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Remarks on the Generalized Reflectionless Schrödinger Potentials

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Abstract We discuss certain compact, translation-invariant subsets of the set $\mathcal{R}$ of the generalized reflectionless potentials for the one-dimensional Schrödinger operator. We determine a stationary ergodic subset of $\mathcal{R}$ whose Lyapunov exponent is discontinuous at a point. We also determine an almost automorphic, non-almost periodic minimal subset of $\mathcal{R}$.

Keywords Generalized reflectionless potential · Stationary ergodic flow · Lyapunov exponent · Almost automorphic minimal set

Mathematics Subject Classification 37B55 · 34L40 · 31A35 · 30F35

1 Introduction

Let $q : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, and consider the differential expression

$$L = -\frac{d^2}{dx^2} + q(x).$$

It is well-known that L defines an unbounded self-adjoint operator (Schrödinger operator) on $L(\mathbb{R})$, which we will also denote by L, or by L_q if it is useful to specify q. The spectral properties of L are well-understood when q decays rapidly at $x = \pm\infty$, and when q is a periodic function of x. In the first case, one has the scattering theory [36,39,44]; in the second case, the operator L has band spectrum [42].

Less information is available in other cases. There are examples of Bohr almost periodic "potentials" $q(x)$ for which one has detailed knowledge concerning the spectrum of L. The

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algebro-geometric potentials of Dubrovin–Matveev–Novikov [9] are the best-known among these; see the lecture notes of Moser [48] for a description of their basic properties. Levitan-Savin [40], Moser [47] and Egorova [12] have studied Bohr almost periodic potentials q for which the operator L has Cantor spectrum. Sodin–Yuditskii [61] consider potentials q for which L has homogeneous spectrum; their q's are Bohr almost periodic.

In this paper, we will consider the class $\mathcal{R}$ of generalized reflectionless Schrödinger potentials, which were introduced by Lundina in 1985 [41]. The class $\mathcal{R}$ was subsequently studied by Marchenko [43], Gesztesy–Karwowski–Zhao [14], Kotani [34], and Johnson–Zampogni [28]. To define $\mathcal{R}$, let us recall that $q(x)$ is a soliton potential if it decays rapidly as $x \to \pm\infty$, and if its (left or right) reflection coefficient vanishes. A soliton potential has a very special structure; in fact it can be viewed as a Dyson determinant [11]:

$$q(x) = -2\frac{d^2}{dx^2}\ln\det(I + A(x))$$

where $A(x) = \left(A_{ij}(x)\right)_{1\le i,j\le n}$ and $A_{ij(x)} = \dfrac{\sqrt{l_i l_j}}{\eta_i + \eta_j}e^{-(\eta_i+\eta_j)x}$. Here $l_1, \ldots, l_n, \eta_1, \ldots, \eta_n$ are positive numbers. If $q(x)$ is a soliton potential, then the spectrum of L consists of $[0, \infty)$ together with finitely many eigenvalues $-\eta_1^2, \ldots, -\eta_n^2$. A generalized reflectionless potential is a compact uniform limit of soliton potentials: thus $q(x) = \lim_{n\to\infty} q_n(x)$, where each q_n is a soliton potential, and the convergence is uniform on compact subsets of $\mathbb{R}$.

It is natural and convenient to fix a negative constant c, and restrict attention to solitons q for which the spectrum of L is contained in $[c, \infty)$. Ler $\mathcal{R}_c$ be the set of all potentials q which are obtained as compact uniform limits of such solitons. Then $\mathcal{R}_c$ is compact in the compact-open topology [41]. Moreover, each $q \in \mathcal{R}_c$ is analytic in a horizontal strip centered on the x-axis whose width depends only on c [34]. Later we will fix $c = -1$.

Kotani ([34]; see also Marchenko [44]) characterized the spectral measures of the half-line operators defined by $q \in \mathcal{R}_c$, and also showed that a potential in $\mathcal{R}_c$ is of Sato–Segal–Wilson type. This implies, among other things, that q admits a meromorphic extension to the entire complex x-plane, and moreover defines an element of Sato's Grassmaniann $\mathfrak{Gr}_2$ [55,56]. As a consequence of the Sato–Segal–Wilson theory, the Korteweg-de Vries hierarchy of nonlinear evolution equations with initial datum $q \in \mathcal{R}_c$ admits a (meromorphic) global solution. To round-out this picture, it was proved in [28] that a bounded Sato–Segal–Wilson potential is, after appropriate dilatation and translation, an element of $\mathcal{R}_c$.

In this paper, we will analyze certain compact, translation-invariant subsets $\mathcal{Q} \subset \mathcal{R}_c$. We set essentially three goals. The first is to introduce a "divisor map" π for appropriate sets $\mathcal{Q}$, which assigns to a given element $q \in \mathcal{Q}$ a finite or countably infinite set of "poles" (of the Weyl m-functions; see below). We will state and prove a result—Theorem 3.3—which gives sufficient conditions that π be a homeomorphism.

We will then study certain (compact, translation-invariant) subsets $\mathcal{Q} \subset \mathcal{R}_c$ having dynamical properties such as minimality and ergodicity. There are well-known examples of sets $\mathcal{Q}$ of this sort which consist of Bohr almost-periodic functions [18,60]. Our second goal is to construct a minimal subset $\mathcal{Q} \subset \mathcal{R}_c$ which is almost automorphic in the sense of Bochner-Veech [63], but is not almost periodic. Since any minimal subset $\mathcal{Q} \subset \mathcal{R}_c$ supports at least one ergodic measure, this will also give an example of a stationary ergodic flow $\mathcal{Q} \subset \mathcal{R}_c$ which is not almost periodic.

We wish to point out that, in a recent paper, Volberg and Yuditskii [64] have constructed translation invariant minimal sets of Jacobi matrices which are not almost periodic. In a succeeding paper, Damanik and Yuditskii [8] carried out an analogous construction for continuous Schrödinger operators. Our result is parallel to theirs. The proofs in [8,64] rely on a detailed study of character-automorphic Hardy spaces. We will also make use of some elements of this beautiful theory, but rather less than in [64] and [8]. In fact, at certain points we will make use of alternate, dynamical methods. On the other hand, the main goal in [8,64] was to obtain counterexamples to the Kotani-Last conjecture, which is a question in spectral theory and not dynamics. We will comment on the Kotani-Last conjecture later.

Our example has the following feature. First, the spectrum of each $q \in \mathcal{Q}$ is a fixed closed set $E \subset \mathbb{R}$ of the form $[-1,\infty)\setminus\bigcup_{j=1}^{\infty}(a_j,b_j)$ where the open intervals (a_j,b_j) are all contained in $(-1,0)$. Second, it turns out that each $q \in \mathcal{Q}$ is uniquely determined by a "pole divisor" d and vice-versa; here d is a sequence of pairs $\{(y_j,\varepsilon_j)\}$ such that $y_j \in [a_j,b_j]$ and $\varepsilon_j = \pm 1$ $(j = 1,2,\ldots)$. Our proof of this fact makes use of the theory of exponential dichotomies for linear differential systems [5,54]. Third, there is an Abel map $\mathfrak{a}$ which sends divisors d to unitary homomorphisms α (characters) defined on the fundamental group Γ_E of the multiply-connected domain $\Omega_E = \mathbb{C}\setminus E$. This Abel map is that of Sodin-Yuditskii ([60,61]; also [18]). When E is chosen appropriately, it is surjective and continuous but not injective; however it admits a right inverse $\mathfrak{i}$ which, although discontinuous, is a pointwise limit of a sequence of continuous functions. That is, $\mathfrak{i}$ is of first Baire class. This fact has dynamical consequences for $\mathcal{Q}$ as we will discuss. We mention that the properties of the Abel map $\mathfrak{a}$ are related to those of the domain Ω_E itself: it is of Parreau-Widom type but does not satisfy the Direct Cauchy Theorem [8,21,50,60,64,65].

One may ask numerous other interesting questions concerning the (compact, translation invariant) stationary ergodic subsets $\mathcal{Q} \subset \mathcal{R}_c$. One of these regards the "random" nature of the potentials $q \in \mathcal{Q}$, as measured by the positivity of the Lyapunov exponent $\beta(\lambda)$ for points $\lambda \in E$. This question is a version of one posed by Zakharov (see [34]). One would like to construct examples of stationary ergodic $\mathcal{Q} \subset \mathcal{R}_c$ for which $\beta(\lambda) > 0$ for a.a. $\lambda \in E$. We have not done this, but we do give an example of such a $\mathcal{Q}$ for which $\beta(\lambda) > 0$ at a non-zero set of points $\{\lambda\} \subset E$. For this construction we do not (cannot) use the theory of Parreau-Widom domains, since we will require the existence of points in E which are not regular for the Dirichlet problem in Ω_E. The construction of this example constitutes the third goal of this paper.

Here is an outline of the contents of the present paper. In Sect. 2, we review some basic facts concerning the class $\mathcal{R}$ of generalized reflectionless Schrödinger potentials. We will restrict attention to the subset $\mathcal{R}_c \subset \mathcal{R}$ obtained by setting $c = -1$, then abuse notation by writing $\mathcal{R} = \mathcal{R}_{-1}$. So if $q \in \mathcal{R}$, then the spectrum of $L = -\dfrac{d^2}{dx^2} + q(x)$ is contained in $[-1,\infty)$ and contains $[0,\infty)$. In this latter interval it is purely absolutely continuous, and moreover a basic non-reflection condition is satisfied; see below. So the "interesting" part of the spectrum of L is contained in $[-1,0]$. We will review results of Kotani [34] and Marchenko [43] concerning the spectral measure of L_q, and other aspects of its spectral theory.

We will also review some facts concerning Parreau-Widom domains in $\mathbb{C}$ [21,60,65]. We consider domains of the form $\Omega_E = \mathbb{C}\setminus E$ where $E \subset \mathbb{R}$ is a closed subset with appropriate properties. We outline the construction of the Abel map $\mathfrak{a}$ from the set $\mathcal{D}_E$ of real divisors of Ω_E onto the set $\mathcal{J}_E$ of unimodular characters on the fundamental group Γ_E of Ω_E [60,61].

In Sect. 3 we will present our main results. Let $E \subset [-1,\infty)$ be a closed subset of the real line which has locally positive measure and which contains $[0,\infty)$. We first study the set $\mathcal{Q}_E$ of bounded continuous potentials q for which L_q has spectrum E and which are reflectionless in the sense of Craig [6]. It turns out that $\mathcal{Q}_E$ is contained in $\mathcal{R} = \mathcal{R}_{-1}$; moreover it is compact and translation invariant. We determine conditions under which $\mathcal{Q}_E$ is homeomorphic to the space of divisors $\mathcal{D}_E$ corresponding to E; here $\mathcal{D}_E$ is given the Tychonov topology. Our proof of this basic fact relies on the theory of exponential dichotomies [5,53,54].

We then consider sets E of the above type which contain points λ which are irregular in the sense of the Dirichlet problem. Let ν be a measure on $\mathcal{Q}_E$ which is ergodic with respect to the translation flow. We show that the ν-Lyapunov exponent β is positive in λ: $\beta(\lambda) > 0$. For the proof, we use the most basic properties of the ν-Floquet exponent. We remark that this method does not give rise to a subset E of positive measure on which $\beta > 0$: this is because of a classical result of Kellogg [30] which implies that the set of irregular points in E has zero logarithmic capacity.

Finally we consider a set E as above which is also of Parreau-Widom type (in particular is regular for the Dirichlet problem). Let $\mathcal{J}_E$ be the set of unitary characters defined on the fundamental group Γ_E of the domain $\Omega_E = \mathbb{C}\backslash E$. There is a natural group operation on $\mathcal{J}_E$ (namely pointwise multiplication) which we denote by $+$. We study the Abel map $\mathfrak{a} : \mathcal{D}_E \to \mathcal{J}_E$ which is continuous and surjective but need not be injective [21]. Since $\mathcal{D}_E$ is homeomorphic to $\mathcal{Q}_E$, the translation flow on $\mathcal{Q}_E$ induces a flow on $\mathcal{D}_E$. There is a fixed character $\delta \in \mathcal{J}_E$ such that, if one defines a translation flow on $\mathcal{J}_E$ by $\alpha \mapsto \alpha + \delta x$, then $\mathfrak{a}$ is a homomorphism of flows. Now, if E is a homogeneous set in the sense of Carleson [2], then $\mathfrak{a}$ is actually an isomorphism of flows [61]. We will produce an example of a set E of Parreau-Widom type, for which $\mathfrak{a}$ is not injective but admits a right inverse of the first Baire class. One can arrange that the translation flow on $\mathcal{J}_E$ is minimal. It then follows that $\mathcal{Q}_E$ contains a minimal subset M which is almost automorphic in the sense of Bochner–Veech, but not almost periodic. This is a nontrivial dynamical fact concerning the flow on $\mathcal{Q}_E$. We will compare our construction with those of [8]. We will also offer some speculation to the effect that the flow on $\mathcal{Q}_E$ appears to be of "Denjoy type" in a sense to be explained.

We close this Introduction with a review of some basic information concerning nonautonomous dynamics and its application to the theory of the Schrödinger operator with a bounded continuous potential. We also review the basic facts concerning the algebro-geometric potentials.

Let P be a topological space. A real flow on P is defined by a jointly continuous map $\tau : P \times \mathbb{R} \to P : \tau(p, x) = \tau_x(p)$ with the following properties: (i) $\tau_0(p) = p$ for all $p \in P$; (ii) $\tau_x \circ \tau_t = \tau_{x+t}$ for all $x, t \in \mathbb{R}$. Suppose that P is compact. A regular Borel probability measure ν on P is said to be τ-invariant if $\nu(\tau_x(B)) = \nu(B)$ for each Borel set $B \subset P$. It is said to be ν-ergodic if in addition the condition $\nu(\tau_x(B)\Delta B) = 0$ for all $x \in \mathbb{R}$ implies that $\nu(B) = 0$ or $\nu(B) = 1$. Here Δ is the symmetric difference of sets. If P, $\{\tau_x\}$, and ν are as above then we will sometimes refer to the triple $(P, \{\tau_x\}, \nu)$ as a stationary ergodic process. See [13,49].

We recall some standard terminology concerning flows. Let P be a topological space, and let $\{\tau_x\}$ be a flow on P. If $p \in P$, the orbit through p is $\{\tau_x(p) \mid x \in \mathbb{R}\}$. A subset $Q \subset P$ is said to be invariant if $\tau_x(Q) = Q$ for all $x \in \mathbb{R}$. Suppose now that P is a compact metric space. A flow $\{\tau_x\}$ on P is said to be minimal or Birkhoff recurrent if for every $p \in P$, the orbit through p is dense in P. The flow is said to be (Bohr) almost periodic if there is a metric d on P, which is compatible with the topology on P, such that $d(\tau_x(p), \tau_x(q)) = d(p, q)$ for all $p, q \in P$ and all $x \in \mathbb{R}$. If $(P, \{\tau_x\})$ is almost periodic and $p \in P$, the orbit closure $cls\{\tau_x(p) \mid x \in \mathbb{R}\} \subset P$ is minimal. A minimal flow $(P, \{\tau_x\})$ is said to be almost automorphic if there is a point $p \in P$ such that, whenever $|x_n| \to \infty$ and $\tau_{x_n}(p) \to p$, one has also $\tau_{-x_n}(p) \to p$. If a flow $(P, \{\tau_x\})$ is minimal almost automorphic, then a Baire residual set of $p \in P$ has the above property (see [13,63]).

Next let $q : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, and let $L_q = L = -\dfrac{d^2}{dx^2} + q(x)$ denote the corresponding self-adjoint Schrödinger operator on $L^2(\mathbb{R})$. The spectrum E of L

is a closed subset of $\mathbb{R}$ which is bounded below and unbounded above. If $\lambda \in \mathbb{C}$, consider the equation $L\varphi = \lambda\varphi$, or equivalently

$$-\varphi'' + q(x)\varphi = \lambda\varphi.$$

Introducing the phase coordinates $u = \binom{\varphi}{\varphi'}$, one can also write

$$u' = \begin{pmatrix} 0 & 1 \\ -\lambda + q(x) & 0 \end{pmatrix} u. \tag{1}$$

If $\Im\lambda \neq 0$, then it is well-known that (1) admits nonzero solutions $u_\pm(x,\lambda) = \binom{\varphi_\pm(x)}{\varphi'_\pm(x)}$ which decay exponentially as $x \to \pm\infty$ respectively, and these are unique up to a constant multiple. We introduce the Weyl m-functions [4]

$$m_\pm(\lambda) = \frac{\varphi'_\pm(0)}{\varphi_\pm(0)}.$$

Then $m_\pm$ are holomorphic in $\mathbb{C}\backslash\mathbb{R}$, and satisfy

$$sgn \frac{\Im m_\pm(\lambda)}{\Im\lambda} = \pm 1.$$

Moreover, if $I \subset \mathbb{R}\backslash E$ is a nondegenerate open interval in the resolvent set of L (a spectral gap), then $m_\pm$ extend meromorphically through I. It is not hard to show that at most one of the functions $m_\pm$ can admit a pole in I: this is because the extended real-valued functions $m_\pm : I \to \mathbb{R} \cup \{\infty\}$ are strictly monotone on I, with m_+ increasing and m_- decreasing. See [23].

We will work in the context of compact translation-invariant sets of potentials [1]. It is convenient to introduce the following setup. Let $M > 0$, and let $\mathcal{P}_M = \{q : \mathbb{R} \to \mathbb{R} \mid q$ is continuous and $\sup_{x\in\mathbb{R}} |q(x)| \leq M\}$. Give $\mathcal{P}_M$ the topology of uniform convergence on compact sets. There is a natural flow $\{\tau_x \mid x \in \mathbb{R}\}$ on $\mathcal{P}_M$, defined by translation: $\tau_x(q)(\cdot) = q(x+\cdot)$.

Let $\mathcal{Q} \subset \mathcal{P}_M$ be a compact translation-invariant set. For example, if $q \in \mathcal{P}_M$ is uniformly continuous, then $cls\{\tau_x(q) \mid x \in \mathbb{R}\}$ is compact and invariant. For fixed $\lambda \in \mathbb{C}$, consider the family of differential systems

$$u' = \begin{pmatrix} 0 & 1 \\ -\lambda + q(x) & 0 \end{pmatrix} u \qquad (q \in \mathcal{Q}). \tag{1_q}$$

The family (1_q) induces a flow on $\mathcal{Q} \times \mathbb{C}^2$, as follows. Let $\mathcal{U}(x,\lambda,q)$ denote the fundamental matrix solution of (1_q). Define

$$\hat{\tau}_x(q, u_0) = (\tau_x(q), \mathcal{U}(x,\lambda,q)u_0) \qquad (q \in \mathcal{Q},\ u_0 \in \mathbb{C}^2,\ x \in \mathbb{R}).$$

Then in fact $\{\hat{\tau}_x \mid x \in \mathbb{R}\}$ defines a so-called skew-product flow on $\mathcal{Q} \times \mathbb{C}^2$. If $\lambda \in \mathbb{R}$, then $\mathcal{Q} \times \mathbb{R}^2 \subset \mathcal{Q} \times \mathbb{C}^2$ is an invariant subset.

Again fix $\lambda \in \mathbb{C}$. Let $\mathcal{L}(\mathbb{C}^2)$ be the set of complex 2×2 matrices with the usual operator norm $|\cdot|$. Recall that the family (1_q) is said to have an exponential dichotomy over $\mathcal{Q}$ if there are constants $C > 0$, $\eta > 0$ together with a continuous family of projections $P^2 = P : \mathcal{Q} \to \mathcal{L}(\mathbb{C}^2)$ such that

$$\left|\mathcal{U}(x,q,\lambda)P(q)\mathcal{U}(t,x,\lambda)^{-1}\right| \leq Ce^{-\eta(x-t)}, \qquad x > t,$$
$$\left|\mathcal{U}(x,q,\lambda)(I-P(q))\mathcal{U}(t,x,\lambda)^{-1}\right| \leq Ce^{\eta(x-t)}, \qquad x < t.$$

We have the following result [24].

Proposition 1.1 *Let $\tilde{q} \in \mathcal{Q}$ be a point whose orbit is dense in $\mathcal{Q}$: $cls\{\tau_x(\tilde{q}) \mid x \in \mathbb{R}\} = \mathcal{Q}$. Then the resolvent set of $L_{\tilde{q}}$ equals $\{\lambda \in \mathbb{C} \mid$ the family (1_q) admits an exponential dichotomy over $\mathcal{Q}\}$. If $\lambda \in \mathbb{C}$, and if the family (1_q) admits an exponential dichotomy over $\mathcal{Q}$, then λ is in the resolvent set of $L_{\tilde{q}}$ for each $\tilde{q} \in \mathcal{Q}$.*

Let $\mathbb{P}^1(\mathbb{C})$ be the usual projective space of complex lines through the origin in $\mathbb{C}^2$. We can and will view the real projective space $\mathbb{P}^1(\mathbb{R})$ of lines through the origin in $\mathbb{R}^2$ as a subset of $\mathbb{P}^1(\mathbb{C})$. Let $\lambda \in \mathbb{C}$ be fixed. Then the family (1_q) induces a flow $\{\tilde{\tau}_x\}$ on $\mathcal{Q} \times \mathbb{P}^1(\mathbb{C})$, via the simple construction $\tilde{\tau}_x(q,l) = (\tau_x(q), \mathcal{U}(x,\lambda,q)\cdot l)$ for $(q,l) \in \mathcal{Q} \times \mathbb{P}^1(\mathbb{C})$. If $\lambda \in \mathbb{R}$, then $\{\tilde{\tau}_x\}$ leaves invariant $\mathcal{Q} \times \mathbb{P}^1(\mathbb{R})$. One has a convenient geometric interpretation of the Weyl functions when $\lambda \in \mathbb{C}$ is such that the family (1_q) has an exponential dichotomy over $\mathcal{Q}$. Namely,

$$\operatorname{Im} P(q) = Span\begin{pmatrix} 1 \\ m_+(q,\lambda)\end{pmatrix}, \qquad \operatorname{Ker} P(q) = Span\begin{pmatrix} 1 \\ m_-(q,\lambda)\end{pmatrix}.$$

If $\lambda \in \mathbb{R}$ then we make the proviso that, if m_+ or m_- should take on the value ∞, then $\begin{pmatrix} 1 \\ m_+\end{pmatrix}$ resp. $\begin{pmatrix} 1 \\ m_-\end{pmatrix}$ is identified with the “vertical” vector $\begin{pmatrix} 0 \\ 1\end{pmatrix}$.

We make a simple but important remark. Let $q \in \mathcal{Q}$ and $x \in \mathbb{R}$. Let us write $m_\pm(x,\lambda)$ for the Weyl functions $m_\pm(\tau_x(q),\lambda)$ of the translated potential $\tau_x(q)$. Let $u_\pm(x,\lambda) = \begin{pmatrix}\varphi_\pm(x,\lambda) \\ \varphi'_\pm(x,\lambda)\end{pmatrix}$ be the solutions of the system (1_q) which decay exponentially as $x \to \pm\infty$ respectively. Then it is easy to check that

$$m_\pm(x,\lambda) = \frac{\varphi'_\pm(x,\lambda)}{\varphi_\pm(x,\lambda)}.$$

Next, let ν be a $\{\tau_x\}$-ergodic measure on $\mathcal{Q}$. We introduce the Lyapunov exponent, the rotation number and the Floquet exponent which are determined by ν. We briefly describe the relevant constructions, as follows [26].

If $\mathcal{U}(x,\lambda,q)$ is the fundamental matrix solution of (1_q), set

$$\beta_q(\lambda) = \limsup_{x\to\infty} \frac{1}{x} \ln|\mathcal{U}(x,\lambda,q)|.$$

Then $\beta_q(\lambda)$ is constant for ν-a.a. q, and the common value $\beta(\lambda)$ is called the ν-Lyapunov exponent. The function $\beta(\cdot)$ is subharmonic in $\mathbb{C}$ [4], and is harmonic on $\mathbb{C}\backslash\mathbb{R}$ [23].

Let $\lambda \in \mathbb{R}$, and let $u = \begin{pmatrix} r\cos\theta \\ r\sin\theta\end{pmatrix} \in \mathbb{R}^2$ be written in polar form. If $q \in \mathcal{Q}$, and if $u(x) = \begin{pmatrix} r(x)\cos\theta(x) \\ r(x)\sin\theta(x)\end{pmatrix}$ is a nonzero solution of (1_q), set

$$\rho_q(\lambda) = \lim_{x\to\infty} -\frac{\theta(x)}{x}.$$

The limit need not exist for all $q \in \mathcal{Q}$, but it does exist for ν-a.a. $q \in \mathcal{Q}$, where it takes on a common value $\rho(\lambda)$, called the ν-rotation number of the family (1_q). The function $\lambda \mapsto \rho(\lambda)$ is continuous, monotone nondecreasing, and its set of increase points equals the spectrum of L_q, for ν-a.a. $q \in \mathcal{Q}$ [26].

If $\lambda \in \mathbb{C}$ has non-zero imaginary part, define the Floquet exponent

$$w(\lambda) = \int_{\mathcal{Q}} m_+(q, \lambda)\nu(dq).$$

Then $w(\cdot)$ is holomorphic in $\mathbb{C}\backslash\mathbb{R}$, and $sgn\dfrac{\Im w(\lambda)}{\Im\lambda} = 1$. Let us introduce the convenient notation $w(\lambda + i0) = \lim_{\varepsilon\to 0^+} w(\lambda + i\varepsilon)$ when $\lambda \in \mathbb{R}$; then one has

$$w(\lambda + i0) = -\beta(\lambda) + i\rho(\lambda) \qquad (\lambda \in \mathbb{R}).$$

This shows that $\rho(\cdot)$ (which is initially defined on $\mathbb{R}$) admits a harmonic extension into the upper half-plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \Im\lambda > 0\}$. Moreover $-\beta(\lambda)$ and $\rho(\lambda)$ are harmonic conjugates in $\mathbb{C}_+$ [26].

Let us emphasize that there is a fixed closed set $E \subset \mathbb{R}$ such that, for ν-a.a. $q \in \mathcal{Q}$, the spectrum of L_q equals E. In fact this may be proved by using the result that, for ν-a.a. q, the rotation number $\alpha(\cdot)$ increases exactly in the spectrum of L_q. One has the following additional facts: the Lyapunov exponent $\beta(\lambda)$ is harmonic in $\mathbb{C}\backslash E$; the rotation number assumes a constant value in each spectral gap, i.e. in each open interval in $\mathbb{R}\backslash E$.

Next we discuss the properties of the algebro-geometric potentials of Dubrovin–Matveev–Novikov [9]; see also McKean–van Moerbeke [46]. For this let us fix numbers $-\infty < b_0 < a_1 < b_1 < \cdots < a_g < b_g < \infty$, and consider the Schrödinger potentials $q(x)$ which are bounded and continuous, and have the property that the spectrum of $L = -\dfrac{d^2}{dx^2} + q(x)$ equals

$$E = [b_0, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_g, \infty).$$

Then the resolvent of L is formed by the interval $(-\infty, b_0)$ together with the union of the spectral gaps $(a_1, b_1), \ldots, (a_g, b_g)$. Suppose further that, for a.a. λ in the spectrum, the following nonreflection condition holds:

$$m_+(\lambda + i0) = \overline{m_-(\lambda + i0)} \qquad \text{for a.a. } \lambda \in E. \tag{2}$$

Then the set $\mathcal{Q}_E$ of all such potentials q forms a translation-invariant set which is compact in the topology of uniform convergence on $\mathbb{R}$, and which is homeomorphic to a real torus of dimension g. In fact, for a given q for which the condition (2) holds, one shows that the m-functions $m_\pm$ "glue together" to form a single meromorphic function M on the Riemann surface $\mathcal{R}$ of the quadratic relation

$$\kappa^2 = -(\lambda - b_0)(\lambda - a_1)\ldots(\lambda - b_g).$$

To prove this statement, one uses a standard extension of the Schwarz reflection principle [10] together with the big Picard theorem, as explained in ([23], Prop. 6.7). The meromorphic function M has exactly one pole μ_j in the closure of the resolvent interval $[a_j, b_j]$, lifted to $\mathcal{R}$ via the projection $\pi : (\kappa, \lambda) \mapsto \lambda$ $(1 \le j \le g)$. The same holds for each translate $\tau_x(q)$, so there is a pole motion

$$x \mapsto (\mu_1(x), \ldots, \mu_g(x)).$$

Expanding M at the point $\infty \in \mathcal{R}$ and using the Riccati equation $m' = -\lambda + q(x) - m^2$ which is satisfied by $m_\pm$, one determines the trace formula

$$q(x) = \frac{1}{2}\sum_{j=1}^{g}[a_j + b_j - 2\mu_j(x)].$$

Further manipulations show that the $\mu_j(\cdot)$ satisfy the system of differential equations (Dubrovin equations)

$$\mu_j' = \frac{2\kappa(\mu_j)}{\prod_{k\neq j}(\mu_k - \mu_j)}. \tag{3}$$

Continuing, let $\mathcal{D}_E = \{(y_1, \varepsilon_1), \ldots, (y_g, \varepsilon_g) \mid y_j \in [a_j, b_j],\ \varepsilon_j = \pm 1\}$, where $(a_j, 1)$ is identified with $(a_j, -1)$ and $(b_j, 1)$ is identified with $(b_j, -1)$ $(1 \leq j \leq g)$. Then $\mathcal{D}_E$ can be viewed as the product as the product of circles $\mathfrak{c}_1 \times \mathfrak{c}_2 \times \cdots \times \mathfrak{c}_g$, where in turn each circle $\mathfrak{c}_j$ can if desired be identified with the inverse image $\pi^{-1}[a_j, b_j] \subset \mathcal{R}$ $(1 \leq j \leq g)$.

At this point one can introduce a homology basis on $\mathcal{R}$ consisting of simple closed curves $\mathfrak{c}_1, \mathfrak{d}_1; \mathfrak{c}_2, \mathfrak{d}_2; \ldots, \mathfrak{c}_g, \mathfrak{d}_g$ on $\mathcal{R}$ satisfying the usual intersection relations $\mathfrak{c}_i \circ \mathfrak{c}_j = 0$, $\mathfrak{d}_i \circ \mathfrak{d}_j = 0$, $\mathfrak{c}_i \circ \mathfrak{d}_j = \delta_{ij}$ $(1 \leq i, j \leq g)$. One further introduces a normalized basis $dw_1, \ldots, dw_g$ of holomorphic differentials on $\mathcal{R}$, a base point $p \in \mathcal{R}$, and the Abel map

$$\mathfrak{a} : (p_1, \ldots, p_q) = \sum_{j=1}^{g} \int_p^{p_j} (dw_1, \ldots, dw_g) \in \mathbb{C}^g.$$

Let us restrict attention to points of the type $p_1 = (y_1, \varepsilon_1), \ldots, p_g = (y_g, \varepsilon_g)$. In particular we have

$$\mathfrak{a}(\mu_1(x), \ldots, \mu_g(x)) = \mathfrak{a}(\mu_1(0), \ldots, \mu_g(0)) + \delta x. \tag{4}$$

Here δ is a real vector whose components can be described as the $\mathfrak{d}_j$-periods of the integral of the second kind w on $\mathcal{R}$ which is normalized so that $\int_{\mathfrak{c}_j} dw = 0$ $(1 \leq j \leq g)$ and so that $w(b_0) = 0$ and $w(\lambda) \sim -\sqrt{-\lambda}$ as $\lambda \to -\infty$. See [9] and especially Moser [48], who makes the following very significant additional observation. Let ν be a $\{\tau_x\}$-ergodic measure on $\mathcal{Q}_E$, and let $w_\nu(\lambda)$ be the corresponding Floquet exponent. Then (identifying $\lambda \in \mathbb{C}_+$ with the appropriate point in the upper sheet of the Riemann surface $\mathcal{R}$) one has

$$w(\lambda) = w_\nu(\lambda). \tag{5}$$

One usually puts the relation (4) in the context of the Jacobi variety $\mathcal{J}(\mathcal{R})$ of the Riemann surface $\mathcal{R}$: this object is a complex torus defined using the period matrix $\left(\int_{\mathfrak{d}_j} dw_i\right)_{1\leq i,j\leq g}$ of $\mathcal{R}$. For the developments of this paper, it is better to de-emphasize this aspect, since we will have available only a generalized version of the "real part" of $\mathcal{J}(\mathcal{R})$. See [60,61] for discussion of this point.

2 Preliminaries

We review some properties of the generalized reflectionless Schrödinger potentials. As was indicated in the Introduction, we will consider only those generalized reflectionless potentials q lying in $\mathcal{R} = \mathcal{R}_{-1}$; that is, we assume that the spectrum of $L = L_q = -\frac{d^2}{dx^2} + q(x)$ is contained in $[-1, \infty)$ (and contains $[0, \infty)$). As stated earlier, $\mathcal{R}$ is compact in the topology of uniform convergence on compact sets [41]. It turns out that each $q \in \mathcal{R}$ is in fact analytic on some strip $\{x + iy \mid x \in \mathbb{R},\ |y| < \eta\}$ where $\eta > 0$ does not depend on q. It also turns out that $\mathcal{R}$ contains the algebro-geometric potentials q whose spectrum is of the form $[b_0, a_1] \cup \cdots \cup [b_g, \infty)$ where $-1 \leq b_0 < a_1 < \vdots < b_g \leq 0$, $g = 1, 2, \ldots$.

Kotani [34] showed that each $q \in \mathcal{R}$ is of Sato–Segal–Wilson (SSW) type. We will not stop here to discuss the structure of the SSW potentials. Let us just note that each such potential admits a meromorphic extension to the complex x-plane. Moreover the Korteveg-de Vries (K-dV) equation with initial datum $q(x)$:

$$\frac{\partial u}{\partial t} = 3u\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial^3 u}{\partial x^3}$$

admits a solution $u(t, x)$ which is real analytic for $(t, x) \in \mathbb{R}^2$, and which admits a meromorphic extension to $\mathbb{C}^2$. The solution can be constructed using the theory of a certain infinite-dimensional Grassmann manifold $\mathcal{G}r_2$. It is also known that, if a SSW potential $q(x)$ is bounded on $\mathcal{R}$, then a suitable translate and dilatation $c + q(\eta x)$ lies in $\mathcal{R}$ [28].

One can actually characterize the elements of $\mathcal{R}$, in the following way. Let $q \in \mathcal{R}$, and let $m_\pm(\lambda)$ be the Weyl functions of L. Then $q \in \mathcal{R}$ if and only if q is continuous and bounded on $\mathbb{R}$, and the following conditions hold. First,

$$m_+(\lambda + i0) = \overline{m_-(\lambda + i0)} \quad \text{for a.a. } \lambda > 0;$$

see condition (2) above. Second, there is a regular Borel measure σ, which is supported on $[-1, 1]$, such that

$$\int_{-1}^{1} \frac{\sigma(dt)}{1 - t^2} \leq 1 \tag{6_a}$$

and

$$m_\pm(-z^2) = -z - \int_{-1}^{1} \frac{\sigma(dt)}{\pm t - z} \qquad -z^2 \in \mathbb{C}\backslash[-1, \infty). \tag{6_b}$$

In fact. the set of all such measures $\{\sigma\}$ parametrizes $\mathcal{R}$ [34].

It is interesting to consider the subset of $\mathcal{R}$ consisting of potentials which are reflectionless in the sense of Craig [6]. For this, let $q \in \mathcal{R}$, and let $g_q(x, y, \lambda) = q(x, y, \lambda)$ be the Green's function (integral kernel) of $(L - \lambda)^{-1}$, where λ takes values in the resolvent of $L = L_q = -\dfrac{d^2}{dx^2} + q(x)$.

Definition 2.1 Let $q : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Let E be the spectrum of $L = -\dfrac{d^2}{dx^2} + q(x)$ acting on $L^2(\mathbb{R})$. We say that q is reflectionless if there is a set $E' \subset E$ of Lebesgue measure zero such that, for all $x \in \mathbb{R}$ and all $\lambda \in E\backslash E'$:

$$\Re g(x, x, \lambda + i0) = 0. \tag{7}$$

This turns out to be quite a restrictive condition on the potential q. For example, if E consists of a finite number of nondegenerate closed intervals, and if q is reflectionless, then q is of algebro-geometric type (because it follows from (7) that condition (2) is valid; see [61] and Proposition 2.2 below). Let us note that there exist potentials $q \in \mathcal{R}$ which are not reflectionless; this is because the measure σ in (6_a)–(6_b) can be any Borel regular measure supported on $[-1, 1]$. On the other hand, there seem to be no examples of stationary ergodic processes $(\mathcal{Q}, \{\tau_x\}, \nu)$ with $\mathcal{Q} \subset \mathcal{R}$ which consist (ν-a.e.) of non reflectionless potentials.

At several points in subsequent developments we will have to check that a potential $q \in \mathcal{R}$ is reflectionless. The following result will be useful in doing this.

Let $\{q_r \mid r \geq 1\}$ and q be bounded continuous functions with corresponding Schrödinger operators $L_r = -\dfrac{d^2}{dx^2} + q_r(x)$ $(r \geq 1)$ and $L = -\dfrac{d^2}{dx^2} + q(x)$ on $L^2(\mathbb{R})$. Suppose that $q_r \to q$ uniformly on compact subsets of $\mathbb{R}$. Let $E \subset \mathbb{R}$ be a set of locally positive measure which is contained in the spectrum of all L_r $(r \geq 1)$. Let $g_r(x, x, \lambda)$ resp. $g(x, x, \lambda)$ be the diagonal Green's function of L_r resp. L; these functions are all defined and holomorphic in $\mathbb{C}\backslash\mathbb{R}$, in particular in $\mathbb{C}_+$. If $\lambda \in \mathbb{C}_+$, let $m_{\pm r}(x, \lambda)$ resp. $m_\pm(x, \lambda)$ be the Weyl functions of L_r resp. L. It is well-known that

$$\begin{cases} g_r(x, x, \lambda) = \dfrac{1}{m_{-r}(x, \lambda) - m_{+r}(x, \lambda)} \\ \\ g(x, x, \lambda) = \dfrac{1}{m_-(x, \lambda) - m_+(x, \lambda)} \end{cases} \qquad \Im\lambda > 0.$$

It is also well-known that, for each fixed $x \in \mathbb{R}$, one has $m_{\pm r}(x, \lambda) \to m_\pm(x, \lambda)$, $g_r(x, x, \lambda) \to g(x, x, \lambda)$ uniformly on compact subsets of $\mathbb{C}_+$.

As functions of $\lambda \in \mathbb{C}_+$, all the maps $g_r, g, \pm m_{\pm r}, \pm m_\pm$ take values in $\mathbb{C}_+$, i.e. are Herglotz functions. So they all have nontangential boundary values for a.a. $\lambda \in \mathbb{R}$; we denote the corresponding functions by $g_r(x, x, \lambda + i0)$, $g(x, x, \lambda + i0)$, etc. In what follows we will use with little comment the standard properties of these boundary value functions [10].

Proposition 2.2 *Under the above conditions: suppose that, for a.a.* $(x, \lambda) \in \mathbb{R} \times E$, *there holds* $\Re g_r(x, x, \lambda) = 0$ $(r = 1, 2, \ldots)$. *Then q is reflectionless in E. That is, there is a set* $E_* \subset E$, *whose complement* $E\backslash E_*$ *has Lebesgue measure zero, such that if* $\lambda \in E_*$, *then*

$$\Re g(x, x, \lambda) = 0 \qquad \textit{for all } x \in \mathbb{R}.$$

Further, one has

$$m_+(x, \lambda + i0) = \overline{m_-(x, \lambda + i0)} \qquad (x \in \mathbb{R}, \lambda \in E_*).$$

Proof First we stipulate that the term "almost all (a.a.)" refers to Lebesgue measure of the appropriate dimension.

Suppose without loss of generality that E is contained in a bounded interval I. Let $D \subset \mathbb{C}_+$ be the open half-disc with diameter I. Using Fubini's theorem, we can state that, for a.a. $x \in \mathbb{R}$, there is a set $E_x \subset E$ of zero measure such that the nontangential limit $g_r(x, x, \lambda)$ is defined for all $r \geq 1$, for all $\lambda \in E\backslash E_x$, and $\Re g_r(x, x, \lambda + i0) = 0$. Using an argument of Kotani ([32]; see also [27]) one can show that, for each such x, there is a subset $E_{xx} \subset E$ of zero measure such that $g(x, x, \lambda + i0)$ is defined for $\lambda \in E\backslash E_{xx}$ and $\Re g(x, x, \lambda + i0) = 0$.

Using Fubini's theorem again, we conclude that $\{(x, \lambda) \in \mathbb{R} \times E \mid \Re g(x, x, \lambda + i0) = 0\}$ has full product measure. So there is a subset $E_* \subset E$ of full measure such that, if $\lambda \in E_*$, then for a.a. $x \in \mathbb{R}$ one has $\Re g(x, x, \lambda + i0) = 0$. There is no loss of generality in assuming that, for each $\lambda \in E_*$, the nontangential limits $m_\pm(x, \lambda + i0)$ exist as complex numbers for a.a. $x \in \mathbb{R}$.

Fix $\lambda_* \in E_*$. Then for a.a. $x \in \mathbb{R}$, one has $\Re m_-(x, \lambda_* + i0) = \Re m_+(x, \lambda_* + i0)$. Let us assume that $\Re m_-(0, \lambda_* + i0) = \Re m_+(0, \lambda_* + i0)$; if necessary this can be arranged by substituting a translate $\tau_x(q)$ for q (note that the thesis of Proposition 2.2 holds for q if and only if it holds for each translate $\tau_x(q)$). Now, let $u_\pm(x, \lambda_*) = \begin{pmatrix} \varphi_\pm(x, \lambda_*) \\ \varphi'_\pm(x, \lambda_*) \end{pmatrix}$ be the solution of equation (1_q) which satisfies $u_\pm(0, \lambda_*) = \begin{pmatrix} 1 \\ m_\pm(0, \lambda_* + i0) \end{pmatrix}$. Let $\mathcal{U}(x, \lambda)$ be the fundamental matrix solution of (1_q), where now λ ranges over $\mathbb{C}$. Using the joint continuity of $\mathcal{U}(x, \lambda)$ in

$(x, \lambda) \in \mathbb{R} \times \mathbb{C}$, we see that $m_{\pm}(x, \lambda_* + i0)$ must agree with $\dfrac{\varphi'_{\pm}(x, \lambda_*)}{\varphi_{\pm}(x, \lambda_*)}$ for a.a. $x \in \mathbb{R}$. Even more, $m_{\pm}(x, \lambda_* + i0)$ exists as an extended complex number for all $x \in \mathbb{R}$, and agrees with $\dfrac{\varphi'_{\pm}(x, \lambda_*)}{\varphi_{\pm}(x, \lambda_*)}$ for all $x \in \mathbb{R}$, in particular $m_{\pm}(x, \lambda_* + i0) = \infty$ if and only if $\varphi_{\pm}(x, \lambda_*) = 0$.

Now however, $\Re g(x, x, \lambda_* + i0) = 0$ for a.a. $x \in \mathbb{R}$, and there is no loss of generality in assuming that $\Re g(0, 0, \lambda_* + i0) = 0$. It is clear that at least one of the quantities $\Im m_{\pm}(0, \lambda_* + i0)$ is different from zero. Suppose, e.g., that $\Im m_{+}(0, \lambda_* + i0) \neq 0$, hence is > 0. Then since $\mathcal{U}(x, \lambda_*) : \mathbb{C}^2 \to \mathbb{C}^2$ leaves invariant $\mathbb{R}^2$ and also $\mathbb{C}^2 \backslash \mathbb{R}^2$ for all $x \in \mathbb{R}$, we have that $m_{+}(x, \lambda_* + i0) > 0$ for all $x \in \mathbb{R}$.

We must rule out the possibility that $m_{-}(0, \lambda_* + i0) \in \mathbb{R}$. Suppose without loss of generality that λ_* is not the left end-point of the spectrum of L, so that solutions of (1_q) oscillate as $x \to \infty$ [20]. This means that there exists $x_* > 0$ such that $\varphi_{-}(x_*, \lambda_*) = 0$, so $m_{-}(x, \lambda_* + i0) \to \infty$ as $x \to x_*$. But then $\Re m_{-}(x, \lambda_* + i0) \neq \Re m_{+}(x, \lambda_* + i0)$ for all x sufficiently near x_*, and this is inconsistent with the condition that $\Re g(x, x, \lambda_* + i0) = 0$ for a.a. x.

We thus have that $\Im m_{-}(x, \lambda_* + i0) < 0$ for all $x \in \mathbb{R}$. We claim that $m_{-}(x, \lambda_* + i0) = \overline{m_{+}(x, \lambda_* + i0)}$ for all $x \in \mathbb{R}$. It is sufficient to prove this for $x = 0$ because $\mathcal{U}(x, \lambda_*)$ is real. Now both the functions $m_{\pm}(x) = m_{\pm}(x, \lambda_* + i0)$ satisfy the Riccati equation $m' + m^2 = -\lambda_* + q(x)$. So $v = m_{+} - m_{-}$ satisfies $v' = -(m_{+} + m_{-})v$. Since $\Re g(0, 0, \lambda_* + i0) = 0$ we have $\Re v(0) = 0$. It follows that $\Re v'(0) = \Im v(0) \cdot \Im[m_{+}(0) + m_{-}(0)]$. If $\Im[m_{+}(0) + m_{-}(0)] \neq 0$ then $\Re v(x) \neq 0$ for all small x, which is inconsistent with the condition $\Re g(x, x, \lambda + i0) = 0$. So in fact $m_{-}(0) = \overline{m_{+}(0)}$. This completes the proof of Proposition 2.2 □

Note that, under the hypotheses of Proposition 2.2, the set E is contained in the spectrum of L.

One can sometimes use Proposition 2.2 to show that a bounded continuous potential q is reflectionless in E if only $\Re(0, 0, \lambda + i0) = 0$ for a.a. $\lambda \in E$. We give a result along these lines which seems (to us) illuminating.

Corollary 2.3 *Let $\mathcal{Q} = \{q\}$ be a translation-invariant set of bounded continuous potentials which is compact in the topology of uniform convergence on compact subsets of $\mathbb{R}$. Let ν be a $\{\tau_x\}$-ergodic measure on $\mathcal{Q}$. Let $g_q(x, x, \lambda)$ be the diagonal Green's function of $L_q = -\dfrac{d^2}{dx^2} + q(x)$, and let E be the spectrum of L_q for ν-a.a q (thus E depends only on ν).*

Suppose that, for ν-a.a. q, $\{\lambda \in E \mid \Re g_q(0, 0, \lambda + i0) = 0\}$ has full (Lebesgue) measure. Then for ν-a.a. q: q is reflectionless and $m_{+}(q, \lambda + i0) = \overline{m_{-}(q, \lambda + i0)}$ for a.a. $\lambda \in E$. Moreover, if the topological support Supp ν of ν equals $\mathcal{Q}$, then q is reflectionless for all $q \in \mathcal{Q}$.

Proof We use the fact that $g_q(x, x, \lambda) = g_{\tau_x(q)}(0, 0, \lambda)$ and the invariance of ν to conclude that, for each $x \in \mathbb{R}$, the set $\{q \in \mathcal{Q} \mid \Re g_q(x, x, \lambda + i0) = 0\}$ has ν-measure 1. So by Fubini's theorem we have that $\{(x, q, \lambda) \in \mathbb{R} \times \mathcal{Q} \times E \mid \Re g_q(x, x, \lambda + i0) = 0\}$ has full measure, or again using Fubini's theorem

$$\{q \in \mathcal{Q} \mid \Re g_q(x, x, \lambda) = 0 \ \text{ for a.a. } (x, \lambda) \in \mathbb{R} \times E\}$$

has ν-measure 1. By Proposition 2.2, ν-almost every $q \in \mathcal{Q}$ is reflectionless.

To prove the second statement, recall that the support *Supp* ν of the regular Borel measure ν is the complement in $\mathcal{Q}$ of the largest open subset of $\mathcal{Q}$ which has ν-measure zero. We know that ν-a.a. $q \in \mathcal{Q}$ are reflectionless, or more precisely reflectionless in E. If $q \in \mathcal{Q}$ is a generic point, there is a sequence $\{q_r\} \subset \mathcal{Q}$ of reflectionless potentials which converges to q. So by Proposition 2.2, q is reflectionless in E.

To finish the proof, we must show that E equals the spectrum of L_q for all $q \in \mathcal{Q}$. So fix $q \in \mathcal{Q}$. By what we have just shown and standard facts, the spectrum of L_q contains E. Let $\{q_r\} \subset \mathcal{Q}$ be a sequence of points such that $q_r \to q$ and the spectrum of L_{q_r} equals E $(r = 1, 2, \ldots)$. A standard argument using spectral measures shows that the spectrum of L_q is contained in E. So in fact the spectrum of L_q equals E, for all q in $Supp\ \nu = \mathcal{Q}$. □

Let E be a closed subset of the real line which is bounded below by $\lambda = -1$, and which contains the semi-infinite interval $[0, \infty)$. Let $b_0 = \inf E$; we assume that $-1 \le b_0 < 0$. We can write $E = \mathbb{R}\backslash\left\{\bigcup_{j=1}^{\infty}(a_j, b_j) \cup (-\infty, b_0)\right\}$ where the open intervals $(a_j, b_j) \subset (-1, 0)$ are pairwise disjoint (and may or may not be finite in number). We will assume that E has locally positive measure.

It will be convenient to consider the closed intervals $[a_j, b_j]$, $j = 1, 2, \ldots$. We imagine that two copies of each such interval, labelled with $\varepsilon_j \in \{-1, 1\}$, are glued together at the endpoints $\{a_j, b_j\}$ to form a circle $\mathfrak{c}_j$. The set of divisors $\mathcal{D}_E$ is by definition the product space $\prod_{J=1}^{\infty} \mathfrak{c}_j$ with the Tychonov (product) topology. We label points of $\mathcal{D}_E$ by $\{(y_j, \varepsilon_j) \mid j \ge 1\}$ where $y_j \in [a_j, b_j]$, $\varepsilon_j = \pm 1$, and ε_j indicates the corresponding copy of $[a_j, b_j]$ (upper/lower copy). If $y_j = a_j$ or $y_j = b_j$, then $(y_j, \pm 1)$ are identified.

Now we review some basic facts concerning Parreau–Widom domains [50,65], in the context considered in this paper. Let E be as above, and let $\Omega_E = \mathbb{C}\backslash E$. Fix $\lambda_0 = -2 \in \Omega_E$. Let $\mathcal{G}(\lambda) = \mathcal{G}(\lambda, \lambda_0)$ be the Green's function of Ω_E with logarithmic pole at λ_0. We assume that Ω_E is regular for the Dirichlet problem, which means that $\mathcal{G}$ assumes continuously the boundary value 0 at each $\lambda \in E$. Let $\{c_j \mid j \ge 1\}$ be the points in Ω_E where the gradient $\nabla\mathcal{G}(c_j) = 0$; there is exactly one such point in each interval (a_j, b_j), and there are no other such points in Ω_E. One says that Ω_E is of Parreau-Widom type if

$$\sum_{j=1}^{\infty} \mathcal{G}(c_j) < \infty. \tag{8}$$

Let Γ_E be the fundamental group of Ω_E, and let $\mathcal{J}(E)$ be the compact abelian topological group of characters on Γ_E. Thus $\mathcal{J}(E)$ is the set of group homomorphisms from Γ_E to the unit circle $\mathbb{T}$, with the topology of pointwise convergence. Another way to say this is that one gives the abelianized group $\Gamma_E/[\Gamma_E, \Gamma_E]$ the discrete topology, and $\mathcal{J}(E)$ the dual topology.

The elements of $\mathcal{J}(E)$ can be described fairly explicitly, as follows. For each $k \ge 1$, let γ_k be a circle contained in Ω_E which is orthogonal to the real axis and intersects it in $\lambda_0 = -2$ and in a point of (a_k, b_k). Let γ_k have the clockwise orientation. Then α is determined by the values $\alpha(\gamma_k)$, $k = 1, 2, \ldots$.

Next we define the Abel map $\mathfrak{a} : \mathcal{D}_E \to \mathcal{J}(E)$. For this, we associate with each divisor $d \in \mathcal{D}_E$ a character, in the following way. Fix the divisor $d_0 = \{b_1, b_2, \ldots, b_j, \ldots\}$ of right endpoints of the intervals $[a_j, b_j]$. for each $k \ge 1$, let $\omega(\lambda, E_k)$ be the harmonic measure on Ω_E of $E_k = E \cap Int\ \gamma_k$, i.e. the points in E which lie in the unbounded component of $\mathbb{C}\backslash\gamma_k$ in the complex plane. If $d = \{(y_1, \varepsilon_1, \ldots, (y_j, \varepsilon_j), \ldots\}$ we define

$$\mathfrak{a}(d)(\gamma_k) = \frac{1}{2}\sum_{j=1}^{\infty} \varepsilon_j \int_{y_j}^{b_j} \omega(dt, E_k) \qquad \text{mod } \mathbb{Z}$$

where here we have identified $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$. It can be shown that the sum on the right-hand side converges absolutely, and that $\mathfrak{a}$ is a continuous map. See ([60], p. 422); one uses the Parreau-Widom condition (8).

We will see in Sect. 3 that, when the Parreau-Widom condition holds, the Abel map $\mathfrak{a}$ is surjective. We will discuss examples of PW domains Ω_E for which $\mathfrak{a}$ is not injective, but admits a left inverse $\mathfrak{i} : \mathcal{J}(E) \to \mathcal{D}(E)$ which is of the first Baire class, i.e., is a pointwise limit of continuous functions. We will see that this property allows one to obtain interesting information concerning the translation flow $\{\tau_x\}$ on the set of reflectionless potentials $q \in \mathcal{R}$ for which the spectrum of L_q equals E.

3 Results

We adopt the notation introduced above. In particular, $\mathcal{R}$ represents the set of generalized reflectionless potentials q such that the spectrum of $L_q = -\dfrac{d^2}{dx^2} + q(x)$ is contained in $[-1, \infty)$; one then has that $m_+(\lambda + i0) = \overline{m_-(\lambda + i0)}$ for a.a. $\lambda > 0$. We recall that $\mathcal{R}$ is compact in the topology of uniform convergence on compact sets, and that it contains the algebro-geometric potentials q such that the spectrum of L_q contains $[0, \infty)$ and is contained in $[-1, \infty)$. Let $E \subset \mathbb{R}$ be a closed set of the type $\mathbb{R}\setminus \left\{\bigcup_{j=1}^{\infty}(a_j, b_j) \cup (-\infty, b_0)\right\}$ where $-1 \le b_0 < a_j < b_j$ for all $j = 1, 2, \ldots$ and the pairwise disjoint open intervals (a_j, b_j) are all contained in $(-1, 0)$. Note that then $\sum_{j=1}^{\infty}(b_j - a_j) < \infty$. We assume that E has locally positive measure. We refer to these conditions as the "Standing Hypotheses" on E. For the moment we do not impose further hypotheses on E, in particular we do not assume that $\Omega_E = \mathbb{C}\setminus E$ is a Parreau-Widom domain.

Let us write

$$E = \bigcap_{g=1}^{\infty} E_g$$

where $E_g = \mathbb{R}\setminus \left\{\bigcup_{j=1}^{g}(a_j, b_j) \cup (-\infty, b_0)\right\}$ for $g = 1, 2, \ldots$. Then the closed sets E_g decrease with g, and E_g is the union of exactly $g + 1$ pairwise disjoint nondegenerate closed intervals (or bands). If g is fixed, one can relabel the points $b_0, a_1, b_1, \ldots, b_g$ and write

$$E_g = [b_0, a_1] \cup [b_1, a_2] \cup \ldots \cup [b_g, \infty)$$

where $-1 \le b_0 < a_1 < a_2 < \cdots < b_g \le 0$.

Let $\mathcal{Q}_{E_g}$ be the set of bounded continuous potentials q which are reflectionless; i.e., satisfy Definition 2.1. As noted in Sect. 2, each $q \in \mathcal{Q}_{E_g}$ is of algebro-geometric type, and one can also check that each algebro-geometric potential satisfies Definition 2.1. That is, $\mathcal{Q}_{E_g}$ coincides with the set of algebro-geometric Schrödinger potentials q such that the spectrum of L_q equals E_g.

Our first goal is to study the reflectionless elements $q \in \mathcal{R}$ such that the spectrum of L_q equals E. We state and prove a preliminary proposition.

Proposition 3.1 *Suppose that E satisfies the Standing Hypotheses. Let $1 \le g_1 < g_2 < \cdots < g_r < \cdots \to \infty$, and let $q_r \in \mathcal{Q}_{g_r}$. Suppose that $q_r \to q$ uniformly on compact subsets of $\mathbb{R}$. Then the spectrum of L_q equals E, and q is reflectionless. Moreover, if $\mathcal{Q}_E$ is defined to be $\{q \in \mathcal{R} \mid q$ is reflectionless and has spectrum $E\}$, then $\mathcal{Q}_E$ is compact (in the compact-open topology).*

Proof The first statement follows immediately from the preceding remarks and from Proposition 2.2. As for the second statement, $\mathcal{Q}_E$ is contained in the compact set $\mathcal{R}$ and so it is sufficient to check that $\mathcal{Q}_E$ is closed. But this is also a consequence of Proposition 2.2; see the arguments used in proving the second statement of Corollary 2.3. This completes the proof of Proposition 3.1 □

It is not immediately clear that $\mathcal{Q}_E$ is "filled out" by limits of sequences $\{q_r\}$, $q_r \in \mathcal{Q}_{E_{g_r}}$, $g_1 < g_2 < \cdots < g_r \to \infty$. That is, it is not clear (as yet) that $\mathcal{Q}_E \subset \limsup_{g\to\infty} \mathcal{Q}_{E_g}$ where the limsup is calculated in the Hausdorff topology on the set of (nonempty) compact subsets of $\mathcal{R}$. We will see later that this is indeed the case for some interesting sets E. So using Proposition 3.1, it is then actually the case that $\mathcal{Q}_E = \limsup_{g\to\infty} \mathcal{Q}_{E_g}$.

Next, recall that $\mathcal{D}_E$ is the set of divisors $\{(y_j, \varepsilon_j) \mid y_j \in [a_j, b_j, \ \varepsilon_j = \pm 1\}$, where $(a_j \pm 1)$ resp. $(b_j, \pm 1)$ are identified. So $\mathcal{D}_E$ is the product of circles $\mathfrak{c}_j$, $j = 1, 2, \ldots$; $\mathcal{D}_E$ is compact in the Tychonov (product) topology. We look for conditions on E which ensure that $\mathcal{Q}_E$ and $\mathcal{D}_E$ are homeomorphic. We will consider sets which have the following "Property P".

Definition 3.2 Let $E \subset \mathbb{R}$ be a closed set of the form

$$\mathbb{R} \setminus \bigcup_{j=0}^{\infty} (a_j, b_j)$$

where $a_0 = -\infty$, and the nonempty open intervals (a_j, b_j) are pairwise disjoint and satisfy $b_0 < a_j < b_j < \infty$ ($j \geq 1$). Let ρ_{jk} be the distance between $[a_j, b_j]$ and $[a_k, b_k]$ for $1 \leq j \neq k < \infty$. Say that E has Property P if

$$\sigma_k = \sum_{j \neq k} \frac{b_j - a_j}{\rho_{jk}} < \infty$$

for each k (it is not assumed that $\{\sigma_k\}$ is bounded).

Variants of Property P were introduced and discussed in [6,27]. One can see that if, for each $j \geq 1$, each a_j resp. b_j is the right resp. left endpoint of a nondegenerate closed interval contained in E, then $\{\rho_{jk} \mid j \geq 1\}$ is bounded away from zero for each k, so Property P holds if $\sum_{j=1}^{\infty}(b_j - a_j) < \infty$. Also it is easy to construct "fat" Cantor sets which have Property P.

Theorem 3.3 *Let $E \subset \mathbb{R}$ be a closed subset which satisfies the Standing Hypotheses and has Property P. Then there is a natural map $\pi : \mathcal{Q}_E \to \mathcal{D}(E) : q \mapsto d_q$ which is continuous and surjective. If in addition, for each $q \in \mathcal{Q}_E$ the half-line operators $L_q^{\pm}$ have purely absolutely continuous spectrum, then π is a homeomorphism.*

Proof By Proposition 3.1, $\mathcal{Q}_E$ is a compact metric space which is clearly invariant with respect to the translation flow on $\mathcal{R}$.

Consider the family of differential systems

$$u' = \begin{pmatrix} 0 & 1 \\ -\lambda + q(x) & 0 \end{pmatrix} u, \qquad u = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \in \mathbb{C}^2 \tag{9_q}$$

where $q \in \mathcal{Q}_E$ and $\lambda \in \mathbb{C}$ is fixed. We claim that, if $\lambda \in \mathbb{C} \setminus E$, then this family admits an exponential dichotomy over $\mathcal{Q}_E$. This holds if $\Im\lambda \neq 0$ by general results of [24]. If $\lambda \in \mathbb{R} \setminus E$ we argue as follows. If for some $q \in \mathcal{Q}_E$ the system (9_q) admits a non-zero solution $u(x) = \binom{\varphi(x)}{\varphi'(x)}$ which is bounded on $-\infty < x < \infty$, then $\varphi(x)$ is a non-zero solution of $L_q\varphi = \lambda\varphi$ which is bounded on $-\infty < x < \infty$. But then λ is in the spectrum of L_q, which equals E, a contradiction. By a well-known result of Sacker–Sell–Selgrade [53] the family (9_q) admits an exponential dichotomy over each minimal subset $M \subset \mathcal{Q}_E$. Furthermore, the dichotomy projections $P(q)$ have rank one for each $q \in M$. Hence by a further result of Sacker and Sell [53], the family (9_q) has an exponential dichotomy over all $\mathcal{Q}_E$.

The dichotomy projections $P(q,\lambda)$ are parametrized by the m-function $m_+(q,\lambda)$ for $\Im\lambda \neq 0$, in the sense that Im $P(q,\lambda) = Span\begin{pmatrix}1\\ m_+(q,\lambda)\end{pmatrix}$ when $q \in \mathcal{Q}_E$. See Sect. 1; one also has that Ker $P(q,\lambda) = Span\begin{pmatrix}1\\ m_-(p,\lambda)\end{pmatrix}$ for $q \in \mathcal{Q}_E$, $\Im\lambda \neq 0$. The functions $m_\pm(q,\cdot)$ extend meromorphically through the interval $(-\infty, b_0)$ and through each interval (a_j, b_j), $j \geq 1$. Moreover $sgn\ \dfrac{dm_\pm(q,\lambda)}{d\lambda} = \pm 1$ if $\lambda \in (a_j, b_j)$ is not a pole of $m_\pm(x,\cdot)$. Geometrically, this means that the lines through the origin in $\mathbb{R}^2$ which are parametrized by $m_\pm(q,\lambda)$ move in opposite directions on the projective circle $\mathbb{P}^1(\mathbb{R}) \simeq \mathbb{R}\cup\{\infty\}$ as λ increases in (a_j, b_j), $j = 0, 1, \ldots$. Again, see the discussion in Sect. 1.

Next fix $q \in \mathcal{Q}_E$ and consider an interval (a_k, b_k), $k \geq 1$. We claim that, if $\lambda \uparrow b_k$ or if $\lambda \downarrow a_k$ for $\lambda \in (a_k, b_k)$, then $\lim m_+(q,\lambda) = \lim m_-(q,\lambda)$ in $\mathbb{P}^1(\mathbb{R})$, or equivalently in the one-point compactification $\mathbb{R}\cup\{\infty\}$ of $\mathbb{R}$. [We observe parenthetically that these limits exist, because of the monotone, opposing motions of $\lambda \mapsto m_\pm(q,\lambda)$ for $a_k < \lambda < b_k$, and because of the fact that $m_+(q,\lambda) \neq m_-(q,\lambda)$ for $\lambda \in (a_k, b_k)$.]

Suppose, e.g., that $\lambda \downarrow a_k$. We prove that $m_+(a_k^+) = \lim_{\lambda\downarrow a_k} m_+(\lambda)$ and $m_-(a_k^+) = \lim_{\lambda\downarrow a_k} m_-(\lambda)$ are equal, where we have omitted the q-dependence since $q \in \mathcal{Q}_E$ is fixed. Let $g_q(x,x,\lambda)$ be the diagonal Green's function of L_q, and set $g(\lambda) = g_q(0,0,\lambda)$. Then $g(\lambda)$ admits the following product representation:

$$g(\lambda) = \frac{i}{2\sqrt{\lambda - b_0}} \prod_{j=1}^{\infty} \frac{\lambda - \mu_j}{\sqrt{(\lambda - a_j)(\lambda - b_j)}} \qquad \mu_j \in [a_j, b_j] \tag{10}$$

where each μ_j is uniquely determined and the product converges absolutely for $\lambda \in \mathbb{C}\setminus E$ (because $\sum_{j=1}^{\infty}(b_j - a_j) < \infty$). The square root is defined by continuation beginning with $\sqrt{1} = 1$. One can prove (10) using a result of Krein–Nudelman [35]. In the present circumstances one can actually prove (10) by hand, using basic properties of $g(\lambda)$, in particular

$$g(\lambda) = \frac{1}{m_-(\lambda) - m_+(\lambda)}, \tag{11}$$

together with a standard formula for the Herglotz function $h(\lambda) = \ln g(\lambda)$, namely

$$h(\lambda) = \Re h(i) + \int_{-\infty}^{\infty} \Im \ln g(\lambda) \cdot \left[\frac{1}{t-\lambda} - \frac{1}{1+t^2}\right] dt.$$

This last relation implies (10) if one takes account of the non-reflection property of q.

We return to $m_\pm(a_k^+)$. Suppose that $a_k < \mu_k < b_k$. Then as λ increases from a_k to b_k, $g(\lambda)$ is strictly monotone increasing and assumes all values in $(-\infty, \infty)$. This is proved using (10) together with Property P. Using (11), we see that $\lim_{\lambda\downarrow a_k}[m_-(\lambda) - m_+(\lambda)] = 0$, and that $m_-(\mu_k) - m_+(\mu_k) = \infty$. By the monotone movement of m_+ and m_-, we see that $m_+(a_k^+)$ and $m_-(a_k^+)$ assume the same finite value. We say that $m_\pm(\lambda)$ "glue together" at $\lambda = a_k$. In a similar way, if $a_k < \mu_k < b_k$, then $m_\pm(\lambda)$ glue together at $\lambda = b_k$.

Let us assume that $\mu_k = a_k$. In this case, $g(\lambda)$ assumes positive values in (a_k, b_k), and limits to 0 as $\lambda \to a_k^+$ and to ∞ as $\lambda \to b_k^-$. This follows from Property (P) and the representation (10). We wish to show that $m_\pm(\lambda)$ glue together at $\lambda = a_k$ and at $\lambda = b_k$. First of all, since $g(\lambda) \to 0$ as $\lambda \to a_k^+$, we certainly have that either $m_+(\lambda)$ or $m_-(\lambda)$ tends to $\infty \in \mathbb{P}^1(\mathbb{C}) \cong \mathbb{R} \cup \{\infty\}$ as $\lambda \to a_k^+$ (or both). If they both tend to ∞, then in fact $m_\pm(\lambda)$ glue together at $\lambda = a_k$.

Suppose, say, that $m_+(a_k^+) = \infty$ and $m_-(a_k^+) \in \mathbb{R}$. We introduce the m-functions $m_\pm(x, \lambda)$ of $\tau_x(q)$ for $x \in \mathbb{R}$. Using $\mathcal{U}(x, \lambda)$ of (9_q), we see that the quantities $m_\pm(x, a_k^+)$ are continuous functions of x, in fact $m_\pm(x, a_k^+) = \mathcal{U}(x, a_k) \cdot m_\pm(a_k^+)$ where on the right-hand side one has the projective action of $\mathcal{U}(x, a_k)$. So if $\lambda = a_k$ and $x \neq 0$ is small, we have $m_+(x, a_k^+) \neq m_-(x, a_k^+)$. If $x \neq 0$ is small, we also have $m_+(x, a_k^+) \neq \infty$ by the identical normality of (9_q) (that is, there is a 1 in the upper right-hand corner of the coefficient matrix). This all means that, for small $x \neq 0$, $m_\pm(x, a_k^+) \in \mathbb{R}$.

We now consider the product expansion of $g(x, x, \lambda)$:

$$g(x, x, \lambda) = \frac{i}{2\sqrt{\lambda - b_0}} \prod_{j=1}^{\infty} \frac{\lambda - \mu_j(x)}{\sqrt{(\lambda - a_j)(\lambda - b_j)}}. \tag{10_{bis}}$$

Clearly $\mu_k(x) \neq a_k$ for small $x \neq 0$, say for $0 < |x| < \eta$. On the other hand $\mu_k(x)$ is continuous in $x = 0$. This can be proved using an argument of Craig ([6], p. 397). We give another proof. Let $\varepsilon > 0$ be small and let $\lambda = a_k + \varepsilon$. Since $g(0, 0, \lambda) > 0$ for all $a_k < \lambda < b_k$, we can find $\eta_1 \in (0, \eta)$ such that, if $|x| < \eta_1$, then $g(x, x, \lambda) > 0$. This implies that $\mu_k(x) \in [a_k, a_k + \varepsilon)$ if $0 < |x| < \eta_1$.

Now let $\{x_n\}$ be a sequence of nonzero numbers such that $x_n \to 0$. Then $\mu_k(x_n) \to a_k$ and $a_k < \mu_k(x) < b_k$. By the reasoning carried out earlier, $m_-(x_n, a_k^+) = m_-(x_n, a_k^+)$ for each $n = 1, 2, \ldots$. So letting $n \to \infty$ we see that $m_-(a_n^+) = m_+(a_n^+) = \infty$. This contradiction shows that in fact $m_\pm(\lambda)$ glue together at $\lambda = a_k$.

We continue to assume that $\mu_k = a_k$. Then as $\lambda \uparrow b_k$, one has that $g(\lambda) \to \infty$, as follows from Property P together with the representation (10). This means that $m_-(\lambda) - m_+(\lambda)$ tends to zero as $\lambda \uparrow b_k$, that is $m_-(b_k^-) = m_+(b_k^-)$. And this means that $m_\pm(\lambda)$ glue together at $\lambda = b_k$.

If $\mu_k = b_k$, we can argue as above with minor variations to prove that $m_\pm(\lambda)$ glue together at $\lambda = a_k$ and at $\lambda = b_k$.

Next let $q \in \mathcal{Q}_E$. We associate to q a divisor $d \in \mathcal{D}_E$, as follows. Write again

$$g_q(0, 0, \lambda) = \frac{1}{2\sqrt{\lambda - b_0}} \prod_{j=1}^{\infty} \frac{\lambda - \mu_j}{\sqrt{(\lambda - a_j)(\lambda - b_j)}}.$$

If $\mu_k \in (a_k, b_k)$, then exactly one of the functions $m_\pm(\lambda)$ has a pole at μ_k. We set $\varepsilon_k = \pm 1$ accordingly. If $\mu_k = a_k$ or $\mu_k = b_k$, we identify $(a_k, 1)$ with $(a_k, -1)$ resp. $(b_k, 1)$ with $(b_k, -1)$. Define $d = d_q = \{(\mu_k, \varepsilon_k) \mid k \geq 1\}$. This is the divisor corresponding to q.

We claim that the map $q \to d_q$ is continuous. For this, let $q_r \to q$ in $\mathcal{Q}_E$. It is sufficient to show that, for each $k \geq 1$, one has convergence of the k^{th} component $(\mu_{k,r}, \varepsilon_{k,r})$ of d_{q_r} to the k^{th} component (μ_k, ε_k) of d_q. Precisely, we must show that $\mu_{k,r} \to \mu_k$, and that, if $\mu_k \neq a_k$ or b_k, then $\varepsilon_{k,r}$ is equal to ε_k for all large r. To do this, recall that the family (9_q) has an exponential dichotomy over $\mathcal{Q}_E$ for all $\lambda \in \mathbb{C} \backslash E$. Recall also that there is a close relation between the dichotomy projections $P(q, \lambda)$ and the Weyl functions $m_\pm(q, \lambda)$:

$$\operatorname{Im} P(q, \lambda) = Span \begin{pmatrix} 1 \\ m_+(q, \lambda) \end{pmatrix}, \qquad \operatorname{Ker} P(q, \lambda) = Span \begin{pmatrix} 1 \\ m_-(q, \lambda) \end{pmatrix}$$

for all $q \in \mathcal{Q}_E$, $\lambda \in \mathbb{C}\backslash E$. Suppose first that $\mu_k \in (a_k, b_k)$. Then, using the continuity in (q, λ) of the dichotomy projections together with the glueing property of the Weyl functions in $\lambda = a_k$, $\lambda = b_k$, one can show that in fact $\mu_{k,r} \to \mu_k$ and that $\varepsilon_{k,r} = \varepsilon_k$ for all large r.

We must use a bit more care if $\mu_k \in \{a_k, b_k\}$, say $\mu_k = b_k$. Let $\varepsilon > 0$ be small; we must show that, if r is sufficiently large, then $\mu_{k,r} \in (b_k - \varepsilon, b_k]$. To do this, we apply a variant of an argument used above. First of all, if $\mu_k = b_k$, then the glueing property of the Weyl functions shows that $m_+(q, b_k^-) = \lim_{\lambda \uparrow b_k} m_+(q, \lambda)$ and $m_-(q, b_k^-) = \lim_{\lambda \uparrow b_k} m_-(q, \lambda)$ are both equal to ∞. Set $\lambda = b_k - \varepsilon$, and consider the values $m_+(q, \lambda)$ resp. $m_-(q, \lambda)$, which are large positively resp. negatively when viewed as elements of $\mathbb{R}$. If r is sufficiently large, then $m_+(q_r, \lambda)$ is large positively and $m_-(q_r, \lambda)$ is large negatively (and neither value is ∞). Since $m_\pm(q_r, \lambda)$ close monotonely to $m_+(q_r, b_k^-) = m_-(q_r, b_k^-)$ as $\lambda \uparrow b_k$, we see that there is exactly one $\lambda \in (b_k - \varepsilon, b_k]$ such that either $m_+(q, \lambda) = \infty$ or $m_-(q, \lambda) = \infty$; the sign $\pm$ is uniquely determined if $\lambda < b_k$, while $m_\pm(q, \lambda)$ are both equal to ∞ if $\lambda = b_k$. In either case, one has that $\mu_{k,r} \in (b_k - \varepsilon, b_k]$, and so $q \to d_q$ is indeed continuous.

Let us write $\pi : \mathcal{Q}_E \to \mathcal{D}_E : q \mapsto d_q$ for the map we have just constructed. Then π is continuous. We will now show that π is surjective, and that, if the half-line operators $L_q^\pm$ have purely a.c. spectrum, then it is injective as well.

We proceed in two steps. The first is to show that, if $d \in \mathcal{D}_E$, then there exists $q \in \mathcal{Q}_E$ such that $d_q = d$. The second is to show that there is at most one $q \in \mathcal{Q}_E$ with this property.

For the first step, write $d = \{(\mu_1, \varepsilon_1), \ldots, (\mu_j, \varepsilon_j), \ldots\}$. For each $g \geq 1$, set $d_g = \{(\mu_1, \varepsilon_1), \ldots, (\mu_g, \varepsilon_g), b_{g+1}, b_{g+2}, \ldots\}$. Let q_g be the algebro-geometric potential which has divisor $\tilde{d}_g = \{(\mu_1, \varepsilon_1), \ldots, (\mu_g, \varepsilon_g)\}$. Then $q_g \in \mathcal{Q}_g := \mathcal{Q}_{E_g}$. By passing to a subsequence indexed by $g_r \to \infty$, we can assume that $q_r := q_{g_r}$ converges to $q \in \mathcal{R}$. By Proposition 3.1, $q \in \mathcal{Q}_E$. It is clear that $d_r := d_{g_r} \to d$ in $\mathcal{D}_E$.

We claim that the divisor of q equals d, that is $\pi(q) = d$. To see this, let $d' = \pi(q)$, so that a priori d' may differ from d. Write $d' = \{(\mu_1', \varepsilon_1'), \ldots, (\mu_j', \varepsilon_j'), \ldots\}$. The components of d' can be read off from the product expansion (10) of the diagonal Green's function $g(\lambda) = q_g(0, 0, \lambda)$, together with the knowledge of "which m-function" has a pole at μ_j' if $a_j < \mu_j' < b_j$.

Set $\mathcal{Q}_r = \mathcal{Q}_{g_r}$, $r = 1, 2, \ldots$. By Proposition 3.1, the compact set $\limsup_{r\to\infty} \mathcal{Q}_r$ is contained in $\mathcal{Q}_E$. It is clearly $\{\tau_x\}$-invariant. Let $V \subset \mathcal{R}$ be a closed neighborhood of $\mathcal{Q}_E$. Then there exists $r_V \geq 1$ such that, if $r \geq r_V$, then $\mathcal{Q}_r \subset V$. This means that, if $q_r \in \mathcal{Q}_r$, then $q_r \in V$, but more importantly the orbit closure $cls\{\tau_x(q_r) \mid x \in \mathbb{R}\}$ is contained in V.

Let $K \subset \mathbb{C}\backslash E$ be a compact set. We apply the perturbation theorem of Sacker and Sell [54] together with a compactness argument to draw the following conclusion. –There is a closed neighborhood V_k of $\mathcal{Q}_E$ in $\mathcal{R}$ such that, if $M \subset V_K$ is a compact invariant set, then equations (9_q) admit an exponential dichotomy over M for all $\lambda \in K$. Moreover, the "dichotomy projections over M are close to those over $\mathcal{Q}_E$". More precisely, if $M \subset V_K$ is the maximal compact $\{\tau_x\}$-invariant set in V_K, then the dichotomy projections vary continuously with $q \in M$, $\lambda \in K$.

We can now reason as follows. Let $k \geq 1$ be fixed. Suppose that $\mu_k' \in (a_k, b_k)$. Let $K \subset \mathbb{C}\backslash E$ be a compact set containing μ_k' in its interior. Let $r' = r_{V_K}$ be determined in correspondence to the compact neighborhood V_K of $\mathcal{Q}_E$. If $r' \leq r \to \infty$, we have that $m_\pm(q_r, \lambda) \to m_\pm(q, \lambda)$ uniformly on K. One now uses the continuity of the dichotomy projections and the monotonicity properties of $m_\pm(q_r, \cdot), m_\pm(q, \cdot)$ to show that the k-component of d_r converges to (μ_k', ε_k'). In fact, if $g_r \geq k$, then the k-component of d_r equals (μ_k, ε_k), so in fact $(\mu_k', \varepsilon_k') = (\mu_k, \varepsilon_k)$.

If $\mu'_k \in \{a_k, b_k\}$, then one can still apply the Sacker–Sell perturbation theorem and the monotonicity properties of the Weyl functions to conclude that the sequence of k-components of d_r converges to (μ'_k). Putting this fact together with the result of the previous paragraph, we see that $\{d_r\}$ converges to d'. Since $d_r \to d$, we have $d = d' = \pi(q)$. We have succeeded in carrying out the first step.

There remains to show that, if $d \in \mathcal{D}_E$, then there is at most one $q \in \mathcal{Q}_E$ with divisor d, at least if the representing measures $\sigma_\pm(d\lambda)$ of the Weyl functions $m_\pm(\lambda)$ are purely absolutely continuous on E. To prove this inverse spectral result, we first use the classical Gel'fand-Levitan theory [39] to show that, if the divisor d determines uniquely the representing measures $\sigma_\pm(d\lambda)$, then it actually determines $q(x)$. In fact, if $\sigma_\pm(d\lambda)$ are known, then $q(x)$ is determined for a.a. $x \in (-\infty, 0)$ and for a.a. $x \in (0, \infty)$. Since q is continuous (actually real-analytic), it is determined by the measures $\sigma_\pm(d\lambda)$.

We will show that the divisor d determines the measures $\sigma_\pm(d\lambda)$. To do this, note first that the relations (10) and (11) imply that the difference $m_+(\lambda) - m_-(\lambda)$ is determined by d ($\Im\lambda > 0$). Since q is reflectionless, we also have that $\Im m_+(\lambda + i0) = -\Im m_-(\lambda + i0)$ for a.a. $\lambda \in E$. Hence $\Im m_\pm(\lambda + i0) = \pm\frac{1}{2}[m_+(\lambda + i0) - m_-(\lambda + i0)]$ a.e. on E. This means that $\Im m_\pm(\lambda + i0)$ are determined by d for a.a. $\lambda \in E$. We conclude that, if the representing measures $\sigma_\pm$ of $m_\pm$ are both purely absolutely continuous on E, then they are determined uniquely on E by the divisor d.

Finally, the poles of $m_\pm$ in $\mathbb{R}\backslash E$ are determined by d, and the residues at these poles can be computed from (11), hence are also determined by d. All this means that, if $L_q^\pm$ have absolutely continuous spectrum on E, then q is uniquely determined by d. This concludes the proof of Theorem 3.3. $\square$

Remark 3.4 (1) Suppose that the closed set $E \subset \mathbb{R}$ satisfies the Standing Hypotheses but that property P does not hold. Then if $q \in \mathcal{Q}_E$, the diagonal Green's function $g_q(x, x, \lambda)$ still admits a product expansion (10_{bis}). The functions $x \mapsto \mu_j(x)$ are continuous ($j \geq 1$). However it is no longer clear that the m-functions $m_\pm(x, \lambda)$ glue together at the endpoints a_j, b_j of the spectral gaps. Indeed it seems possible that $\mu_j(x)$ can oscillate in a complex way for x near x_0 if $\mu_j(x_0) = a_j$ or b_j. Using the theory of the ξ-function [16,17,58,59], one can still prove the trace formula

$$q(x) = \frac{1}{2}\sum_{j=1}^{\infty}(a_j + b_j - 2\mu_j(x)),$$

as well as the higher-order trace formulas obtained using the so-called K-dV invariants [15].

(2) One can ask when a set E which satisfies the Standing Hypotheses gives rise to a set $\mathcal{Q}_E$ such that, for each $q \in \mathcal{Q}_E$, the half-line operators $L_q^\pm$ have purely absolutely continuous spectrum in E. First of all, if E is homogeneous in the sense of Carleson [60], then each $q \in \mathcal{Q}_E$ has the desired property. Indeed q is reflectionless, which means that $g_q(\lambda)$ has the property that $\Re g_q(\lambda + i0) = 0$ for a.a. $\lambda \in E$. This implies that $\Re\left(\frac{-1}{g_q(\lambda+i0)}\right) = 0$ for a.a. $\lambda \in E$. By a theorem of Zinsmeister [66], the representing measure $\sigma(d\lambda)$ of the Herglotz function $\frac{-1}{g_q}$ is purely absolutely continuous on E (see [18], Theorem 3.3). One can actually say more: in fact, $\sigma(d\lambda)$ is purely absolutely continuous on E if only E is weakly homogeneous in the sense of Poltoratski–Remling [51]. Since $\frac{-1}{g_q} = m_+ - m_-$, we see that the operators $L_q^\pm$ have purely absolutely continuous spectrum on E.

We do not wish to assume that E is (weakly) homogeneous; instead we suppose that for each $j \geq 1$, a_j (resp. b_j) is the endpoint of a nondegenerate closed interval contained in E. Let us call this condition Property P_*. Let $q \in \mathcal{Q}_E$; then the singular spectrum (if any) of $L_q^\pm$ is contained in the set H of accumulation points of $\{a_j\} \cup \{b_j\}$. This is because each point in $E \backslash H$ is contained in a nondegenerate closed interval $I \subset E$ where $m_\pm(\lambda + i0) = 0$ a.e., and using the Schwarz reflection principle and the Picard theorem (as were used in the discussion of the algebro-geometric potentials) one checks that the spectrum of $L_q^\pm$ is purely a.c. on each such interval (in fact the only questionable points in this regard are the endpoints of I, but near such an endpoint λ_* the Weyl functions $m_\pm(\lambda)$ glue together to form a meromorphic function M of $\sqrt{\lambda - \lambda_*}$ with at most a simple pole at λ_*).

Now, even if E has property P_*, we have no general criterion for excluding the presence of singular spectrum in H. However, consider the case when H reduces to $\{b_0\}$. That is, the intervals (a_j, b_j) can be labeled so that $0 \geq b_1 > a_1 > b_2 > a_2 > \cdots$ and $a_j \to b_0$, $b_j \to b_0$ as $j \to \infty$. Let $q \in \mathcal{Q}_E$. To prove that $L_q^\pm$ has no singular spectrum it is sufficient to prove that b_0 is not an eigenvalue of $L_q^\pm$. But this follows from the classical theory of the principal solutions of $L_q \varphi = b_0 \varphi$ at $x = \infty$; see, e.g., Hartman [20]. So, if E has the above structure, then $\pi : \mathcal{Q}_E \to \mathcal{D}_E$ is a homeomorphism.

One can study the structure of π in certain other cases, using the theory of subordinate solutions of Gilbert and Pearson [19].

(3) Let us suppose that $\pi : \mathcal{Q}_E \to \mathcal{D}_E$ is a homeomorphism. Then each $q \in \mathcal{Q}_E$ can be approximated (in the compact-open topology) by algebro-geometric potentials q_g, where the spectrum of $L_g = L_{q_g}$ equals E_g. To see this, let $d_q = \pi(q) = \{(\mu_1, \varepsilon_1), \ldots, (\mu_j, \varepsilon_j \ldots\}$ be the divisor of q. Let q_g $(g = 1, 2, \ldots)$ be the algebro-geometric potential with divisor $\{(\mu_1, \varepsilon_1), \ldots (\mu_g, \varepsilon_g)\}$. Choose a sequence $\{q_r\}$ such that the sequence $\{q_r = q_{g_r}\}$ converges in $\mathcal{R}$, say to $\tilde{q}$. Then $\tilde{q} \in \mathcal{Q}_E$ has divisor d, as follows from Proposition 3.1 and the proof of Theorem 3.3. But $q \in \mathcal{Q}_E$ is uniquely determined by d, so $\tilde{q} = q$, and $q_g \to q$. This means that (if π is a homeomorphism) $\mathcal{Q}_E = \limsup_{g \to \infty} \mathcal{Q}_g$.

(4) Theorem 3.3 is clearly related to Theorems 8.2 and 8.3 in Kotani [33]. In particular the relation between the results stated in Theorem 8.3 of [33] and our result concerning injectivity of π merits further study.

Using the results obtained so far, we can give an example of a closed set $E \subset \mathbb{R}$ for which $\pi : \mathcal{Q}_E \to \mathcal{D}_E$ is a homeomorphism, and a stationary ergodic process $(M, \{\tau_x\}, \nu)$ with $M \subset \mathcal{Q}_E$ such that the Lyapunov exponent β_ν is positive in $b_0 = \min E$.

Example 3.5 Let $E = \mathbb{R} \backslash \bigcup_{j=1}^\infty (a_j, b_j)$ where $a_0 = -\infty$, $-1 \leq b_0 < 0$, and $0 \geq b_1 > a_1 > b_2 > a_2 > \cdots \to b_0$. According to Remark 3.4 (2), the map $\pi : \mathcal{Q}_E \to \mathcal{D}_E$ is a homeomorphism. Let us choose the intervals (a_j, b_j) so that $\lambda = b_0$ is a non-regular point of E in the sense of potential theory. This can be done by choosing the lengths and positions of the intervals (a_j, b_j) in an appropriate way, and applying the Wiener criterion [22,37].

Let $M \subset \mathcal{Q}_E$ be a minimal set, and let ν be a $\{\tau_x\}$-ergodic measure on M. Introduce the ν-Lyapunov exponent $\beta_\nu(\lambda)$; see Sect. 1. Then β_ν is harmonic and positive on $\Omega_E = \mathbb{C} \backslash E$. If $\beta_\nu(b_0)$ were zero, then β_ν would be a weak harmonic barrier at b_0, contradicting the non-regularity of b_0 [22]. Hence $\beta_\nu(b_0) > 0$.

It can be shown that, for ν-a.a $q \in M$, the Marchenko-Kotani measure σ of the relations (6_a), (6_b) satisfies

$$\int_{-1}^{1} \frac{\sigma(dt)}{1 - t^2} < 1;$$

see [34]. It can also be shown that $\{q \in M \mid$ the operator L_q has an eigenvalue at $\lambda = b_0\}$ is "small" in the sense that it has ν-measure zero and is of the first Baire category (we do not know if it is empty). This indicates that our example is of a different nature than that of Gesztesy–Yuditskii [18]; we omit a detailed discussion of this issue. One final remark: the Lyapunov exponent β_ν is clearly discontinuous at $\lambda = b_0$ in this example.

Now we turn out attention to certain issues concerning Parreau–Widom domains Ω_E, the corresponding Abel map $\mathfrak{a}$, and the dynamical properties of the flow $(\mathcal{Q}_E, \{\tau_x\})$. In what follows, we will at numerous points fail to give details concerning interesting and important facts related to the function theory of infinitely-connected domains. For more information one can refer to the excellent text of Hasumi [21], and to the papers of Sodin–Yuditskii [61], Gesztesy–Yuditskii [18], Volberg–Yuditskii [64] and Damanik–Yuditskii [8].

It will simplify matters to assume from the outset that E has the structure indicated in Remark 3.4 (2). We take a moment to list the conditions which E will be required to satisfy from this point on.

Hypotheses 3.6 *The closed set $E \subset \mathbb{R}$ has the form $E = \mathbb{R}\backslash\bigcup_{j=1}^{\infty}(a_j, b_j)$ where $a_0 = -\infty$, $-1 \le b_0 < 0$, and $0 \ge b_1 > a_1 > a_2 > b_2 > \cdots \to b_0$. Moreover Ω_E is of Parreau-Widom type, but the Abel map $\mathfrak{a} : \mathcal{D}_E \to \mathcal{J}_E$ is not invertible.*

As we saw in Remark 3.4 (2), the first hypothesis implies that the divisor map $\pi : \mathcal{Q}_E \to \mathcal{D}_E$ is a homeomorphism. The hypothesis that Ω_E be of Parreau–Widom type ensures that the Abel map $\mathfrak{a}$ is well-defined, continuous and surjective. We will take for granted that the points a_j, b_j $(j \ge 1)$ can be chosen so that $\mathfrak{a}$ is not invertible. See [18] for indications on how to do this.

Note that the homeomorphism π induces a flow $\{\hat{\tau}_x\}$ on $\mathcal{D}_E$, as follows: if $d \in \mathcal{D}_E$, $x \in \mathbb{R}$, and $q = \pi^{-1}(d)$, then $\hat{\tau}_x(d) = \pi(\tau_x(q))$. We abuse notation and write $\{\tau_x\}$ for this flow on $\mathcal{D}_E$. There is no "natural" flow on the space of divisors $\mathcal{D}_E$. We now investigate the composition of the Abel map $\mathfrak{a}$ with the flow $\{\tau_x\}$ on $\mathcal{D}_E$. We will show that there is a particular character $\delta \in \mathcal{J}_E$ such that

$$\mathfrak{a}(\tau_x(d)) = \mathfrak{a}(d) + \delta x \qquad (d \in \mathcal{D}_E,\ x \in \mathbb{R}). \tag{12}$$

This is a significant relation, for the following reason. The translation $\alpha \mapsto \alpha + \delta x$ ($\alpha \in \mathcal{J}_E,\ x \in \mathbb{R}$) defines an almost periodic flow on the compact Abelian topological group $\mathcal{J}_E$; we denote this flow by $(\mathcal{J}_E, \delta)$. Then (12) states that $\mathfrak{a}$ is a (surjective) homomorphism of the flow $(\mathcal{D}_E, \{\tau_x\})$ onto $(\mathcal{J}_E, \delta)$.

To identify the character δ, we use the relation between the Parreau-Widom domain Ω_E and a certain comb domain $K_E = \{z = u + iv \in \mathcal{C} \mid u < 0,\ v > 0\}\backslash \bigcup_{j=1}^{\infty}\{z = h_j + i\delta_j\} \mid -h_j < u < 0\}$, which is contained in the second quadrant. Here δ_j, h_j are positive numbers with $\sum_{j=1}^{\infty} h_j < \infty$. The point is that, for a certain choice of the numbers h_j, δ_j, there is a holomorphic map w taking $\mathbb{C}_+$ onto K_E with the following properties. First, w extends continuously to the real axis; it maps b_0 to 0 and sends (a_j, b_j) to the tooth of the comb which has height $v = \delta_j$ and length h_j. Second, $w(\lambda) \sim -\sqrt{-\lambda}$ as $\lambda \to \infty$. It turns out that these conditions uniquely determine the numbers h_j, δ_j and the function w.

It also turns out that w extends holomorphically through the interval $(-\infty, b_0)$, and the extension is holomorphic on $\mathbb{C}\backslash(b_0, \infty)$. it is not holomorphic in Ω_E (though $\Re w$ is a well-defined, negative harmonic function on Ω_E). Rather $\Im w$ has periods which correspond to the resolvent intervals (a_j, b_j). In fact, let $k \ge 1$ and let γ_k be the curve introduced in Sect. 1; this γ_k is orthogonal to the real axis, contains $\lambda_0 = -2$ and a point of (a_k, b_k), and is oriented clockwise. Let $-2\delta_k$ be the change of argument of $w(\lambda)$ as λ traverses γ_k:

$$2\delta_k = -\Delta \arg w|_{\gamma_k} \qquad (k = 1, 2, \dots).$$

Then $\delta = (\delta_1, \delta_2, \dots)$ determines a character on Γ_E; i.e., an element of $\mathcal{J}_E$. We will prove that (12) holds for this character δ.

This can be done using arguments of [61] as we now discuss. Let $d = \{(\mu_1, \varepsilon_1), \dots, (\mu_j, \varepsilon_j), \dots\} \in \mathcal{D}_E$ and let $q \in \pi^{-1}(d)$. According to Remark 3.4 (3), for any sequence $g_r \to \infty$, there is a unique sequence $q_r = q_{g_r} \in \mathcal{Q}_{E_{g_r}} = \mathcal{Q}_r$ such that q_r has the divisor $d_r = \{(\mu_1, \varepsilon_1), \dots, (\mu_{g_r}, \varepsilon_{g_r})\}$. We can view d_r as an element not only of $\mathcal{D}_r = \mathcal{D}_{E_{g_r}}$ but also of $\mathcal{D}_E$, if we identify $d_r \in \mathcal{D}_r$ with $d_r = \{(\mu_1, \varepsilon_1), \dots, (\mu_{g_r}, \varepsilon_{g_r}), (b_{g_r+1}), \dots\} \in \mathcal{D}_E$.

Let $\mathfrak{a}_r$ be the Abel map from $\mathcal{D}_r$ to $\mathcal{J}_r = \mathcal{J}_{E_{g_r}}$ $(r \geq 1)$. The base point of $\mathfrak{a}_r$ is $\{(b_1), \dots, (b_{g_r})\}$ as in Sect. 1. We abuse notation and write $\tau_x(d_r)$ for the unique divisor in $\mathcal{D}_r$ which corresponds to $\tau_x(q_r)$ $(x \in \mathbb{R})$. Then from Sect. 1:

$$\mathfrak{a}_r(\tau_x(d_r)) = \mathfrak{a}_r(d_r) + \delta^{(r)} x$$

where $\delta^{(r)}$ is an element of $\mathcal{J}_r$ which is in the first moment the vector of periods of the normalized integral of the second kind on the Riemann surface corresponding to $E_r = E_{g_r}$. Now however, Moser [48] showed that if w_r is the normalized conformal mapping of $\mathbb{C}_+$ onto the slit domain $K_r = K_{E_r}$ corresponding to E_r, then

$$\delta^{(r)}(\gamma_k) = \Im w_r|_{(a_k, b_k)}, \qquad 1 \leq k \leq g_r. \tag{13}$$

We identify a character $\alpha_r \in \mathcal{J}_r$ with an element $\tilde{\alpha}_r$ of $\mathcal{J}_E$ by setting $\tilde{\alpha}_r(\gamma_k) = 1$ if $k > g_r$. We abuse notation and write α_r instead of $\tilde{\alpha}_r$. Moreover, if $\alpha \in \mathcal{J}_E$ we define a corresponding character $\hat{\alpha}_r \in \mathcal{J}_E$ by restricting α to the closed curves $\gamma_1, \dots, \gamma_{gr}$, then extending to all Γ_{E_r}. We now proceed in two steps: First, it can be shown that $\mathfrak{a}_r(d_r) \to \mathfrak{a}(d)$ when the above identifications are made. This is done making use of the Parreau-Widom condition (8), by repeating the arguments of ([61], pp. 650–651). Second, it can be shown that $\delta^{(r)} \to \delta$ in the sense that $\delta^{(r)}(\gamma_k) \to \delta(\gamma_k)$ for all $k \geq 1$.

This second step is carried out in [61] by appealing to a relation between conformal maps on comb domains and subharmonic majorants (see Levin [38]). We sketch an alternate procedure. Let ν_r be a $\{\tau_x\}$-ergodic measure on $\mathcal{Q}_r$. According to Moser [48], the Floquet exponent $w_{\nu_r} = \tilde{w}_r$ equals the conformal map w_r. Let ν be a weak-$*$ limit point of (ν_r) in the space of regular Borel probability measures on $\mathcal{R}$. Using Remark 3.4 (3), one can show that ν is supported on $\mathcal{Q}_E$. The measure ν is $\{\tau_x\}$-invariant but not necessarily ergodic. Nevertheless one can define $\tilde{w}(\lambda) = \int_{\mathcal{Q}_E} m_+(q, \lambda)\nu(dq)$ for all $\Im\lambda > 0$, and then show that one obtains the same function if any $\{\tau_x\}$-invariant measure ν_1 which is supported on $\mathcal{Q}_E$ is substituted for ν. This is done by introducing an ergodic decomposition for ν, namely $\nu = \int_H \Lambda(d\nu_e)$ where Λ is a measure concentrated on the set $H = \{\nu_e\}$ of $\{\tau_x\}$- ergodic measures on $\mathcal{Q}_E$; for an application of this method see [28]. It turns out that, for each $\nu_e \in H$, w_{ν_e} equals the conformal map w for $\Im\lambda > 0$, and hence $\tilde{w}$ equals w for $\Im\lambda > 0$. Finally, noting that $\tilde{\rho}_r(\lambda) = \Im\tilde{w}(\lambda)$ is the ν_r-rotation number for $\lambda \in \mathbb{R}$, and using certain continuity property of the rotation number under weak-$*$ convergence [26], one can show that in fact $\tilde{\rho}_r(\lambda) \to \tilde{\rho}(\lambda) = \Im\tilde{w}(\lambda)$ for $\lambda \in (a_k, b_k)$. Using (13), one sees that $\delta^{(r)}(\gamma_k) \to \delta(\gamma_k)$ for all $k \geq 1$.

Summarizing all these considerations, we have proved

Proposition 3.7 *If $d \in \mathcal{D}_E$, then $\mathfrak{a}(\tau_x(d)) = \mathfrak{a}(d) + \delta x$ for all $x \in \mathbb{R}$.*

We turn to a discussion of the right inverse $\mathfrak{i}$ of the Abel map $\mathfrak{a} : \mathcal{D}_E \to \mathcal{J}_E$. This object exists whenever $E \subset \mathbb{R}$ is a closed set and Ω_E is a Parreau-Widom domain. It is constructed in [60] and we will use the results proved there with limited discussion. We will, however, want to show that $\mathfrak{i}$ enjoys two additional properties for the special sets E which we consider. We list those properties now.

(i) The map $\mathfrak{i}$ is of the first Baire class; that is, $\mathfrak{i}$ is the pointwise limit of a sequence $\{\mathfrak{i}_g\}$ of continuous maps $\mathfrak{i}_g : \mathcal{J}_E \to \mathcal{D}_E$.
(ii) The map $\mathfrak{i}$ commutes with the translation flows: $\mathfrak{i}(\alpha + \delta x) = \tau_x(\mathfrak{i}(\alpha))$ for all $\alpha \in \mathcal{J}_E$, $x \in \mathbb{R}$.

The results of [60] are stated for a Parreau-Widom domain $\Omega_{\hat{E}}$ where $\hat{E} \subset \mathbb{R}$ is compact. We can put ourselves in this situation by introducing the linear fractional transformation

$$z = \psi(\lambda) = \frac{\lambda - b_0}{\lambda + 2} \Leftrightarrow \lambda = \frac{2z + b_0}{1 - z}.$$

Note that $\psi(-2) = \infty$, and that the image set $\hat{E} = \psi(E)$ is a compact subset of $[0, 1]$. The domain $\Omega_{\hat{E}} = (\mathbb{C} \cup \{\infty\})\backslash \hat{E}$ is of Parreau-Widom type [61]. Write

$$\mathbb{R}\backslash \hat{E} = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{j=1}^{\infty} (\hat{a}_j, \hat{b}_j)$$

where $\hat{a}_j = \psi(a_j)$, $\hat{b}_j = \psi(b_j)$ and $\dfrac{-b_0}{2} \geq \hat{b}_1 > \hat{a}_1 > \hat{b}_2 > \cdots \to 0$.

Let $z_0 \in \Omega_{\hat{E}}$, and let $\hat{\mathcal{G}}(z, z_0)$ be the Green's function of $\Omega_{\hat{E}}$ with logarithmic pole at z_0. If $z_0 = \psi(\lambda_0)$ with $\lambda_0 \in \Omega_E$, then the Green's function satisfies $\mathcal{G}(\lambda, \lambda_0) = \hat{\mathcal{G}}(z, z_0)$. Setting $\lambda_0 = -2$ and $\hat{c}_j = \psi(c_j)$, one verifies the Parreau-Widom condition (8) with $z_0 = \infty$:

$$\sum_{j=1}^{\infty} \hat{\mathcal{G}}(c_j, \infty) < \infty.$$

We introduce the closed curves $\hat{\gamma}_k = \psi \circ \gamma_k$ $(k \geq 1)$, so that each $\hat{\gamma}_k$ contains $z_0 = \infty$. We also introduce the sets $\hat{E}_k = \psi(E_k)$, and observe that $\hat{E}_k = E \cap [b_k, 1]$. Since a conformal map transforms harmonic functions to harmonic functions, the harmonic measure $\hat{\omega}(z, \hat{E}_k)$ satisfies $\hat{\omega}(z, \hat{E}_k) = \omega(\lambda, E_k)$ where $\lambda = \psi^{-1}(z)$. A character $\alpha \in \mathcal{J}_E$ gives rise to a unique character $\hat{\alpha} \in \mathcal{J}_{\hat{E}}$ via the formula $\hat{\alpha}(\hat{\gamma}_k) = \alpha(\gamma_k)$ $(k = 1, 2, \ldots)$. Introduce the set $\mathcal{D}_{\hat{E}}$ of divisors $\hat{d} = \{(\hat{\mu}_1, \varepsilon_1), \ldots, (\hat{\mu}_j, \varepsilon_j), \ldots\}$ with the identifications $(\hat{a}_j, +1) = (\hat{a}_j, -1)$ and $(\hat{b}_j, +1) = (\hat{b}_j, -1)$. We introduce the Abel map

$$\hat{\mathfrak{a}}(\hat{d}) = \sum_{j \geq 1} \frac{\varepsilon_j}{2} \int_{\hat{\mu}_j}^{\hat{b}_j} \hat{\omega}(dt, \hat{E}_k) \qquad (\hat{d} \in \mathcal{D}_{\hat{E}}).$$

We then have $\hat{\mathfrak{a}}(\hat{d}) = \mathfrak{a}(d)$ when $d = \{(\mu_1, \varepsilon_1), \ldots, (\mu_j, \varepsilon_j), \ldots\} \in \mathcal{D}_E$ and $\hat{\mu}_j = \psi(\mu_j)$.

We thus have a good correspondence between quantities related to E and quantities related to $\hat{E}$. We will write down the inverse map $\hat{\mathfrak{i}} : \mathcal{J}_{\hat{E}} \to \mathcal{D}_{\hat{E}}$, then pull it back to $\mathcal{J}_E$ via ψ. In the next lines we will discuss constructions related to $\hat{E} \to \Omega_{\hat{E}}$, so we will omit the $\hat{}$ until further notice.

We pass to the construction of a right inverse $\mathfrak{i} : \mathcal{J}_E \to \mathcal{D}_E$ of $\mathfrak{a}$ which has the properties (i) and (ii). We follow Hasumi [21] and especially Sodin–Yuditskii [60]. For each $\alpha \in \mathcal{J}_E$ consider the set $\{F\}$ of multiple-valued locally holomorphic functions on Ω_E which have a single-valued modulus $|F|$ and which have character α. This means that, if F is continued along a curve $\gamma \in \Gamma_E$, then F is multiplied by $e^{2\pi i \alpha(\gamma)}$. It is further required that $|F|^2$ admit a harmonic majorant in Ω_E. One defines $||F||^2$ to be the value at ∞ of the least harmonic majorant of $|F|^2$. It turns out that $||\cdot||$ is a norm on the space H^2_α of all such functions F. It further turns out that the inner product

$$\langle F_1, F_2\rangle = \int_E F_1(z)\overline{F_2(z)}\omega(dz, \infty)$$

gives rise to the norm $||\cdot||$. Here the boundary values $F_1(z)$, $F_2(z)$ $(z \in E)$ must be appropriately defined.

It is convenient to study H^2_α by "lifting" it to the unit disc $U = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$, via the introduction of a certain Fuchsian group Γ of linear fractional transformations of U. This group is chosen so as to be group isomorphic to the fundamental group Γ_E of Ω_E, and so that Ω_E is analytically diffeomorphic to the quotient space U/Γ. In fact one can give a fairly concrete description of Γ and of the appropriate covering map $\zeta \mapsto z(\zeta) : U \to \Omega_E$; see ([60], pp. 405–406). Namely, introduce the boundary $\mathbb{T} = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$ of U, together with the lower half $U_- = \{\zeta \in U \mid |\Im\zeta \le 0\}$ of U and the lower half $\mathbb{T}_- = \{\zeta \in \mathbb{T} \mid \Im\zeta \le 0\}$. For each $j \ge 1$, let $U_j \subset \mathbb{C}$ be a disc with center ζ_j whose boundary $\mathbb{T}_j$ is orthogonal to $\mathbb{T}$ and intersect it in points $\tilde{a}_j, \tilde{b}_j \in \mathbb{T}_-$. One can choose the discs U_j in such a way that the conformal mapping

$$z : U_- \backslash \bigcup_{j=1}^{\infty} U_j \to \mathbb{C}_+ \quad \text{satisfying} \;\; z(-1) = 0, z(1) = 1, z(0) = \infty$$

maps the arcs $\mathbb{T}_j \cap U$ onto the intervals $(a_j, b_j) \subset (-b_0/2, 1)$ $(j = 1, 2, \ldots)$. Let $\tilde{\gamma}_j$ be the linear fractional transformation defined by first reflecting in the real axis, then reflecting in the circle $\mathbb{T}_j$: thus

$$\tilde{\gamma}_j(\zeta) = \frac{\zeta\zeta_j - 1}{\zeta - \overline{\zeta_j}}. \tag{14}$$

Let Γ be the (free) group generated by $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_j, \ldots\}$. Then Γ_E and Γ are isomorphic if one identifies the closed curve γ_j with the transformation $\tilde{\gamma}_j$ $(j \ge 1)$. Moreover, it turns out that z extends to a covering map of U onto Ω_E, which induces an analytic diffeomorphism of U/Γ onto Ω_E.

Let us agree to identify $\gamma \in \Gamma_E$ with the corresponding element $\tilde{\gamma} \in \Gamma$. Then a character $\alpha \in \mathcal{J}_E$ can be viewed as defined on Γ. In this way, one identifies H^2_α with a closed subspace of the Hardy space $H^2(U)$:

$$H^2_\alpha = \left\{ f \in H^2(U) \mid f \circ \tilde{\gamma} = e^{2\pi i \alpha(\tilde{\gamma})} f \;\text{ for all } \tilde{\gamma} \in \Gamma \right\}.$$

One speaks of elements of H^2_α as character-automorphic functions. Now let us pose an extremal problem in H^2_α. We first view H^2_α as a set of multivalued functions on Ω_E. By a theorem of Widom [65], the hypothesis that Ω_E is of Parreau-Widom domain implies that H^2_α contains a non-constant function $\mathcal{K}^\alpha(z)$ which does not vanish at $z = \infty$ and which satisfies $|\mathcal{K}^\alpha(\infty)| = \sup\{|F(\infty) \mid F \in H^2_\alpha\}$. It turns out that $\mathcal{K}^\alpha(z)$ is uniquely defined if it is required that, in addition, $\mathcal{K}^\alpha(\infty) > 0$. In the $H^2(U)$-picture, one sets $\mathfrak{K}^\alpha(\zeta) = \mathcal{K}^\alpha(z(\zeta))$ $(\zeta \in U)$. It turns out that

$$\mathfrak{K}^\alpha = P_{H^2_\alpha}\mathbf{1}/||P_{H^2_\alpha}\mathbf{1}|| \tag{15}$$

where $P_{H^2_\alpha}$ is the orthogonal projection of $H^2(U)$ onto H^2_α, and $\mathbf{1}$ is the function which is identically equal to 1 on U.

Let $\mathcal{G}(z, z_0)$ be the Green's function of Ω_E with logarithmic pole at $z_0 \in \Omega_E$. Let $\tilde{\mathcal{G}}$ be a harmonic conjugate of $\mathcal{G}$, and let $\Phi(z, z_0) = \exp[-\mathcal{G}(z, z_0) - i\tilde{\mathcal{G}}(z, z_0)]$, so that $\Phi(z)$ has a zero at z_0. We normalize $\tilde{\mathcal{G}}$ so that $\Phi(z, z_0) = (z - z_0) + \cdots$. Write $\Phi(z) = \Phi(z, \infty)$. Let $\alpha \in \mathcal{J}_E$. Via an elegant detour by way of the theory of reflectionless Jacobi operators, Sodin and Yuditskii [60] find a divisor $d = \{(y_1, \varepsilon_1), \ldots.(y_j, \varepsilon_j), \ldots\} \in \mathcal{D}_E$ such that

$$\mathcal{K}^\alpha(z) = \left\{\prod_{j\geq 1} \frac{z - y_j}{z - c_j} \cdot \frac{\Phi(z, c_j)}{\Phi(x, y_j)}\right\}^{1/2} \left\{\prod_{j\geq 1} \Phi(z, y_j)^{(1+\varepsilon_j)/2}\right\}. \tag{16}$$

This divisor d is uniquely determined by α. Let us introduce the fixed divisor

$$d_c = \{(c_1, -1), (c_2, -1), \ldots, (c_j, -1), \ldots\} \in \mathcal{D}_E$$

and set $\alpha_c = \mathfrak{a}(d_c)$. We define

$$\mathfrak{i}(\alpha + \alpha_c) = d,$$

or in other words $\mathfrak{i}(\alpha)$ is the divisor determined from $\alpha - \alpha_c$ by the above procedure. It turns out that [60]

$$\mathfrak{a} \circ \mathfrak{i}(\alpha) = \alpha \qquad (\alpha \in \mathcal{J}_E).$$

So $\mathfrak{i}$ is a right inverse of $\mathfrak{a}$. The map $\mathfrak{i}$ is continuous if E is a homogeneous set [60]. However it is not continuous for all Parreau-Widom domains.

We want to show that $\mathfrak{i}$ has the properties listed in (i) and (ii) above. For this, we must write out the coefficients of the Jacobi operator which corresponds to $\alpha \in \mathcal{J}_E$. Referring to ([60], p. 418), this operator is

$$P(\alpha + \mu(n-1))x_{n-1} + Q(\alpha + \mu n)x_n + P(\alpha + \mu n)x_{n+1} = zx_n$$

with the following notation. First,

$$P(\alpha) = cap\ E \cdot \frac{\mathcal{K}^{\alpha+\mu}(0)}{\mathcal{K}^\alpha(0)} \tag{19_1}$$

where $cap\ E$ is the logarithmic capacity of E. Second,

$$Q(\alpha) = c_0 + c_{-1}\left\{\frac{(\mathcal{K}^\alpha)'(0)}{\mathcal{K}^\alpha(0)} - \frac{(\mathcal{K}^{\alpha-\mu})'(0)}{\mathcal{K}^{\alpha-\mu}(0)}\right\} \tag{19_2}$$

where c_0, c_{-1} are defined by the relation $z(\zeta) = \dfrac{c_{-1}}{\zeta} + c_0 + \cdots$. Third, μ is a fixed element of $\mathcal{J}_E$, namely $-\mu$ is the character of the function $B(\zeta) = \Phi(z(\zeta))$. It is clear from these formulas that we must express $\alpha \mapsto \mathcal{K}^\alpha(0),\ \alpha \mapsto (\mathcal{K}^\alpha)'(0)$ as pointwise limits of sequences of continuous functions on $\mathcal{J}_E$.

For this we first replace E by $E_g = E \cup \bigcup_{j=g+1}^{\infty}(a_j, b_j)$. A character $\alpha \in \mathcal{J}_E$ defines a character $\alpha_g \in \mathcal{J}_{E_g} = \mathcal{J}_g$ by restriction onto the subgroup of Γ_E generated by the closed curves $\gamma_1, \ldots, \gamma_g$ contained in Ω_E; this subgroup is isomorphic of Γ_{E_g}. It is convenient to identify $\gamma_1, \ldots, \gamma_g$ with the corresponding linear fractional transformations $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_g$ defined in (14). Recall that $\tilde{\gamma}_j$ is obtained by composing the reflection in $\mathbb{R}$ with the reflection in a certain circle $\mathbb{T}_j$ which is orthogonal to $\mathbb{T} = \partial U$ and which is the boundary of a disc U_j $(1 \leq j \leq g)$.

Next we introduce the conformal map $z_g : U_- \setminus \bigcup_{j=1}^{g} U_j \to \mathbb{C}_+$ which satisfies $z_g(-1) = 0$, $z_g(1) = 1$ and $z_g(0) = \infty$. Then z_g maps the set $\mathbb{T}_j \cap U$ onto an interval $(a_{j,g}, b_{j,g})$

which need not be equal to (a_j, b_j). However it is clear that, as $g \to \infty$, one has $a_{j,g} \to a_j$, $b_{j,g} \to b_j$ for each fixed $j = 1, 2, \ldots$. The point is that we have fixed the linear fractional transformations $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_g$ to be the same maps as in (14). Let $\tilde{\Gamma}_g$ be the Fuchsian group of linear fractional transformations generated by $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_g$. If Γ is the Fuchsian group defined previously, then $\tilde{\Gamma}_g \subset \Gamma$. We view α_g as a homomorphism from $\tilde{\Gamma}_g$ to the unit circle $\mathbb{R}/\mathbb{Z}$.

Now introduce the space $H^2_{\alpha_g} = H^2_g$ $(g = 1, 2, \ldots)$. We can view H^2_g in two ways. The first way is as a set of multivalued analytic functions on $\tilde{\Omega}_g = \mathbb{C} \cup \{\infty\} \backslash \tilde{E}_g$ where $\tilde{E}_g = [0, 1] \backslash \bigcup_{j=1}^{g} (a_{j,g}, b_{j,g})$. Let $\tilde{\mathcal{K}}^g$ be the solution of the corresponding extremal problem $(z \in \tilde{\Omega}_g)$. The second way is as the subset of $H^2(U)$ consisting of those functions f satisfying $f \circ \tilde{\gamma} = \alpha_g(\tilde{\gamma}) f$ for all $\tilde{\gamma} \in \tilde{\Gamma}_g$. We set $\tilde{\mathfrak{K}}^g(\zeta) = \tilde{\mathcal{K}}^g(z_g(\zeta))$ for $\zeta \in U$. We note that, if $P_{H^2_g}$ is the orthogonal projection of $H^2(U)$ onto H^2_g, then the following analogue of (15) is valid

$$\tilde{\mathfrak{K}}^g = P_{H^2_g} \mathbf{1} / || P_{H^2_g} \mathbf{1} ||. \tag{17_g}$$

We note further that, when H^2_g is viewed in this way, then

$$H^2_1 \supset H^2_2 \supset \cdots \supset H^2_g \supset \cdots \quad \text{and} \quad H^2_\alpha = \bigcap_{g=1}^{\infty} H^2_g. \tag{18}$$

We now arrive at the main point of this discussion.

Lemma 3.8 *With notation as above: $\tilde{\mathfrak{K}}^g(\zeta) \to \mathfrak{K}^\alpha(\zeta)$ uniformly for ζ in compact subsets of U.*

Proof First of all, it follows from a direct argument that the sequence of projections $\{P_{H^2_g} \mid g \geq 1\}$ converges strongly to $P_{H^2_\alpha}$. Thus $P_{H^2_g} \mathbf{1} \to P_{H^2_\alpha} \mathbf{1}$ in $H^2(U)$.

Next, a theorem of Widom [65] concerning Parreau-Widom domains states that, for each $\alpha \in \mathcal{J}_E$, $\mathfrak{K}^\alpha(\cdot)$ does not vanish identically; in other words, $P_{H^2_\alpha} \mathbf{1} \neq 0$. Of course this statement is true for each $\tilde{\mathfrak{K}}^g$ as well. Using (15) and (17_g), we see that $\{\tilde{\mathfrak{K}}^g\}$ converges to $\mathfrak{K}^\alpha$ in $H^2(U)$. This implies that $\{\tilde{\mathfrak{K}}^g\}$ converges to $\mathfrak{K}^\alpha$ uniformly on compact subsets of U, and this completes the proof. $\square$

We note that the classical Weierstrass theorem implies that $\dfrac{d^s}{d\zeta^s} \tilde{\mathfrak{K}}^g(\zeta) \to \dfrac{d^s}{d\zeta^s} \mathfrak{K}^\alpha(\zeta)$ uniformly on compact subsets of U for all $s = 0, 1, \ldots$.

This lemma allows one to prove that the map $\imath$ is of the first Baire class. First we need to introduce a map which is analogous to that used in the context of comb domains in [18]. Return to the map z_g defined earlier. There is another map z'_g of a similar sort which can be defined as follows. Namely, there are discs $U'_1, \ldots, U'_g$ with boundaries $\mathbb{T}'_1, \ldots, \mathbb{T}'_g$ which are orthogonal to $\mathbb{T} = \partial U$, and a conformal map $z'_g : U_- \setminus \bigcup_{j=1}^{g} U_j \to \mathbb{C}_+$ such that the image under z'_g of $\mathbb{T}'_j \cap U$ equals (a_j, b_j) $(1 \leq j \leq g)$. The discs $U'_1, \ldots, U'_g$ and the map z'_g are uniquely determined by specifying that $z'_g(-1) = 0$, $z'_g(1) = 1$, $z'_g(0) = \infty$.

Let us set $U_{-,g} = U_- \setminus \bigcup_{j=1}^{g} U_j$ and $U'_{-,g} = U_- \setminus \bigcup_{j=1}^{g} U'_j$. There is a unique conformal map h_g of $U_{-,g}$ onto $U'_{-,g}$ which satisfies $h_g(0) = 0$, $h'_g(0) > 0$. One can prove that $z'_g \circ h_g = z_g$.

Now return to Lemma 3.8. There is a divisor $\tilde{d}_g = \{(\tilde{y}_1, \tilde{\varepsilon}_1), \ldots (\tilde{y}_g, \tilde{\varepsilon}_g)\}$ with $\tilde{y}_j \in [a_{j,g}, b_{j,g}]$ such that (16) holds for $\tilde{\mathcal{K}}^g$. Let $y'_j \in [a_j, b_j]$ be the image of $\tilde{y}_j$ under h_g: $y'_j = h_g(\tilde{y}_j)$ $(1 \leq j \leq g)$. Also set $\varepsilon'_j = \tilde{\varepsilon}_j$, $1 \leq j \leq g$.

Proposition 3.9 *Let $\alpha \in \mathcal{J}_E$. For each $g = 1, 2, \ldots$ let $d'_g = \{(y'_1, \varepsilon'_1), \ldots, (y'_g, \varepsilon'_g)\}$ be the divisor constructed above. Let us identify d'_g with $d_g = \{(y'_1, \varepsilon'_1), \ldots, (y'_g, \varepsilon'_g), (b_{g+1}), \ldots\} \in \mathcal{D}_E$. Set*

$$\mathfrak{i}_g(\alpha - \alpha_c) = d_g \qquad (\alpha \in \mathcal{J}_E).$$

Then the map $\mathfrak{i}_g : \mathcal{J}_E \to \mathcal{D}_E$ is continuous, and $\mathfrak{i}_g(\alpha) \to \mathfrak{i}(\alpha)$ for all $\alpha \in \mathcal{J}_E$.

To put a bound in the present discussion, we are going to omit the proof of Proposition 3.9, which is based on Lemma 3.8 and the relations (19_1) and (19_2). The basic idea is to study the sequence of Jacobi operators determined by α_g via (19_1) and (19_2) for $g = 1, 2, \ldots$. Using Lemma 3.8 one proves the strong resolvent convergence of this sequence to the Jacobi operator determined by α. One shows that this implies that $\tilde{d}_g$ converges componentwise to $d = \mathfrak{i}(\alpha)$, and then that $d_g \to d$. At one point in the proof of the strong resolvent convergence one uses the Hypotheses 3.6 on the set E.

Let us now return to the Schrödinger situation, that is, we pull back the preceding constructions via the map ψ to the preceding problem with $E = [b_0, \infty)\backslash\bigcup_{j=1}^{\infty}(a_j, b_j)$, etc. In this way we obtain a right inverse $\mathfrak{i}$ of the Abel map $\mathfrak{a} : \mathcal{D}_E \to \mathcal{J}_E$, together with a sequence of continuous maps $\mathfrak{i}_g : \mathcal{J}_E \to \mathcal{D}_E$ such that $\mathfrak{i}_g(\alpha) = \mathfrak{i}(\alpha)$ for all $\alpha \in \mathcal{J}_E$. We must show that $\mathfrak{i}(\alpha + \delta x) = \tau_x(\mathfrak{i}(\alpha))$ for all $\alpha \in \mathcal{J}_E$, $x \in \mathbb{R}$. Here δ is the character defined previously, using the conformal map w of Ω_E onto a comb domain.

One way to do this is to pull back via ψ the intermediate construction of the intervals $(a_{j,g}, b_{j,g})$. We obtain certain intervals which we denote by $(\hat{a}_{j,g}, \hat{b}_{j,g})$; these are contained in $(b_0, 0)$, and one has that $\hat{a}_{j,g} \to a_j$, $\hat{b}_{j,g} \to b_j$ as $g \to \infty$ $(j \geq 1)$. Let us consider the set $\hat{\mathcal{Q}}_g$ of algebro-geometric potentials for which the spectrum is

$$\hat{E}_g = \mathbb{R}\backslash \bigcup_{j=1}^{g} (\hat{a}_{j,g}, \hat{b}_{j,g}).$$

The classical Abel map $\hat{\mathfrak{a}}_g$ maps $\hat{\mathcal{D}}_g = \mathcal{D}_{\hat{E}_g}$ homeomorphically onto $\hat{\mathcal{J}}_g = \mathcal{J}_{\hat{E}_g}$. The inverse map $\hat{\mathfrak{i}}_g$ maps $\hat{\mathcal{J}}_g$ homeomorphically onto $\hat{\mathcal{D}}_g$, which we identify via the divisor map with $\hat{\mathcal{Q}}_g$ in the next lines. We also use Hypotheses 3.6 and Remark 3.4 (2) to identify $\mathcal{D}_E$ with $\mathcal{Q}_E$ via the divisor map π.

Now one can show that, due to the fact that $\tilde{\Gamma}_g$ is a subgroup of Γ in the previous construction, it is the case that

$$\hat{\mathfrak{a}}_g(\tau_x(q_g)) = \hat{\mathfrak{a}}_g(q_g) + \delta_g x \qquad (q_g \in \hat{\mathcal{Q}}_g,\ x \in \mathbb{R}).$$

Here δ_g is the restriction of the character δ to $\tilde{\Gamma}_g \cong \Gamma_{\hat{E}_g}$. It follows that

$$\hat{\mathfrak{i}}_g(\alpha_g + \delta x) = \tau_x(\hat{\mathfrak{i}}_g(\alpha_g)) \in \hat{\mathcal{Q}}_g$$

for all $\alpha \in \mathcal{J}_E$; we have written α_g for the restriction of α to $\tilde{\Gamma}_g \cong \Gamma_{\hat{E}_g}$. Next let $g \to \infty$: if $\hat{\mathfrak{i}}_g(\alpha_g) = q_g$, then one can show that the sequence $\{q_g\}$ converges to an element $q \in \mathcal{Q}_E$ (by modifying a bit the argument used in Remark 3.4 (3)). Moreover, using the continuity of the flow $\{\tau_x\}$ on $\mathcal{R}$: $\hat{\mathfrak{i}}_g(\alpha_g + \delta x) \to \tau_x(q)$ as $g \to \infty$. Finally $\hat{\mathfrak{i}}_g(\alpha_g) \to \hat{\mathfrak{i}}(\alpha) = q$ and $\hat{\mathfrak{i}}_g(\alpha_g + \delta_g x) \to \hat{\mathfrak{i}}(\alpha + \delta x) = \tau_x(q)$, so in fact we have proved:

Proposition 3.10 *Let $\alpha \in \mathcal{J}_E$ and $x \in \mathbb{R}$; then $\hat{\mathfrak{i}}(\alpha + \delta x) = \tau_x(\hat{\mathfrak{i}}(\alpha))$.*

An alternative proof can be based on the method of Gesztesy–Yuditskii ([18], pp 496–498).

Let us now draw conclusions concerning the dynamics of $\{\tau_x\}$ on $\mathcal{Q}_E$ when $\mathfrak{i}$ is of the first Baire class but is not continuous. We will assume that the flow $(\mathcal{J}_E, \delta)$ which is defined by the map $(\alpha, x) \mapsto \alpha + \delta x$ is minimal; this can be arranged by choosing the points a_j, b_j in an appropriate way.

Theorem 3.11 *Let* $E = [b_0, \infty)\setminus\bigcup_{j=1}^{\infty}(a_j, b_j)$ *satisfy Hypotheses 3.6. Suppose that the flow* $(\mathcal{J}_E, \delta)$ *is minimal. Then there exists a minimal almost automorphic subflow* $(M, \{\tau_x\})$ *of* $(\mathcal{Q}_E, \{\tau_x\})$ *which is not almost periodic.*

Proof First we recall that, when Hypotheses 3.6 are valid, the divisor map $\pi : \mathcal{Q}_E \to \mathcal{D}_E$ is a homeomorphism.

We consider the right inverse $\mathfrak{i} : \mathcal{J}_E \to \mathcal{D}_E$ and compose it with π^{-1}: in this way we obtain a (discontinuous) flow homomorphism which is of the first Baire class. We write $\mathfrak{i}$ instead of $\pi^{-1} \circ \mathfrak{i}$ in what follows. Using Proposition 3.10, we see that the set $\mathcal{Q}_0 = cls\{\mathfrak{i}(\alpha) \mid \alpha \in \mathcal{J}_E\}$ is a compact $\{\tau_x\}$-invariant subset of $\mathcal{Q}_E$. The Abel map $\mathfrak{a} \circ \pi : \mathcal{Q}_0 \to \mathcal{J}_E$ is a flow homomorphism which is clearly surjective. We again abuse notation and write $\mathfrak{a}$ instead of $\mathfrak{a} \circ \pi$.

By a well-known theorem, the map $\mathfrak{i}$ has a residual set of continuity points [3]. Let α_0 be a continuity point of $\mathfrak{i}$, and let $q_0 = \mathfrak{i}(\alpha_0)$. Note that, if $\{x_n\} \subset \mathbb{R}$ is a sequence such that $\alpha_0 + \delta x_n \to \alpha_0$ in $\mathcal{J}_E$, then $\mathfrak{i}(\alpha_0 + \delta x_n) \to q_0$ in $\mathcal{Q}_0$. Using this observation together with the continuity in α_0 of $\mathfrak{i}$ and the minimality of $(\mathcal{J}_E, \delta)$, one can show that q_0 is a Birkhoff almost periodic point of $\{\tau_x\}$. This is equivalent to saying that the orbit closure $M = cls\{\tau_x(q_0) \mid x \in \mathbb{R}\} \subset \mathcal{Q}_0$ is minimal. It is clear that M contains $\mathfrak{i}(\alpha)$ whenever $\alpha \in \mathcal{J}_E$ is a continuity point of $\mathfrak{i}$.

Let us also note that, if $\alpha \in \mathcal{J}_E$ is a continuity point of $\mathfrak{i}$, then the fiber in $\mathcal{Q}_0$ over α of $\mathfrak{a}$ reduces to $\mathfrak{i}(\alpha)$: $\{q \in \mathcal{Q}_0 \mid \mathfrak{a}(q) = \alpha\} = \{\mathfrak{i}(\alpha)\}$. This property holds a fortiori if M replaces $\mathcal{Q}_0$. This implies that M is an almost automorphic extension of $(\mathcal{J}_E, \delta)$.

Let us now suppose for contradiction that $(M, \{\tau_x\})$ is Bohr almost periodic. Then there is a well-known method to impose the structure of a compact Abelian topological group on M so that M contains a dense subgroup isomorphic to $\mathbb{R}$; moreover τ_x becomes translation by $x \in \mathbb{R} \subset \mathbb{M}$ with respect to this group structure. In this way, $\mathfrak{a}$ extends to a homomorphism of the compact Abelian topological groups M and $\mathcal{J}_E$. Since $\mathfrak{a}^{-1}(\alpha)$ has cardinality one for some $\alpha \in \mathcal{J}_E$, it has cardinality one for all $\alpha \in \mathcal{J}_E$. But then $\mathfrak{i}$ agrees on a residual subset of $\mathcal{J}_E$ with a continuous map $\hat{\mathfrak{i}} : \mathcal{J}_E \to \mathcal{Q}_E$, which means that $\mathfrak{a}$ is a homeomorphism. That contradicts one of Hypotheses 3.6. So $(M, \{\tau_x\})$ cannot be Bohr almost periodic. This completes the proof of Theorem 3.11. □

It follows from the proof of Theorem 3.11 that, if $q \in \mathcal{Q}_E$, then q is not almost periodic. We can relate this observation to the Kotani-Last conjecture, according to which a stationary ergodic family of Schrödinger potentials with absolutely continuous spectrum should consist a.e. of almost periodic potentials. This is not the case for $q \in \mathcal{Q}_E$. It seems that we obtain this conclusion with less information on character-automorphic Hardy spaces than is needed in [8,64].

As noted previously, it is shown in [8] that $\mathcal{Q}_E$ contains an almost automorphic minimal set which is not almost periodic (although it seems that the authors do not state this fact explicitly). One may ask whether $M = \mathcal{Q}_E$. This cannot be excluded from what has been said so far. However, the developments in [64] and [8] seem to indicate that $M \neq \mathcal{Q}_E$. Moreover, it is proved in [8,64] that M is the unique minimal subset of $\mathcal{Q}_E$. So it seems that one has a sort of "Denjoy picture" for the flow homomorphism $\mathfrak{a} : \mathcal{Q}_E \to \mathcal{J}_E$ (think of a Denjoy flow on a 2-torus).

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