SOLUTIONS OF ELLIPTIC EQUATIONS WITH A LEVEL SURFACE PARALLEL TO THE BOUNDARY: STABILITY OF THE RADIAL CONFIGURATION

By

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Abstract. A positive solution of a homogeneous Dirichlet boundary value problem or initial-value problems for certain elliptic or parabolic equations must be radially symmetric and monotone in the radial direction if *just one* of its level surfaces is parallel to the boundary of the domain. Here, for the elliptic case, we prove the stability counterpart of that result. We show that if the solution is almost constant on a surface at a fixed distance from the boundary, then the domain is almost radially symmetric, in the sense that is contained in and contains two concentric balls B_{r_e} and B_{r_i}, with the difference $r_e - r_i$ (linearly) controlled by a suitable norm of the deviation of the solution from a constant. The proof relies on and elaborates arguments developed by Aftalion, Busca, and Reichel.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. It has been noticed in [MS2]-[MS3], [Sh], and [CMS] that a positive solution of a homogeneous Dirichlet boundary value problem or initial-boundary value problem for certain elliptic or parabolic equations must be radially symmetric (and Ω must be a ball) if *just one* of its level surfaces is parallel to $\partial\Omega$ (i.e., if the distance of its points from $\partial\Omega$ remains constant).

This property was first identified in [MS2], motivated by the study of invariant isothermic surfaces of a nonlinear non-degenerate fast diffusion equation (designed on the heat equation), and was used to extend to nonlinear equations the symmetry results obtained in [MS1] for the heat equation. The proof hinges on the method of moving planes developed by J. Serrin in [Se] and based on A. D. Aleksandrov's reflection principle [Al]. Under slightly different assumptions and

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with a different proof (also based on the method of moving planes), a similar result was independently obtained in [Sh]; see also [GGS]. Further extensions can be found in [MS3] and [CMS].

To clarify matters, we consider a simple (possibly the simplest) situation. Let G be a C^1-smooth domain in $\mathbb{R}^N$, denote by B_R the ball centered at the origin of radius R [1], and define the **Minkowski sum** of G and B_R

$$\Omega = G + B_R = \{y + z : y \in G,\ |z| < R\}.$$

Consider the solution $u = u(x)$ of the **torsion boundary value problem**

$$-\Delta u = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

As shown in the aforementioned references, if there exists a positive constant c such that

$$u = c \text{ on } \partial G, \tag{1.2}$$

then G and Ω must be concentric balls.

The aim of this paper is to investigate the stability of the radial configuration. In other words, we suppose that u is *almost constant* on ∂G by assuming that the semi-norm

$$[u]_{\partial G} = \sup_{\substack{x,y\in\partial G,\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|}$$

is *small*, and want to show quantitatively that Ω is close to a ball. The following theorem is a result in this direction that concerns the torsion problem (1.1).

Theorem 1.1. *Let G be a bounded domain in $\mathbb{R}^N$ with boundary ∂G of class $C^{2,\alpha}$, $0 < \alpha \leq 1$, and set $\Omega = G + B_R$ for some $R > 0$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be the solution of* (1.1). *There exist constants $\varepsilon, C > 0$ such that if $[u]_{\partial G} \leq \varepsilon$, then there exist two concentric balls B_{r_i} and B_{r_e} such that*

$$B_{r_i} \subset \Omega \subset B_{r_e} \quad \text{and} \tag{1.3}$$

$$r_e - r_i \leq C\, [u]_{\partial G}. \tag{1.4}$$

The constants ε and C depend only on N, the $C^{2,\alpha}$-regularity of ∂G (see [ABR, Remark 1, p. 909], the diameter of G and, most importantly, R.

A result similar to Theorem 1.1 was proved by Aftalion, Busca, and Reichel in [ABR]. There, under further suitable assumptions, the stability estimate

$$r_e - r_i \leq C \,\big|\log \|u_\nu - d\|_{C^1(\partial\Omega)}\big|^{-1/N} \tag{1.5}$$

[1] In the sequel, $B_R(x)$ denotes the ball of radius R centered at x.

is proved, where u_ν denotes the (exterior) normal derivative of the solution of (1.1), d is a constant, and $\|\cdot\|_{C^1(\partial\Omega)}$ is the usual C^1 norm on ∂G; (1.5) is the quantitative version of Serrin's symmetry result [Se], which states that the solutions of (1.1) whose normal derivative is constant on $\partial\Omega$, i.e.,

$$u_\nu = d \ \text{ on } \partial\Omega, \tag{1.6}$$

must be radially symmetric (and Ω must be a ball). In [ABR], (1.5) is a particular case of a similar estimate that holds for general semilinear equations, i.e., when solutions of the problem

$$-\Delta u = f(u) \ \text{ and } \ u \geq 0 \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega, \tag{1.7}$$

are considered (here, f is a locally Lipschitz continuous function).

The logarithmic dependence appears in (1.5) due to the method of proof employed, which is based on the idea of refining the method of moving planes from a quantitative point of view. As that method is based on the maximum (or comparison) principle, its quantitative counterpart relies on Harnack's inequality and quantitative versions of Hopf's boundary lemma and Serrin's corner lemma (which involve the second derivatives of the solution on $\partial\Omega$). The exponential dependence of the constant involved in Harnack's inequality leads to the logarithmic dependence in (1.5).

Estimate (1.5) was improved significantly in [BNST2] (see also [BNST1]), but only for the case of the torsion problem (1.1); a version of this result is also available for Monge-Ampère equations [BNST3]. In [BNST2], (1.5) is enhanced in two respects: the logarithmic dependence on the right-hand side of (1.5) is replaced by a power law of Hölder type, and an estimate is also given in which the C^1-norm is replaced by the L^1-norm. The results in [BNST1] are obtained by remodeling in a quantitative manner Weinberger's proof [We] of Serrin's result, which is grounded on integral formulas such as the Rellich-Pohozaev identity. Unfortunately, that approach does not seem to work in our setting, since the overdetermining condition (1.2) does not conveniently match the underlying integral formulas of variational type.

The proof of Theorem 1.1 adapts the approach used in [ABR] but, unlike that paper, results in the *linear* estimate (1.3)-(1.4). The reason for this gain can be ascribed to the fact that, in our setting, we deal with the values of u on ∂G, which lie in the *interior* of Ω, and dispense with the use of estimates *up to the boundary of* $\partial\Omega$, which account for the logarithmic behavior in (1.5). However, the benefit obtained in Theorem 1.1 has a cost: the constant C in (1.4) blows up exponentially as $R \to 0$.

The proof can be outlined as follows. For any fixed direction ω, the method of moving planes determines a hyperplane π orthogonal to ω and in *critical*[2] *position* and a domain X contained in G and symmetric about π. Since $\overline{X} \subseteq \overline{G} \subset \Omega$, we can use interior and boundary Harnack inequalities to estimate in terms of $[u]_{\partial G}$ the values of the harmonic function $w(x) = u(x') - u(x)$ (here, x' is the mirror image in π of x) in the half of X where w is non-negative. Such an estimate gives a bound on the distance of ∂X from ∂G; observe that such a bound does not depend on the direction ω. By repeating this argument for N orthogonal directions, an approximate center of symmetry — at which the two balls B_{r_i} and B_{r_e} in (1.3) are centered — can be determined. The radii of these balls then satisfy (1.4).

We conclude this introduction with some remarks on the relationship between problem (1.1)-(1.2) and Serrin's problem (1.1)-(1.6). In both cases, a solution exists (if and) only if the domain is a ball. It is natural to ask whether the symmetry obtained for one of these problems implies that for the other problem.

We begin by observing that conditions (1.1)-(1.2) neither imply nor are implied by conditions (1.1)-(1.6), directly. However, (1.6) seems to be a stronger constraint in the sense that in order to obtain it, one has to require that (1.2) hold (with different constants) for at least a sequence of parallel surfaces clustering around $\partial\Omega$. On the other hand, if (1.1)-(1.6) holds, we cannot claim that its solution u is constant on surfaces parallel to $\partial\Omega$, but we can surely say that the oscillation of u on a parallel surface becomes smaller the closer the surface is to $\partial\Omega$; in fact, an easy Taylor-expansion argument shows that if $\Gamma_t = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) = t\}$, then

$$\max_{\Gamma_t} u - \min_{\Gamma_t} u = O(t^2) \ \text{ as } \ t \to 0^+. \tag{1.8}$$

Theorem 1.1 suggests the possibility that Serrin's symmetry result may be obtained, so to speak, by *stability*, via (1.4) (conveniently remodeled) and (1.8) in the limit as $t \to 0^+$. This strategy was proved to be successful in a case study by the first two authors [CM]. If the strategy were successful for fairly general domains, the method of moving planes could be employed to prove Serrin's symmetry result with no need of the aforementioned corner lemma and under weaker assumptions on $\partial\Omega$ and u; see also [Sh]. To our knowledge, this issue has been addressed only in [Pr] (for domains with one possible corner or cusp) and in [GL] (by arguments based on [We]). Currently, the blowing up dependence of the constant C in (1.4) on the distance of the relevant parallel surface from $\partial\Omega$ is an obstacle to this line of reasoning.

The paper is organized as follows. In Section 2, we introduce some notation and, for the reader's convenience, recall the proof of the relevant symmetry result.

[2]See Section 2 for the meaning of this word.

In Section 3, we carry out the proof of Theorem 1.1. In Section 4, we explain how our result can be extended to the classical case of semilinear equations (1.7).

2 Parallel surfaces and symmetry

To help the reader understand the proof of Theorem 1.1, here we summarize the arguments developed in [MS2], [MS3], and [CMS] to prove the symmetry of a domain Ω admitting a solution u of (1.1) and (1.2). This also gives us an opportunity to introduce notation.

In this section, we assume that the boundary of G is of class C^1; this assumption guarantees that the method of moving planes can be implemented; see [Fr].

For a unit vector $\omega \in \mathbb{R}^N$ and parameter $\lambda \in \mathbb{R}$, define

$$(2.1)\qquad \begin{aligned} \pi_\lambda &= \{x \in \mathbb{R}^N : x \cdot \omega = \lambda\} && \text{a hyperplane orthogonal to } \omega,\\ A_\lambda &= \{x \in A : x \cdot \omega > \lambda\} && \text{the right-hand cap of a set } A,\\ x^\lambda &= x - 2(x \cdot \omega - \lambda)\,\omega && \text{the reflection of } x \text{ about } \pi_\lambda,\\ A^\lambda &= \{x \in \mathbb{R}^N : x^\lambda \in A_\lambda\} && \text{the reflected cap about } \pi_\lambda. \end{aligned}$$

Define the **extent of G in direction** ω by $\sup\{x \cdot \omega : x \in G\}$, and denote it by Λ. Then $G^\lambda \subset G$ if $\lambda < \Lambda$ is close to Λ; see [Fr, Theorem 5.7, p. 149]. Set

$$(2.2)\qquad m = \inf\{\mu : G^\lambda \subset G \text{ for all } \lambda \in (\mu, \Lambda)\}.$$

Then for $\lambda = m$ at least one of the following two cases occurs:

(i) G^λ becomes internally tangent to ∂G at some point $P \in \partial G \setminus \pi_\lambda$;

(ii) π_λ is orthogonal to ∂G at some point $Q \in \partial G \cap \pi_\lambda$.

It follows from an easy adaptation of [MS3, Lemmas 2.1 and 2.2] that if $\Omega = G + B_R = \{x + y : x \in G, \ y \in B_R\}$, then $G^\lambda \subset G$ implies $\Omega^\lambda \subset \Omega$.

Theorem 2.1. *Let G be a bounded domain in $\mathbb{R}^N$ with boundary ∂G of class C^1 and set $\Omega = G + B_R$ for some $R > 0$. Suppose there exists $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfying* (1.1) *and* (1.2). *Then G (and hence Ω) is a ball, and u is radially symmetric.*[3]

Proof. Assume that G is not a ball. Let $\omega \in \mathbb{S}^{N-1}$ and consider the function defined on the right-hand cap Ω_m at critical position $\lambda = m$ by

$$(2.3)\qquad w^m(x) = u(x^m) - u(x), \ x \in \Omega_m.$$

[3]Note that if G is a spherical annulus in $\mathbb{R}^N$, then the solution of (1.1) is radially symmetric but does not satisfy (1.2).

It is clear that w^m satisfies

$$\Delta w^m = 0 \text{ and } w^m \geq 0 \text{ on } \partial\Omega_m. \tag{2.4}$$

Moreover, since G is not a ball, there is at least one direction ω such that w^m is not identically 0 on $\partial\Omega_m$. Hence, by the Strong Maximum Principle,

$$w^m > 0 \text{ in } \Omega_m. \tag{2.5}$$

Also, applying Hopf's boundary point lemma to points in $\partial\Omega^m \cap \pi_m$, we have

$$\frac{\partial w^m}{\partial\omega} > 0 \text{ on } \partial\Omega_m \cap \pi_m,$$

and hence

$$\frac{\partial u}{\partial\omega} < 0 \text{ on } \partial\Omega_m \cap \pi_m, \tag{2.6}$$

since

$$\frac{\partial w^m}{\partial\omega} = -2\frac{\partial u}{\partial\omega} \text{ on } \Omega \cap \pi_m.$$

If (i) holds, we get a contradiction to (2.5) by observing that

$$w^m(P) = u(P^m) - u(P) = 0,$$

since (1.2) holds and both P and P^m belong to ∂G. If (ii) holds, ω is tangent to ∂G at Q and (1.2) implies that

$$\frac{\partial u}{\partial\omega}(Q) = 0,$$

contradicting (2.6).

Thus, we conclude that G must be a ball, and hence $\Omega = G + B_R$ is also a ball.□

3 Stability estimates for the torsion problem

In this section, we present a proof of Theorem 1.1 that relies on the ideas in [ABR] and the Harnack-type and tangential stability estimates contained respectively in Lemmas 3.1 and 3.2 below.

The following two lemmas analyze the two critical situations (i) and (ii) of Section 2 from a quantitative point of view. We consider two domains D_1 and D_2 in $\mathbb{R}^N$ containing the origin and such that $D_1 \subset D_2$. We also assume that there exists $R > 0$ such that $B_R(x) \subset D_2$ for all $x \in \overline{D}_1$ and define

$$\begin{aligned} D_j^+ &= \{x \in D_j : x_1 > 0\}, \ j = 1, 2, \\ E_R &= \{x \in D_1 + B_{R/2} : x_1 > R/2\}. \end{aligned} \tag{3.1}$$

Without loss of generality, we can assume that D_1^+ is connected; this is easily seen to imply that E_R is also connected.

Lemma 3.1. *Assume that* $w \in C^0(D_2) \cap C^2(D_2^+)$ *satisfies*

$$\Delta w = 0 \ \text{ and } \ w \geq 0 \ \text{ in } \ D_2^+, \quad w = 0 \ \text{ on } \ \partial D_2^+ \cap \{x_1 = 0\},$$

and let $z = (z_1, \dots, z_N) \in \overline{D_1^+}$. *Then, for every* $x \in \overline{E_R}$,

$$w(x) \leq C\,\frac{w(z)}{z_1} \qquad \text{if } z_1 > 0, \tag{3.2}$$

$$w(x) \leq C\,\frac{\partial w}{\partial x_1}(z) \quad \text{if } z_1 = 0, \tag{3.3}$$

where C *is a constant depending explicitly on* N*, the diameters of* D_1 *and* D_2*, and* R.

Proof. We can always assume that $w > 0$ in D_2^+. We distinguish the three cases: (i) $z_1 \geq R/2$; (ii) $0 < z_1 < R/2$; (iii) $z_1 = 0$.

(i) Let $B_r(x_0)$ be a ball contained in D_2^+. Harnack's inequality (see [GT, Problem 2.6, p. 29]) gives

$$r^{N-2}\,\frac{r - |x - x_0|}{(r + |x - x_0|)^{N-1}}\,w(x_0) \leq w(x) \leq r^{N-2}\,\frac{r + |x - x_0|}{(r - |x - x_0|)^{N-1}}\,w(x_0) \tag{3.4}$$

for all $x \in B_r(x_0)$. By the monotonicity of the two ratios in $|x - x_0|$, we have

$$2^{N-2}3^{1-N}\,w(x_0) \leq w(x) \leq 3 \cdot 2^{N-2} w(x_0) \quad \text{for all } x \in \overline{B_{r/2}(x_0)}. \tag{3.5}$$

Let $y \in \overline{E_R}$ be such that $w(y) = \sup_{E_R} w$. Since $\operatorname{dist}(y, \partial D_2^+) \geq R/2$ and $(D_1 + B_{R/2}) \cap \{x_1 > s\}$ is connected for each $0 \leq s \leq R/2$, there exists, by [ABR, Lemma A.1], a chain $\{B_{R/4}(p_i)\}_{i=1}^n$ of n pairwise disjoint balls such that

$$\operatorname{dist}(p_i, \partial D_2^+) \geq R/2, \quad i = 1, \dots, n,$$
$$y, z \in \bigcup_{i=1}^n \overline{B_{R/4}(p_i)},$$

and

$$\overline{B_{R/4}(p_i)} \cap \overline{B_{R/4}(p_{i+1})} \neq \varnothing, \quad i = 1, \dots, n-1.$$

Since $B_{R/2}(p_i) \subset D_2^+$, applying (3.5) to each ball and combining the resulting inequalities yields

$$\sup_{E_R} w \leq (3 \cdot 2^{N-2})^{n+1} w(z). \tag{3.6}$$

Since $z_1 \leq \operatorname{diam} D_1^+ \leq \operatorname{diam} D_1$, it follows easily that

$$\sup_{E_R} w \leq (\operatorname{diam} D_1)(3 \cdot 2^{N-2})^{n+1} \frac{w(z)}{z_1}. \tag{3.7}$$

An upper bound for the optimal number of balls n is clearly $(2 \operatorname{diam} D_2)^N R^{-N}$; the last inequality then implies (3.2).

(ii) Since $0 < z_1 < R/2$, the point $\hat{z} = (R/2, z_2, \ldots, z_N)$ is such that $z \in B_{R/2}(\hat{z}) \subset D_2^+$. Applying (3.4) with $x_0 = \hat{z}$, $x = z$ and $r = R/2$, we obtain

$$\left(\frac{R}{2}\right)^{N-2} \frac{z_1}{(R - z_1)^{N-1}}\, w(\hat{z}) \leq w(z);$$

hence $w(\hat{z}) \leq 2^{N-2} R w(z)/z_1$. Observing that $\hat{z} \in \overline{E_R}$, we use the same Harnack-type argument as used for case (i) (see formula (3.6)) to obtain

$$\sup_{E_R} w \leq (3 \cdot 2^{N-2})^{[1+(2 \operatorname{diam} D_2)^N / R^N]} w(\hat{z}),$$

and hence

$$\sup_{E_R} w \leq 2^{N-2} R (3 \cdot 2^{N-2})^{[1+(2 \operatorname{diam} D_2)^N / R^N]} \frac{w(z)}{z_1}. \tag{3.8}$$

(iii) Taking the limit as $z_1 \to 0$ in (3.8) and noting that $w = 0$ for $x_1 = 0$, we obtain (3.3) in the form

$$\sup_{E_R} w \leq 2^{N-2} R (3 \cdot 2^{N-2})^{[1+(2 \operatorname{diam} D_2)^N / R^N]} \frac{\partial w}{\partial x_1}(z). \tag{3.9}$$

An inspection of (3.7), (3.8), and (3.9) informs us that the constant C in (3.2) and (3.3) is given by

$$C = 3 \max(2^{N-2} R, \operatorname{diam} D_1) 2^{N-2} C_N^{(\operatorname{diam} D_2/R)^N} \quad \text{with} \quad C_N = 3^{2^N} 2^{(N-2)2^N}. \tag{3.10}$$

□

Now, we extend estimates (3.2) and (3.3) to the case x is any point in $\overline{D_1^+} \setminus \{x_1 = 0\}$.

Lemma 3.2. *Let D_1, D_2, R, w and z be as in Lemma 3.1. Then, for all $x \in \overline{D_1^+} \setminus \{x_1 = 0\}$,*

$$w(x) \leq M\, C \frac{w(z)}{z_1}, \qquad \text{if } z_1 > 0,$$
$$w(x) \leq M\, C \frac{\partial w}{\partial x_1}(z), \qquad \text{if } z_1 = 0,$$

where C is given by (3.10) *and M is a constant depending only on N.*

Proof. Clearly, $w(x) \le \sup_{E_R} w$ for $x \in \overline{D_1^+} \cap \overline{E_R}$.

For $x \in \overline{D_1^+}$ and $0 < x_1 < R/2$, we estimate $w(x)$ in terms of $w(\hat{x})$ with $\hat{x} = (R/2, x_2, \ldots, x_N) \in \overline{E_R}$ using the "boundary Harnack inequality" [CS, Theorem 11.5] to obtain

$$\sup_{B^+_{R/2}(x_0)} w \le M\, w(\hat{x}); \tag{3.11}$$

here $x_0 = (0, x_2, \ldots, x_N)$ and $M \ge 1$ is a constant depending only on N. Since $M \ge 1$ and $w(\hat{x}) \le \sup_{E_R} w$, Lemma 3.1 yields the conclusion. □

Notice that since ∂G is of class $C^{2,\alpha}$, it satisfies a uniform interior ball condition of (optimal) radius $\rho > 0$ at each point of the boundary. The following result is our analog of [ABR, Proposition 1].

Theorem 3.3. *Let Ω and G be as in Theorem 1.1 and $u \in C^{2,\alpha}(\Omega)$ be a solution of* (1.1). *Let $\omega \in \mathbb{R}^N$ be a unit vector and G_m be the maximal cap of G in the direction ω as defined by* (2.1) *and* (2.2). *Then for (a connected component of) G_m, we have*

$$w^m \le MC\,[u]_{\partial G} \quad in \quad G_m, \tag{3.12}$$

where w^m is defined by (2.3) *and C is an explicit constant depending on N, R, the diameter of G, and the C^2-regularity of ∂G.*

Proof. We adopt the same notation introduced for the proof of Theorem 2.1 and assume, without loss of generality, that $\omega = e_1$. The function $w = w^m$ satisfies (2.4) and $w = 0$ on $\partial G_m \cap \pi_m$. Let us assume that case (i) of the proof of Theorem 2.1 occurs. Without loss of generality, we use the same symbols G_m and Ω_m to denote, respectively, the connected component of G_m which intersects $B_R(P^m)$ and the corresponding connected component of Ω_m. Modulo a translation in the direction of e_1, we can apply Lemma 3.2 with $D_1^+ = G_m$ and $D_2^+ = \Omega_m$ and $z = P^m$ to obtain

$$w \le MC' \frac{w(P^m)}{\operatorname{dist}(P^m, \pi_m)} \quad \text{in} \quad G_m, \tag{3.13}$$

where C' is computed by means of (3.10), viz.,

$$C' = \max(2^{N-2}R, \operatorname{diam} G)\, C_N^{(2+\operatorname{diam} G/R)^N};$$

here, we have used the fact that $\operatorname{diam} \Omega = 2R + \operatorname{diam} G$.

To estimate $w(P^m)/\operatorname{dist}(P^m, \pi_m)$, we note that $|P - P^m| = 2 \operatorname{dist}(P^m, \pi_m)$ and distinguish two cases.

Case $|P - P^m| \geq \rho$. Since P and P^m lie on ∂G,

$$w(P^m) = u(P) - u(P^m) \leq \operatorname{diam}(G)\,[u]_{\partial G},$$

and we easily obtain

$$\frac{w(P^m)}{\operatorname{dist}(P^m, \pi_m)} \leq \frac{2\operatorname{diam}(G)}{\rho}\,[u]_{\partial G}.$$

Case $|P - P^m| < \rho$. Every point of the segment joining P to P^m is at distance less than ρ from a connected component of ∂G. The curve γ obtained by projecting that segment on ∂G has length bounded by $C''\,|P - P^m|$, where C'' is a constant depending on ρ and on the regularity of ∂G. An application of the Mean Value Theorem to the function u restricted to γ gives that $u(P) - u(P^m)$ can be estimated by the length of γ times the maximum of the tangential gradient of u on ∂G. Thus

$$w(P^m) \leq 2\,C''\operatorname{dist}(P^m, \pi_m)\,[u]_{\partial G}. \tag{3.14}$$

From (3.10) and (3.13)–(3.14) we then get (3.12).

Now, let us assume that case (ii) of the proof of Theorem 2.1 occurs. Again, without loss of generality, we denote by the same symbol G_m the connected component of G_m which intersects $B_R(Q)$, and Ω_m accordingly. Then, applying Lemma 3.2 with $D_1^+ = G_m$, $D_2^+ = \Omega_m$ and $z = Q$, we obtain $w(x) \leq MC\partial w(Q)/\partial\omega$ for all $x \in G_m$. Since ω belongs to the tangent hyperplane to ∂G at Q, we again obtain (3.12).

An inspection of the calculations in this proof shows us that the constant C in (3.12) can be chosen as

$$C = \max\left(1, \frac{2\operatorname{diam} G}{\rho}, 2\rho\, C''\right)\max\left(R, \frac{\operatorname{diam} G}{2^{N-2}}\right) C_N^{[2+\frac{\operatorname{diam} G}{R}]^N}. \tag{3.15}$$

From here on, the proof of Theorem 1.1 follows the arguments used in [ABR, Sections 3 and 4] with one major change (we use Theorem 3.3 instead of [ABR, Proposition 1]) and some minor changes that we sketch below. The key idea is to use the smallness of w^m in (the relevant connected component of) G_m to show that G is almost equal to the symmetric open set X which is defined as the interior of $G_m \cup G^m \cup (\partial G_m \cap \pi_m)$. In order to do that, we need a priori bounds on u from below in terms of the distance function from ∂G. Such bounds are derived in [ABR, Proposition 4], where it is shown that if $u = 0$ on $\partial\Omega$, then there exist positive constants K_1 and K_2 such that

$$K_1 \operatorname{dist}(x, \partial\Omega) \leq u(x) \leq K_2 \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \Omega; \tag{3.16}$$

$1/K_1$ and K_2 are bounded by a constant which depends on the diameter of Ω and the $C^{2,\alpha}$-regularity of $\partial\Omega$.

In our situation, unlike in [ABR], the symmetrized set X is the reflection of a connected component of G, compactly contained in Ω at distance R from $\partial\Omega$, so we need not extend the relevant estimates up to the boundary of Ω. Thus we obtain a better rate of stability. On the other hand, since we are dealing with the distance from ∂G instead of that from $\partial\Omega$ and u, which *is not constant on* ∂G, we need a stability version of the first inequality in (3.16). This is given in the following lemma.

Lemma 3.4. *There exists a positive constant K depending on* diam G *and the $C^{2,\alpha}$ regularity of ∂G such that*

$$K\,\text{dist}(x,\partial G) + \min_{\partial G} u \le u(x), \tag{3.17}$$

for every $x \in \overline{G}$. Moreover, if $x \in \partial X$, then

$$\text{dist}(x,\partial G) \le C_*[u]_{\partial G}, \tag{3.18}$$

with

$$C_* = \frac{C + \text{diam}(G)}{K}, \tag{3.19}$$

and where C is given by (3.15)*.*

Proof. Let v be a solution of

$$\Delta v = -1 \ \text{ in } \ G, \qquad v = \min_{\partial G} u \ \text{ on } \ \partial G.$$

By the comparison principle, $v \le u$ on $\overline{G}$. Since $v - \min_{\partial G} u = 0$ on ∂G, applying (3.16) to ∂G instead of $\partial\Omega$ yields

$$K\,\text{dist}(x,\partial G) \le v(x) - \min_{\partial G} u \ \text{ for any } \ x \in G,$$

where we have set $K = K_1$. Since $v \le u$, we obtain (3.17).

Now we prove (3.18). If $x \cdot \omega \ge m$, then $x \in \partial G$ and (3.18) clearly holds. If $x \cdot \omega < m$, then $x^m \in \partial G$ and, by Theorem 3.3,

$$u(x) = w^m(x^m) + u(x^m) \le C\,[u]_{\partial G} + u(x^m).$$

Since $x^m \in \partial G$,

$$u(x^m) - \min_{\partial G} u \le \text{diam}(G)\,[u]_{\partial G}.$$

These last two inequalities give

$$u(x) - \min_{\partial G} u \leq (C + \operatorname{diam}(G))[u]_{\partial G}$$

which, together with (3.17) gives (3.18) immediately. □

For $s > 0$, let $G^s_{\shortparallel}$ be the subset of G **parallel** to ∂G at distance s, i.e.,

$$G^s_{\shortparallel} = \{x \in G : \operatorname{dist}(x, \partial G) > s\}.$$

Note that $G^s_{\shortparallel}$ is connected for $s < \rho/2$; indeed, any path in G connecting any two points $x, y \in G^s_{\shortparallel}$ can be moved inwards into $G^s_{\shortparallel}$ by the normal field on ∂G.

Our next result is crucial to the proof of Theorem 1.1.

Theorem 3.5. *Suppose that* $[u]_{\partial G} < \rho/4C_*$. *Then*

$$G^s_{\shortparallel} \subset X \subset G \tag{3.20}$$

for all $s \in (C_*[u]_{\partial G}, \rho/2)$.

Proof. We argue by contradiction. Since the maximal cap G_m contains a ball of radius $\rho/2$, X intersects $G^s_{\shortparallel}$. Assume that there exists $y \in G^s_{\shortparallel} \setminus X$, and take $x \in X \cap G^s_{\shortparallel}$. Since $G^s_{\shortparallel}$ is connected, we can join x to y with a path contained in $G^s_{\shortparallel}$. Let z be the first point on this path that falls outside X; then $z \in \partial X \cap G^s_{\shortparallel}$.

If $z \cdot \omega \geq m$, then $z \in \partial G$, contradicting the fact that $\operatorname{dist}(z, \partial G) > s$. If, instead, $z \cdot \omega < m$, a contradiction is reached by observing that $s < \operatorname{dist}(z, \partial G) \leq C_*[u]_{\partial G}$ by Lemma 3.4. □

We now have all the ingredients needed to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. An admissible choice of the parameter s in Theorem 3.5 is

$$s = 2C_* \, [u]_{\partial G}. \tag{3.21}$$

Theorem 3.5 then implies that for each $x \in \partial G$, there exists $y \in \partial G$ such that

$$|x^m - y| \leq 2s; \tag{3.22}$$

see [ABR, Corollary 1]. It is important to observe that s does not depend on the direction ω. Thus we can choose ω to be each one of the coordinate directions $e_1, \dots, e_N$. By (2.1) and (2.2), these choices define N hyperplanes $\pi_{m_1}, \dots, \pi_{m_N}$, each in critical position with respect to the corresponding direction.

Let $O = \pi_{m_1} \cap \cdots \cap \pi_N$, and denote by x^O the reflection $2O - x$ of x in O. Since x^O is obtained by N successive reflections with respect to the planes $\pi_{m_1}, \dots, \pi_{m_N}$, it follows from (3.22) that for each $x \in \partial G$ there exists $y \in \partial G$ such that $|x^O - y| \leq 2Ns$; see [ABR, Corollary 2].

The point O can now be chosen as the center of the balls B_{r_i} and B_{r_e} in (1.3). In fact, since (3.22) holds, [ABR, Proposition 6] guarantees that, for any direction ω, $\text{dist}(O, \pi_m) \leq 4N(1 + \text{diam}\, G)\, s$, with s given by (3.21). It is clear that (1.3) holds with $r_i = \min_{x\in\partial G} |x - O|$ and $r_e = \max_{x\in\partial G} |x - O|$. Finally, [ABR, Proposition 7] states that $r_e - r_i \leq 8N(1 + \text{diam}\, G)\, s$; and this last estimate and (3.21) give (1.4). □

Remark 3.6. Observe that the constant C in Theorem 1.1 is given by $C = 16N(1 + \text{diam}\, G)\, C_*$. Hence, the dependence of C on R is of the type $C = O(A^{R^{-N}})$ as $R \to 0^+$, where $A > 1$ is a constant and $C = O(R)$ as $R \to +\infty$.

4 Semilinear equations

In this section, we show how Theorem 1.1 can be extended to the case in which (1.1) is replaced by (1.7). The structure of the proofs is the same; for this reason, we limit ourselves to identifying only the relevant passages that need to be modified. In what follows, we use the standard norms

$$\|u\|_{C^1(\Omega)} = \max_{\overline{\Omega}} |u| + \max_{\overline{\Omega}} |Du|,$$
$$\|u\|_{C^2(\Omega)} = \|u\|_{C^1(\Omega)} + \max_{\overline{\Omega}} |D^2 u|.$$

Let f be a locally Lipschitz continuous function with $f(0) \geq 0$, and consider the problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega = G + B_R$. If we add the overdetermination (1.2), then the G must be a ball and u is radially symmetric. The details of this result can be reconstructed from [CMS].

Theorem 4.1. *Let G be a domain as in Theorem 1.1. Let $u \in C^2(\overline{\Omega})$ be the solution of* (4.1) *with $-\partial u/\partial \nu \geq d_0 > 0$ on $\partial\Omega$.*

If $R < \frac{1}{2} d_0 \|u\|^{-1}_{C^2(\Omega)}$, then there exist constants $\varepsilon, C > 0$ such that if $[u]_{\partial G} \leq \varepsilon$, there exist concentric balls B_{r_i} and B_{r_e} such that (1.3) *and* (1.4) *hold. The constants ε and C depend only on N, R, the $C^{2,\alpha}$-regularity of ∂G, and on upper bounds for the diameter of G, $\max_{\overline{\Omega}} |u|$, and the Lipschitz constant of f.*

As for Theorem 1.1, the proof of Theorem 4.1 is based on a quantitative study of the method of moving planes. In this case, the function w^m defined by (2.3) satisfies

$$\Delta w^m + c(x)\, w^m = 0 \text{ and } w^m \geq 0 \text{ in } \Omega_m,$$

where for $x \in \Omega_m$,

$$c(x) = \begin{cases} \dfrac{f(u(x^m)) - f(u(x))}{u(x^m) - u(x)} & \text{if } u(x^m) \neq u(x), \\ 0 & \text{if } u(x^m) = u(x). \end{cases}$$

Notice that $c(x)$ is bounded in the interval $[0, \max_{\overline{\Omega}} u]$ by the Lipschitz constant L of f.

The following lemma summarizes and generalizes the contents of Lemmas 3.1 and 3.2 to the present case.

Lemma 4.2. *Let D_1, D_2 and R be as in Lemma 3.1. Assume that a function $w \in C^0(D_2) \cap C^2(D_2^+)$ satisfies the conditions*

$$\begin{cases} \Delta w + c(x) w = 0, \quad w \geq 0, & \text{in } D_2^+, \\ w = 0, & \text{on } \partial D_2^+ \cap \{x_1 = 0\}, \end{cases}$$

with $|c| \leq L$ in D_2^+. Let $z \in \overline{D_1^+}$. Then

$$w(x) \leq C \frac{w(z)}{z_1}, \quad \text{if } z_1 > 0, \tag{4.2}$$

$$w(x) \leq C \frac{\partial w}{\partial x_1}(z), \quad \text{if } z_1 = 0, \tag{4.3}$$

for all $x \in \overline{D_1^+} \setminus \{x_1 = 0\}$, where C is a constant depending only on N, R, L, and the diameters of D_1 and D_2.

Proof. Let E_R be defined by (3.1). First we prove (4.2) and (4.3) for $x \in E_R$ (as done in Lemma 3.1), and then we extend such estimates to $x \in \overline{D_1^+} \setminus \{x_1 = 0\}$ (as in Lemma 3.2). We follow step by step the proofs of Lemmas 3.1 and 3.2.

The main ingredient in the proof of Lemma 3.1 was Harnack's inequality. In the present case, [GT, Theorem 8.20] gives

$$\sup_{B_{r/4}} w \leq C \inf_{B_{r/4}} w, \tag{4.4}$$

for any $B_r \subset D_1^+$. Here, the constant C can be bounded by $C_N^{\sqrt{N}+\sqrt{LR}}$, where C_N is a constant depending only on N. In the present setting, inequality (4.4) replaces (3.5).

In case (i) of the proof of Lemma 3.1, we readily obtain

$$\sup_{E_R} w \leq (\operatorname{diam} D_1) C^{(2 \operatorname{diam} D_2/R)^N} \frac{w(z)}{z_1}.$$

In case (ii) or case (iii), the conclusion follows from an application of [ABR, Proposition 2] or [ABR, Lemma 2], respectively.

To prove the analog of Lemma 3.2, we need the equivalent of (3.11) for the semilinear case. This can be found, for instance, in [BCN, Theorem 1.3], which gives

$$\sup_{B^+_{R/2}} w \leq M u(\hat{x});$$

here, the constant M depends only on R, N and L. The rest of the proof is completely analogous to that of Lemma 3.2. □

Theorem 3.3 and its proof apply also to the semilinear case. It is clear that the constant MC appearing in (3.12) now depends on N, R, $\operatorname{diam} G$, the C^2 regularity of ∂G, and L.

Lemma 3.4 is the last ingredient that needs some remodeling.

Lemma 4.3. *Let $u \in C^2(\overline{\Omega})$ be a solution of* (4.1) *with $-\partial u/\partial \nu \geq d_0 > 0$ on $\partial\Omega$, where ν is the normal exterior to $\partial\Omega$. If $R < \frac{1}{2} d_0 \, \|u\|^{-1}_{C^2(\Omega)}$, there exists a positive constant K depending on* $\operatorname{diam} G$, $\max_{\overline{\Omega}} |u|$*, the $C^{2,\alpha}$ regularity of ∂G, and the Lipschitz constant L such that*

$$K \operatorname{dist}(x, \partial G) + \min_{\partial G} u \leq u(x)$$

for every $x \in \overline{G}$.

Proof. The proof follows that of [ABR, Proposition 4]. Let $x \in G$ and assume that $\operatorname{dist}(x, \partial G) < \rho$, where ρ is the optimal radius of the interior touching ball to ∂G. For such x, there exist $y \in \partial G$ and $z \in \partial\Omega$ such that

$$y - x = \operatorname{dist}(x, \partial G)\, \nu(y), \quad z - x = (\operatorname{dist}(x, \partial G) + R)\, \nu(z).$$

We notice that $\nu(y) = \nu(z)$, $|y - z| = R$ and $-\nabla u(z) \cdot \nu(z) \geq d_0$. The Mean Value Theorem implies that

$$-\nabla u(y) \cdot \nu(y) \geq d_0 - \|u\|_{C^2(\Omega)} R \geq \frac{d_0}{2}, \tag{4.5}$$

for every $R \leq \frac{1}{2} d_0 / \|u\|_{C^2(\Omega)}$. Again, by Taylor expansion, we have

$$u(x) \geq u(y) + \nabla u(y) \cdot (x - y) - \frac{1}{2} \|u\|_{C^2(\Omega)} |x - y|^2;$$

and from (4.5), we obtain

$$u(x) \geq u(y) + \frac{1}{4}\, d_0\, |x - y|, \quad \text{for every } |x - y| \leq \frac{1}{2}\, d_0\, \|u\|_{C^2(\Omega)}^{-1},$$

which implies

$$u(x) \geq \min_{\partial G} u + \frac{1}{4}\, d_0 \operatorname{dist}(x, \partial G) \tag{4.6}$$

for every x such that $\operatorname{dist}(x, \partial G) \leq \frac{1}{2}\, d_0\, \|u\|_{C^2(\Omega)}^{-1}$.

Now, (4.6) replaces formula (28) in the proof of [ABR, Proposition 4], and each argument of that proof can be repeated in our case, leading to the conclusion. □

Proof of Theorem 4.1. The proof follows that of Theorem 1.1.

For fixed direction ω, the moving plane method provides a maximal cap G_m where, thanks to Lemma 4.2, we have the bound

$$w^m \leq C\, [u]_{\partial G} \ \text{ in } \ G_m,$$

which is the analog of (3.12). Here, w^m is defined by (2.3) and C is a constant depending on N, R, $\operatorname{diam} G$, the C^2 regularity of ∂G, and L.

Next, we define a symmetric set X, as done in Section 3. We use Lemma 4.3 and prove that

$$\operatorname{dist}(x, \partial G) \leq C_*\, [u]_{\partial G}$$

for all $x \in \partial X$, where C is as in (3.19). This says that if $[u]_{\partial G}$ is small, X is almost equal to G; in particular, $G_{||}^s \subset X \subset G$ for all $s \in (C_*[u]_{\partial G}, \rho/2)$ whenever $[u]_{\partial G} < \rho/4C_*$. Note that such estimates do not depend on the direction ω. Choosing ω to be each of the coordinate directions $e_1, \dots, e_N$, we define the approximate center of symmetry. The conclusion follows by repeating the argument of the proof of Theorem 1.1. □

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