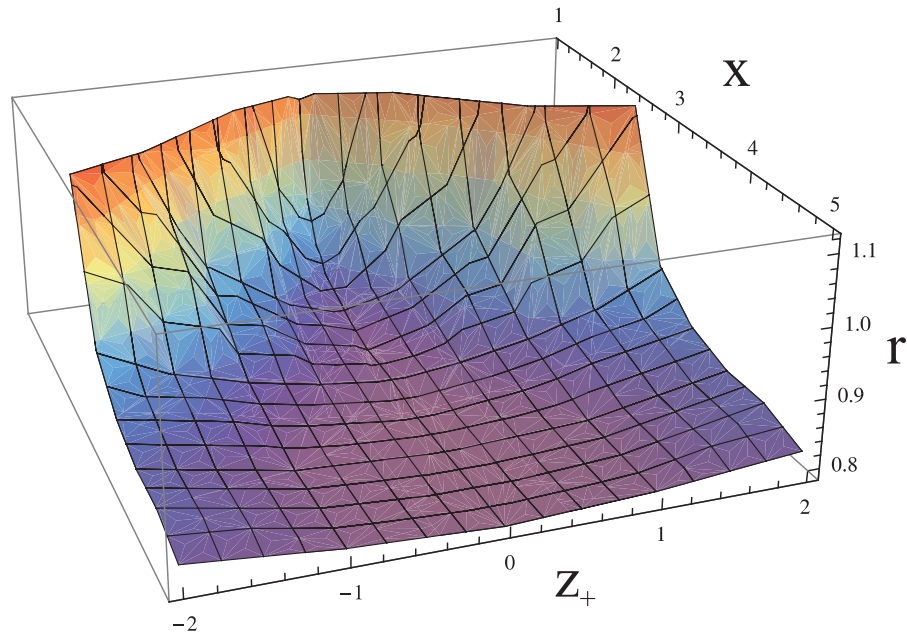
$$\Gamma_1 - (\mathbb{1}_2 - J^{-T})\Gamma_1(\mathbb{1}_2 - J^{-1}) \geq 0 \quad (108)$$

and

$$J\Gamma_2J^T - (\mathbb{1}_2 - J)\Gamma_2(\mathbb{1}_2 - J^T) - (J^{-1} - 2\mathbb{1}_2) \left[ \Gamma_1 - (\mathbb{1}_2 - J^{-T})\Gamma_1(\mathbb{1}_2 - J^{-1}) \right]^{-1} (J^{-T} - 2\mathbb{1}_2) \geq 0. \quad (109)$$

In the same way, the anti-degradability is obtained when both these quantities are non-positive. As concerns the environmental initial state of the unitary dilation, one can consider a generic two-mode state as in equation (105). On one hand, we find that, if  $[J, R] = 0$ , this two-mode squeezing transformation  $V(\epsilon)$  can be simply ‘absorbed’ in local symplectic operations to the output states and then it does not affect the weak-degradability properties. On the other hand, if  $[J, R] \neq 0$ , we find numerically that the introduction of correlations between the two environmental modes contrasts with the presence of (anti-) weak-degradability features. Therefore, one can consider the particular case in which the environment is initially in a state with a symmetric covariance matrix  $\gamma_E$  as in equation (103) with  $x = y = 2N + 1$  and





**Figure 1.** Relation between the parameters  $x$ ,  $z_+$  and the minimum value of  $r$  in the initial environmental state such that the two-mode channel with  $X = \mathbb{1}_2 \oplus J_1$  reduces to the decoupled case  $X' = \mathbb{1}_2 \oplus J_0$  with the same interaction parameter  $a$  for the two system modes.

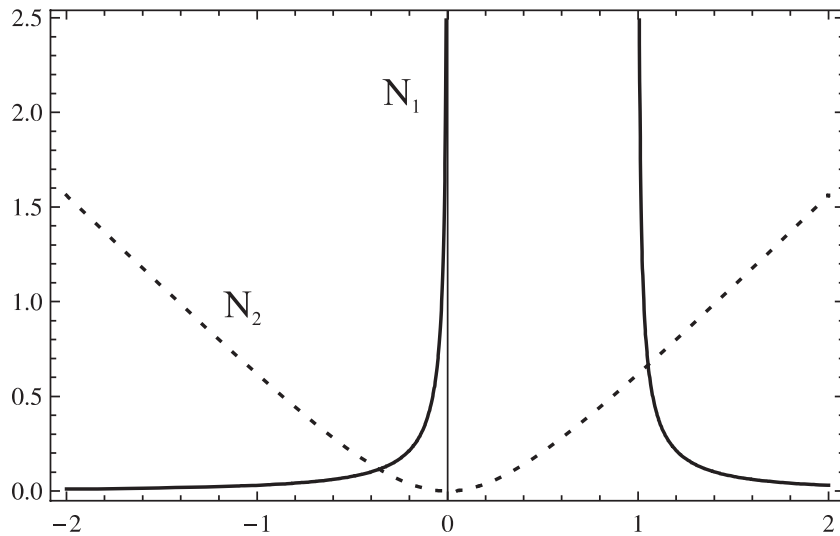
$z_- = z_+ = 0$  where  $N \geq 0$ . In this case,  $\gamma_E = (2N + 1)\mathbb{1}_2$  corresponds to a thermal state of two uncoupled environmental modes with the same photon average number  $N$  and it is possible to see the results above easily through analytical details. In fact, we study analytically the positivity condition in equation (91) in the three possible forms of the real Jordan block  $J_i$ .

In the uncoupled case,  $J_0$  as in equation (100), substituting in equation (91), we find that these two-mode BGCs are WD if  $a, b \geq 1/2$  and AD for  $a, b \leq 1/2$  (any  $N \geq 0$ ). In other words, in the case of two uncoupled modes, the weak-degradability properties can be derived from the results for one-mode BGCs: tensoring two WD (AD) one-mode Gaussian channels with WD (AD) one-mode Gaussian channels yield two-mode Gaussian channels which are WD (AD).

In the case of defective  $J$ , i.e.  $J_1$  as in equation (101), corresponding to noisy interaction coupling the two system modes, substituting in equation (91), we find that, on one hand, these two-mode BGCs are WD if  $a > 1$  and

$$N \geq N_1 := \frac{1}{2} \left[ -1 + \frac{1}{2} \frac{|2a - 1|}{\sqrt{a(a - 1)}} \right]. \quad (110)$$

On the other hand, it is AD if  $a < 0$  and  $N \geq N_1$  (see figure 2). Note that the defective Jordan blocks are not usually stable with respect to perturbations [20]. Indeed, we find numerically that, applying proper two-mode squeezing transformations to the environmental input, these weak-degradability conditions reduce to the decoupled case ones. In figure 1, we consider, for simplicity, a symmetric environmental initial state  $\gamma'_E$  as in equation (105) with  $x = y$ ,  $z_- = 0$  and  $\epsilon = r$ , and we plot the relation between  $x$ ,  $z_+$  and the minimum value of  $r$  such that  $J := J_1$  reduces to  $J := J_0$  corresponding to the decoupled case. One realizes that a squeezing parameter  $r$  close to 1 is enough to decouple the two modes representing the system carrying quantum



**Figure 2.** Continuous line:  $N_1$  as function of  $a$  in the case of  $J_1$ . For  $N \geq N_1$  the map is WD if  $a > 1$  and AD if  $a < 0$ . Dashed line:  $N_2$  as function of  $b$  when  $a = 0$  in the case of  $J_2$ . For  $N \geq N_2$  the channel is WD (AD) if  $a > 1/2$  ( $a < 1/2$ ).

information. Moreover, let us point out that this squeezing threshold ( $r$ ) increases slightly with the presence of correlations ( $z_+$ ) while it decreases when increasing the level of noise ( $x$ ) in the initial environmental state  $\gamma'_E$ .

Finally, in the case of a real Jordan block with complex eigenvalues, i.e.  $J_2$  as in equation (102), the corresponding two-mode BGCs are WD if  $a > 1/2$  and

$$N \geq N_2 := \frac{1}{2} \left[ -1 + \left( 1 + \frac{4b^2}{(1-2a)^2} \right)^{1/2} \right], \quad (111)$$

while they are AD if  $a < 1/2$  and  $N \geq N_2$  (see figure 2). In both of these cases (real and complex eigenvalues), in which the interaction term couples the two bosonic modes, there is the (apparently) counter-intuitive fact that above a certain environmental noise threshold the weak-degradability features appear, while for one-mode BGCs they do not depend on the initial state of the environment. Actually, one would expect at most that, when the level of the environmental noise increases, the coherence progressively decreases until it is destroyed. This would mean that it becomes more and more difficult to recover the environment (system) output from the system (environment) output after the noisy evolution. However, things go the other way around when multi-mode BGCs are considered.

#### 4.2. Channels with zero quantum capacity

Analogously to [17] where the one-mode case is investigated, one can enlarge (other than the AD maps) the class of two-mode BGCs with  $Q = 0$ , composing a generic channel with an AD one. First of all, consider a channel  $\Phi$  as in section 2.3, but being AD (not necessarily minimal noise), then the maps  $\Phi'$ , defined in equation (63), have zero quantum capacity, i.e. they cannot be used to transfer quantum information. For instance, one can choose  $\gamma_E = (2N_c + 1)\mathbb{1}_n$ , i.e. the

environmental initial state of the map  $\Phi$  is a multi-mode thermal state with  $N_c$  being the average photon number for each mode, such that  $\Phi$  is AD or simply with zero capacity; therefore, for any  $\gamma'_E \geq \gamma_E = (2N_c + 1)\mathbb{1}_n$ , as in equation (65), the map  $\Phi'$  of equation (63) has  $Q = 0$ . Particularly, for  $n = 2$ , using these observations and choosing  $N_c$  equal to either  $N_1$  (and  $a < 0$ ) or  $N_2$  (and  $a < 1/2$ ) as in equations (110) and (111), one obtains that for  $X = \mathbb{1}_2 \oplus J_{1,2}$  and  $Y' = s_2 \gamma'_E s_2^T$  (with  $s_2$  as in equation (82)) the resulting channel  $\Phi'$  always has zero capacity. In this way, one extends considerably the set of two-modes maps with zero capacity, other than the very particular cases of two-mode environmental thermal states studied above and shown in figure 2. For instance, two-mode squeezing can be applied to the thermal state  $\gamma_E$  including not only states with  $N > N_c$  but also with non-trivial two-mode correlations such that  $\gamma'_E \geq (2N_c + 1)\mathbb{1}_2$ . Therefore, just considering this last simple inequality one includes a larger set of maps that have zero quantum capacity.

Moreover, we observe that, according to the composition rules above, the combination  $\Phi = \Phi_{II} \circ \Phi_I$  of two channels  $\Phi_I$  and  $\Phi_{II}$  of class  $A_2$  and  $C$ , respectively, with Jordan blocks  $J_I$  as in equation (100) with  $a_I = b_I$  and  $J_{II}$  as in equation (102) with  $a_{II}$  and  $b_{II} \neq 0$ , gives  $J = a_I J_{II}$  which is in the class  $C$ . Now, since we have  $N_1 \geq 0$ ,  $N_2 \geq 0$  and assuming  $a_I \leq 1/2$ , the channel  $\Phi_I$  is AD and the resulting channel  $\Phi$  must have  $Q = 0$ . Varying the parameters but keeping the product  $a_I a_{II} = a$  and  $a_I b_{II} = b$  fixed, the parameter  $N$  can assume any value satisfying the inequality

$$N \geq \frac{1}{4} \left[ \left( \frac{5(1 - 4a + 8a^2 + 8b^2)}{b^2 + (a - 1)^2} \right)^{1/2} - 2 \right]. \quad (112)$$

Note that  $a_I$  has been chosen equal to  $1/2$  and  $\Phi_I$  corresponds to two uncoupled beam-splitter maps with transmissivity  $1/2$ . We can therefore conclude that all channels of the form  $C$  with  $N$  as in equation (112) have zero quantum capacity—see figure 3.

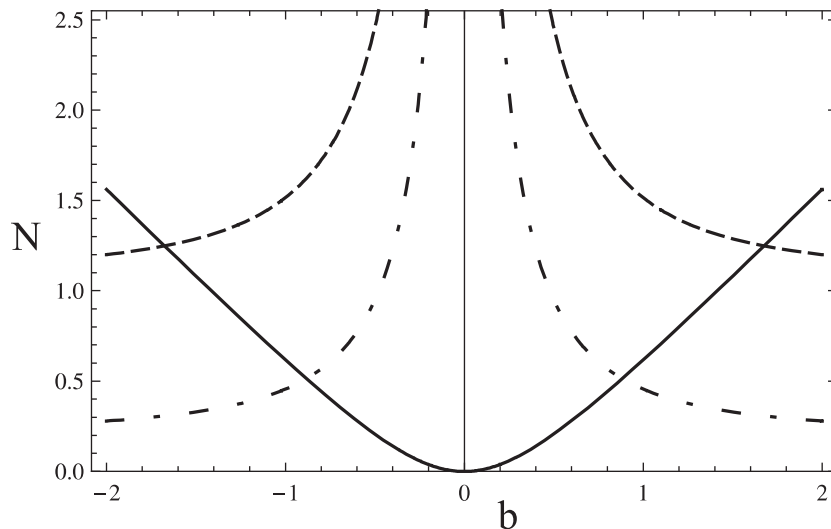
Consider now the composition  $\Phi = \Phi_{II} \circ \Phi_I$  of two channels  $\Phi_I$  and  $\Phi_{II}$  of class  $C$  and  $A_2$  (i.e. in the opposite order with respect to above), respectively, with Jordan blocks  $J_I$  as in equation (102) with  $a_I$  and  $b_I \neq 0$  and  $J_{II}$  as in equation (100) with  $a_{II} = b_{II}$ , giving  $J = a_{II} J_I$  which is in the class  $C$ . As before, since we have  $N_1 \geq 0$ ,  $N_2 \geq 0$  and assuming again  $a_{II} \leq 1/2$ , the channel  $\Phi_2$  is AD and the resulting channel has  $Q = 0$ . Varying the parameters but keeping the product  $a_I a_{II} = a$  and  $b_I a_{II} = b$  fixed, the parameter  $N$  can assume any value satisfying the inequality

$$N \geq \frac{1}{4} \left[ \left( \frac{(1 + 4a^2 + 4b^2)(1 - 4a + 8a^2 + 8b^2)}{4(b^2 + (a - 1)^2)(a^2 + b^2)} \right)^{1/2} - 2 \right], \quad (113)$$

where again  $a_{II}$  is chosen equal to  $1/2$ . Again, we can conclude that all class  $C$  channels with  $N$  as in equation (113) have zero quantum capacity. However, notice that the constraint in equation (113) is an improvement with respect to the constraint of equation (112)—see figure 3.

## 5. Conclusions

In this work, we have presented a complete analysis of generic multi-mode Gaussian channels by proving a unitary dilation theorem and by finding their canonical form. This is a simple form that can be achieved for any Gaussian quantum channel, as a convenient starting



**Figure 3.** The continuous line depicts  $N_2$  as in equation (111) versus  $b$ , with  $a = 1$  in  $J_2$  of equation (102). For  $N \geq N_2$  the channel is WD (AD) if  $a > 1/2$  ( $a < 1/2$ ). The dashed line refers to the bound in equation (113), while the dashed-dotted line to that in equation (112); above these bounds the class  $C$  map is WD but with  $Q = 0$ . Note that equation (113) is an improvement with respect to the constraint of equation (112). Similar bounds can be obtained in the case  $a < 1/2$ , enlarging the group of AD maps with other channels with  $Q = 0$ .

point for various considerations. For instance, it allows us to simplify the analysis of the weak-degradability properties of multi-mode BGCs. Minimal output entropies, or quantum and classical information capacities and other difficult questions might be tackled using the canonical form of multi-mode Gaussian channels shown in this paper. Here, we investigated in detail the two-mode scenario that is relevant since any  $n$ -mode channel can always be reduced to single-mode and two-mode parts [20]. Furthermore, the results of this paper could play a basic role in characterizing the efficiency of continuous-variable quantum information processing, quantum communication and quantum key distribution protocols.

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### Appendix A. Proof of lemma 1

Note that it does not restrict generality to take  $\sigma_{2\ell}^E = \sigma_{2\ell}$ , as this can always be accompanied by an appropriate similarity transform. Our problem at hand of extending a symplectic form is

then equivalent to the following problem: suppose we are given column vectors  $e_1, \dots, e_n$  and  $f_1, \dots, f_n$  from  $\mathbb{R}^{2(n+\ell)}$  that satisfy

$$e_j^T \sigma_{2(n+\ell)} e_k = 0, \quad (\text{A.1})$$

$$f_j^T \sigma_{2(n+\ell)} f_k = 0, \quad (\text{A.2})$$

$$e_j^T \sigma_{2(n+\ell)} f_k = \delta_{j,k} \quad (\text{A.3})$$

for  $j, k = 1, \dots, n$ . The procedure continues by identifying vectors  $e_{n+1}$  and  $f_{n+1}$  such that  $e_{n+1}^T \sigma_{2(n+\ell)} f_{n+1} = 1$  and

$$e_{n+1}^T \sigma_{2(n+\ell)} w = f_{n+1}^T \sigma_{2(n+\ell)} w = 0 \quad (\text{A.4})$$

for all

$$w \in W_n := \text{span}(e_1, \dots, e_n, f_1, \dots, f_n). \quad (\text{A.5})$$

Now define

$$W_n^\perp = \{w : w^T \sigma_{2(n+\ell)} v = 0 \forall v \in W_n\}. \quad (\text{A.6})$$

It is now not difficult to see that  $W_n \cap W_n^\perp = \{0\}$  and  $\mathbb{R}^{2(n+\ell)} = W_n \oplus W_n^\perp$ : suppose that the vector  $v$  has  $v^T \sigma_{2(n+\ell)} e_j =: \alpha_j$  and  $v^T \sigma_{2(n+\ell)} f_j =: \beta_j$  for  $j = 1, \dots, n$ . Then

$$v = \left[ \sum_{j=1}^n (-\alpha_j f_j + \beta_j e_j) \right] + \left[ v + \sum_{j=1}^n (\alpha_j f_j - \beta_j e_j) \right], \quad (\text{A.7})$$

where the first term is an element of  $W_n$  and the second of  $W_n^\perp$ . Following a symplectic Gram–Schmidt procedure, the symplectic basis can hence be completed, which is equivalent to extending the matrices  $s_1$  and  $s_2$  to a symplectic

$$S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \in Sp(2(n+\ell), \mathbb{R}). \quad (\text{A.8})$$

## Appendix B. Derivation of equation (35)

Here, we show that equation (35) admits a solution for  $s'_2$  as in equation (41). In fact, assuming  $\sigma_{4n}^E = \sigma_{2n} \oplus \sigma_{2n}$  with  $\sigma_{2n}$  as in equation (1), one has

$$\begin{aligned} s'_2 \sigma_{4n}^E (s'_2)^T - \Sigma' &= [K^{-1} \mid O^T A] \left[ \begin{array}{c|c} \sigma_{2n} & 0 \\ \hline 0 & \sigma_{2n} \end{array} \right] \left[ \begin{array}{c} K^{-T} \\ \hline A^T O \end{array} \right] - \Sigma' \\ &= K^{-1} \sigma_{2n} K^{-T} + O^T A \sigma_{2n} A^T O - \Sigma' \\ &= K^{-1} (K \Sigma' K^T + B) K^{-T} + O^T A \sigma_{2n} A^T O - \Sigma' \\ &= K^{-1} B K^{-T} + O^T A \sigma_{2n} A^T O \\ &= O (M^{1/2} B M^{1/2} + A \sigma_{2n} A^T) O^T, \end{aligned} \quad (\text{B.1})$$

where we used equation (40) to write  $\sigma_{2n} = K \Sigma' K^T + B$ , with  $B$  being the  $2n \times 2n$  matrix

$$B := \left[ \begin{array}{cc|cc} 0 & & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbb{1}_{n-r/2} \\ \hline 0 & -\mathbb{1}_{n-r/2} & & 0 \end{array} \right]. \quad (\text{B.2})$$

The identity (35) finally follows by noticing that the last term in equation (B.1) cancels since  $M^{1/2} B = B M^{1/2} = B$  and  $A \sigma_{2n} A^T = -B$ .

### Appendix C. Properties of the environmental states

In this appendix, we first give an explicit derivation of equation (43). Then, we analyze in detail the property of the state  $\hat{\rho}_E$  associated with the covariance matrix  $\gamma_E$  defined by equations (45) and (46). Substituting equation (41) into (26), we get

$$\begin{aligned} \mathbb{1}_{2n} &= s_2' \gamma_E (s_2')^T = \left[ K^{-1} \mid O^T A \right] \left[ \begin{array}{c|c} \alpha & \delta \\ \hline \delta^T & \beta \end{array} \right] \left[ \begin{array}{c} K^{-T} \\ \hline A^T O \end{array} \right] \\ &= K^{-1} \alpha K^{-T} + O^T A \delta^T K^{-T} + K^{-1} \delta A^T O + O^T A \beta A^T O \\ &= O^T (M^{1/2} \alpha M^{1/2} + A \delta^T M^{1/2} + M^{1/2} \delta A^T + A \beta A^T) O, \end{aligned}$$

which leads to

$$M^{-1} = \alpha + M^{-1/2} A \delta^T + \delta A^T M^{-1/2} + M^{-1/2} A \beta A^T M^{-1/2} \quad (\text{C.1})$$

and hence to equation (43) by the fact  $M^{-1/2} A = A^T M^{-1/2} = A = A^T$ . Such an equation admits the solution given in equations (45) and (46). Explicitly this corresponds to the  $4n \times 4n$  covariance matrix  $\gamma_E$  of the form

$$\left[ \begin{array}{c|c|c|c} \frac{\mu^{-1}}{0} \mid \frac{0}{\xi \mathbb{1}} & 0 & 0 & \frac{f(\mu^{-1})}{0} \mid \frac{0}{f(\xi \mathbb{1})} \\ \hline 0 & \frac{\mu^{-1}}{0} \mid \frac{0}{\xi \mathbb{1}} & \frac{f(\mu^{-1})}{0} \mid \frac{0}{f(\xi \mathbb{1})} & 0 \\ \hline 0 & \frac{f(\mu^{-1})}{0} \mid \frac{0}{f(\xi \mathbb{1})} & \frac{\mu^{-1}}{0} \mid \frac{0}{\xi \mathbb{1}} & 0 \\ \hline \frac{f(\mu^{-1})}{0} \mid \frac{0}{f(\xi \mathbb{1})} & 0 & 0 & \frac{\mu^{-1}}{0} \mid \frac{0}{\xi \mathbb{1}} \end{array} \right],$$

where for easy of notation  $\mathbb{1} := \mathbb{1}_{n-r/2}$ . By looking at the structure of this covariance matrix, one realizes that it is composed of two independent sets formed by  $r$  and  $2n-r$  modes, respectively. The first set describes  $r/2$  thermal states characterized by the matrices  $\mu^{-1}$  which have been purified by adding further  $r/2$  modes. The second set instead describes a collection of  $2(n-r/2) = 2n-r$  modes prepared in a pure state formed by  $n-r/2$  independent pairs of modes which are entangled. By reorganizing its rows and columns this can be cast into the simpler form

$$\gamma_E = \left[ \begin{array}{c|c} \frac{\bar{\mu}^{-1}}{f(\bar{\mu}^{-1})} \mid \frac{f(\bar{\mu}^{-1})}{\bar{\mu}^{-1}} & 0 \\ \hline 0 & \frac{\xi \mathbb{1}_{2n-r}}{f(\xi \mathbb{1}_{2n-r})} \mid \frac{f(\xi \mathbb{1}_{2n-r})}{\xi \mathbb{1}_{2n-r}} \end{array} \right] \begin{array}{l} \} r, \\ \} r, \\ \} 2n-r, \\ \} 2n-r, \end{array} \quad (\text{C.2})$$

where we have used  $\bar{\mu}$  to indicate the  $r \times r$  matrix  $\bar{\mu} = \mu \oplus \mu$ .

#### C.1. Solution for $\ell_{\text{pure}} = 2n - r'/2$ environmental modes

Defining  $r'$  as in equation (57) we choose the environmental commutation matrix to be  $\sigma_{2\ell}^E = \sigma_{2n} \oplus \sigma_{2n-r'}$  with  $\sigma_{2n}$  and  $\sigma_{2n-r'}$  as in equation (1). A unitary dilation with  $\ell_{\text{pure}} = 2n - r'/2$  environmental modes in a pure state is obtained by having  $s_2 = Y^{1/2} s_2'$  with  $s_2'$  as

in equation (41). In this case, however,  $A$  is a rectangular matrix  $2n \times 2(n - r'/2)$  of the form

$$A = \left[ \begin{array}{cc|cc} & & 0 & 0 \\ & 0 & \hline 0 & 0 & 0 & 0 \\ & & 0 & \mathbb{1}_{n-r/2} \\ \hline 0 & 0 & & \\ \hline 0 & 0 & & \\ \hline 0 & \mathbb{1}_{n-r/2} & & \end{array} \right] \begin{array}{l} \} r'/2, \\ \} (r - r')/2, \\ \} n - r/2, \\ \} r'/2, \\ \} (r - r')/2, \\ \} n - r/2. \end{array} \quad (\text{C.3})$$

Similarly, the covariance matrix  $\gamma_E$  can be still expressed as in equation (44). In this case, however,  $\alpha$  is a  $2n \times 2n$  matrix of block form

$$\alpha = \left[ \begin{array}{ccc|ccc} \mathbb{1}_{r'/2} & 0 & 0 & & & \\ \hline 0 & \mu_o^{-1} & 0 & & & \\ \hline 0 & 0 & \xi \mathbb{1}_{n-r/2} & & & \\ \hline & & & 0 & & \\ \hline & & & \mathbb{1}_{r'/2} & 0 & 0 \\ & & & \hline 0 & \mu_o^{-1} & 0 \\ & & & \hline 0 & 0 & \xi \mathbb{1}_{n-r/2} \end{array} \right] \begin{array}{l} \} r'/2, \\ \} (r - r')/2, \\ \} n - r/2, \\ \} r'/2, \\ \} (r - r')/2, \\ \} n - r/2, \end{array} \quad (\text{C.4})$$

where  $\xi = 5/4$  and  $\mu_o$  is the  $(r - r')/2 \times (r - r')/2$  diagonal matrix formed by the elements of  $\mu$  which are strictly smaller than 1.  $\beta$  is the  $(2n - r') \times (2n - r')$  matrix

$$\beta = \left[ \begin{array}{cc|cc} \mu_o^{-1} & 0 & & \\ \hline 0 & \xi \mathbb{1}_{n-r/2} & & \\ \hline & & 0 & \\ \hline & & \mu_o^{-1} & 0 \\ & & \hline 0 & \xi \mathbb{1}_{n-r/2} \end{array} \right] \begin{array}{l} \} (r - r')/2, \\ \} n - r/2, \\ \} (r - r')/2, \\ \} n - r/2 \end{array} \quad (\text{C.5})$$

and

$$\delta = \left[ \begin{array}{cc|cc} & & 0 & 0 \\ & 0 & \hline 0 & 0 & f(\mu_o^{-1}) & 0 \\ & & 0 & f(\xi \mathbb{1}_{n-r/2}) \\ \hline 0 & 0 & & \\ \hline f(\mu_o^{-1}) & 0 & & \\ \hline 0 & f(\xi \mathbb{1}_{n-r/2}) & & \end{array} \right] \begin{array}{l} \} r'/2, \\ \} (r - r')/2, \\ \} n - r/2, \\ \} r'/2, \\ \} (r - r')/2, \\ \} n - r/2 \end{array} \quad (\text{C.6})$$

with  $f$  as in equation (46).

By looking at the structure of this covariance matrix, one realizes that it is composed by three independent pieces. The first one describes a collection of  $r'/2$  vacuum states. The second one, in turn, describes  $(r - r')/2$  thermal states characterized by the matrices  $\mu_o^{-1}$  which have been purified by adding further  $(r - r')/2$  modes. The third one, finally, reflects a collection of  $2(n - r/2) = 2n - r$  modes prepared in a pure state formed by  $n - r/2$  independent pairs of modes which are entangled.



### C.2. Solution for $\ell = 2n - r/2$ not necessarily pure environmental modes

In this subsection, we present the alternative derivation of a dilation that does not necessarily involve an environment prepared in a pure state. Choosing the commutation matrix  $\sigma_{2\ell}^E = \sigma_{2n} \oplus \sigma_{2n-r}$  with  $\sigma_{2n}$  and  $\sigma_{2n-r}$  as in equation (1), the matrix  $s'_2$  can still be expressed as in equation (41). In this case, however,  $A$  is a rectangular matrix  $2n \times (2n - r)$  of the form

$$A = \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_{n-r/2} \\ \hline \mathbb{1}_{n-r/2} & 0 \end{array} \right] \begin{array}{l} \} r/2, \\ \} n - r/2, \\ \} r/2, \\ \} n - r/2. \end{array} \quad (\text{C.7})$$

Similarly,  $\gamma_E$  has the block form (44), where  $\alpha$  is still the  $2n \times 2n$  matrix of equation (45), whereas  $\beta$  and  $\delta$  are, respectively, the following  $(2n - r) \times (2n - r)$  and  $2n \times (2n - r)$  real matrices:

$$\beta = \left[ \begin{array}{c|c} \xi \mathbb{1}_{n-r/2} & 0 \\ \hline 0 & \xi \mathbb{1}_{n-r/2} \end{array} \right] \begin{array}{l} \} n - r/2, \\ \} n - r/2, \end{array} \quad (\text{C.8})$$

$$\delta = \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & f(\xi \mathbb{1}_{n-r/2}) \\ \hline 0 & 0 \\ \hline f(\xi \mathbb{1}_{n-r/2}) & 0 \end{array} \right] \begin{array}{l} \} r/2, \\ \} n - r/2, \\ \} r/2, \\ \} n - r/2 \end{array} \quad (\text{C.9})$$

with  $\xi$  and  $f$  as in equation (46). That is,

$$\gamma_E = \left[ \begin{array}{c|c|c|c|c} \mu^{-1} & 0 & & 0 & 0 \\ \hline 0 & \xi \mathbb{1} & & 0 & f(\xi \mathbb{1}) \\ \hline & & \mu^{-1} & 0 & 0 \\ \hline 0 & & 0 & \xi \mathbb{1} & f(\xi \mathbb{1}) \\ \hline 0 & 0 & 0 & f(\xi \mathbb{1}) & \xi \mathbb{1} \\ \hline 0 & f(\xi \mathbb{1}) & 0 & 0 & \xi \mathbb{1} \end{array} \right] \begin{array}{l} \} r/2, \\ \} n - r/2, \\ \} r/2, \\ \} n - r/2, \\ \} n - r/2, \\ \} n - r/2 \end{array} \quad (\text{C.10})$$

with  $\mathbb{1} = \mathbb{1}_{n-r/2}$ . This covariance matrix now consists of two independent parts: the first one describes a collection of  $r/2$  thermal states described by the matrices  $\mu^{-1}$ . The second instead reflects a collection of  $2(n - r/2) = 2n - r$  modes prepared in a pure state formed by  $n - r/2$  independent couples of modes which are entangled. The covariance matrix given in theorem 1 can be recovered from the one given above by adding  $r$  modes to purify the thermal states  $\mu^{-1}$ .

## Appendix D. Equivalent unitary dilations

Let

$$S = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \quad (\text{D.1})$$

and  $\gamma_E$  define a unitary dilation for a BGC  $\Phi$  characterized by matrices  $X$  and  $Y$ . Then a full class of unitary dilations

$$S' = \begin{bmatrix} s'_1 & s'_2 \\ s'_3 & s'_4 \end{bmatrix} \quad (\text{D.2})$$

can be obtained by taking  $\gamma'_E = V\gamma_E V^T$  and

$$s'_1 = s_1, \quad s'_2 = s_2 V, \quad s'_3 = W s_3, \quad s'_4 = W s_4 V \quad (\text{D.3})$$

with  $V \in Sp(2\ell, \mathbb{R})$  and  $W \in Sp(2n, \mathbb{R})$  being symplectic transformations of  $\ell$  and  $n$  modes, respectively. With this choice in fact  $\gamma'_E$  is still a covariance matrix while the conditions (23) and (24) are automatically satisfied. From a physical point of view the symplectic transformations  $V$  and  $W$  correspond to unitary local operations applied to the environmental input and output states, respectively, by virtue of the metaplectic representation. Consequently, the weak complementary channels  $\tilde{\Phi}$  and  $\tilde{\Phi}'$  associated with these two representations are unitarily equivalent and the weak-degradability properties one can determine for  $\Phi$  will be the same when studied for  $\Phi'$ .

Conversely, let us suppose we have two unitary dilations of  $\Phi$ , realized with  $\ell = n$  environmental modes and characterized by the symplectic matrices  $S$  and  $S'$  as in equations (D.1) and (D.2), respectively, with  $s_i$  and  $s'_i$  being  $2n \times 2n$  square matrices. Then it is possible to show that they must be related as in equation (D.3) under the hypothesis that  $s_2$  and  $s_3$  are non-singular. First of all, since equation (24) must be satisfied for all the input covariance matrices  $\gamma$ , we have  $s_1 = X^T = s'_1$ . Define then  $V = s_2^{-1} s'_2$  and  $W = s'_3 s_3^{-1}$ . By using the first part of equation (23) and exploiting the non-singularity of  $s_2$  one has

$$s_2 V \sigma_{2\ell}^E V^T s_2^T = s_2 \sigma_{2n} s_2^T \implies V \sigma_{2n} V^T = \sigma_{2n}, \quad (\text{D.4})$$

which implies that  $V$  is a symplectic matrix (we are assuming  $\sigma_{2\ell}^E = \sigma_{2n}$ ). Moreover, from the second condition in equation (23) for  $S$  and  $S'$ , we obtain

$$s_2 \sigma s_4^T W^T = s_2 V \sigma s_4'^T, \quad \implies s'_4 = W s_4 V, \quad (\text{D.5})$$

because  $s_2$  is non-singular and  $V$  is symplectic. By considering the third condition (23) one then has

$$W(s_3 \sigma_{2n} s_3^T + s_4 \sigma_{2n} s_4^T) W^T = W \sigma_{2n} W^T = \sigma_{2n}, \quad (\text{D.6})$$

which proves that  $W$  is a symplectic. Finally, let us observe that the proof above does not use the non-singularity of  $s_3$ . Indeed, one can relax this hypothesis and assume more simply that there exists a  $W$  such that  $s'_3 = W s_3$ ; from equation (23)  $W$  has to still be a symplectic matrix but  $s_3$  and  $s'_3$  may be singular.

As an application of these equivalent unitary dilation results, we can find an alternative canonical form to the one in section 2.5 with the same  $s_1$  and  $s_4$  but with  $s_2$  and  $s_3$  of the following anti-diagonal block form:

$$s_j = \begin{bmatrix} 0 & F_j \\ G_j & 0 \end{bmatrix}, \quad (\text{D.7})$$

where, for  $j = 2, 3$ ,  $F_j, G_j$  are  $n \times n$  real matrices. Imposing equation (23), one obtains the following relations:

$$J^T - F_2 G_2^T = \mathbb{1}_n, \quad J^T - F_3 G_3^T = \mathbb{1}_n, \quad (\text{D.8})$$

$$F_2 - F_3^T = 0, \quad J G_3^T - G_2 J^T = 0,$$

the solution of which provides the following unitary dilation:

$$S = \left[ \begin{array}{cc|cc} \mathbb{1}_n & 0 & 0 & -(\mathbb{1}_n - J^T) G_2^{-T} \\ 0 & J & G_2 & 0 \\ \hline 0 & -G_2^{-1}(\mathbb{1}_n - J) & \mathbb{1}_n & 0 \\ G_2^T & 0 & 0 & G_2^T J^T G_2^{-T} \end{array} \right], \quad (\text{D.9})$$

where again  $G_2$  is an arbitrary (non-singular) matrix and the eigenvalues of  $J$  are assumed to be different from 1. This solution is unitarily equivalent to the one in equation (79) by applying  $V = -\sigma_{2n}$  and

$$W = \begin{bmatrix} 0 & G_2^{-1} J^{-1} G_2 \\ -G_2^T J^T G_2^{-T} & 0 \end{bmatrix}, \quad (\text{D.10})$$

as above.

### Appendix E. The ideal-like quantum channel

Here, we consider a quantum channel with  $X = \mathbb{1}_{2n}$  but  $Y \geq 0$  with rank less than  $2n$ , which can be described in terms of only  $n$  additional (environmental) modes. We call it the ideal-like quantum channel. Accordingly, the canonical unitary transformation  $\hat{U}$  of equation (20) will be uniquely determined by a  $4n \times 4n$  real matrix  $S \in Sp(4n, \mathbb{R})$  of block form in equation (22), where  $s_i$  are  $2n \times 2n$  real matrices. Particularly,  $s_1 = s_4 = \mathbb{1}_{2n}$ ,

$$s_3 = \begin{bmatrix} F_3 & 0 \\ 0 & G_3 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -G_3^T & 0 \\ 0 & -F_3^T \end{bmatrix} \quad (\text{E.1})$$

with  $F_3$  and  $G_3$  being  $n \times n$  real matrices such that  $F_3 G_3^T = G_3^T F_3 = 0$ , in order to satisfy the symplectic conditions in equation (23). Taking advantage of the freedom in the choice of the unitary dilation shown in appendix D, the matrix  $S$  can be put in the form of equation (22) in which  $s'_1 = s'_4 = \mathbb{1}_{2n}$ ,

$$s'_2 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1}_n \end{bmatrix}, \quad s'_3 = \begin{bmatrix} -\mathbb{1}_n & 0 \\ 0 & 0 \end{bmatrix}, \quad (\text{E.2})$$

where  $F_3$  is assumed non-singular. In this respect, one uses  $V, W \in Sp(2n, \mathbb{R})$  (of appendix D) of the following form:

$$V = \begin{bmatrix} -F_3 & 0 \\ 0 & -F_3^{-T} \end{bmatrix} \quad (\text{E.3})$$

and  $W = V^{-1}$ . Similarly, one can proceed, if  $G_3$  is non-singular, and obtain a similar structure for  $S$  as above. As concerns the weak-degradability properties, if one assumes the initial environmental input state as  $\gamma_E = \text{diag}(2N+1, 2M+1, 2N+1, 2M+1)$ , the eigenvalues of  $\tilde{Y} - \tilde{X}^T X^{-T} (Y + i\sigma) X^{-1} \tilde{X} + i\sigma$  are  $\{2M, 2(M+1), 2N, 2(N+1)\}$ , which are always positive for any  $N \geq 0$  and  $M \geq 0$ ; hence, this channel with  $\gamma_E$  as above is always WD.

Finally, one may consider another ideal-like channel with  $X = \mathbb{1}_{2n}$  and  $Y = [(1 - \sigma_3)/4]^{\otimes n}$ , i.e.  $\Phi_{X,Y} = \otimes_{i=1}^n (B_1)_i$ , where the single-mode  $B_1$  channel is defined in [17] as  $X = \mathbb{1}_2$  and  $Y = (1 - \sigma_3)/4$ . Trivially, this multi-mode channel is always WD (like  $B_1$ ) and is able to transfer a quantum state without decoherence with the maximum quantum capacity (like for the single-mode case [17]).

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