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Global analysis and indeterminacy in a two-sector growth model with human capital

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Abstract

The purpose of the present paper is to highlight some features of global dynamics of the two-sector growth model with accumulation of human and physical capital analyzed by Brito and Venditti, which is a specialization of the model proposed by Mulligan and Sala-i-Martín. In particular, our analysis focuses on the context in which the Brito–Venditti system admits two balanced growth paths, each corresponding, after a change of variables, to an equilibrium point of a three-dimensional system, and proves the possible existence of points $\bar{P}$ such that in any neighborhood of $\bar{P}$ lying on the plane corresponding to fixed values of the state variables there exist points $\bar{Q}$ whose positive trajectories tend to either equilibrium point. This implies that equilibrium selection in the Brito–Venditti system may depend on the expectations of economic agents rather than on the history of the economy. That is, economies with identical technologies and preferences, starting from the same initial values of the state variables (history), may follow rather different equilibrium trajectories according to the economic agents' choices of the initial values of the jumping variables (expectations). Moreover, we prove that the basins of attraction (two or three-dimensional) of locally indeterminate equilibrium points may be very large, as they can extend up to the boundary of the system phase space.

1 Introduction

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Equilibrium selection in dynamic optimization models with externalities may depend on the expectations of economic agents rather than on the history of the economy, as Krugman (1991) and Matsuyama (1991) pointed out in their seminal papers. Economies with identical technologies and preferences, starting from the same initial values of the state variables (history), may follow rather different equilibrium trajectories according to the economic agents' choices of the initial values of the jumping variables (expectations). A well-known context in which expectations matter is that in which the dynamic system describing the evolution of the economy admits a locally attracting equilibrium point (which may correspond to a balanced growth path). In such a case, if the initial values of the state variables are close enough to the equilibrium values, the transition dynamics depends on the initial choice of the jumping variables and so there exists a continuum of equilibrium trajectories that the economy could follow to approach the equilibrium point. There is an enormous literature on this type of indeterminacy, which is known as "local indeterminacy."[1] The analysis of the linearization of a dynamic system around an equilibrium point gives all the information required to detect local indeterminacy (if the equilibrium point is hyperbolic).[2] The relative simplicity of local analysis explains why many works in the literature focus on local indeterminacy issues. However, a fast-growing number of contributions suggest caution in drawing predictions on the future evolution of the economy based exclusively on local analysis; in fact, local stability analysis refers to a *small* neighborhood of an equilibrium point, whereas the initial values of the jumping variables do not have to belong to such a neighborhood (see, for example, Matsuyama 1991; Raurich-Puigdevall 2000; Boldrin *et al.* 2001); Benhabib and Eusepi 2005; Benhabib *et al.* 2008; Coury and Wen 2009; Mattana *et al.* 2009) These works stress the relevance of global analysis in order to get satisfactory information about the equilibrium selection process: in fact, global analysis allows us to highlight more complex contexts in which equilibrium selection is not unequivocally determined by the initial values of the state variables.[3] The indeterminacy of equilibrium selection occurs, for example, when there exists an attracting limit cycle around a non-attractive equilibrium point (see, for example, Mattana and Venturi, 1999; Nishimura and Shigoka, 2006; Slobodyan, 2007). Another context in which global analysis techniques allow us to detect cases of indeterminacy is that in which multiple equilibrium trajectories exist, starting from the same initial values of state variables, approaching different equilibrium points or, in general, different ω-limit sets (see, for example, Matsuyama, 1991, Benhabib *et al.* 2008; Mattana *et al.* 2009; Antoci *et al.* 2011;, Antoci *et al.* 2014; Carboni and Russu, 2013). In such a context, given the initial values of the state variables, the economy can follow equilibrium trajectories along which the long run behavior of the state variables is rather different, in that the trajectories converge to different ω-limit sets.[4] In all cases in which indeterminacy of equilibrium selection is observed outside the "small neighborhood" of an equilibrium point to which local analysis techniques refer, the indeterminacy is called "global." In Mattana *et al.* (2009), it is simply stated that: "If equilibrium is indeterminate for a reason different from the case of local indeterminacy, it is said that equilibrium is globally indeterminate."

The purpose of the present paper is to show examples proving the occurrence of global indeterminacy in the two-sector growth model with accumulation of human and physical capital analyzed by Brito and Venditti (2010), which is a specification of the more general model proposed by Mulligan and Sala-i-Martín (1993). The Brito–Venditti model can admit two balanced growth paths leading to two equilibrium points in the correspondent three-dimensional system, which can be either simultaneously locally indeterminate (one with a two-dimensional stable manifold, the other with a three-dimensional one) or only one indeterminate and the other determinate (i.e., with a one-dimensional stable manifold or repelling). Therefore, the system offers a particularly rich environment in which global analysis techniques can be applied. Obviously, our analysis is not exhaustive. In fact, we limit ourselves to exploring two cases where the Brito–Venditti system admits two equilibrium points. In one, the two equilibrium points have, respectively, a two-dimensional and a one-dimensional stable manifold (i.e., they are, respectively, in the Brito–Venditti terminology, locally indeterminate of order 2 and determinate). In the other, the stable manifolds of the two equilibria have, respectively, dimension 2 and 3 (i.e., they are locally indeterminate of order 2 and 3). In both cases we provide examples where we prove the existence of points $\bar{P}$ such that in any neighborhood of $\bar{P}$ lying on the plane corresponding to fixed values of the state variables there exist points Q whose positive trajectories tend to either equilibrium point (these results are illustrated in Figures 1, 2, 5, and 6). In such a context we prove that the two-dimensional stable manifold of the order 2 locally indeterminate equilibrium, in the former case, and the basin of the attracting equilibrium, in the latter case, are both unbounded (i.e., they extend to the boundary of the original phase space).

The results concerning the former case are obtained assuming that the quantity of externalities is the same in both sectors (i.e., $b_1 = b_2$ in the Brito–Venditti model). Under this assumption, there exists an invariant plane and the dynamics is completely described by a two-dimensional system. In such a simplified context, it is also possible to prove that when the locally indeterminate equilibrium point becomes a source, a supercritical Hopf bifurcation occurs giving rise to an attracting (i.e., endowed with a two-dimensional stable manifold) limit cycle. When this happens, global indeterminacy is observed (see Figure 2) in a context where no equilibrium point is locally indeterminate (an analogous result is obtained by Benhabib *et al.*,2008; Coury and Wen, 2009; Gori and Sodini, 2011, 2014; Antoci *et al.*, 2011; Carboni and Russu, 2013).

In the latter case, the dimension of the Brito–Venditti system cannot be reduced and consequently global analysis of the system becomes more complex. In such a context, our result, that is, the unboundedness, for suitable values of the parameters, of the basin of the attracting equilibrium, appears to contain more information than other global indeterminacy results, where the equilibrium is shown to be globally indeterminate in the interior of a two-dimensional invariant region enclosed by a periodic or homoclinic orbit (see, for example, Benhabib *et al.*, 2008; Mattana *et al.*, 2009).

Very few authors have engaged in the investigation of global indeterminacy in continuous-time two-sector models. In a context in which a unique balanced growth path exists, Mattana and Venturi (1999), Mattana (2004), Nishimura and Shigoka (2006), and Slobodyan (2007) obtain a global indeterminacy result proving the existence of periodic orbits arising via a Hopf bifurcation. In a context in which two balanced growth paths coexist, Mino (2004) shows that, in a Lucas–Uzawa framework, both balanced growth paths may be selected starting from the same initial values of the state variables. He obtains his results by imposing specific conditions on parameter values which enable him to reduce the three-dimensional dynamic system to a two-dimensional one. Besides the cases in which the dynamics can be fully analyzed by imposing specific conditions on parameter values, only Mattana *et al.* (2009) (to the best of our knowledge) use global analysis techniques to prove the existence of global indeterminacy. In particular, the authors analyze a model in which physical capital is not an input in the production process of human capital and apply the Kopell–Howard theorem Kopell and Howard (1975) to show that, under some conditions on parameter values, one of the two equilibria is surrounded (in an ambient space represented by a two-dimensional manifold) by either a homoclinic orbit or a periodic orbit (or both).

The present paper has the following structure. Section 2 briefly presents the set-up of the Brito–Venditti model and the associated dynamic system. Section 3 introduces a change of variables into the system and retrieves, in a more direct way, some local analysis results of Brito and Venditti which are useful for global analysis. Sections 4 and 5 deal with global analysis. Section 6 provides concluding comments. A mathematical appendix contains the proofs of several of the results stated in the paper.

2 The model

Brito and Venditti have analyzed a two-sector endogenous growth model in which the representative agent solves the following optimization problem:

$$\max_{C(t),K_{11}(t),K_{21}(t),K_{12}(t),K_{22}(t)} \int_0^{+\infty} \frac{C(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt$$

subject to

$$\begin{aligned} \dot{K_1}(t) &= Y_1(t) - C(t), \\ \dot{K_2}(t) &= Y_2(t), \\ Y_j(t) &= e_j(t) K_{1j}(t)^{\beta_{1j}} K_{2j}(t)^{\beta_{2j}}, \quad j=1,2, \\ K_i(t) &= K_{i1}(t) + K_{i2}(t), \quad i=1,2, \\ K_j(0) &> 0, \{e_j(t)\}_{t=0}^{+\infty}, \quad j=1,2, \quad \text{given}, \end{aligned} \tag{1}$$

where $K_1(t)$ and $K_2(t)$ represent physical and human capital, respectively; $K_{ij}(t)$ is the amount of capital good $i=1,2$ used in sector $j=1,2$; $\sigma>0$ is the inverse of the elasticity of intertemporal substitution in consumption; and $\rho>0$ is the discount rate.

Each technology $Y_j(t)$ is characterized by constant returns at the private level, that is, $\sum_{i=1}^{2} \beta_{ij}=1$, $j=1,2$, $\beta_{ij}>0$. The functions $e_1(t)$ and $e_2(t)$ represent productive externalities, assumed to depend on physical capital by unit of efficient labor, that is,

$$e_j(t) = \bar{k}(t)^{b_j}, \quad j=1,2, \tag{2}$$

where $\bar{k}(t)=\bar{K}_1(t)/\bar{K}_2(t)$, $\bar{K}_1(t)$ and $\bar{K}_2(t)$ are the economy-wide average stocks of physical and human capital, and $b_j \in [0,1]$. Therefore, Brito and Venditti assume external effects derived from a “knowledge-based” definition of physical capital.

The representative agent considers $\bar{K}_1(t)$ and $\bar{K}_2(t)$ as exogenously determined; however, along the equilibrium trajectories, $\bar{K}_i=K_i$ and $\bar{k}(t)=k(t)=K_1(t)/K_2(t)$ hold and the technologies $Y_1(t)$ and $Y_2(t)$ at the social level are

$$\begin{aligned} Y_1(t) &= K_{11}(t)^{\beta_{11}} K_{21}(t)^{\beta_{21}} k(t)^{b_1} = K_{11}(t)^{\beta_{11}} K_{21}(t)^{\beta_{21}} \left(\frac{K_{11}(t)+K_{12}(t)}{K_{21}(t)+K_{22}(t)}\right)^{b_1}, \\ Y_2(t) &= K_{12}(t)^{\beta_{12}} K_{22}(t)^{\beta_{22}} k(t)^{b_2} = K_{12}(t)^{\beta_{12}} K_{22}(t)^{\beta_{22}} \left(\frac{K_{11}(t)+K_{12}(t)}{K_{21}(t)+K_{22}(t)}\right)^{b_2}. \end{aligned}$$

Notice that $Y_1(t)$ and $Y_2(t)$ represent constant returns technologies. Therefore, the economy-wide external effects are formulated in such a way that the returns to scale in both sectors are constant at the private and social levels. This assumption agrees with the empirical findings of Basu and Fernald (1997) on the aggregate returns to scale in US production and avoids the existence of private positive profits, which would stimulate entry of new firms (see Benhabib and Nishimura, 1998, p. 69).

It is worth stressing that $K_1(t)$ and $K_2(t)$ could be interpreted as other forms of capital. The key distinction between these capital goods is that $K_1(t)$ is a perfect substitute for consumption, while this is not the case for $K_2(t)$ (see Mulligan and Sala-i-Martín, 1993, p. 742). Furthermore, notice that, in the general model proposed by Mulligan and Sala-i-Martín (1993), constant returns to scale at the private and social levels can be obtained only by setting

$$e_j(t) = \left(\frac{\bar{K}_1(t)}{\bar{K}_2(t)}\right)^{b_j} \quad \text{or} \quad e_j(t) = \left(\frac{\bar{K}_2(t)}{\bar{K}_1(t)}\right)^{b_j}.$$

That is, it is necessary to assume some type of “congestion effect” produced by one capital good on the other, as done by Brito and Venditti.

The *Hamiltonian* and *Lagrangian* in current value terms associated with problem (1) are respectively

$$\begin{aligned} \mathcal{H} &= \tfrac{C(t)^{1-\sigma}-1}{1-\sigma} + P_1(Y_1 - C) + P_2 Y_2, \\ \mathcal{L} &= \mathcal{H} + R_1(K_1 - K_{11} - K_{12}) + R_2(K_2 - K_{21} - K_{22}), \end{aligned}$$

where P_i is the utility price and R_i the rental rate of good $i=1,2$.

Applying the Pontryagin *maximum principle* and using the normalization of variables introduced by Caballé and Santos (1993):

$$\begin{aligned} k_1(t) &:= K_1(t)e^{-\gamma t}, \\ k_2(t) &:= K_2(t)e^{-\gamma t}, \\ c(t) &:= C(t)e^{-\gamma t}, \\ p_1(t) &:= P_1(t)e^{-\gamma_P t}, \\ p_2(t) &:= P_2(t)e^{-\gamma_P t}, \end{aligned}$$

where $\gamma > 0$ and $\gamma_P = -\sigma\gamma < 0$ represent, respectively, the (constant) rate of growth of $K_1(t), K_2(t), C(t)$ and the rate of decrease of $P_1(t), P_2(t)$ along a balanced growth path, Brito and Venditti obtain the four-dimensional dynamic system given by

$$\begin{cases} \dot{p}_1 = p_1\left(\rho + \sigma\gamma - r_1(\pi, k)\right) \\ \dot{p}_2 = p_2\left(\rho + \sigma\gamma - r_2(\pi, k)\right) \\ \dot{k}_1 = (\alpha_{11}(\pi, k) - \gamma)k_1 + \alpha_{12}(\pi, k)k_2 - p_1^{-1/\sigma} \\ \dot{k}_2 = \alpha_{21}(\pi, k)k_1 + \alpha_{22}(\pi, k)k_2 - \gamma k_2 \end{cases} \tag{3}$$

where k_1 and k_2 are the state variables, while p_1 and p_2 are the jumping variables, with $\pi := \frac{p_2}{p_1} = \frac{P_2}{P_1}$ and $k := \frac{k_1}{k_2} = \frac{K_1}{K_2}$. The transversality conditions are

$$\lim_{t\to+\infty} p_1(t)\,k_1(t)\,e^{[\gamma(1-\sigma)-\rho]t} = \lim_{t\to+\infty} p_2(t)\,k_2(t)\,e^{[\gamma(1-\sigma)-\rho]t} = 0, \tag{4}$$

with the assumption $\gamma(1-\sigma) - \rho < 0$. Any solution $(k_1(t), k_2(t), p_1(t), p_2(t))$ of system (3) satisfying the transversality conditions (4) and initial conditions $(k_1(0), k_2(0)) = (k_1^0, k_2^0)$ is an optimal solution of problem (1) in that problem (1) satisfies Arrow's condition.

At an equilibrium point of (3), in particular, $r_1(\pi, k) = r_2(\pi, k) = r(\pi, k)$ holds and thus $\gamma = \frac{r(\pi,k)-\rho}{\sigma}$. The transversality conditions imply $0 < \gamma < r$. Furthermore, $r_1(\pi, k) := c_1\pi^{\psi_{21}}k^{b_1\psi_{11}+b_2\psi_{21}}$, $r_2(\pi, k) := c_2\pi^{-\psi_{12}}k^{b_1\psi_{12}+b_2\psi_{22}}$, $\alpha_{ij}(\pi, k) := \psi_{ij}r_j(\pi, k)\pi^{j-i}$, $c_i := (\beta_i^*)^{\psi_{ii}}\left(\beta_j^*\right)^{\psi_{ji}}$, $i \neq j$, $\beta_i^* := \beta_{1i}^{\beta_{1i}}\beta_{2i}^{\beta_{2i}}$, $b_1, b_2 \in [0, 1]$.[5] The coefficients ψ_{ij} are the components of the matrix

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \frac{1}{\beta_{11}-\beta_{12}}\begin{pmatrix} \beta_{22} & -\beta_{12} \\ -\beta_{21} & \beta_{11} \end{pmatrix} = B^{-1}$$

where

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

is the matrix of private Cobb–Douglas coefficients satisfying $\beta_{11} + \beta_{21} = \beta_{12} + \beta_{22} = 1$, $\beta_{11} - \beta_{12} \neq 0$. Consequently, the components of Ψ satisfy the conditions $\psi_{11} + \psi_{21} = \psi_{12} + \psi_{22} = 1$, $\psi_{11} \cdot \psi_{22} > 0$, $\psi_{ii} \cdot \psi_{ij} < 0$ for $i \neq j$. Furthermore, $\psi_{12}, \psi_{21} > 0 \iff \beta_{11} < \beta_{12}$, $\psi_{12} = \psi_{21} \iff \beta_{12} = \beta_{21}$ and $\psi_{11} = \psi_{22} \iff \beta_{11} = \beta_{22}$.

3 A change of variables

By setting $\pi = e^u$, $k = e^v$, $p_1^{-1/\sigma}k_2^{-1} = e^w$ (i.e., $u = \ln\pi = \ln\frac{p_2}{p_1} = \ln\frac{P_2}{P_1}$, $v = \ln k = \ln\frac{k_1}{k_2} = \ln\frac{K_1}{K_2}$, $w = \ln\left(p_1^{1/\sigma}k_2\right)^{-1}$, we obtain, after multiplying the equations by e^v (change of time), a three-dimensional system defined in $\mathbb{R}^3$, whose trajectories generate those of (3). This system is given by

$$\begin{cases} \dot{u} = e^v\left(r_1(u, v) - r_2(u, v)\right) = f(u, v) \\ \dot{v} = e^v\left(\psi_{11}r_1(u, v) - \psi_{22}r_2(u, v) + \psi_{12}r_2(u, v)e^{u-v}\right. \\ \qquad \left.-\psi_{21}r_1(u, v)e^{v-u}\right) - e^w = g(u, v) - e^w \\ \dot{w} = e^v\left(-\frac{\rho}{\sigma} + \frac{r_1(u,v)}{\sigma} - \psi_{22}r_2(u, v) - \psi_{21}r_1(u, v)e^{v-u}\right) = h(u, v) \end{cases} \tag{5}$$

where, in an abuse of notation, $r_i(u, v) := r_i(e^u, e^v)$.

An equilibrium point $(\bar{u}, \bar{v}, \bar{w})$ of system (5) corresponds to a one-dimensional manifold of equilibrium points of the Brito–Venditti system (3) defined, in the space $(\gamma, p_1, p_2, k_1, k_2)$, via the equations

$$\begin{aligned} \gamma &= \frac{r(\bar{u},\bar{v})-\rho}{\sigma}, \\ p_1 &= \left(e^{\bar{w}}k_2\right)^{-\sigma}, \\ p_2 &= \bar{\pi}p_1 = e^{\bar{u}}p_1 = e^{\bar{u}}\left(e^{\bar{w}}k_2\right)^{-\sigma}, \\ k_1 &= \bar{k}k_2 = e^{\bar{v}}k_2. \end{aligned}$$

The local analysis results of Brito and Venditti can be retrieved by analyzing (5). In the rest of this section we focus on those our global analysis is built on.

Let

$$\begin{aligned} \tau &:= \frac{b_1\psi_{12}+b_2\psi_{21}}{\psi_{12}+\psi_{21}}, \\ \delta &:= \frac{(b_1-b_2)(\psi_{12}+\psi_{21}-1)}{\psi_{12}+\psi_{21}}, \end{aligned} \tag{6}$$

implying $0 \le \tau \le 1$, $\mathrm{sgn}\,(\delta) = \mathrm{sgn}\,(b_1 - b_2)$. Since $\tau = 0 \Longleftrightarrow b_1 = b_2 = 0$, we assume in what follows that $\tau > 0$.

It is easily computed that the possible equilibrium points of (5) lie on the plane $u = \delta v + \bar{d}$, with $\bar{d} := (\psi_{12} + \psi_{21})^{-1} \ln \frac{c_2}{c_1}$. Then

$$r_1(\delta v + \bar{d}, v) = r_2(\delta v + \bar{d}, v) = r(v) = ce^{\tau v}, \quad c > 0. \tag{7}$$

Moreover, two equilibrium points can exist only if the function $\widetilde{h}(v) = h(\delta v + \bar{d})$ has one (necessarily unique) relative extremum, which implies that the system parameters satisfy one of the following conditions:

1. $\psi_{12}, \psi_{21} > 0$ (implying $\psi_{12}, \psi_{21} > 1$ and therefore $|\delta| < 1$);
2. $\psi_{12}, \psi_{21} < 0, \delta > 1 + \tau, \sigma^{-1} - \psi_{22} > 0$;
3. $\psi_{12}, \psi_{21} < 0, 1 < \delta < 1 + \tau, \sigma^{-1} - \psi_{22} < 0$.

System (5) has at most one equilibrium in all the other cases except when $\delta = 1 + \tau$ and $\psi_{21} ce^{-\bar{d}} + \frac{\rho}{\sigma} = 0$ or $\delta = 1$ and $\sigma^{-1} - \psi_{22} - \psi_{21} e^{-\bar{d}} \le 0$. In such cases, (5) has no equilibrium, except for $\delta = 1 + \tau$ and $\psi_{21} ce^{-\bar{d}} + \frac{\rho}{\sigma} = \sigma^{-1} - \psi_{22} = 0$, when (5) has infinite equilibria.

Actually in their article Brito and Venditti (theorem 1) provide necessary and sufficient conditions for the existence of two balanced growth paths (i.e., two equilibria of system (5)). That result can be rephrased in our notation after observing that, denoting by $\bar{v}$ and v^* the values of v where respectively $\widetilde{g}(v) = g(\delta v + \bar{d}, v) = 0$ and $\widetilde{h}'(v) = 0$, the existence of two equilibria requires $v^* < \bar{v}$ and $\widetilde{h}(\bar{v})\widetilde{h}(v^*) < 0$. Hence, after computing $\bar{v} = (1-\delta)^{-1} (\bar{d} + \ln \frac{\psi_{21}}{\psi_{12}})$ and $v^* = (1-\delta)^{-1} (\bar{d} + \tau + \ln \frac{\sigma^{-1} - \psi_{22}}{\psi_{12(1+\tau-\delta)}})$, we can state the following proposition:

Proposition 1. System (5) admits two equilibrium points if and only if one of the following three cases occurs:

1. $\psi_{12}, \psi_{21} > 0$ (implying $|\delta| < 1$), $0 < \sigma^{-1} < 1 + \psi_{12} \frac{1-\delta}{\tau}$, $\max(0, r(\bar{v})(1-\sigma)) < \rho < \frac{(1-\delta)(1-\psi_{22}\sigma)}{1+\tau-\delta} r(v^*)$;
2. $\psi_{12}, \psi_{21} < 0, \delta > 1 + \tau, \sigma^{-1} > \max(\psi_{22}, 1 - \psi_{12} \frac{\delta - 1}{\tau})$, $\frac{(\delta-1)(1-\psi_{22}\sigma)}{\delta - 1 - \tau} r(v^*) < \rho < r(\bar{v})(1-\sigma)$ (which requires $\frac{r(\bar{v})}{r}(v^*) = e^{\tau(\bar{v} - v^*)} > \frac{(\delta-1)(1-\psi_{22}\sigma)}{(\delta-1-\tau)(1-\sigma)}$);
3. $\psi_{12}, \psi_{21} < 0, 1 < \delta < 1 + \tau, 0 < \sigma^{-1} < \min(\psi_{22}, 1 - \psi_{12} \frac{\delta-1}{\tau})$, $\max(0, r(\bar{v})(1-\sigma)) < \rho < \frac{(\delta-1)(\psi_{22}\sigma - 1)}{1+\tau-\delta} r(v^*)$.

Remember that $\psi_{12}, \psi_{21} > 0 \Longleftrightarrow \beta_{11} < \beta_{12}$, where β_{11} and β_{12} measure the physical capital intensity in sectors 1 (final good sector) and 2 (human capital sector), respectively. Then the above results show that, as stressed by Brito and Venditti, multiple equilibrium points (i.e., multiple balanced growth paths) can arise in both contexts $\beta_{11} < \beta_{12}$ (i.e., the final good is intensive in human capital at the private level) and $\beta_{11} > \beta_{12}$ (i.e., the final good is intensive in physical capital at the private level). [6]

Now let $P_0 = (u_0, v_0, w_0)$ be an equilibrium point of (5) and set $r(v_0) = r_0$. Then its Jacobian matrix is

$$J(P_0) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & 0 \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & -e^w \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & 0 \end{pmatrix} (P_0) \tag{8}$$

where $\frac{\partial f}{\partial u} = e^{v_0} r_0 (\psi_{12} + \psi_{21})$, $\frac{\partial f}{\partial v} = -\delta e^{v_0} r_0 (\psi_{12} + \psi_{21})$, while $\frac{\partial g}{\partial u} < 0$. Then set $\bar{h}(v) := h(\delta v + \bar{d}, v)$. It easily follows that

$$\mathrm{sgn}\,[\det J(P_0)] = \mathrm{sgn}\,[\bar{h}'(v_0)(\psi_{12} + \psi_{21})]\,. \tag{9}$$

In particular, assume that $\psi_{12}, \psi_{21} > 0$ and two equilibria exist, $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, with $v_1 < v_2$. Then $\det J(P_1) > 0 > \det J(P_2)$.

On the other hand, suppose that $\psi_{12}, \psi_{21} < 0$ and $\delta \le 1$. In this case at most one equilibrium P_0 exists, where $\det J(P_0) < 0$.

The following proposition rephrases one of Brito and Venditti's results:

Proposition 2. Let *P* be one of the equilibria of (5). Then $\delta \ge 0$ (i.e., $b_1 \ge b_2$) implies $\mathrm{trace}\,[J(P)] > 0$.

Proof 1. See the Appendix. □

In particular, if $\delta \ge 0$ (i.e., $b_1 \ge b_2$: the quantity of externalities in the final good sector is greater than that in the human capital sector), *P* cannot be an attractor. Hence, as underlined by Brito and Venditti, the coexistence of two locally indeterminate equilibria (of order 2 and 3, respectively) can occur only if $b_1 < b_2$ and $\psi_{12}, \psi_{21} > 0$ (thus greater than 1). In fact in Sections 4 and 5 we will study examples of global indeterminacy with two equilibrium points in the context of Case 1 of Proposition 1 (i.e., when, in particular, $\psi_{12}, \psi_{21} > 0$). As to Cases 2 and 3, we reformulate results stated in theorem 5 Brito and Venditti (2010), illustrating the local stability of the two equilibria, in the following proposition:

Proposition 3. Assume Cases 2 or 3 of Proposition 1 hold and denote the two equilibria by $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, with $v_1 < v_2$. Then, in Case 2, P_1 is a source while P_2 is a saddle with a one-dimensional stable manifold. In Case 3, P_1 is a saddle with a one-dimensional stable manifold while P_2 can be either repelling or locally indeterminate of order 2 (i.e., its stable manifold is two-dimensional).

Proof 2. See the Appendix. □

Example 1. In system (5) , let $c_1 = c_2 = 1$ *(this can always be obtained by a suitable translation of* (u, v, w) *and a rescaling of the parameter* ρ *and the time variable t*). Set $\psi_{21} = -\varepsilon - \varepsilon^3$, $\psi_{12} = -\varepsilon^2$, $\sigma^{-1} = 1 - \varepsilon^2$, $\rho = 2\exp(\tau v_2)\sigma\varepsilon^4$, $b_1 = 1$, $b_1 - b_2 = \varepsilon(1+\varepsilon)(1+\varepsilon+\varepsilon^2)/(1+\varepsilon+\varepsilon^2+\varepsilon^3)$, where $\varepsilon > 0$ is sufficiently small. Then the conditions of Case 3 are satisfied and there exist two equilibria, $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, with $v_1 < v_2$ and $\exp(v_2 - u_2) = 2\varepsilon$. Hence it is easy to check that P_1 is a saddle with a one-dimensional stable manifold, while P_2 is a saddle with a two-dimensional stable manifold.

We end this section by the following observation. As we have seen above, system (5) passes from zero to two equilibrium points whenever there exists, in the notation of Proposition 1, $v^* < \bar{v}$ such that $\widetilde{h}(v^*) = \widetilde{h}'(v^*) = 0$. Moreover, it is easy to compute that $\widetilde{h}''(v^*) \ne 0$. Hence in such cases we can take ρ as the bifurcation parameter and in fact, thanks to the local analysis conducted by Brito and Venditti (see, in particular, their theorems 4 and 5), it is possible to affirm that what takes place is precisely a saddle-node bifurcation (for a definition see, for example, Guckenheimer and Holmes, 1997, theorem 3.4.1). Actually we will exploit such a saddle-node bifurcation in order to prove, in Section 5, the occurrence of global indeterminacy in a case where two locally indeterminate equilibrium points of order 2 and 3 coexist.

4 Global analysis in a two-dimensional context

Our aim is to show, via global analysis of system (5), examples proving the occurrence of specific patterns of global indeterminacy in the above model. In fact we will consider two cases where system (5) exhibits two equilibrium points. In the first, the object of the present section, the two equilibria will have a two-dimensional and a one-dimensional stable manifold, respectively (i.e., they will be, in the Brito–Venditti terminology, locally indeterminate of order 2 and determinate).[7] In the second case, the stable manifolds of the two equilibria will have dimension 2 and 3, respectively (in the Brito–Venditti terminology the equilibria will be locally indeterminate of order 2 and 3). In both cases we will show the occurrence of global indeterminacy in the following form. We will prove that, for suitable values of the parameters, the two-dimensional stable manifold of the order 2 locally indeterminate equilibrium, in the first case, and the basin of the sink, that is, of the order 3 locally indeterminate equilibrium, in the second case, are both unbounded. Moreover, we will show in both cases the existence of points $\bar{P} = (\bar{u}, \bar{v}, \bar{w})$ such that in any neighborhood of $\bar{P}$ lying on the plane $v = \bar{v}$ (corresponding to a fixed value of the state variable $k = k_1/k_2 = K_1/K_2 = e^v$) there exist points Q whose positive trajectories tend to either equilibrium.

We start by stating the following result:

Proposition 4. When $\delta = 0$, the plane $u = \bar{d}$ (recall $\bar{d} = (\psi_{12} + \psi_{21})^{-1} \ln \frac{c_2}{c_1}$) is invariant.

Proof 3. Recall that $u = \delta v + \bar{d}$ implies $r_1(\delta v + \bar{d}, v) = r_2(\delta v + \bar{d}, v)$ and thus (see system (5)) $\dot{u} = 0$. Hence, when $\delta = 0$, $u = \bar{d}$ is invariant. □

Therefore we first assume $\delta = 0$ (i.e., $b_1 = b_2$: the amount of externalities is the same in both sectors). In such a context, if $\psi_{12}, \psi_{21} < 0$ (i.e., $\beta_{11} > \beta_{12}$: the final good sector is physical capital intensive at the private level), there exists at most one equilibrium P_0, lying on $u = \bar{d}$, such that $\det J(P_0) < 0 < \operatorname{trace}[J(P_0)]$. Thus P_0 is locally determinate. If, instead, $\psi_{12}, \psi_{21} > 0$ (i.e., $\beta_{11} < \beta_{12}$: the final good sector is human capital intensive at the private level), there can exist up to two equilibria lying on the invariant plane $u = \bar{d}$. Suppose this is the case, that is, the conditions of Case 1 in Proposition 1 are satisfied, and denote the two equilibria by $P_1 = (\bar{d}, v_1, w_1)$ and $P_2 = (\bar{d}, v_2, w_2)$, with $v_1 < v_2$ (note that, by (7), the growth rate γ associated with P_2 is higher than that associated with P_1). Then $\det J(P_1) > 0 > \det J(P_2)$, while $\operatorname{trace}[J(P_1)], \operatorname{trace}[J(P_2)] > 0$. Therefore P_2 is locally determinate, whereas P_1 can be either repelling or locally indeterminate of order 2. Moreover, by a translation $u \to u + \bar{u}$, where $\exp(\psi_{12} + \psi_{21})\bar{u} = \frac{c_2}{c_1}$, the original c_1 and c_2 are changed into $\overline{c_1} = \overline{c_2}$, so that $\bar{d} = 0$. Consequently, the system on the invariant plane $u = 0$ reduces to

$$\begin{cases} \dot{v} = \tilde{g}(v) - e^w \\ \dot{w} = \tilde{h}(v) \end{cases} \tag{10}$$

where $\tilde{g}(v) = g(\delta v, v)$, $\tilde{h}(v) = h(\delta v, v)$. So, since $\delta = 0$, it follows that, on the plane $u = 0$, $\tilde{g}'(v) = \frac{\partial g}{\partial v}$ and $\tilde{h}'(v) = \frac{\partial h}{\partial v}$. Therefore P_1 is locally indeterminate of order 2 if and only if $\frac{\partial g}{\partial v}(0, v_1) < 0$.

In fact it is easy to compute that

$$\begin{aligned} \tilde{g}(v) &= r(v)(1 + e^v)(\psi_{12} - \psi_{21}e^v), \\ \tilde{h}(v) &= e^v \left[-\tfrac{\rho}{\sigma} + r(v)(\tfrac{1}{\sigma} + \psi_{12} - 1 - \psi_{21}e^v)\right], \end{aligned}$$

where $r(v) = ce^{\tau v}$, $c > 0$, $\tau = b_1 = b_2$. Assuming $\psi_{12}, \psi_{21} > 0$ (and thus greater than 1), we recall that $\tilde{h}(v)$ has two zeros, $v_1 < v_2$, if and only if $\tilde{h}(v^*) > 0$, where $v^* = \ln \frac{\tau\left(\frac{1}{\sigma} + \psi_{12} - 1\right)}{(1+\tau)\psi_{21}}$. On the other hand, the function $w = \ln \tilde{g}(v)$ is defined for $v < \bar{v} = \ln \frac{\psi_{12}}{\psi_{21}}$ and has a maximum at the point v_0, where e^{v_0} is the positive solution of the equation $\psi_{21}(2 + \tau)x^2 - [\psi_{12}(1 + \tau) - \psi_{21}]x - \tau\psi_{12} = 0$. Hence two equilibria exist if and only if there exist $v_1 < v_2$ such that $\tilde{h}(v_1) = \tilde{h}(v_2) = 0$ and $v_2 < \bar{v}$. Moreover, $P_1 = (0, v_1, w_1)$ has a two-dimensional stable manifold if and only if $v_0 < v_1$.

Remark 1. Suppose all the previous conditions are satisfied. Then, by observing the phase portrait of system (10), defined on $u = 0$, it easily follows that $\tilde{P}_1 = (v_1, w_1)$ is a sink (in the plane $u = 0$) and $\tilde{P}_2 = (v_2, w_2)$ is a saddle; moreover, there is a repeller at the boundary point $v = +\infty$, $w = +\infty$ and an attractor at the boundary point $v = \bar{v}$, $w = -\infty$. Consequently, if for suitable values of the parameters system (10) has no limit cycle, then it is shown through straightforward arguments that the basin of attraction of $\tilde{P}_1$ (i.e., the two-dimensional stable manifold of P_1) is limited by the stable manifold of $\tilde{P}_2$, connecting $\tilde{P}_2$ to the repeller $(+\infty, +\infty)$, and thus is unbounded. This will prove the global indeterminacy result stated at the beginning of this section.

In the following we provide conditions for the above situation to occur.

First of all, we observe that system (10) can be regarded as a Liénard system when $v \in (-\infty, v_2)$. To fix ideas, let us take $\tau = 0.5$. We also assume, for sake of simplicity, that $\psi_{12} = \psi_{21} = \psi > 1$.[8] Then $v_0 = -\frac{1}{2}\ln 5$. If $\tilde{h}(v_0) < 0$ and the parameters ρ, σ, ψ are suitably chosen, an important theorem on the uniqueness of limit cycles for Liénard systems (see Zhou *et al.*, 2005) can be applied. Specifically, consider the new variables $x = v - v_1$, $y = w - w_1$ and change t into $-t$. Then the following Liénard system is defined in the strip $-\infty < x < \bar{x}$, where $\bar{x} = v_2 - v_1$:

$$\begin{cases} \dot{x} = \lambda(y) - \Phi(x) \\ \dot{y} = -\gamma(x) \end{cases} \tag{11}$$

where $\lambda(y) = e^{w_1}(e^y - 1)$, $\Phi(x) = \tilde{g}(v_1 + x) - e^{w_1}$, $\gamma(x) = \tilde{h}(v_1 + x)$. Then, setting $\bar{x} = v_2 - v_1$, $x_0 = v_0 - v_1 < 0$, $\varphi(x) = \Phi'(x)$, and $\Gamma(x) = \int_0^x \gamma(z)dz$, it is easy to check that the smooth system (11), defined in the strip $x \in (-\infty, \bar{x})$, satisfies:

1. $\lambda(y)$ is increasing and $y \cdot \lambda(y) > 0$ when $y \neq 0$;
2. $(x - x_0) \cdot \varphi(x) < 0$ when $x \neq x_0$;
3. $x \cdot \gamma(x) > 0$ when $x \neq 0$.

Moreover, by (Zhou *et al.*, 2005, theorem 3), if two further conditions are met:

1. $\frac{\varphi(x)}{\gamma(x)}$ is non-decreasing in $(-\infty, b)$, where $b \in (-\infty, x_0)$ is defined by $\Phi(b) = 0$ (i.e., $\tilde{g}(v_1 + b) = \tilde{g}(v_1)$);
2. the system of equations $\Phi(x) = \Phi(z)$, $\Gamma(x) = \Gamma(z)$ has at most one solution for $x \in (-\infty, b)$, $z \in (0, \bar{x})$;

then (11) has at most one limit cycle, which, if it exists, is simple (hence does not generate several limit cycles).

Example 2. In system (5), let $\delta = 0$, $\tau = 0.5$, $\frac{\rho}{c} = \frac{1}{\sqrt[4]{5}}$, $\sigma = \frac{1}{3}$, $\psi_{12} = \psi_{21} = 1.698$. Then (5) *has two equilibria,* P_1 *and* P_2, *lying on the invariant plane* $u = 0$ *and the planar system* (11) *satisfies the above conditions 1–5.*

The following theorem gives, on the basis of Remark 1, sufficient conditions for the basin of attraction of P_1 to be unbounded.

Theorem 1. Assume system (5) has parameters $\delta = 0$, $\tau = 0.5$, $\psi_{12} = \psi_{21} = \psi > 1$. Assume there exist two equilibrium points P_1 and P_2 which are, for the system (10) defined on the invariant plane $u = 0$, respectively a sink and a saddle. Then, if the planar system (11) satisfies conditions 1–5, no limit cycle exists and consequently the basin of attraction of P_1, lying on the plane $u = 0$, is unbounded.

Proof 4. See the Appendix. □

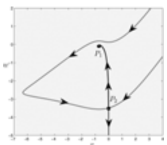

Figure 1. Numerical simulation of the phase portrait of system (10) with parameter values satisfying the conditions of Theorem 1. The unbounded basin of attraction in the invariant plane $u = 0$ of the equilibrium P_1 (which is a poverty trap) is limited by the one-dimensional stable manifold of the determinate equilibrium P_2. Parameter values: $\psi = 1.698$, $\sigma = \frac{1}{3}$, $\tau = 0.5$, $\rho = \frac{1}{\sqrt[4]{5}}$.

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Figure 1 shows a numerical simulation of the phase portrait of system (10) with parameter values satisfying the conditions of Theorem 1. The unbounded basin of attraction of the sink P_1 (which is a poverty trap) is limited by the one-dimensional stable manifold of P_2. Notice that, if the initial value $v(0)$ of the predetermined variable v is high enough, then there exists a continuum of initial values $w^* = w(0)$ of the jumping variable w such that the trajectory starting from $(v(0), w^*)$ approaches P_1, while the stable manifold of P_2 can be selected by choosing two different initial values of w^*. This is an interesting example of indeterminacy because, given the initial value of

v, the economy can approach the locally determinate equilibrium P_2 by following rather different transition paths. Observe that in this case we possess a full description of the unbounded basin of P_1 (on the plane $u = 0$) and therefore of the global indeterminacy scenario. Finally, notice that, as in Matsuyama (1991) and Antoci *et al.* (2011), the poverty trap P_1 can be reached even if the initial value $v(0)$ coincides with the value assumed by the predetermined variable v at the locally determined equilibrium P_2; symmetrically, P_2 can be reached even if the economy starts with an initial value of v coinciding with that of the poverty trap P_1.

In the above context, still denoting by $v_1 < v_2$ the zeros of $\tilde{h}(v)$ for $v \in (v_0, 0)$, we can move v_1 (e.g., by suitably varying ρ and/or σ) until it crosses the value v_0, causing (generically) a Hopf bifurcation to occur. The following proposition holds:

Proposition 5. Under our assumptions, when v_1 crosses the value v_0, the Hopf bifurcation occurs and is supercritical (i.e., an attracting limit cycle arises around $\tilde{P}_1$ when it becomes a source).

Proof 5. See the Appendix. □

Notice that, according to such a proposition, the two coexisting ω limit sets, $\tilde{P}_2$ and the limit cycle around $\tilde{P}_1$, have respectively one-dimensional and two-dimensional stable manifolds lying in the plane $u = 0$. It is worth noting that this global indeterminacy scenario occurs in a context in which $\tilde{P}_1$ is a repeller and $\tilde{P}_2$ is locally determinate, that is, in a context in which no equilibrium point is locally indeterminate (a similar result is obtained by Benhabib *et al.*, 2008; Mattana *et al.*, 2009; Coury and Wen, 2009; Antoci *et al.*, 2011; Carboni and Russu, 2013). Figure 2 shows a numerical simulation of the phase portrait of system (10). Observe that there exists an interval (which is, in fact, unbounded) of values of the predetermined variable v from which the economy can approach either $\tilde{P}_2$ or the limit cycle around $\tilde{P}_1$, according to the initial choice of the jumping variable w (the initial value of the other jumping variable u is fixed at the value $u = \bar{d}$). In $\tilde{P}_1$ the value of v (and consequently, by (7), the value of the growth rate γ) is lower than in $\tilde{P}_2$. However, even if the equilibrium $\tilde{P}_1$ is not (generically) reachable by the economy, there exists a continuum of equilibrium growth trajectories approaching the cycle around $\tilde{P}_1$. The basin of attraction of the cycle is limited by the one-dimensional stable manifold of the locally determinate point $\tilde{P}_2$. In particular, if the initial value $v(0)$ of the predetermined variable v is high enough, then there always exists an interval of initial values w^* of the jumping variable w such that the trajectory starting from $(v(0), w^*)$ approaches the limit cycle and there exist two values w_1^*, w_2^* of w such that the points $(v(0), w_1^*)$ and $(v(0), w_2^*)$ belong to the stable manifold of $\tilde{P}_2$. Again, in such a context, the economy may approach the locally determinate point $\tilde{P}_2$ by following rather different transition paths according to the initial choice (w_1^* or w_2^*) of w.

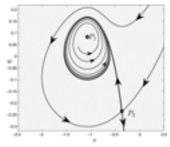

Figure 2. Global indeterminacy scenario in which two ω-limit sets exist, $\tilde{P}_2$ and the limit cycle around $\tilde{P}_1$, having respectively one-dimensional and two-dimensional stable manifolds lying in the plane $u = 0$. The (unbounded) basin of attraction of the limit cycle is limited by the one-dimensional stable manifold of the locally determinate point $\tilde{P}_2$. Parameter values: $\tau = 0.3$, $c = 1$, $\psi = 1.698$, $\sigma = \frac{1}{3}$, $\rho = 0.752877378571337$.

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5 Global analysis in a three-dimensional context

In this section we consider the scenario, already emphasized in Brito and Venditti, where a saddle with a two-dimensional stable manifold (local indeterminacy of order 2) and a sink (local indeterminacy of order 3) coexist. The above discussion shows that such a situation can occur only if two equilibria exist with $\psi_{12}, \psi_{21} > 0$ (thus $\psi_{11}, \psi_{22} < 0$) and $\delta < 0$.[9] Then it can happen that the equilibria $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, $u_1 > u_2$ and $v_1 < v_2$, are respectively, as we said, a saddle endowed with a two-dimensional stable manifold and a sink.

Our goal is to prove, in such a context, a global analysis result analogous to that of the previous section (but now for a three-dimensional phase space): that is, that the basin of attraction of the sink P_2 can be unbounded. To this end we will consider the case where P_1 and P_2 *split*, through a saddle-node bifurcation, from a degenerate equilibrium P_0. More precisely, we can take ρ as a bifurcation parameter. Then Example 3 below shows that for a suitable $\rho = \rho_0$ and for suitable values of the other parameters, system (5) has a unique equilibrium P_0 endowed with one zero eigenvalue and two complex conjugate eigenvalues with negative real part. Hence it follows from Brito and Venditti's analysis, as remarked at the end of Section 3, that a saddle-node bifurcation occurs. Specifically, formulas (5) and (9) yield that, for $\rho < \rho_0$ lying in a sufficiently small neighborhood of ρ_0, there exist two equilibria with the above described features. Then we will prove (see Lemma 1) that for $\rho = \rho_0$ the degenerate equilibrium P_0 has an unbounded *basin of attraction* (defined as the totality of the trajectories having P_0 as ω-limit set). Next we will prove (see Theorem 1) that such a property is *conserved*, after the bifurcation, by the sink P_2.

Finally, we will show (see Proposition 6) that, in the above context, when $\bar{v} \in (v_1, v_2)$, for $v_2 - v_1$ sufficiently small, there exists an open interval I contained in the line $\{u = \delta\bar{v} + \bar{d},\ v = \bar{v}\}$ whose trajectories converge to P_2 (as $t \to +\infty$), while the trajectory starting at one extreme of I tends to P_1 (as $t \to +\infty$). Hence starting from any initial value $v(0) = \bar{v}$ of the state variable v belonging to the interval (v_1, v_2), the economy may approach either the poverty trap [10] P_1 or the equilibrium point P_2, according to the choice of the initial value of the jumping variable w. We observe that such a result, although it may appear intuitive, is not at all obvious in a three-dimensional phase space.

Let us start with the following example:

Example 3. Let us take a system (5) where $\psi_{12} = 1.1$, $\psi_{22} = -0.1$, $\psi_{21} = 2$, $\psi_{11} = -1$, *having an equilibrium* $P_0 = (u_0, v_0, w_0)$. *Hence* $r_1(u_0, v_0) = r_2(u_0, v_0) = r_0$. *For the sake of simplicity, let* $\rho = r_0$ *(the transversality conditions require* $\rho < r_0$*). Then, if* $\psi_{21}e^{v_0 - u_0} = \psi_{12} - 1$, $\sigma^{-1} = \frac{(\psi_{12}-1)(1-\delta)}{\tau}$, $\delta = -0.615$, $\tau = 0.645$, *it is easy to check that* $b_1, b_2 \in (0, 1)$, *that* P_0 *is the unique equilibrium of (5) and, finally, that the Jacobian matrix* $J(P_0) = J_0$ *has one zero eigenvalue and two complex conjugate eigenvalues with negative real part.*

Lemma 1. Assume a system (5) where $\psi_{12}, \psi_{21} > 0$ and $\delta < 0$ has a unique equilibrium P_0 satisfying the conditions described in the above example. Then P_0 is a saddle-node, that is, there exists a two-dimensional smooth manifold through P_0, whose trajectories converge to P_0, separating a region R_1 constituted by trajectories tending to P_0 (as $t \to +\infty$) from a region R_2 whose trajectories do not converge to P_0. Moreover, R_1 is unbounded.

Proof 6. First of all, the existence of the degenerate equilibrium $P_0 = (u_0, v_0, w_0)$ implies $u_0 = \delta v_0 + \bar{d}$ and, setting $\tilde{h}(v) = h(\delta v + \bar{d}, v)$, we have $\tilde{h}(v_0) = \tilde{h}'(v_0) = 0$, while $\tilde{h}''(v_0) < 0$ (as is easy to compute). Moreover, referring to the expression (8) for $J(P_0)$, we have $\frac{\partial h/\partial u}{\partial f/\partial u}(P_0) = \frac{\partial h/\partial v}{\partial f/\partial v}(P_0) = m$. Consider, then, the change of coordinates

$$x = u - u_0,\ y = v - v_0,\ z = w - w_0 - m(u - u_0). \tag{12}$$

In the new coordinates, $P_0 = O = (0, 0, 0)$ and

$$J(O) = \begin{pmatrix} a & b & 0 \\ -c & -d & -l \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

where $a, b, c, d, e > 0$, $a < d$ and $(d-a)^2 < 4(bc-ad)$. In fact, multiplying the vector field of the system, in the new coordinates, by e^{-mx}, we obtain a system similar to (5):

$$\begin{cases} \dot{x} = p(x, y) \\ \dot{y} = q(x, y) - le^z \\ \dot{z} = s(x, y) \end{cases} \tag{14}$$

where $O = (0, 0, 0)$ is the unique equilibrium, $\frac{\partial s}{\partial x}(0, 0) = \frac{\partial s}{\partial y}(0, 0) = 0$ and, since $\tilde{h}''(v_0) < 0$,

$$\left(\frac{\partial^2 s}{\partial x^2}\delta^2 + 2\frac{\partial^2 s}{\partial x \partial y}\delta + \frac{\partial^2 s}{\partial y^2}\right)(0, 0) < 0. \tag{15}$$

Moreover, when $z = -\ln l + \ln q(\delta y, y) = \varphi(y)$, it is easy to check that inline image and inline image.

From straightforward computations it follows that the eigenline associated with the zero eigenvalue of inline image is given by inline image, while the eigenplane associated with the complex conjugate eigenvalues of inline image is inline image. Now take a sufficiently small neighborhood *N* of *O*. From previous considerations it follows that there exists a two-dimensional smooth manifold *S*, whose trajectories converge to *O*, which separates *N* into two disjoint open subsets inline image and inline image, containing, say, the intersections of *N* with the positive and negative *z*-semiaxis, respectively. Therefore the intersection with *N* of a center manifold at *O* of (14), tangent to inline image in *O*, can be written as inline image, inline image, inline image. Besides, straightforward calculations show that, if *N* is small enough, the coordinates of inline image satisfy

display math (16)

where inline image. More precisely, it can be shown that for a sufficiently small *N* the equations of a center manifold inline image (i.e., of an invariant manifold tangent in *O* to the line *L*) are of the type inline image, inline image, with inline image and inline image smooth in a neighborhood of inline image. Moreover, the center manifold is proven to be unique (see the Appendix).

It follows that along inline image, inline image increases while inline image and inline image decrease (recall that (15) holds).

Consider now a point inline image and a sufficiently small disk *D* centered in *Q* and lying in inline image. From what we have seen it follows that all the trajectories starting in *D* converge to *O* and those from inline image do so spiraling. In particular, along them inline image changes sign infinitely many times and thus they intersect infinitely many times the plane inline image (corresponding to inline image). Moreover, all the trajectories in inline image converge to *O* (if *N* is small enough), as they cross inline image alternately on each *side* of the line *L* and therefore eventually wind around inline image and so spiral toward *O*.

Our final step is to prove that inline image decreases along the negative trajectory starting from a point of inline image, where inline image. Suppose, by contradiction, that this is not the case. Then there should exist a first point inline image on the above-mentioned trajectory such that inline image (i.e., inline image) and inline image for inline image in a right neighborhood of inline image. Since it follows from (16) that inline image when inline image, we have:

display math

Hence inline image, that is, inline image. Suppose that inline image. Then, by the continuous dependence of trajectories on initial conditions, there exists a small disk inline image centered in *R* and contained in the planar region inline image, such that all the positive trajectories starting from inline image enter into inline image and then converge to *O*. Besides, all the positive trajectories from inline image cross inline image again for the first time at some positive value of *t*. This way we can define a map inline image from inline image into the plane inline image, which can be extended to *R* by setting inline image. Therefore inline image is a homeomorphism mapping inline image onto an open neighborhood of *O*, which is clearly impossible, since in any neighborhood of *O* on the plane inline image there exist points (with inline image) whose trajectories do not converge to *O*. Hence inline image. Therefore, since inline image, it follows that inline image, while

display math (17)

Hence inline image both in a left and a right neighborhood of inline image, which leads to a contradiction. Consequently it can be proven (see the Appendix) that along the above trajectory (say, the *continuation* of inline image) *x*, *y* and *z* are all unbounded: specifically, coming back to the original time *t*,

display math

This completes the proof of the lemma. inline image

Figures 3 and 4 show, by utilizing the parameter values of Example 3, the dynamics of trajectories converging to inline image in the half-space inline image. Figure 4 zooms in on a small region of Figure 3, showing how the generic trajectory converging to inline image winds around the center manifold through inline image.

image

Figure 3. The dynamics of trajectories converging to inline image in the half-space inline image, utilizing the parameter values of Example 3. Parameter values: inline image, inline image, inline image, inline image, inline image, inline image, inline image.

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image

Figure 4. Close-up of the small region indicated in Figure 3, showing how the generic trajectory converging to inline image winds around the central manifold.

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We can now prove our main theorem:

Theorem 2. Assume, in system (5), that inline image, inline image, and that there exist two equilibria, inline image and inline image, with inline image and inline image sufficiently small. Moreover, suppose that inline image has a two-dimensional stable manifold, inline image is a sink and both the Jacobian matrices inline image and inline image have two complex conjugate eigenvalues. Then the basin of attraction of inline image is unbounded.

Proof 7. Let inline image. Then we can assume that the system originates from a saddle-node bifurcation where inline image. Consider a point inline image, the intersection of the above center manifold with inline image, Q being also in the attractive basin of inline image in the bifurcated system. Then, if inline image is small enough, the negative trajectory of Q remains *close* to inline image for a sufficiently long time. More precisely, setting inline image, we can take, for a sufficiently small inline image, some inline image sufficiently large that, denoting inline image, the following inequalities hold:

1. inline image;
2. inline image (recall that inline image and inline image when v is large enough);
3. inline image.

Then inequalities 1 and 2 imply, respectively, inline image and inline image. Hence, for inline image, inline image keeps increasing and, because of inequality 2, so does inline image, while inline image decreases. It follows, by the same arguments used in the Appendix, that such a trajectory is unbounded and, therefore, so is the basin of attraction of inline image. inline image

Notice that the value of v (and consequently, by (7), the value of the growth rate inline image) in inline image is higher than in inline image. Moreover, as the proof of the above theorem shows, there exists a continuum of trajectories approaching the *virtuous* equilibrium inline image if the initial value of the predetermined variable v (remember that inline image) is high enough, that is, if the initial ratio between physical capital inline image and human capital inline image is high enough.

Finally, we prove the following proposition:

Proposition 6. With the assumptions of Theorem 1, if inline image is sufficiently small, there exists on every line inline image, inline image, an interval inline image such that all the trajectories starting from I converge to inline image, while the trajectory starting from either A or B converges to inline image.

Proof 8. Consider the strip inline image. Taking coordinates inline image, inline image, inline image, inline image, it follows from the proof of Theorem 1 that the stable manifold of inline image is tangent, at inline image, to a plane close, if inline image is small enough, to inline image in a neighborhood of inline image. Hence the manifold intersects each line inline image. Therefore on each such line there exists an interval with the properties described in the statement of the proposition. inline image

Example 4. Consider system (5) *with* inline image, inline image, inline image, inline image, inline image, inline image, inline image. *By a suitable translation of* inline image *and a rescaling of t*, inline image *we can assume* inline image. *Take* inline image, *where* inline image *satisfies* inline image, inline image. *Then, if* inline image *is sufficiently small the system has two equilibrium points* inline image *and* inline image, *with* inline image, *satisfying the conditions of Theorem* 1. *Hence* inline image *is a saddle with a two-dimensional stable manifold and* inline image *is a sink*.

Referring to this example, we can consider a further linear change of coordinates, namely inline image inline image inline image inline image. This way inline image is translated to the origin and inline image, inline image lie on inline image. On such a plane a line inline image represents a fixed choice of the state variable. Then let z vary on a line inline image: for a suitable value of z close to 0, say inline image, the trajectory starting at inline image spirals toward inline image, while the trajectories starting from points of the line with inline image, up to a certain value of z, converge to inline image (see Figures 5 and 6).

image

Figure 5. Two trajectories starting from the same initial value of the state variable inline image (remember that inline image). The trajectory starting from inline image, with inline image, approaches the saddle inline image; the trajectory converging to the locally attractive equilibrium inline image starts from inline image, with inline image. The parameter values are those given in Example 4 with inline image.

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image

Figure 6. Phase portrait of system (5) obtained with the same parameter values of the simulation in Figure 5. Only one trajectory (the same as illustrated in Figure 5) approaches the saddle inline image; the other trajectories, starting from inline image with inline image, belong to the basin of attraction of the equilibrium inline image.

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Figures 5 and 6 illustrate the phase portrait of system (5) with the parameter values of the above example. Figure 5 shows two trajectories starting from the same initial value of the state variable inline image (remember that inline image), one approaching inline image and the other converging to inline image. Figure 6 is obtained with the same parameter values; however, more trajectories are plotted, all starting from the same value of the state variable y. In Figure 6 only one trajectory approaches inline image, while the others belong to the basin of attraction of the virtuous equilibrium inline image.

6 Concluding remarks

1. Top of page

We have explored two cases where the Brito–Venditti model (5) admits two balanced growth paths, each of them corresponding, after a change of variables, to an equilibrium point of a three-dimensional system. In one case the two equilibrium points inline image (poverty trap) and inline image have, respectively, a two-dimensional and a one-dimensional stable manifold (i.e., they are, respectively, locally indeterminate of order 2 and determinate). In the other case the stable manifolds of the two equilibria inline image (poverty trap) and inline image have, respectively, dimension 2 and 3 (i.e., they are locally indeterminate of order 2 and 3). The results concerning the first case are obtained assuming that the quantity of externalities is the same in both sectors (i.e., inline image) and consequently the dynamics is fully described by a two-dimensional system (there exists an invariant plane). In the second case the dimension of system (5) cannot be reduced. In both cases, for suitable values of the parameters, we have proved:

1. The existence of points inline image inline image such that in any neighborhood of inline image lying on the plane inline image (corresponding to a fixed value of the state variable inline image) there exist points *Q* whose positive trajectories tend to either equilibrium (these results are illustrated in Figures 1, 2, 5, and 6).
2. The unboundedness of the basin of attraction of the locally indeterminate second-order equilibrium in the first case, and of the locally indeterminate third-order equilibrium in the second case. This result appears to contain more information than other global indeterminacy results, where the equilibrium is shown to be globally indeterminate in the interior of a two-dimensional invariant region enclosed by a periodic or homoclinic orbit (see, for example, Benhabib *et al.*, 2008; Mattana *et al.*, 2009).
3. The basins of attraction of the equilibrium point inline image (in the first case) and of the point inline image (in the second case) are limited respectively by the one-dimensional stable manifold of inline image and by the two-dimensional stable manifold of inline image.

Finally, in the first case, we have also shown that when the locally indeterminate equilibrium point becomes a source, a supercritical Hopf bifurcation occurs giving rise to an attracting (i.e., endowed with a two-dimensional stable manifold) limit cycle (see Figure 2). When this happens, global indeterminacy is observed in a context where no equilibrium point is locally indeterminate.

Appendix

Proof of Proposition 2

Let inline image be an equilibrium of (5). Then inline image, as can easily be checked. From straightforward computations it follows that

display math

Hence the coefficient of inline image is positive, being equal to inline image, and so is the coefficient of inline image, as inline image. Moreover,

display math

This proves the proposition. inline image

Proof of Proposition 3

Let us assume that either Case 2 or 3 of Proposition 1 is satisfied, so that two equilibrium points exist, inline image and inline image, with inline image. Then, in Case 2, inline image, and the reverse holds in Case 3. While inline image implies in both cases that inline image, inline image, it follows that inline image in Case 2 and inline image in Case 3 are saddles with a one-dimensional stable manifold, hence locally determinate. When inline image, writing the characteristic polynomial as inline image, where, of course, inline image, it follows from straightforward computations that two negative real part eigenvalues exist if and only if inline image. In fact inline image can be written as a function of inline image, that is, inline image. Then it can be calculated that in Case 2, when inline image, inline image as inline image. Hence it follows that inline image is a source. In Case 3, on the other hand, inline image can be locally indeterminate of order 2, as Example 1 shows. inline image

Proof of Theorem 1

If system (11) satisfies conditions 1–5, there exists at most one simple, repelling limit cycle surrounding inline image. Suppose, by contradiction, that this is the case and move inline image toward inline image, where inline image. More precisely, setting inline image, inline image, we choose smooth functions inline image, inline image, inline image, inline image, starting from the original parameters inline image, with inline image, such that, for any inline image, (10) has equilibria inline image and inline image. Moreover, as the original system (inline image) possesses a limit cycle, the trajectory inline image, from a point of the unstable manifold of inline image in the half-plane inline image, intersects the line inline image (if at all) at a point inline image, inline image. Therefore we can choose the functions inline image, inline image, inline image, inline image in such a way that, for any inline image, inline image, defined analogously as inline image, may intersect the line inline image at a point inline image, inline image. Thus no saddle connection occurs as inline image moves to inline image. As a consequence, for any inline image, system (10) has an odd number of limit cycles. In fact, consider an intermediate equilibrium point inline image and the analytical Poincaré return map inline image defined on an open interval inline image of the half-line inline image, where inline image, inline image being the intersection of the half-line with the unstable manifold of inline image. Hence, if a bifurcation occurs, there is some inline image, where inline image and inline image has the same sign, positive or negative, in a neighborhood of inline image for inline image. Thus an even number of limit cycles is possibly generated or removed. Moreover, by setting inline image, inline image, system (10) is equivalent to a polynomial system defined in the invariant half-plane inline image, which has a finite number of limit cycles (see, for example, Arnold and Il'yashenko, 1994). Finally, as proven in Proposition 5, a further limit cycle is generated by the Hopf bifurcation when inline image is sufficiently small. Hence system (10) must have an even number greater than zero of limit cycles when such inline image is sufficiently small. In this case the two equilibria can be written as inline image, inline image, inline image. Again, we observe that, by the change of variables inline image, inline image, system (10) gives rise to a Liénard system of the type (11) defined in inline image, where inline image. It follows that, when inline image is small enough, this system has at most one simple limit cycle if:

1. inline image is non-increasing in inline image;
2. the system of equations inline image, inline image has at most one solution for inline image, inline image.

As straightforward, even if lengthy, calculations show the two conditions to be satisfied, we get a contradiction, implying that the original system (10) has no limit cycle. Therefore Remark 1, as we have recalled, implies the statement of the theorem. inline image

Proof of Proposition 5

We want to prove that the bifurcation of system (10), occurring when inline image, is Hopf supercritical. To this end consider system (10) and the equilibrium inline image with inline image lying in a sufficiently small neighborhood of inline image. Then the eigenvalues at inline image are given, as can easily be computed, by

display math

As inline image, so is inline image for inline image sufficiently close to inline image, while, since inline image, inline image changes sign as inline image crosses inline image. Moreover, when inline image is close to inline image, the eigenvalues inline image are complex conjugate. Hence, referring to (Guckenheimer and Holmes, 1997, theorem 3.4.2), we can take as bifurcation parameter inline image, that is, we can set inline image. Therefore

display math

so that condition inline image of the above-mentioned theorem is satisfied. Consider now the bifurcation situation when inline image. Since inline image, we can replace *w* by *kw*, where inline image, so that (10) becomes

display math (18)

This way the system is in the form

display math

where inline image. Then we can apply (Guckenheimer and Holmes, 1997, formula (3.4.11)) in order to calculate the quantity *a* which yields a Hopf supercritical (inline image) or a Hopf subcritical (inline image) bifurcation. In fact straightforward computations lead to

display math (19)

As it can easily be checked that

display math

this proves that a Hopf supercritical bifurcation occurs. inline image

Uniqueness of the center manifold in Lemma 1

As we have seen, a center manifold in a neighborhood of inline image can be represented as inline image inline image, with inline image, inline image, inline image (using the notation of (13)). First of all, we check that inline image are inline image in a suitable interval inline image, inline image. In fact, by induction, let

display math

inline image. Then, differentiating, we have

display math (20)

where (see (14)) inline image, inline image, inline image. Hence, after straightforward computations, display math

where inline image are determined by inline image. So also inline image are unequivocally determined.

However, this does not guarantee that the center manifold is analytic and thus unique. Let us therefore assume, by contradiction, that there exist infinitely many center manifolds. In fact we can confine ourselves to considering inline image, as for inline image a trajectory lying on the center manifold tends to O as inline image, which implies that the center manifold in such half-space is unique Sijbrand (1985).

Our first observation is that the *pencil* of center manifolds is bounded, that is, when inline image, inline image sufficiently small, all the center manifolds lie in a parallelepiped inline image, inline image. This follows from the fact that the trajectory starting at a point Q of the half-plane inline image sufficiently close to O spirals toward O (as inline image), crossing infinitely many times the plane inline image alternately on each side of the curve inline image and thus of the line inline image. Hence the pencil of center manifolds *lies* inside this spiral.

Next we show that each center manifold inline image satisfies, in a suitable interval inline image, a second-order differential equation inline image. To this end, from (20) we derive

display math

that is,

display math (21)

On the other hand, inline image yields inline image, from which we get inline image as a function of inline image. Therefore, differentiating with respect to y, a series of easy steps leads to

display math

Next we want to show that we can write

display math (22)

where inline image and inline image are smooth and non-zero in a neighborhood of inline image. In fact, for any inline image, let

display math

where inline image are the same for any inline image. From what we have observed, we can consider, in a suitable interval inline image, the *lowest* center manifold with respect to x, that is, inline image such that inline image for any inline image when inline image. From the theory on center manifolds (see Sijbrand, 1985) it follows that there exist, for each inline image, two constants, inline image and inline image, such that

display math

where inline image and inline image. By differentiating with respect to y, we can calculate inline image and inline image, and in fact we can write

display math

Analogously,

display math

where the functions inline image, inline image are positive. Moreover, as inline image and inline image are uniformly bounded when inline image, we can extend inline image as functions of inline image defined in a suitable neighborhood of inline image. Clearly these functions may not be continuous in inline image. However, for any inline image, the functions defined as

display math

when inline image, and 0 when inline image, are smooth in a neighborhood of inline image. Then, recalling inline image, (22) follows from straightforward computations. Hence

display math (23)

where inline image is smooth in a neighborhood of inline image. But this implies the existence of a unique solution of (23) satisfying inline image, hence yielding a contradiction. Therefore the center manifold is unique.

Unbounded trajectory converging to inline image in Lemma 1

Let inline image, the intersection of the unique center manifold with the half-space inline image. Exchanging t with inline image, we have proved that the trajectory starting at Q satisfies inline image when inline image. Suppose, by contradiction, that inline image. On the other hand, inline image implies inline image, that is, inline image, while, since inline image is bounded, for any inline image there exists inline image such that inline image as inline image, except possibly in an interval of amplitude inline image. Consider, now, inline image. From straightforward calculations it follows that, when inline image, except possibly in an interval of amplitude inline image

display math

where [inline image] are suitably defined and, by our assumptions, [inline image], [inline image], [inline image] when [inline image]. Then, by taking [inline image] sufficiently small, it follows, for [inline image], outside a possible interval of amplitude [inline image], [inline image] for a suitable k, implying [inline image]. Consequently, [inline image] and [inline image], yielding a contradiction. Therefore, as [inline image], [inline image], so that [inline image] tends to [inline image], and it can be easily seen that [inline image] does likewise.

Notes

1 See Benhabib and Farmer (1999). Although the main body of the literature on local indeterminacy is concerned with economies with increasing social returns (see, for example, Benhabib and Farmer, 1994; Boldrin and Rustichini, 1994), a growing proportion of articles deal with models where indeterminacy is obtained under the assumption of social constant return technologies (see Benhabib and Farmer, 1999; Mino, 2001; Mino *et al.*, 2008).

2 In particular, local indeterminacy occurs if the number of eigenvalues with negative real parts of the linearization matrix evaluated at the equilibrium point is greater than the number of state variables. So, in a two-dimensional system, we have local indeterminacy if and only if the equilibrium point is a sink.

3 In the present work we do not deal with another important problem in economic dynamics, namely the existence of indifference points in an optimal control problem. Starting from these points, more than one optimal solution exists, giving rise to the same value of the objective function (see the seminal contributions of Skiba, 1978; Dechert and Nishimura, 1983; Sethi, 1997). In our context, the trajectories followed by the economy do not represent optimal solutions, the dynamics being conditioned by externalities. Therefore, when multiple equilibrium trajectories exist, starting from the same initial values of the state variables, economic agents may select one trajectory along which the value of the objective function is lower than along other admissible trajectories, due to coordination problems.

4 The long run behaviors of the state variables may be different also when there exists a chaotic attractor (see, e.g., Boldrin *et al.*, 2001; Antoci *et al.*, 2010, 2014; Gori and Sodini, 2011, 2014).

5 Where [inline image] represents the equilibrium rental rate [inline image], [inline image].

6 The relevance, with respect to the existing literature, of the local analysis results illustrated in this section is exhaustively discussed in Brito and Venditti's article.

7 Notice that, in system (5), v is a state variable while u and w can be considered as jumping variables. So, an equilibrium point is locally determinate if it has a one-dimensional stable manifold or is repelling.

8 Remember that [inline image] and [inline image].

9 Remember that [inline image] if and only if [inline image], that is, if the final good is intensive in human capital at the private level.

10 Remember that, by (7), the growth rate [inline image] associated with each equilibrium point is positively correlated with the equilibrium value of v.

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