Monotone valuations on the space of convex functions

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Abstract

We consider the space $\mathcal{C}^n$ of convex functions u defined in $\mathbf{R}^n$ with values in $\mathbf{R} \cup \{\infty\}$, which are lower semi-continuous and such that $\lim_{|x|\to\infty} u(x) = \infty$. We study the valuations defined on $\mathcal{C}^n$ which are invariant under the composition with rigid motions, monotone and verify a certain type of continuity. We prove integral representations formulas for such valuations which are, in addition, simple or homogeneous.

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Contents

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1 Introduction

The aim of this paper is to begin an exploration of the valuations defined on the space of convex functions, having as a model the valuations of convex bodies.

The theory of valuations is currently a significant part of convex geometry. We recall that, if $\mathcal{K}^n$ denotes the set of convex bodies (compact and convex sets) in $\mathbf{R}^n$, a (real-valued) valuation is a function $\sigma : \mathcal{K}^n \to \mathbf{R}$ that verifies the following (restricted) additivity condition

$$\sigma(K \cup L) + \sigma(K \cap L) = \sigma(K) + \sigma(L) \quad \forall\, K, L \in \mathcal{K}^n \text{ such that } K \cup L \in \mathcal{K}^n, \tag{1.1}$$

together with

$$\sigma(\emptyset) = 0. \tag{1.2}$$

In the realm of convex geometry the most familiar examples of valuations are the so-called *intrinsic volumes* V_k, $k \in \{0, 1, \dots, n\}$, which have many additional properties such as: invariance under rigid motions, continuity with respect to the Hausdorff metric, homogeneity and monotonicity. Note that intrinsic volumes include the volume (here denoted by V_n) itself, i.e. the Lebesgue measure, which is clearly a valuation.

A celebrated result by Hadwiger (see [7], [8], [9], [17]) provides a characterization of an important class of valuations on $\mathcal{K}^n$.

Theorem 1.1. *A functional $\sigma : \mathcal{K}^n \to \mathbf{R}$ is a rigid motion invariant, continuous (or monotone) valuation if and only if there exist $c_0, \dots, c_n \in \mathbf{R}$, such that*

$$\sigma(K) = \sum_{i=0}^{n} c_i V_i(K) \quad \textit{for every } K \in \mathcal{K}^n.$$

Moreover, σ is increasing (resp. decreasing) if and only if $c_i \geq 0$ (resp. $c_i \leq 0$) for every $i \in \{0, \dots, n\}$.

A special case of this theorem is known as the *volume theorem*.

Corollary 1.2. *Let σ be a rigid motion invariant and continuous (or monotone) valuation on $\mathcal{K}^n$, which is simple, i.e. $\sigma(K) = 0$ for every K such that $\dim(K) < n$. Then σ is a multiple of the volume: there exists a constant $c \in \mathbf{R}$ such that*

$$\sigma(K) = cV_n(K) \quad \forall\, K \in \mathcal{K}^n.$$

These deep results gave a strong impulse to the development of the theory which, in the last decades, was enriched by a wide variety of new results and counts now a considerable number of prolific ramifications. A survey on the state of the art of this subject is presented in the monograph [17] by Schneider (see chapter 6), along with a detailed list of references.

Recently the study of valuations was extended from spaces of sets, like $\mathcal{K}^n$, to spaces of real functions. The condition (1.1) is adapted to this situation replacing union and intersection by "max" and "min". In other words, if $\mathcal{X}$ is a space of functions, a function $\mu : \mathcal{X} \to \mathbf{R}$ is called a valuation if

$$\mu(u \vee v) + \mu(u \wedge v) = \mu(u) + \mu(v) \tag{1.3}$$

for every $u, v \in \mathcal{X}$ such that $u \wedge v \in \mathcal{X}$ and $u \vee v \in \mathcal{X}$. Here $u \vee v$ and $u \wedge v$ denote the point-wise maximum and minimum of u and v, respectively. To motivate this definition one can observe that the epigraphs of $u \vee v$ and $u \wedge v$ are the intersection and the union of the epigraphs of u and v, respectively. The same property is shared by sub-level sets: for every $t \in \mathbf{R}$ we have

$$\{u \vee v < t\} = \{u < t\} \cap \{v < t\} \quad \text{and} \quad \{u \wedge v < t\} = \{u < t\} \cup \{v < t\}.$$

This fact will be particularly important in the present paper.

Valuations defined on the Lebesgue spaces $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, were studied by Tsang in [18], while the case $p = \infty$ is considered in [4]. In relation to some of the results presented in our paper it is interesting to mention that one of the results of Tsang asserts that any translation invariant and continuous valuation μ on $L^p(\mathbf{R}^n)$ can be written in the form

$$\mu(u) = \int_{\mathbf{R}^n} f(u(x))\, dx \quad \forall\, u \in L^p(\mathbf{R}^n) \tag{1.4}$$

where f is a continuous function subject to a suitable growth condition at infinity. The results of Tsang have been extended to Orlicz spaces by Kone, in [10]. Valuations of different types (taking values in $\mathcal{K}^n$ or in spaces of matrices, instead of $\mathbf{R}$), defined on Lebesgue, Sobolev and BV spaces, have been considered in [19], [11], [13], [14], [20], [21] and [15] (see also [12] for a survey).

Wright, in his PhD Thesis [22] and subsequently in collaboration with Baryshnikov and Ghrist in [3], studied a rather different class, formed by the so-called *definable functions*. We cannot give here the details of the construction of these functions, but we mention that the main result of these works is a characterization of valuations as suitable integrals of intrinsic volumes of level sets. This type of integrals will have a crucial role in our paper, too.

The class of functions that we are considering is

$$\mathcal{C}^n = \{u : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\},\ u \text{ convex, l.s.c., } \lim_{|x|\to\infty} u(x) = \infty\}.$$

It includes the so-called indicatrix functions of convex bodies, i.e. functions of the form

$$I_K : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}, \quad I_K(x) = \begin{cases} 0 \text{ if } x \in K, \\ \infty \text{ if } x \notin K, \end{cases}$$

where K is a convex body. Note that the function $\boldsymbol{\infty}$, identically equal to ∞, belongs to $\mathcal{C}^n$. This element will play in some sense the role of the empty set. If $u \in \mathcal{C}^n$ we will denote by $\mathrm{dom}(u)$ the set where u is finite; if $u \neq \boldsymbol{\infty}$, this is a non-empty convex set, and then its dimension $\dim(\mathrm{dom}(u))$ is well defined and it is an integer between 0 and n.

We say that a functional $\mu : \mathcal{C}^n \to \mathbf{R}$ is a valuation if it verifies (1.3) for every $u, v \in \mathcal{C}^n$ such that $u \wedge v \in \mathcal{C}^n$ (note that $\mathcal{C}^n$ is closed under "$\vee$"), and $\mu(\boldsymbol{\infty}) = 0$. We are interested in valuations which are rigid motion invariant (i.e. $\mu(u) = \mu(u \circ T)$ for every $u \in \mathcal{C}^n$ and every rigid motion T of $\mathbf{R}^n$), and monotone decreasing (i.e. $\mu(u) \leq \mu(v)$ for every $u, v \in \mathbf{R}^n$ such that $u \geq v$ point-wise in $\mathbf{R}^n$). As $\mu(\boldsymbol{\infty}) = 0$, we immediately have that they are non-negative in $\mathcal{C}^n$. We will also need to introduce a notion of continuity of valuations. In this regard, note that for (rigid motion invariant) valuations on the space of convex bodies, continuity and monotonicity are conditions very close to each other. In particular monotonicity implies continuity, as Hadwiger's theorem shows. The situation on $\mathcal{C}^n$ is rather different. To the best of our knowledge, there are no standard topologies on the space of convex functions, moreover it is easy to provide examples of monotone valuations which are not continuous with respect to any reasonable notion of convergence on $\mathcal{C}^n$. In section 4 we introduce the following notion of continuity. We say that a valuation on $\mathcal{C}^n$ is *monotone continuous* if

$$\lim_{i\to\infty} \mu(u_i) = \mu(u)$$

whenever u_i, $i \in \mathbf{N}$, is a decreasing sequence in $\mathcal{C}^n$ converging to $u \in \mathcal{C}^n$ point-wise in the relative interior of $\mathrm{dom}(u)$, and such that $u_i \geq u$ in $\mathbf{R}^n$ for every i. A discussion concerning the advantages of this definition and the relations of the induced topology with other possible topologies on $\mathcal{C}^n$ is deferred to section 4. As monotone continuity is the only notion of continuity that we will use in this paper, for simplicity we will refer to it simply by continuity.

How does a "typical" valuation of this kind look like? A first answer is provided by functionals of type (1.3); indeed we will see in section 6 that if $f : \mathbf{R} \to \mathbf{R}$ is a non-negative decreasing function having finite $(n-1)$-st moment

$$\int_0^\infty f(t)t^{n-1}dt < \infty \tag{1.5}$$

then the functional

$$\mu(u) = \int_{\mathrm{dom}(u)} f(u(x))dx \tag{1.6}$$

is a rigid motion invariant, monotone decreasing valuation on $\mathcal{C}^n$, which is moreover continuous if f is right-continuous. A valuation of this form vanishes obviously on every function $u \in \mathcal{C}^n$ such that $\dim(\mathrm{dom}(u)) < n$:

$$\dim(\mathrm{dom}(u)) < n \quad \Rightarrow \quad \mu(u) = 0. \tag{1.7}$$

When μ has this property we will say that it is *simple*. Our first characterization result is the following theorem, proven in section 8.

Theorem 1.3. *A functional $\mu : \mathcal{C}^n \to \mathbf{R}$ is a rigid motion invariant, decreasing, continuous and simple valuation if and only if there exists a decreasing, right-continuous function $f : \mathbf{R} \to \mathbf{R}$ with finite $(n-1)$-st moment such that*

$$\mu(u) = \int_{\mathrm{dom}(u)} f(u(x))dx$$

for every $u \in \mathcal{C}^n$.

The proof of this fact is based on a rather simple idea, even if there are several technical points to transform it into a rigorous argument. First, μ determines the function f as follows: for $t \in \mathbf{R}$ let $\sigma_t : \mathcal{K}^n \to \mathbf{R}$ be defined as

$$\sigma_t(K) = \mu(t + I_K).$$

It is straightforward to check that this is a rigid motion and monotone increasing valuation, so that by the volume theorem there exists a constant, which will depend on t and which we call $f(t)$, such that

$$\mu(t + I_k) = \sigma_t(K) = f(t)V_n(K) \quad \forall\, K \in \mathcal{K}^n, \forall\, t \in \mathbf{R}.$$

As μ is decreasing, f is decreasing too. Now for every $u \in \mathcal{C}^n$ and $K \subseteq \mathrm{dom}(u)$, by monotonicity we obtain

$$f(\max_K u)V_n(K) = \mu(\max_K u + I_K) \le \mu(u + I_k) \le f(\min_K u)V_n(K) = \mu(\min_K u + I_K).$$

This chain of inequalities, and the fact that μ is simple, permit to compare easily the value of $\mu(u)$ with upper and lower Riemann sums of $f \circ u$, over suitable partitions of subsets of $\mathrm{dom}(u)$. This leads to the proof of (1.6). Monotonicity is an essential ingredient of this argument. It would be very interesting to obtain a similar characterization of simple valuations without this assumption.

If we apply the layer cake (or Cavalieri) principle, we obtain a second way of writing the valuation μ defined by (1.6):

$$\mu(u) = \int_{\mathbf{R}} V_n(\mathrm{cl}(\{u < t\}))d\nu(t) \tag{1.8}$$

where V_n is the n-dimensional volume, "cl" is the closure, and ν is a Radon measure on $\mathbf{R}$ defined by

$$f(t) = \nu((t, \infty)) \quad \forall\, t \in \mathbf{R}. \tag{1.9}$$

Note that the set $\mathrm{cl}(\{u < t\})$ is a compact convex set, i.e. a convex body. Formula (1.8) suggests to consider the more general expression

$$\mu(u) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\}))d\nu(t) \tag{1.10}$$

where, for $k \in \{0, \ldots, n\}$, V_k is the k-th intrinsic volume and ν is a Radon measure on $\mathbf{R}$. We will see (still in section 6) that the integral in (1.10) is finite for every $u \in \mathcal{C}^n$ if and only if

$$\int_0^\infty t^k d\nu(t) < \infty \tag{1.11}$$

(which is equivalent to (1.5) when $k = n$) and in this case it defines a rigid motion invariant, decreasing and continuous valuation on $\mathcal{C}^n$. Valuations of type (1.10) are homogeneous of order k in the following sense. For $u \in \mathcal{C}^n$ and $\lambda > 0$, let $u_\lambda : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ be defined by

$$u_\lambda(x) = u\left(\frac{x}{\lambda}\right) \quad \forall\, x \in \mathbf{R}^n$$

(note that $u_\lambda \in \mathcal{C}^n$). Then

$$\mu(u_\lambda) = \lambda^k \mu(u).$$

In section 9 we will prove the following fact.

Theorem 1.4. *A functional $\mu : \mathcal{C}^n \to \mathbf{R}$ is a non-trivial, rigid motion invariant, decreasing, continuous and k-homogeneous valuation if and only if $k \in \{0, 1, \ldots, n\}$ and there exists a Radon measure ν on $\mathbf{R}$ with finite k-th moment such that*

$$\mu(u) = \int_{\mathbf{R}} V_k(\mathrm{cl}\{u < t\})d\nu(t)$$

for every $u \in \mathcal{C}^n$.

Theorems 1.3 and 1.4 may suggest that valuations of type (1.10) could form a sort of generators for invariant, monotone and continuous valuations on $\mathcal{C}^n$, playing a similar role to intrinsic volumes for convex bodies (with the difference that the dimension of the space of valuations on $\mathcal{C}^n$ is infinite). On the other hand in the conclusive section of the paper we show the existence of valuations on $\mathcal{C}^n$ with the above properties, which cannot be decomposed as the sum of homogeneous valuations, and hence are not the sum of valuations of the form (1.10).

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2 Preliminaries

We work in the n-dimensional Euclidean space $\mathbf{R}^n$, $n \geq 1$, endowed with the usual Euclidean norm $|\cdot|$ and scalar product $(\cdot,\cdot)$. For $x_0 \in \mathbf{R}^n$ and $r > 0$, $B_r(x_0)$ denotes the closed ball centred at x_0 with radius r; when $x_0 = 0$ we simply write B_r. For $k \in [0,n]$, the k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. In particular $\mathcal{H}^n$ denotes the Lebesgue measure in $\mathbf{R}^n$ (which, as we said, will be often written as V_n, especially when referred to convex bodies). Integration with respect to such measure will be always denoted simply by dx, where x is the integration variable. Given a subset A of $\mathbf{R}^n$ we denote by $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ its interior and its closure, respectively.

As usual, we will denote by $\mathbf{O}(n)$ and $\mathbf{SO}(n)$ respectively, the group of rotations and of proper rotations of $\mathbf{R}^n$. By a *rigid motion* we mean the composition of a rotation and a translation, i.e. a mapping $T : \mathbf{R}^n \to \mathbf{R}^n$ such that there exist $R \in \mathbf{O}(n)$ and $x_0 \in \mathbf{R}^n$ for which

$$T(x) = R(x) + x_0, \quad \forall\, x \in \mathbf{R}^n.$$

2.1 Convex bodies

A *convex body* is a compact convex subset of $\mathbf{R}^n$. We will denote by $\mathcal{K}^n$ the family of convex bodies in $\mathbf{R}^n$. For all the notions and results concerning convex bodies we refer to the monograph [17]. The set $\mathcal{K}^n$ can be endowed with a metric, induced by the Hausdorff distance (see [17] for the definition).

Let $K \in \mathcal{K}^n$; if $\mathrm{int}(K) = \emptyset$, then K is contained in some k-dimensional affine sub-space of $\mathbf{R}^n$, with $k < n$; the smallest k for which this is possible is called the dimension of K, and is denoted by $\dim(K)$. Clearly, if K has non-empty interior we set $\dim(K) = n$. Using this notion we can define the relative interior of K as the subset of those points x of K for which there exists a k-dimensional ball centred at x and contained in K, where $k = \dim(K)$. The relative interior will be denoted by $\mathrm{relint}(K)$. The notion of relative interior can be given in the same way for every convex subset of $\mathbf{R}^n$.

$\mathcal{K}^n$ can be naturally equipped with an addition (Minkowski, or vector, addition) and a multiplication by non-negative reals. Given $K, L \in \mathcal{K}^n$ and $s \geq 0$ we set

$$K + L = \{x + y \,:\, x \in K, \quad y \in L\}$$

and

$$sK = \{sx \,:\, x \in K\}.$$

$\mathcal{K}^n$ is closed with respect to these operations.

To every convex body $K \in \mathcal{K}^n$ we may assign a sequence of $(n+1)$ numbers, $V_k(K)$, $k = 0, \dots, n$, called the *intrinsic volumes* of K; for their definition see [17, Chapter 4]. We recall in particular that $V_n(K)$ is the volume, i.e. the Lebesgue measure, of K, while $V_0(K) = 1$ for every $K \in \mathcal{K}^n \setminus \{\emptyset\}$. More generally, if K is a convex body in $\mathbf{R}^n$ having dimension $k \in \{0, 1, \dots, n\}$, then $V_k(K)$ is the k-dimensional Lebesgue measure of K as a subset of $\mathbf{R}^k$. As real-valued functionals defined on $\mathcal{K}^n$, intrinsic volumes are continuous, monotone increasing with respect to set inclusion and invariant under the action of rigid motions: $V_i(T(K)) = V_i(K)$ for every $i \in \{0, \dots, n\}$, $K \in \mathcal{K}^n$ and for every rigid motion

T. Moreover, the intrinsic volumes are special and important examples of valuations on the space of convex bodies. The characterization theorem of Hadwiger (Theorem 1.1) will be a crucial tool in this paper.

3 The space $\mathcal{C}^n$

Let us a consider a function $u : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$, which is *convex*. We denote the so-called *domain* of u as

$$\mathrm{dom}(u) = \{x \in \mathbf{R}^n : u(x) < \infty\}.$$

By the convexity of u, $\mathrm{dom}(u)$ is a convex set. By standard properties of convex functions, u is continuous in the interior of $\mathrm{dom}(u)$ and it is Lipschitz continuous in any compact subset of $\mathrm{int}(\mathrm{dom}(u))$.

In this work we focus in particular on the following space of convex functions:

$$\mathcal{C}^n = \{u : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\},\ u \text{ convex, l.s.c., } \lim_{|x|\to\infty} u(x) = \infty\}. \tag{3.12}$$

Here by l.s.c. we mean *lower semi-continuous*, i.e.

$$\liminf_{x\to x_0} u(x) \ge u(x_0) \quad \forall x_0 \in \mathbf{R}^n.$$

Note that the function $\boldsymbol{\infty}$ (which, we recall, is identically equal to ∞ on $\mathbf{R}^n$) belongs to our functions space. As it will be clear in the sequel, this special function plays the role that the empty set has for valuations defined on families of sets (instead of functions).

Remark 3.1. Let $u \in \mathcal{C}^n$. As a consequence of convexity and the behavior at infinity we have that

$$\inf_{\mathbf{R}^n} u > -\infty.$$

Moreover, by the lower semi-continuity, u admits a minimum in $\mathbf{R}^n$. We will often use the notation

$$m(u) = \min_{\mathbf{R}^n} u\,.$$

Let $A \subseteq \mathbf{R}^n$; we denote by $I_A : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ the so-called *indicatrix function* of A, which is defined by

$$I_A = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{if } x \notin A. \end{cases}$$

If $K \subset \mathbf{R}^n$ is a convex body, then $I_K \in \mathcal{C}^n$.

Sub-level sets of functions belonging to $\mathcal{C}^n$ will be of fundamental importance in this paper. Given $u \in \mathcal{C}^n$ and $t \in \mathbf{R}$ we set

$$K_t := \{u \le t\} = \{x \in \mathbf{R}^n : u(x) \le t\},$$

and

$$\Omega_t := \{u < t\} = \{x \in \mathbf{R}^n : u(x) < t\}.$$

Both sets are empty for $t < m(u)$. K_t is a convex body for all $t \in \mathbf{R}$, by the properties of u. For all real t, Ω_t is a bounded (possibly empty) convex set, so that its closure $\mathrm{cl}(\Omega_t)$ is a convex body, obviously contained in K_t.

Lemma 3.2. *Let $u \in \mathcal{C}^n$; for every $t > m(u)$*

$$\mathrm{relint}(K_t) \subseteq \Omega_t.$$

Proof. We start by considering the case in which $\dim(K_t) = n$. Assume by contradiction that there exists a point $x \in \mathrm{int}(K_t)$ such that $u(x) = t$. Then x is a local maximum for u but, by convexity, this is possible only if $u \equiv t$ in K_t, which, in turn implies that $t = m(u)$, a contradiction.

If $\dim(K_t) = k < n$ then, by convexity, $\mathrm{dom}(u)$ is contained in a k-dimensional affine subspace H of $\mathbf{R}^n$, and we can apply the previous argument to u restricted to H to deduce the assert of the lemma. □

Corollary 3.3. *Let $u \in \mathcal{C}^n$; for every $t > m(u)$*

$$\mathrm{cl}(\Omega_t) = K_t.$$

3.1 On the intrinsic volumes of sub-level sets

As we have just seen, if $u \in \mathcal{C}^n$ and $t \in \mathbf{R}$, the set

$$\Omega_t = \{u < t\}$$

is empty for $t \leq m(u)$ and it is a bounded convex set for $t > m(u)$. For $k \in \{0, \dots, n\}$, we define the function $v_k(u;\cdot) : \mathbf{R} \to \mathbf{R}$ as follows

$$v_k(u;t) = V_k(\mathrm{cl}(\Omega_t)).$$

As intrinsic volumes are non-negative and monotone with respect to set inclusion and the set Ω_t is increasing with respect to inclusion as t increases, $v_k(u;\cdot)$ is a non-negative increasing function. In particular it is a function of bounded variation, so that there exists a (non-negative) Radon measure on $\mathbf{R}$, that we will denote by $\beta_k(u;\cdot)$, which represents the weak, or distributional, derivative of v_k (see for instance [2]).

We want to describe in a more detailed way the structure of the measure β_k. In general, the measure representing the weak derivative of a non-decreasing function consists of three parts: a jump part, a Cantor like part and an absolutely continuous part (with respect to Lebesgue measure). We will see that β_k does not have a Cantor part and its jump part, if any, is a single Dirac delta at $m(u)$.

As a starting point, note that as v_i is identically zero in $(-\infty, m(u)]$ then $\beta_k(u;\eta) = 0$ for every measurable set $\eta \subseteq (-\infty, m(u))$. On the other hand, in $(m(u), \infty)$, due to the Brunn-Minkowski inequality for intrinsic volumes, the functions v_k have a more regular behavior than than a general non-decreasing function. Indeed, for $k \geq 1$, let $t_0, t_1 \in (m(u), \infty)$ and consider, for $\lambda \in [0,1]$, $t_\lambda = (1-\lambda)t_0 + \lambda t_1$. Then we have the set inclusion

$$K_{t_\lambda} \supseteq (1-\lambda)K_{t_0} + \lambda K_{t_1},$$

which follows from the convexity of u. By the monotonicity of intrinsic volumes and the Brunn-Minkowski inequality for such functionals (see [17, Chapter 7]), and by Corollary

3.3, we have

$$\begin{aligned} v_k(u;t_\lambda) &= V_k(K_{t_\lambda}) \geq V_k((1-\lambda)K_{t_0} + \lambda K_{t_1}) \\ &\geq [(1-\lambda)V_k(K_{t_0})^{1/k} + \lambda V_k(K_{t_1})^{1/k}]^k \\ &= [(1-\lambda)v_k(u;t_0)^{1/k} + \lambda v_k(u;t_1)^{1/k}]^k. \end{aligned}$$

In other words, the function v_k to the power $1/k$ is concave in $(m(u),\infty)$. This implies in particular that v_k is absolutely continuous in $(m(u),\infty)$ so that the measure β_k is absolutely continuous with respect to the Lebesgue measure in this interval, and its density is given by the point-wise derivative of v_k, which exists a.e. (see [2, Chapter 3]). Next we examine the behavior at $m(u)$; as v_k is constantly zero in $(-\infty, m(u)]$

$$\lim_{t\to m(u)^-} v_k(u;t) = 0$$

(in particular v_k is left-continuous at $m(u)$). On the other hand let t_i, $i \in \mathbf{N}$, be a decreasing sequence converging to $m(u)$, with $t_i > m(u)$ for every i, and consider the corresponding sequence of convex bodies $L_i = \mathrm{cl}(\Omega_{t_i})$, $i \in \mathbf{N}$. This is a decreasing sequence and $L_i \supseteq K_{m(u)}$ for every i. Moreover, trivially

$$K_{m(u)} = \bigcap_{i\in\mathbf{N}} L_i.$$

This implies in particular that $K_{m(u)}$ is the limit of the sequence L_i with respect to the Hausdorff metric (see [17, Section 1.8]). Then

$$\lim_{i\to\infty} V_k(L_i) = V_k(K_{m(u)})$$

so that

$$\lim_{t\to m(u)^+} v_k(u;t) = V_k(K_{m(u)}) = V_k(\{u = m(u)\}).$$

If

$$V_k(K_{m(u)}) > 0$$

then $v_k(u;t)$ has a jump discontinuity at $m(u)$ of amplitude $V_k(K_{m(u})$. In other words

$$\beta_k(u;\{m(u)\}) = V_k(K_{m(u)}).$$

The case $k=0$ can be treated as follows: as $V_0(K)$ is the Euler characteristic of K for every K, i.e. is constantly 1 on $\mathcal{K}^n \setminus \{\emptyset\}$, $v_0(u;t)$ equals 0 for $t \leq m(u)$ and equals 1 for $t > m(u)$; hence β_0 is just the Dirac point mass measure concentrated at $m(u)$.

The following statement collects the facts that we have proven so far in this part.

Proposition 3.4. *Let $u \in \mathcal{C}^n$ and $k \in \{0,\ldots,n\}$; let K_t, v_k be defined as before. Define the measure β_k as*

$$\beta_k(u;\cdot) = V_k(K_{m(u)})\delta_{m(u)}(\cdot) + \frac{dv_k}{dt}\mathcal{H}^1(\cdot),$$

where δ denotes the Dirac point-mass measure (and $\mathcal{H}^1$ is the Lebesgue measure on $\mathbf{R}$). Then $\beta_k(u;\cdot)$ is the distributional derivative of v_k, more precisely

$$v_k(u;t) = \beta_k(u;(m(u),t]) \quad \forall\, t \geq m(u) \quad \text{and } v_k(u;t) = 0\ \forall t \leq m(u).$$

In particular $v_k(u;\cdot)$ is left-continuous at $m(u)$.

3.2 Max and min operations in $\mathcal{C}^n$

As we will see, the definition of valuations on $\mathcal{C}^n$ is based on the point-wise minimum and maximum of convex functions. This part is devoted to some basic properties of these operations.

Given u and v in $\mathcal{C}^n$ we set, for $x \in \mathbf{R}^n$,

$$(u \vee v)(x) = \max\{u(x), v(x)\} = u(x) \vee v(x), \quad (u \wedge v)(x) = \min\{u(x), v(x)\} = u(x) \wedge v(x).$$

Hence $u \vee v$ and $u \wedge v$ are functions defined in $\mathbf{R}^n$, with values in $\mathbf{R} \cup \{\infty\}$.

Remark 3.5. If $u, v \in \mathcal{C}^n$ then $u \vee v$ belongs to $\mathcal{C}^n$ as well. Indeed convexity and behavior at infinity are straightforward. Concerning lower semicontinuity of $u \vee v$, this is equivalent to saying that $\{u \vee v \leq t\}$ is closed for every $t \in \mathbf{R}$, which follows immediately from the equality

$$\{u \vee v \leq t\} = \{u \leq t\} \cup \{v \leq t\}.$$

However, $u, v \in \mathcal{C}^n$ does not imply, in general, that $u \wedge v \in \mathcal{C}^n$ (a counterexample is given by the indicatrix functions of two disjoint convex bodies).

For $u, v \in \mathcal{C}^n$ and $t \in \mathbf{R}$ the following relations are straightforward:

$$\{u \leq t\} \cap \{v \leq t\} = \{u \vee v \leq t\}, \quad \{u \leq t\} \cup \{v \leq t\} = \{u \wedge v \leq t\}; \tag{3.13}$$

$$\{u < t\} \cap \{v < t\} = \{u \vee v < t\}, \quad \{u < t\} \cup \{v < t\} = \{u \wedge v < t\}. \tag{3.14}$$

In the sequel we will also need the following result.

Proposition 3.6. *Let $u, v \in \mathcal{C}^n$ be such that $u \wedge v \in \mathcal{C}^n$. Then, for every $t \in \mathbf{R}$,*

$$\mathrm{cl}(\{u < t\}) \cap \mathrm{cl}(\{v < t\}) = \mathrm{cl}(\{u \vee v < t\}), \tag{3.15}$$

$$\mathrm{cl}(\{u < t\}) \cup \mathrm{cl}(\{v < t\}) = \mathrm{cl}(\{u \wedge v < t\}). \tag{3.16}$$

Proof. Equality (3.16) comes directly from the second equality in (3.14), passing to the closures of the involved sets. As for the proof of (3.15), we first observe that $u \wedge v \in \mathcal{C}^n$ implies

$$m(u \vee v) = m(u) \vee m(v)$$

(see the next lemma). Let $t > m(u \vee v)$; then, by Corollary 3.3 and (3.13):

$$\mathrm{cl}(\{u < t\}) \cap \mathrm{cl}(\{v < t\}) = \{u \leq t\} \cap \{v \leq t\} = \{u \vee v \leq t\} = \mathrm{cl}(\{u \vee v < t\}).$$

If we assume that $t \leq m(u \vee v)$, then we have $t \leq m(u)$ or $t \leq m(v)$. In the first case $\{u < t\} = \emptyset$, so that the left hand-side of (3.15) is empty. On the other hand $u \vee v \leq u$ implies that the right hand-side is empty as well. The case $t \leq m(v)$ is completely analogous. □

Lemma 3.7. *If $u, v \in \mathcal{C}^n$ are such that $u \wedge v \in \mathcal{C}^n$, then*

$$m(u \vee v) = m(u) \vee m(v).$$

Proof. The inequality $m(u \vee v) \geq m(u) \vee m(v)$ is obvious. To prove the reverse inequality, let $t \geq m(u) \vee m(v)$; hence

$$\{u \leq t\} \neq \emptyset, \quad \{v \leq t\} \neq \emptyset$$

and

$$\{u \leq t\} \cup \{v \leq t\} = \{u \wedge v \leq t\} \in \mathcal{K}^n,$$

where the last relation comes from the assumption $u \wedge v \in \mathcal{C}^n$. Hence $\{u \leq t\}$ and $\{v \leq t\}$ are non-empty convex bodies such that their union is also a convex body. This implies that they must have a non-empty intersection. But then

$$\{u \vee v \leq t\} = \{u \leq t\} \cap \{v \leq t\} \neq \emptyset,$$

i.e. $m(u \vee v) \leq t$. □

We conclude this section with a proposition (see [5, Lemma 2.5]) which will be frequently used throughout the paper.

Proposition 3.8. *If $u \in \mathcal{C}^n$ there exist two real numbers a and b, with $a > 0$, such that*

$$u(x) \geq a|x| + b \text{ for every } x \in \mathbf{R}^n.$$

4 Valuations on $\mathcal{C}^n$

Definition 4.1. *A valuation on $\mathcal{C}^n$ is a map $\mu : \mathcal{C}^n \to \mathbf{R}$ such that $\mu(\infty) = 0$ and*

$$\mu(u \vee v) + \mu(u \wedge v) = \mu(u) + \mu(v)$$

for every $u, v \in \mathcal{C}^n$ such that $u \wedge v \in \mathcal{C}^n$.

A valuation μ is called:

- ***rigid motion invariant**, if $\mu(u) = \mu(u \circ T)$ for every $u \in \mathcal{C}^n$ and for every rigid motion T;*
- ***monotone decreasing** (or just monotone), if $\mu(u) \leq \mu(v)$ whenever $u, v \in \mathcal{C}^n$ and $u \geq v$ point-wise in $\mathbf{R}^n$;*
- ***k-simple** ($k \in \{1, \dots, n\}$) if $\mu(u) = 0$ for every $u \in \mathcal{C}^n$ such that $\dim(\mathrm{dom}(u)) < k$;*
- ***simple**, if μ is n-simple, i.e. if $\mu(u) = 0$ for every u such that $\mathrm{dom}(u)$ has no interior points.*

The following simple observation will turn out to be very important.

Remark 4.2. Every monotone decreasing valuation μ on $\mathcal{C}^n$ is non-negative. If we set $\infty(x) = \infty$ for all $x \in \mathbf{R}^n$, then $u \leq \infty$ holds for each $u \in \mathcal{C}^n$, which in turn leads to $\mu(u) \geq \mu(\infty) = 0$ by monotonicity.

In the sequel other features of valuations will be considered, like *monotone-continuity* and homogeneity. Concerning the latter, the definition is the following.

Definition 4.3. *Let μ be valuation on $\mathcal{C}^n$ and let $\alpha \in \mathbf{R}$; we say that μ is positively homogeneous of order α, or simply α-homogeneous, if for every $u \in \mathcal{C}^n$ and every $\lambda > 0$ we have*

$$\mu(u_\lambda) = \lambda^\alpha \mu(u)$$

where $u_\lambda : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ is defined by

$$u_\lambda(x) = u\left(\frac{x}{\lambda}\right) \quad \forall\, x \in \mathbf{R}^n$$

(note that $u \in \mathcal{C}^n$ implies $u_\lambda \in \mathcal{C}^n$).

Remark 4.4. Other definitions of homogeneous valuations are possible. For instance one could consider valuations for which there exists $\alpha \in \mathbf{R}$ such that

$$\mu(\lambda u) = \lambda^\alpha \mu(u), \quad \forall\, u \in \mathcal{C}^n, \forall\, \lambda > 0.$$

This corresponds to homogeneity with respect to a vertical stretching of the graph of u, while Definition 4.3 involves a horizontal stretching. In addition, one could consider a more general type of homogeneity where both types of dilations (vertical and horizontal) are simultaneously in action. Definition 4.3 is more natural from the point of view of convex bodies. Indeed, if $u = I_K$ with $K \in \mathcal{K}^n$, then u_λ is the indicatrix function of the dilated body λK.

The next one is the definition of monotone-continuous valuations.

Definition 4.5. *Let μ be a valuation on $\mathcal{C}^n$; μ is called* monotone-continuous*, or simply* continuous*, if the following property is verified: given a sequence $u_i \in \mathcal{C}^n$, $i \in \mathbf{N}$, and $u \in \mathcal{C}^n$, such that:*

$$u_i \geq u_{i+1} \geq u \quad \text{in } \mathbf{R}^n, \text{ for every } i \in \mathbf{N},$$

and

$$\lim_{i\to\infty} u_i(x) = u(x) \quad \forall\, x \in \text{relint}(\text{dom}(u))$$

we have

$$\lim_{i\to\infty} \mu(u_i) = \mu(u).$$

We recall that “relint” denotes the relative interior of a convex set (see the definition given in section 2.1).

As mentioned in the introduction, we are not aware of any standard notion of convergence on $\mathcal{C}^n$ comparable, for instance with the one induced by the Hausdorff distance for convex bodies. As we will see in the rest of this paper, the notion of monotone continuity has several advantages when it is used jointly with the monotonicity of a valuation. The following two remarks aim to shed some light on two other possible notions of continuity.

Remark 4.6. A natural choice of convergence on $\mathcal{C}^n$ would be the point-wise convergence. On the other hand notice that this would be too restrictive; indeed the functional

$$\mu(u) = \int_{\mathrm{dom}(u)} e^{-u} dx \tag{4.17}$$

is not continuous with respect to this convergence. On the other hand, as we will extensively see in section 6, (4.17) is a sort of paradigmatic example of a class of valuations, i.e. integral valuations, which are of crucial importance in our characterization theorems.

An example showing that (4.17) is not continuous with respect to point-wise convergence, for $n = 2$, can be constructed as follows. Let (x, y) denote a point in $\mathbf{R}^2$. For $r > 0$ we consider the strip

$$S_r = \mathbf{R} \times [-r, r]$$

and the function $g_{\alpha,r} : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$v_r(x, y) = r|x| + I_{S_r}(x, y)$$

(remember that I denotes the indicatrix function); note that $v \in \mathcal{C}^n$. In particular we will choose $r = i \in \mathbf{N}$, which will generate a sequence v_i of elements in $\mathcal{C}^n$. For $i \in \mathbf{N}$, let $T_i : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation defined by

$$T_i(x, y) = \left(x \cos \frac{1}{\sqrt{i}} + y \sin \frac{1}{\sqrt{i}}, -x \sin \frac{1}{\sqrt{i}} + y \cos \frac{1}{\sqrt{i}} \right).$$

Finally we define $u_i : \mathbf{R}^2 \to \mathbf{R}$ by

$$u_i = v_i \circ T_i.$$

More explicitly:

$$u_i(x, y) = i \, |x \cos \frac{1}{\sqrt{i}} + y \sin \frac{1}{\sqrt{i}}| + I_{T_i^{-1}(S_i)}(x, y).$$

It is easy to check that, for every $(x, y) \in \mathbf{R}^2$

$$\lim_{i \to \infty} u_i(x, y) = u(x, y),$$

where

$$u(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ +\infty & \text{if } (x, y) \neq (0, 0). \end{cases}$$

For every $i \in \mathbf{N}$ we have

$$\int_{\mathbf{R}^2} e^{-u_i} dxdy \;\; = \;\; \int_{\mathbf{R}^2} e^{-v_i} dxdy = \int_{\mathbf{R}} e^{-i|x|} dx \; \int_{-i}^{i} dy = 4.$$

In particular

$$4 = \lim_{i \to \infty} \int_{\mathbf{R}^2} e^{u_i} dxdy \neq \int_{\mathbf{R}^2} e^{-u} dxdy = 0.$$

Remark 4.7. A consequence of Proposition 3.8 is that

$$\int_{\mathrm{dom}(u)} e^{-u}dx < \infty \quad \forall\, u \in \mathcal{C}^n.$$

Hence $\mathcal{C}^n$ can be viewed as a subset of $L^1(\mathbf{R}^n)$ if each of its element u is identified with e^{-u}. This observation gives another option for a topology on $\mathcal{C}^n$: a sequence u_i, $i \in \mathbf{N}$, in $\mathcal{C}^n$ converges to $u \in \mathcal{C}^n$ if $e^{-u_i} \to u$ in $L^1(\mathbf{R}^n)$. Exploiting the monotone convergence theorem, it is easy to see that continuity with respect to this topology implies monotone continuity.

Remark 4.8. Let K_i, $i \in \mathbf{N}$, be a sequence converging to a convex body K in the Hausdorff metric. Assume moreover that the sequence is monotone increasing:

$$K_i \subseteq K_{i+1} \subseteq K \quad \forall\, i \in \mathbf{N}.$$

Then the corresponding sequence of indicatrix functions I_{K_i}, $i \in \mathbf{N}$, is decreasing, it verifies $I_{K_i} \geq I_K$ point-wise in $\mathbf{R}^n$, for every i, and

$$\lim_{i\to\infty} I_{K_i}(x) = I_K(x) \quad \forall\, x \in \mathrm{relint}(K).$$

Hence, if μ is a continuous valuation on $\mathcal{C}^n$,

$$\lim_{i\to\infty} \mu(I_{K_i}) = \mu(I_K).$$

5 Geometric densities

Throughout this section, μ will be a rigid motion invariant and monotone decreasing valuation on $\mathcal{C}^n$.

Let $t \in \mathbf{R}$ be fixed, and consider the following function σ_t defined on $\mathcal{K}^n$:

$$\sigma_t \,:\, \mathcal{K}^n \to \mathbf{R}, \quad \sigma_t(K) = \mu(t + I_K) \quad \forall\, K \in \mathcal{K}^n.$$

It is straightforward to check that σ_t is a rigid motion invariant valuation on $\mathcal{K}^n$. Moreover, if $K \subseteq L$, then $I_K \geq I_L$, so that $\sigma_t(K) \leq \sigma_t(L)$, i.e. σ_t is monotone increasing. By Theorem 1.1 there exist $(n+1)$ non-negative coefficients, that we will denote by $f_0, f_1, \dots, f_n$, such that

$$\sigma_t(K) = \sum_{k=0}^{n} f_k\, V_k(K) \quad \forall\, K \in \mathcal{K}^n.$$

The numbers f_k's clearly depend on t, i.e. they are real-valued functions defined on $\mathbf{R}$; we will refer to these functions as the *geometric densities* of μ.

We prove that the monotonicity of μ implies that these functions are monotone decreasing. Fix $k \in \{0, \dots, n\}$ and let B^k be a closed k-dimensional ball of radius 1 in $\mathbf{R}^n$; note that

$$V_j(B^k) = 0 \quad \forall\, j = k+1, \dots, n,$$

while

$$V_k(B^k) =: \kappa_k = (\text{the } k\text{-dimensional volume of } B^k) > 0.$$

Fix $r \geq 0$; V_j is positively homogeneous of order j, hence for every $t \in \mathbf{R}$ we have

$$\mu(t + I_{rB^k}) = \sum_{j=1}^{k} r^j V_j(B^k) f_j(t).$$

Hence we get

$$f_k(t) = V_k(B^k) \cdot \lim_{r\to\infty} \frac{\mu(t + I_{rB^k})}{r^k}.$$

On the other hand, as μ is decreasing, the function $t \to \mu(t + I_{rB^k})$ is decreasing for every $r \geq 0$; this proves that f_k is decreasing.

Proposition 5.1. *Let μ be a rigid motion invariant and decreasing valuation defined on $\mathcal{C}^n$. Then there exists $(n+1)$ functions $f_0, f_1, \dots, f_n$, defined on $\mathbf{R}$, non-negative and decreasing, such that for every convex body $K \in \mathcal{K}^n$ and for every $t \in \mathbf{R}$*

$$\mu(t + I_K) = \sum_{k=0}^{n} f_k(t) V_k(K). \tag{5.18}$$

If in addition to the previous assumption the valuation μ is continuous, then all its geometric densities are *right-continuous*, i.e.

$$\lim_{t\to t_0^+} f_i(t) = f_i(t_0) \quad \forall t_0 \in \mathbf{R},\ i \in \{0, \dots, n\}.$$

Indeed, for every convex body K the function

$$t \mapsto \mu(t + I_K)$$

is right-continuous, by the definition of continuity. If we chose $K = \{0\}$, as $V_k(K) = 0$ for $k \geq 1$ and $V_0(K) = 1$, we have, by (5.18)

$$\mu(t + I_K) = f_0(t) \quad \forall\, t \in \mathbf{R}.$$

This proves that f_0 is right-continuous. If we now take K to be a one-dimensional convex body, such that $V_1(K) = 1$, we have that $V_k(K) = 0$ for every $k \geq 2$, hence

$$\mu(t + I_K) = f_0(t) + f_1(t) \quad \forall\, t \in \mathbf{R}.$$

As the left hand-side is right-continuous and f_0 is also right-continuous (by the previous step) then f_1 must have the same property. Proceeding in a similar way we obtain that each f_k is right-continuous.

Proposition 5.2. *Let μ be a rigid motion invariant, monotone and continuous valuation on $\mathcal{C}^n$. Then its geometric densities f_i, $i \in \{1, \dots, n\}$, are right-continuous in $\mathbf{R}$.*

Assume now that μ is positively homogeneous of some order α; then it is readily checked that for every $t \in \mathbf{R}$ the valuation σ_t defined at the beginning of this section is positively homogeneous of the same order, i.e.

$$\sigma_t(s \cdot K) = s^\alpha \sigma_t(K) \quad \forall\, K \in \mathcal{K}^n,\, s \geq 0.$$

On the other hand, each σ_t is a linear combination of the intrinsic volumes V_k's, and V_k is positively homogenous of order k. We are led to the following conclusion.

Corollary 5.3. *Let μ be a rigid motion invariant and monotone decreasing valuation on $\mathcal{C}^n$ and assume that it is α-homogeneous for some $\alpha \in \mathbf{R}$. Then necessarily $\alpha \in \{0, 1, \dots, n\}$ and $f_k \equiv 0$ for every $k \neq \alpha$.*

We are in position to prove a characterization result for 0-homogeneous valuations which are also monotone and continuous. We recall that, for $u \in \mathcal{C}^n$, $m(u)$ is the minimum of u on $\mathbf{R}^n$.

Proposition 5.4. *Let μ be a rigid motion invariant, monotone decreasing and continuous valuation on $\mathcal{C}^n$ and assume that it is 0-homogeneous. Then, for every $u \in \mathcal{C}^n$ we have*

$$\mu(u) = f_0(m(u)).$$

Proof. We first prove the claim of this proposition under the additional assumption that $\mathrm{dom}(u)$ is bounded; let K be a convex body containing $\mathrm{dom}(u)$. Moreover, let $x_0 \in \mathbf{R}^n$ be such that $u(x_0) = m(u)$. Then

$$m(u) + I_K(x) \leq u(x) \leq m(u) + I_{\{x_0\}}(x) \quad \text{for every } x \text{ in } \mathbf{R}^n.$$

As μ is monotone decreasing

$$\mu(m(u) + I_K) \geq \mu(u) \geq \mu(m(u) + I_{\{x_0\}}).$$

On the other hand, by Poposition 5.1 and Corollary 5.3 we have

$$\mu(m(u) + I_{\{x_0\}}) = \mu(m(u) + I_K) = f_0(m(u)),$$

hence $\mu(u) = f_0(m(u))$. To extend the result to the general case, for $u \in \mathcal{C}^n$ and $i \in \mathbf{N}$ let

$$u_i = u + I_{B_i}.$$

The sequence u_i is contained in $\mathcal{C}^n$, is monotone decreasing and converges point-wise to u in $\mathbf{R}^n$. As μ is continuous we have, by the previous part of the proof,

$$\mu(u) = \lim_{i\to\infty} \mu(u_i) = \lim_{i\to\infty} f_0(m(u_i)).$$

On the other hand, as $m(u) = \min_{\mathbf{R}^n} u$, by the point wise convergence we have that for i sufficiently large $m(u_i) = m(u)$. □

Another special case in which more information can be deduced on geometric densities, is when the valuation μ is simple.

Proposition 5.5. *Let μ be a rigid motion invariant, monotone decreasing and simple valuation on $\mathcal{C}^n$. Then, for each $k \in \{0, \dots, n-1\}$, the k-th geometric density f_k of μ is identically zero.*

Proof. Fix $t \in \mathbf{R}$; the valuation $\sigma_t : \mathcal{K}^n \to \mathbf{R}$ defined by

$$\sigma_t = \mu(t + I_K)$$

is monotone, rigid motion invariant and simple; the volume theorem (Corollary 1.2) and the definition of geometric densities yield

$$\sigma_t = f_n(t) V_n.$$

In other words, $f_k(t) = 0$ for every $k = 0, \dots, n-1$ and $t \in \mathbf{R}$. □

The following result relates homogeneity and simplicity, and its proof makes use of geometric densities.

Proposition 5.6. *Let $\mu : \mathcal{C}^n \to \mathbf{R}$ be a valuation with the following properties:*

- *μ is rigid motion invariant;*
- *μ is monotone decreasing;*
- *μ is k-homogeneous;*
- *μ is continuous.*

Then μ is k-simple.

Proof. Let $f_0, f_1, \dots, f_n$ be the geometric densities of μ. As μ is k-homogeneous, $f_i \equiv 0$ for every $i \neq k$. Let $u \in \mathcal{C}^n$ be such that $\dim(\mathrm{dom}(u)) < k$. For $i \in \mathbf{N}$ let $Q_i = [-i, i]^n = [-i, i] \times \cdots \times [-i, i]$, and set

$$u_i = u + I_{Q_i}.$$

Clearly $\mathrm{dom}(u_i) = \mathrm{dom}(u) \cap Q_i$; in particular $\dim(\mathrm{dom}(u_i)) < k$ for every i. Let

$$m_i = m(u_i), \quad i \in \mathbf{N}.$$

As μ is monotone we have

$$0 \le \mu(u_i) \le \mu(m_i + I_{\mathrm{cl}(\mathrm{dom}(u_i))}) = f_k(m_i) V_k(\mathrm{cl}(\mathrm{dom}(u_i))).$$

On the other hand $V_k(\mathrm{cl}(\mathrm{dom}(u_i))) = 0$, as $\dim(\mathrm{dom}(u_i)) < k$. Hence $\mu(u_i) = 0$ for every i. To conclude, note that u_i, $i \in \mathbf{N}$, is a decreasing sequence of functions in $\mathcal{C}^n$, converging to u point-wise in $\mathbf{R}^n$. As a consequence of continuity of μ we have

$$\mu(u) = \lim_{i \to \infty} \mu(u_i) = 0.$$

□

5.1 Regularization of geometric densities

In the sequel sometimes it will be convenient to work with valuations having geometric densities with more regularity than that of a decreasing function. In this section we describe a procedure which allows to approximate (in a suitable sense) a valuation with a sequence of valuations having smooth densities.

Let $g : \mathbf{R} \to \mathbf{R}$ be a standard mollifying kernel, i.e. g has the following properties: $g \in C^\infty(\mathbf{R})$, $g(t) \geq 0$ for every $t \in \mathbf{R}$, the support of g is contained in $[-1, 1]$ and

$$\int_{\mathbf{R}} g(t)dt = 1.$$

For $\epsilon > 0$ let $g_\epsilon : \mathbf{R} \to \mathbf{R}$ be defined by

$$g_\epsilon(t) = \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}\right).$$

Then $g_\epsilon \in C^\infty(\mathbf{R})$; $g_\epsilon(t) \geq 0$ for every $t \in \mathbf{R}$; the support of g_ϵ is contained in $[-\epsilon, \epsilon]$ and

$$\int_{\mathbf{R}} g_\epsilon(t)dt = 1.$$

Now let $u \in \mathcal{C}^n$ and consider the function $\phi : \mathbf{R} \to \mathbf{R}$ defined by

$$\phi(t) = \mu(u + t).$$

By the properties of μ, this is a non-negative and decreasing function. For $\epsilon > 0$ and $t \in \mathbf{R}$ set:

$$\phi_\epsilon(t) = (\phi \star g_\epsilon)(t) = \int_{\mathbf{R}} \phi(t - s)g_\epsilon(s)ds = \int_{\mathbf{R}} \mu(u + t - s)g_\epsilon(s)ds.$$

Then ϕ_ϵ is a non-negative decreasing function of class $C^\infty(\mathbf{R})$. Moreover, by the properties of the kernel g,

$$\lim_{\epsilon \to 0^+} \phi_\epsilon(t) = \phi(t)$$

for every $t \in \mathbf{R}$ where ϕ is continuous (in fact, for every Lebesgue point t of ϕ, see for instance [6]); in particular $\phi_\epsilon \to \phi$ a.e. in $\mathbf{R}$ as $\epsilon \to 0^+$.

For $\epsilon > 0$ we define $\mu_\epsilon : \mathcal{C}^n \to \mathbf{R}$ as

$$\mu_\epsilon(u) = \phi_\epsilon(0) = \int_{\mathbf{R}} \phi(-s)g_\epsilon(s)ds = \int_{\mathbf{R}} \mu(u - s)g_\epsilon(s)ds.$$

It is a straightforward exercise to verify that μ_ϵ inherits most of the properties of μ: it is a valuation, rigid motion invariant, non-negative and decreasing. Moreover, if $f_{k,\epsilon}$, for $k \in \{0, \ldots, n\}$, denote the geometric densities of μ_ϵ, we have

$$f_{k,\epsilon} = f_k \star g_\epsilon$$

for every $k \in \{0, \ldots, n\}$, where $f_0 \ldots, f_n$ are the densities of μ. Indeed, for every convex body $K \in \mathcal{K}^n$ and every $t \in \mathbf{R}$ we have

$$\begin{aligned} \mu_\epsilon(t + I_K) &= \int_{\mathbf{R}} \mu(t - s + I_K) g_\epsilon(s) ds = \int_{\mathbf{R}} \sum_{k=0}^{n} f_k(t-s) V_k(K) g_\epsilon(s) ds \\ &= \sum_{k=0}^{n} (f_k \star g_\epsilon)(t) V_k(K). \end{aligned}$$

The core properties of the above construction can be summed up in the following proposition.

Proposition 5.7. *Let $\mu : \mathcal{C}^n \to \mathbf{R}$ be a rigid motion invariant and decreasing valuation. Then there exists a family of rigid motion invariant and decreasing valuations μ_ϵ, $\epsilon > 0$, such that:*

1. *for every $\epsilon > 0$ the geometric densities $f_{0,\epsilon}, \ldots, f_{n,\epsilon}$ of μ_ϵ belong to $C^\infty(\mathbf{R})$;*
2. *for every $k \in \{0, \ldots, n\}$, $f_{k,\epsilon} \to f_k$ a.e. in $\mathbf{R}$, as $\epsilon \to 0^+$, where $f_0, \ldots, f_n$ are the geometric densities of μ;*
3. *for every $u \in \mathcal{C}^n$, $\mu_\epsilon(u + t) \to \mu(u + t)$ for a.e. $t \in \mathbf{R}$, as $\epsilon \to 0^+$.*

6 Integral valuations

In this section we introduce a class of integral valuations which will turn out to be crucial in the characterization results that we will present in the sequel. As we will see, they are similar to those introduced by Wright in [22] and subsequently studied by Baryshnikov, Ghrist and Wright in [3].

Let ν be a (non-negative) Radon measure on the real line $\mathbf{R}$ and fix $k \in \{0, \ldots, n\}$. For every $u \in \mathcal{C}^n$ we set

$$\mu(u) := \int_{\mathbf{R}} V_k(\mathrm{cl}(\Omega_t)) d\nu(t), \tag{6.19}$$

where $\Omega_t = \{u < t\}$ for every $t \in \mathbf{R}$. As noted in sub-section 3.1, the function $t \mapsto V_k(\mathrm{cl}(\Omega_t))$ vanishes on $(-\infty, m(u)]$ and, for $k \neq 0$ its k-th root is concave in $(m(u), \infty)$, while for $k = 0$ it is simply constantly 1 in $(m(u), \infty)$; hence it is Borel measurable. Moreover it is non-negative, so that it is integrable with respect to ν. On the other hand its integral (6.19) might be ∞. We first find equivalent conditions on ν such that (6.19) is finite for every $u \in \mathcal{C}^n$.

Proposition 6.1. *Let ν be a non-negative Radon measure on the real line. The integral* (6.19) *is finite for every $u \in \mathcal{C}^n$ if and only if ν has finite k-th moment:*

$$\int_{(0,\infty)} t^k d\nu(t) < \infty. \tag{6.20}$$

Proof. Assume that $\mu(u)$ is finite for every u. Choosing in particular $u \in \mathcal{C}^n$ defined by $u(x) = |x|$ we have that $V_k(\mathrm{cl}(\Omega_t))$ is zero for every $t \leq 0$. For $t > 0$, $\mathrm{cl}(\Omega_t)$ is a ball centred at the origin with radius t, hence $V_k(\mathrm{cl}(\Omega_t)) = c(n,k)t^k$ with $c(n,k) > 0$. Therefore

$$\mu(u) = \int_{(0,\infty)} c(n,k)t^k d\nu(t) < \infty.$$

Vice versa, assume that ν has finite k-th moment. Given $u \in \mathcal{C}^n$ there exists $a > 0$ and $b \in \mathbf{R}$ such that $u(x) \geq a|x| + b$ for every x (see Proposition 3.8). Hence, for $t \in \mathbf{R}$, $t \geq b$,

$$\Omega_t \subseteq \{x \in \mathbf{R}^n \,:\, a|x| + b < t\} = \left\{x \in \mathbf{R}^n \,:\, |x| < \frac{t-b}{a}\right\},$$

while Ω_t is empty for $t \leq b$. By the monotonicity of intrinsic volumes

$$\mu(u) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\Omega_t))d\nu(t) \leq c(n,k) \int_{(b,\infty)} \left(\frac{t-b}{a}\right)^k d\nu(t),$$

and the last integral is finite by (6.20). □

Proposition 6.2. *Let $k \in \{0, 1, \ldots, n\}$ and let t be a fixed real number. Then the function $u \mapsto V_k(\mathrm{cl}(\{u < t\}))$*

i) is a valuation;

ii) is rigid motion invariant;

iii) is monotone;

iv) is k-homogeneous;

v) is continuous.

Proof. i) The condition on $\boldsymbol{\infty}$ is easily verified, as a matter of fact

$$V_k(\mathrm{cl}(\{\boldsymbol{\infty} < t\})) = V_k(\emptyset) = 0.$$

Let now $u, v \in \mathcal{C}^n$ be such that $u \wedge v \in \mathcal{C}^n$. By Proposition 3.6, for every $t \in \mathbf{R}$ we have

$$\mathrm{cl}(\{u \wedge v < t\}) = \mathrm{cl}(\{u < t\}) \cup \mathrm{cl}(\{v < t\}), \quad \mathrm{cl}(\{u \vee v < t\}) = \mathrm{cl}(\{u < t\}) \cap \mathrm{cl}(\{v < t\}).$$

Consequently, as intrinsic volumes are valuations, we get

$$\begin{aligned} V_k(\mathrm{cl}(\{u \wedge v < t\})) \;&+\; V_k(\mathrm{cl}(\{u \vee v < t\})) = \\ &= V_k(\mathrm{cl}(\{u < t\}) \cup \mathrm{cl}(\{v < t\})) + V_k(\mathrm{cl}(\{u < t\}) \cap \mathrm{cl}(\{v < t\})) \\ &= V_k(\mathrm{cl}(\{u < t\})) + V_k(\mathrm{cl}(\{v < t\})). \end{aligned}$$

ii) Let $u \in \mathcal{C}^n$ and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a rigid motion; let moreover $v := u \circ T \in \mathcal{C}^n$, i.e. $v(y) = u(T(y))$ for every $y \in \mathbf{R}^n$. Then, for every $t \in \mathbf{R}$,

$$\{y \,:\, v(y) < t\} = \{y \,:\, u(T(y)) < t\} = T^{-1}(\{x \,:\, u(x) < t\}).$$

As intrinsic volumes are invariant with respect to rigid motions, we have

$$V_k(\{u < t\}) = V_k(\{v < t\}),$$

for every t.

iii) As for monotonicity, if $u, v \in \mathcal{C}^n$ and $u \le v$ point-wise on $\mathbf{R}^n$, then for every $t \in \mathbf{R}$,

$$\{v < t\} \subseteq \{u < t\},$$

and thus

$$\mathrm{cl}(\{v < t\}) \subseteq \mathrm{cl}(\{u < t\}).$$

By the monotonicity of intrinsic volumes

$$V_k(\mathrm{cl}(\{v < t\})) \le V_k(\mathrm{cl}(\{u < t\})).$$

iv) Let $u \in \mathcal{C}^n$ and $\lambda > 0$, and define u_λ by

$$u_\lambda(x) = u\left(\frac{x}{\lambda}\right).$$

For $t \in \mathbf{R}$ we have

$$\{x \,:\, u_\lambda(x) < t\} = \left\{x \,:\, u\left(\frac{x}{\lambda}\right) < t\right\} = \lambda\{x \,:\, u(x) < t\}.$$

Then, by homogeneity of intrinsic volumes

$$V_k(\mathrm{cl}(\{u_\lambda < t\})) = V_k(\lambda\,\mathrm{cl}(\{u < t\})) = \lambda^k V_k(\mathrm{cl}(\{u < t\})).$$

v) Let $u \in \mathcal{C}^n$ and let u_i, $i \in \mathbf{N}$, be a sequence in $\mathcal{C}^n$, point-wise decreasing and converging to u in the relative interior of $\mathrm{dom}(u)$. We want to prove that

$$\lim_{i\to\infty} V_k(\mathrm{cl}(\{u_i < t\})) = V_k(\mathrm{cl}(\{u < t\})).$$

Let $j \in \{0, 1, \dots, n\}$ be the dimension of $\mathrm{dom}(u)$; for every $i \in \mathbf{N}$, as $u_i \ge u$ we have that $\mathrm{dom}(u_i) \subseteq \mathrm{dom}(u)$, and, in particular, the dimension of the domain of u_i is less than or equal to j. If $k > j$, then $V_k(\mathrm{cl}(\{u < t\})) = 0$ for every $t \in \mathbf{R}$ and analogously $V_k(\mathrm{cl}(\{u_i < t\})) = 0$ for every i, so that the assert of the proposition holds true. Hence we may assume that $k \le j$ and, up to restricting all involved functions to a j-dimensional affine subspace of $\mathbf{R}^n$ containing the domain of u, we may assume without loss of generality that $j = n$.

As usual, we denote $\min_{\mathbf{R}^n} u$ by $m(u)$. If $t \le m(u)$, then, for all i, $\{u_i < t\} = \{u < t\} = \emptyset$ and the claim holds trivially. Let now $t > m(u)$, then, by Corollary 3.3

$$\mathrm{cl}(\{u < t\}) = K_t := \{u \le t\}.$$

As $\dim(\mathrm{dom}(u)) = n$ and $t > m(u)$, K_t is a convex body with non-empty interior (this follows from the convexity of u). Let

$$\Omega_t^i = \{u_i < t\}, \quad K_t^i = \mathrm{cl}(\Omega_t^i), \quad i \in \mathbf{N}.$$

Clearly $K_t^i \subseteq K_t$ for every i. On the other hand, if x is an interior point of K_t, then $u(x) < t$ (see Lemma 3.2). Hence $u_i(x) < t$ for sufficiently large i, which leads to

$$\bigcup_{i \in \mathbf{N}} K_t^i \supseteq \mathrm{int}(K_t).$$

As a consequence, K_t^i converges to K_t in the Hausdorff metric, and then (by the continuity of intrinsic volumes)

$$\lim_{i\to\infty} V_k(K_t^i) = V_k(K_t), \quad \forall\, t > m(u),$$

as we wanted. □

Corollary 6.3. *Let $k \in \{0, 1, \dots, n\}$ and let ν be a Radon measure on $\mathbf{R}$ with finite k-th moment. Then the function $\mu : \mathcal{C}^n \to \mathbf{R}$ defined by*

$$\mu(u) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\}))d\nu(t), \quad u \in \mathcal{C}^n$$

has the following properties:

i) it is a valuation;

ii) it is rigid motion invariant;

iii) it is monotone;

iv) it is k-homogeneous;

v) it is continuous.

Proof. Claims *i) - iv)* follow easily from Proposition 6.2 by integration. The proof of the continuity of μ is a bit more delicate. Let $u \in \mathcal{C}^n$ and let u_i, $i \in \mathbf{N}$, be a sequence in $\mathcal{C}^n$, point-wise decreasing and converging to u in relint(dom(u)).

As $u_i \geq u$ we have, for every $t \in \mathbf{R}$,

$$\{u_i < t\} \subseteq \{u < t\} \;\Rightarrow\; V_k(\mathrm{cl}(\{u_i < t\})) \leq V_k(\mathrm{cl}(\{u < t\}))$$

for every $i \in \mathbf{N}$.

By Proposition 6.2 we know that

$$\lim_{i\to\infty} V_k(\mathrm{cl}(\{u_i < t\})) = V_k(\mathrm{cl}(\{u < t\})), \quad \forall\, t.$$

This fact and the monotone convergence theorem imply

$$\lim_{i\to\infty} \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u_i < t\}))d\nu(t) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\}))d\nu(t).$$

□

Let μ be a valuation of the form (6.19); by Proposition 5.1 and Corollary 5.3, μ has exactly one geometric density which is not identically zero, i.e. f_k; this can be explicitly computed in terms of the measure ν. Let $K \in \mathcal{K}^n$ be such that $V_k(K) > 0$, then, for $t \in \mathbf{R}$

$$\begin{aligned}\mu(t + I_K) = f_k(t)V_k(K) &= \int_{\mathbf{R}} V_k(\mathrm{cl}(\{x : t + I_K(x) < s\}))d\nu(s) \\ &= V_k(K)\int_{\{s : t<s\}} d\nu(s) = V_k(K)\int_{(t,\infty)} d\nu(s);\end{aligned}$$

i.e.

$$f_k(t) = \nu((t,\infty)) \quad \forall\, t \in \mathbf{R}. \tag{6.21}$$

We observe that this is a non-increasing function and, by the basic properties of measures, it is right-continuous.

6.1 An equivalent representation formula

As in the previous part of this section, ν will be a non-negative Radon measure on $\mathbf{R}$ with finite k-th moment, where k is a fixed integer in $\{0, 1, \dots, n\}$. Moreover, $f : \mathbf{R} \to \mathbf{R}$ is defined by

$$f(t) = \int_{(t,\infty)} d\nu(t) = \nu((t,\infty)). \tag{6.22}$$

We first consider the case $k \geq 1$. Note that

$$\int_0^\infty t^{k-1} f(t)dt < \infty. \tag{6.23}$$

Indeed

$$\int_0^\infty t^{k-1} f(t)dt \;=\; \int_0^\infty t^{k-1}\int_{(t,\infty)} d\nu(s)dt = \int_{(0,\infty)}\int_0^s t^{k-1}dtd\nu(s) = \frac{1}{k}\int_{(0,\infty)} s^k d\nu(s).$$

Now let $u \in \mathcal{C}^n$ and let $v_k(u;\cdot)$ be the function defined in section 3.1

$$v_k(u;t) = V_k(\mathrm{cl}(\{u < t\})) \quad t \in \mathbf{R}.$$

For simplicity we set $h(t) = v_k(u;t)$ for every $t \in \mathbf{R}$; h is a monotone non-decreasing function identically vanishing on $(-\infty, m(u)]$ and $h^{1/k}$ is concave in $(m(u),\infty)$, as pointed out in section 3.1; in particular h is locally Lipschitz. This implies (see for instance [2, Chapter 3]) that the product fh is a function of bounded variation in $(m(u),\infty)$ and its weak derivative is the measure

$$-h\nu + h' f \mathcal{H}^1$$

(we recall that $\mathcal{H}^1$ is the one-dimensional Lebesgue measure). Note also that fh is right-continuous, as f has this property. Hence for every $t_0, t \in \mathbf{R}$, with $m(u) < t_0 \leq t$,

$$f(t)h(t) = f(t_0)h(t_0) + \int_{(t_0,t)} h'(s)f(s)ds - \int_{(t_0,t)} h(s)d\nu(s).$$

If we let $t_0 \to m(u)^+$ we get

$$f(m(u))V_k(\{u = m(u)\}) + \int_{(m(u),t)} f(s)h'(s)ds = f(t)h(t) + \int_{(m(u),t)} f(s)d\nu(t). \quad (6.24)$$

Indeed, as we proved in section 3.1,

$$\lim_{t \to m(u)^+} h(t) = V_k(\{u = m(u)\}).$$

By Lemma 3.8, there is a constants $a > 0$ such that

$$h(t) \le at^k$$

for t sufficiently large. Hence $h(t)f(t) \le at^k f(t)$. On the other hand, the integrability condition (6.23) implies

$$\liminf_{t \to \infty} t^k f(t) = 0,$$

so that

$$\liminf_{t \to \infty} f(t)h(t) = 0.$$

Hence, passing to the limit for $t \to \infty$ in (6.24), we get

$$f(m(u))V_k(\{u = m(u)\}) + \int_{(m(u),\infty)} f(s)h'(s)ds = \int_{(m(u),\infty)} h(s)d\nu(s) = \int_{\mathbf{R}} h(s)d\nu(s)$$

(the last equality is due to: $h \equiv 0$ in $(-\infty, m(u)]$). Recalling the structure of the weak derivative of the function h proven in Proposition 3.4, we may write

$$\int_{\mathbf{R}} f(s)d\beta_k(u;s) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < s\}))d\nu(s). \quad (6.25)$$

The above formula is proven for $k \ge 1$. The case $k = 0$ is straightforward, indeed

$$V_0(\{u < t\}) = \begin{cases} 0 \text{ if } t \le m(u), \\ 1 \text{ if } t > m(u) \end{cases}$$

so that (6.25) becomes

$$f(m(u)) = \int_{(m(u),\infty)} d\nu(s),$$

which is true by the definition of f. The previous considerations provide the proof of the following result.

Proposition 6.4. *Let $k \in \{0, \ldots, n\}$, let ν be a non-negative Radon measure on $\mathbf{R}$ with finite k-th moment and let f be defined as in* (6.22). *Let $\mu : \mathcal{C}^n \to \mathbf{R}$ be the valuation defined by* (6.19). *Then for every $u \in \mathcal{C}^n$*

$$\mu(u) = \int_{\mathbf{R}} f(t)d\beta_k(u;t).$$

Valuations expressed as in the above proposition were considered in [22] and [3].

In the remaining part of this section we analyze two special cases of the integral valuations introduced so far, corresponding to the indices $k = 0$ and $k = n$, which can be written in a simpler alternative form.

6.2 The case $k = 0$

Let ν be a Radon measure on $\mathbf{R}$; then the finiteness of the k-th moment is just

$$\int_0^\infty d\nu(s) < \infty,$$

which is equivalent to saying that the function f defined by (6.22) is well defined (i.e. finite) on $\mathbf{R}$. Let $u \in \mathcal{C}^n$; as we pointed out before,

$$V_0(\mathrm{cl}(\{u < t\})) = \begin{cases} 0 \text{ if } t \leq m(u), \\ \\ 1 \text{ if } t > m(u). \end{cases}$$

Then

$$\mu(u) = \int_{(m(u),\infty)} d\nu(t) = f(m(u))$$

for every $u \in \mathcal{C}^n$.

6.3 The case $k = n$

Proposition 6.5. *Let ν be a Radon measure on $\mathbf{R}$ with finite n-th moment, let f be defined as in* (6.22)*, and let μ be the valuation:*

$$\mu(u) = \int_{\mathbf{R}} V_n(\mathrm{cl}(\{u < t\}))d\nu(t), \quad u \in \mathcal{C}^n.$$

Then

$$\mu(u) = \int_{\mathrm{dom}(u)} f(u(x))dx \quad \forall\, u \in \mathcal{C}^n.$$

Proof. Let us extend f to $\mathbf{R} \cup \{\infty\}$ setting $f(\infty) = 0$. As a direct consequence of the so-called layer cake (or Cavalieri's) principle and of the definition of f, we have that

$$\int_{\mathbf{R}} \mathcal{H}^n(\{u < t\})d\nu(t) = \int_{\mathbf{R}^n} f(u(x))dx = \int_{\mathrm{dom}(u)} f(u(x))dx,$$

where $\mathcal{H}^n$ denotes the Lebesgue measure in $\mathbf{R}^n$. On the other hand, for every $t \in \mathbf{R}$ the set $\{u < t\}$ is convex and bounded, so that its boundary is negligible with respect to the Lebesgue measure. Hence $\mathcal{H}^n(\{u < t\}) = V_n(\mathrm{cl}(\{u < t\}))$ for every t. □

We can change the point of view and take the function f as a starting point, instead of the measure ν.

Proposition 6.6. *Let $f : \mathbf{R} \to \mathbf{R}$ be non-negative, decreasing, and right-continuous. Define the map $\mu : \mathcal{C}^n \to \mathbf{R}$ by*

$$\mu(u) = \int_{\mathrm{dom}(u)} f(u(x))dx$$

for every $u \in \mathcal{C}^n$. Then:

i) μ *is well defined (i.e.* $\mu(u) \in \mathbf{R}$ *for every* $u \in \mathcal{C}^n$*) if and only if* f *has finite* $(n-1)$*-st moment:*

$$\int_0^\infty t^{n-1} f(t)dt < \infty;$$

ii) μ *is a valuation on* $\mathcal{C}^n$*, and it is rigid motion invariant, simple and decreasing;*

iii) μ *is* n*-homogeneous;*

iv) μ *is continuous.*

The proof follows directly from the previous considerations and the dominated convergence theorem. A typical example in this sense is given by the function μ defined by

$$\mu(u) = \int_{\mathrm{dom}(u)} e^{-u(x)} dx.$$

7 A decomposition result for simple valuations

Later on in this paper we will need to approximate integrals by Riemann sums, and generic convex functions by piecewise linear functions; in both cases it will be important to use an effective notion of partition. This role will be played by *inductive partitions*, introduced in this section.

7.1 Inductive partitions

Given $K, K_1, \dots, K_N \in \mathcal{K}^n$, the collection

$$\mathcal{P} := \{K_1, \dots, K_N\}$$

is called a *convex partition* of K if

$$K = \bigcup_{i=1}^N K_i \quad \text{and} \quad \mathrm{int}(K_i \cap K_j) = \emptyset \quad \forall\, i \neq j.$$

Definition 7.1. (Inductive partition). *Let* $K \in \mathcal{K}^n$*. A convex partition* $\mathcal{P}$ *of* K *is called an* inductive partition *if there exists a sequence of convex bodies* $H_1, \dots, H_l$*, with* $H_l = K$*, such that for all* $i = 1, \dots, l$ *one of the two following conditions holds true:*

- $H_i \in \mathcal{P}$*,*
- $\exists j, k < i$ *such that* $H_i = H_j \cup H_k$ *and* $\mathrm{int}(H_j \cap H_k) = \emptyset$*.*

The idea behind this definition is the following. $\mathcal{P}$ is an inductive partition of K if K can be split into two parts, $K = K' \cup K''$ (where K' and K'' are convex bodies with disjoint interiors), and $\mathcal{P}$ is the union of two partitions $\mathcal{P}'$ and $\mathcal{P}''$ of K' and K'' respectively. Most importantly, $\mathcal{P}'$ and $\mathcal{P}''$ are again inductive, so that the process can be repeated for K' and K''. This is particularly helpful when one wants to apply the induction principle.

Note first of all that if we fix an index $i \leq l$; if H_i is not an element of $\mathcal{P}$, then $H_i = H_j \cup H_k$, for some $j \neq k$. If we apply the same argument to H_j and H_k, after a finite number of steps we conclude that H_i can be written as the union of elements of $\mathcal{P}$. In other words $\mathcal{P}$ contains a convex partition of any H_i.

Moreover, if we assume that l and the number of elements of $\mathcal{P}$ are at least 2, then $K = H_l$ can be written as $H_j \cup H_j$, and $\mathcal{P}$ provides (i.e. it contains) a partition of H_j and H_k, which is still inductive.

Example 1. An illustrative example of an inductive partition are rectangular grids, described for convenience in the plane. Let K be a rectangle in $\mathbf{R}^2$

$$K = [a, b] \times [c, d], \quad a \leq b, \quad c \leq d.$$

For $N \in \mathbf{N}$ let

$$\mathcal{P} = \{R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \ : i,\, j = 1, \dots, N\},$$

where

$$x_i = a + \frac{i(b-a)}{N} \quad y_j = b + \frac{j(d-c)}{N},$$

be the standard regular partition of K. Then $\mathcal{P}$ is inductive. Indeed, consider the family $\mathcal{H}$ formed by all possible convex subsets of K, which can be written as union of elements of $\mathcal{P}$. Then $\mathcal{H}$ is a finite family

$$\mathcal{H} = \{H_1 \dots H_l\}$$

and the definition of inductive partition is easily verified.

Example 2. It is immediate to prove that convex partitions made of one or two elements are inductive partitions. On the other hand it is easy to construct a convex partition of three elements which is not inductive. Let K be a disk in the plane, centred at the origin, and consider three rays from the origin such that the angle between any two of them is $2\pi/3$. These rays divide K into three subsets K_1, K_2 and K_3 which form a convex partition $\mathcal{P}$ of K. $\mathcal{P}$ is not an inductive partition.

7.2 Complete partitions

The definition of inductive partitions could be sometimes hard to verify. The main point of this part is to show that every partition of a *polytope*, whose elements are polytopes as well, can be refined so that it becomes an inductive partition.

From now on in this section P will be a polytope of $\mathbf{R}^n$, i.e. the convex hull of finitely many points of $\mathbf{R}^n$. Note in particular that $P \in \mathcal{K}^n$; moreover we will always assume that *P has non-empty interior.* We consider a convex partition

$$\mathcal{P} = \{P_1, \dots, P_N\}$$

of P whose elements are all polytopes, with non-empty interior; we will refer to such partitions as *polytopal* partitions.

If H is a hyperplane (i.e. an affine subspace of dimension $(n-1)$) of $\mathbf{R}^n$, we can refine $\mathcal{P}$ by H in the usual way. Let H^+ and H^- be the closed half-spaces determined by H and set

$$P_i^+ = P_i \cap H^+, \quad P_i^- = P_i \cap H^-.$$

Then

$$\mathcal{P}_H = \{P_i^+ \,:\, i \in \{1, \ldots, N\}, \, \mathrm{int}(P_i^+) \neq \emptyset\} \cup \{P_i^- \,:\, i \in \{1, \ldots, N\}, \, \mathrm{int}(P_i^-) \neq \emptyset\}$$

is still a polytopal partition of K that we call the refinement of $\mathcal{P}$ by H.

Definition 7.2. *A polytopal partition is called complete if for every hyperplane H which contains an $(n-1)$-dimensional facet of at least one element of $\mathcal{P}$ we have*

$$\mathcal{P}_H = \mathcal{P}.$$

Remark 7.3. Every polytopal partition $\mathcal{P}$ can be successively refined until it becomes, in a finite number of steps, a complete partition. Indeed, let $\{H_1, \ldots, H_R\}$ be the collection of all hyperplanes containing a $(n-1)$-dimensional facet of at least one element of $\mathcal{P}$. Then

$$(\ldots((\mathcal{P})_{H_1})_{H_2} \ldots)_{H_R}$$

is a complete partition.

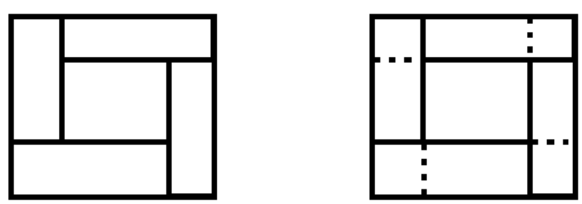

Figure 1: A partition being completed

Proposition 7.4. *Let P be a polytope with non-empty interior and let $\mathcal{P} = \{P_1, \ldots, P_N\}$ be a complete partition of P. Then $\mathcal{P}$ is an inductive partition of P.*

Proof. The proof proceeds by induction on N. For $N = 1$ the assert is true as any partition consisting of one element is trivially an inductive partition. Let $N \geq 2$ and assume that the claim is true for every integer up to $(N-1)$. Let $\mathcal{P}$ be a complete polytopal partition of P. Let H be a hyperplane containing a $(n-1)$-dimensional facet of an element of $\mathcal{P}$, intersecting the interior of P. Such a hyperplane exists because P has non-empty interior and $N \geq 2$. Let H^+ and H^- be the closed half-spaces determined by H. Then, as $\mathcal{P}$ is complete (and each P_i has non-emtpy interior), each P_i is contained either in H^+ or in H^-. Moreover, as $N \geq 2$ and $H \cap \mathrm{int}(P) \neq \emptyset$, there exist at least one element of $\mathcal{P}$ contained in H^+ and at least one element in H^-. Then clearly

$$\begin{aligned} \mathcal{P}^+ &= \{Q \in \mathcal{P} \,:\, Q \subseteq H^+\} \text{ is a complete partition of } P \cap H^+, \\ \mathcal{P}^- &= \{Q \in \mathcal{P} \,:\, Q \subseteq H^-\} \text{ is a complete partition of } P \cap H^-. \end{aligned}$$

Each of these partitions has a number of elements which is strictly less than N. Consequently, by the induction hypothesis, $\mathcal{P}^+$ is also an inductive partition of $P \cap H^+$ and $\mathcal{H}^-$ is an inductive partition of $P \cap H^-$. Therefore, by Definition 7.1, there exist two sequences

$$P_1^+, \ldots, P_j^+ = P \cap H^+ \quad \text{and} \quad P_1^-, \ldots, P_k^- = P \cap H^-$$

that fulfill the required properties. We claim that such a sequence can be formed for the partition $\mathcal{P}$ as well: as a matter of fact consider the following

$$P_1^+, \ldots, P_j^+, P_1^-, \ldots, P_k^-, P.$$

As $P_j^+ \cup P_k^- = P$ and $\mathrm{int}(P_j^+ \cap P_k^-) = \emptyset$ we conclude that $\mathcal{P}$ is an inductive partition too. □

7.2.1 Rectangular partitions

A *rectangle* R in $\mathbf{R}^n$ is a set of the form

$$R = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \,:\, a_j \le x_j \le b_j \text{ for every } j = 1, \ldots, n\},$$

where, for $j = 1, \ldots, n$, a_j and b_j are real numbers such that $a_j < b_j$. In particular, R is a convex polytope, and each of its facets is parallel to a hyperplane of the form $e_j^\perp$, for some $j \in \{1, \ldots, n\}$ (where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbf{R}^n$). This property characterizes rectangles.

A *rectangular partition* of a rectangle R is a partition

$$\mathcal{P} = \{R_1, \ldots, R_N\}$$

of R such that each R_k is itself a rectangle.

If $\mathcal{P}$ is a rectangular partition of a rectangle R, and we refine it so that its refinement $\mathcal{P}'$ is complete, as explained in Remark 7.3, then $\mathcal{P}'$ is still a rectangular partition; indeed each facet of each element of $\mathcal{P}'$ is contained in a hyperplane parallel to $e_j^\perp$, for some j.

7.3 A decomposition result for simple valuations

The following result is the main motivation for the definition of inductive partitions.

Lemma 7.5. *Let μ be a simple valuation on $\mathcal{C}^n$. Let K be a convex body and let*

$$\mathcal{P} = \{K_1, \ldots, K_N\}$$

be an inductive partition of K. Then, for every $u \in \mathcal{C}^n$,

$$\mu(u + I_K) = \sum_{i=1}^{N} \mu(u + I_{K_i}).$$

Proof. Since $\mathcal{P}$ is an inductive partition we can find a sequence of convex bodies $H_1, \dots, H_l = K$ with the properties stated in Definition 7.1. We argue by induction on l. If $l = 1$ the claim holds trivially. Assume now that the claim is true up to $l-1$. If $H_l \in \mathcal{P}$ we can conclude as in the case $l = 1$. Therefore we may assume $\exists j, k < l$ such that $H_j \cup H_k = K$ and $\text{int}(H_j \cap H_k) = \emptyset$. As H_j and H_k are convex bodies, $u + I_{H_j}$ and $u + I_{H_k}$ belong to $\mathcal{C}^n$. Moreover

$$(u + I_{H_j}) \wedge (u + I_{H_k}) = u + I_{H_j \cup H_k} = u + I_K,$$

while

$$(u + I_{H_j}) \vee (u + I_{H_k}) = u + I_{H_j \cap H_k}.$$

In particular, as $\text{int}(H_j \cap H_k)$ is empty, $\dim(\text{dom}(u + I_{H_j \cap H_k})) < n$. Hence, as μ is a simple valuation, we get

$$\mu(u + I_K) = \mu(u + I_{H_j}) + \mu(u + I_{H_k}).$$

Now, if we set $\mathcal{P}' = \{P \in \mathcal{P} : P \subseteq H_j\}$ and $\mathcal{P}'' = \{P \in \mathcal{P} : P \subseteq H_k\}$ and apply the inductive hypothesis to the just defined partitions we get

$$\mu(u + I_{H_j}) + \mu(u + I_{H_k}) = \sum_{P \in \mathcal{P}'} \mu(u + I_P) + \sum_{P \in \mathcal{P}''} \mu(u + I_P) = \sum_{P \in \mathcal{P}} \mu(u + I_P).$$

□

8 Characterization results I: simple valuations

Our first characterization result is a converse of Proposition 6.6.

Theorem 8.1. *Let $\mu : \mathcal{C}^n \to \mathbf{R}$ be a rigid motion invariant, monotone decreasing, simple and continuous valuation. Then there exists a function $f : \mathbf{R} \to \mathbf{R}$, non-negative, decreasing, right-continuous, with finite n-th moment:*

$$\int_0^\infty t^n f(t) dt < \infty,$$

such that for every $u \in \mathcal{C}^n$

$$\mu(u) = \int_{\text{dom}(u)} f(u(x)) dx. \tag{8.26}$$

Equivalently

$$\mu(u) = \int_{\mathbf{R}} V_n(\text{cl}(\{u < t\})) d\nu(t), \tag{8.27}$$

where ν is the Radon measure related to f by:

$$f(t) = \nu((t, \infty)) \quad \forall\, t \in \mathbf{R}.$$

The function f coincides with the geometric density f_n of μ, determined by Proposition 5.1.

Let us begin with some considerations preliminary to the proof. As μ is rigid motion invariant and monotone, its geometric densities f_i, $i = 0, \dots, n$ are defined (see Proposition 5.1). On the other hand, by Proposition 5.5, the only non-zero geometric density of μ is f_n. Recall that f_n is a non-negative decreasing function defined on $\mathbf{R}$, moreover, as μ is continuous f_n is right-continuous. Let $f : (-\infty, +\infty] \to [0, +\infty)$ be the extension of f_n, with the additional condition $f(\infty) := 0$.

We will need the following Lemma.

Lemma 8.2. *Assume that μ is as in Theorem 8.1 and let f be the extension of its geometric density defined as above. Let K be a convex body and*

$$\mathcal{P} = \{K_1, \dots, K_N\}$$

be an inductive partition of K.

Let $u \in \mathcal{C}^n$ be such that $L = \mathrm{dom}(u)$ is a convex body, the restriction of u to L is continuous and $L \subseteq \mathrm{int}(K)$. Then

$$\sum_{i=1}^{N} m_i \, V_n(K_i) \le \mu(u + I_K) \le \sum_{i=1}^{N} M_i \, V_n(K_i),$$

where

$$m_i = \inf\{f(u(x)) \,|\, x \in K_i\}, \quad M_i = \sup\{f(u(x)) \,|\, x \in K_i\}.$$

Proof. First we prove that u attains a maximum and a minimum when restricted to K_i.

Since u is lower semi-continuous and K_i is compact and non-empty, we have $\inf_{K_i} u = \min_{K_i} u$. Suppose now that there exists a point $x \in \mathcal{K}_i$ such that $u(x) = \infty$: in this case $\sup_{K_i} u = \infty$ is attained at x; on the other hand, if $K_i \subseteq L$, as the restriction of u to L (and thus to K_i) is continuous, $\sup_{K_i} u = \max_{K_i} u$.

Therefore, as f is decreasing,

$$M_i = f\left(\min_{K_i} u\right) \quad \text{and} \quad m_i = f\left(\max_{K_i} u\right).$$

Using the monotonicity of μ and the definition of geometric densities we obtain

$$\begin{aligned} \mu(u + I_{K_i}) &\le \mu\left(\min_{K_i} u + I_{K_i}\right) = M_i \, V_n(K_i), \\ \mu(u + I_{K_i}) &\ge \mu\left(\max_{K_i} u + I_{K_i}\right) = m_i \, V_n(K_i). \end{aligned}$$

Then

$$m_i V_n(K_i) \le \mu(u + I_{K_i}) \le M_i V_n(K_i). \tag{8.28}$$

By Lemma 7.5 we have that

$$\mu(u + I_K) = \sum_{i=1}^{N} \mu(u + I_{K_i}).$$

This equality and (8.28) conclude the proof. □

We will also need a well known theorem relating Riemann integrability and Lebesgue measure (known as Lebesgue-Vitali theorem). The proof of this result in the one-dimensional case can be found in standard texts of real analysis. The reader interested in the proof for general dimension may consult [1].

Theorem 8.3. *Let $g : \mathbf{R}^n \to \mathbf{R}$ be a bounded function which vanishes outside a compact set. Then g is Riemann integrable in $\mathbf{R}^n$ if and only if the set of discontinuities of g has Lebesgue measure zero.*

Proof of Theorem 8.1. Let us consider an arbitrary $u \in \mathcal{C}^n$ which, from now on will be fixed. If $\dim(\mathrm{dom}(u)) < n$ then $f(u(x)) = 0$ for $\mathcal{H}^n$-a.e. $x \in \mathbf{R}^n$; hence, as μ is simple,

$$0 = \mu(u) = \int_{\mathrm{dom}(u)} f(u(x))dx,$$

i.e. the theorem is proven. Therefore in the remaining part of the proof we will assume that $\dim(\mathrm{dom}(u)) = n$.

Initially, we will assume that $\mathrm{dom}(u) = L$ is a convex body (with non-empty interior) and that the restriction of u to L is continuous. This implies in particular that $g := f \circ u$ has compact support. We claim that the function g is Riemann integrable on $\mathbf{R}^n$. This will follow from Theorem 8.3 if we show that $g = f \circ u$ is bounded on $\mathbf{R}^n$ and the set of its discontinuities is a Lebesgue-null set.

It is easy to prove that g is bounded, since it is non-negative by construction and, as f is monotone decreasing, $\max_{\mathbf{R}^n}(g) = f(\min_{\mathbf{R}^n}(u)) < \infty$.

Since f is monotone decreasing, the set of its discontinuities is countable: let us call it $\{\xi_i\}_{i\in\mathbf{N}}$ and set $\xi_0 = \infty$. We claim that the set of discontinuities of g is contained in

$$\bigcup_{i\geq 0} \partial\{x \in \mathbf{R}^n \,:\, u(x) = \xi_i\}.$$

Let $C \subseteq \mathbf{R}^n$ denote the set of points where g is continuous. We therefore aim to prove the following:

$$C^{\complement} \subseteq \bigcup_{i\geq 0} \partial(u^{-1}(\xi_i)) = \bigcup_{i\geq 0} \big(\mathrm{cl}(u^{-1}(\xi_i)) \setminus \mathrm{int}(u^{-1}(\xi_i))\big) = \bigcup_{i\geq 0} \Big(\mathrm{cl}(u^{-1}(\xi_i)) \cap (\mathrm{int}(u^{-1}(\xi_i)))^{\complement}\Big),$$

which in turn is equivalent to

$$A := \bigcap_{i\geq 0} \Big((\mathrm{cl}(u^{-1}(\xi_i)))^{\complement} \cup \mathrm{int}(u^{-1}(\xi_i))\Big) \subseteq C.$$

Let us take a fixed $x \in A$; for every choice of $i \geq 0$ there are two possibilities:

(a) $x \notin \mathrm{cl}(u^{-1}(\xi_i))$,

(b) $x \in \mathrm{int}(u^{-1}(\xi_i))$.

Suppose **(b)** holds for two integers i, j; then

$$\left.\begin{array}{l} x \in \operatorname{int}(u^{-1}(\xi_i)) \subseteq u^{-1}(\xi_i) \implies u(x) = \xi_i \\ x \in \operatorname{int}(u^{-1}(\xi_j)) \subseteq u^{-1}(\xi_j) \implies u(x) = \xi_j \end{array}\right\} \text{ then } i = j.$$

Which means that **(b)** can happen at most once for every choice of x. Let us prove $x \in C$ in case **(b)** never holds. As $x \notin \operatorname{cl}(u^{-1}(\xi_0))$, and $\xi_0 = \infty$, x is an interior point of the domain of u, so that u is continuous at x. Moreover, for every $i \geq 0$ we have $x \notin \operatorname{cl}(u^{-1}(\xi_i))$ which implies $x \notin u^{-1}(\xi_i)$, i.e. f is continuous at $u(x)$. It follows that g is continuous at x. It remains to prove $x \in C$ in case **(b)** holds for a specific $j \geq 0$. Since $x \in \operatorname{int}(u^{-1}(\xi_j))$ there exists a neighborhood B of x such that $B \subseteq \operatorname{int}(u^{-1}(\xi_j)) \subseteq u^{-1}(\xi_j)$, which means $u(B) = \{\xi_j\}$. Thus u, and also g, is constant on a neighborhood of x and hence continuous at x.

Having finally proven $A \subseteq C$ it remains to show that $A^{\complement}$ is a Lebesgue-null set. Since $\partial u^{-1}(\xi_i)$ is the boundary of a convex body (possibly being empty) for every i, all these sets must be null sets, and so is their countable union (and therefore C itself is a null set, being a subset of a null set).

Now let R be a closed rectangle such that $\operatorname{int}(L) \subset R$; in particular g vanishes in the complement of R. As f is Riemann integrable, for every $\epsilon > 0$ there exists a rectangular partition (see section 7.2.1) of R

$$\mathcal{P} = \{R_1, \ldots, R_N\}$$

such that

$$\sum_{i=1}^{N} \left(\sup_{R_i} g - \inf_{R_i} g \right) V_n(R_i) \leq \epsilon,$$

and, clearly,

$$\sum_{i=1}^{N} \inf_{R_i} g \, V_n(R_i) \leq \int_{\mathbf{R}^n} g \, dx \leq \sum_{i=1}^{N} \sup_{R_i} g \, V_n(R_i).$$

Without loss of generality we could assume that $\mathcal{P}$ is an inductive partition and thus, by Lemma 8.2 we have

$$\sum_{i=1}^{N} \inf_{R_i} g \, V_n(R_i) \leq \mu(u) \leq \sum_{i=1}^{N} \sup_{R_i} g \, V_n(R_i)$$

(note that $u + I_R = u$, as R contains the domain of u). Hence

$$\left| \mu(u) - \int_{\mathbf{R}^n} f(u(x)) \, dx \right| \leq \epsilon$$

and, as ϵ is arbitrary,

$$\mu(u) = \int_{\mathbf{R}^n} f(u(x)) \, dx,$$

i.e. (8.26) is true for every $u \in \mathcal{C}^n$ such that the domain of u is a compact convex set and u is continuous on its domain. To prove the same equality for a general u, let L_i, $i \in \mathbf{N}$, be an increasing sequence of convex bodies such that

$$\bigcup_{i=1}^{\infty} L_i = \operatorname{dom}(u),$$

and consider the sequence of functions u_i, $i \in \mathbf{N}$, defined by $u_i = u + I_{L_i}$. This is a decreasing sequence of elements of $\mathcal{C}^n$ converging point-wise to u in the interior of $\mathrm{dom}(u)$. By the continuity of μ we have

$$\mu(u) = \lim_{i\to\infty} \mu(u_i) = \lim \int_{\mathbf{R}^n} f(u_i)\, dx$$

where, in the second equality, we have used the first part of the proof. On the other hand the sequence of functions $f \circ u_i$, $i \in \mathbf{N}$, is increasing and converges point-wise to $f \circ u$ in $\mathbf{R}^n$. Hence, by the monotone convergence theorem,

$$\lim_{i\to\infty} \int_{\mathbf{R}^n} f(u_i)\, dx = \int_{\mathbf{R}^n} f(u)\, dx.$$

The proof of (8.26) is complete. As for (8.27), it follows from Proposition 6.5.

□

Remark 8.4. It is clear from the previous proof that the representation formula (8.26) of Theorem 8.1 remains valid for those functions $u \in \mathcal{C}^n$, such that: $\mathrm{dom}(u) = L \in \mathcal{K}^n$ and the restriction of u to L is continuous, even if we drop the assumption of continuity of μ.

9 Characterization results II: homogeneous valuations

9.1 Part one: n-homogeneous valuations

The following result is a direct consequence of Theorem 8.1 and Proposition 5.6.

Theorem 9.1. *Let $\mu : \mathcal{C}^n \to \mathbf{R}$ be a rigid motion invariant, monotone decreasing, n-homogeneous and continuous valuation. Then there exists a function $f : \mathbf{R} \to \mathbf{R}$, coinciding with the geometric density f_n of μ, non-negative, decreasing, right-continuous, with finite n-th moment:*

$$\int_0^\infty t^n f(t) dt < \infty,$$

such that for every $u \in \mathcal{C}^n$

$$\mu(u) = \int_{\mathrm{dom}(u)} f(u(x)) dx.$$

Definition 9.2. (Extensions and restrictions of convex functions). *Let $k < n$. Let $u \in \mathcal{C}^k$. We can now extend u to the whole $\mathbf{R}^n$ in a canonical way by assigning the value ∞ where u was otherwise undefined:*

$$u|^n(x) = u|^n(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = \begin{cases} u(x_1, \dots, x_k) & \text{if } x_{k+1} = \dots = x_n = 0 \\ \infty & \text{otherwise} \end{cases}.$$

If $u \in \mathcal{C}^k$, then, it can be shown that $u|^n \in \mathcal{C}^n$. On the other hand, the so-called restriction of a convex function $u \in \mathcal{C}^n$ can be defined in the following way:

$$u|_k(x_1, \dots, x_k) = u(x_1, \dots, x_k, 0, \dots, 0).$$

It is immediate to show that $u|_k$ belongs to $\mathcal{C}^k$ for every choice of $u \in \mathcal{C}^n$.

Definition 9.3. (Restrictions of valuations). *Let $k < n$ as above. Let μ be a real valuation on $\mathcal{C}^n$, then we can define the restriction of μ to $\mathcal{C}^k$ as*

$$\mu|_k(u) = \mu(u|^n) \ \ \forall u \in \mathcal{C}^k.$$

It is easy to verify that $\mu|_k$ defined as above is a valuation on $\mathcal{C}^k$. Moreover, the valuation $\mu|_k$ inherits the following properties from μ: rigid motion invariance, monotonicity, continuity and homogeneity. Let us now consider a valuation μ on $\mathcal{C}^n$ and a convex function $u \in \mathcal{C}^n$ such that

$$\mathrm{dom}(u) \subseteq \{(x_1, \ldots, x_n) \in \mathbf{R}^n \ : \ x_{k+1} = \cdots = x_n = 0\}. \tag{9.29}$$

Under these assumptions we have that

$$\mu(u) = \mu|_k(u|_k). \tag{9.30}$$

The previous equality is an immediate consequence of Definition 9.3 and of the following consideration:

$$(u|_k)|^n = u$$

for every $u \in \mathcal{C}^n$ which satisfies (9.29). Restricted valuations also share geometric densities up to the suitable dimension. To be more precise, if $f_0, \ldots, f_n$ are the geometric densities of μ, then $f_0, \ldots, f_k$ are the geomtric densities of $\mu|_k$. To prove this, let t be a real number and H an arbitrary convex body in $\mathcal{K}^k$. Then,

$$K \times P \in \mathcal{K}^n$$

where $P = \{(0, \ldots, 0)\} \in \mathcal{K}^{n-k}$. We have

$$\mathrm{dom}(t + I_{K \times P}) = K \times P,$$

in other words, the function $t + I_{K \times P}$ satisfies (9.29). We deduce that (9.30) holds for $u = t + I_{K \times P}$, thus

$$\mu|_k\left((t + I_{K\times P})|_k\right) = \mu(t + I_{K \times P}) = \sum_{i=0}^{n} f_i(t) V_i(K \times P) = \sum_{i=0}^{k} f_i(t) V_i(K).$$

A simple calculation yields

$$(t + I_{K \times P})|_k = t + I_K \in \mathcal{C}^k.$$

Therefore

$$\mu|_k(t + I_K) = \sum_{i=0}^{k} f_i(t) V_i(K),$$

we conclude by the arbitrariness of t and K.

The following corollary of Theorem 9.1 will be important in the sequel.

Corollary 9.4. *Let $\mu : \mathcal{C}^n \to \mathbf{R}$ be a rigid motion invariant, monotone decreasing, k-homogeneous (for some in$\{0, 1, \dots, n\}$) and continuous valuation. Let f_k denote the k-th geometric density of μ. Then f_k has finite $(k-1)$-st moment:*

$$\int_0^\infty f_k(t) t^{k-1} dt < \infty.$$

Moreover, for every $u \in \mathcal{C}^n$ such that $\dim(\mathrm{dom}(u)) \leq k$

$$\mu(u) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\})) d\nu(t)$$

where ν is a Radon measure on $\mathbf{R}$ and f_k and ν are related by the identity

$$f_k(t) = \int_{(t,\infty)} d\nu(s) \quad \forall\, t \in \mathbf{R}.$$

Proof. Starting from μ we define its restriction to $\mathcal{C}^k$, $\mu|_k$.

As remarked, $\mu|_k$ is a valuation on $\mathcal{C}^k$ with the following properties: it is rigid motion invariant, monotone decreasing, continuous and k-homogeneous. Denote by g_i, $i \in \{0, 1, \dots, k\}$ its geometric densities; then $g_0 \equiv \dots g_{k-1} \equiv 0$ and $g_k = f_k$. By Theorem 9.1 we have that f_k has finite $(k-1)$-st moment and

$$\mu|_k(v) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{v < t\})) d\nu(t) \tag{9.31}$$

for every $v \in \mathcal{C}^k$, where ν and f_k are related as usual by $f_k(t) = \nu((t, \infty))$. Now let $u \in \mathcal{C}^n$ be such that $\dim(\mathrm{dom}(u)) \leq k$; we want to compute $\mu(u)$. As μ is rigid motion invariant, without loss of generality we may assume that

$$\mathrm{dom}(u) \subseteq \{(x_1, \dots, x_n) \in \mathbf{R}^n \,:\, x_{k+1} = \dots = x_n = 0\}.$$

Then $\mu(u) = \mu|_k(u|_k)$. If we set $P = \{(0, \dots, 0)\} \in \mathcal{K}^{n-k}$ as before, then it is simple to verify that

$$\{u < t\} = \{u|_k < t\} \times P \implies \mathrm{cl}(\{u < t\}) = \mathrm{cl}(\{u|_k < t\}) \times P,$$

so that $V_k(\mathrm{cl}(\{u < t\})) = V_k(\mathrm{cl}(\{u|_k < t\}))$. The claimed representation formula for $\mu(u)$ follows from the previous considerations and (9.31) specialized to the case $v = u|_k$. □

9.2 Part two: the general case

Theorem 9.5. *Let $\mu : \mathcal{C}^n \to \mathbf{R}$ be a rigid motion invariant, monotone decreasing, k-homogeneous (for some $k \in \{0, \dots, n\}$) and continuous valuation. Then there exists a Radon measure ν defined on $\mathbf{R}$ with finite k-th moment, such that*

$$\mu(u) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\})) d\nu(t),$$

for every $u \in \mathcal{C}^n$. Moreover, the measure ν is determined by the unique non-vanishing geometric density f_k of μ as follows:

$$f_k(t) = \int_{(t,\infty)} d\nu(s) \quad \forall\, t \in \mathbf{R}.$$

The rest of this section is devoted to the proof of this result; throughout, μ will be a valuation with the properties listed in the previous theorem. Note that the validity of the Theorem for $k = n$ is established by Theorem 9.1.

By Proposition 5.1 we may assign to μ its geometric densities f_j, $j = 0, \dots, n$. By homogeneity we have $f_j \equiv 0$ for every $j \neq k$. In other words, the only density which can be non-identically zero is f_k. For simplicity we will call this function f. By the properties of μ, this is a non-negative decreasing function; moreover, as μ is continuous, f is right-continuous on $\mathbf{R}$ and, by Corollary 9.4, it has finite $(k-1)$-st moment:

$$\int_0^\infty t^{k-1} f(t) dt < \infty.$$

We proceed by induction on the dimension n. Let us then start from the case $n = 1$. As the theorem is already proven for $k = n = 1$ we only need to consider the case $k = 0$; but this follows from Proposition 5.4: Theorem 9.5 is proven in dimension $n = 1$.

To continue with the induction argument, we assume that the theorem holds up to dimension $(n-1)$ and we are going to prove it in the n-dimensional case. We may assume that $1 \leq k \leq n - 1$.

In the next part of the proof we will assume, in addition to the above properties, that the only non-zero density f of μ is smooth: $f \in C^\infty(\mathbf{R})$. As in the previous sections, we introduce the Radon measure ν related to f by the identity

$$f(t) = \int_{(t,\infty)} d\nu(s) \quad \forall\, t \in \mathbf{R}.$$

Based on ν, we construct an auxiliary valuation $\mu_a : \mathcal{C}^n \to \mathbf{R}$ defined as follows

$$\mu_a(u) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\})) d\nu(t).$$

By the results of section 5, μ_a is a well defined valuation and it is rigid motion invariant, decreasing, k-homogeneous and continuous. Moreover, its geometric density of order k is precisely f, i.e. the same as μ.

We also set

$$\mu_r = \mu_a - \mu.$$

The idea is to prove that μ_r is identically zero. Note that μ_r inherits most of the properties of μ and μ_a: it is a valuation, rigid motion invariant, k-homogeneous and continuous. We cannot infer in general that μ_r is monotone.

Claim 1. *The valuation μ_r "vanishes horizontally", i.e. for every convex body $K \in \mathcal{K}^n$ and every $t \in \mathbf{R}$ we have*

$$\mu_r(t + I_K) = 0.$$

The proof is a straightforward consequence of the fact that μ and μ_a have the same geometric densities.

Claim 2. *The valuation μ_r is simple.*

Proof. Let $u \in \mathcal{C}^n$ be a convex function whose domain has dimension strictly less than n. As μ_r is rigid motion invariant, we might assume without loss of generality that

$$u \subseteq \{(x_1, \ldots, x_n) \in \mathbf{R}^n \,:\, x_n = 0\}.$$

As remarked after Definition 9.3, $\mu(u) = \mu|_{n-1}(u|_{n-1})$. By the induction hypothesis we have

$$\mu|_{n-1}(u|_{n-1}) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u|_{n-1} < t\}))d\nu(t). \tag{9.32}$$

As $\{u < t\} = \{u|_{n-1} < t\} \times \{0\}$, (9.32) can be rewritten as

$$\mu(u) = \mu|_{n-1}(u|_{n-1}) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\}))d\nu(t) = \mu_a(u).$$

Therefore $\mu_r(u) = 0$, we conclude that μ_r is simple as claimed. □

We will now introduce a construction which is going to help us evaluate a valuation on piecewise linear functions. We fix

$$e \in \mathbf{R}^n \text{ s.t. } |e| = 1, \quad p \geq 0.$$

Let μ_0 be a valuation on $\mathcal{C}^n$ and consider the linear function $w : \mathbf{R}^n \to \mathbf{R}$ defined by

$$w(x) = (x, pe), \quad x \in \mathbf{R}^n.$$

Then we define a mapping on the family of convex bodies of $\mathbf{R}^n$, $\sigma_{\mu_0,e,p} : \mathcal{K}^n \to \mathbf{R}$, as follows

$$\sigma_{\mu_0,e,p}(K) = \mu_0(w + I_K) \quad \forall\, K \in \mathcal{K}^n.$$

It is easy to check that $\sigma_{\mu_0,V}$ is a valuation.

From now on throughout this paper, we will consider the previous construction specialized to valuations which are rigid motion invariant, hence we will assume without loss of generality that $e = e_n = (0, \ldots, 0, 1)$. Moreover, for the sake of brevity, we will introduce the following simplified notation:

$$\sigma := \sigma_{\mu,e,p}, \quad \sigma_a := \sigma_{\mu_a,e,p}, \quad \sigma_r := \sigma_{\mu_r,e,p} = \sigma - \sigma_a.$$

The following claim collects some of the properties of σ_r that will be used in the sequel.

Claim 3. *σ_r has the following properties:*

1. *it is a valuation on $\mathcal{K}^n$;*
2. *it is simple;*
3. *it is invariant with respect to every rigid motion T of $\mathbf{R}^n$ such that*

$$T(x) = T(x_1, \ldots, x_n) = (T'(x_1, \ldots, x_{n-1}), x_n) \quad \forall\, x \in \mathbf{R}^n, \tag{9.33}$$

where T' is a rigid motion of $\mathbf{R}^{n-1}$.

Proof. Let $K, L \in \mathcal{K}^n$ be such that $K \cup L \in \mathcal{K}^n$. Then

$$I_{K\cup L} = I_K \wedge I_L, \quad I_{K\cap L} = I_K \vee I_L.$$

These relations remain valid if we add w as follows

$$w + I_{K\cup L} = (w + I_K) \wedge (w + I_L), \quad w + I_{K\cap L} = (w + I_K) \vee (w + I_L).$$

Using the valuation property of μ_r we easily deduce the valuation property for σ_r. Moreover

$$\sigma_r(\emptyset) = \mu_r(w + I_\emptyset) = \mu_r(\infty) = 0.$$

We conclude that σ_r is a valuation.

If $K \in \mathcal{K}^n$ has no interior point, the domain of I_K, and consequently that of $w + I_K$, have the same property. Then,

$$\sigma_r(K) = \mu_r(w + I_K) = 0.$$

Next we prove 3. Let T be a rigid motion of $\mathbf{R}^n$ of the form (9.33), and let $K \in \mathcal{K}^n$. Then

$$\begin{aligned}(w + I_{T(K)})(x) &= px_n + I_{T(K)}(x) = px_n + I_K(T'^{-1}(x_1, \dots, x_{n-1}), x_n) \\ &= (w + I_K)(T'^{-1}(x_1, \dots, x_{n-1}), x_n) = (w + I_K)(T^{-1}(x)).\end{aligned}$$

Therefore

$$\sigma_r(T(K)) = \mu_r(w + I_{T(K)}) = \mu_r((w + I_K) \circ T^{-1}) = \mu_r(w + I_K) = \sigma_r(K),$$

where we have used the invariance if μ_r. □

We anticipate that the following step is one of the most delicate in the proof.

Claim 4. *The valuation σ_r is non-negative, i.e.*

$$\sigma_r(K) \geq 0 \quad \forall\, K \in \mathcal{K}^n.$$

Proof. We first treat the easier case $p = 0$, which leads to $w \equiv 0$ so that

$$\sigma(K) = \mu(I_K) = f(0)V_k(K) = \sigma_a(K) \Rightarrow \sigma_r(K) = \sigma_a(K) - \sigma(K) = 0,$$

where we have used Claim 1. Next we assume $p > 0$. Given two real numbers α, β with $\alpha \leq \beta$, we define the strip:

$$S[\alpha, \beta] := \{(x_1, \dots, x_n) \in \mathbf{R}^n \,:\, \alpha \leq x_n \leq \beta\}.$$

Let $K \in \mathcal{K}^n$ and let y_m and y_M be such that the hyperplanes with equations $x_n = y_m$ and $x_n = y_M$ are the supporting hyperplanes to K with outer unit normals $-e_n$ and e_n respectively. In other words,

$$K \subset S[y_m, y_M]$$

and $S[y_m, y_M]$ is the intersection of all possible sets of the form $S[\alpha, \beta]$ containing K. We define a function $\phi : [y_m, y_M] \to \mathbf{R}$:

$$\phi(y) = \sigma_r(K \cap S[y_m, y]).$$

Let $y \in (y_m, y_M)$ and $h \geq 0$ be sufficiently small so that $y + h \leq y_M$. As, trivially,

$$S[y_m, y] \cup S[y, y+h] = S[y_m, y+h], \quad S[y_m, y] \cap S[y, y+h] = S[y, y],$$

using the valuation property of σ_r and the fact that it is simple, we get

$$\begin{aligned} \phi(y+h) - \phi(y) &= \sigma_r(K \cap S[y, y+h]) \\ &= \sigma_a(K \cap S[y, y+h]) - \sigma(K \cap S[y, y+h]). \end{aligned}$$

Next we use the monotonicity of μ. Note that

$$\min_{K \cap S[y,y+h]} w = py, \qquad \max_{K \cap S[y,y+h]} w = p(y+h).$$

Hence

$$\begin{aligned} \sigma(K \cap S[y, y+h]) &= \mu(w + I_{K \cap S[y,y+h]}) \leq \mu(py + I_{K \cap S[y,y+h]}) \\ &= f(py)\, V_k(K \cap S[y, y+h]). \end{aligned}$$

And similarly

$$\sigma(K \cap S[y, y+h]) \geq f(p(y+h))\, V_k(K \cap S[y, y+h]).$$

On the other hand

$$\begin{aligned} \sigma_a(K \cap S[y, y+h]) &= \mu_a(w + I_{K \cap S[y,y+h]}) \\ &= -\int_{\mathbf{R}} V_k(\mathrm{cl}(\{w + I_{K \cap S[y,y+h]} < t\})) f'(t) dt, \end{aligned}$$

where we have used the assumption that f is smooth. Now

$$\mathrm{cl}(\{w + I_{K \cap S[y,y+h]} < t\}) = \begin{cases} \emptyset & \text{if } t \leq py, \\ K \cap S[y, t/p] & \text{if } py < t < p(y+h), \\ K \cap S[y, y+h] & \text{if } t \geq p(y+h). \end{cases}$$

Hence

$$\begin{aligned} \sigma_a(K \cap S[y, y+h]) &= -\int_{py}^{p(y+h)} V_k(K \cap S[y, t/p]) f'(t) dt - V_k(K \cap S[y, y+h]) \int_{p(y+h)}^{\infty} f'(t) dt \\ &= -p \int_{y}^{y+h} V_k(K \cap S[y, s]) f'(ps) ds + f(p(y+h)) V_k(K \cap S[y, y+h]), \end{aligned}$$

where, for the second term we have used the equality, due to the condition $k \geq 1$ and the integrability condition on f,

$$\lim_{t \to \infty} f(t) = 0.$$

Consequently we have the following bounds:

$$\phi(y+h) - \phi(y) \leq -p \int_y^{y+h} V_k(K \cap S[y, y+h]) f'(ps) ds, \tag{9.34}$$

and

$$\begin{aligned} \phi(y+h) - \phi(y) \ &\geq \ V_k(K \cap S[y, y+h]) \left(f(p(y+h)) - f(py) \right) \\ &- \ p \int_y^{y+h} V_k(K \cap S[y, y+h]) f'(ps) ds = 0. \end{aligned} \tag{9.35}$$

Note that the function

$$\tau \mapsto V_k(K \cap S[y, y+\tau])$$

is Lipschitz continuous in a neighborhood of $\tau = 0$ (indeed, as already remarked before, its $1/k$ power is concave in $[y_m, y_M]$) and, by monotonicity of intrinsic volumes, it is bounded by $V_k(K)$, i.e. a constant independent of y and h. Then, as f is smooth, it follows from (9.34) and (9.35) that ϕ is Lipschitz continuous in $[y_m, y_M]$; in particular (9.35) implies that

$$\phi'(y) \geq 0$$

for every y for which ϕ' is defined. As

$$\phi(y_m) = \sigma_r(K \cap S[y_m, y_m]) = 0$$

(recall that σ_r is simple), we have that

$$\sigma_r(K) = \phi(y_M) = \phi(y_m) + \int_{y_m}^{y_M} \phi'(t) dt \geq 0.$$

The proof is complete. □

For the sequel we will need the following result (which could be well-known in the theory of valuations on convex bodies).

Lemma 9.6. *Let $\sigma : \mathcal{K}^n \to \mathbf{R}$ be a valuation which is non-negative and simple. Then σ is monotone increasing on the class of polytopes, i.e. for every P and Q polytopes in $\mathbf{R}^n$ such that $P \subseteq Q$ we have*

$$\sigma(P) \leq \sigma(Q).$$

Proof. Let P and Q be polytopes such that $P \subseteq Q$; let $\mathcal{F}$ be a family of hyperplanes in $\mathbf{R}^n$, defined as follows:

$$\mathcal{F}(P, Q) = \{H \text{ is a hyperplane containing a facet of } P \text{ and } H \cap \text{int}(Q) \neq \emptyset\},$$

and let $N(P,Q) \geq 0$ be the cardinality of $\mathcal{F}(P,Q)$. We will prove that

$$\sigma(P) \leq \sigma(Q),$$

by induction on $N(P,Q)$. If $N(P,Q) = 0$ we have that $P = Q$ so that there is nothing to prove. Assume that the claim is true up to $(n-1)$, for some $n \in \mathbf{N}$, and that $N(P,Q) = n$. Let $H \in \mathcal{F}$ and let H^+ and H^- be the closed half-spaces determined by H. We may assume that $P \subseteq H^+$. Let $Q^+ = Q \cap H^+$ and $Q^- = Q \cap H^-$ (which are still polytopes); as

$$Q = Q^+ \cup Q^- \quad \text{and} \quad Q^+ \cap Q^- \subset H,$$

and as σ is simple and non-negative

$$\sigma(Q) = \sigma(Q^+) + \sigma(Q^-) \geq \sigma(Q^+).$$

On the other hand $Q^+ \supseteq P$ and

$$\mathcal{F}(P, Q^+) \subset \mathcal{F}(P,Q),$$

in particular $N(P,Q^+) < N(P,Q)$ so that, by the induction assumption,

$$\sigma(P) \leq \sigma(Q^+) \leq \sigma(Q).$$

□

Claim 5. *The valuation σ_r is monotone increasing.*

Proof. Let $K, L \in \mathcal{K}^n$ be such that $K \subseteq L$; there exist two sequences of polytopes P_i, Q_i, $i \in \mathbf{N}$, with the following properties:

1. they are increasing with respect to set inclusion;
2. $P_i \to K$ and $Q_i \to L$ as i tends to infinity, in the Hausdorff metric;
3. $P_i \subseteq Q_i$ for every $i \in \mathbf{N}$.

In particular $\sigma_r(P_i) \leq \sigma_r(Q_i)$. Now, recalling the definition of σ_r we have that

$$\sigma_r(P_i) \quad = \quad \sigma_a(P_i) - \sigma(P_i) = \mu_a(u_i) - \mu(u_i)$$

where

$$u_i(x) = I_{P_i}(x) + (x, V), \quad x \in \mathbf{R}^n.$$

Note that u_i is a decreasing sequence, and it converges point-wise to $u : \mathbf{R}^n \to \mathbf{R}$ defined by

$$u(x) = I_K(x) + (x, V)$$

in the relative interior of K. As μ_a and μ are continuous we have

$$\sigma_r(P_i) = \mu_r(u_i) = \mu_a(u_i) - \mu(u_i) \to \mu_a(u) - \mu(u) = \mu_r(u) = \sigma_r(K).$$

In a similar way we can prove that

$$\lim_{i \to \infty} \sigma_r(Q_i) = \sigma_r(L).$$

Since, as already pointed out, $\sigma_r(P_i) \leq \sigma_r(Q_i)$ for all $i \in \mathbf{N}$, passing to the limit for $i \to \infty$ yields the claimed $\sigma_r(K) \leq \sigma_r(L)$. □

Let us make a further step to investigate the behavior of μ_r on restrictions of linear functions. Given a function $u \in \mathcal{C}^{n-1}$, we may consider the set

$$\mathrm{epi}(u) = \{(x', y) \in \mathbf{R}^{n-1} \times \mathbf{R} \,:\, y \geq u(x')\}.$$

Claim 6. *Let $p > 0$. For every $u \in \mathcal{C}^{n-1}$ the function $w + I_{\mathrm{epi}(u)}$ belongs to $\mathcal{C}^n$.*

Proof. The set epi(u) is convex and closed (by the semi-continuity of u). Hence the function $v = w + I_{\mathrm{epi}(u)}$ is lower semi-continuous and convex. Let x_i, $i \in \mathbf{N}$, be a sequence in $\mathbf{R}^n$ such that

$$\lim_{i\to\infty} |x_i| = \infty.$$

From any subsequence of x_i we may extract a further subsequence (let us call it $\bar{x}_i$) such that either $\bar{x}_i \in \mathrm{epi}(u)$ for every i or $\bar{x}_i \in \mathbf{R}^n \setminus \mathrm{epi}(u)$ for every i. In the second case we have $v(\bar{x}_i) = \infty$ for every i. In the first case we have, setting $\bar{x}_i = (\bar{x}_i', \bar{y}_i) \in V^\perp \times \mathbf{R}$, there exists constants $a > 0$, $b \in \mathbf{R}$ such that

$$\bar{y}_i \geq u(\bar{x}_i) \geq a|\bar{x}_i'| + b \quad \forall\, i \in \mathbf{N}$$

(see Proposition 3.8). As $|\bar{x}_i|$ is unbounded, we must have that $\bar{y}_i$ is not bounded from above and, up to extracting a further subsequence we may assume that

$$\lim_{i\to\infty} \bar{y}_i = \infty.$$

This implies that $v(\bar{x}_i) = py_i$ tends to infinity as well. Hence from any subsequence x_i such that $|x_i| \to \infty$ we may extract a subsequence $\hat{x}_i$ such that $v(\hat{x}_i)$ tends to infinity. Hence

$$\lim_{|x|\to\infty} v(x) = \infty$$

and we conclude that $v \in \mathcal{C}^n$. □

Let us define $\bar{\mu} : \mathcal{C}^{n-1} \to \mathbf{R}$ by

$$\bar{\mu}(u) = \mu_r(w + I_{\mathrm{epi}(u)}).$$

Claim 7. *Let $p > 0$. The function $\bar{\mu}$ has the following properties:*

1. *it is a rigid motion invariant valuation;*
2. *it is simple;*
3. *it is monotone decreasing.*

Proof. We will denote a point in $\mathbf{R}^n$ by (x', y), with $x' \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$. For $u \in \mathcal{C}^{n-1}$ and $t \in \mathbf{R}$ set

$$\mathrm{epi}_t(u) = \{(x', y) \in \mathbf{R}^{n-1} \times \mathbf{R} \,:\, u(x') \leq y \leq t\} = \mathrm{epi}(u) \cap \{(x', y) \,:\, y \leq t\}.$$

By the continuity of μ_r,

$$\bar{\mu}(u) = \lim_{t\to\infty} \mu_r(w + I_{\mathrm{epi}_t(u)}) = \lim_{t\to\infty} \sigma_r(\mathrm{epi}_t(u)).$$

We will see that properties 1 - 3 follow easily from this characterization of $\bar{\mu}$ and Claim 3. Assume that T is a rigid motion of $\mathbf{R}^{n-1}$. Define

$$\bar{T} : \mathbf{R}^n \to \mathbf{R}^n, \quad \bar{T}(x', y) = (T(x'), y).$$

$\bar{T}$ is a rigid motion of $\mathbf{R}^n$ and it verifies $(\bar{T}(x), V) = (x, V)$ for every $x \in \mathbf{R}^n$. Then, by item 3 in Claim 3,

$$\sigma_r(\mathrm{epi}_t(u)) = \sigma_r(\bar{T}(\mathrm{epi}_t(u))).$$

On the other hand

$$\bar{T}(\mathrm{epi}_t(u)) = \mathrm{epi}_t(u \circ T^{-1}).$$

Replacing this equality in the previous one, and letting $t \to \infty$, we get

$$\bar{\mu}(u) = \bar{\mu}(u \circ T^{-1}),$$

which proves that $\bar{\mu}$ is rigid motion invariant.

To prove that $\bar{\mu}$ is simple, let $u \in \mathcal{C}^{n-1}$ be such that $\dim(\mathrm{dom}(u)) \le (n-2)$. Then $\dim(\mathrm{epi}(u)) \le (n-1)$ and

$$\dim(\mathrm{epi}_t(u)) \le n-1 \quad \forall\, t \in \mathbf{R}.$$

As μ_r is simple, σ_r is simple, by Claim 3. Hence

$$\sigma_r(\mathrm{epi}_t(u)) = 0 \quad \forall\, t.$$

Letting t tend to infinity we get $\bar{\mu}(u) = 0$.

As for monotonicity, if u and v belong to $\mathcal{C}^{n-1}$ and are such that $u \le v$ in $\mathbf{R}^{n-1}$, then

$$\mathrm{epi}(u) \supseteq \mathrm{epi}(v) \;\Rightarrow\; \mathrm{epi}_t(u) \supseteq \mathrm{epi}_t(v) \quad \forall\, t \in \mathbf{R}.$$

As σ_r is monotone increasing we get

$$\sigma_r(\mathrm{epi}_t(u)) \ge \sigma_r(\mathrm{epi}_t(v)) \quad \forall\, t \in \mathbf{R}.$$

The conclusion follows letting t to ∞. □

Claim 8. *Let $p > 0$. There exists a function $\bar{f} : \mathbf{R} \times (0, \infty)$, $\bar{f} = \bar{f}(t, p)$, such that*

$$\bar{\mu}(u) = \int_{\mathrm{dom}(u)} \bar{f}(u(x'), p)\, dx' \tag{9.36}$$

for every $u \in \mathcal{C}^{n-1}$ such that $\mathrm{dom}(u) \in \mathcal{K}^n$ and the restriction of u to $\mathrm{dom}(u)$ is continuous (here dx' denotes the usual integration in $\mathbf{R}^{n-1}$).

Proof. By Claim 7 we may apply Theorem 8.1 and subsequent Remark 8.4 to deduce (9.36). □

Given $K \in \mathcal{K}^{n-1}$ and $t_1, t_2 \in \mathbf{R}$, with $t_1 \le t_2$, we consider the cylinder:

$$K \times [t_1, t_2] \in \mathcal{K}^n.$$

Evaluating σ_r on cylinders is a crucial step, as we will see in the following claim.

Claim 9. *Let $p > 0$. For every $K \in \mathcal{K}^{n-1}$, and every $t_1, t_2 \in \mathbf{R}$ with $t_1 \le t_2$ we have:*

$$\sigma_r(K \times [t_1, t_2]) = \mu_r(w + I_{K\times[t_1,t_2]}) = V_{n-1}(K)(\bar{f}(t_1, p) - \bar{f}(t_2, p)). \tag{9.37}$$

Proof. We have, for every $t_1 \in \mathbf{R}$,

$$K \times [t_1, \infty) = \mathrm{epi}(u) \quad \text{where} \quad u = t_1 + I_K.$$

$$\begin{aligned} \mu_r(w + I_{K\times[t_1,\infty)}) &= \mu_r(w + I_{\mathrm{epi}(u)}) \\ &= \bar{\mu}(u) = \int_{\mathrm{dom}(u)} \bar{f}(u(x'), p)dx' \\ &= \int_K \bar{f}(t_1, p)dx' = V_{n-1}(K)f(t_1, p). \end{aligned} \tag{9.38}$$

On the other hand, for $t_1, t_2 \in \mathbf{R}$ with $t_1 \le t_2$ we have

$$K \times [t_1, t_2] \cup K \times [t_2, \infty) = K \times [t_1, \infty), \quad K \times [t_1, t_2] \cap K \times [t_2, \infty) = K \times \{t_2\};$$

so that

$$\begin{aligned} (w + I_{K\times[t_1,t_2]}) \wedge (w + I_{K\times[t_2,\infty)}) = w + I_{K\times[t_1,\infty)}, \\ (w + I_{K\times[t_1,t_2]}) \vee (w + I_{K\times[t_2,\infty)}) = w + I_{K\times\{t_2\}}. \end{aligned}$$

Hence, as μ_r is a valuation and it is simple, and as $\dim(K \times \{t_2\}) \le n - 1$, we obtain

$$\begin{aligned} \mu_r(w + I_{K\times[t_1,t_2]}) &= \mu_r(w + I_{K\times[t_1,\infty)}) - \mu_r(w + I_{K\times[t_2,\infty)}) \\ &= V_{n-1}(K)(\bar{f}(t_1, p) - \bar{f}(t_2, p)). \end{aligned}$$

□

The next step is to deduce further information about $\bar{f}$ exploiting the homogeneity of μ_r (recall that μ_r is homogeneous of order k).

Claim 10. *There exists a non-negative decreasing function $\phi : \mathbf{R} \to \mathbf{R}$ such that*

$$\bar{f}(t, p) = p^{n-1-k}\phi(tp) \quad \forall\, (t, p) \in \mathbf{R} \times (0, \infty).$$

Proof. We recall that $w_p(x) = p(x, e_n)$ for every choice of $p \geq 0$ and $x \in \mathbf{R}^n$. As before, let $K \in \mathcal{K}^{n-1}$ and let $\lambda > 0$; we have, for $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$,

$$\begin{aligned} w_p\left(\frac{x}{\lambda}\right) + I_{K\times[t,\infty)}\left(\frac{x}{\lambda}\right) &= w_{p/\lambda}(x) + I_{\lambda(K\times[t,\infty))}(x) \\ &= w_{p/\lambda}(x) + I_{\lambda K\times[\lambda t,\infty)}(x). \end{aligned}$$

By the homogeneity of μ_r

$$\begin{aligned} \mu_r\left((w_p + I_{K\times[t,\infty)})\left(\frac{\cdot}{\lambda}\right)\right) &= \lambda^k \mu_r(w_p + I_{K\times[t,\infty)}) \\ &= \lambda^k V_{n-1}(K)\bar{f}(t,p), \end{aligned}$$

and

$$\begin{aligned} \mu_r(w_{p/\lambda} + I_{\lambda K\times[\lambda t,\infty)}) &= V_{n-1}(\lambda K)\bar{f}\left(\lambda t, \frac{p}{\lambda}\right) \\ &= \lambda^{n-1}V_{n-1}(K)\bar{f}\left(\lambda t, \frac{p}{\lambda}\right) \end{aligned}$$

by the homogeneity of intrinsic volumes. Hence, as we may chose K so that $V_{n-1}(K) > 0$, we obtain that for every $t \in \mathbf{R}$, $p > 0$ and $\lambda > 0$ we have

$$\bar{f}(t,p) = \lambda^j \bar{f}\left(\lambda t, \frac{p}{\lambda}\right).$$

with

$$j = n - 1 - k.$$

Taking $\lambda = p$ yields

$$\bar{f}(t,p) = p^j \bar{f}(tp, 1) = p^j \phi(tp),$$

where we have set

$$\phi(s) = \bar{f}(s, 1)$$

for all real s. As $\bar{f}$ is non-negative and decreasing with respect to t for every $p > 0$ the claim follows. □

In the next step we prove that the continuity of μ_r implies that the function ϕ is constant.

Claim 11. *The function ϕ introduced in the previous step is constant in $\mathbf{R}$ (in particular σ_r vanishes on cylinders).*

Proof. By the previous steps we have that for every $K \in \mathcal{K}^{n-1}$ and for every $t_1, t_2 \in \mathbf{R}$ with $t_1 \leq t_2$,

$$\mu_r(w + I_{K\times[t_1,t_2]}) = V_{n-1}(K)(\phi(pt_1) - \phi(pt_2))\, p^j. \tag{9.39}$$

Let K be the $(n-1)$-dimensional unit cube with centre at the origin and let

$$D = \{x = (x_1, \dots, x_n) \,:\, (x_1, \dots, x_{n-1}) \in K,\, -1 \leq x_n \leq 0\}.$$

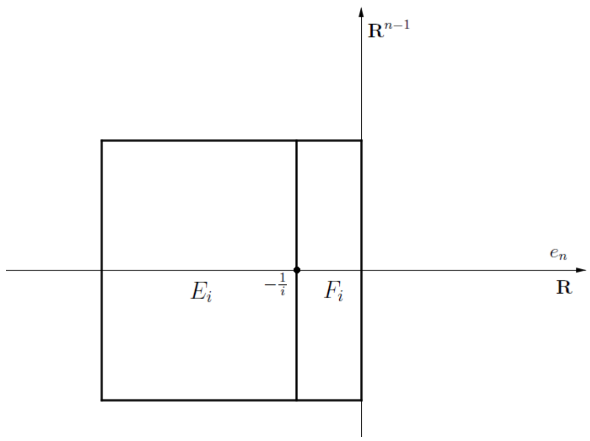

Figure 2: The sets E_i and F_i

We also set, for $i \in \mathbf{N}$,

$$E_i = D \cap \left\{ x \,:\, -1 \leq x_n \leq -\frac{1}{i} \right\}, \quad F_i = D \cap \left\{ x \,:\, -\frac{1}{i} \leq x_n \leq 0 \right\}.$$

In particular, for every i,

$$E_i, F_i \in \mathcal{K}^n, \quad E_i \cup F_i = D, \quad \dim(E_i \cap F_i) = n - 1.$$

Let $s > 0$. For $i \in \mathbf{N}$ define the function $v_i \,:\, \mathbf{R}^n \to \mathbf{R}$ as

$$\bar{v}_i(x) = \bar{v}_i(x_1, \ldots, x_n) = s \cdot i \cdot \left(x_n + \frac{1}{i} \right)$$

and

$$v_i = \bar{v}_i \vee I_D.$$

Note that

$$v_i = \infty \text{ in } \mathbf{R}^n \setminus D, \quad v_i = 0 \text{ in } E_i, \quad v_i = \bar{v}_i \text{ in } F_i.$$

In particular v_i is a decreasing sequence of functions in $\mathcal{C}^n$ converging to I_D in the relative interior of D, so that by continuity we have

$$\lim_{i\to\infty} \mu_r(v_i) = \mu_r(I_D) = 0,$$

where we have used the fact that μ_r vanishes horizontally (Claim 1). We may also write

$$v_i = (\bar{v}_i + I_{F_i}) \vee I_{E_i},$$

and using the fact that μ_r is a simple valuation, and Claim 1 again, we get that $\mu_r(v_i) = \mu_r(\bar{v}_i + I_{F_i})$ so that

$$\lim_{i\to\infty} \mu_r(\bar{v}_i + I_{F_i}) = 0.$$

On the other hand, by translation invariance, if we set

$$u_i(x) = u_i(x_1, \ldots, x_n) = (\bar{v}_i + I_{F_i})\left(x_1, \ldots, x_{n-1}, x_n - \frac{1}{i}\right),$$

we find that

$$u_i = w_i + I_{K\times[0,t_i]},$$

where

$$w_i(x) = si\, x_n \quad \text{and} \quad t_i = \frac{1}{i}.$$

Consequently, by (9.39)

$$\mu_r(u_i) = (si)^j\, V_{n-1}(K)(\phi(0) - \phi(s)) \quad \forall\, i \in \mathbf{N}.$$

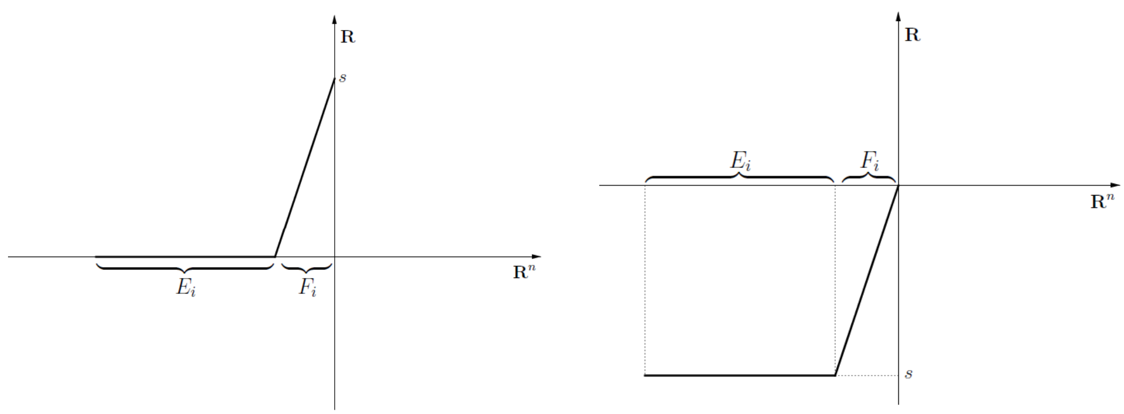

Figure 3: The construction of v_i for $s > 0$ and $s < 0$

Letting i tend to infinity this quantity must tend to zero, by the previous part of the proof; as $j \geq 0$ and $V(K) > 0$, the only possibility is $\phi(s) = \phi(0)$. This proves that ϕ is constantly equal to $\phi(0)$ in $[0, \infty)$.

To achieve the same result in $(-\infty, 0]$ we may argue in a similar way. Let $s < 0$ and K, D, E_i as above. Set

$$\bar{v}_i(x_1, \ldots, x_n) = -si\, x_n$$

and

$$v_i = (I_D + s) \vee \bar{v}_i.$$

This is again a decreasing sequence in $\mathcal{C}^n$, converging to $s + I_D$ in the relative interior of D; by Claim 1:

$$\lim_{i\to\infty} \mu_r(v_i) = 0.$$

On the other hand

$$\mu_r(v_i) = \mu_r(\bar{v}_i + I_{K\times[-1/i,0]}) = (-si)^j V_{n-1}(K)(\phi(s) - \phi(0)).$$

The conclusion $\phi(s) = \phi(0)$ follows as above.

□

Claim 12. *Let $e \in \mathbf{R}^n$ be a unit vector, $p \geq 0$, $c \in \mathbf{R}$ and $K \in \mathcal{K}^n$; define*

$$u : \mathbf{R}^n \to \mathbf{R}, \quad u(x) = (x, pe) + c + I_K(x).$$

Then

$$\mu_r(u) = 0.$$

Proof. Assume first that $c = 0$. If $p = 0$ the assert follows from Claim 1. Assume $p > 0$; recalling the definition of $\sigma_{\mu_r,e,p}$ we have:

$$\mu_r(u) = \sigma_{\mu_r,e,p}(K).$$

On the other hand, since μ_r is rigid motion invariant, we can assume, without loss of generality, that $e = e_n$ and, as remarked in Claim 11, setting as before $w : \mathbf{R}^n \to \mathbf{R}$ defined by $w(x) = (x, pe_n)$, we get

$$\sigma_{\mu_r,e,p}(H \times [t_1, t_2]) = \mu_r(w + I_{H\times[t_1,t_2]}) = 0.$$

for every $H \in \mathcal{K}^{n-1}$, $t_1, t_2 \in \mathbf{R}$ such that $t_1 \leq t_2$. Let us choose H, t_1 and t_2 such that

$$K \subseteq H \times [t_1, t_2].$$

Then, as $\sigma_{\mu_r,e,p}$ is non-negative and monotone increasing (Claims 4 and 5),

$$0 \leq \sigma_{\mu_r,V}(K) \leq \sigma_{\mu_r,e,p}(H \times [t_1, t_2]) = 0.$$

The case $c \neq 0$ is readily recovered by the previous one using the translation invariance of μ_r. □

The last result will open the way to prove that μ_r vanishes on piecewise linear functions and, eventually, it vanishes identically on $\mathcal{C}^n$.

Definition 9.7. *A function $u \in \mathcal{C}^n$ is called piecewise linear if:*

- $\mathrm{dom}(u) = P$ *is a polytope;*
- *there exists a polytopal partition $\mathcal{P} = \{P_1, \dots, P_N\}$ of P such that for every $i \in \{1, \dots, N\}$ there exists $\xi_i \in \mathbf{R}^n$ and $c_i \in \mathbf{R}$ such that*

$$u(x) = c_i + (x, \xi_i) \quad \forall\, x \in P_i.$$

Claim 13. *The valuation μ_r vanishes on piecewise linear functions.*

Proof. As any polytopal partition admits a refinement which is a complete partition (see Remark 7.3 in section 7.2), without loss of generality we may assume that $\mathcal{P}$ is complete, so that in particular it is an inductive partition (see Proposition 7.4). The claim follows immediately from Claim 12, the fact that μ_r is simple, and Lemma 7.5. □

Claim 14. *The valuation μ_r vanishes on $\mathcal{C}^n$.*

Proof. Let $u \in \mathcal{C}^n$; if the dimension of $\mathrm{dom}(u)$ is strictly less than n, $\mu_r(u) = 0$ as μ_r is simple. So, assume that $\Omega = \mathrm{int}(\mathrm{dom}(u)) \neq \emptyset$. Let P be a polytope contained in Ω, and let u_i, $i \in \mathbf{N}$, be a sequence of piecewise linear functions of $\mathcal{C}^n$, such that for every i: $\mathrm{dom}(u_i) = P$, $u_i \geq u_{i+1}$ in P, and the sequence u_i converges uniformly to u in P; such a sequence exists by standard approximation results of convex functions by piecewise linear functions. Using the continuity of μ_r and the previous Claim 13, we obtain

$$\mu_r(u + I_P) = \lim_{i\to\infty} \mu_r(u_i) = 0.$$

Now take a sequence of polytopes P_i, $i \in \mathbf{N}$, such that: $P_i \subseteq P_{i+1} \subseteq \Omega$ for every i and

$$\Omega = \bigcup_{i\in\mathbf{N}} P_i.$$

Then the sequence

$$u + I_{P_i}, \quad i \in \mathbf{N},$$

is formed by elements of $\mathcal{C}^n$, is decreasing, and converges point-wise to u in Ω; by continuity and the previous part of this proof

$$\mu_r(u) = \lim_{i\to\infty} \mu_r(u + I_{P_i}) = 0.$$

□

The proof of Theorem 9.5 is complete, under the additional assumption that the density f of μ is smooth. The next and final step explains how to deduce the theorem in the general case.

Claim 15. *The assumption that f is smooth can be removed.*

Proof. Let μ be as in the statement of Theorem 9.5, and let μ_i, $i \in \mathbf{N}$, be the sequence of valuations determined by Proposition 5.7 (taking for example $\epsilon = 1/i$, $i \in \mathbf{N}$). It follows from the definition of μ_i given in section 5.1 that, as μ is k-homogeneous, μ_i is k-homogeneous as well. Moreover, the only non-vanishing geometric density of μ_i, that we will denote by f_i, is smooth. Hence, for every i we may apply the previous part of the proof to μ_i and deduce that

$$\mu_i(u) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\}))d\nu_i(t),$$

where ν_i is a Radon measure on $\mathbf{R}$ and it is related to f_i by the equality

$$f_i(t) = \int_{(t,\infty)} d\nu_i(s), \quad \forall\, t \in \mathbf{R}.$$

We apply Proposition 6.4 to get

$$\mu_i(u) = \int_{\mathbf{R}} f_i(t)d\beta_k(u;t) \quad \forall\, i \in \mathbf{N},\, u \in \mathcal{C}^n,$$

we recall that $\beta_k(u;\cdot)$ is the distributional derivative of the increasing function

$$\mathbf{R} \ni t \mapsto V_k(\mathrm{cl}(\{u < t\})).$$

From Proposition 3.4 we know that $\beta_k(u;\cdot)$ can be decomposed as the sum of a part which is absolutely continuous with respect to the one-dimensional Lebesgue measure and a Dirac point-mass measure having support at $m(u)$ and weight $V_k(\{x : u(x) = m(u)\})$. If in particular we assume that $u \in \mathcal{C}^n$ is such that

$$\{x : u(x) = m(u)\} \text{ consists of a single point,} \tag{9.40}$$

we have that (as $k \geq 1$)

$$V_k(\{u = m(u)\}) = 0,$$

so that $\beta_k(u;\cdot)$ is absolutely continuous with respect to the Lebesgue measure on the real line.

Our next move is to prove that, under the assumption (9.40)

$$\lim_{i\to\infty} \int_{\mathbf{R}} f_i(t) d\beta_k(u;t) = \int_{\mathbf{R}} f(t) d\beta_k(u;t). \tag{9.41}$$

We know that the sequence f_i converges to f almost everywhere on $\mathbf{R}$ with respect to the Lebesgue measure, and hence with respect to $\beta_k(u;\cdot)$. Note also that

$$f_i(t) = \int_{\mathbf{R}} f(t-s) g_{1/i}(s) ds = \int_{-1}^{1} f(t-s) g_{1/i}(s) ds$$

where g is the mollifying kernel introduced in section 5.1 (which in particular is supported in $[-1,1]$) and

$$g_\epsilon(s) = \frac{1}{\epsilon} g\left(\frac{s}{\epsilon}\right), \quad \forall\, \epsilon > 0.$$

As f is decreasing (and non-negative)

$$0 \leq f_i(t) \leq \int_{\mathbf{R}} f(t-1) g_{1/i}(s) ds = f(t-1) \quad \forall t \in \mathbf{R},\ i \in \mathbf{N}.$$

On the other hand

$$\begin{aligned}\int_{\mathbf{R}} f(t-1) d\beta_k(u;t) &= \int_{\mathbf{R}} f(t) d\beta_k(u;t+1) \\ &= \int_{\mathbf{R}} f(t) d\beta_k(\bar{u};t) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{\bar{u} < t\})) d\nu(t) < \infty,\end{aligned}$$

where $\bar{u} = u - 1$ and the last inequality is due to the integrability condition on f (Proposition 6.1 and Corollary 9.4). Hence we may apply the dominated convergence theorem and obtain (9.41). Note that if u verifies condition (9.40), then so does the function $u+s$, for every $s \in \mathbf{R}$. By Proposition 5.7 we conclude that

$$\mu(u+s) = \int_{\mathbf{R}} f(t) d\beta_k(u+s;t) = \int_{\mathbf{R}} f(t) d\beta_k(u;t-s) = \int_{\mathbf{R}} f(t+s) d\beta_k(u;t), \quad \text{for a.e. } s \in \mathbf{R}. \tag{9.42}$$

Let s_i, $i \in \mathbf{N}$, be a decreasing sequence of real numbers converging to zero such that (9.42) holds true; then by m continuity of μ

$$\lim_{i\to\infty} \mu(u + s_i) = \mu(u).$$

The continuity implies also that f is right-continuous (see right after Corollary 6.3), hence

$$\lim_{i\to\infty} f(t + s_i) = f(t) \quad \forall\, t \in \mathbf{R}.$$

Using again the monotonicity of f and the monotone convergence theorem we obtain

$$\lim_{i\to\infty} \int_{\mathbf{R}} f(t + s_i) d\beta_k(u; t) = \int_{\mathbf{R}} f(t) d\beta_k(u; t).$$

Putting the last equalities together we arrive to

$$\mu(u) = \int_{\mathbf{R}} f(t) d\beta_k(u; t) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\})) d\nu(t), \tag{9.43}$$

for every $u \in \mathcal{C}^n$ verifying (9.40). The last step will be to prove that this equality is true for every $u \in \mathcal{C}^n$. For $i \in \mathbf{N}$ set

$$u_i : \mathbf{R}^n \to \mathbf{R} \quad u_i(x) = u(x) + \frac{|x|^2}{i}.$$

Clearly $u_i \in \mathcal{C}^n$ and, as u_i is strictly convex it verifies condition (9.40) and, consequently, (9.43). By continuity

$$\lim_{i\to\infty} \mu(u_i) = \mu(u).$$

We need to prove that

$$\lim_{i\to\infty} \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u_i < t\})) d\nu(t) = \int_{\mathbf{R}} V_k(\mathrm{cl}(\{u < t\})) d\nu(t). \tag{9.44}$$

As $u_i \geq u$ in $\mathbf{R}^n$ fo every i we have that $\{u_i < t\} \subseteq \{u < t\}$ for every t. We have already proven that

$$\lim_{i\to\infty} \mathrm{cl}(\{u_i < t\}) = \mathrm{cl}(\{u < t\}) \quad \forall\, t \in \mathbf{R},$$

where the limit is intended in the Hausdorff metric on $\mathcal{K}^n$. Then (9.44) follows by the monotone convergence theorem, and Theorem 9.5 is finally proven in the general case as well. □

10 A non level-based valuation

In this section we will present a way to construct monotone valuations on $\mathcal{C}^n$ which are moreover rigid motion invariant and continuous and, despite verifying all these desirable properties, cannot be expressed as a linear combination of homogeneous valuations on $\mathcal{C}^n$.

Fix $n, m \in \mathbf{N}$, for all $u \in \mathcal{C}^n$ we set $\hat{u}(x, y) = u(x) + |y|$ for all $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$, where $|\cdot|$ is to be interpreted as the Euclidean norm in $\mathbf{R}^m$. Note that if $u \in \mathcal{C}^n$, then $\hat{u} \in \mathcal{C}^{n+m}$.

We are now ready to define the prototype of the valuations described at the beginning of this section.

Proposition 10.1. *Let n, m be fixed natural numbers and let $k \in \{0, \dots, n+m\}$. Let $t \in \mathbf{R}$. Then the map $\mathcal{C}^n \to \mathbf{R}$, defined as $u \mapsto V_k(\mathrm{cl}(\{\hat{u} < t\}))$,*

i) *is a valuation,*

ii) *is monotone decreasing,*

iii) *is rigid motion invariant,*

iv) *is continuous.*

Proof. First of all, for ease of notation, set $\mu(\cdot) = V_k(\mathrm{cl}(\{\cdot < t\}))$. By Proposition 6.2, μ verifies all the properties *i) - iv)*.

i) As a preliminary step to prove the condition on $\boldsymbol{\infty}$, notice that $\widehat{\boldsymbol{\infty}} = \boldsymbol{\infty} \in \mathcal{C}^{n+m}$. As a matter of fact, for all $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$ we have

$$\widehat{\boldsymbol{\infty}}(x, y) = \boldsymbol{\infty}(x) + |y| = \infty + |y| = \infty.$$

As a consequence

$$\mu(\widehat{\boldsymbol{\infty}}) = \mu(\boldsymbol{\infty}) = 0.$$

Let now $u, v \in \mathcal{C}^n$, we have

$$\widehat{u \wedge v} = \hat{u} \wedge \hat{v}, \tag{10.45a}$$

$$\widehat{u \vee v} = \hat{u} \vee \hat{v}. \tag{10.45b}$$

We are going to prove (10.45a) only, as (10.45b) is completely analogous. Let $u, v \in \mathcal{C}^n$, then, for all $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$ we get

$$\hat{u}(x, y) \wedge \hat{v}(x, y) = (u(x) + |y|) \wedge (v(x) + |y|) = u(x) \wedge v(x) + |y| = \widehat{u \wedge v}(x, y),$$

and (10.45a) is proven. Let now $u, v \in \mathcal{C}^n$ with $u \wedge v \in \mathcal{C}^n$:

$$\mu(\widehat{u \vee v}) + \mu(\widehat{u \wedge v}) = \mu(\hat{u} \vee \hat{v}) + \mu(\hat{u} \wedge \hat{v}) = \mu(\hat{u}) + \mu(\hat{v}),$$

where in the last equality we have employed the valuation property of μ.

ii) Let $u, v \in \mathcal{C}^n$ with $u \le v$. It is immediate to verify that $\hat{u} \le \hat{v}$. Indeed, for all $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$,

$$\hat{u}(x, y) = u(x) + |y| \le v(x) + |y| = \hat{v}(x, y).$$

As μ is monotone we have

$$u \le v \implies \hat{u} \le \hat{v} \implies \mu(\hat{u}) \ge \mu(\hat{v}).$$

iii) Let T be a rigid motion of $\mathbf{R}^n$ and let u be a convex function in $\mathcal{C}^n$. We define $u_T(x) = u(T(x))$ for all $x \in \mathbf{R}^n$. We have

$$\widehat{u_T}(x, y) = u_T(x) + |y| = u(T(x)) + |y| = \hat{u}_{\hat{T}}(x, y)$$

for all $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$ where $\hat{T}$ is the rigid motion of $\mathbf{R}^n \times \mathbf{R}^m$ defined by $(x, y) \mapsto (T(x), y)$. Therefore, as μ is rigid motion invariant,

$$\mu(\widehat{u_T}) = \mu(\hat{u}_{\hat{T}}) = \mu(\hat{u}).$$

In other words, the map $u \mapsto \mu(\hat{u})$ is rigid motion invariant as claimed.

iv) Let $u \in \mathcal{C}^n$ and let u_i, $i \in \mathbf{N}$, be a point-wise decreasing sequence of convex functions in $\mathcal{C}^n$ converging to u point-wise in $\operatorname{relint}(\operatorname{dom}(u))$. We want to show that $\lim_{i\to\infty} \mu(\widehat{u_i}) = \mu(\hat{u})$. In order to prove it, we will use the continuity of μ and show that the sequence $\widehat{u_i}$, $i \in \mathbf{N}$, is also a point-wise decreasing sequence of convex functions (this time in $\mathcal{C}^{n+m}$) that converges to $\hat{u}$ in the relative interior of its domain. By the reasoning used to prove *ii)* we deduce that $\widehat{u_i}$, $i \in \mathbf{N}$, is point-wise decreasing as well. Notice also that

$$\begin{aligned}\operatorname{dom}(\hat{u}) &= \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^m \, : \, u(x) + |y| < \infty\} \\ &= \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^m \, : \, u(x) < \infty\} = \operatorname{dom}(u) \times \mathbf{R}^m;\end{aligned}$$

so that

$$\operatorname{relint}(\operatorname{dom}(\hat{u})) = \operatorname{relint}(\operatorname{dom}(u)) \times \mathbf{R}^m.$$

Let now $(x,y) \in \operatorname{relint}(\operatorname{dom}(u))$, then

$$\widehat{u_i}(x,y) = u_i(x) + |y| \to u(x) + |y| = \hat{u}(x,y).$$

We conclude by the continuity of μ. □

Let us specialize the valuation of Proposition 10.1 to the case $n = m = k = 1$. In this case we can provide a simple geometric explanation: $V_1(\operatorname{cl}(\{\hat{u} < t\}))$ is equal to the length of that portion of the graph of u that lies strictly under the level t and therefore we will refer to it as *undergraph-length*.

To see that, first consider $t \leq m(u) = m(\hat{u})$: in this case the set $\{\hat{u} < t\}$ is empty and so $V_1(\operatorname{cl}(\{\hat{u} < t\}))$ trivially equals the length of the graph lying strictly under t, the latter being 0 as well. On the other hand, let $t > m(u) = m(\hat{u})$; then, by Corollary 3.3 we have that $\operatorname{cl}(\{\hat{u} < t\}) = \{\hat{u} \leq t\}$. Note that this set can be rewritten as

$$\{(x,y) \in \mathbf{R}^2 \, : \, u(x) + |y| \leq t\} = \{(x,y) \in \mathbf{R}^2 \, : \, |y| \leq t - u(x)\}.$$

In other words, $\operatorname{cl}(\{\hat{u} < t\})$ be obtained as a result of the following process: take the part of $\operatorname{epi}(u)$ that lies below the line $\{(x,y) \in \mathbf{R}^2 \, : \, y = t\}$, translate it "vertically" so that the flat top is now lying on the x-axis $H := \{(x,0) \in \mathbf{R}^2\}$, finally symmetrize it with respect to H. We recall that $V_1(K)$ coincides with the length (1-dimensional Lebesgue-measure) in case $\dim(K) = 1$ and with $\frac{1}{2}\mathcal{H}^1(\partial K)$ when $\dim(K) = 2$ (see [17]). If $\operatorname{dom}(u)$ has dimension 1, as $t > m(u)$, the $\operatorname{epi}(u) \cap \{(x,y) \in \mathbf{R}^2 \, : \, y \leq t\}$ is 2-dimensional and therefore $V_1(\operatorname{cl}(\{\hat{u} < t\}))$ is equal to the length of the graph of u that lies strictly under the level t for every choice of $t \in \mathbf{R}$.

The undergraph-length is not a level based valuation. By these words we mean that we could actually take a convex function $u \in \mathcal{C}^1$, rearrange its levels using translations and obtain another convex function v such that $V_1(\operatorname{cl}(\{\hat{u} < t\})) \neq V_1(\operatorname{cl}(\{\hat{v} < t\}))$ for all $t > m(u)$. Take for instance $u(x) = |x|$ and $v(x) = x/2 + I_{[0,\infty)}$ for all $x \in \mathbf{R}$; we have $\{u < t\} = (-t,t)$ and $\{v < t\} = (0,2t) = t + (-t,t)$ for all positive real t. On the other hand, their undergraph-lengths differ: a quick use of the Pythagorean theorem reveals that $V_1(\operatorname{cl}(\{\hat{u} < t\})) = 2\sqrt{2}t$ while $V_1(\operatorname{cl}(\{\hat{v} < t\})) = \sqrt{5}t$.

The length of the undergraph is a valuation which is completely different from the ones we have studied so far: not only it is not α-homogeneous for any real α, it turns out that $V_1(\mathrm{cl}(\{\widehat{\cdot} < t\}))$ cannot even be written as a finite sum of homogeneous functions. To prove this consider the following $u \in \mathcal{C}^1$, defined as $u(x) = |x|$ for all $x \in \mathbf{R}$. For all $\lambda > 0$ we get $V_1(\mathrm{cl}(\{\widehat{u_\lambda} < 1\})) = 2\sqrt{1+\lambda^2}$. Since $V_1(\mathrm{cl}(\{\widehat{u_\lambda} < 1\}))$ is not a polynomial in λ, $V_1(\mathrm{cl}(\{\widehat{u_\lambda} < 1\}))$ cannot be decomposed into the (finite) sum of homogeneous functions. This implicitly tells us that under these assumptions (monotonicity, rigid motion invariance and continuity), homogeneous valuations do not form a basis for the vector space of valuations on $\mathcal{C}^n$.

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