The copula as an instrument to evaluate the quasi-static combination of wind loads

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ABSTRACT

In many instances related to wind engineering, the problem of the combination of wind loads or wind effects has to be faced. As an example, when dealing with high- and middle-rise buildings, it is important to determine which are the possible instantaneous maximum values of along-wind, across-wind and torque to be considered as acting simultaneously on the building, so to get a proper estimation of maximum stresses. In order to reach a complete description of the wind loads, the knowledge of some joint probability density functions (JPDFs) of the load components and in particular of their extreme values would be required. In order to estimate these JPDFs from the marginal distributions of the single variables, a copula-based approach is introduced; this approach, commonly used in several engineering and statistical fields, has practically never been used in wind engineering. In the paper the copula-based approach is applied to a tall building, in order to determine the wind load combination coefficients maximizing a generic effect given by the linear combination of two or more load components; analyses were performed on experimental data measured in the wind tunnel on a rigid model, so neglecting the resonant part of the structural response. The reliability of the method has been evaluated by comparing the results obtained with those determined directly on the linear combination of the time histories.

1 INTRODUCTION

In several cases the correct determination of the performances of a structure subjected to wind loading has to be achieved through a probabilistic analysis of the structural response. This is the case, for example, of analyses performed in the framework of the Performance-Based Design of wind sensitive structures (Petrini et al., 2009). In particular, if aeroelastic effects are not taken into account and the resonant behavior of the structure can be neglected, a complete definition of the response would require the knowledge of some joint probability distribution functions (JPDFs) of external actions, i.e. along- and across-wind forces, torque, and so on. In contrast, in the case of flexible structures characterized by small levels of damping, the JPDFs of the external actions would be enough to describe the quasi-static response of the structure but not its complete dynamic behavior. Moreover, in many cases attention can be limited to extreme values of some selected actions only, namely those which are relevant with respect to the structural design.

In the present paper, a method for estimating the JPDFs of the extremes of some variables is presented. The proposed approach is based on the copula, a mathematical tool fairly common in other engineering branches, which allows to estimate the joint distribution of several random variables, given their marginal distributions and some information about their mutual correlation. In the particular case where these variables are the extreme values of random processes, a copula-based approach can be useful to integrate the experimental data, as usually the load time histories available are not long enough to properly define the JPDFs of maxima. Similarly, such an approach may allow a significant reduction of the computational burden of Monte Carlo simulations.

As a simple working example, also in order to validate the method, the problem of the determination of the maximum value of the sum of two or three variables was analyzed, but the same procedure could be used for different combinations, either linear or nonlinear, even stochastic, of every set of random variables.

2 WIND ACTIONS COMBINATION

In the scientific literature there are reported different approaches in order to obtain a criterion for combining either wind actions or wind response maxima in different directions, such as along- and across-wind.

Isyumov (1982) proposed a simple approach to estimate the maximum acceleration of a building, given the along-wind and across-wind components, by using the square root sum of squares rule (SRSS) applied to the peak values of the accelerations in the two directions; the values obtained are then reduced through an empirical factor, which experimental evidences show to range from 0.7 to 1.0, depending on the relative magnitudes of the two components.

Solari et al. (1998) and Solari and Pagnini (1999) considered uncorrelated processes and defined an elliptical threshold with small probability that a vectorial combination of the processes can cross it. The axes of the ellipse are defined by the minimum and maximum values of the single processes. In order to minimize the risk of underestimating the effect, a dodecagon is constructed outside the ellipse and the largest distance of its vertices from the origin represents the mean value of the maximum of the resultant effect. The same approach is extended to the case of a scalar combination of the processes.

In the AIJ Recommendations for wind loads on buildings (Tamura et al., 2003), in the case of not very slender and flexible buildings, it is suggested to combine the along-wind load with an across-wind load as large as 35% of the former scaled with the ratio of the along-wind to the across-wind dimension of the building, following empirical evidences (Tamura et al., 2008). In contrast, for slender and flexible buildings the AIJ Recommendations provide a more detailed method to combine along-wind, across-wind and torsional loads, which follows the approach of Solari et al. (1998) and Solari and Pagnini (1999), though the eight load combinations (instead of twelve) account for the correlation between cross-wind and torsional actions.

Chen and Huang (2009) addressed the more complex case of the vectorial resultant response, which presents non-Gaussian features. In particular, the problem of the estimation of the maximum acceleration at the top of a tall building was studied. By looking for the maximum of the response via the upcrossing theory for non-Gaussian processes and for vector-valued Gaussian processes, a parametric study showed that the relevant parameters controlling the peak factor of the resultant response are the variances of the single process components, their mean zero-upcrossing rates, the correlation coefficient and the time duration. In the same paper the authors proposed an extension of the combination rule proposed by Solari and Pagnini (1999) to the case of correlated processes and its satisfactory accuracy was shown.

Naess et al. (2009) proposed a method to estimate extreme values of combined stochastic load effects based on Monte Carlo simulation. The realizations of the process are used to estimate the mean upcrossing rate for different levels of the response. Computational efficiency is ensured by means of the assumption of exponential tail for the mean upcrossing rate, which reduces the problem to the optimization of four constants. Applications in the case of linear and nonlinear combinations of Gaussian and non-Gaussian processes are shown.

3 COPULA

One method of modelling dependencies which has recently become very popular is the copula. Copula-based approaches have been used for different statistical problems in many fields, such as economics, finance and hydrology. The word copula is a Latin noun which means ‘a link, tie or bond’, and it was firstly formulated in a mathematical or statistical sense by A. Sklar (see Nelsen, 2006). The fundamental theorem for copulas is the so-called Sklar’s theorem, which states that for a given joint multivariate distribution function and the corresponding marginal distributions, there exists a copula function that relates them and this function is unique, if the variables are continuous.

From a mathematical point of view, if $F(x_1,x_2,\ldots,x_n)$ denotes a n-dimensional joint distribution with margins $F_1(x_1)$, $F_2(x_2)$, …, $F_n(x_n)$, the copula C is represented by a function $C: [0,1]^n \rightarrow [0,1]$ such that

$$F\left(x_1,x_2,\ldots,x_n\right)=C\left[F_1\left(x_1\right),F_2\left(x_2\right),\ldots,F_n\left(x_n\right)\right] \tag{1}$$

If $x_1,x_2,\ldots,x_n$ are continuous, then C is unique; otherwise, C is uniquely determined on the (range of x_1) × (range of x_2)× …×(range of x_n). Conversely if C is a copula and $F_1(x_1)$, $F_2(x_2)$, …, $F_n(x_n)$ are distribution functions, then the function $F(x_1,x_2,\ldots,x_n)$ previously defined is a joint distribution with margins $F_1(x_1)$, $F_2(x_2)$, …, $F_n(x_n)$. At the same time, Sklar’s theorem not only defines the copula but directly gives a way to build it. As a matter of fact, by considering a random vector $\mathbf{x} = (x_1,x_2,\ldots,x_n)^T$, whose continuous and strictly increasing marginal cumulative distributions are denoted by $F_1(x_1)$, $F_2(x_2)$, …, $F_n(x_n)$, then the copula is such that:

$$C(\mathbf{u}):=C\left(u_1,u_2,\ldots,u_n\right)=F\left[F_1^{-1}\left(u_1\right),F_2^{-1}\left(u_2\right),\ldots,F_n^{-1}\left(u_n\right)\right] \tag{2}$$

where the inverse operator is to be thought as a left continuous generalized inverse, i.e., for every function J:

$$J^{-1}\left(y\right)=\inf\left\{x:J\left(x\right)\geq y\right\} \tag{3}$$

The copula then describes how the marginals are tied together into the joint distribution. The use of copulas to build multivariate distributions is a flexible and powerful technique, because it separates the choice of dependence from the choice of the marginals, on which no restrictions are placed (e.g. Clemen and Reilly, 1999).

As no theoretical results are available to determine the copula (given the marginal distributions, an infinity of multivariate joint distributions can be derived), normally a parametric family of functions (depending upon a parameter) can be chosen to model the dependence. The so-called Archimedean copulas are built starting from a continuous decreasing convex function. Different choices for this function lead to different copula families (e.g. Clayton, Frank or Gumbel-Hougaard copulas). The parameter summarizing the dependence can be related to non-parametric correlations (rank correlations) such as Spearman’s or Kendall’s rank correlation coefficients (see e.g. Grimaldi and Serinaldi, 2006).

As one parameter only is often not enough to characterize the dependence when dealing with more than two variables, a second class of copulas can be introduced, the so-called Elliptical copulas. Within this class, one of the most widely used is the Gaussian Copula, which requires the estimation of the covariance matrix that can be done through a maximum likelihood approach (e.g. Renard and Lang, 2007).

4 APPLICATION TO MULTIVARIATE EXTREMES

As a simple working example, the copula-based approach was applied to the problem dealing with the estimation of the expected maxima of a linear combination of two or more processes. Given the extreme values of the single processes with a certain exceedance probability p, the aim is to determine the design value of one variable to be combined with the maximum of the other ones, in order their sum to have the same exceedance probability p. As these design values can be thought as fractions of the corresponding extreme values, the problem is reduced to the determination of some combination coefficients. Once the JPDF of the maxima is available, these design values can be obtained through successive integrations over a domain defined by the sum of the variables.

The application herein was performed by using the wind forces measured in the CRIACIV Boundary Layer Wind Tunnel (Fig. 1) at the base of the 1:500 scaled model of a square-prismatic tall building, with dimensions 103.6 mm (b) × 103.6 mm (d) × 610.2 mm (h).

The time histories of five generalized forces (two shear forces F_X and F_Y, two bending moments M_X and M_Y and the torque M_Z, where X denotes the wind mean direction and Z the vertical axis) were obtained by integrating the pressures simultaneously measured over the model; the values obtained were then converted into force and moment coefficients by using the relationships reported in Eq. (4), being U_{ref} the mean wind speed at the top of the model.

$$c_{F\alpha}(t)=\frac{F_\alpha(t)}{\frac{1}{2}\cdot\rho\cdot U_{ref}^2\cdot b\cdot h}$$
$$c_{M\alpha}(t)=\frac{M_\alpha(t)}{\frac{1}{2}\cdot\rho\cdot U_{ref}^2\cdot b\cdot h^2}\quad[\alpha=X,Y]\qquad c_{MZ}(t)=\frac{M_Z(t)}{\frac{1}{2}\cdot\rho\cdot U_{ref}^2\cdot b^3}\tag{4}$$

These time histories were subdivided into 20 windows corresponding approximately to 10 minutes at full scale and a Gumbel distribution was fitted to the maximum values extracted in these windows. Then the extreme values were determined according to the Cook and Mayne (1980, 1981) approach as the quantiles with non-exceedance probability of 78%. In Tables 1 and 2 the main results obtained are summarized.

Table 1. Base reaction coefficients: correlation coefficients

	c_{FX}	c_{FY}	c_{MX}	c_{MY}	c_{MZ}
c_{FX}	1	0.0239	0.0007	0.9246	0.0304
c_{FY}	0.0239	1	-0.9492	0.0183	-0.5657
c_{MX}	0.0007	-0.9492	1	-0.0038	0.4948
c_{MY}	0.9246	0.0183	-0.0038	1	0.0285
c_{MZ}	0.0304	-0.5657	0.4948	0.0285	1

Table 2. Base reaction coefficients (μ_c and σ_c denote respectively the mean and standard-deviation values of the variables)

	μ_c	σ_c	min	max	min - μ_c	max - μ_c
c_{FX}	0.861	0.094	0.580	1.116	-0.282	0.255
c_{FY}	0.018	0.162	-0.484	0.511	-0.502	0.493
c_{MX}	-0.012	0.084	-0.286	0.263	-0.274	0.275
c_{MY}	0.472	0.044	0.325	0.586	-0.147	0.114
c_{MZ}	-0.014	0.090	-0.331	0.281	-0.317	0.296

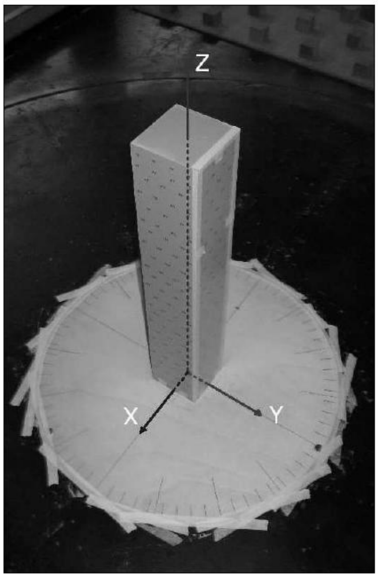

Figure 1. Model in the wind tunnel

4.1 *Application to two variables*

As shown in Figure 2 for the force coefficients c_{FY} and c_{MZ}, the values assumed by a process while the other reaches its maxima are also extracted inside each window (synchronous values). The pairs of values obtained allow to study the correlation between the extreme values of the two processes and therefore to determine the JPDF through a copula (see right-hand side of Figure 2 for the particular case of a Gaussian copula).

The JPDF is the one obtained by analyzing the values that one variable can assume given that the other reaches its maximum value. It should be denoted as $f_{Y1\vee Y2}$ where the symbol "or" between the two variables indicates that either the first or the second one can reach its maximum value. In other words

$$f_{Y_1\vee Y_2}\left(y_1,y_2\right)dy_1dy_2=\text{Prob}\left[y_1\le Y_1<y_1+dy_1,y_2\le Y_2<y_2+dy_2\,|\,Y_1=y_{1,\max}\vee Y_2=y_{2,\max}\right] \quad (5)$$

It is worth noting that the experimental data available would be by far not enough to build empirically this joint PDF. As validation of the approach proposed, the combination coefficients obtained through the JPDF (as schematized in the right frame of Fig. 3) were compared with those obtained through a "direct" method, that is by summing the processes, building the distribution of its maxima and then extracting the 78%-quantiles (Tables 3-4).

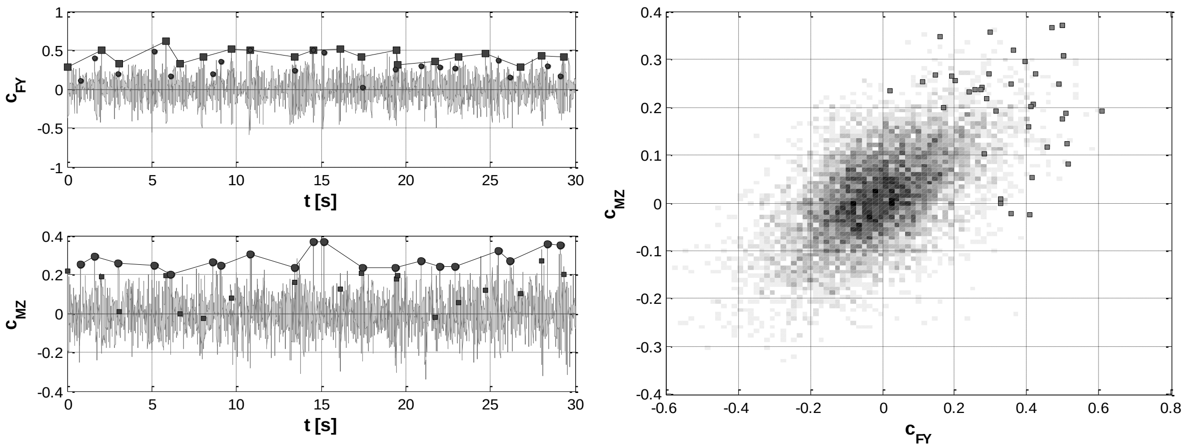

Figure 2. In the left frame: time histories of c_{FY} and c_{MZ} and synchronous values for c_{FY} maxima (squares) and c_{MZ} maxima (circles); In the right frame: distribution of c_{FY} and c_{MZ} and synchronous values for maxima (squares).

In particular, the following procedure was employed: once the margins of the previously mentioned synchronous values has been obtained separately for the two variables, the JPDF is built by using a copula. The probability of exceedance of the sum of the variables is then obtained by the following integral:

$$p=\text{Prob}\left[Y_1+Y_2\ge s_{\max}\,|\,Y_1=y_{1,\max}\vee Y_2=y_{2,\max}\right]=1-\int_{-\infty}^{s_{\max}}\int_{-\infty}^{\infty} f_{Y_1\vee Y_2}\left(s-y,y\right)dy\,ds \quad (6)$$

By varying the value of s_{max}, is possible to build the cumulative distribution function of the sum of the extremes of the variables, that is the probability of the maximum value of the sum of the two variables Y_1 and Y_2. The design value for the sum is the value of s_{max} corresponding to the desired level of probability *p*.

In this way, only the maximum value of the sum of the variables can be determined, but there are infinite pairs of values for the variables giving the same value of their sum. In order to define some possible design values of the variables to be used jointly in order to determine a combination leading to the maximum value of the sum, the following procedure can be used. Let us suppose that $y_{1,max}$, $y_{2,max}$ and s_{max} are characterized by the same probability *p* of exceedance ($y_{1,max}$ and $y_{2,max}$ are determined by considering the two processes separately, that is without taking into account their mutual relationship). Two possible pairs of combination factors can be obtained by supposing that one of the two variables assumes its maximum value, while the other (unless their maxima are perfectly correlated) reaches a value less than its maximum. In other words, there can be sought two values γ_1 and γ_2 in such a way that the pairs ($y_{1,max}$, $\gamma_2 \cdot y_{2,max}$) and ($\gamma_1 \cdot y_{1,max}$, $y_{2,max}$) give the same sum s_{max}. The coefficients γ_1 and γ_2 are then easily given by the following relationship:

$$\gamma_1 = \frac{s_{\max} - y_{2,\max}}{y_{1,\max}}, \quad \gamma_2 = \frac{s_{\max} - y_{1,\max}}{y_{2,\max}} \tag{7}$$

As an example, for the two force coefficients c_{FY} and c_{MZ}, the joint probability density of the synchronous values is reported on the left-hand side of Figure 3, while the design values of the two variables obtained as above are represented by the two grey circles on the right-hand side of Figure 3.

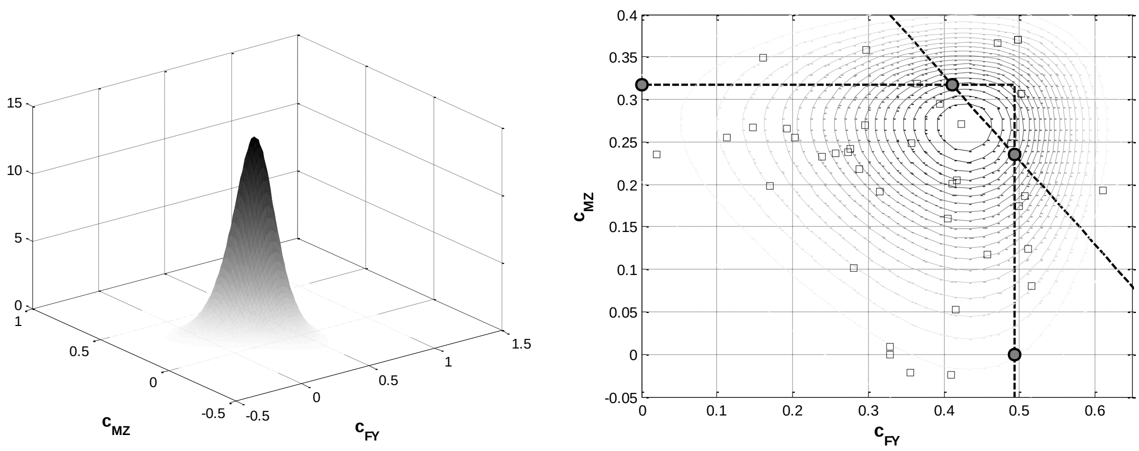

Figure 3. JPDF of the maxima of the variables c_{FY} and c_{MZ} from a Gaussian copula on Gumbel marginal distributions (left); determination of the design values for c_{FY} and c_{MZ} (right).

Table 3. Combination coefficients for c_{FY} and c_{MZ}

Comb. coeff	"direct" method	Gumbel-Hougaard copula		Gaussian copula	
		value	err.	value	err.
γ_1 (c_{FY})	0.759	0.747	-1.62%	0.742	-2.27%
γ_2 (c_{MZ})	0.845	0.837	-0.93%	0.834	-1.30%

Table 4. Combination coefficients for c_{FX} and c_{FY}

Comb. coeff	"direct" method	Gumbel-Hougaard copula		Gaussian copula	
		value	err.	value	err.
γ_1 (c_{FX})	0.725	0.731	0.91%	0.604	-16.63%
γ_2 (c_{FY})	0.467	0.480	2.72%	0.234	-49.94%

It is worth noting that even when the parent data show a very little correlation (as in the case of c_{FX} and c_{FY}), maxima can be more correlated, due to the possible correlation of derivative processes (e.g. Chen and Huang, 2009), so that one variable assumes non-negligible values when the other reaches its maximum value.

Regarding the best copula model describing the JPDF of maxima, in the present example it can be observed that the higher is the level of correlation, the most negligible seems to be the influence of the choice of the copula model (as it can be observed from the combination coefficients in the case of the sum of the two variables c_{FY} and c_{MZ}). In contrast, the more uncorrelated are the two variables, the less adequate is the choice of a Gaussian copula to model mutual maxima dependence (see Bartoli et al., 2011, for further details): in this case, the Gumbel-Hougaard copula seems to produce more reliable results (as in the case of the sum of the two variables c_{FX} and c_{FY}).

It is also to be noticed that, as expected, the combination coefficients are generally non-symmetric; a certain symmetry is attained only if the two variables have similar variances. As a consequence, "rules of thumb" can hardly be used when dealing with the combination of variables with fairly different parent and extreme distributions.

Finally, it is interesting to remark that the combination coefficient γ_2 for c_{FX} and c_{FY}, that is the portion of the maximum value of the across-wind force c_{FY} to be combined with the maximum value of the along-wind force c_{FX}, is about 0.47 (Table 4), which is only slightly larger than 0.4, as suggested by the AIJ Japanese Recommendations (Tamura et al., 2003). Similarly, the combination coefficients between the across-wind force and the torque (c_{FY} and c_{MZ}) are about 0.76 and 0.85, which are reasonably close to the value 0.77 proposed by the AIJ Recommendations for a correlation coefficient $|\rho| = 0.57$, although the norm does not distinguish between the cases wherein either c_{FY} or c_{MZ} is considered with its full design value.

4.2 *Application to three variables*

The same procedure can be used for more than two variables, by considering only Gaussian copulas as in this case more parameters describing the mutual dependence have to be used. Referring to the same dataset previously introduced, the combination coefficients for the maximum values of the three variables c_{FX}, c_{FY} and c_{MZ} (along- and across-wind resultant force coefficients and base torque coefficient), in order to obtain the maximum value of their sum, is looked for. The trivariate distribution is now defined by:

$$f_{Y_1 \vee Y_2 \vee Y_3}\left(y_1, y_2, y_3\right) dy_1 dy_2 dy_3 = \text{Prob}\begin{bmatrix} y_1 \le Y_1 < y_1 + dy_1, y_2 \le Y_2 < y_2 + dy_2, \\ y_3 \le Y_3 < y_3 + dy_3 \,|\, Y_1 = y_{1,\max} \vee Y_2 = y_{2,\max} \vee Y_3 = y_{3,\max} \end{bmatrix} \tag{8}$$

By adopting the same procedure previously described, once the JPDF has been determined, the following integrals have to be evaluated:

$$\begin{aligned} p &= \text{Prob}\left[Y_1 + Y_2 + Y_3 \ge s_{\max} \,|\, Y_1 = y_{1,\max} \vee Y_2 = y_{2,\max} \vee Y_3 = y_{3,\max}\right] = \\ &= 1 - \iiint_{\Omega_{s_{\max}}} f_{Y_1 \vee Y \vee Y_3}\left(x, y, z\right) dx\, dy\, dz \end{aligned} \tag{9}$$

where the integration domain Ω_{smax} is the one defined by

$$\Omega_{s_{\max}} = \left[x, y, z \,|\, x + y + z \le s_{\max}\right] \tag{10}$$

Once the value $s_{\max}$ has been determined (by choosing the proper value of the exceedance probability p), the combination coefficients can be evaluated. For instance, by assuming that two out of three variables assume simultaneously their maximum value, coefficients reported in Table 5 are obtained.

Table 5. Combination coefficients for c_{FX}, c_{FY} and c_{MZ}

γ_1 (c_{FY})	γ_2 (c_{FY})	γ_3 (c_{FY})
1.000	1.000	0.285
1.000	0.548	1.000
0.108	1.000	1.000

5 CONCLUDING REMARKS

A possible method to determine the JPDFs of the maxima of two variables when their marginal distributions are known was introduced.

The proposed approach was used for estimating the design values to be given to the base resultant forces on a tall building subjected to simultaneous wind forces. Results from wind-tunnel tests were used for this purpose. Two different copulas were tested and the difference in the results was highlighted. The results obtained confirm the complexity of the problem and the difficulty to define general values of the combination coefficients.

Extension to three variables was also presented, since in some cases it could be necessary to deal with the combination of more variables (for instance, to determine the maximum stresses on a structural component, a combination of all the external actions could be necessary).

6 ACKNOWLEDGEMENT

The research has been supported by a grant of MIUR (Italian Ministry of Education, University and Research), within the PRIN07 project “Wind effects on slender structures: Performance-based Optimal Design (Wi-POD)” coordinated by Prof. Giuliano Augusti and Prof. Marcello Ciampoli of the University of Roma “La Sapienza”.

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