Elastic energy of a convex body

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Abstract

In this paper a Blaschke-Santaló diagram involving the area, the perimeter and the elastic energy of planar convex bodies is considered. More precisely we give a description of set

$$\mathcal{E} := \left\{ (x, y) \in \mathbb{R}^2, x = \frac{4\pi A(\Omega)}{P(\Omega)^2}, y = \frac{E(\Omega)P(\Omega)}{2\pi^2}, \Omega \text{ convex} \right\},$$

where A is the area, P is the perimeter and E is the elastic energy, that is a Willmore type energy in the plane. In order to do this, we investigate the following shape optimization problem:

$$\min_{\Omega \in \mathcal{C}} \{E(\Omega) + \mu A(\Omega)\},$$

where $\mathcal{C}$ is the class of convex bodies with fixed perimeter and $\mu \geqslant 0$ is a parameter. Existence, regularity and geometric properties of solutions to this minimum problem are shown.

*K*ey words: Elastic energy, Willmore type energy, convex geometry, Blaschke diagram, shape optimization.
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Contents

1 Introduction

For a regular planar convex body Ω, that is a planar convex compact set, we introduce the three geometric quantities $A(\Omega)$, $P(\Omega)$, $E(\Omega)$ where $A(\Omega)$ is the area, $P(\Omega)$ is the perimeter and $E(\Omega)$ is the elastic energy defined by

$$E(\Omega) = \frac{1}{2} \int_{\partial\Omega} k^2(s)\, ds$$

where k is the curvature and s is the arc length. The elastic energy of a curve seems to have been introduced by L. Euler in 1744 who studied the *elasticae*. These curves are critical points of the elastic energy which satisfy some boundary conditions. This question has been widely studied and has many applications in geometry, in kinematics (the ball-plate problem), in numerical analysis (non-linear splines), in computer vision (reconstruction of occluded edges) etc. For a good overview and historical presentation, we refer e.g. to [10].

The aim of this paper is to study the links between $E(\Omega)$, $A(\Omega)$ and $P(\Omega)$. This can be done by investigating the set of points in $\mathbb{R}^3$ corresponding to the triplet $(A(\Omega), E(\Omega), P(\Omega))$ or a planar scale invariant version as $(A(\Omega)/P(\Omega)^2, E(\Omega)P(\Omega))$. The first one who studied the diagram of these points is probably W. Blaschke in [3] where the three quantities in consideration were the volume, the surface area and the integral of the mean curvature of a three-dimensional convex body. Later on, L. Santaló in [12] proposed a systematic study of this kind of diagrams for planar convex body and geometric quantities like the area, the perimeter, the diameter, the minimal width, the inradius and the circumradius. From that time, this kind of diagram is often called *Blaschke-Santaló diagram*.

Our aim is to study the following Blaschke-Santaló diagram involving area, elastic energy and perimeter:

$$\mathcal{E} := \left\{ (x,y) \in \mathbb{R}^2, x = \frac{4\pi A(\Omega)}{P(\Omega)^2}, y = \frac{E(\Omega)P(\Omega)}{2\pi^2}, \, \Omega \text{ convex} \right\}. \tag{1.1}$$

This will be done in Section 5.

For this analysis, we recall an important geometric inequality due to Gage in [4]:

Theorem 1.1 (Gage)**.** *For any planar convex body of class C^1 and piecewise C^2, the following inequality holds*

$$\frac{E(\Omega)A(\Omega)}{P(\Omega)} \geqslant \frac{\pi}{2} \tag{1.2}$$

with equality if and only if Ω is a disk.

In other words, the disk minimizes the product $E(\Omega)A(\Omega)$ among convex bodies with given perimeter. More general inequalities involving different functions of the curvature, area and perimeter have been proved in [5]. Notice that Gage's result implies that the points (x, y) in $\mathcal{E}$ satisfy the inequality $xy \geqslant 1$. In order to describe the diagram $\mathcal{E}$ we need additional relations which lead us to consider the following minimization problem:

$$\min_{\Omega \in \mathcal{C}} \left(E(\Omega) + \mu A(\Omega) \right), \tag{1.3}$$

where $\mu \geqslant 0$ is a parameter and $\mathcal{C}$ is the class of regular planar convex bodies Ω such that $P(\Omega) = P_0$. We stress that there is a competition between the two terms since the disk minimizes $E(\Omega)$, see below (1.4) while it maximizes $A(\Omega)$ by the isoperimetric inequality. Thus we can expect that the penalization parameter μ plays an important role and that the solution is close to the disk when μ is small while it is close to the segment when μ is large. More precisely, we will present several results in Section 4. Our objective is to describe the boundary of the set $\mathcal{E}$ defined in (1.1) by solving this minimization problem.

Before tackling this minimization problem, let us make some observations about the minimization of the elastic energy $\min\{E(\Omega)\}$. Without any constraint this problem has no solution. Indeed if we consider a disk D_r of radius r, the curvature k is constant equal to $1/r$ so that

$$\int_{\partial D_r} k^2 \, ds = \frac{2\pi}{r} \to 0 \quad \text{as } r \to \infty.$$

Now, if we add a constraint of perimeter $P(\Omega) = P_0$, and if we consider that Ω is a bounded simply connected domain then by using the Cauchy-Schwarz inequality we deduce

$$2\pi = \int_{\partial\Omega} k \, ds \leqslant \left(\int_{\partial\Omega} k^2 \, ds \right)^{1/2} (P(\Omega))^{1/2} \tag{1.4}$$

with equality only in the case of a disk. Thus the disk solves

$$\min\{E(\Omega), P(\Omega) = P_0\} \tag{1.5}$$

among simply connected domains. Let us remark that the equality constraint $P(\Omega) = P_0$ can be replaced by an inequality $P(\Omega) \leqslant P_0$ since $E(t\Omega) = E(\Omega)/t$. Moreover, if Ω is not simply connected, the result still holds true since removing extra parts of the boundary makes the perimeter and the elastic energy lower.

Now if we consider the minimization of $E(\Omega)$ with a constraint on the area, there is no minimum. Indeed we can take the annulus of radii r and $r+\delta_r$ so that the area constraint is satisfied, then

$$\int_{\partial\Omega} k^2\, ds = 2\pi\left(\frac{1}{r}+\frac{1}{r+\delta_r}\right) \to 0 \quad \text{as } r\to\infty.$$

On the other hand, this problem has a solution in the class of convex bodies. This is an easy consequence of Gage's inequality together with the isoperimetric inequality: the disk is the unique minimizer of the elastic energy under a constraint of area among convex bodies.

To our knowledge, the question to look for a minimizer for the elastic energy among simply connected sets of given area remains open. Let us also mention some related works. In [11], Yu. L. Sachkov studies the "closed elasticae", that is the closed curves which are stationary points of the elastic energy. He obtains only two possible curves: the disk or the "eight elasticae" which is a local minimum. His method relies on optimal control theory and Pontryagin Maximum Principle. The problem of minimizing $E(\Omega)$ among sets with given $P(\Omega)$ and $A(\Omega)$ has also been studied. Indeed, this problem is related to the modelling of vesicles which attracts much attention recently. For a study of critical points of the functional and some numerical results, we refer to [14].

Let us mention that this kind of problem has a natural extension in higher dimension, the elastic energy being replaced by the Willmore functional. This one being scale invariant, the nature of the problem is different. For a physical point of view, it is a much more realistic model for vesicles. For example, the problem of minimizing the Willmore functional (or the Helfrich functional which is very similar) among three-dimensional sets, with constraints on the volume and the surface area, is a widely studied problem.

The plan of the paper is as follows: in Sections 2, 3, 4 we study the minimization problem (1.3). First existence and C^2 regularity of a minimizer is proved, then some geometric properties are given: symmetry, possibilities of segments on the boundary and the case of the disk is investigated (for what values of μ is it solution or not). In the two last sections, the Blaschke-Santaló diagram of the set $\mathcal{E}$ is investigated, first from a theoretical point of view in Section 5 and then from a numerical point of view in Section 6.

1.1 Notations

For points M, Q in the plane, we indicate by $\overrightarrow{QM}$ the planar vector joining these two points and we denote by $\|\cdot\|$ the Euclidean norm in R^N.

For an integer $p \geqslant 1$ and a real number $q \geqslant 1$, the Sobolev space $W^{p,q}(a,b)$ is the subset of functions f in $L^q(a,b)$ such that the function f and its weak derivatives up to the $p-th$ order belong to the space $L^q(a,b)$. By $W_0^{p,q}(a,b)$ we indicate the closure in $W^{p,q}(a,b)$ of the infinitely differentiable functions compactly supported in (a,b). We indicate by $\langle\cdot,\cdot\rangle_{L^2(a,b)}$ the scalar products in the Hilbert space $L^2(a,b)$ and by $\|\cdot\|_{L^2(a,b)}$ its operator norm.

2 Existence and regularity

2.1 Existence

We recall that A, P and E can be expressed in different ways depending on which parametrization is considered. Indeed choosing the arc length s parametrization, the area and the elastic energy can be written in terms of the angle $\theta(s)$ (angle between the tangent and the horizontal axis) in the following way:

$$E(\Omega) = \frac{1}{2}\int_0^P {\theta'}^2(s)\, ds \qquad A(\Omega) = \int\int_T \cos(\theta(u))\sin(\theta(s))\, du\, ds \tag{2.1}$$

where T is the triangle $T = \{(u,s)\in\mathbb{R}^2\ ;\ 0\leqslant u\leqslant s\leqslant P\}$. In this case, we recall that

$$\partial\Omega = \Big\{(x(s),y(s)),\quad s\in[0,P]\Big\}, \tag{2.2}$$

and

$$x'(s) = \cos\theta(s),\ \ y'(s) = \sin\theta(s). \tag{2.3}$$

The convexity of the set Ω is expressed by the fact that the function $s \mapsto \theta(s)$ is non-decreasing. Notice that expression (2.1) for the elastic energy leads us to impose the following regularity condition on (the boundary of) the convex set Ω; that is the function $\theta(s)$ has to belong to the Sobolev space $W^{1,2}(0,P)$ Let us remark that if θ is given, we recover the boundary of the convex set by integrating $\cos\theta$ and $\sin\theta$.

Hence let us consider the following class of convex sets:

$$\mathcal{C} := \left\{\Omega \subset \mathbb{R}^2 \text{ bounded and open set such that (2.2) and (2.3) hold and } \theta \in \mathcal{M}\right\}, \tag{2.4}$$

where

$$\mathcal{M} := \left\{\theta \in W^{1,2}(0,P)\ ;\ \theta(0)+2\pi = \theta(P),\ \theta' \geqslant 0\ a.e.,\quad \int_0^P \cos(\theta(s))\,ds = \int_0^P \sin(\theta(s))\,ds = 0\right\}. \tag{2.5}$$

On the other hand, choosing the parametrization of the convex set by its support function $h(t)$ ($t \in [0,2\pi]$) and its radius of curvature $\phi = h'' + h \geqslant 0$, with $\phi = 1/k$, we have

$$P(\Omega) = \int_0^{2\pi} h(t)dt = \int_0^{2\pi} \phi(t)dt, \quad A(\Omega) = \frac{1}{2}\int_0^{2\pi} h(t)\phi(t)\,dt, \quad E(\Omega) = \frac{1}{2}\int_0^{2\pi} \frac{1}{\phi(t)}\,dt; \tag{2.6}$$

this last expression being valid as soon as Ω is C^2_+ meaning that the radius of curvature is a positive continuous function and the fact that $ds = \phi(t)\,dt$ (where s is the curvilinear abscissa).

Remark 2.1. *We underline that if the domain Ω is not strictly convex or not of class C^2, it is well known that its convexity is just expressed by the fact that $h'' + h$ is a non-negative measure. This is actually a consequence of the Minkowski existence Theorem, see [13, Section 7.1].*

In this general case the expression of $E(\Omega)$ in (2.6) *is no longer valid. For more results and properties of the support function, we refer again to [13].*

Let us remark also that, for a regular convex body, the radius of curvature ϕ is positive (because $\phi = 0$ would mean that the curvature k is infinite). In what follows, we use the operator G defined by $G\phi = h$ where h is the solution of

$$h'' + h = \phi \text{ in } (0,2\pi),\ h\ 2\pi-\text{periodic},\ \int_0^{2\pi} h(t)\cos(t)\,dt = \int_0^{2\pi} h(t)\sin(t)\,dt = 0. \tag{2.7}$$

Hence the area of Ω can be rewritten as

$$A(\Omega) = \frac{1}{2}\int_0^{2\pi} G\phi(t)\phi(t)\,dt. \tag{2.8}$$

Without loss of generality and to simplify the presentation, we assume from now on that the perimeter constraint is

$$P(\Omega) = 2\pi. \tag{2.9}$$

Using the parametrization in θ, Problem (1.3) can be written as

$$\inf_{\theta\in\mathcal{M}} j_\mu(\theta), \tag{2.10}$$

where $\mathcal{M}$ is defined by (2.5) with $P = 2\pi$ and

$$j_\mu(\theta) := \frac{1}{2}\int_0^{2\pi} {\theta'}^2(s)\,ds + \mu \int\int_T \cos(\theta(u))\sin(\theta(s))\,du\,ds$$

with

$$T = \{(u,s) \in \mathbb{R}^2\ ;\ 0 \leqslant u \leqslant s \leqslant 2\pi\}.$$

Classical arguments allow to prove the existence of a minimum to problem (1.3):

Theorem 2.2. *For all $\mu \geqslant 0$, there exists $\Omega^* \in \mathcal{C}$ which minimizes $J_\mu(\Omega) = E(\Omega) + \mu A(\Omega)$.*

Proof. Let $\theta_n \in \mathcal{M}$ corresponding to a minimizing sequence of domains Ω_n. Necessarily θ_n is bounded in $W^{1,2}(0,2\pi)$, therefore we can extract a subsequence which converges weakly to some θ^* in $W^{1,2}(0,2\pi)$ and uniformly in $C^0([0,2\pi])$ (the embedding $W^{1,2}(0,2\pi) \hookrightarrow C^0([0,2\pi])$ being compact). Thus θ^* is non-decreasing, $\theta^*(0)+2\pi = \theta^*(2\pi)$ and

$$\int_0^{2\pi} \cos(\theta^*(s))\ ds = \int_0^{2\pi} \sin(\theta^*(s))\ ds = 0.$$

Moreover, $J_\mu(\theta^*) \leqslant \liminf J_\mu(\theta_n)$ which proves the result. □

2.2 Optimality conditions and regularity

We want to characterize the optimum of the problem

$$\min_{\Omega\in\mathcal{C}} E(\Omega) + \mu A(\Omega), \tag{2.11}$$

where $\mathcal{C}$ is defined by (2.4) and where (2.9) holds.

We first write optimality condition for (2.11) by considering the parametrization $(x(s), y(s))$ of $\partial\Omega$ such that

$$x'(s) = \cos\theta(s), \quad y'(s) = \sin\theta(s), \quad (s \in [0,2\pi]). \tag{2.12}$$

Without loss of generality, we may assume

$$x(0) = 0, \quad y(0) = 0. \tag{2.13}$$

The following result gives the main properties of a function θ associated to an optimal domain Ω.

Theorem 2.3. *Assume θ is associated to an optimal domain Ω solution of (2.11). Then $\theta \in W^{2,\infty}(0,2\pi)$ and there exist Lagrange multipliers λ_1, λ_2 and a constant C such that, for all $s \in [0,2\pi]$*

$$\theta'(s) = \frac{\mu}{2}\left(\frac{\lambda_1^2+\lambda_2^2}{\mu^2} + \frac{2C}{\mu} - \left[x(s) - \frac{\lambda_2}{\mu}\right]^2 - \left[y(s) + \frac{\lambda_1}{\mu}\right]^2\right)^- \tag{2.14}$$

where $(\cdot)^-$ denotes the negative part of a real number.

We postpone the proof of Theorem 2.3 at the end of the section.

Remark 2.4. *On the strictly convex parts of the boundary of Ω, (2.14) writes*

$$\theta'(s) = \frac{\mu}{2}\left(\left[x(s) - \frac{\lambda_2}{\mu}\right]^2 + \left[y(s) + \frac{\lambda_1}{\mu}\right]^2 - \frac{\lambda_1^2+\lambda_2^2}{\mu^2} - \frac{2C}{\mu}\right) \tag{2.15}$$

and by a classical bootstrap argument, this shows that θ is indeed C^∞. In the non-strictly convex case there may be a lack of regularity due to the connection points between segments and strictly convex parts.

For similar regularity results for shape optimization problems with convexity constraints, in a more general context, we refer to [8].

By setting

$$R_0^2 := \frac{\lambda_1^2+\lambda_2^2}{\mu^2} + \frac{2C}{\mu}, \quad Q = \left(\frac{\lambda_2}{\mu}, -\frac{\lambda_1}{\mu}\right) \tag{2.16}$$

and

$$M(s) := \left(\int_0^s \cos(\theta(t))\ dt,\ \int_0^s \sin(\theta(t))\ dt\right) \in \partial\Omega, \tag{2.17}$$

we can write (2.14) as

$$k(s) = \frac{\mu}{2}\left(R_0^2 - \left\|\overrightarrow{QM(s)}\right\|^2\right)^- \quad \forall s \in [0,2\pi]. \tag{2.18}$$

In particular, if $k(s) > 0$, then

$$k(s) = \frac{\mu}{2}\left(\left\|\overrightarrow{QM(s)}\right\|^2 - R_0^2\right). \tag{2.19}$$

Using the tools of shape derivative, we can also write the optimality condition for the curvature k of an optimal domain in a different way.

Proposition 2.5. *Let k be the curvature associated to an optimal domain Ω solution of* (2.11)*. Then, on the strictly convex parts it holds*

$$k''(s) = -\frac{1}{2}k^3 - \lambda k + \mu, \tag{2.20}$$

where

$$\lambda := \frac{2\mu A(\Omega) - E(\Omega)}{2\pi} \tag{2.21}$$

Moreover, using the notations introduced in (2.16) *and* (2.17)*, and denoting by* n *the unit normal exterior vector to* Ω*, it holds at the point* $M(s)$

$$\langle \overrightarrow{QM}, \mathsf{n}\rangle = \frac{\lambda}{\mu} + \frac{1}{2\mu}\, k^2 \quad \text{in } \partial\Omega. \tag{2.22}$$

Remark 2.6. *One can wonder whether a relation like* $\langle \overrightarrow{QM}, \mathsf{n}\rangle = a + bk^2$ *implies that the domain is a disk. According to Andrews, [2, Theorem 1.5], this is certainly true if* $a \leqslant 0$ *and* $b \geqslant 0$*, since it is possible to prove that the isoperimetric ratio* P^2/A *decreases under a flow driven by such a relation. As we will see below (Sections 5 and 6) this is not true in general if both coefficients* a *and* b *are positive.*

The proof of Proposition 2.5 makes the use of shape derivatives. For the reader convenience we present in the following lemma the shape derivative of the area, of the perimeter and of the elastic energy. We postpone the proof to the Appendix.

Lemma 2.7. *The shape derivatives of the three quantities* A, P, E *are given by*

$$dA(\Omega; V) = \int_{\partial\Omega} \langle V, \mathsf{n}\rangle\, ds,$$

$$dP(\Omega; V) = \int_{\partial\Omega} k\langle V, \mathsf{n}\rangle\, ds,$$

$$dE(\Omega; V) = -\int_{\partial\Omega} (k'' + \frac{1}{2}k^3)\langle V, \mathsf{n}\rangle\, ds.$$

where V *is any deformation field and* n *the exterior normal vector.*

We are now in position to prove Proposition 2.5.

Proof of Proposition 2.5. From Lemma 2.7, we deduce that for any admissible V, a solution Ω of (2.11) satisfies

$$-\int_{\partial\Omega} (k'' + \frac{1}{2}k^3)\langle V, \mathsf{n}\rangle\, ds + \mu \int_{\partial\Omega} \langle V, \mathsf{n}\rangle\, ds = \lambda \int_{\partial\Omega} k\langle V, \mathsf{n}\rangle\, ds, \tag{2.23}$$

for some Lagrange multiplier λ associated to the perimeter constraint. In particular, on any part of $\partial\Omega$ where the domain is strictly convex,

$$\theta'''(s) = k''(s) = -\frac{1}{2}k^3 - \lambda k + \mu. \tag{2.24}$$

On the other hand, on any part of $\partial\Omega$ where the domain is strictly convex we can differentiate the optimality condition (2.15) obtaining

$$\theta'''(s) = \mu + \mu\left[\left(x(s) - \frac{\lambda_2}{\mu}\right)x''(s) + \left(y(s) + \frac{\lambda_1}{\mu}\right)y''(s)\right]. \tag{2.25}$$

Combining the above relation with (2.24), we deduce that on the part of $\partial\Omega$ where $k > 0$ a.e. (namely on a strictly convex part):

$$\mu\,\langle \overrightarrow{QM}, \mathsf{n}\rangle = \lambda + \frac{1}{2}\,k^2, \tag{2.26}$$

where Q denotes the point defined by (2.16) and the exterior normal vector is $\mathsf{n} = (\sin\theta, -\cos\theta)$.

By continuity the above relation still holds true on segments of $\partial\Omega$ and it writes

$$\langle \overrightarrow{QM}, \mathsf{n}\rangle = \frac{\lambda}{\mu}. \tag{2.27}$$

Therefore for an optimal domain condition (2.26) holds true on the whole boundary. Integrating (2.26) on $\partial\Omega$, we obtain

$$2\mu A(\Omega) = 2\pi\lambda + E(\Omega).$$

This proves the proposition. ☐

Remark 2.8. *Let us focus on Equation* (2.20)*. This differential equation has a central role in the analysis of* $\partial\Omega$ *as it can be seen in the proof of the next proposition.*

Notice that it can be explicitly solved by quadrature using Jacobian elliptic functions as shown in Section 6, Lemma 6.1.

Proposition 2.9. *Assume that* Ω *is an optimal domain. Then* $\partial\Omega$ *is periodic and is the union of suitably rotated and translated copies of a symmetric curve.*

In particular, in the strictly convex parts, it holds

$$\overrightarrow{QM}(s) = \left(\frac{\lambda}{\mu} + \frac{1}{2\mu}\,k(s)^2\right)\mathsf{n}(s) + \frac{1}{\mu}k'(s)\tau(s). \tag{2.28}$$

If the boundary of Ω *contains segments, then they have the same length*

$$L = 2\sqrt{R_0^2 - \left(\frac{\lambda}{\mu}\right)^2}.$$

Proof. First of all, let us consider the strictly convex case. Let us assume that $k(s)$ attains its maximum k_M at $s = 0$ and its minimum k_m at $s = s_1$. Using equation (2.20) and the Cauchy–Lipschitz Theorem, one can deduce that k is symmetric with respect to s_1 and attains k_M also at $s = 2s_1$. Using again the Cauchy–Lipschitz Theorem we find $k(s) = k(s + 2s_1)$ for $s \in [0, 2s_1]$ that concludes the first part of the proposition.

In the case where there is a segment, we assume again that $k(s)$ attains its maximum k_M at $s = 0$. Let us call b the first positive zero of k so that $k > 0$ in the interval $[0, b)$. Assume $k(s) = 0$ for $s \in [b, b + L]$ and $k(s) > 0$ for s in a right neighbourhood of $s > b + L$. Using (2.19) and the continuity of $k(s)$, we deduce

$$\left\|\overrightarrow{QM(b)}\right\| = R_0. \tag{2.29}$$

Using (2.22) we obtain

$$\langle\overrightarrow{QM(b)}, \mathsf{n}(b)\rangle = \frac{\lambda}{\mu}. \tag{2.30}$$

As a consequence,

$$\left|\langle\overrightarrow{QM(b)}, \tau(b)\rangle\right| = \sqrt{R_0^2 - \left(\frac{\lambda}{\mu}\right)^2}.$$

Differentiating (2.19) in the strictly convex part $s \in (0, b)$, we deduce

$$k'(s) = \mu\,\langle\overrightarrow{QM(s)}, \tau(s)\rangle. \tag{2.31}$$

Thus

$$\lim_{s\to b^-} k' = -\mu\sqrt{R_0^2 - \left(\frac{\lambda}{\mu}\right)^2}. \tag{2.32}$$

Following the above calculation, we can show

$$\lim_{s\to(b+L)^+} k' = \mu\sqrt{R_0^2 - \left(\frac{\lambda}{\mu}\right)^2}.$$

Using the Cauchy–Lipschitz Theorem, we deduce that $k(s+b+L) = k(b-s)$ for $s \in [0, b]$. That is the boundary of Ω is composed by symmetric curves with a segment of length L. To conclude it remains to estimate L. Equations (2.29) and (2.30) entail $(L/2)^2 = R_0^2 - (\lambda/\mu)^2$.

In the strictly convex part, we combine (2.31) and (2.22) to obtain (2.28). □

Here below, we present the proof of Theorem 2.3.

Proof of Theorem 2.3. The function $\theta(s)$ corresponds to a solution of (2.11) if and only if it is a solution of

$$\inf_{\theta\in\mathcal{M}} j_\mu(\theta). \tag{2.33}$$

Using classical theory for this kind of optimization problem with constraints in a Banach space (see, for instance, Theorem 3.2 and Theorem 3.3 in [9]), we can derive the optimality conditions. More precisely, let us introduce the closed convex cone K of $L^2(0,2\pi)\times\mathbb{R}^3$ defined by

$$K := L^2_+(0,2\pi)\times\{(0,0,0)\},$$

where

$$L^2_+(0,2\pi) := \left\{\ell\in L^2(0,2\pi)\ ;\ \ell\geqslant 0\right\}.$$

We also set for $\theta\in W^{1,2}(0,2\pi)$

$$m(\theta) = \left(\theta',\quad \int_0^{2\pi}\cos(\theta(s))\ ds,\quad \int_0^{2\pi}\sin(\theta(s))\ ds,\quad \theta(2\pi)-\theta(0)-2\pi\right).$$

Then, Problem (2.33) can be written as

$$\inf\left\{j_\mu(\theta),\quad \theta\in W^{1,2}(0,2\pi),\quad m(\theta)\in K\right\}.$$

As a consequence, for a solution θ of (2.33) there exist $\ell\in L^2_+(0,2\pi)$, $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3$ such that the two following conditions hold:

$$\begin{aligned} j'_\mu(\theta)(v) = \langle(\ell,\lambda_1,\lambda_2,\lambda_3), m'(\theta)(v)\rangle_{L^2(0,2\pi)\times\mathbb{R}^3}\quad \forall v\in W^{1,2}(0,2\pi),\\ ((\ell,\lambda_1,\lambda_2,\lambda_3), m(\theta))_{L^2(0,2\pi)\times\mathbb{R}^3} = 0.\end{aligned}$$

The two above conditions can be written as

$$\begin{aligned}\int_0^{2\pi}\theta' v'\ ds + \frac{\mu}{2}\int_0^{2\pi}\int_0^{s}\cos(\theta(s)-\theta(t))(v(s)-v(t))\ ds\ dt\\ = \int_0^{2\pi}\ell v'\ ds - \lambda_1\int_0^{2\pi}\sin(\theta)v\ ds + \lambda_2\int_0^{2\pi}\cos(\theta)v\ ds + \lambda_3(v(2\pi)-v(0)),\end{aligned}\tag{2.34}$$

$$\int_0^{2\pi}\ell\theta'\ ds = 0.\tag{2.35}$$

In (2.34), we have used that, due to the constraints of θ in (2.5) we have

$$\begin{aligned}\int\int_T \sin(\theta(s))\cos(\theta(t))\,ds\,dt = \int_0^{2\pi}\left(\int_0^s\cos(\theta(t))\ dt\right)\sin(\theta(s))\ ds =\\ -\int_0^{2\pi}\left(\int_0^s\sin(\theta(t))\ dt\right)\cos(\theta(s))\ ds = \frac{1}{2}\int\int_T\sin(\theta(s)-\theta(t))\,ds\,dt.\end{aligned}\tag{2.36}$$

Standard calculation gives

$$\int_0^{2\pi}\int_0^s\cos(\theta(s)-\theta(t))(v(s)-v(t))\ dt\ ds = 2\int_0^{2\pi}\left(\int_0^s\cos(\theta(s)-\theta(t))\ dt\right)v(s)\ ds.$$

We thus define

$$f(s) = \mu\int_0^s\cos(\theta(s)-\theta(t))\ dt + \lambda_1\sin(\theta(s)) - \lambda_2\cos(\theta(s))\quad \text{for } s\in[0,2\pi]\tag{2.37}$$

and we rewrite (2.34) as

$$\int_0^{2\pi}\theta' v'\ ds + \int_0^{2\pi} f v\ ds = \int_0^{2\pi}\ell v'\ ds\qquad v\in W^{1,2}_0(0,2\pi).\tag{2.38}$$

Let us consider the continuous function $F\in W^{1,\infty}(0,2\pi)$ defined by

$$F(s) := -\int_0^s f(\alpha)\ d\alpha.\tag{2.39}$$

Then integrating by parts in (2.38) yields (for some constant C)

$$\theta' = -F + \ell - C \quad \text{in } (0, 2\pi). \tag{2.40}$$

The above equation implies that

$$\ell - F - C \geqslant 0 \quad \text{in } (0, 2\pi).$$

On the other hand condition (2.35) yields $\ell\theta' = 0 \quad \text{in } (0, 2\pi)$ which implies $\ell(\ell - F - C) = 0 \quad \text{in } (0, 2\pi)$, thanks to relation (2.40).

We rewrite the above equality by using the decomposition $F + C = g^+ - g^-$, (where g^+ and g^- are the positive and negative parts of $F + C$):

$$\begin{aligned} \ell(\ell - F + C) = (\ell - g^+ + g^+)(\ell - g^+ + g^-) = (\ell - g^+)^2 + g^-(\ell - g^+) + \\ + g^+(\ell - g^+) + g^+g^- = (\ell - g^+)^2 + g^-\ell + g^+(\ell - F - C) \end{aligned}$$

which is the sum of three non-negative terms. Thus

$$\ell = (F + C)^+ \tag{2.41}$$

and in particular, from (2.40),

$$\theta' = (F + C)^- \quad \text{in } (0, 2\pi) \tag{2.42}$$

We deduce that $\theta \in W^{2,\infty}(0, 2\pi)$.

Using (2.12) and (2.13), we can write

$$\int_0^s \cos(\theta(s) - \theta(t)) \; dt = x(s)x'(s) + y(s)y'(s). \tag{2.43}$$

Moreover, the function F defined by (2.39) and (2.37) can be rewritten as

$$F(s) = -\int_0^s \left[\mu \int_0^\alpha \cos(\theta(\alpha) - \theta(t)) \; dt + \lambda_1 \sin(\theta(\alpha)) - \lambda_2 \cos(\theta(\alpha)) \right] \; d\alpha$$

and combining this relation with (2.43), we obtain

$$\begin{aligned} F(s) = -\int_0^s \left[\mu(x(\alpha)x'(\alpha) + y(\alpha)y'(\alpha)) + \lambda_1 y'(\alpha) - \lambda_2 x'(\alpha) \right] \; d\alpha \\ = \frac{\mu}{2} \left(\frac{\lambda_1^2 + \lambda_2^2}{\mu^2} - \left[x(s) - \frac{\lambda_2}{\mu} \right]^2 - \left[y(s) + \frac{\lambda_1}{\mu} \right]^2 \right). \end{aligned}$$

The above relation and (2.42) yield (2.14). □

3 Geometric properties

3.1 Symmetries

Using a classical *reflexion* method, we can prove that there always exists a minimizer with a central symmetry:

Theorem 3.1. *There exists at least one minimizer of Problem* (1.3) *that has a center of symmetry.*

Remark 3.2. *Notice that the symmetry result in Proposition 2.9 does not imply the centrally symmetric result, as it can be seen considering a smooth approximation of an equilateral triangle.*

Proof of Theorem 3.1. Let m_μ denotes the value of the minimum of problem (1.3): $m_\mu := \min_{\Omega \in \mathcal{A}} E(\Omega) + \mu A(\Omega)$ and let us consider a minimizer Ω, which exists thanks to Theorem 2.2. For any direction (unit vector) η let us denote by $X(\eta)$ a point on the boundary of Ω whose exterior normal vector is η. By a continuity argument (change η in $-\eta$), there exists at least one direction η such that the segment joining $X(\eta)$ to $X(-\eta)$ (if not unique, choose one) cuts the boundary in two parts $\Gamma_+(\eta)$ and $\Gamma_-(\eta)$ having the same length π. Let us denote

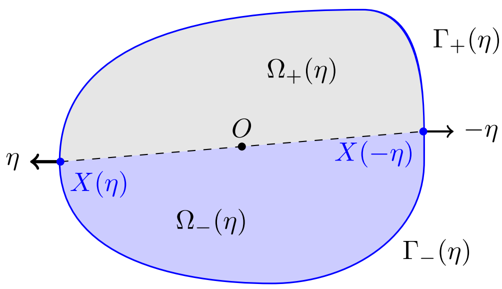

Figure 3.1: The set Ω separated into the sets $\Omega_+(\eta)$ and $\Omega_-(\eta)$.

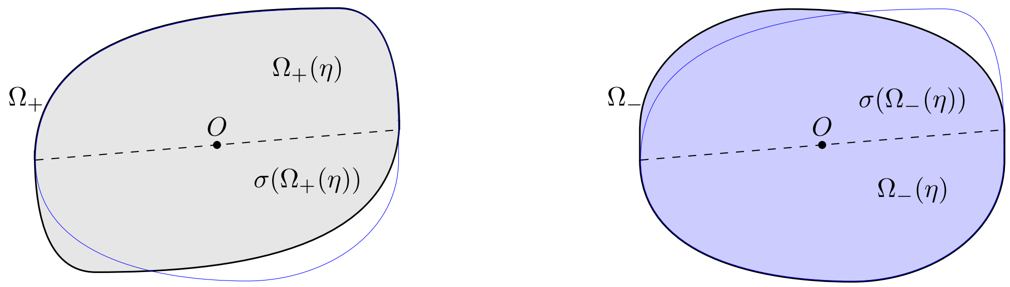

Figure 3.2: The centrally symmetric sets Ω_+ and Ω_-.

by $\Omega_+(\eta)$ (resp. $\Omega_-(\eta)$) the part of Ω bounded by the segment $[X(\eta), X(-\eta)]$ and $\Gamma_+(\eta)$ (resp. $\Gamma_-(\eta)$), see Figure 3.1 and Figure 3.2.

Let us denote by O the middle of the segment $[X(\eta), X(-\eta)]$ and let σ be the central symmetry with respect to O; define

$$\Omega_+ := \Omega_+(\eta) \cup \sigma(\Omega_+(\eta)) \quad \text{and} \quad \Omega_- := \Omega_-(\eta) \cup \sigma(\Omega_-(\eta)).$$

By construction Ω_+ and Ω_- have perimeter 2π and then they are admissible. It follows

$$E(\Omega_+) + \mu A(\Omega_+) \geqslant m_\mu,$$
$$E(\Omega_-) + \mu A(\Omega_-) \geqslant m_\mu \ .$$

Adding these two inequalities yields

$$2m_\mu = 2E(\Omega) + 2\mu A(\Omega) \geqslant 2m_\mu.$$

Therefore, we have equality everywhere and both Ω_+ and Ω_- solve the minimization problem. Moreover they are centrally symmetric (and coincide if Ω is itself centrally symmetric). □

Remark 3.3. *We emphasize that, thanks to Proposition 2.9, Ω is locally axially symmetric. Indeed the boundary of Ω can be decomposed in the union of suitably rotated and translated copies γ_i of a symmetric curve. In particular, for each of these copies γ_i, there exists a point $M_i \in \gamma_i$ such that the curve γ_i is symmetric with respect to the line QM_i.*

Proposition 3.4. *Assume that Ω is a centrally symmetric minimizer of Problem* (1.3) *and locally symmetric with respect to lines passing by Q. Then Ω is axially symmetric.*

More precisely, up to translation the boundary of Ω can be decomposed in the union of suitable rotated copies γ_i of a symmetric curve γ, that is

$$\partial\Omega = \cup_{i=1}^p \gamma_i \qquad \text{with} \qquad \gamma_i(s) = \rho_i \gamma(s), \qquad \text{for} \qquad s \in [0, 2s_1], \tag{3.1}$$

where ρ_i are rotations of the plane and $2p\, s_1 = 2\pi$. Moreover Ω has p axis of symmetry.

Proof. Thanks to Proposition 2.9, Ω is locally axially symmetric and $\partial\Omega$ can be decomposed in the union of suitable rotated and translated copies γ_i of a symmetric curve γ, that is $\partial\Omega = \cup_{i=1}^p (\rho_i\gamma(s) + b_i)$, for planar rotations ρ_i and vectors b_i. In particular, for each of these copies γ_i, there exists a point $M_i \in \gamma_i$ such that the curve γ_i is symmetric with respect to the line QM_i; $M_i = \gamma(s_1)$. Up to translations we can assume that Q coincides with the origin.

Fix $i \in \{1, ..., p\}$ and let us consider the curve γ_i. Since Ω is centrally symmetric and γ is symmetric with respect to the point $\gamma(s_1)$ there exists a corresponding index $l \in \{1, ..., p\}$ such that the curves γ_i and γ_l are axially symmetric with respect to the line $\gamma_i(s_1)\gamma_l(s_1)$, through the origin (we can assume this line to be the axis $\{x = 0\}$). Since this property holds true for each $j \in \{1, ..., p\}$, we have that the points $\gamma_j(0), \gamma_j(2s_1)$ belongs to a common circle of radius $\|\gamma_1(0)\|$. This entails that the decomposition in (3.1) holds true.

More precisely by the symmetry of the curve γ (and hence that of γ_j), the curves γ_{i-1} and γ_{i+1} are axially symmetric too, since it holds $\gamma_{i-1}(2s_1) = \gamma_i(0) = (x_i(0), y_i(0)) = (-x_i(2s_1), y_i(2s_1)) = (-x_{i+1}(0), y_{i+1}(0))$, where $(x_j(s), y_j(s))$ denotes the point $\gamma_j(s)$ and $\gamma_{i-1}(0)$,$\gamma_{i+1}(2s_1)$ both belong to the common circle. □

3.2 Segments

We are interested in the analysis of existence of segments for minimizers of Problem (1.3), i.e. non-empty intervals (a, b) of $[0, 2\pi]$ such that $k(s) = 0$ on (a, b).

Lemma 3.5. *Assume that Ω is a centrally symmetric minimizer of Problem* (1.3)*. If $\partial\Omega$ has at least one segment, then*

$$E(\Omega) \leqslant \mu A(\Omega) \leqslant 2E(\Omega). \tag{3.2}$$

Proof. Assume that $\partial\Omega$ has at least one segment. Since Ω has a center of symmetry, $\partial\Omega$ has at least two parallel segments hence Ω is contained in the infinite strip corresponding to the two segments. By (2.22), the width of this strip is

$$\frac{2\mu A(\Omega) - E(\Omega)}{\pi\mu}.$$

As a consequence,

$$A(\Omega) \leqslant \mathrm{diam}(\Omega)\frac{2\mu A(\Omega) - E(\Omega)}{\pi\mu} \leqslant 2A(\Omega) - \frac{E(\Omega)}{\mu},$$

which implies the first inequality in (3.2).

In order to prove the second inequality, we make a perturbation Ω_ε of Ω. First we increase the size of a segment by ε and we modify in a symmetric way the opposite segment. Then we perform an homothety of center Q and ratio $1/(1 + \varepsilon/\pi)$ so that the perimeter of Ω_ε remains equal to 2π.

The domain Ω_ε satisfies

$$\begin{aligned} E(\Omega_\varepsilon) + \mu A(\Omega_\varepsilon) &= E(\Omega)\left(1 + \frac{\varepsilon}{\pi}\right) + \mu\left(A(\Omega) + \frac{2\mu A(\Omega) - E(\Omega)}{\pi\mu}\varepsilon\right)\frac{1}{\left(1 + \frac{\varepsilon}{\pi}\right)^2} \\ &= E(\Omega) + \mu A(\Omega) + \frac{\varepsilon^2}{\pi^2}\left(2E(\Omega) - \mu A(\Omega)\right) + o(\varepsilon^2). \end{aligned}$$

This ends the proof since $E(\Omega_\varepsilon) + \mu A(\Omega_\varepsilon) \geqslant E(\Omega) + \mu A(\Omega)$. □

Lemma 3.6. *For any $\mu > 1$, the following inequalities hold for any optimal domain Ω.*

$$2\pi\sqrt{\mu} \leqslant E(\Omega) + \mu A(\Omega) \leqslant 3\pi\sqrt{\mu} - \pi. \tag{3.3}$$

Proof. Thanks to Theorem 1.1 by Gage [4],

$$E(\Omega)A(\Omega) \geqslant \pi^2.$$

On the other hand relation (1.4) entails a lower bound for the elastic energy: $E(\Omega) \geqslant \pi$. As a consequence

$$E(\Omega) + \mu A(\Omega) \geqslant E(\Omega) + \mu\frac{\pi^2}{E(\Omega)} \geqslant 2\pi\sqrt{\mu}$$

by using that $\mu > 1$.

To prove the second inequality, we consider the admissible stadium Ω_S composed by a rectangle of lengths $2/\sqrt{\mu}$ and $\pi(1-1/\sqrt{\mu})$ and by two half disks of radius $1/\sqrt{\mu}$. For this stadium, $P(\Omega_S) = 2\pi$, and

$$E(\Omega_S) + \mu A(\Omega_S) = \pi\sqrt{\mu} + \mu\left(\frac{\pi}{\mu} + \frac{2}{\sqrt{\mu}}\pi(1-\frac{1}{\sqrt{\mu}})\right) = 3\pi\sqrt{\mu} - \pi.$$

□

Theorem 3.7. *Assume that Ω is a minimizer of Problem* (1.3) *with a center of symmetry. Then $\partial\Omega$ contains either* 0 *or* 2 *segments.*

Proof. Assume that the boundary of Ω contains m segments, hence thanks to Proposition 2.9, $m = 2N$ and suppose $m \geqslant 4$ (that is $N \geqslant 2$). By Theorem 3.1 and Proposition 3.4 the set Ω is contained in the union of $2N$ copies of isosceles triangles with common vertex in Q, height equal to λ/μ, angle at the vertex equal to π/N, moreover by its convexity Ω is contained in a regular $2N$-gon of inradius λ/μ.

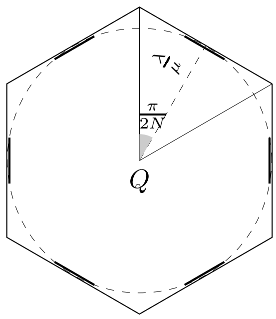

Figure 3.3: The set Ω is contained in a regular $2N$-gon of inradius λ/μ

Hence, comparing the perimeters, we deduce that

$$2\pi \leqslant \frac{\lambda}{\mu}4N\tan\left(\frac{\pi}{2N}\right). \tag{3.4}$$

On the other hand, we deduce from (3.3) and (3.2)

$$E(\Omega) \geqslant \frac{2\pi\sqrt{\mu}}{3}. \tag{3.5}$$

Combining again (3.3) and (3.2) we obtain

$$\frac{3}{2}\mu A(\Omega) \leqslant E(\Omega) + \mu A(\Omega) \leqslant 3\pi\sqrt{\mu} - \pi, \tag{3.6}$$

and thus

$$\mu A(\Omega) \leqslant 2\pi\sqrt{\mu} - \frac{2}{3}\pi. \tag{3.7}$$

Relations (3.5) and (3.7) imply

$$\frac{\lambda}{\mu} = \frac{2\mu A(\Omega) - E(\Omega)}{2\pi\mu} \leqslant \frac{5}{3\sqrt{\mu}} - \frac{2}{3\mu}. \tag{3.8}$$

On the other hand, since $E(\Omega) \geqslant \pi$ and $A(\Omega) \leqslant \pi$,

$$\frac{\lambda}{\mu} = \frac{2\mu A(\Omega) - E(\Omega)}{2\pi\mu} \leqslant 1 - \frac{1}{2\mu}. \tag{3.9}$$

Consequently, by (3.4) and (3.8)

$$\mu \leqslant \left(\frac{10N}{3\pi}\tan\left(\frac{\pi}{2N}\right)\right)^2 \tag{3.10}$$

and by (3.4) and (3.9)

$$\frac{1}{2\left(1-\frac{\pi}{2N}\operatorname{cotan}\left(\frac{\pi}{2N}\right)\right)} \leqslant \mu. \tag{3.11}$$

We can notice that the sequences

$$\left\{\left(\frac{10n}{3\pi}\tan\left(\frac{\pi}{2n}\right)\right)^2\right\}_{n\geqslant 2}, \qquad \left\{\frac{1}{2\left(1-\frac{\pi}{2n}\operatorname{cotan}\left(\frac{\pi}{2n}\right)\right)}\right\}_{n\geqslant 2}$$

are decreasing and increasing respectively and since for $n=3$,

$$\left(\frac{10}{\pi}\tan\left(\frac{\pi}{6}\right)\right)^2 < \frac{1}{2\left(1-\frac{\pi}{6}\operatorname{cotan}\left(\frac{\pi}{6}\right)\right)}$$

we deduce from (3.10) and from (3.11) that $N \leqslant 2$.

We also deduce that if $N=2$, then

$$2.3 \leqslant \mu \leqslant 4.6.$$

If $N=2$, then we deduce that

$$A(\Omega) \leqslant \left(2\frac{\lambda}{\mu}\right)^2. \tag{3.12}$$

Using (3.8), we deduce

$$A(\Omega) \leqslant 4\left(\frac{5}{3\sqrt{\mu}} - \frac{2}{3\mu}\right)^2. \tag{3.13}$$

The above relation and (3.5) imply

$$\frac{\lambda}{\mu} = \frac{2\mu A(\Omega) - E(\Omega)}{2\pi\mu} \leqslant \frac{4}{\pi}\left(\frac{5}{3\sqrt{\mu}} - \frac{2}{3\mu}\right)^2 - \frac{1}{3\sqrt{\mu}}. \tag{3.14}$$

Since (3.4) writes

$$\frac{\lambda}{\mu} \geqslant \frac{\pi}{4}$$

we deduce from (3.14) that

$$\frac{\pi}{4} \leqslant \frac{4}{\pi}\left(\frac{5}{3\sqrt{\mu}} - \frac{2}{3\mu}\right)^2 - \frac{1}{3\sqrt{\mu}}. \tag{3.15}$$

That yields

$$\mu < 2.$$

□

Lemma 3.8. *Assume that Ω is a minimizer of Problem (1.3). For $\mu \geqslant 1$, we have*

$$\lambda \geqslant \frac{\sqrt{1+16\mu}-1}{4} - \sqrt{\frac{\mu}{2}}, \tag{3.16}$$

and

$$A(\Omega) \geqslant \frac{\pi}{4\mu}\left(\sqrt{1+16\mu}-1\right). \tag{3.17}$$

Proof. Multiplying the optimality condition (2.22) by k and integrating on the boundary yields

$$2\pi\mu = \mu\int_0^{2\pi} k\langle\overrightarrow{QM}, \mathsf{n}\rangle\, ds = 2\pi\lambda + \frac{1}{2}\int_0^{2\pi} k^3\, ds. \tag{3.18}$$

The Cauchy-Schwarz inequality and the previous equality (3.18) give

$$(2E(\Omega))^2 = \left(\int_0^{2\pi} k^2 ds\right)^2 \leqslant \int_0^{2\pi} k\, ds \int_0^{2\pi} k^3\, ds = 8\pi^2(\mu-\lambda) \leqslant 8\pi^2\mu, \tag{3.19}$$

the last inequality coming from the fact that λ is necessarily positive if $\mu \geqslant 1$. Therefore, from (3.19) we have

$$E(\Omega) \leqslant \pi\sqrt{2\mu}. \tag{3.20}$$

We use the Green-Osher inequality, valid for any (regular) convex domain, see [5] and we obtain

$$\int_0^{2\pi} k^3\,ds \geqslant \frac{P(\Omega)^2\pi - 2A(\Omega)\pi^2}{A(\Omega)^2}. \tag{3.21}$$

Plugging (3.21) into (3.18), we obtain

$$4\pi(\mu-\lambda) = \int_0^{2\pi} k^3\,ds \geqslant \frac{4\pi^3 - 2A(\Omega)\pi^2}{A(\Omega)^2} \tag{3.22}$$

and hence

$$\mu A(\Omega)^2 + \frac{\pi}{2}A(\Omega) - \pi^2 \geqslant (\mu-\lambda)A(\Omega)^2 + \frac{\pi}{2}A(\Omega) - \pi^2 \geqslant 0. \tag{3.23}$$

Considering the sign of the polynomial, this implies

$$\mu A(\Omega) \geqslant \frac{\pi}{4}\left(\sqrt{1+16\mu} - 1\right). \tag{3.24}$$

The proof concludes by using (2.21), (3.20) and (3.24). □

Let us now prove that, for sufficiently large μ, the optimal domains are not strictly convex.

Proposition 3.9. *If $\mu > 47.7750$, then the boundary of an optimal domain Ω contains segments.*

Proof. Let us multiply (2.24) by k':

$$\frac{(k')^2}{2} = -\frac{k^4}{8} - \frac{\lambda}{2}k^2 + \mu k + C. \tag{3.25}$$

where λ is defined in (2.21) and

$$C = \frac{k_M^4}{8} + \frac{\lambda}{2}k_M^2 - \mu k_M. \tag{3.26}$$

where $k_M > 0$ is the maximum of k.

Notice that equation (3.25) can be written as

$$\frac{(k')^2}{2} = \mathscr{P}(k),$$

where $\mathscr{P}$ is a concave polynomial function (using the fact that $\lambda > 0$). As a consequence, either $\mathscr{P}$ has 2 distinct roots $k_m < k_M$ or $\mathscr{P}$ has a double root $k_m = k_M$ (and $\mathscr{P} \leqslant 0$). Therefore $\partial\Omega$ has a segment if and only if $k_m < 0$. Note that this condition is equivalent to $\mathscr{P}(0) = C > 0$.

Relation (2.26) evaluated at the point s_M such that $k(s_M) = k_M$, entails

$$\|\overrightarrow{QM(s_M)}\| = \frac{\lambda}{\mu} + \frac{1}{2\mu}k_M^2. \tag{3.27}$$

Notice that $\|\overrightarrow{QM(s_M)}\| \geqslant 1$ otherwise the whole domain will be included in the disk of center Q and of radius one (because $\|\overrightarrow{QM(s_M)}\|$ is the radius of the circumscribed disk according to (2.19), which leads to contradiction since the set Ω has perimeter 2π. Therefore, we deduce

$$k_M^2 \geqslant 2\mu - 2\lambda. \tag{3.28}$$

Let us denote by E_μ and A_μ the elastic energy and the area of an optimal domain. We use inequality (3.3) and Gage's inequality (1.2) to obtain

$$\frac{\pi^2}{A_\mu} + \mu A_\mu \leqslant E_\mu + \mu A_\mu \leqslant 3\pi\sqrt{\mu} - \pi.$$

Assuming $\mu > 1$, we deduce from the above estimate that

$$A_\mu \leqslant \mathcal{A}_M(\mu) := \frac{\pi}{2\mu}\left((3\sqrt{\mu}-1)+\sqrt{5\mu-6\sqrt{\mu}+1}\right). \tag{3.29}$$

The above estimate, the definition of λ in (2.21) and Gage's inequality (1.2) yield

$$\lambda = \frac{2\mu A_\mu - E_\mu}{2\pi} \leqslant \lambda_M(\mu) := \frac{2\mu\mathcal{A}_M(\mu) - \pi^2/\mathcal{A}_M(\mu)}{2\pi}. \tag{3.30}$$

Therefore (3.28) implies

$$k_M^2 \geqslant 2\mu - 2\lambda_M(\mu). \tag{3.31}$$

Since $\mu > \lambda_M(\mu)$ for $\mu > 3$, we deduce

$$k_M^4 \geqslant (2\mu - 2\lambda_M(\mu))^2. \tag{3.32}$$

At last, we use the fact that the half-diameter of the optimal set is less than $\pi/2$ (because the perimeter is 2π) and the optimality condition (3.27) to get

$$\frac{\lambda}{\mu} + \frac{1}{2\mu}k_M^2 = \|\overrightarrow{QM(s_M)}\| \leqslant \frac{\pi}{2}.$$

Combining the above estimate with (3.16), it follows

$$k_M(\mu) \leqslant \sqrt{\pi\mu - 2\lambda_m(\mu)}, \tag{3.33}$$

where

$$\lambda_m(\mu) := \frac{\sqrt{1+16\mu}-1}{4} - \sqrt{\frac{\mu}{2}}.$$

Gathering (3.32), (3.16), (3.31), and (3.33), we deduce from the definition of the constant C in (3.26) that

$$C \geqslant \frac{1}{8}(2\mu - 2\lambda_M(\mu))^2 + \frac{1}{2}\lambda_m(\mu)(2\mu - 2\lambda_M(\mu)) - \mu\sqrt{\pi\mu - 2\lambda_m(\mu)}.$$

It turns out that the function of μ in the right-hand side is positive as soon as $\mu > 47.775$ which proves the result. □

4 The disk

As already pointed out, Gage's inequality (Theorem 1.1) asserts that the disk minimizes the product $E(\Omega)A(\Omega)$ among convex bodies with given perimeter. This leads to the following result.

Corollary 4.1. *The disk is the unique minimizer to Problem* (1.3) *for* $\mu \leqslant 1$.

Proof. Let Ω be a convex set of perimeter 2π and let D be the unit disk. Using Gage's inequality (1.2) we have

$$E(\Omega) + A(\Omega) \geqslant 2\sqrt{E(\Omega)A(\Omega)} \geqslant 2\sqrt{E(D)A(D)} = E(D) + A(D),$$

the last equality coming from the fact that $E(D) = A(D) = \pi$. Therefore the disk is the minimizer to Problem (1.3) for $\mu = 1$. For $\mu \leqslant 1$, we use the isoperimetric inequality for the elastic energy, expressed in (1.4), (1.5), to obtain:

$$E(\Omega) + \mu A(\Omega) = \mu(E(\Omega) + A(\Omega)) + (1-\mu)E(\Omega) \geqslant$$
$$\mu(E(D) + A(D)) + (1-\mu)E(D) = E(D) + \mu A(D).$$

□

Notice that the disk cannot be the solution for large μ. Indeed considering the stadium Ω_S obtained as the union of a rectangle of length $\pi/2$ with two half-disks of radius $1/2$ one gets $P(\Omega_S) = 2\pi$ and

$$E(\Omega_S) + \mu A(\Omega_S) = 2\pi + \mu \frac{3\pi}{4}. \tag{4.1}$$

Comparing this with the value of $E(D) + \mu A(D)$ (where D is the unit disk) we obtain an equality for $\mu = 4$ while (4.1) gives a strictly better value for $\mu > 4$. We are going to show that in fact the disk cannot be the solution for $\mu > 3$ since it is no longer a local minimum. Notice that the value $\mu = 3$ is probably optimal since the numerical algorithm presented in Section 6) seems to show that the disk is optimal for $\mu \leqslant 3$.

Theorem 4.2. *The unit disk D is a local strict minimum for Problem* (1.3) *if and only if $\mu \leqslant 3$.*

Proof. We consider small perturbations of the unit disk obtained through perturbations of its support function. First, let $\mu > 3$ and consider the convex body Ω_ε whose support function is

$$h_\varepsilon(t) := 1 + \varepsilon \cos 2t. \tag{4.2}$$

Notice that the set Ω_ε is C^2_+ while $\varepsilon < 1/3$. Moreover, using (2.6), its area is

$$A(\Omega_\varepsilon) = \frac{1}{2}\int_0^{2\pi} (1 + \varepsilon \cos(2t))(1 - 3\varepsilon \cos(2t))\, dt = \pi(1 - \frac{3}{2}\varepsilon^2),$$

and its elastic energy is

$$E(\Omega_\varepsilon) = \frac{1}{2}\int_0^{2\pi} \frac{dt}{1 - 3\varepsilon \cos(2t)} = \frac{\pi}{\sqrt{1 - 9\varepsilon^2}}.$$

Therefore

$$J_\mu(\Omega_\varepsilon) = E(\Omega_\varepsilon) + \mu A(\Omega_\varepsilon) = \pi(1 + \mu) - \pi\frac{\varepsilon^2}{2}(3\mu - 9) + o(\varepsilon^2),$$

which is strictly less than $J_\mu(D) = E(D) + \mu A(D) = (1 + \mu)\pi$ for ε small enough.

Conversely for $\mu \leqslant 3$ let D_ε be a perturbation of the unit disk with $D_\varepsilon \in C^2_+$ and let d_ε denote its support function. Since $P(\Omega_\varepsilon) = 2\pi$ and from (2.7), we can write

$$d_\varepsilon(t) = 1 + \varepsilon \sum_{k=2}^{+\infty} a_k \cos(kt) + b_k \sin(kt). \tag{4.3}$$

Using again (2.6) and since $d_\varepsilon \in C^2$, we have

$$A(D_\varepsilon) = \pi \left(1 - \frac{\varepsilon^2}{2}\sum_{k=2}^{+\infty}(k^2 - 1)[a_k^2 + b_k^2]\right),$$

and

$$E(D_\varepsilon) = \pi \left(1 + \frac{\varepsilon^2}{2}\sum_{k=2}^{+\infty}(k^2 - 1)^2[a_k^2 + b_k^2] + o(\varepsilon^2)\right).$$

Thus

$$J_\mu(D_\varepsilon) - J_\mu(D) = \frac{\pi\varepsilon^2}{2}\sum_{k=2}^{+\infty}\left((k^2 - 1)^2 - \mu(k^2 - 1)\right)[a_k^2 + b_k^2] + o(\varepsilon^2),$$

which is positive for ε small enough when either $\mu < 3$, whatever the a_k, b_k are, or $\mu = 3$ if at least one of the a_k, b_k are non-zero for $k \geqslant 3$.

It remains to consider the case $\mu = 3$, $a_k = b_k = 0$ for $k \geqslant 3$ for which a direct computation gives

$$J_\mu(D_\varepsilon) - J_\mu(D) = \frac{3^5\pi\varepsilon^4}{2^3}[a_2^2 + b_2^2]^2 + o(\varepsilon^4)$$

and the result follows.

Finally, for general perturbation not necessarily C^2_+, we use Theorem 5.2. □

5 Description of the Blaschke-Santaló diagram

We want to study the set

$$\mathcal{E} := \left\{ (x,y) \in \mathbb{R}^2, x = \frac{4\pi A(\Omega)}{P(\Omega)^2}, y = \frac{E(\Omega)P(\Omega)}{2\pi^2}, \ \Omega \in \mathcal{C} \right\}, \tag{5.1}$$

where $\mathcal{C}$ is defined by (2.4).

Notice that by homogeneity the sets Ω and $t\Omega$ correspond to the same point in $\mathcal{E}$. Therefore, without loss of generality we can consider convex sets with fixed perimeter $P(\Omega) = 2\pi$.

According to the classical isoperimetric inequality and inequality (1.4), for any $(x,y) \in \mathcal{E}$ it holds $x \leqslant 1$ and $y \geqslant 1$. Moreover, we emphasize that Gage's inequality (1.2) writes as $xy \geqslant 1$.

Let us now present the main result of this section: a convexity result for the set $\mathcal{E}$.

Theorem 5.1. *The set $\mathcal{E}$ is convex with respect to both the horizontal and the vertical directions. That is: if $(x_0, y_0) \in \mathcal{E}$, then $[x_0, 1) \times [y_0, \infty) \in \mathcal{E}$.*

Moreover, the half-line $(x = 1, y = s)$, for $s \in [1, +\infty)$ is contained in the boundary of $\mathcal{E}$ and it is not included in $\mathcal{E}$ except for the point $(1,1)$.

Proof. Let us first prove the convexity result for the regularized set

$$\mathcal{E}_{reg} := \left\{ (x,y) \in \mathbb{R}^2, x = \frac{4\pi A(\Omega)}{P(\Omega)^2}, y = \frac{E(\Omega)P(\Omega)}{2\pi^2}, \ \Omega \in \mathcal{C}_{reg} \right\}, \tag{5.2}$$

where $\mathcal{C}_{reg}$ is the subset of $\mathcal{C}$ of convex bodies Ω whose radius of curvature $\phi = h + h''$ is positive and of class C^1.

We show that for any $(x_0, y_0) \in \mathcal{E}_{reg}$, the segment $(x = t, y = y_0)$, is contained in $\mathcal{E}_{reg}$, for $t \in [x_0, 1)$; and the half-line $(x = x_0, y = s)$, is contained in $\mathcal{E}_{reg}$ for $s \in [y_0, +\infty)$.

We first show the vertical convexity. Let us take $(x_0, y_0) \in \mathcal{C}_{reg}$ corresponding to a convex set Ω of perimeter 2π. Without loss of generality (up to rotations), we can assume that

$$\xi := \min_{t \in [0,2\pi]} \phi > 0,$$

is attained at $t = 0$. Let us assume that Ω is not the unit disk. Since $P(\Omega) = 2\pi$ and by condition (2.7), we can write

$$h(t) = 1 + \sum_{n \geqslant 2} \alpha_n \cos(nt) + \beta_n \sin(nt) \quad \forall t \in [0, 2\pi].$$

Therefore

$$\phi(0) = 1 - \sum_{n \geqslant 2} \alpha_n (n^2 - 1) = \xi,$$

with $\xi < 1$ thanks to the expression in (2.6) and to $P(\Omega) = 2\pi$. This implies that there exists $m \geqslant 2$ such that

$$\alpha_m = \frac{1}{\pi} \int_0^{2\pi} h(t) \cos mt \, dt > 0.$$

Let us introduce the convex set Ω_1 defined through its support function by

$$h_1(t) = h(t) + a_m \cos(mt),$$

where a_m is a suitable constant such that $|a_m| < \mu/(m^2 - 1)$. We emphasize that this guarantees the convexity of Ω_1 since $h_1 + h_1{}'' > 0$. In particular the perimeter of Ω_1 is 2π and its area is given by

$$A(\Omega_1) = \frac{1}{2} \int_0^{2\pi} h_1(h_1 + h_1{}'') = A(\Omega) + \frac{\pi}{2}(1 - m^2)[a_m^2 + 2a_m \alpha_m].$$

Notice that $A(\Omega_1) < A(\Omega)$ if $0 < a_m < \mu/(m^2 - 1)$.

Let us now denote by I the interval $I = (0, \mu/(m^2 - 1))$. By the isoperimetric inequality $A(\Omega) < \pi$, thus we can choose an integer $p \neq m$, $p \geqslant 2$ such that

$$\frac{\pi}{2(p^2 - 1)} \leqslant \pi - A(\Omega). \tag{5.3}$$

Let a_p be a real number satisfying $|a_p| < 1/(p^2-1)$. We introduce a new convex set Ω_0 through its support function: $h_0(t) = 1 + a_p \cos(pt)$. By construction we have $\phi_0 := h_0'' + h_0 = 1 + (1-p^2)a_p\cos(pt) > 0$, Ω_0 has perimeter 2π and its area is given by $A(\Omega_0) = \pi - \pi(p^2-1)a_p^2/2$. By assumption (5.3), we have $A(\Omega_0) > A(\Omega)$ for any $a_p \in [0, \frac{1}{p^2-1})$. We denote by J the interval $J = [0, \frac{1}{p^2-1})$.

For $\tau \in [0,1]$ let us consider the Minkowski combination $\Omega_\tau := \tau\Omega_1 + (1-\tau)\Omega_0$ whose support function is $h_\tau = \tau h_1 + (1-\tau)h_0$, see [13] for additional properties of the Minkowski sum. We have $P(\Omega_\tau) = 2\pi$ and its area is given by

$$A(\Omega_\tau) = \frac{1}{2}\int_0^{2\pi} h_\tau(h_\tau + h_\tau'')\,dt = \tau^2\left(A(\Omega) + \frac{\pi}{2}(1-m^2)[a_m^2 + 2a_m\alpha_m]\right) + (1-\tau)^2\left(\pi + \frac{\pi}{2}(1-p^2)a_p^2\right) + \tau(1-\tau)(1-p^2)\pi a_p\alpha_p. \quad (5.4)$$

Notice that the right-hand side of formula (5.4) defines a continuous (quadratic) function of τ, say $g(\tau; a_m, a_p)$ such that,

$$\forall a_m, a_p \in I \times J,\ g(0; a_m, a_p) = A(\Omega_0) > A(\Omega) \text{ and } g(1; a_m, a_p) = A(\Omega_1) < A(\Omega).$$

Therefore, for any fixed a_m, a_p in $I \times J$ there exists a value $\tau(a_m, a_p) \in [0,1]$ such that $A(\Omega_\tau) = A(\Omega)$. Moreover, the function $(a_m, a_p) \mapsto \tau(a_m, a_p)$ can be chosen such that τ is continuous and $\tau(0,0) = 1$.

The elastic energy of Ω_τ is given by

$$E(\Omega_\tau) = \frac{1}{2}\int_0^{2\pi} \frac{dt}{(1-\tau)[1 + (1-p^2)a_p\cos(pt)] + \tau[\phi + (1-m^2)a_m\cos(mt)]}. \quad (5.5)$$

If we replace τ by $\tau(a_m, a_p)$ this expression defines a continuous function $E(a_m, a_p)$ of a_m and a_p such that $E(0,0) = E(\Omega)$. Moreover since the denominator of the quotient vanishes at $t = 0$ when a_m approaches $\mu/(m^2-1)$ and a_p approaches $1/(p^2-1)$ and since $\phi \in C^1[0, 2\pi])$, the Fatou lemma yields

$$\lim_{\substack{a_m \to \mu/(m^2-1) \\ a_p \to 1/(p^2-1)}} E(a_m, a_p) = +\infty.$$

Thus the set of values taken by $E(a_m, a_p)$ when (a_m, a_p) varies in $I \times J$ contains $[E(\Omega), +\infty)$ which proves that the whole half line $(x_0, y), y \in [y_0, +\infty)$ is in the domain $\mathcal{E}$.

We prove the horizontal convexity. Let Ω_0 and Ω_1 be two convex domains of perimeter 2π with the same elastic energy. We denote by (x_0, y) and (x_1, y) the corresponding points in $\mathcal{E}_{reg}$. We claim that the elastic energy is convex for the Minkowski sum:

$$\forall \tau \in [0,1],\ E((1-\tau)\Omega_0 + \tau\Omega_1) \leqslant (1-\tau)E(\Omega_0) + \tau E(\Omega_1). \quad (5.6)$$

This can be proven by using (2.6) and the convexity of the function $x \mapsto 1/x$.

Let us now consider the path in the diagram $\mathcal{E}$, joining the two points (x_0, y) and (x_1, y), obtained by the points corresponding to the convex combination $(1-\tau)\Omega_0 + \tau\Omega_1$. Inequality (5.6) implies that the whole path is below the horizontal line of ordinate y. We conclude to the fact that all points (x, y) with $x \in [x_0, x_1]$ belong to $\mathcal{E}_{reg}$ using the vertical convexity.

Let us consider convex domains with a support function defined by $h_{n,a}(t) = 1 + a\cos(nt)$, $n \in \mathbb{N}, n \geqslant 2$, $a \in \mathbb{R}, |a| < 1/(n^2-1)$. The area and elastic energy are given by

$$A(\Omega_{n,a}) = \pi - \frac{\pi(n^2-1)a^2}{2} \qquad E(\Omega_{n,a}) = \frac{\pi}{\sqrt{1-(n^2-1)^2a^2}}$$

This yields a family of parametric curves $x(a;n) = A(\Omega_{n,a})/\pi$ and $y(a;n) = E(\Omega_{n,a})/\pi$ which accumulate on the half-line $x = 1, y \in [1, +\infty)$ when n increases. Finally, this line does not contain any point in $\mathcal{E}_{reg}$ because the isoperimetric inequality gives $x < 1$ except for the ball (among convex domains). Using the horizontal convexity, this also allows to prove that for any $(x_0, y_0) \in \mathcal{E}_{reg}$ the horizontal segment $x = t, y = y_0, t \in [x_0, 1)$ is contained in $\mathcal{E}_{reg}$.

The proof concludes by using Theorem 5.2 below. More precisely, assume $(x_0, y_0) \in \mathcal{E}$. Let us consider (x_0, y) with $y > y_0$. Then there exists $(x_\varepsilon, y_\varepsilon) \in \mathcal{E}_{reg}$ with $x_\varepsilon < x_0$ and $y_\varepsilon < y$. Using the first part of the proof, we deduce that $(x_0, y) \in [x_\varepsilon, 1) \times [y_\varepsilon, \infty) \subset \mathcal{E}_{reg}$.

Consider (x, y_0) with $x > x_0$. We can use relation (5.6) which is valid for $\Omega \in \mathcal{C}$ (using Theorem 5.2) and follow the same proof as in the case of $\mathcal{E}_{reg}$.

$\square$

Theorem 5.2. *Let Ω be a convex domain in the class $\mathcal{C}$ (defined in (2.4)), then there exists a sequence of C^∞ regular strictly convex domains Ω_ε in the class $\mathcal{C}$ such that:*

1. *the support function h_ε of Ω_ε is of class C^∞;*
2. *the curvature of Ω_ε, k_ε satisfies $k_\varepsilon(s) \geqslant \varepsilon > 0$;*
3. *the elastic energy converges: $\lim_{\varepsilon\to 0} E(\Omega_\varepsilon) = E(\Omega)$;*
4. *the area converges: $\lim_{\varepsilon\to 0} A(\Omega_\varepsilon) = A(\Omega)$ and we can assume $A(\Omega_\varepsilon) < A(\Omega)$.*

Proof. Since Ω is an open bounded convex set, there exist a point O and two positive numbers $0 < r_0 < R_0$ such that $B(O, r_0) \subset \Omega \subset B(O, R_0)$. We use the gauge function u which defines the convex domain in polar coordinates (r, τ):

$$\Omega := \left\{(r,\tau) \in (r_0, R_0) \times \mathbb{R} \ ; \ r < \frac{1}{u(\tau)}\right\}$$

where u is a positive and 2π-periodic function. Since $\partial\Omega$ is contained in the ring $B(O, R_0) \setminus B(O, r_0)$ we have

$$R_0^{-1} \leqslant u(\tau) \leqslant r_0^{-1} \tag{5.7}$$

Moreover, it is classical that the convexity of Ω is equivalent to the fact that $u'' + u$ is a non-negative measure. Let us detail the regularity of u when Ω belongs to the class $\mathcal{C}$ and the link between the gauge function u and the support function h. From the parametrization

$$\begin{cases} x(\tau) = (\cos\tau)/u(\tau), \\ y(\tau) = (\sin\tau)/u(\tau) \end{cases}$$

we deduce $ds = \sqrt{u(\tau)^2 + u'(\tau)^2}/u(\tau)^2\, d\tau$ which makes the curvilinear abscissa s an increasing function of the angle τ. The unit tangent vector is given by

$$\begin{cases} \cos\theta = -\frac{(\sin\tau)u(\tau) + (\cos\tau)u'(\tau)}{\sqrt{u(\tau)^2 + u'(\tau)^2}} \\ \sin\theta = \frac{(\cos\tau)u(\tau) - (\sin\tau)u'(\tau)}{\sqrt{u(\tau)^2 + u'(\tau)^2}}, \end{cases}$$

while the exterior normal vector is $\mathsf{n} = (\sin\theta, -\cos\theta)$. Since the boundary of Ω is strictly convex, there is a one-to-one correspondence between τ and θ. We write $h(\tau)$, meaning $h(\theta(\tau))$, given by

$$h(\tau) = \langle \overrightarrow{OM}, \mathsf{n}\rangle = \frac{1}{\sqrt{u(\tau)^2 + u'(\tau)^2}}. \tag{5.8}$$

Since $r_0 \leqslant h(\tau) \leqslant R_0$, we deduce from (5.8) that $u' \in L^\infty(\mathbb{R})$. Moreover the curvature at the point $(x(\tau), y(\tau))$ is given by

$$k(\tau) = \frac{u^3}{(u^2 + u'^2)^{3/2}}\,(u + u''),$$

and hence the elastic energy is

$$E(\Omega) = \frac{1}{2}\int_0^{2\pi} \frac{u^4(u + u'')^2\, d\tau}{(u^2 + u'^2)^{5/2}}. \tag{5.9}$$

Therefore, the fact that Ω belongs to the class $\mathcal{C}$ means that the function

$$\tau \mapsto \frac{u^2(u + u'')}{(u^2 + u'^2)^{5/4}}$$

is in $L^2(0,2\pi)$. More precisely $u \in W^{2,2}(0,2\pi)$, since u and u' are bounded and u is bounded from below, and hence u'' is in $L^2(0,2\pi)$. The converse is also true that is

$$\Omega \in \mathcal{C} \iff u \in W^{2,2}(0,2\pi). \tag{5.10}$$

Let us assume that Ω is strictly convex, of class C^∞, whose curvature is bounded from below: i.e. $k \geqslant \varepsilon > 0$. Then its gauge function is of class C^∞, and the function $\tau \mapsto \theta(\tau)$ is strictly increasing, C^∞ and its derivative is given by

$$\frac{d\theta}{d\tau} = \frac{u(u+u'')}{u^2+u'^2}$$

which is also bounded from below. Therefore its inverse function $\theta \mapsto \tau(\theta)$ is of class C^∞. Notice that the angle of the normal vector n with the x-axis is $t = \theta - \pi/2$, therefore $\tau \mapsto t(\tau)$ is also C^∞. Moreover the support function can be expressed in terms of t by:

$$h(t) = \frac{1}{\sqrt{u(t(\tau))^2 + u'(t(\tau))^2}},$$

which is C^∞.

We now proceed to the approximation result. Let Ω be any convex domain in the class $\mathcal{C}$ and let u be its gauge function. We choose a sequence of (non-negative) C^∞ mollifiers ρ_ϵ (of support of size ϵ) and we consider the regularized functions u_ε defined by $u_\varepsilon := u * \rho_{\varepsilon^4} + a\varepsilon$ where $a := 2R_0^3/r_0^3$. When ε goes to zero, the convolution product $u * \rho_{\varepsilon^4}$ converges to u in $W^{2,2}(0,2\pi)$, therefore u_ε converges to u in $W^{2,2}(0,2\pi)$ and up to subsequence, we can assume that u_ε and u'_ε converge uniformly to u and u', respectively (by the compactness embedding $W^{2,2} \hookrightarrow C^1$). In particular the sequence of convex domains Ω_ε defined by their gauge functions u_ε converges to Ω in the Hausdorff metric and in particular $B(O, r_0/2^{1/6}) \subset \Omega_\varepsilon \subset B(O, 2^{1/6}R_0)$ for ε small enough. Since $u''_\varepsilon + u_\varepsilon = (u''+u) * \rho_{\varepsilon^4} + a\varepsilon \geqslant a\varepsilon$, the curvature k_ε of Ω_ε satisfies

$$k_\varepsilon(\tau) = \frac{u_\varepsilon^3}{(u_\varepsilon^2 + {u'_\varepsilon}^2)^{3/2}}\,(u_\varepsilon + u''_\varepsilon) \geqslant R_0^{-3}2^{-1/2}r_0^3 2^{-1/2}a\varepsilon = \varepsilon.$$

Following the previous discussion, this implies that the support function of Ω_ε is of class C^∞. The L^2 convergence of u''_ε to u'' and the uniform convergence of u_ε and u'_ε to u and u', respectively, ensure that

$$E(\Omega_\varepsilon) = \frac{1}{2}\int_0^{2\pi} \frac{u_\varepsilon^4(u_\varepsilon + u''_\varepsilon)^2\, d\tau}{(u_\varepsilon^2 + {u'_\varepsilon}^2)^{5/2}}$$

converges to $E(\Omega)$. The convergence of the area $A(\Omega_\varepsilon)$ to $A(\Omega)$ follows from the expression of the area in terms of the gauge function:

$$A(\Omega) = \frac{1}{2}\int_0^{2\pi} \frac{d\tau}{u^2(\tau)}, \tag{5.11}$$

and the uniform convergence. Let us consider the perimeter of Ω_ε, given by

$$P(\Omega_\varepsilon) = \int_0^{2\pi} \sqrt{u_\varepsilon^2 + {u'_\varepsilon}^2}\, d\tau; \tag{5.12}$$

it converges to $P(\Omega)$. Hence it suffices to make an homothety of Ω_ε of ratio $P(\Omega)/P(\Omega_\varepsilon)$ to construct a sequence of convex domains

$$\widetilde{\Omega}_\varepsilon = \frac{P(\Omega)}{P(\Omega_\varepsilon)}\Omega_\varepsilon,$$

with fixed perimeter and which fulfills the same properties as Ω_ε.

It remains to show that in the above construction $A(\widetilde{\Omega}_\varepsilon) < A(\Omega)$. For this purpose we need to show that

$$A(\Omega_\varepsilon) < A(\Omega) \quad \text{and} \quad P(\Omega_\varepsilon) > P(\Omega). \tag{5.13}$$

In order to obtain this, we note that

$$u_\varepsilon(x) - u(x) = a\varepsilon + \int_0^{2\pi} \rho_{\varepsilon^4}(y)\,[u(x-y) - u(x)]\;dy.$$

Using the fact that the support of ρ_{ε^4} is of size ε^4, we deduce that

$$\left|\int_0^{2\pi} \rho_{\varepsilon^4}(y)\left[u(x-y)-u(x)\right]\,dy\right| \leqslant \varepsilon^4\|u'\|_{L^\infty(0,2\pi)}.$$

As a consequence, for ε small,

$$u_\varepsilon(x) > u(x),$$

which yields $A(\Omega_\varepsilon) < A(\Omega)$ by formula (5.11).

On the other hand,

$$u_\varepsilon'(x) - u'(x) = \int_0^{2\pi} \rho_{\varepsilon^4}(y)\left[u'(x-y)-u'(x)\right]\,dy.$$

Using again the fact that the support of ρ_{ε^4} is of size ε^4, we deduce that

$$\left|\int_0^{2\pi} \rho_{\varepsilon^4}(y)\left[u'(x-y)-u'(x)\right]\,dy\right| \leqslant \int_0^{2\pi} \rho_{\varepsilon^4}(y)\left|\int_x^{x-y} u''\,da\right|\,dy \leqslant \varepsilon^2\|u''\|_{L^2(0,2\pi)}.$$

As a consequence, for ε small,

$$\sqrt{u_\varepsilon^2 + {u_\varepsilon'}^2} > \sqrt{u^2+u'^2}$$

which yields $P(\Omega_\varepsilon) > P(\Omega)$ by formula (5.12).

□

6 Numerical algorithm

In this section, we show some numerical results regarding the problem

$$\min\left\{E(\Omega)+\mu A(\Omega), \Omega\in\mathcal{C}, P(\Omega)=2\pi\right\} \tag{6.1}$$

where $\mathcal{C}$ is defined by (2.4) and we apply it to plot the convex hull of the Blaschke-Santaló diagram (5.1).

To solve the optimization problem (6.1) we choose to directly consider the optimality conditions (2.20) in term of the curvature $k(s)$. More precisely, we consider the ODE

$$\left\{\begin{array}{rcl} k'' &=& -\frac{1}{2}k^3 - \lambda k + \mu \\ k(0) &=& k_M \\ k'(0) &=& 0 \end{array}\right. \tag{6.2}$$

where λ is the Lagrange multiplier defined in Proposition 2.4, see (2.21), and k_M is the maximum value of the curvature. The first step of the numerical procedure consists in evaluating these two parameters λ and k_M. The ODE (6.2) being valid only on the strictly convex parts of the boundary, we have to decide whether we look for a strictly convex solution (without segments) or for a solution with segments. According to Proposition 3.9, we know that we have segments on the boundary when μ is large enough. It turns out that such segments appear numerically as soon as $\mu \geqslant 3.34...$, that is when the function k vanishes before $s=\pi/2q$ (q being the periodicity).

Let us explain in detail the procedure. First we notice that we can obtain an explicit formula for the solution to (6.2) in term of the elliptic Jacobi function cn (see, for instance, [1, chapter 16]).

Lemma 6.1. *Assume $k\not\equiv 1$ and $\lambda\geqslant 0$. Then the solution of (6.2) can be written as*

$$k(s) = \frac{\alpha\,\mathrm{cn}(\omega s|\tau^2)+\beta}{\gamma\,\mathrm{cn}(\omega s|\tau^2)+1} \tag{6.3}$$

Let us note that this solution may not be the curvature of a domain in $\mathcal{C}$, since for large s it may be negative.

Proof. For any given data (k_M,λ) we integrate once equation (6.2) to get

$$(k')^2 = -\frac{1}{4}k^4 - \lambda k^2 + 2\mu k + C \tag{6.4}$$

where $C = \frac{1}{4}\, k_M^4 + \lambda k_M^2 - 2\mu k_M$. This shows that the solution of (6.2) is global and bounded. Moreover it has a minimum value k_m and thus the polynomial

$$Q(z) = -\frac{1}{4}\, z^4 - \lambda z^2 + 2\mu z + C$$

has two real roots k_M and k_m and two conjugate non-real roots z_0 and $\overline{z_0}$. We make a change of variables to transform (6.4). In order to do this we introduce

$$\sigma = \frac{k_M + k_m}{2}, \;\; \delta = \frac{k_M - k_m}{2}. \tag{6.5}$$

Using the relations between the coefficients and the roots of Q, we check that $\sigma, \delta > 0$.

We can also verify that there exists a unique root $\gamma \in (-1, 0)$ of

$$X^2 + \frac{1}{\sigma\delta}\, \left(3\sigma^2 + \delta^2 + 2\lambda\right) X + 1 = 0. \tag{6.6}$$

We can then define

$$\alpha = \gamma\sigma + \delta, \;\; \beta = \gamma\delta + \sigma \tag{6.7}$$

and perform the change of variables

$$k = \frac{\alpha y + \beta}{\gamma y + 1},$$

that is $y = \frac{\beta - k}{\gamma k - \alpha}$. Tedious calculation and (6.6) yield that y satisfies

$$\left\{ \begin{array}{lcl} (y')^2 & = & \omega^2 (1 - y^2)(1 - \tau^2 + \tau^2 y^2)) \\ y(0) & = & 1\,. \end{array} \right. \tag{6.8}$$

with

$$\omega^2 = \sigma\delta \frac{\gamma^2 - 1}{2\gamma} > 0 \quad \tau^2 = \frac{\gamma^2 + \frac{\delta\gamma}{2\sigma}}{\gamma^2 - 1} \in (0, 1).$$

It is well-known (see, for instance, [1, chapter 16]) that the solution of (6.8) is the Jacobian elliptic function $s \mapsto \mathrm{cn}(\omega s|\tau^2)$. Therefore, we have obtained (6.3) with $\alpha, \beta, \gamma, \omega, \tau$ defined as above. □

According to Remark 2.8 and Proposition 2.9 (see also Proposition 3.4), the curve is $2q$ periodic, for an integer q, $q \geqslant 1$ in the strictly convex case and $1 \leqslant q \leqslant 2$ when there are segments (see Theorem 3.7). We will use two different algorithms when looking for strictly convex solutions, **case a)** and non-strictly convex solutions, **case b)**.

Case a): *strictly convex solutions*: We choose $q \geqslant 1$ and we try to find the parameters k_M, λ such that the two following conditions are satisfied:

$$2K(\tau^2) = \frac{\omega\pi}{2q} \quad \text{(periodicity)} \tag{6.9}$$

$$\int_0^{\pi/2q} k(s)\, ds = \frac{\pi}{2q} \quad \text{(in order to have } \theta(\pi/2q) = \pi/2q\text{)} \tag{6.10}$$

where K is the complete Elliptic integral of the first kind

$$K(m) := \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - mt^2)}} \quad (m \in [0, 1]). \tag{6.11}$$

which defines the periodicity of the Jacobian elliptic function cn, see [1, chapter 17]. This gives a 2×2 non-linear system that can be solved by using the Levenberg-Marquardt algorithm implemented in `Matlab`. Then we compute the angle $\theta(s)$ by integrating $k(s)$ and the curve by

$$x(s) = \int_0^s \cos\theta(u)\, du, \quad y(s) = \int_0^s \sin\theta(u)\, du.$$

The elastic energy is computed by integrating (numerically) the curvature squared, while the area of the domain is computed using (2.21): $A(\Omega) = \frac{2\pi\lambda + E(\Omega)}{2\mu}$.

Case b): *non-strictly convex solutions*: Choose $q = 1, 2$ and consider the first zero of the function $k(s)$, named s_1. Hence $s_1 \leqslant \pi/2q$. We search numerically parameters k_M, λ such that the two following conditions are satisfied:

$$\int_0^{s_1} k(s)\, ds = \frac{\pi}{2q}, \tag{6.12}$$

$$\frac{2}{\mu}\sqrt{C} + 2s_1 = \frac{\pi}{q}; \tag{6.13}$$

notice that (6.13) guarantees $\theta(s_1) = \pi/2q$). Indeed, according to Proposition 2.9 and equality (2.32), any segment have the same length L which is related to the value of $k'(s_1^-)$ by $L = -\frac{2}{\mu}\, k'(s_1^-)$. Using equation (6.4) and the fact that $k(s_1) = 0$ we see that $k'(s_1^-) = -\sqrt{C}$. Thus (6.13) expresses the fact that the total length of the curve has to be 2π. Then one can proceed following the same steps as above.

This method gives the following results: it turns out that a value of q greater than one is never competitive (with respect to $q = 1$), neither in the case of strictly convex domains, nor in the case where segments appear; the disk remains the optimal domain while $\mu \leqslant 3$, showing that Theorem 4.2 is probably optimal. As already mentioned, segments appear as soon as $\mu \geqslant 3.3425$. Figure 6.1 shows three optimal domains obtained for $\mu = 3.2, \mu = 4$ and $\mu = 8$.

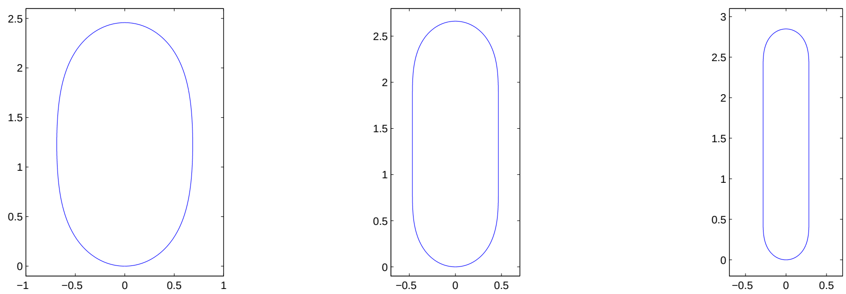

Figure 6.1: Three optimal domains corresponding to $\mu = 3.2, 4, 8$.

We are going to use the previous method to find optimal domains for any value μ starting at $\mu = 1$ in order to plot the unknown part of the boundary of the convex hull of the Blaschke-Santaló diagram $\mathcal{E}$, defined at (5.1), which is contained in the half plane $\{x < 1\}$. This is equivalent to find the point(s) of $\mathcal{E}$ whose supporting line is parallel to $y + \mu x = 0$. Numerically, this process gives a unique continuous family of convex domains, say Ω_μ, which tends to prove that the set $\mathcal{E}$ is indeed convex and can be plotted this way. We show it in Figure 6.2.

The lower point of $\mathcal{E}$ is obviously the disk whose coordinates are $(1, 1)$ due to the normalization. The other point which appears on the boundary corresponds to the last strictly convex solution (obtained for $\mu = 3.3425$). The dotted line is the hyperbola $yx = 1$ which is the lower bound given by Gage's Theorem 1.1. Actually, this is not the asymptotic hyperbola for the set $\mathcal{E}$. Next proposition makes the asymptotic behaviour of the set $\mathcal{E}$ on its left boundary more precise.

Proposition 6.2. *The hyperbola $yx = \rho^2/\pi^2$ is asymptotic to the set $\mathcal{E}$, where ρ is given by*

$$\rho = 2\sqrt{2\pi}\left[2\mathsf{E}\left(\frac{1}{2}\right) - K\left(\frac{1}{2}\right)\right] \simeq 4.2473 \tag{6.14}$$

where $K(\cdot)$ and $\mathsf{E}(\cdot)$ are the complete Elliptic Integral of the first and the second kind, respectively:

$$\mathsf{E}\left(\frac{1}{2}\right) = \int_0^1 \sqrt{\frac{1 - t^2/2}{1 - t^2}}\, dt, \quad K\left(\frac{1}{2}\right) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - t^2/2)}}.$$

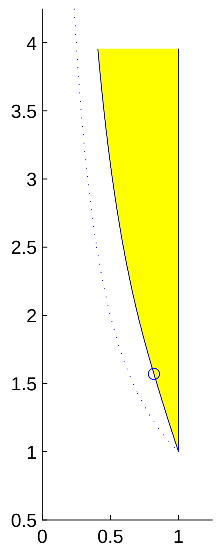

Figure 6.2: The Blaschke-Santalo diagram $\mathcal{E}$

More precisely, the elastic energy and the area of the optimal domains Ω_μ behave, when $\mu \to +\infty$, like

$$E(\Omega_\mu) \sim \rho\sqrt{\mu} \quad A(\Omega_\mu) \sim \rho/\sqrt{\mu}.$$

Proof. Let us denote by $E_\mu = E(\Omega_\mu)$ and $A_\mu = A(\Omega_\mu)$ the elastic energy and the area of an optimal domain. For μ large, the boundary of Ω_μ contains segments (according to Proposition 3.9), therefore, following Lemma 3.5, $E_\mu \leqslant A_\mu \leqslant 2E_\mu$. We plug these inequalities into (3.3) to get

$$\frac{\pi}{\sqrt{\mu}} \leqslant A_\mu \leqslant \frac{2\pi}{\sqrt{\mu}} \quad \text{and} \quad \frac{2\pi\sqrt{\mu}}{3} \leqslant E_\mu \leqslant \frac{3\pi\sqrt{\mu}}{2}.$$

This shows that $\sqrt{\mu}A_\mu$ and $E_\mu/\sqrt{\mu}$ are bounded. Therefore, there exist $\rho_1 \in [\pi, 2\pi]$ and $\rho_2 \in [2\pi/3, 3\pi/2]$ such that, up to some subsequence, $\sqrt{\mu}A_\mu \to \rho_1$ and $E_\mu/\sqrt{\mu} \to \rho_2$. By (2.21) it follows

$$\lambda \sim \frac{2\rho_1 - \rho_2}{2\pi}\sqrt{\mu} \text{ when } \mu \to +\infty. \tag{6.15}$$

Since A_μ goes to 0 when μ goes to $+\infty$, the optimal (convex) domain Ω_μ converges to a segment of length π (and the half-diameter converges to $\pi/2$). Let us denote by H_M the point on the boundary which is at maximum distance of Q. According to (2.19), it corresponds to the point with maximal curvature k_M. Therefore, using (2.14)

$$\frac{\lambda}{\mu} + \frac{1}{2\mu}k_M^2 = \langle\overrightarrow{QH_M}, \mathsf{n}\rangle = \left\|\overrightarrow{QH_M}\right\| \to \frac{\pi}{2}.$$

But $\lambda/\mu \to 0$ by (6.15), therefore

$$k_M \sim \sqrt{\pi\mu} \text{ when } \mu \to +\infty. \tag{6.16}$$

We now use the notation and formula in the proof of Lemma 6.1. By (6.4) and (6.16), we have $C \sim \pi^2\mu^2/4$ while the length of the segments satisfies $L \to \pi$. The optimal domain Ω_μ contains a rectangle of length L and width $2\lambda/\mu$ plus a part that can be included in a rectangle of edges sizes $1/2(\text{diam}(\Omega_\mu) - L)$ and $2\lambda/\mu$. Since $\text{diam}(\Omega_\mu) \to \pi$, this remaining part is of order $o\left(1/\sqrt{\mu}\right)$ Therefore, using (6.15) and the definition of ρ_1 we have $A_\mu \sim \rho_1/\sqrt{\mu}$ together with $A_\mu \sim 2L\lambda/\mu \sim (2\rho_1 - \rho_2)/\sqrt{\mu}$. It follows that $\rho_1 = \rho_2$. We will now denote by ρ this common value.

Straightforward calculations now give (keeping the above notations):

$$k_M k_m \sim -\pi\mu, \; k_M + k_m \sim \frac{4}{\pi}, \; \sigma \sim \frac{2}{\pi}, \; \delta \sim \sqrt{\pi\mu},$$

and

$$\gamma \sim \frac{-2}{\pi\sqrt{\pi\mu}},\ \alpha \sim \sqrt{\pi\mu},\ \beta = o(1),\ \omega^2 \sim \frac{\pi\mu}{2},\ \tau^2 = \frac{1}{2} + o(1).$$

Therefore, at order 1, the curvature behaves like

$$k(s) \sim \sqrt{\pi\mu}\ \mathrm{cn}\left(\omega s \mid \frac{1}{2}\right) \quad \text{with } \omega = \sqrt{\frac{\pi\mu}{2}}. \tag{6.17}$$

and s_1 the first zero of $k(s)$ tends to 0 as $K\left(\frac{1}{2}\right)/\omega$ since $K\left(\frac{1}{2}\right)$ is the first zero of $\mathrm{cn}\left(s \mid \frac{1}{2}\right)$ (the definition of K is recalled in (6.11)). It follows that the elastic energy satisfies

$$E_\mu = 2\int_0^{s_1} k^2(s)ds \sim 2\pi\mu \int_0^{s_1} \mathrm{cn}^2\left(\omega s \mid \frac{1}{2}\right) ds \sim 2\sqrt{2\pi\mu}\int_0^{K(1/2)} \mathrm{cn}^2\left(t \mid \frac{1}{2}\right) dt\,.$$

Now using formulae in [1, 17.2.11] for Elliptic integral of second kind,

$$\int_0^{K(1/2)} \mathrm{cn}^2\left(t \mid \frac{1}{2}\right) dt = 2\mathsf{E}\left(K\left(\frac{1}{2}\right) \mid \frac{1}{2}\right) - K\left(\frac{1}{2}\right),$$

where

$$\mathsf{E}(u \mid m) := \int_0^{\mathrm{sn}(t|m)} \sqrt{\frac{1-mt^2}{1-t^2}}\ dt.$$

Using $\mathrm{sn}(K(0.5) = 1$ and formulae [1, 17.3.3] give the desired result. Finally, since the accumulation point for $\sqrt{\mu}A_\mu$ and $E_\mu/\sqrt{\mu}$ are unique, both sequences converge to ρ. □

7 Appendix

Here below we present the proof of Lemma 2.7. For the shape derivative formulas of the area and of the perimeter, we refer, for instance, to [7].

In order to derive $E(\Omega)$ with respect to the domain, we consider a parametrization of $\partial\Omega$:

$$s \in [0, P(\Omega)] \mapsto (x(s), y(s)).$$

Let us consider a variation of the domain of the form $\Omega_\varepsilon = \Omega + \varepsilon V(\Omega)$, where V is a smooth function. Then a parametrization of $\partial\Omega_\varepsilon$ is

$$\begin{aligned} x_\varepsilon(s) &= x(s) + \varepsilon V_1(x(s), y(s)), \\ y_\varepsilon(s) &= y(s) + \varepsilon V_2(x(s), y(s)), \end{aligned}$$

so that

$$\begin{aligned} x_\varepsilon'(s) &= x'(s) + \varepsilon \frac{d}{ds} V_1(x(s), y(s)) = \cos\theta(s) + \varepsilon\frac{d}{ds}V_1(x(s), y(s)), \\ y_\varepsilon'(s) &= y'(s) + \varepsilon \frac{d}{ds} V_2(x(s), y(s)) = \sin\theta(s) + \varepsilon\frac{d}{ds}V_2(x(s), y(s)), \end{aligned}$$

and

$$\begin{aligned} x_\varepsilon''(s) &= -\theta'(s)\sin\theta(s) + \varepsilon\frac{d^2}{ds^2}V_1(x(s), y(s)), \\ y_\varepsilon''(s) &= \theta'(s)\cos\theta(s) + \varepsilon\frac{d^2}{ds^2}V_2(x(s), y(s)). \end{aligned}$$

We notice that

$$x_\varepsilon'(s)^2 + y_\varepsilon'(s)^2 = 1 + 2\varepsilon\left(\cos\theta(s)\frac{d}{ds}V_1(x(s), y(s)) + \sin\theta(s)\frac{d}{ds}V_2(x(s), y(s))\right) + o(\varepsilon^2).$$

Moreover,

$$
\begin{aligned}
k_\varepsilon(s) &= \left[-x''_\varepsilon(s)y'_\varepsilon(s) + y''_\varepsilon(s)x'_\varepsilon(s)\right] / \left[x'_\varepsilon(s)^2 + y'_\varepsilon(s)^2\right]^{3/2} \\
&= \Bigg[\left(\theta'(s)\sin\theta(s) - \varepsilon\frac{d^2}{ds^2}V_1(x(s),y(s))\right)\left(\sin\theta(s) + \varepsilon\frac{d}{ds}V_2(x(s),y(s))\right) \\
&\quad + \left(\theta'(s)\cos\theta(s) + \varepsilon\frac{d^2}{ds^2}V_2(x(s),y(s))\right)\left(\cos\theta(s) + \varepsilon\frac{d}{ds}V_1(x(s),y(s))\right)\Bigg] \\
&\qquad \times \left(1 - 3\varepsilon\left(\cos\theta(s)\frac{d}{ds}V_1(x(s),y(s)) + \sin\theta(s)\frac{d}{ds}V_2(x(s),y(s))\right)\right) + o(\varepsilon^2)
\end{aligned}
$$

which yields

$$
\begin{aligned}
k_\varepsilon(s) = \theta'(s) + \varepsilon\Bigg(&\theta'(s)\sin\theta(s)\frac{d}{ds}V_2(x(s),y(s)) - \sin\theta(s)\frac{d^2}{ds^2}V_1(x(s),y(s)) \\
&+ \theta'(s)\cos\theta(s)\frac{d}{ds}V_1(x(s),y(s)) + \frac{d^2}{ds^2}V_2(x(s),y(s))\cos\theta(s)\Bigg) \\
&- 3\varepsilon\left(\cos\theta(s)\theta'(s)\frac{d}{ds}V_1(x(s),y(s)) + \theta'(s)\sin\theta(s)\frac{d}{ds}V_2(x(s),y(s))\right) + o(\varepsilon^2)
\end{aligned}
$$

and thus

$$
\begin{aligned}
k_\varepsilon(s) = \theta'(s) + \varepsilon\Bigg(&-2\theta'(s)\sin\theta(s)\frac{d}{ds}V_2(x(s),y(s)) - \sin\theta(s)\frac{d^2}{ds^2}V_1(x(s),y(s)) \\
&-2\theta'(s)\cos\theta(s)\frac{d}{ds}V_1(x(s),y(s)) + \frac{d^2}{ds^2}V_2(x(s),y(s))\cos\theta(s)\Bigg) + o(\varepsilon^2).
\end{aligned}
$$

Consequently, we can write the elastic energy for the perturbation of Ω:

$$
\begin{aligned}
E(\Omega_\varepsilon) = \frac{1}{2}\int_0^{2\pi} \Bigg[&\theta'(s) + \varepsilon\Bigg(-2\theta'(s)\sin\theta(s)\frac{d}{ds}V_2(x(s),y(s)) - \sin\theta(s)\frac{d^2}{ds^2}V_1(x(s),y(s)) \\
&-2\theta'(s)\cos\theta(s)\frac{d}{ds}V_1(x(s),y(s)) + \frac{d^2}{ds^2}V_2(x(s),y(s))\cos\theta(s)\Bigg) + o(\varepsilon^2)\Bigg]^2 \\
&\times \left[1 + 2\varepsilon\left(\cos\theta(s)\frac{d}{ds}V_1(x(s),y(s)) + \sin\theta(s)\frac{d}{ds}V_2(x(s),y(s))\right) + o(\varepsilon^2)\right]^{1/2} ds
\end{aligned}
$$

Thus

$$
\begin{aligned}
E(\Omega_\varepsilon) = \frac{1}{2}\int_0^{2\pi} \theta'(s)^2 + 2\varepsilon\Bigg(&-2\theta'(s)^2\sin\theta(s)\frac{d}{ds}V_2(x(s),y(s)) - \theta'(s)\sin\theta(s)\frac{d^2}{ds^2}V_1(x(s),y(s)) \\
&-2\theta'(s)^2\cos\theta(s)\frac{d}{ds}V_1(x(s),y(s)) + \frac{d^2}{ds^2}V_2(x(s),y(s))\cos\theta(s)\theta'(s)\Bigg) \\
&+ \varepsilon\left(\cos\theta(s)\theta'(s)^2\frac{d}{ds}V_1(x(s),y(s)) + \sin\theta(s)\theta'(s)^2\frac{d}{ds}V_2(x(s),y(s))\right) + o(\varepsilon^2)\ ds
\end{aligned}
$$

Thus

$$
\begin{aligned}
E(\Omega_\varepsilon) = \frac{1}{2}\int_0^{2\pi} \theta'(s)^2 + \varepsilon\Bigg(&-3\theta'(s)^2\sin\theta(s)\frac{d}{ds}V_2(x(s),y(s)) - 2\theta'(s)\sin\theta(s)\frac{d^2}{ds^2}V_1(x(s),y(s)) \\
&-3\theta'(s)^2\cos\theta(s)\frac{d}{ds}V_1(x(s),y(s)) + 2\frac{d^2}{ds^2}V_2(x(s),y(s))\cos\theta(s)\theta'(s)\Bigg) + o(\varepsilon^2)\ ds. \qquad (7.1)
\end{aligned}
$$

Now,

$$
\int_0^{2\pi} \theta'(s)^2\sin\theta(s)\frac{d}{ds}V_2(x(s),y(s))\ ds = -\int_0^{2\pi} \left(2\theta'\theta''\sin\theta + (\theta')^3\cos\theta\right)V_2(x,y)\ ds, \qquad (7.2)
$$

$$\int_0^{2\pi} \theta'(s)^2 \cos\theta(s) \frac{d}{ds} V_1(x(s), y(s))\ ds = -\int_0^{2\pi} \left(2\theta'\theta'' \cos\theta - (\theta')^3 \sin\theta\right) V_1(x,y)\ ds, \tag{7.3}$$

$$\int_0^{2\pi} \theta'(s) \sin\theta(s) \frac{d^2}{ds^2} V_1(x(s), y(s))\ ds = \int_0^{2\pi} \left(\theta''' \sin(\theta) + 3\theta''\theta' \cos\theta - (\theta')^3 \sin\theta\right) V_1(x,y)\ ds, \tag{7.4}$$

$$\int_0^{2\pi} \theta'(s) \cos\theta(s) \frac{d^2}{ds^2} V_2(x(s), y(s))\ ds = \int_0^{2\pi} \left(\theta''' \cos(\theta) - 3\theta''\theta' \sin\theta - (\theta')^3 \cos\theta\right) V_2(x,y)\ ds. \tag{7.5}$$

Gathering (7.1) and (7.2)–(7.5) yields

$$\frac{dE(\Omega_\varepsilon)}{d\varepsilon}|_{\varepsilon=0} = \frac{1}{2}\int_0^{2\pi} \Big(3\left(2\theta'\theta'' \sin\theta + (\theta')^3 \cos\theta\right) V_2(x,y) + 3\left(2\theta'\theta'' \cos\theta - (\theta')^3 \sin\theta\right) V_1(x,y)$$
$$-2\left(\theta''' \sin(\theta) + 3\theta''\theta' \cos\theta - (\theta')^3 \sin\theta\right) V_1(x,y) + 2\left(\theta''' \cos(\theta) - 3\theta''\theta' \sin\theta - (\theta')^3 \cos\theta\right) V_2(x,y)\Big)\ ds. \tag{7.6}$$

Since the normal to Ω is $\mathsf{n} = (\sin\theta, -\cos\theta)$, we deduce from (7.6) that

$$\frac{dE(\Omega_\varepsilon)}{d\varepsilon}|_{\varepsilon=0} = -\int_0^{2\pi} \left(\frac{1}{2}(\theta')^3 + \theta'''\right) \langle V, \mathsf{n}\rangle\ ds. \tag{7.7}$$

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