

ON A GENERALIZATION OF THE CORONA PROBLEM

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ABSTRACT. Let $g, f_1, \dots, f_m \in H^\infty(\Delta)$. We provide conditions on f_1, \dots, f_m in order that $|g(z)| \leq |f_1(z)| + \dots + |f_m(z)|$, for all z in Δ , imply that g , or g^2 , belong to the ideal generated by f_1, \dots, f_m in H^∞ .

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1. INTRODUCTION.

Let $H(\Delta) = H$ be the space of all holomorphic functions on $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and let $H^\infty(\Delta) = H^\infty$ be the subspace of all bounded functions of $H(\Delta)$. Let f_1, \dots, f_m be functions in H^∞ and let $g \in H^\infty$ satisfy the following condition:

$$|g(z)| \leq |f_1(z)| + \dots + |f_m(z)| \quad (\text{any } z \in \Delta). \quad (1.1)$$

As a generalization of the corona problem (which was first solved by Carleson [1]) it is natural to ask if (1.1) implies that g belongs to the ideal $I_{H^\infty}(f_1, \dots, f_m)$ generated in H^∞ by f_1, \dots, f_m , i.e. if (1.1) implies the existence of g_1, \dots, g_m in H^∞ such that, on Δ ,

$$g = f_1 g_1 + \dots + f_m g_m. \quad (1.2)$$

Rao, [2], has shown that the answer to this question is negative in general. On the other hand Wolff (see [3], th. 2.3) has proved that (1.1) implies that g^3 belongs to $I_{H^\infty}(f_1, \dots, f_m)$. The question whether (1.1) implies the existence of g_1, \dots, g_m in H^∞ such that

$$g^2 = f_1 g_1 + \dots + f_m g_m \quad (1.3)$$

is still open, as Garnett has pointed out ([4], problem 8.20).

In this work we obtain some results on this generalized corona problem, making use of techniques which appear in the theory of A_p spaces, the spaces of entire functions with growth conditions introduced by Hörmander [5].

With the same aim of Berenstein and Taylor [6] in A_p , we introduce in H^∞ the notion of jointly invertible functions (definition 3) and prove that if f_1, \dots, f_m are jointly invertible, condition (1.1) implies that g belongs to $I_{H^\infty}(f_1, \dots, f_m)$ (proposition 5). We also prove that if the ideal $I_{H^\infty}(f_1, \dots, f_m)$ contains a weakly invertible

function having simple interpolating zeroes (see [3]), then again (1.1) implies that g belongs to $I_{H^\infty}(f_1, \dots, f_m)$ (theorem 6).

Finally, in the same spirit of Kelleher and Taylor [7] we introduce the notion of congeniality for m -tuples of functions in H^∞ , and give a partial answer to the problem posed by Garnett ([4]): we prove that if $(f_1, \dots, f_m) \in (H^\infty)^m$ is congenial, then (1.1) implies $g^2 \in I_{H^\infty}(f_1, \dots, f_m)$ (theorem 8).

2. WEAK INVERTIBILITY.

We first study some conditions under which (1.1) implies that $g \in I_{H^\infty}(f_1, \dots, f_m)$.

DEFINITION 1. A function f in $H^\infty(\Delta)$ is called weakly invertible if there exists a Blaschke product B such that $f(z) = B(z)\tilde{f}(z)$ (z in Δ) with \tilde{f} invertible in H^∞ .

The reason for this definition is the following simple criterion of divisibility for functions in H^∞ .

PROPOSITION 2. Let $f \in H^\infty$. Then f is weakly invertible if, and only if, for all $g \in H^\infty$ the fact that $g/f \in H$ implies $g/f \in H^\infty$.

-PROOF. Suppose f is weakly invertible: then there exists a Blaschke product B such that $f(z) = B(z)\tilde{f}(z)$, with \tilde{f} invertible in H^∞ . Since g/f is holomorphic and since B contains exactly the zeroes of f , it follows that $g/B \in H$; however, since B is a Blaschke product, $g/B \in H$ implies, [8], that $g/B \in H^\infty$. Since $1/\tilde{f} \in H^\infty$ one has $g/f = (g/B)(1/\tilde{f})$, i.e. $g/f \in H^\infty$. Conversely, suppose that for all $g \in H^\infty$ such that $g/f \in H$, it follows $g/f \in H^\infty$. Write $f(z) = B(z)\tilde{f}(z)$, where B is the Blaschke product of all the zeroes of f (see [8]). Then B/f is holomorphic on Δ and therefore $1/\tilde{f}$ must belong to H^∞ . ¶

An extension of the notion of weak invertibility to m -tuples of functions in H^∞ is given by the following definition, analogous to the one given by Berenstein and Taylor for the spaces A_P in [6].

DEFINITION 3. The functions $f_1, \dots, f_m \in H^\infty$ are called jointly invertible if the ideal generated by f_1, \dots, f_m in H^∞ coincides with $I_{loc}(f_1, \dots, f_m) = \{g \in H^\infty(\Delta) : \text{for any } z \in \Delta, \text{ there exists a neighborhood } U \text{ of } z \text{ and } \lambda_1, \dots, \lambda_m \text{ in } H(U) \text{ such that } g = \lambda_1 f_1 + \dots + \lambda_m f_m \text{ on } U\}$.

In view of Cartan's theorem B, it follows immediately that f_1, \dots, f_m are jointly invertible if, and only if, $I_{H^\infty}(f_1, \dots, f_m) = I_H(f_1, \dots, f_m)$, the latter being the ideal generated by f_1, \dots, f_m in $H(\Delta)$. As a consequence of the corona theorem, all m -tuples f_1, \dots, f_m in H^∞ for which there exists $\delta > 0$ such that $|f_1(z)| + \dots + |f_m(z)| \geq \delta$ for all z in Δ , are jointly invertible ($I_H = I_{H^\infty} = H^\infty$). More generally one has:

PROPOSITION 4. Let $b \in H^\infty$ be weakly invertible, and let $f_1(z) = b(z)\tilde{f}_1(z), \dots, f_m(z) = b(z)\tilde{f}_m(z)$, for $\tilde{f}_1, \dots, \tilde{f}_m$ in H^∞ such that $|\tilde{f}_1(z)| + \dots + |\tilde{f}_m(z)| \geq \delta > 0$ for some δ and all z in Δ . Then f_1, \dots, f_m are jointly invertible.

PROOF. Let $g \in H^\infty$ belong to $I_H(f_1, \dots, f_m)$. There exist $\lambda_1, \dots, \lambda_m$ in $H(\Delta)$ such that

$$g(z) = \lambda_1(z)f_1(z) + \dots + \lambda_m(z)f_m(z) \quad (\text{all } z \in \Delta) \quad (2.1)$$

i.e., for all z in Δ ,

$$g(z) = b(z) [\lambda_1(z)\tilde{f}_1(z) + \dots + \lambda_m(z)\tilde{f}_m(z)]. \quad (2.2)$$

Since b is invertible, and $g/b \in H$, it follows that $\tilde{g} = g/b = \lambda_1\tilde{f}_1 + \dots + \lambda_m\tilde{f}_m \in H^\infty$. By the corona theorem, then, it follows that there are h_1, \dots, h_m in H^∞ such that

$$\tilde{g}(z) = h_1(z)\tilde{f}_1(z) + \dots + h_m(z)\tilde{f}_m(z), \quad (2.3)$$

therefore

$$g(z) = \tilde{g}(z)b(z) = h_1(z)f_1(z) + \dots + h_m(z)f_m(z) \quad (2.4)$$

and the assertion is proved.

Let now $f_1, \dots, f_m, g \in H^\infty(\Delta)$, and suppose that (1.1) holds. It is well known, [2], that in general (1.1) does not imply that $g \in I_{H^\infty}(f_1, \dots, f_m)$. However, (1.1) certainly implies that $g \in I_{\text{loc}}(f_1, \dots, f_m)$ and hence

PROPOSITION 5. Let f_1, \dots, f_m be jointly invertible. Then if g satisfies condition (1.1), it follows that $g \in I_{H^\infty}(f_1, \dots, f_m)$.

A different situation in which (1.1) implies that $g \in I_{H^\infty}(f_1, \dots, f_m)$ occurs when at least one of the f_j 's, say f_1 , is weakly invertible and has simple zeroes which form an interpolating sequence ([3]); this happens, for example, when f_1 is an interpolating Blaschke product with simple zeroes ([3]). Indeed, following an analogous result proved in [7] for the space of entire functions of exponential type, one has:

THEOREM 6. Let $f_1, \dots, f_m \in H^\infty$, and suppose f_1 is weakly invertible with simple, interpolating zeroes. Then if $g \in H^\infty$ satisfies condition (1.1) it follows that g belongs to $I_{H^\infty}(f_1, \dots, f_m)$.

PROOF. Choose $a_{ij} \in \mathbb{C}$, $i=2, \dots, m$, $j \geq 1$, such that for $\{z_j\} = \{z \in \Delta : f_1(z) = 0\}$ it is $|a_{ij}| = 1$ and $a_{ij} f_i(z_j) \geq 0$. Define now $b_{ij} \in \mathbb{C}$ (i, j as before) by

$$b_{ij} = \begin{cases} 0 & \text{if } f_2(z_j) = \dots = f_m(z_j) = 0 \\ a_{ij} g(z_j) / (|f_2(z_j)| + \dots + |f_m(z_j)|) & \text{otherwise.} \end{cases}$$

By (1.1) it follows $|b_{ij}| \leq 1$ (all i, j), and since $\{z_j\}$ is interpolating, one finds h_2, \dots, h_m in H^∞ such that $h_i(z_j) = b_{ij}$. Therefore the function $h = g - (h_2 f_2 + \dots + h_m f_m)$ belongs to H^∞ and vanishes at each z_j . The simplicity of the zeroes of f_1 shows that $h/f_1 \in H$, and the invertibility of f_1 implies $h/f_1 = h_1 \in H^\infty$. The thesis now follows, since $g = f_1 h_1 + \dots + f_m h_m$. \square

It is worthwhile noticing that the hypotheses of Proposition 5 and Theorem 6 are not comparable. Consider, indeed, the following conditions on $f_1, \dots, f_m \in H^\infty$:

(C₁) f_1, \dots, f_m are jointly invertible.

(C₂) there exists j ($1 \leq j \leq m$) such that f_j is invertible, with an interpolating sequence of zeroes, all of which are simple.

Then (C₁) does not imply (C₂): take $m=1$ and f_1 weakly invertible with non-simple zeroes.

On the other hand, also (C₂) does not imply (C₁): consider f_1 invertible with simple interpolating zeroes $\{z_n\}$; let $f_2 \in H^\infty$ be a function such that $f_2(z_n) = 1/n$ (such a function certainly exists since $\{z_n\}$ is an interpolating sequence); now f_1 and f_2 have no common zeroes, and hence $1 \in I_{\text{loc}}(f_1, f_2)$; however $1 \notin I_{H^\infty}(f_1, f_2)$ since if $1 = \lambda_1 f_1 + \lambda_2 f_2$, then it is $\lambda_2(z_n) = n$, i.e. $\lambda_2 \notin H^\infty$. Therefore the pair (f_1, f_2) satisfies (C₂) but not (C₁).

3. CONGENIALITY.

In this section we describe a class of m -tuples of functions in $H^\infty(\Delta)$, for which condition (1.1) implies that $g^2 \in I_{H^\infty}(f_1, \dots, f_m)$.

DEFINITION 7. An m -tuple (f_1, \dots, f_m) of functions in H^∞ is called congenial if, for all $i, j=1, \dots, m$,

$$(f_i f_j' - f_j f_i') / \|f\|^2 \|f'\| \text{ belongs to } L^\infty(\Delta),$$

where $\|f(z)\|^2 = |f_1(z)|^2 + \dots + |f_m(z)|^2$, $\|f'(z)\|^2 = |f_1'(z)|^2 + \dots + |f_m'(z)|^2$, and $f_i' = \partial f_i / \partial z$.

Notice that the class of congenial m -tuples is not empty. Indeed, one might consider pairs f_1, f_2 in H^∞ which, at their common zeroes, satisfy some simple conditions on their vanishing order easily deducible from Definition 7. For example, one can ask that $f_1(z_0)=f_2(z_0)=0, f_2'(z_0)\neq 0, f_1'(z_0)=0$. As a partial answer to problem 8.20 in [4], we prove the following

THEOREM 8. Let $f_1, \dots, f_m, g \in H^\infty(\Delta)$, and suppose (f_1, \dots, f_m) be congenial. If g satisfies (1.1), then $g^2 \in I_{H^\infty}(f_1, \dots, f_m)$, i.e. there are g_1, \dots, g_m in H^∞ such that (on Δ)

$$g^2(z) = f_1(z)g_1(z) + \dots + f_m(z)g_m(z) \tag{3.1}$$

PROOF. We mainly follow the proof due to Wolff, [3], of the fact that (1.1) implies that $g^3 \in I_{H^\infty}$. We can assume $\|f_j\|_\infty \leq 1, \|g\|_\infty \leq 1$, and $f_j, g \in H(\bar{\Delta})$ ($j=1, \dots, m$). Put $\psi_j = g\bar{f}_j / |f|^2$ (ψ_j is bounded and C^∞ on $\bar{\Delta}$) and consider the differential equation

$$\partial b_{j,k} / \partial \bar{z} = \psi_j \partial \psi_k / \partial \bar{z} = g^2 G_{j,k} \quad (1 \leq j, k \leq m) \tag{3.2}$$

for

$$G_{j,k} = \bar{f}_j \sum_{\ell} f_{\ell} \overline{(f_{\ell} f'_k - f'_k f_{\ell})} / |f|^6.$$

If solutions $b_{j,k} \in L^\infty$ exist, then clearly $g_j = g\psi_j + \sum_k (b_{j,k} - b_{k,j}) f_k \in H^\infty$ and (3.1) holds (indeed $\bar{\partial} g_j = 0$ and g_j is bounded on Δ). In order to prove that (3.2) admits a solution in L^∞ it is enough to show that $|g^2 G_{j,k}|^2 \log(1/|z|) dx dy$ and $\partial (g^2 G_{j,k}) / \partial z$ are Carleson measures for $1 \leq j, k \leq m$.

As far as $|g^2 G_{j,k}|^2 \log(1/|z|) dx dy$ is concerned, notice that, by the congeniality of (f_1, \dots, f_m) , it is

$$|g^2 G_{j,k}|^2 \leq |g|^4 |\bar{f}_j|^2 \left| \sum_{\ell} f_{\ell} \overline{(f_{\ell} f'_k - f'_k f_{\ell})} \right|^2 / |f|^{12} \leq C |f'|^2.$$

On the other hand,

$$\partial (g^2 G_{j,k}) / \partial z = 2g g' G_{j,k} + g^2 G_{j,k} / \partial z;$$

again by the congeniality of (f_1, \dots, f_m) , one has

$$\begin{aligned} |g g' G_{j,k}| &\leq |g| |g'| |\bar{f}_j| \left| \sum_{\ell} f_{\ell} \overline{(f_{\ell} f'_k - f'_k f_{\ell})} \right| / |f|^6 \leq C (|g'|^2 + \|f'\|^2) / |f| \leq \\ &\leq C (|g'|^2 / |g| + \|f'\|^2 / |f|), \end{aligned}$$

and

$$\begin{aligned} |g^2 \partial G_{j,k} / \partial z| &= |g|^2 \cdot |\bar{f}_j| \left| \sum_{\ell} \bar{f}_{\ell} f'_{\ell} \right| \cdot \left| \sum_{\ell} f_{\ell} \overline{(f_{\ell} f'_k - f'_k f_{\ell})} \right| / |f|^8 + \\ &+ |g|^2 |\bar{f}_j| / |f|^2 \cdot \left(\left| \sum_{\ell} f_{\ell} \overline{(f_{\ell} f'_k - f'_k f_{\ell})} \right| / |f|^4 + 2 \left| \sum_{\ell} f_{\ell} \bar{f}_{\ell} \right| \left| \sum_{\ell} f_{\ell} \overline{(f_{\ell} f'_k - f'_k f_{\ell})} \right| / |f|^6 \right) \leq \\ &\leq C \sum_{\ell} |f_{\ell}|^2 / |f_{\ell}|. \end{aligned}$$

This concludes the proof. ¶

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