Decaying solutions for discrete boundary value problems on the half line

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Abstract. Some nonlocal boundary value problems, associated to a class of functional difference equations on unbounded domains, are considered by means of a new approach. Their solvability is obtained by using properties of the recessive solution to suitable half-linear difference equations, a half-linearization technique and a fixed point theorem in Fréchet spaces. The result is applied to derive the existence of nonoscillatory solutions with initial and final data. Examples and open problems complete the paper.

Keywords. p-Laplacian difference equations, decaying solutions, recessive solutions, functional equations, fixed point theorems in Frechét spaces.

1 Introduction

Consider the functional difference equation

$$\Delta(a_n|\Delta x_n|^\alpha \operatorname{sgn}(\Delta x_n)) = \lambda F(n, x_{n+q}), \tag{1}$$

where $\lambda > 0$ is a real parameter, $\alpha > 0$, $q \in \{0, 1, 2\}$ and Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$. We assume that $a = \{a_n\}$ is a positive sequence and F is a continuous function on $\mathbb{N} \times \mathbb{R} \mapsto \mathbb{R}$. Moreover, F is not identically zero for large n, that is

$$\sup\{F(k, u) : k \geq n\} > 0 \tag{H$_2$}$$

for any $n \in \mathbb{N}$ and $u \in [0, 1]$, and a sequence $\gamma = \{\gamma_n\}$ exists such that

$$0 \leq F(n, u) \leq \gamma_n u^\alpha, \quad n \in \mathbb{N}, \ u \in [0, 1]. \tag{H$_1$}$$

Discrete boundary value problems (BVPs), associated to equations of type (1), have attracted considerable attention in the last years, especially when they are examined on unbounded domains, see, e.g., [2, 4, 5, 10, 11, 18, 20], the monographies [1, 3] and references therein. Equation (1) appears in the discretization process for searching spherically symmetric solutions of certain nonlinear elliptic differential equations with p-Laplacian, see, e.g., [13]. The case of noncompact domains seems to be of particular interest in view of applications to radially symmetric solutions to PDEs on the exterior of a ball.

For any solution x of (1) denote by $x^{[1]}$ its quasidifference, that is

$$x_n^{[1]} = a_n |\Delta x_n|^\alpha \operatorname{sgn}(\Delta x_n).$$

Our aim here is to study certain global BVPs associated to (1), which include asymptotic boundary conditions at infinity jointly with initial conditions. Their solvability will be obtained as application of a general existence result for solutions of (1), which are, roughly speaking, the minimal nonoscillatory solutions of (1). In particular, the existence of positive solutions to (1), which satisfy one of the following boundary conditions

$$x_1 = c > 0,\ x_n > 0,\ \Delta x_n \le 0,\ \lim_n x_n = 0,$$

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will be considered.

Several approaches can be used for treating BVPs on infinite intervals; beside classical ones, such as, for instance, variational methods, critical point theory and fixed point theorems, recently new methods have been proposed, especially as an extension of the Leray-Schauder continuation principle. The reader can refer to [3, 6, 19, 21] for more details on this topic and to [7, 8, 15, 17, 22] for recent applications to various types of discrete BVPs.

Here we employ a general fixed point theorem for operators defined in a Fréchet space, which is stated in [19]. This approach allows us to treat in a unified way the problems, independently of the presence of the delay argument ($q = 0$) or the advanced one ($q = 2$). More precisely, the considered BVPs are solved by considering an auxiliary BVP on the half-line, associated to a half-linear difference equation without deviating argument. This method does not require the explicit form of the fixed point operator, but only some *a-priori* bounds. For obtaining these estimations, a crucial role is played by the theory of recessive solutions in the half-linear case. Moreover, this approach permit us to consider also a large variety of functional asymptotic BVPs, like, for instance, asymptotic problems in which the initial value of the first difference of the solution is assigned. Finally, our results can be easily formulated for the more general functional equation

$$\Delta(a_n|\Delta x_n|^\alpha \operatorname{sgn}(\Delta x_n)) = F(n, x_{\sigma(n)}), \tag{2}$$

where $\sigma : \mathbb{N} \to \mathbb{N}$, $\lim_n \sigma(n) = \infty$, and σ satisfies either $\sigma(n) \geq n+1$, or $\sigma(n) \leq n$.

We close this section by stating the quoted result from [19], in the form that will be used for the solvability of our BVPs.

Denote by $\mathbb{F}$ the Fréchet space of real sequences $x = \{x_k\}$, $k \in \mathbb{N}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{N}$. We recall that a subset $W \subset \mathbb{F}$ is bounded if and only if it consists of sequences which are equibounded on every discrete interval, i.e., if and only if there exists a positive sequence $z \in \mathbb{F}$ such that $|w_k| \leq z_k$ for each $k \in \mathbb{N}$ and $w \in W$. The following holds.

Theorem 1. *Let $\mathbb{S}$ be a subset of $\mathbb{F}$. Assume that there exists a nonempty, closed, convex and bounded set $\Omega \subset \mathbb{F}$ such that for any $u \in \Omega$ the equation*

$$\Delta(a_n|\Delta y_n|^\alpha \operatorname{sgn}(\Delta y_n)) = \frac{F(n, u_{\tau(n)})}{|u_{n+1}|^\alpha \operatorname{sgn}(u_{n+1})} |y_{n+1}|^\alpha \operatorname{sgn}(y_{n+1})$$

has a unique solution $y = T(u) \in \mathbb{S} \cap \Omega$. If $\overline{T(\Omega)} \subset \mathbb{S}$, then T has a fixed point x in Ω, which is solution of (1) and $x \in \mathbb{S}$.

Some notations are in order.

Denote by J_a, I_1, I_2 the series

$$J_a = \sum_{j=1}^{\infty} \left(\frac{1}{a_j}\right)^{1/\alpha},$$

$$I_1 = \sum_{n=1}^{\infty} \gamma_n \Big(\sum_{j=n+1}^{\infty} \left(\frac{1}{a_j}\right)^{1/\alpha} \Big)^\alpha$$

$$I_2 = \sum_{n=1}^{\infty} \Big(\frac{1}{a_n} \sum_{k=n}^{\infty} \gamma_k \Big)^{1/\alpha}.$$

In this paper, both cases $J_a = \infty$ and $J_a < \infty$ are considered. The notation $\sum_{n=1}^{\infty} f_n \sum_{k=n}^{\infty} g_k$ means $\lim_m \sum_{n=1}^{m} f_n \sum_{k=n}^{m} g_k$, where $f = \{f_n\}, g = \{g_n\}$ are nonnegative sequences.

If $J_a < \infty$, let $A = \{A_n\}$ be the sequence

$$A_n = \sum_{j=n}^{\infty} \left(\frac{1}{a_j}\right)^{1/\alpha}. \tag{3}$$

The following relations between the series J_a, I_1, I_2 hold.

Lemma 1. *(i_1) If $I_1 < \infty$ then $J_a < \infty$.*

(i_2) $I_1 = \infty$ and $I_2 < \infty$ if and only if $J_a = \infty$ and $I_2 < \infty$.

Proof. Fixed $m > 1$, we have

$$\sum_{n=1}^{m} \gamma_n \Big(\sum_{j=n+1}^{m} \left(\frac{1}{a_j}\right)^{1/\alpha} \Big)^{\alpha} \geq \gamma_1 \Big(\sum_{j=2}^{m} \left(\frac{1}{a_j}\right)^{1/\alpha} \Big)^{\alpha} \tag{4}$$

and Claim (i_1) follows. Claim (i_2). A similar argument shows that if $I_2 < \infty$, then $\sum_{i=1}^{\infty} \gamma_i < \infty$. If $I_1 = \infty$, then $J_a = \infty$. The vice-versa follows from (4). □

2 Half-linear equations and recessive solutions

Consider the second order half-linear difference equation

$$\Delta(a_n |\Delta x_n|^{\alpha} \operatorname{sgn} \Delta x_n) = b_n |x_{n+1}|^{\alpha} \operatorname{sgn} x_{n+1}, \tag{5}$$

where $b = \{b_k\}$, $k \in \mathbb{N}$, is a real sequence. When (5) is nonoscillatory, in [14] the concept of recessive solution of (5) has been given, using a certain generalized Riccati difference equation. This notion is the discrete counterpart of the one of principal solution, introduced by Leighton and Morse in studying the qualitative behavior of solutions of second order linear differential equations (see, e.g., [16]) and reads as follows. Consider the generalized Riccati equation

$$\Delta w_n - b_n + \big(1 - S(a_n, w_n)\big) w_n = 0, \tag{6}$$

where

$$S(a_n, w_n) = \frac{a_n}{|(a_n)^{1/\alpha} + |w_n|^{1/\alpha} \operatorname{sgn} w_n|^{\alpha}} \operatorname{sgn} \big((a_n)^{1/\alpha} + |w_n|^{1/\alpha} \operatorname{sgn} w_n\big).$$

According to [14], there exists a solution w^∞ of (6), satisfying $a_n+w_n^\infty > 0$ for large n, with the property that, for any other solution w of (6), with $a_n+w_n > 0$ in some neighborhood of ∞, it holds $w_n^\infty < w_n$ in this neighborhood. Such solution w^∞ is called *the (eventually) minimal solution* of (6) and the solution u of (5), given by

$$\Delta u_n = \big(|w_n^\infty| \operatorname{sgn}(w_n^\infty)/a_n\big)^{1/\alpha} u_n, \tag{7}$$

is called *the recessive solution* of (5). Clearly, the recessive solution of (5) is determined up to a constant factor.

Now, consider the half-linear difference equation

$$\Delta(a_n|\Delta y_n|^\alpha \operatorname{sgn}(\Delta y_n)) = B_n|y_{n+1}|^\alpha \operatorname{sgn}(y_{n+1}), \tag{8}$$

where

$$B_n \geq b_n, \quad n \geq N \geq 1. \tag{9}$$

The following comparison result is an easy consequence of [14, Theorem 1].

Theorem 2. *Assume that (8) is nonoscillatory. Let x and y be the recessive solutions of (5) and (8), respectively, such that $x_n > 0$, $y_n > 0$ for $n \geq N$ and either $x_N \geq y_N$ or $\Delta x_N \leq \Delta y_N \leq 0$. Then we have for any $n \geq N$*

$$x_n \geq y_n. \tag{10}$$

Proof. Since (8) is nonoscillatory, in view of (9) equation (5) is nonoscillatory too. Let w^∞ and v^∞ be the minimal solution of the generalized Riccati equation associated to (5) and (8), respectively. From [14, Theorem 1] and its proof we have for $n \geq N$

$$w^\infty \geq v^\infty.$$

Thus, in virtue of (7), we get for $n \geq N$

$$\frac{\Delta x_n}{x_n} \geq \frac{\Delta y_n}{y_n},$$

which implies for $n \geq N$

$$\frac{x_{n+1}}{x_n} \geq \frac{y_{n+1}}{y_n} > 0.$$

Hence

$$\frac{x_{n+1}}{x_N} = \prod_{k=N}^{n} \frac{x_{k+1}}{x_k} \geq \prod_{k=N}^{n} \frac{y_{k+1}}{y_k} = \frac{y_{n+1}}{y_N},$$

and (10) follows, since $x_N \ge y_N$.

Now, let x and y be the positive recessive solutions of (5) and (8), respectively, such that $\Delta x_N \le \Delta y_N$, and let z be the recessive solution of (5) such that $z_N = y_N$. From the previous part of the proof, we have $y_n \le z_n$ for $n \ge N$, and, in particular $\Delta y_N \le \Delta z_N$. Since z and x are positive recessive solutions of the same equation, there exists $\xi > 0$ such that $z_n = \xi x_n$ for all $n \in \mathbb{N}$. The inequality $\Delta y_N \le \Delta z_N = \xi \Delta x_N \le \xi \Delta y_N \le 0$ implies that $0 < \xi \le 1$ and therefore $x_n \ge z_n \ge y_n$ for $n \ge N$. □

When b is nonnegative and

$$\sup \{b_k : k \ge n\} > 0 \text{ for any } n \in \mathbb{N}, \tag{11}$$

according to [9], equation (5) is nonoscillatory and the recessive solution of (5) can be characterized in a more expressive form. More precisely, the recessive solution of (5) is, roughly speaking, the minimal solution of (5) in a neighborhood of infinity. Moreover, a summation characterization for the recessive solution holds. These properties can be viewed as the discrete counterpart of well-known properties of the principal solution in the continuous case [16].

Lemma 2. *Assume that b is a nonnegative sequence satisfying (11). Then (5) is nonoscillatory. Moreover, a solution x of (5) is the recessive solution if and only if any of the following two conditions is satisfied.*

(i_1) For every solution y of (5) such that $y \ne \mu x$, $\mu \in \mathbb{R}$,

$$\lim_n \frac{x_n}{y_n} = 0. \tag{12}$$

(i_2) There exists $n_0 \ge 1$ such that

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha} x_n x_{n+1}} = \infty. \tag{13}$$

Proof. The assertion follows from Theorem 4 and Remark 3 in [9], with minor changes. □

In case b is nonnegative and satisfies (11), the asymptotic behavior of recessive solutions of (5) is described by the series

$$W_1 = \sum_{n=1}^{\infty} b_n \Big(\sum_{k=n}^{\infty} \Big(\frac{1}{a_{k+1}}\Big)^{1/\alpha}\Big)^{\alpha}, \quad W_2 = \sum_{n=1}^{\infty} \Big(\frac{1}{a_n}\Big)^{1/\alpha} \Big(\sum_{k=n}^{\infty} b_k\Big)^{1/\alpha}.$$

Lemma 3. *Assume that b is a nonnegative sequence satisfying (11), and let x be the recessive solution of (5) such that $x_1 > 0$. Then the following statements hold:*

(i_1) *The sequence x is positive decreasing for any $n \geq 1$, that is $x_n > 0$, $\Delta x_n < 0$ for any $n \geq 1$.*

(i_2) *If $\lim_n x_n = \ell_x > 0$, then $\lim_n x_n^{[1]} = 0$.*

(i_3) *We have*

$$\lim_n x_n = 0, \quad \lim_n x_n^{[1]} = \ell_x^{[1]} < 0$$

if and only if $W_1 < \infty$.

(i_4) *We have*

$$\lim_n x_n = \ell_x > 0, \quad \lim_n x_n^{[1]} = 0$$

if and only if $W_1 = \infty$ and $W_2 < \infty$ or, equivalently, if and only if $J_a = \infty$ and $W_2 < \infty$.

Proof. The assertion follows from [12, Theorem A, Theorems 2 and 3]. See also [12, page 12]. □

Remark 1. Observe that in [9, 12], the positivity of the sequence b is assumed. Nevertheless, we can verify that the above Lemmas 2 and 3 continue to hold also when b is a nonnegative sequence satisfying (11).

3 Main results

Our main results are two existence theorems for positive solutions to the following nonlocal boundary value problem

$$\begin{cases} \Delta(a_n|\Delta x_n|^\alpha \operatorname{sgn}(\Delta x_n)) = \lambda F(n, x_{n+q}), \quad n \in \mathbb{N} \\ x_1 = c, \quad x_n > 0, \ \displaystyle\sum_1^\infty \frac{1}{a_n^{1/\alpha} x_n x_{n+1}} = \infty. \end{cases} \tag{14}$$

In our approach, an important role is played by the half-linear equation

$$\Delta(a_n|\Delta z_n|^\alpha \operatorname{sgn}(\Delta z_n)) = 0. \tag{15}$$

The recessive solution z of (15), satisfying $z_1 = c$, is the sequence

$$z_n = \begin{cases} c & \text{if } \ J_a = \infty, \\ cA_n/J_a & \text{if } \ J_a < \infty. \end{cases} \tag{16}$$

Clearly, z satisfies

$$\sum_{1}^{\infty} \frac{1}{a_n^{1/\alpha} z_n z_{n+1}} = \infty, \tag{17}$$

as it follows using the equality

$$\Delta \frac{1}{z_n} = \frac{-\Delta z_n}{z_{n+1} z_n}.$$

In the next two subsections, existence results are given for problem (14), obtained by using Theorem 1 and a half-linearization approach. More precisely, the solution of (14) is obtained as fixed point of an operator, which is defined via recessive solutions of a class of suitable half-linear difference equations.

3.1 Case $q \in \{1, 2\}$

We start with the cases $q \in \{1, 2\}$, which yields an equation without deviating argument or with advanced argument, respectively.

Theorem 3. *Fixed $c \in (0, 1]$, the problem (14) with $q = 1, 2$, has at least a solution for every $\lambda > 0$.*

Proof. Let z be the recessive solutions of (15) satisfying $z_1 = c$, and let v be the recessive solution of

$$\Delta(a_n |\Delta v_n|^\alpha \operatorname{sgn}(\Delta v_n)) = \lambda \gamma_n |v_{n+1}|^\alpha \operatorname{sgn} v_{n+1}, \tag{18}$$

satisfying $v_1 = c$.

In view of (H_1) and (H_2), the sequence $\{\gamma_n\}$ satisfies

$$\sup \{\gamma_k : k \geq n\} > 0 \text{ for any } n \in \mathbb{N}.$$

Hence, from Theorem 2 and Lemma 3, the recessive solutions z and v are positive nonincreasing and satisfy the inequality

$$0 < v_n \leq z_n \leq c.$$

Denote by Ω the subset of $\mathbb{F}$ given by

$$\Omega = \{u \in \mathbb{F} : v_n \leq u_n \leq z_n,\ \Delta u_n \leq 0,\ \text{for } n \in \mathbb{N}\}. \tag{19}$$

Fixed $u \in \Omega$, consider the half-linear equation

$$\Delta(a_n |\Delta y_n|^\alpha \operatorname{sgn}(\Delta y_n)) = \lambda \frac{F(n, u_{n+q})}{u_{n+1}^\alpha} |y_{n+1}|^\alpha \operatorname{sgn}(y_{n+1}) \tag{20}$$

and let T be the map which associates to every $u \in \Omega$ the recessive solution y of (20), that is $T(u) = y$. Taking into account (H_2) and using Lemma 2, we have

$$\sum_1^\infty \frac{1}{a_n^{1/\alpha} y_n y_{n+1}} = \infty.$$

Thus, denoting by $\mathbb{S}$ the subset of $\mathbb{F}$ given by

$$\mathbb{S} = \left\{ \varphi \in \mathbb{F} : \varphi_1 = c, \quad \sum_1^\infty \frac{1}{a_n^{1/\alpha} \varphi_n \varphi_{n+1}} = \infty \right\}, \tag{21}$$

the solution $y = T(u)$ belongs to $\mathbb{S}$. Moreover, in virtue of Lemma 3, y is positive nonincreasing, that is $\Delta y_n \leq 0$. Since $0 < u_n \leq c$ for $u \in \Omega$, and $q \geq 1$, in view of (H_1) we have

$$0 \leq F(n, u_{n+q}) \leq \gamma_n u_{n+q}^\alpha \leq \gamma_n u_{n+1}^\alpha \tag{22}$$

or

$$0 \leq \frac{F(n, u_{n+q})}{u_{n+1}^\alpha} \leq \gamma_n. \tag{23}$$

From Theorem 2, the inequality (23) yields $v_n \leq y_n \leq z_n$. Therefore T maps Ω into itself and so we have

$$T(u) \in \mathbb{S} \cap \Omega.$$

To apply Theorem 1, we have to show that $\overline{T(\Omega)} \subset \mathbb{S}$. Let $\bar{y} \in \overline{T(\Omega)}$. Thus, there exists $\{u^h\} \subset \Omega$, $u^h = \{u_n^h\}$, such that $\{T(u^h)\}$ uniformly converges to $\bar{y}$ on every bounded subset of $\mathbb{N}$. Since Ω is closed and $T(\Omega) \subset \Omega$, we obtain $\bar{y} \in \Omega$. Hence we have for $n \in \mathbb{N}$

$$0 < v_n \leq \bar{y}_n \leq z_n. \tag{24}$$

In particular, being $v_1 = z_1 = c$, we obtain $\bar{y}_1 = c$. Moreover, taking into account that z is the positive recessive solution of (15), using (17) we get

$$\sum_{n=1}^\infty \frac{1}{a_n^{1/\alpha} \bar{y}_n \bar{y}_{n+1}} \geq \sum_{n=1}^\infty \frac{1}{a_n^{1/\alpha} z_n z_{n+1}} = \infty,$$

that is $\bar{y} \in \mathbb{S}$. Thus, Theorem 1 assures that T has at least a fixed point $x \in \Omega$, which is a solution of (14). □

Remark 2. Theorem 3 can be easily extended, with minor changes, to the equation (2) where σ is a general advanced argument, i.e. $\sigma(n) \geq n+1$. The details are left to the reader. Notice that no monotonicity is assumed on the advanced argument.

3.2 Case $q = 0$

Now, consider the case $q = 0$, which yields an equation with delay. In this case, a slightly different approach is needed for solving the problem (14). Indeed the argument in the proof of Theorem 3 cannot be used because for any $u \in \Omega$ we have $u_{n+1} \geq u_n$, that is $\gamma_n u_{n+1}^{\alpha} \geq \gamma_n u_n^{\alpha}$, i.e., the inequality (22) is not satisfied when u is not constant. The following holds.

Theorem 4. *Let $q = 0$. Assume that one of the following condition is satisfied.*

(i) $I_1 < \infty$ and

$$\limsup_n \frac{a_{n+1}}{a_n} < \infty. \tag{25}$$

(ii) $I_1 = \infty$ and $I_2 < \infty$.

Then, fixed $c \in (0, 1]$, the problem

$$\begin{cases} \Delta(a_n|\Delta x_n|^{\alpha} \operatorname{sgn}(\Delta x_n)) = \lambda F(n, x_n), \quad n \in \mathbb{N} \\ x_1 = c, \ x_n > 0 \ \displaystyle\sum_1^{\infty} \frac{1}{a_n^{1/\alpha} x_n x_{n+1}} = \infty \end{cases} \tag{26}$$

has at least a solution for every $\lambda > 0$ sufficiently small.

Proof. Let z be the recessive solutions of (15) satisfying $z_1 = c$, and let Ω be the set

$$\Omega = \{u \in \mathbb{F} : \ z_n/2 \leq u_n \leq z_n, \ \Delta u_n \leq 0, \ \text{for } n \in \mathbb{N}\}.$$

Since z is nonincreasing, we have $0 < z_n/2 \leq u_n \leq z_n \leq c$. For any $u \in \Omega$, consider the half-linear equation

$$\Delta(a_n|\Delta y_n|^{\alpha} \operatorname{sgn}(\Delta y_n)) = \lambda \frac{F(n, u_n)}{u_{n+1}^{\alpha}} |y_{n+1}|^{\alpha} \operatorname{sgn}(y_{n+1}), \tag{27}$$

and let $y = T(u)$ be the recessive solution of (27) satisfying $y_1 = c$. Hence, in view of Lemma 2, $y = T(u) \in \mathbb{S}$, where $\mathbb{S}$ is the set defined in (21).

Now, let us show that $T(\Omega) \subseteq \Omega$. From Theorem 2 we have $y_n \le z_n$. To prove the inequality $y_n \ge z_n/2$ we distinguish two cases, according to the condition (i) or (ii) holds.

Case I). Assume that condition (i) is fulfilled. From Lemma 1 we get $J_a < \infty$ and, in view of (16), $z_n = cA_n/J_a$. By the discrete l'Hospital rule, (25) implies that $H > 0$ exists, such that

$$H = \sup_n \frac{A_n}{A_{n+1}}. \tag{28}$$

In view of (H1), (H2) and Lemma 3, for any $u \in \Omega$, we get

$$0 < \frac{F(n, u_n)}{u_{n+1}^\alpha} \le \gamma_n \Big(\frac{u_n}{u_{n+1}}\Big)^\alpha \le \gamma_n \Big(\frac{2z_n}{z_{n+1}}\Big)^\alpha. \tag{29}$$

Since $z_n/z_{n+1} = A_n/A_{n+1}$, we obtain

$$\frac{F(n, u_n)}{u_{n+1}^\alpha} \le 2^\alpha \gamma_n \Big(\frac{A_n}{A_{n+1}}\Big)^\alpha \le \gamma_n (2H)^\alpha. \tag{30}$$

Thus, Theorem 2 yields $y_n \ge w_n$, where $w = \{w_n\}$ is the recessive solution of

$$\Delta(a_n |\Delta w_n|^\alpha \operatorname{sgn}(\Delta w_n)) = \lambda (2H)^\alpha \gamma_n |w_{n+1}|^\alpha \operatorname{sgn} w_{n+1}, \tag{31}$$

such that $w_1 = c$. Since $I_1 < \infty$, from Lemma 3 we get $\lim_n w_n = 0$, $\lim_n w_n^{[1]} = w_\infty^{[1]} < 0$. By summation of (31) we obtain

$$w_n = \sum_{k=n}^{\infty} \left(\frac{1}{a_k}\right)^{1/\alpha} \left(-w_\infty^{[1]} + \lambda (2H)^\alpha \sum_{j=k}^{\infty} \gamma_j w_{j+1}^\alpha\right)^{1/\alpha} \ge A_n (-w_\infty^{[1]})^{1/\alpha}.$$

The initial condition $w_1 = c$ gives

$$c = \sum_{k=1}^{\infty} \left(\frac{1}{a_k}\right)^{1/\alpha} \left(-w_\infty^{[1]} + \lambda (2H)^\alpha \sum_{j=k}^{\infty} \gamma_j w_{j+1}^\alpha\right)^{1/\alpha},$$

and, taking into account that $w_n \le z_n$, we get

$$\begin{aligned} c &\le \sum_{k=1}^{\infty} \left(\frac{1}{a_k}\right)^{1/\alpha} \left(-w_\infty^{[1]} + \lambda (2H)^\alpha \left(\frac{c}{J_a}\right)^\alpha \sum_{j=1}^{\infty} \gamma_j A_{j+1}^\alpha\right)^{1/\alpha} \\ &= J_a \left(-w_\infty^{[1]} + \lambda (2H)^\alpha \left(\frac{c}{J_a}\right)^\alpha I_1\right)^{1/\alpha}. \end{aligned}$$

Thus

$$\left(\frac{c}{J_a}\right)^{\alpha} \le -w_{\infty}^{[1]} + \lambda(2H)^{\alpha}\left(\frac{c}{J_a}\right)^{\alpha} I_1.$$

Let $\lambda \le \lambda_0$, where

$$\lambda_0 = \frac{2^{\alpha}-1}{2^{2\alpha}H^{\alpha}I_1}. \tag{32}$$

Hence $\lambda(2H)^{\alpha}I_1 < 1$ and

$$-w_{\infty}^{[1]} \ge \left(\frac{c}{J_a}\right)^{\alpha}(1-\lambda(2H)^{\alpha}I_1) \ge \left(\frac{c}{2J_a}\right)^{\alpha}.$$

Thus

$$y_n \ge w_n \ge (-w_{\infty}^{[1]})^{1/\alpha}A_n \ge \frac{c}{2J_a}A_n = \frac{z_n}{2},$$

and $y = T(u) \in \Omega$.

Case II). Assume now that condition (ii) is fulfilled. Again from Lemma 1 we get $J_a = \infty$ and, in view of (16), $z_n \equiv c$. Therefore from (29) we obtain

$$\frac{F(n,u_n)}{u_{n+1}^{\alpha}} \le \gamma_n 2^{\alpha},$$

and Theorem 2 assures that $y_n \ge v_n$, where $v = \{v_n\}$ is the recessive solution of

$$\Delta(a_n|\Delta v_n|^{\alpha}\operatorname{sgn}(\Delta v_n)) = 2^{\alpha}\lambda\gamma_n|v_{n+1}|^{\alpha}\operatorname{sgn} v_{n+1}, \tag{33}$$

such that $v_1 = c$. Since $I_1 = \infty$ and $I_2 < \infty$, in view of Lemma 3 we have $\lim_n v_n = \ell_v > 0$, $\lim_n v_n^{[1]} = 0$. Therefore, the summation of (33) gives

$$v_{n+1} = c - 2\lambda^{1/\alpha}\sum_{k=1}^{n}\left(\frac{1}{a_k}\sum_{j=k}^{\infty}\gamma_j v_{j+1}^{\alpha}\right)^{1/\alpha}.$$

Since v is nonincreasing, we get

$$v_{n+1} \ge c(1-2\lambda^{1/\alpha}I_2).$$

Thus, for every $\lambda \le \lambda_1$, where

$$\lambda_1 = (4I_2)^{-\alpha}, \tag{34}$$

we have

$$y_n \ge v_n \ge c/2 = z_n/2,$$

and $y = T(u) \in \Omega$.

A similar argument to that given in the proof of Theorem 3, shows that $\overline{T(\Omega)} \subset \mathbb{S}$. Hence, from Theorem 1, the map T has at least a fixed point in Ω, which is a solution of (26) for λ sufficiently small. $\square$

Remark 3. The upper value of the parameter λ, for which (26) has solution, depends on the set Ω, in which we look for the fixed point of the operator T. In the proof of Theorem 4, we could more generally choose

$$\Omega = \{u \in \mathbb{F} : \ z_n/B \le u_n \le z_n, \ \Delta u_n \le 0, \ \text{for } n \in \mathbb{N}\},$$

where $B > 1$. When the assumptions (i) in Theorem 4 are satisfied, the same argument to the one given in the proof gives that the upper bound for λ is

$$\lambda_0 = \frac{B^\alpha - 1}{B^{2\alpha} H^\alpha I_1},$$

where H is given in (28). An easy calculation shows that λ_0 attains its maximum for $B = 2^{1/\alpha}$, and $\lambda_0 = (4H^\alpha I_1)^{-1}$ in this case, while with the choice $B = 2$ we get (32). Similarly, when assumptions (ii) are satisfied, the upper bound for λ is

$$\lambda_1 = \left(\frac{B-1}{B^2 I_2}\right)^\alpha$$

and λ_1 attains its maximum (34) exactly for $B = 2$.

Remark 4. Similarly to Remark 1, also Theorem 4 can be extended, with minor changes, to the equation (2) where σ is a delay argument, i.e. $\sigma(n) \le n$. In particular, assumption (25) has to be replaced with the following one

$$\limsup_n \, a_{n+1}^{1/\alpha} \sum_{\sigma(n)}^{n} \left(\frac{1}{a_j}\right)^{1/\alpha} < \infty,$$

which implies that $H_1 > 0$ exists, such that $\sup_n A_{\sigma(n)}/A_{n+1} \le H_1$. The details are left to the reader. Notice that no monotonicity is assumed on the delay argument. The case of a general deviating argument will be considered in a forthcoming paper.

4 Application to BVP's

The solvability of certain BVPs associated to (1) can be easily obtained from the general existence Theorems 3, 4 and their proofs.

For instance, let $c \in (0,1]$ and consider the subsets of $\mathbb{F}$:

$$\mathbb{S}_0 = \left\{ \varphi \in \mathbb{F} : \varphi_1 = c, \quad \varphi_n > 0, \ \Delta\varphi_n \le 0, \ \lim_n \varphi_n = 0 \right\},$$

$$\mathbb{S}_1 = \left\{ \varphi \in \mathbb{F} : \varphi_1 = c, \quad \varphi_n > 0, \ \Delta\varphi_n \le 0, \ \lim_n \varphi_n^{[1]} = 0 \right\}.$$

Corollary 1. *Let $q \in \{1,2\}$ and $c \in (0,1]$. The following hold.*

(i_1) If $J_a < \infty$, then equation (1) has at least a solution in the set $\mathbb{S}_0$ for every $\lambda > 0$.

(i_2) If $J_a = \infty$ and $I_2 < \infty$, then equation (1) has at least a solution in the set $\mathbb{S}_1$ for every $\lambda > 0$.

Proof. Claim (i_1). Since $J_a < \infty$, in view of (16), the recessive solution z of (15) satisfies $\lim_n z_n = 0$. By Theorem 3 and its proof, equation (1) has a positive solution x such that $x_1 = c$, and $\Delta x_n \le 0$, $x_n \le z_n$ for any $n \ge 1$. Hence, $x \in \mathbb{S}_0$.

Claim (i_2). Let v be the recessive solution of (18). In view of Lemma 3 we have

$$\lim_n v_n = \ell_v > 0, \quad \lim_n v_n^{[1]} = 0. \tag{35}$$

By Theorem 3 and its proof, equation (1) has a positive solution x such that $x_1 = c$, and $\Delta x_n \le 0$, $x_n \ge v_n$. Moreover, x is also a recessive solution of the half-linear equation (20), with $u \equiv x$. Since x is positive for $n \ge 1$, from [14, Theorem 1] we have

$$\frac{v_n^{[1]}}{v_n^\alpha} \le \frac{x_n^{[1]}}{x_n^\alpha} < 0, \quad n \in \mathbb{N},$$

or, because v is nonincreasing,

$$\frac{v_n^{[1]}}{v_\infty^\alpha} \le \frac{v_n^{[1]}}{v_n^\alpha} \le \frac{x_n^{[1]}}{x_n^\alpha} < 0, \quad n \in \mathbb{N}.$$

Thus, (35) yields $\lim_n x_n^{[1]} = 0$ and so $x \in \mathbb{S}_1$. □

Corollary 2. *Let $q = 0$ and $c \in (0, 1]$. The following hold.*

(i_1) If $I_1 < \infty$ and (25) holds, then equation (1) has at least a solution in the set $\mathbb{S}_0$ for every $\lambda \in (0, \lambda_0]$, where λ_0 is given by (32).

(i_2) If $J_a = \infty$ and $I_2 < \infty$, then equation (1) has at least a solution in the set $\mathbb{S}_1$ for every $\lambda \in (0, \lambda_1]$, where λ_1 is given by (34).

Proof. The assertion follows from Theorem 4 and its proof, by using a similar argument to the one given in Corollary 1.

Claim (i_1). Since $I_1 < \infty$, from Lemma 1 we get $J_a < \infty$. Thus, in view of (16), the recessive solution z of (15) satisfies $\lim z_n = 0$. From Theorem 4 and its proof, equation (1) has a positive solution x such that $x_1 = c$, and $\Delta x_n \le 0$, $x_n \le z_n$ for any $n \ge 1$ and $\lambda \in (0, \lambda_0]$, where λ_0 is given by (32). Hence, $x \in \mathbb{S}_0$.

Claim (i_2). In view of Lemma 1, we have $I_1 = \infty$ and $I_2 < \infty$. Moreover, since $J_a = \infty$, in view of (16), the recessive solution z is the constant sequence $z_n = c$. By Theorem 4 and its proof, equation (1) has a positive solution x such that $x_1 = c$, and $\Delta x_n \le 0$, $x_n \ge c/2$ for any $n \ge 1$. Moreover, x is also a recessive solution of the half-linear equation (27), with $u \equiv x$ and $\lambda \in (0, \lambda_1]$, where λ_1 is given by (34). Since $\lim_n x_n > 0$, from Lemma 3 we get $\lim_n x_n^{[1]} = 0$. Hence, $x \in \mathbb{S}_1$. □

Corollaries 1, 2 can be easily extended to the case in which the boundary conditions are one of the following.

$$\mathbb{S}_2 = \left\{ \varphi \in \mathbb{F} : \varphi_1 = c, \quad \varphi_n > 0,\ \Delta\varphi_n \le 0,\ \lim_n \varphi_n = 0,\ \lim_n \varphi_n^{[1]} = \ell_\varphi^{[1]} < 0 \right\},$$

$$\mathbb{S}_3 = \left\{ \varphi \in \mathbb{F} : \varphi_1 = c, \quad \varphi_n > 0,\ \Delta\varphi_n \le 0,\ \lim_n \varphi_n = \varphi_\infty > 0,\ \lim_n \varphi_n^{[1]} = 0 \right\}.$$

Corollary 3. *Let $q \in \{1, 2\}$ and $c \in (0, 1]$. The following hold.*

(i_1) If $I_1 < \infty$, then equation (1) has at least a solution in the set $\mathbb{S}_2$ for every $\lambda > 0$.

(i_2) If $I_1 = \infty$ and $I_2 < \infty$, then equation (1) has at least a solution in the set $\mathbb{S}_3$ for every $\lambda > 0$.

Proof. Claim (i_1). Fixed $c \in (0, 1]$, in view of Lemma 1, Corollary 1 and its proof, equation (1) has at least a solution $x \in \mathbb{S}_0$ for every $\lambda > 0$. Moreover, x is also a recessive solution of the half-linear equation (20), with $u \equiv x$.

From $I_1 < \infty$ and (23) we have

$$\sum_{n=1}^{\infty} \frac{F(n, x_{n+q})}{x_{n+1}^{\alpha}} \Big(\sum_{j=n+1}^{\infty} \Big(\frac{1}{a_j}\Big)^{1/\alpha} \Big)^{\alpha} < \infty.$$

Thus, applying Lemma 3 to (20) with $u \equiv x$, we get that the quasi-difference of the recessive solution of (20) tends to a nonzero limit as $n \to \infty$ and the assertion follows.

Claim (i_2). The assertion follows from Lemma 1, Corollary 1 and its proof, by using a similar argument to the one in Claim (i_1). □

In the case $q = 0$, Corollary 2 gives sufficient conditions for the existence of a solution x to (1) in the set $\mathbb{S}_0$ or $\mathbb{S}_1$. A closer examination of its proofs jointly with Lemma 3 yields $\lim_n x_n^{[1]} = x_\infty^{[1]} < 0$ or $\lim_n x_n = x_\infty > 0$, according to (i_1) or (i_2) holds. Thus, we have the following.

Corollary 4. *Let $q = 0$ and $c \in (0, 1]$.*

(i_1) If $I_1 < \infty$ and (16) holds, then equation (1) has at least a solution in the set $\mathbb{S}_2$ for every $\lambda \in (0, \lambda_0]$, where λ_0 is given by (32).

(i_2) If $J_a = \infty$ and $I_2 < \infty$, then equation (1) has at least a solution in the set $\mathbb{S}_3$ for every $\lambda \in (0, \lambda_1]$, where λ_1 is given by (34).

5 Examples and open problems

Independently of the convergence of J_a, equation (1) can have positive solutions x such that $\lim_n x_n = \lim_n x_n^{[1]} = 0$, as the following example illustrates.

Example 1. Consider the equation

$$\Delta^2 x_n = 2^{2n+1} x_{n+1}^3, \tag{36}$$

It is easy to verify that $x = \{2^{-n}\}$ is a solution of (36) and

$$x_1 = 2^{-1}, \quad x_n > 0, \;\; \Delta x_n \le 0, \quad \lim_n x_n = \lim_n x_n^{[1]} = 0. \tag{37}$$

Similarly, for the equation

$$\Delta(a_n \Delta x_n) = b_n x_{n+1}^3, \tag{38}$$

where

$$a_n = (n-1)!, \quad b_n = \frac{4(n+1)^2(n!)^3}{n+2},$$

the sequence x, where

$$x_n = \frac{1}{2}\frac{1}{n!},$$

is a solution of (38) and the boundary conditions (37) are satisfied. Moreover for (36) we have $J_a = \infty$ and for (38) it holds $J_a < \infty$.

Thus, the existence of solutions to (1) in the set

$$\mathbb{S}_4 = \left\{\varphi \in \mathbb{F} : \varphi_1 = c, \quad \varphi_n > 0, \ \Delta\varphi_n \le 0, \ \lim_n \varphi_n = \lim_n \varphi_n^{[1]} = 0\right\}$$

is an open problem. Moreover, the existence of solutions to (1) in $\mathbb{S}_4$ has to be independent of the convergence of J_a, as Example 1 shows.

The following example illustrates our results.

Example 2. Consider the equation

$$\Delta(n^2(\Delta x_n)) = 2(2n^2+2n+1)x_{n+q}^3, \quad q \in \{0,1,2\}. \tag{39}$$

For (39) we have $J_a < \infty$ and $I_1 = \infty$. When $q \in \{0,2\}$, one can check that $\{(-1)^n\}$ is an oscillatory solution of (39). Nevertheless, the corresponding equation without deviating argument, that is (39) with $q = 1$, is nonoscillatory, see, e.g., [1, Lemma 5.3.1.]. Moreover, when $q \in \{1,2\}$, in view of Corollary 1, equation (39) has at least a solution in the set $\mathbb{S}_0$ for every $c \in (0,1]$. Hence in the advanced case ($q = 2$) nonoscillatory solutions, converging to zero, coexist with oscillatory (periodic) solutions. This fact is impossible for the corresponding equation without deviating argument.

Further, for the equation with delay ($q = 0$), since $I_1 = \infty$, Corollary 2 cannot be applied. Thus, when $I_1 = \infty$, can the delay equation (1) with $q = 0$ admit solutions which satisfy the boundary conditions $\mathbb{S}_0$?

6 A further application

Our approach can be used to solve a wide range of BVPs. For example, existence results for BVPs associated to (1), in which the initial value of the first difference of the solution is fixed, can be obtained.

Now, we present the main ideas, which lead to the existence of solutions of the nonlocal BVP

$$\begin{cases} \Delta(a_n|\Delta x_n|^\alpha \operatorname{sgn}(\Delta x_n)) = \lambda F(n, x_{n+q}), & n \in \mathbb{N} \\ \Delta x_1 = -d,\ x_n > 0,\ \displaystyle\sum_1^\infty \frac{1}{a_n^{1/\alpha} x_n x_{n+1}} = \infty, & \end{cases} \tag{40}$$

where d is a given positive constant. The following holds.

Theorem 5. *Assume $J_a < \infty$, and let $d \in (0, 1 - A_2/A_1]$ be fixed.*

i) If $q \in \{1, 2\}$, then the problem (40) has at least a solution for every $\lambda > 0$.

ii) If $q = 0$, assume further $I_1 < \infty$ and that (25) holds. Then the problem (40) has at least a solution for every $\lambda > 0$ sufficiently small.

Proof. Let z be the recessive solution of (15), satisfying $\Delta z_1 = -d$. Notice that the assumption $J_a < \infty$ assures the existence of such a solution, and $z_n = dA_n a_1^{1/\alpha} = dA_n/(A_1 - A_2) \le 1$.

If $q \in \{1, 2\}$, let v be the recessive solution of (18) satisfying $\Delta v_1 = -d$. From Theorem 2 we have $0 < v_n \le z_n \le z_1 = 1$, $n \in \mathbb{N}$. Put $\Omega = \{u \in \mathbb{F} : v_n \le u_n \le z_n,\ \Delta u_n \le 0,\ \text{for } n \in \mathbb{N}\}$. Thus, inequality (22) holds for every $u \in \Omega$, and the half-linear equation (20) has a unique recessive solution $y = T(u)$ such that $\Delta y_1 = -d$. Further, in view of Lemma 2, the solution y satisfies the boundary conditions in (40). Hence, reasoning as in the proof of Theorem 3, we get $T(\Omega) \subseteq \Omega$.

If $q = 0$, the argument is very similar. Set $\Omega = \{u \in \mathbb{F} : z_n/2 \le u_n \le z_n,\ \Delta u_n \le 0,\ \text{for } n \in \mathbb{N}\}$. For any $u \in \Omega$, the half-linear equation (20) has a unique recessive solution $y = T(u)$, such that $\Delta y_1 = -d$. Let H be the constant given in (28). Since inequality (30) holds, Lemma 2 assures that $y_n \ge w_n$, where w is the recessive solution to (31) such that $\Delta w_1 = -d$. Hence, y satisfies the conditions in (40). By summation of (31), taking into account that $w_n \le z_n$, we obtain

$$-w_\infty^{[1]} \ge a_1 d^\alpha [1 - \lambda (2H)^\alpha I_1].$$

Thus, if $\lambda \le \lambda_0$, where λ_0 is given by (32), it holds

$$w_n \ge (-w_\infty^{[1]})^{1/\alpha} A_n \ge a_1^{1/\alpha} d A_n \left(1 - \frac{2^\alpha - 1}{2^\alpha}\right)^{1/\alpha} = \frac{z_n}{2},$$

which implies that $y_n \geq z_n/2$, and $T(\Omega) \subseteq \Omega$ follows.

A similar argument to that given in the proof of Theorem 3 leads to the existence of a fixed point $x = T(x)$ of the operator T in Ω, which is a solution of (40). □

An immediate consequence of the above theorem is an existing result for the solutions of (1) in the set

$$\mathbb{S}_5 = \{\varphi \in \mathbb{F} : \Delta\varphi_1 = -d < 0,\ \varphi_n > 0,\ \Delta\varphi_n \leq 0,\ \lim_n \varphi_n = 0\}.$$

Corollary 5. *Let $J_a < \infty$ and $d \in (0, 1 - A_2/A_1]$. The following hold.*

i) If $q \in \{1, 2\}$, then equation (1) has at least a solution in the set $\mathbb{S}_5$ for every $\lambda > 0$.

ii) If $q = 0$, assume further $I_1 < \infty$ and that (25) holds. Then equation (1) has at least a solution in the set $\mathbb{S}_5$ for every $\lambda \in (0, \lambda_0]$, with λ_0 given by (32).

The proof of the above result is similar to that of Corollary 1, claim (i_1).

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