Proceedings of the Royal Society of Edinburgh, **146A**, 213–223, *2016*
DOI:10.1017/S030821051500044X

Characterization of ellipsoids as K-dense sets

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(MS received 27 February 2014; accepted 7 January 2015)

Let $K \subset \mathbb{R}^N$ be any convex body containing the origin. A measurable set $G \subset \mathbb{R}^N$ with finite and positive Lebesgue measure is said to be K-dense if, for any fixed $r > 0$, the measure of $G \cap (x + rK)$ is constant when x varies on the boundary of G (here, $x + rK$ denotes a translation of a dilation of K). In a previous work, we proved for the case $N = 2$ that if G is K-dense, then both G and K must be homothetic to the same ellipse. Here, we completely characterize K-dense sets in $\mathbb{R}^N$: if G is K-dense, then both G and K must be homothetic to the same ellipsoid. Our proof, which builds upon results obtained in our previous work, relies on an asymptotic formula for the measure of $G \cap (x + rK)$ for large values of the parameter r and a classical characterization of ellipsoids due to Petty.

Keywords: uniformly dense sets; convex bodies; affine inequalities

2010 *Mathematics subject classification:* Primary 52A10; 52A20; 52A39; 52A40

1. Introduction

Let K be a convex body containing the origin of $\mathbb{R}^N$ and let G be a measurable subset of $\mathbb{R}^N$ with finite and positive Lebesgue measure $V(G)$. We say that G is *K-dense* if there is a function $c\colon (0,\infty) \to (0,\infty)$ such that

$$V(G \cap (x + rK)) = c(r) \quad \text{for } x \in \partial G,\ r > 0. \tag{1.1}$$

Here, ∂G is the topological boundary of G and $x + rK$ denotes the translation by a vector x of a dilation of K by a factor $r > 0$. For the case when K is the unit Euclidean ball B in $\mathbb{R}^N$ such sets have been studied in [6], in which it is proved that any B-dense set G is a ball and some characterizations are also given for B-dense sets when $V(G) = \infty$.

In [6], B-dense sets were studied in connection with the so-called stationary isothermic (or time-invariant level) surfaces of solution of the heat equation (see also the related paper [5]). More precisely, it was noted that a set G is B-dense if and only the solution $U = U(x,t)$ of the following Cauchy problem:

$$U_t = \Delta U \quad \text{in } \mathbb{R}^N \times (0,\infty), \qquad U = \mathcal{X}_G \quad \text{on } \mathbb{R}^N \times \{0\}, \tag{1.2}$$

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is such that

$$U(x,t) = a(t) \quad \text{for } (x,t) \in \partial G \times (0,\infty),$$

for some function $a\colon (0,\infty) \to (0,1)$ (here, $\mathcal{X}_G$ denotes the characteristic function of the set G). If that is the case, ∂G is called a *stationary isothermic* or *time-invariant* level surface for U.

While in the case of the (linear) Laplace operator Δ it is relatively easy to establish this connection, which is made possible by the fact that U can be obtained as the convolution of $\mathcal{X}_G$ with the heat kernel, it is still an open problem to confirm such a link between K-dense sets and time-invariant level surfaces of the solutions of the problem

$$U_t = \Delta_K U \quad \text{in } \mathbb{R}^N \times (0,\infty), \qquad U = \mathcal{X}_G \quad \text{on } \mathbb{R}^N \times \{0\}, \tag{1.3}$$

where $\Delta_K U = \operatorname{div}(\nabla h_K(\nabla U) h_K(\nabla U))$ denotes the *Finsler Laplacian* of U generated by the support function h_K of K. We hope that this paper may shed some light on this issue.

Plane K-dense sets have been characterized in [1,4]. They cannot exist unless they are homothetic to K itself and, if this is the case, they must be ellipses (together with K). In this paper, we shall extend that characterization to general dimensions by proving the following result.

Theorem 1.1. *Let $K \subset \mathbb{R}^N$ be a convex body and assume that there is a set $G \subset \mathbb{R}^N$ of finite positive measure such that (1.1) holds.*

Then, both K and G must be homothetic to the same ellipsoid.

The case $N = 2$ was first settled in [1] under some smoothness assumptions (∂K of class C^2 and ∂G of class C^4). It should also be noted that the proof in [1] works even if condition (1.1) holds when r takes a sufficiently small interval $(0, r_0)$, since it only uses local information on ∂G.

In [4], we were able to remove such regularity assumptions. In fact, we showed that in the plane the occurrence of property (1.1) implies that both ∂G and ∂K are necessarily of class C^∞. Moreover, we gave an alternative proof of the characterization, which is based on some local information on ∂G derived from (1.1) and classical affine inequalities for convex bodies.

In [4], we also established some facts that hold in general dimensions and will be useful in the remainder of this paper: let $K \subset \mathbb{R}^N$ be a convex body and assume that property (1.1) holds. Then

(i) G is strictly convex,

(ii) ∂G is at least of class $C^{1,1}$,

(iii) if K is centrally symmetric (i.e. $-K = K$), then $K = G - G$ up to dilations, K is strictly convex and ∂K is at least of class $C^{1,1}$,

(iv) if ∂G is differentiable at x, then

$$V(G \cap (x + rK)) = V_0(x) r^N + o(r^N) \quad \text{as } r \to 0^+,$$

(v) if ∂G is of class C^2 in a neighbourhood of x, then

$$V(G \cap (x + rK)) = V_0(x)r^N + V_1(x)r^{N+1} + o(r^{N+1}) \quad \text{as } r \to 0^+.$$

The coefficients $V_0(x)$ and $V_1(x)$ are explicitly computed; $G - G$ denotes the *Minkowski sum* of G and $-G$: $G - G = G + (-G) = \{x - y\colon x, y \in G\}$.

It will be useful to understand the mechanism of our proof in [4]. Since (1.1) holds, (ii) and (iv) imply that the function V_0 is constant on ∂G. By the explicit expression of $V_0(x)$, one gets that

$$V(\{y \in K\colon y \cdot \nu(x) \geqslant 0\}) = \tfrac{1}{2}V(K) \quad \text{for every } x \in \partial G, \tag{1.4}$$

where $\nu(x)$ denotes the exterior unit normal to ∂G at x. When $N = 2$, due to (i), it is not difficult to show that (1.4) implies that K is centrally symmetric; indeed, that is also true for $N \geqslant 3$, by a non-trivial result of [8]. Thus, (iii) comes into play and we can infer further regularity ($C^{2,1}$) for ∂G. Hence, (v) can be used: the function V_1 must also be constant on ∂G. This condition gives a pointwise constraint on the curvature of ∂G (see [4, (1.8)]) that, for $N = 2$, ensures that $K = 2G$ up to homotheties and, with the help of Minkowski's inequality for mixed volumes and an inequality involving the *affine surface area* of ∂G, gives the desired conclusion.

Now, let us look at the case in which $N \geqslant 3$. Of course, (i) and (ii) still hold if G is K-dense. Thus, the formula in (iv) still makes sense, and hence, by the aforementioned result [8], K is centrally symmetric; consequently, (iii) holds too. Therefore, (v) also makes sense and, even now, we can deduce that V_1 must be constant on ∂G. Unfortunately, the pointwise constraint on the principal curvatures [4, (1.8)] is no longer enough to deduce that $K = 2G$ and to conclude.

In this paper, we succeed in our aim by changing strategy: we give up the asymptotic expansion for $r \to 0^+$ in (v) in favour of an expansion such as

$$V(G \cap (x + rK)) = V(G) + W(x)(r_G - r)^{(N+1)/2} + o((r_G - r)^{(N+1)/2}) \tag{1.5}$$

as $r \to r_G^-$, where

$$r_G = \inf\{r > 0\colon G \subseteq x + rK\}, \quad x \in \partial G.$$

Note that, if G is K-dense, then r_G is independent on $x \in \partial G$; since our problem is invariant with respect to dilations of K, throughout the paper, we shall assume that $r_G = 1$.

The computation of the coefficient $W(x)$ is carried out in § 2 and involves the *support function* $h_K\colon \mathbb{S}^{N-1} \to \mathbb{R}$ of the convex body K with respect to the origin, and the *shape operators* S_G and S_K are of G and K, respectively. In fact, it turns out that for $x \in \partial G$

$$W(x) = -\frac{2\omega_{N-1} h_K(u)^{(N+1)/2}}{(N+1)\det[\frac{1}{2}S_G(u) - \frac{1}{2}S_K(u)]^{1/2}} \quad \text{with } u = \nu(\bar{x}); \tag{1.6}$$

here, $\{\bar{x}\} = \partial G \cap (x + K)$,[1] $\nu(\bar{x})$ is the exterior unit normal to ∂G at $\bar{x}$, and ω_{N-1} denotes the measure of the unit ball of $\mathbb{R}^{N-1}$.

[1] It will be made clear in § 2 that $\bar{x}$ is uniquely determined.

Properties (i) and (1.1) imply that the right-hand side of (1.6) must be constant as a function of $u \in \mathbb{S}^{N-1}$. A first consequence of this fact is that $K = 2G$ up to homotheties; a second consequence is that

$$\kappa_G(u) = c h_G(u)^{N+1} \quad \text{for every } u \in \mathbb{S}^{N-1}, \tag{1.7}$$

for some positive constant c; here, κ_G denotes the *Gauss curvature* of ∂G at the (unique) point $x \in \partial G$ having normal equal to u.

The identity (1.7) is well known in the theory of convex bodies: in fact, Petty proved in [7] that it characterizes G as an ellipsoid.

Section 2 contains all the details.

2. The proof of theorem 1.1

Let $G \subset \mathbb{R}^N$ be a C^2 convex body. In a sufficiently small neighbourhood of a point $x \in \partial G$, the set ∂G is the graph of a C^2-regular convex function over the tangent space to ∂G at x; we denote by S_G the Hessian of this function (the associated bilinear form is often called the *shape operator*); it is well known that its determinant κ_G is the *Gaussian curvature* of ∂G at that point. When G is strictly convex, without any ambiguity we can think of S_G as a function over the unit sphere, so that, for a given $u \in \mathbb{S}^{N-1}$, $S_G(u)$ denotes the shape operator at the only point $x \in \partial G$ with outward unit normal equal to u.

We know from [4] that if G is K-dense, then ∂G is of class C^2, but, unfortunately, we cannot assert that ∂K is of class C^2, even if we know that $K = G - G$ (see, for example, [2]). Nevertheless, in [3] it is shown that if G is strongly convex,[2] then K has the same regularity as G; in particular, the following result holds.

THEOREM 2.1 (Krantz and Parks [3]). *If A is a strongly convex body with boundary of class C^∞ and B is a convex body with boundary of class C^2, then the Minkowski sum $A + B$ has boundary of class C^∞.*

Moreover, the shape operator of $A + B$ can be expressed by the following formula:

$$S_{A+B}(u) = [I + S_A(u)^{-1} S_B(u)]^{-1} S_B(u). \tag{2.1}$$

The proof of this theorem can also be repeated step by step in the case in which the C^∞-regularity of A is replaced by its C^2-regularity: one then gets that the boundary of $A + B$ is of class C^2 and that (2.1) holds as well.

Thus, our aim is now to show that K-dense bodies are strongly convex; then, by theorem 2.1, we will gain the necessary regularity of K that gives a meaning to (2.1) with $A = G$ and $B = -G$.

In order to do this, for $x \in \partial G$ we shall study the asymptotic behaviour of $V(G \setminus (x + rK))$ as $r \to 1^-$. As we shall see, if we want to express $V(G \setminus (x + rK))$ in terms of the shape operator of ∂G at some point $\bar{x} \in \partial G$, it is important to make sure that G shares only one point with the boundary of $x + K$. We observe that this is not always the case: indeed, consider the Releaux triangle as the set G, and let x denote one of its vertices; then, $K = G - G$ is a ball and $G \cap (x + K)$ is one of the arcs constituting the triangle's boundary; hence, $G \setminus (x + rK)$ cannot be localized, so to speak, around any point of ∂G.

[2] That is, $S_K(u) > 0$ for every $u \in \mathbb{S}^{N-1}$, by which we mean that $S_K(u)$ is positive definite for every $u \in \mathbb{S}^{N-1}$.

Note that such a G is strictly convex, but ∂G is not differentiable. Likewise, if we consider differentiable bodies that are not strictly convex, we can still provide an example of the same phenomenon: in fact, it is enough to set $G = B + Q$, where B is the unit ball and Q is the unit square.

The following lemma shows that we can get the desired result if we assume that G is both differentiable and strictly convex.

LEMMA 2.2. *Let G be a strictly convex body with differentiable boundary and set $K = G - G$. Then, for each $x \in \partial G$, the set $\partial(x + K) \cap G$ consists of only one point, $\bar{x} \in \partial G$, characterized by $\nu_K(\bar{x} - x) = -\nu_G(x)$.*

Proof. Let $z \in \partial K \cap (G - x)$ and let $u = \nu_K(z)$. Clearly, $z + x \in \partial G$ and, since $G - x$ is contained in K and touches K at z from inside, then $\nu_G(z + x) = u$. Since $K = G - G$, we have

$$\begin{aligned} h_G(u) + h_G(-u) = h_K(u) &= \langle z, u\rangle \\ &= \langle z + x, u\rangle + \langle x, -u\rangle \\ &= h_G(u) + \langle x, -u\rangle. \end{aligned}$$

Thus, $h_G(-u) = \langle x, -u\rangle$, that is $\nu_G(x) = -u$. It is then enough to set $\bar{x} = z + x$.

Now, suppose that there exists another point, z', such that $z' \in \partial K \cap (G - x)$ and set $u' = \nu_K(z')$; by the same argument as above, we get that $\nu_G(x) = -u'$, and hence $u = u'$. Since K is strictly convex (as G is strictly convex), we finally find $z = z'$. □

The following lemma is helpful to prove that a K-dense set is positively curved.

LEMMA 2.3. *Let G be a strictly convex body with boundary of class C^2 and let $K = G - G$. For $x \in \partial G$ and $\bar{x} \in \partial G$ such that $u = \nu_G(x) = -\nu_G(\bar{x})$, the following hold:*

(i) *if $\kappa_G(u) = 0$, then*

$$\liminf_{r\to 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{(N+1)/2}} = +\infty;$$

(ii) *if $\kappa_G(u) > 0$, then*

$$\limsup_{r\to 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{(N+1)/2}} \leqslant \frac{2^{(N+1)/2}\omega_{N-1}}{N+1}\kappa_G(u) h_K(u)^{(N+1)/2}(1 + \varLambda)^{(N-1)/2},$$

where $\varLambda$ is the maximal principal curvature of ∂G at x.

Proof. First, note that, by the above lemma, our choice of x and $\bar{x}$ ensures that $\{\bar{x}\} = \partial(x + K) \cap G$. Without loss of generality, we can always assume that $\bar{x} = 0$ and that $u = (0, 0, \ldots, -1)$; then, in a neighbourhood of $\bar{x}$, ∂G can be parametrized by

$$y_N = \tfrac{1}{2}\langle S_G(u)y, y\rangle + o(|y|^2) \quad \text{as } |y| \to 0, \tag{2.2}$$

where $y = (y_1, \ldots, y_{N-1})$ ranges in the tangent space to ∂G at $\bar{x}$.

(i) Set $\varepsilon = 1 - r$. Let ε_n be an infinitesimal sequence of positive numbers such that

$$\liminf_{r\to 1^-} \frac{V(G\setminus(x+rK))}{(1-r)^{(N+1)/2}} = \lim_{n\to\infty} \frac{V(G_n)}{\varepsilon_n^{(N+1)/2}},$$

where $G_n\colon = G\setminus(x+(1-\varepsilon_n)K)$; then (2.2) suggests that, by possibly extracting a subsequence from ε_n, we can fit in G_n the set E_n bounded by the paraboloid

$$y_N = \tfrac{1}{2}\langle S_G(u)y, y\rangle + \frac{1}{n}|y|^2$$

and the hyperplane $\varepsilon_n h_K(u)u + u^\perp$ supporting the set $x+(1-\varepsilon_n)K$ at the point whose outer unit normal coincides with u. In our coordinates,

$$E_n = \left\{(y, y_N)\colon \tfrac{1}{2}\langle S_G(u)y, y\rangle + \frac{1}{n}|y|^2 < y_N < \varepsilon_n h_K(u)\right\}$$

and $E_n \subseteq G_n$.

Thus, by Fubini's theorem and some calculations, we get

$$\begin{aligned} V(G_n) \geqslant V(E_n) &= \int_0^{\varepsilon_n h_K(u)} \mathcal{H}^{N-1}\left(\left\{y\colon \left\langle\left[\frac{S_G(u)}{2} + \frac{1}{n}I\right]y, y\right\rangle \leqslant t\right\}\right)\mathrm{d}t \\ &= \frac{\omega_{N-1}}{\det[S_G(u)/2 + I/n]^{1/2}} \int_0^{\varepsilon_n h_K(u)} t^{(N-1)/2}\,\mathrm{d}t \\ &= \frac{2\omega_{N-1}\varepsilon_n^{(N+1)/2} h_K(u)^{(N+1)/2}}{(N+1)\det[S_G(u)/2 + I/n]^{1/2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \liminf_{\varepsilon\to 0^+} \frac{V(G\setminus(x+(1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} &= \lim_{n\to\infty} \varepsilon_n^{-(N+1)/2} V(G_n) \\ &\geqslant \lim_{n\to\infty} \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\sqrt{\det[S_G(u)/2 + I/n]}} \\ &= +\infty, \end{aligned}$$

since $\det S_G(u) = \kappa_G(u) = 0$.

(ii) We obtain the desired inequality by observing that the domain $G\setminus(x+(1-\varepsilon)K)$ can be contained in the region $F_{\varepsilon,\delta}$ bounded by two paraboloids: one outside G and tangential to ∂G at $\bar{x}$, the other tangent to the boundary of $x+(1-\varepsilon)K$ from inside. In order to show this, we assume as before that $\bar{x} = 0$ and $u = -e_N$ and, moreover, that $S_G(u) = I$ (this can be done since the affine transformation $S_G(u)$ is invertible, as $\det S_G(u) = \kappa_G(u) > 0$): the desired formula will then be obtained by multiplying the right-hand side of (2.3) by the factor $\kappa_G(u)$.

We proceed to construct $F_{\varepsilon,\delta}$. We choose any number $\lambda > 0$ such that $\lambda I > S_G(-u)$, i.e. such that $\lambda > \Lambda$. Since $\kappa_G(u) > 0$, theorem 2.1 implies that ∂K is twice differentiable at $\bar{x} - x$; moreover, (2.1) turns into

$$S_K(u) < \frac{\lambda}{1+\lambda} I;$$

hence,

$$S_{(1-\varepsilon)K}(u) < \frac{\lambda}{(1+\lambda)(1-\varepsilon)} I.$$

For $\varepsilon > 0$ sufficiently small, we define $F_{\varepsilon,\delta}$ as

$$F_{\varepsilon,\delta} = \left\{ (y, y_N)\colon \tfrac{1}{2}\delta|y|^2 \leqslant y_N \leqslant \varepsilon h_K(u) + \frac{\lambda}{2(1+\lambda)(1-\varepsilon)}|y - \varepsilon x_*|^2 \right\},$$

where δ is chosen in the interval $(\lambda/(1+\lambda)(1-\varepsilon), 1)$ and x_* is the projection of x on the tangent space to ∂G at $\bar{x}$; in this way,

$$G \setminus (x + (1-\varepsilon)K) \subset F_{\varepsilon,\delta}.$$

Indeed, (2.2) guarantees that the above inclusion holds, at least inside a small neighbourhood of $\bar{x}$; however, by lemma 2.2, we know that $G \setminus (x + (1-\varepsilon)K)$ is contained in a ball B_r around $\bar{x}$ whose radius $r = r(\varepsilon)$ tends to 0 as $\varepsilon \to 0$.

By using the rescaling $(y, y_N) = (\sqrt{\varepsilon}\xi, \varepsilon\xi_N)$, we obtain that

$$V(F_{\varepsilon,\delta}) = \varepsilon^{(N+1)/2} V(F'_{\varepsilon,\delta}),$$

where

$$F'_{\varepsilon,\delta} = \left\{ (\xi, \xi_N)\colon \tfrac{1}{2}\delta|\xi|^2 \leqslant \xi_N \leqslant h_K(u) + \frac{\lambda}{2(1+\lambda)(1-\varepsilon)}|\xi - \sqrt{\varepsilon} x_*|^2 \right\},$$

and it is easy to show that $V(F'_{\varepsilon,\delta}) \to V(F'_{0,\delta})$. By a straightforward computation of $V(F'_{0,\delta})$, we get that

$$\limsup_{\varepsilon \to 0} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \leqslant \frac{\omega_{N-1}}{N+1} \times \frac{2^{(N+1)/2} h_K(u)^{(N+1)/2}}{(\delta - \lambda/(1+\lambda))^{(N-1)/2}},$$

and minimizing the right-hand side of this formula for $\lambda/(1+\lambda) < \delta < 1$ and $\lambda > \Lambda$ then gives

$$\limsup_{\varepsilon \to 0} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \leqslant \frac{\omega_{N-1}}{N+1} \times 2^{(N+1)/2} h_K(u)^{(N+1)/2} (1+\Lambda)^{(N-1)/2}. \tag{2.3}$$

□

Corollary 2.4. *If G is K-dense, then ∂K is of class C^2 and every point of ∂G is a point of strong convexity. The latter conditions and the fact that $K = G - G$ allow us to write, due to theorem 2.1,*

$$S_K(u) = [I + S_G(u)^{-1} S_G(-u)]^{-1} S_G(-u). \tag{2.4}$$

Proof. Since G is K-dense, the limits in lemma 2.3(i) and (ii) do not depend on the particular point $x \in \partial G$; in other words, they must be constant functions on ∂G. Since G is a convex body and ∂G is of class C^2, κ_G is not identically zero; hence, the limit in lemma 2.3(ii) is a finite constant. As a consequence, (i) of the same lemma implies that $\kappa_G > 0$ (and hence $S_G > 0$) on ∂G. Formula (2.4) is then a straightforward consequence of theorem 2.1. □

THEOREM 2.5. *Let G be a strongly convex body with boundary of class C^2, and set $K = G - G$. Choose x, u and $\bar{x}$ as in lemma 2.3; then*

$$\lim_{r\to 1^-} \frac{V(G\setminus(x+rK))}{(1-r)^{(N+1)/2}} = \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\det[\frac{1}{2}S_G(u)-\frac{1}{2}S_K(u)]^{1/2}}.$$

Proof. Again we set $\varepsilon = 1-r$. We begin by showing that

$$\limsup_{\varepsilon\to 0^+} \frac{V(G\setminus(x+(1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \leqslant \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\det[\frac{1}{2}S_G(u)-\frac{1}{2}S_K(u)]^{1/2}}. \tag{2.5}$$

As in the proof of lemma 2.3, without loss of generality, we can set $u = -e_N$ and $\bar{x} = 0$.

We recall that $0 = \bar{x} \in \partial(x+K)$. Thus, $-x \in \partial K$ and $-(1-\varepsilon)x \in \partial((1-\varepsilon)K)$; namely, $\varepsilon x \in \partial(x+(1-\varepsilon)K)$, and u is the unit normal to $\partial(x+(1-\varepsilon)K)$ at that point; also, by a scaling argument, we know that

$$S_{x+(1-\varepsilon)K}(u) = \frac{S_K(u)}{1-\varepsilon}.$$

Note that (2.4) implies that $S_G(u) > S_K(u)$; hence, we can choose $\bar{n}\in\mathbb{N}$ such that

$$\frac{S_G(u)-S_K(u)}{4} > \frac{I}{\bar{n}}. \tag{2.6}$$

In order to get an estimate from above for $V(G\setminus(x+(1-\varepsilon)K))$ we construct a set $C_{\varepsilon,n}$ containing $G\setminus(x+(1-\varepsilon)K)$. In fact, for $n > \bar{n}$ we set

$$C_{\varepsilon,n} = \bigg\{(y,y_N)\colon \bigg\langle\bigg(\frac{S_G(u)}{2}-\frac{1}{n}I\bigg)y,y\bigg\rangle < y_N < \varepsilon h_K(u) + \bigg\langle\bigg[\frac{S_K(u)}{2(1-\varepsilon)}+\frac{1}{n}I\bigg](y-\varepsilon x_*),(y-\varepsilon x_*)\bigg\rangle\bigg\},$$

where x_* denotes the projection of x on $u^\perp$; $C_{\varepsilon,n}$ is the region bounded by two paraboloids, one touching ∂G at $\bar{x}$ from below, the other touching $\partial(x+(1-\varepsilon)K)$ at εx from above and, for ε small enough, we have

$$G\setminus(x+(1-\varepsilon)K) \subset C_{\varepsilon,n}.$$

Also, condition (2.6) guarantees that

$$\frac{S_G(u)}{2} - \frac{I}{n} > \frac{S_K(u)}{2(1-\varepsilon)} + \frac{I}{n} > 0$$

for ε small enough, thus forcing $C_{\varepsilon,n}$ to be bounded.

The usual change of variables $(y,y_N) = (\sqrt{\varepsilon}\xi, \varepsilon\xi_N)$ gives that

$$V(C_{\varepsilon,n}) = \varepsilon^{(N+1)/2}V(C'_{\varepsilon,n}),$$

where

$$C'_{\varepsilon,n} = \left\{(\xi,\xi_N)\colon \left\langle\left[\frac{S_G(u)}{2} - \frac{1}{n}I\right]\xi,\xi\right\rangle < \xi_N < h_K(u) + \left\langle\left[\frac{S_K(u)}{2(1-\varepsilon)} + \frac{1}{n}I\right](\xi-\sqrt{\varepsilon}x_*),(\xi-\sqrt{\varepsilon}x_*)\right\rangle\right\}.$$

Since clearly $V(C'_{\varepsilon,n}) \to V(C'_{0,n})$ as $\varepsilon \to 0$, a straightforward computation gives

$$\limsup_{\varepsilon\to 0^+} \frac{V(G\setminus(x+(1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \leqslant V(C'_{0,n}) = \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\det[S_G(u)/2 - S_K(u)/2 - 2I/n]^{1/2}}. \tag{2.7}$$

Since (2.7) holds for all n large enough, (2.5) follows at once by taking the limit for $n\to\infty$.

The converse inequality,

$$\liminf_{\varepsilon\to 0^+} \frac{V(G\setminus(x+(1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \geqslant \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\det[\frac{1}{2}S_G(u) - \frac{1}{2}S_K(u)]^{1/2}},$$

is proved by using the same strategy used for (2.5): we choose $\bar{n}$ such that

$$S_G(u) > \frac{I}{\bar{n}}$$

and then we construct, for $n > \bar{n}$ and ε small, a set $D_{\varepsilon,n} \subseteq G\setminus(x+(1-\varepsilon)K)$:

$$D_{\varepsilon,n} = \left\{(y,y_N)\colon \left\langle\left(\frac{S_G(u)}{2} + \frac{1}{n}I\right)y,y\right\rangle < y_N < \varepsilon h_K(u) + \left\langle\left[\frac{S_K(u)}{2(1-\varepsilon)} - \frac{1}{n}I\right](y-\varepsilon x_*),(y-\varepsilon x_*)\right\rangle\right\}.$$

As before, the usual rescaling gives

$$\liminf_{\varepsilon\to 0^+} \frac{V(G\setminus(x+(1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \geqslant \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\det[S_G(u)/2 - S_K(u)/2 + 2I/n]^{1/2}}.$$

Again, we conclude by taking the limit for $n\to\infty$. □

Corollary 2.6. *Let G be a K-dense body. Then (1.5) holds with the coefficient $W(x)$ given by (1.6). Moreover, the function defined by*

$$\frac{h_K(u)^{(N+1)/2}}{\det[S_G(u) - S_K(u)]^{1/2}}, \quad u\in\mathbb{S}^{N-1}, \tag{2.8}$$

is constant.

Proof. Corollary 2.4 ensures that G satisfies the assumptions of theorem 2.5. Since $V(G\cap(x+rK)) = V(G) - V(G\setminus(x+rK))$, the following equality holds for the function given in (1.6):

$$W(x) = -\lim_{r\to 1^-} \frac{V(G\setminus(x+rK))}{(1-r)^{(N+1)/2}} = \lim_{r\to 1^-} \frac{V(G\cap(x+rK)) - V(G)}{(1-r)^{(N+1)/2}}. \tag{2.9}$$

Hence, (1.5) holds. Observe that $W|_{\partial G}$ has to be constant, by (2.9) and the K-density assumption. Finally, since G is strictly convex, the last assertion follows from the surjectivity of the Gauss map. □

Now, we shall show that if G is K-dense, then G and K must be equal up to homotheties.

PROPOSITION 2.7. *Let G be a K-dense body. Then $\kappa_G(u) = \kappa_G(-u)$.*

Proof. Let $u \in \mathbb{S}^{N-1}$ and let $L = L_u$ be a linear map of $\mathbb{R}^N$ in itself, which leaves unchanged the unit vector u and whose restriction to $u^\perp$ equals $S_G(u)^{-1/2}$.

Firstly, note that, as an easy consequence of (1.1), the set LG is LK-dense, so that corollary 2.6 holds for this set; in particular, (2.8) implies

$$\begin{aligned} &h_{LK}(-u)^{(N+1)/2}\{\det[S_{LG}(-u) - S_{LK}(-u)]\}^{-1/2} \\ &\qquad = h_{LK}(u)^{(N+1)/2}\{\det[S_{LG}(u) - S_{LK}(u)]\}^{-1/2}. \end{aligned} \tag{2.10}$$

Secondly, we know that K is centrally symmetric, and so must be LK; then, $S_{LK}(u) = S_{LK}(-u)$ and $h_{LK}(u) = h_{LK}(-u)$. Hence, by (2.10),

$$\det[S_{LG}(-u) - S_{LK}(u)] = \det[S_{LG}(u) - S_{LK}(u)]. \tag{2.11}$$

As we shall see, this condition, together with (2.1), is enough to prove that

$$\det[S_{LG}(u)] = \det[S_{LG}(-u)].$$

Indeed, by plugging (2.1) into (2.11) we get

$$\begin{aligned} &\det(S_{LG}(-u) - [I + S_{LG}(u)^{-1}S_{LG}(-u)]^{-1}S_{LG}(-u)) \\ &\qquad = \det(S_{LG}(u) - [I + S_{LG}(u)^{-1}S_{LG}(-u)]^{-1}S_{LG}(-u)); \end{aligned} \tag{2.12}$$

furthermore, our choice of the affine transformation L ensures that

$$S_{LG}(u) = I$$

and

$$S_{LG}(-u) = S_G(u)^{-1/2}S_G(-u)S_G(u)^{-1/2}. \tag{2.13}$$

Equation (2.12) then turns into

$$\det(S_{LG}(-u) - [I + S_{LG}(-u)]^{-1}S_{LG}(-u)) = \det(I - [I + S_{LG}(-u)]^{-1}S_{LG}(-u)); \tag{2.14}$$

by multiplying both sides of (2.14) by $\det[I + S_{LG}(-u)]$ and using Binet's identity, we get

$$\det[S_{LG}(-u)^2] = 1.$$

Hence, (2.13) yields $\det[S_G(u)] = \det[S_G(-u)]$, i.e. $\kappa_G(u) = \kappa_G(-u)$. □

Corollary 2.8. *Let G be K-dense. Then G is symmetric and $K = 2G$.*

Proof. The two bodies $G - G$ and $2G$ have the same Gaussian curvature as a function on $\mathbb{S}^{N-1}$; thus, they only differ by a translation. □

The following theorem and Petty's characterization of ellipsoids [7] complete the proof of theorem 1.1.

Theorem 2.9. *Let G be a K-dense set. Then, for every $x \in \partial G$ it holds that*

$$\lim_{r \to 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{(N+1)/2}} = \frac{2^N \omega_{N-1} h_K(u)^{(N+1)/2}}{(N+1) \det[S_G(u)]^{1/2}} \quad \text{with } u = \nu(\bar{x})$$

and $\{\bar{x}\} = \partial G \cap (x + K)$.

In particular, there exists a positive constant c, depending only on N, such that

$$\kappa_G(u) = c h_G(u)^{N+1} \quad \text{for every } u \in \mathbb{S}^{N-1}.$$

Therefore, G must be an ellipsoid.

Acknowledgements

The authors are deeply indebted to the anonymous referee for the careful revision and comments that greatly improved the paper.

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