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Exact solutions for symmetric magnetohydrodynamic equilibria with mass flow

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Abstract. In this paper the problem of ideal magnetohydrodynamic equilibria with mass flow is treated. Under the assumption of general symmetry (i.e. one ignorable spatial coordinate) we derive a generalized Grad-Shafranov equation in an unspecified curvilinear coordinate system. If incompressibility is assumed an elliptical equation is derived and a new, *totally analytical* method of solution is proposed. This is based on a particular *self-similar* separation of the variables in the unknown flux function and leads to an ordinary, non-linear differential equation for the profile of the magnetic and flow surfaces.

Three novel classes of solutions are derived in different geometries, all being flexible (they contain a minimum of three free functions) and regular, which makes them suitable for astrophysical applications. These are flows in magnetic flux tubes with non-circular section, flows in magnetic arcades above the solar surface and collimated, axisymmetric outflows.

Key words: magnetohydrodynamics (MHD) – plasmas – methods: analytical – Sun: corona – ISM: jets and outflows

1. Introduction

The purpose of this paper is to present a general and completely analytical method of solution to the set of stationary, symmetric, ideal magnetohydrodynamic (MHD) equations with mass flow, where symmetric means that one of the three spatial coordinates is ignorable. Symmetric configuration of plasmas with steady mass flows occur both in laboratory applications (generally in axial symmetry) and in a great variety of astrophysical situations, such as stellar and extra-galactic winds or collimated outflows (again in axial symmetry) and flows within coronal loops or arcades in the solar atmosphere (usually with translational symmetry).

Symmetric MHD flows were first considered by Chandrasekhar (1956), who treated the case of an axisymmetric, incompressible plasma. After the formalism of the magnetic flux function had been introduced by Shafranov (1957) for static equilibria, Woltjer (1959a, 1959b) was the first to apply this formalism to the dynamic problem, deriving some integrals of the system. Helical symmetry, the most general kind of spatial invariance, including translational and azimuthal invariances as sub-cases, was first treated by Morozov & Solov'ev (1963), who derived the set of reduced equations for an adiabatic flow in the non-orthogonal helical coordinate system. A systematic review of the equations of MHD symmetric equilibria in the three different geometries can be found in the series of papers by Tsinganos (1981, 1982a, 1982b, 1982c), whereas the relativistic case is treated in Lovelace et al. (1986) and in Bogovalov (1994). Unified treatments in a general curvilinear coordinate system are given by Edenstrasser (1980a) for the static case and by Agim & Tataronis (1985) for the dynamic problem.

The analytical resolution of the set of ideal MHD equations, despite the hypotheses of stationarity and symmetry, is still a hard task. As we shall see, the MHD equations can be reduced to a second order, quasi-linear, partial differential equation for a magnetic flux function and a non-linear, algebraic, Bernoulli-type equation for the mass density, strongly coupled together. Unlike the classical Grad-Shafranov equation for static equilibria, which is always elliptical, the equation for dynamic equilibria may become hyperbolic at certain critical speeds. Usually a semi-analytical approach is followed: a self-similar behaviour for the flux function is assumed and then the two coupled equations are solved numerically. To this class belong works on polytropic MHD jets from disks in axisymmetry (Bardeen & Berger 1978; Blandford & Payne 1982; Contopoulos & Lovelace 1994; Ferreira & Pelletier 1994) or isothermal flows in uniform gravity for solar applications (Tsinganos et al. 1993; Del Zanna & Hood 1995). A way to avoid the difficulties related with the coupling of the two equations consists in not specifying any energy equation and in deriving the temperature and the other thermodynamical quantities only a-posteriori. Examples of this approach are models of non-polytropic, MHD, axisymmetric winds and outflows from a central object (Trussoni & Tsinganos

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1993; Lima & Priest 1993; Sauty & Tsinganos 1994). Again, the solutions found in the cited works are not fully analytical, in the sense that a final numerical integration is required. Exact solutions for symmetric MHD equilibria with mass flows have been found so far only in the particular case of flow along the invariance direction for applications to laboratory plasmas (Masche & Perrin 1980; Agim & Tataronis 1985) or assuming an incompressible plasma (Bacciotti & Chiuderi 1992, from now on BC; Villata & Ferrari 1994b). An interesting general method for generating incompressible non-linear solutions from given static equilibria is presented in Gebhardt & Kiessling (1992). Finally, some simple solutions with constant density for the linearized Grad-Shafranov equation are given in the cited series by Tsinganos.

The method of solution proposed here, a generalization of that used first in BC, is applied in three different geometries to the case of an incompressible mass flow with a non-constant density. In this situation the two equations are uncoupled and the second order differential equation is always elliptical, as in the static case (apart from a critical magnetic surface). Our method does not require the linearization of the equations and it is based on the assumption of self-similarity for the shape of the magnetic and flow surfaces. Hence, the solutions we find all show a regular nesting around the magnetic axis, and this makes them suitable for physical applications. Namely we derive exact solutions for flows inside a cylindrical magnetic flux tube with a non-circular section, exact solutions for flows in magnetic arcades in uniform gravity (for solar applications) and axisymmetric jet-type MHD structures.

However, apart from the obvious use of the solutions as basic models for the astrophysical situations in which incompressibility may be considered as a realistic approximation (such as some kinds of flows in solar magnetic flux tubes or loops and slow stellar outflows), simple and general exact solutions such as described here can be very useful in testing numerical MHD codes or in stability calculations.

The paper is structured as follows. In Sect. 2 the integrals and the reduced equations for the adiabatic or isothermal cases are derived and the problems that arise due to the coupling of the equations discussed. Then the analysis is restricted to the incompressible case, where the coupling is removed and a single elliptical equation is derived. Sect. 3 is devoted to the presentation of the method of solution, which is then applied to three different kinds of geometries and a correspondent number of novel classes of exact solutions are presented. Finally Sect. 4 contains a discussion of the work and the conclusions.

2. Steady MHD flows for systems with spatial symmetry

In this paper we shall consider a magnetized, inviscid, perfectly conducting plasma in a stationary and non relativistic state. Thus the basic equations are:

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1}$$

$$\nabla \cdot (\rho \boldsymbol{V}) = 0, \tag{2}$$

$$\boldsymbol{V} \times \boldsymbol{B} = \nabla \Phi, \tag{3}$$

$$\rho(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\nabla P - \rho \nabla U + (1/4\pi)(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{4}$$

where Φ is proportional to the electrostatic potential, U is the gravitational potential energy and all the other symbols have their usual meaning.

Consider now a physical system described by a set of coordinates (x^1, x^2, x^3), provided with invariance along the third direction:

$$\frac{\partial}{\partial x^3} \equiv 0. \tag{5}$$

For the sake of generality we shall carry on the calculations in a general curvilinear (not necessarily orthogonal) coordinate system. For a review on tensor notation and for the mathematical demonstrations the reader is referred to Agim & Tataronis (1985). As it is well known, in the Euclidean ordinary space there are only three kinds of spatial symmetry: translational, rotational and helical, the last one being the most general as it includes the first two as subcases (e.g. Morozov & Solov'ev 1963; Edenstrasser 1980b).

Thanks to the symmetry assumption it is possible to write the two components normal to the invariance direction of a divergence-free vector in terms of the spatial derivatives of a scalar function, so that

$$\boldsymbol{B} = \nabla A \times \boldsymbol{g}^3 + B^3 \boldsymbol{g}_3, \tag{6}$$

$$\boldsymbol{V} = \frac{\nabla \Psi}{4\pi\rho} \times \boldsymbol{g}^3 + V^3 \boldsymbol{g}_3, \tag{7}$$

where $\boldsymbol{g}^3$ and $\boldsymbol{g}_3$ are the third components of respectively the contravariant and covariant vector bases (which are not necessarily non-dimensional or normalized). Since $\boldsymbol{B} \cdot \nabla A = \boldsymbol{V} \cdot \nabla \Psi = 0$, the magnetic and flow surfaces are characterized respectively by the conditions $A(x^1, x^2) = \text{const}$ and $\Psi(x^1, x^2) = \text{const}$. Besides, since they are related to the flux of the corresponding vector field, they are generally referred to as *flux functions*.

From the third covariant component of Eq. (3) it is obvious that these two scalar functions are not independent but each of them may be considered as a function of the other. However, instead of setting $\Psi = \Psi(A)$ and considering $A(x^1, x^2)$ as the unknown function, as usually done, a new function $\xi(x^1, x^2)$ is introduced, so that

$$A = A(\xi), \quad \Psi = \Psi(\xi). \tag{8}$$

In this way, symmetry between $\boldsymbol{B}$ and $\boldsymbol{V}$ is maintained and besides, as clarified in the following section, the freedom in choosing the new unknown function ξ will be very helpful for the application of our method of solution. The two functions $A(\xi)$ and $\Psi(\xi)$ are integrals since they do not simply depend on the coordinates x^1 and x^2, but on their combination $\xi(x^1, x^2)$. Note that magnetic and flow surfaces necessarily coincide and are characterized by $\xi = \text{const}$. From now on, all the scalar functions of ξ only will be called *surface functions*.

Other two integrals of ξ may be found from the third covariant component of Eq. (4) and from the component of Eq. (3) parallel to $\nabla\xi$, namely

$$\dot{A}B_3 - \dot{\Psi}V_3 = \chi(\xi), \tag{9}$$

$$\dot{A}\frac{V_3}{g_{33}} - \frac{\dot{\Psi}}{4\pi\rho}\frac{B_3}{g_{33}} = \dot{\Phi}(\xi), \tag{10}$$

where the dot implies differentiation with respect to ξ. In axisymmetry these functions are usually replaced respectively by the *total* angular momentum per mass unit $L(\xi) = -\chi/\dot{\Psi}$ and by the *total* angular velocity $\Omega(\xi) = \dot{\Phi}/\dot{A}$ (for a purely azimuthal flow, $\dot{\Psi} = 0$, Eq. (10) is known as isorotation law).

Making use of the definitions of the four surface functions, the vectors $\boldsymbol{B}$ and $\boldsymbol{V}$ can be rewritten in terms of new orthogonal vectors with components that lie on surfaces with ξ = const. These vectors are $\nabla\xi \times \boldsymbol{g}_3$ and $\boldsymbol{g}_3$ (the third component of this basis is obviously $\nabla\xi$, which is normal to the surface containing $\boldsymbol{B}$ and $\boldsymbol{V}$), so the new expressions for the two vectors are

$$\boldsymbol{B} = \dot{A}\nabla\xi \times \frac{\boldsymbol{g}_3}{g_{33}} + \frac{\chi\dot{A} + g_{33}\dot{\Phi}\dot{\Psi}}{h}\frac{\boldsymbol{g}_3}{g_{33}}, \tag{11}$$

$$\boldsymbol{V} = \frac{\dot{\Psi}}{4\pi\rho}\nabla\xi \times \frac{\boldsymbol{g}_3}{g_{33}} + \frac{\chi\dot{\Psi}/4\pi\rho + g_{33}\dot{\Phi}\dot{A}}{h}\frac{\boldsymbol{g}_3}{g_{33}}, \tag{12}$$

where

$$h(\xi,\rho) = \dot{A}^2 - \dot{\Psi}^2/4\pi\rho, \tag{13}$$

that, in general, it is not a surface function. From Eqs. (11) and (12) the relation

$$\dot{A}\boldsymbol{V} - \frac{\dot{\Psi}}{4\pi\rho}\boldsymbol{B} = \dot{\Phi}\boldsymbol{g}_3 \tag{14}$$

is derived, from which it is clear that the two vectors are parallel only if $\dot{\Phi} = 0$ (i.e. if $\boldsymbol{E} = 0$, according to Ohm's law Eq. (3)). The function h defined in Eq. (13) has a remarkable physical meaning, being related to the Alfvénic Mach number M_A for the components of velocity normal to the invariance direction $\boldsymbol{g}_3$:

$$M_A(\xi,\rho) = \frac{|\boldsymbol{V}\times\boldsymbol{g}_3|}{|\boldsymbol{V}_A\times\boldsymbol{g}_3|} = \frac{\dot{\Psi}}{\dot{A}\sqrt{4\pi\rho}}, \quad \dot{A}\neq 0, \tag{15}$$

where $\boldsymbol{V}_A \equiv \boldsymbol{B}/\sqrt{4\pi\rho}$ is the Alfvén velocity (we note that when Φ = const M_A becomes the *real* Alfvénic Mach number $|\boldsymbol{V}|/|\boldsymbol{V}_A|$). The relation between h and M_A is simply $h = \dot{A}^2(1 - M_A^2)$, so, always referring to the normal components, h is positive for sub-Alfvénic flows and negative for super-Alfvénic flows. When the values of ξ and ρ are such as to satisfy $\dot{A}^2 = \dot{\Psi}^2/4\pi\rho \Leftrightarrow h = 0 \Leftrightarrow M_A = 1$, in order to avoid physical discontinuities it becomes necessary to impose a condition on the surface functions involved in the definitions of the third components of the magnetic and velocity fields, otherwise completely arbitrary.

In order to close our system, two scalar equations for the unknown functions $\xi(x^1,x^2)$ and $\rho(x^1,x^2)$ are required and these can only be derived from Euler's equation, Eq. (4). This can be done if an equation of state connecting P and ρ is introduced. Consider, for example, the adiabatic case $P = P(\rho, S)$, $\boldsymbol{V}\cdot\nabla S = 0$, where S is the entropy per mass unit. Because of the symmetry we have $S = S(\xi)$, that is the entropy is another free surface function. Introducing the enthalpy per mass unit w, defined by $dw = dP + TdS$ ($w \equiv \int_\xi dP/\rho$), the component along $\nabla\xi$ of Eq. (4) is found to be, after some lengthy algebra:

$$h\nabla\cdot\left(\frac{\nabla\xi}{g_{33}}\right) + \frac{1}{2}\frac{\partial h}{\partial\xi}\frac{|\nabla\xi|^2}{g_{33}} + 4\pi\left(\frac{\dot{\Psi}}{4\pi\rho}\right)^2\left(\frac{\nabla\rho\cdot\nabla\xi}{g_{33}}\right) -$$
$$\chi\nabla\cdot\left(\frac{\boldsymbol{g}_3\times\boldsymbol{g}^3}{g_{33}}\right) + \frac{1}{2g_{33}}\frac{\partial}{\partial\xi}\left(\frac{\chi^2}{h}\right) + \frac{g_{33}}{2}\frac{\partial}{\partial\xi}\left(\frac{4\pi\rho\dot{\Phi}^2}{h}\right) +$$
$$\frac{\partial}{\partial\xi}\left(\frac{\chi\dot{\Phi}\dot{\Psi}}{h\dot{A}}\right) + 4\pi\rho\dot{W} - 4\pi\rho T\dot{S} = 0. \tag{16}$$

The sixth surface function $W(\xi)$ is defined as

$$w + \frac{1}{2}\left[\left(\frac{\dot{\Psi}}{4\pi\rho}\right)^2\frac{|\nabla\xi|^2}{g_{33}} + \frac{1}{g_{33}}\left(\frac{\chi\dot{\Psi}/4\pi\rho + g_{33}\dot{\Phi}\dot{A}}{h}\right)^2\right] +$$
$$U - \frac{\dot{\Phi}}{\dot{A}}\left(\frac{\chi\dot{\Psi}/4\pi\rho + g_{33}\dot{\Phi}\dot{A}}{h}\right) = W(\xi), \tag{17}$$

where the terms in round brackets are V_3, as may be recognized from Eq. (12), while that within square brackets is simply $|\boldsymbol{V}|^2$. It is worth noticing that the cross product vanishes in an orthogonal coordinate system. Moreover, in Eqs. (16) and (17) the ratios $\dot{\Psi}/\dot{A}$ may be replaced by $4\pi\rho\dot{A}/\dot{\Psi}$, thus the symmetry between $\boldsymbol{B}$ and $\boldsymbol{V}$ is maintained and both the limits to $\dot{A} = 0$ and $\dot{\Psi} = 0$ can be taken. Finally, when the isothermal assumption is made, Gibbs' energy per mass unit g has to replace the enthalpy w; using then the relation $\mathrm{d}g = \mathrm{d}P/\rho - S\mathrm{d}T$ it is easy to see that the last term in Eq. (16) becomes $+4\pi\rho S\dot{T}$. It is clear that in these equations w and T (or g and S in the isothermal case) must be expressed as functions of ξ and ρ. Another possibility is to assume a barotropic relation $P = P(\rho)$ as equation of state. In this situation the last term in Eq. (16) vanishes and the enthalpy in Eq. (17) has to be replaced by $\int dP(\rho)/\rho$. For example, assuming a polytropic relation $P = k\rho^\gamma$, where γ can be different from the adiabatic index, the enthalpy is replaced by $\frac{\gamma}{\gamma-1}k\rho^{\gamma-1} = \frac{\gamma}{\gamma-1}P/\rho$. Eq. (16) is commonly known as *generalized Grad-Shafranov equation*, as it represents the generalization to dynamical equilibria of the equation obtained for the static case. Once the geometry of the system and the analytical form of the six arbitrary functions of ξ are chosen, Eq. (16) becomes a non-linear, second order partial differential equation for the unknown function $\xi(x^1,x^2)$.

A great simplification of the problem is introduced by assuming incompressibility. Because of the symmetry we have

$$\nabla\cdot\boldsymbol{V} = 0 \Leftrightarrow \rho = \rho(\xi) \tag{18}$$

and the density is now a free surface function, replacing the entropy $S(\xi)$ or the temperature $T(\xi)$. In this case the last term in Eq. (16) vanishes and the enthalpy in Eq. (17) is to be replaced by its incompressible limit P/ρ. In the absence of a-priori relations between P and ρ the pressure is simply derived from the Bernoulli equation so that the two equations are finally decoupled. Moreover, the function h, and consequently also the Alfvénic Mach number M_A defined in Eq. (15), are now surface functions, so that $h = h(\xi)$, $M_A = M_A(\xi)$. Physically this is very important because it means that the critical surface, i.e. the surface on which $|\boldsymbol{V} \times \boldsymbol{g}_3| = |\boldsymbol{V}_A \times \boldsymbol{g}_3|$, coincides both with a magnetic and a flow surface.

In the incompressible case it is convenient to introduce three new surface functions, namely $H(\xi) = \chi/h$, $G(\xi) = \Phi/h$, $\Pi(\xi) = \rho(W + \chi\Phi\dot{\Psi}/4\pi\rho h\dot{A})$. Choosing then as our basic set of surface functions $\rho(\xi)$, $H(\xi)$, $G(\xi)$ and $\Pi(\xi)$, together with $A(\xi)$ and $\Psi(\xi)$, Eq. (16) reduces to:

$$h\nabla\cdot\left(\frac{\nabla\xi}{g_{33}}\right)+\frac{1}{2}\frac{\mathrm{d}h}{\mathrm{d}\xi}\frac{|\nabla\xi|^2}{g_{33}}-hH\nabla\cdot\left(\frac{\boldsymbol{g}_3\times\boldsymbol{g}^3}{g_{33}}\right)+4\pi\frac{\mathrm{d}\Pi}{\mathrm{d}\xi}+$$

$$\frac{1}{2g_{33}}\frac{\mathrm{d}}{\mathrm{d}\xi}(hH^2)+\frac{g_{33}}{2}\frac{\mathrm{d}}{\mathrm{d}\xi}(4\pi\rho hG^2)-4\pi U\frac{\mathrm{d}\rho}{\mathrm{d}\xi}=0. \qquad (19)$$

The vector fields $\boldsymbol{B}$ and $\boldsymbol{V}$ are still given by Eqs. (11) and (12), but now

$$B_3 = H\dot{A} + g_{33}G\dot{\Psi}, \qquad V_3 = H\frac{\dot{\Psi}}{4\pi\rho} + g_{33}G\dot{A}. \qquad (20)$$

Finally, recalling that in the incompressible limit the enthalpy is given by $w = P/\rho$, the pressure can be obtained directly from the definition of Π:

$$P = \Pi - \frac{1}{2}\rho|\boldsymbol{V}|^2 - \rho U + g_{33}\rho hG^2 > 0. \qquad (21)$$

We note that the expression for Π, that can be regarded as a generalized pressure, as well as the incompressible Grad-Shafranov Eq. (19), retain their validity for $\dot{A} = 0$.

From the general theory of second order differential operators, it is now easy to draw another fundamental property that distinguishes it from the barotropic, adiabatic and isothermal cases. The discriminant of the matrix associated with the operator in Eq. (19) turns out to be $\Delta = h^2/gg_{33}$. Since the metric quantities g and g_{33} are always positive in a finite domain, the equation for incompressible flows is always elliptical, exactly as the original Grad-Shafranov equation for the static case. Although Eq. (19) is in general non-linear, it can easily be made linear by properly choosing the arbitrary surface functions. This is the most frequently used method to find analytical solutions to the generalized Grad-Shafranov equation for ξ, but not the only one. In the next sections we shall in fact propose a different approach which will allow to find several new classes of solutions of physical interest without recourse to linearization.

The necessity of having the Grad-Shafranov equation decoupled from the Bernoulli equation when looking for totally analytical solutions to symmetric MHD equilibria is universally recognized. With *totally* analytical we mean that the solutions for all the physical quantities can be derived without the aid of *any* numerical integration, so that they can be written in terms of analytical functions or series. The assumption of incompressibility is a simple way in which this can be achieved but its physical meaning should be always kept in mind. For example, in Villata & Tsinganos (1993) and Villata & Ferrari (1994a) the two relations $P = P(\rho)$ and $\rho = \rho(\xi)$ are assumed together. This means that the pressure is a free surface function, but this is impossible in the general dynamical case because of the velocity (and gravity) terms in the Bernoulli equation. Moreover, in Villata & Ferrari (1994b) the discovery of novel exact *non-barotropic* solutions is announced, without specifying that they simply refer to the incompressible case. However, this paper is very interesting since it presents general and completely analytical solutions for the linearized Grad-Shafranov equation, both in helical and axial symmetry. These solutions are applied in a model for extra-galactic jets in Villata & Ferrari (1995), again without recognizing the assumption of incompressibility.

Another situation in which the relation $\rho = \rho(\xi)$ can be freely assumed is when a flow along the direction of invariance is considered, that is $\dot{\Psi} = 0 \Leftrightarrow \boldsymbol{V} = \Omega(\xi)\boldsymbol{g}_3$. In this case the equation of continuity (2) is automatically satisfied by any functional form of ρ, exactly as in the static case. However, all the relations and the equation for ξ are the same as in the incompressible case with $\dot{\Psi} = 0$.

3. Exact solutions for incompressible flows

In this section we shall present a method of solution of the generalized, incompressible, non-linear Grad-Shafranov equation (19). This method was first applied to the axisymmetric case in BC, in absence of gravity and assuming that the magnetic and velocity fields are parallel. We now generalize that approach to different geometries and relax the two assumptions, establishing the existence of novel classes of exact analytical solutions.

The standard technique, adopted by many other authors (for example the works cited just above), consists in taking advantage of the freedom in the form of the surface functions to linearize the generalized Grad-Shafranov equation, which is further assumed to be separable in terms of the two spatial coordinates left x^1 and x^2. Such an approach has the advantage that the unknown magnetic flux function A can be often expressed in terms of known functions of mathematical physics, but the resulting solutions may be affected by singularities or divergences in the analytical shapes of the magnetic surfaces, which makes them unsuitable for physical applications. To avoid these difficulties we impose right from the start that the magnetic and flow surfaces are regularly nested one inside the other. In other words, let us suppose that the profile of the magnetic and flow surfaces in the $x^1 - x^2$ plane (i.e. the surface x^3 = const) can be expressed as

$$x^1 = \mu f(x^2),$$

where f is any regular and limited function of x^2, while μ is the scale factor characterizing the surface. Now, it is clear that

μ must be a surface quantity, thus, due to the freedom in the definition of ξ, we are free to choose $\xi \equiv \mu$. Then, it suits us to substitute the two geometrical coordinates x^1 and x^2 with two new *magnetic coordinates*, defined as

$$\xi = x^1/f(x^2), \quad X = f(x^2),$$

with X having the same dimensions as x^2 and so letting ξ to be an nondimensional variable. The meaning of these magnetic coordinates is rather simple: given a point P in the $x^1 - x^2$ plane, ξ determines the magnetic surface containing P while X determines the position of P along the profile of the surface. At this stage, all the physical and geometrical quantities can be rewritten in terms of ξ and X; proceeding in this way, it will be shown that Eq. (19) becomes a second order differential equation for the unknown function $X(x^2)$ with coefficients depending on ξ and on the surface functions. Since the solution for X has to be valid for every ξ, we must then require the coefficients of the equation to be constant. This produces a number of *compatibility conditions* for the functions of ξ, reducing at the same time the original Grad-Shafranov equation to an ordinary, non-linear, differential equation for the function X. Obviously, boundary conditions must be restricted to be consistent with the assumed separation.

Finally, note that the separation adopted for the flux function ξ is *self-similar*, as ξ is written as product of two functions where one of them is given analytically. We shall see that our method, outlined above, allows to derive simple analytical solutions also for the unknown function $X(\xi)$, so that all the physical quantities will be given in a completely analytical form. This will be done both in translational and azimuthal symmetry. Unfortunately, our particular separation does not apply to the more general helical symmetry.

3.1. Translational symmetry: flows in a magnetic flux tube with non-circular section

The translational symmetry is the simplest to treat, since the geometrical factor g_{33} equals unity. The first consequence is that, from Eq. (20), the third components of the magnetic and velocity fields become surface functions, so that $H(\xi)$ and $G(\xi)$ can be replaced by $B_3(\xi)$ and $V_3(\xi)$, with a more immediate physical meaning. Moreover, the structure of Eq. (19) suggests, in the present symmetry, to define the new surface function $\tilde{\Pi}(\xi) = \Pi + (1/2)\rho h G^2 + (1/8\pi) h H^2$, which turns out to be:

$$\tilde{\Pi}(\xi) = P + \frac{1}{8\pi}\left(B_3^2 + \frac{\dot{\Psi}^2}{4\pi\rho}|\nabla\xi|^2\right) + \rho U. \tag{22}$$

Therefore, using this new function $\tilde{\Pi}$ instead of Π, the equation to solve reduces to

$$h\nabla^2\xi + \frac{1}{2}\frac{\mathrm{d}h}{\mathrm{d}\xi}|\nabla\xi|^2 + 4\pi\frac{\mathrm{d}\tilde{\Pi}}{\mathrm{d}\xi} - 4\pi U\frac{\mathrm{d}\rho}{\mathrm{d}\xi} = 0. \tag{23}$$

In the present section our method of solution is applied to a case with no external gravitational field ($U = 0$ in Eqs. (22) and (23)), with magnetic and flow surfaces nested around a straight magnetic axis parallel with the direction of invariance. Let z be the ignorable coordinate ($x^3 = z, \partial/\partial z \equiv 0$) in a cylindrical set of coordinates (r, φ, z). Thus we write the unknown function $\xi(r, \varphi)$ as

$$\xi = r/R(\varphi),$$

where $R(\varphi)$ is an unspecified function. The profile of the intersections of the magnetic surfaces with the planes of constant z are given in the form $r = \xi R(\varphi)$, for every positive value of ξ ($\xi = 0$ is the magnetic axis). For its geometrical meaning, it is clear that $R(\varphi)$ has to be a continuous function, limited within two positive values R_{min} and R_{max}, and periodic with period $2\pi/n$, where n is a natural number.

In order to get an equation for $R(\varphi)$ from Eq. (23), the differential operators need to be rewritten in terms of the new magnetic coordinates. These are ($g_{33} = 1$, $\sqrt{g} = r$):

$$\nabla\cdot\left(\frac{\nabla\xi}{g_{33}}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\xi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\xi}{\partial\varphi^2},$$

$$|\nabla\xi|^2 = \left(\frac{\partial\xi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\xi}{\partial\varphi}\right)^2;$$

after changing into the new coordinates ξ and R, Eq. (23) becomes

$$\frac{R''}{R} - \left[2 + \frac{\xi}{2h}\frac{\mathrm{d}h}{\mathrm{d}\xi}\right]\frac{R'^2}{R^2} = \left[4\pi\frac{\xi}{h}\frac{\mathrm{d}\tilde{\Pi}}{\mathrm{d}\xi}\right]R^2 + \left[1 + \frac{\xi}{2h}\frac{\mathrm{d}h}{\mathrm{d}\xi}\right],$$

where, according to the discussion made in the previous section, the terms in square brackets must be constant since the profile $R(\varphi)$ has to be the same for every value of ξ. The results of these compatibility conditions are

$$h(\xi) = h_1\xi^{2\nu}, \quad \tilde{\Pi}(\xi) = \tilde{\Pi}_0 + \frac{\mu h_0}{8\pi\nu}\xi^{2\nu}, \tag{24}$$

where $h_0, \tilde{\Pi}_0, \mu$ and ν are arbitrary constants, with $\nu > 0$ to avoid divergences in Eqs. (24) for $\xi \to 0$. The equation for R now becomes

$$\frac{R''}{R} - (\nu+2)\frac{R'^2}{R^2} = \mu R^2 + (\nu+1),$$

which can be rewritten in the form of a simple first order linear differential equation for R'^2, with the general solution

$$R'^2 = -CR^{2(\nu+2)} + DR^4 - R^2, \tag{25}$$

where C is an arbitrary constant, while $D = -\mu/\nu$. A careful examination of Eq. (25) would show that both C and D have to be positive in order to have closed magnetic surfaces.

A simple solution of Eq. (25) with the right periodicity is easily found by setting $\nu = 1$:

$$R(\varphi) = \frac{R_0}{\sqrt{1 - \beta\cos 2\varphi}}, \tag{26}$$

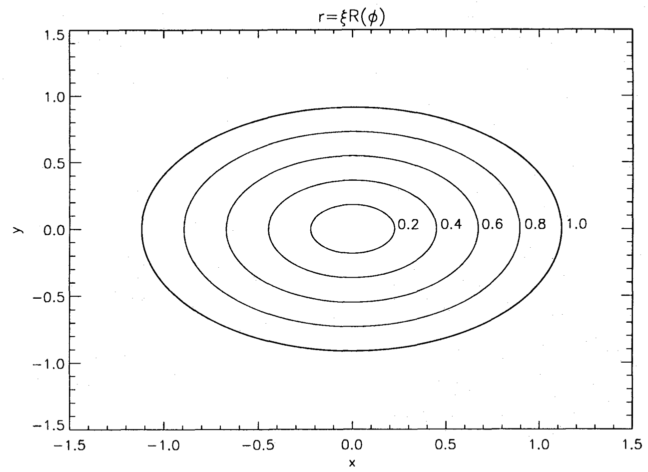

Fig. 1. Projection of field lines in the $x-y$ plane [from Eq. (26) with $R_0=1, \beta=0.2$].

where $R_0=\sqrt{2/D}$ and $\beta=\sqrt{1-4C/D^2}$, with $C \leq D^2/4$. The profiles $r=\xi R(\varphi)$, on the planes z = const, are then ellipses (see Fig. 1) with eccentricity $e=\sqrt{2\beta/(\beta+1)}$ and semiaxes $a=\xi R_{max}, b=\xi R_{min}$, where $R_{max}=R_0/\sqrt{1-\beta}, R_{min}=R_0/\sqrt{1+\beta}$. Consequently, the magnetic and flow surfaces are cylinders with elliptical section, nested one inside the other around the z axis. Note that only in the case $\nu=1$ it is possible to derive a simple analytical form of the function $R(\varphi)$; apart from the two cases $\nu=1/2$ and $\nu=2$, for which expressions involving elliptical functions can be found, Eq. (25) must be solved numerically.

It is now practical to obtain the expressions for the physical quantities involved in the problem. Together with $B_z(\xi)$ and $V_z(\xi)$, other two free surface functions may be chosen in order to satisfy Eq. (24) (let $\nu=1$), for example the density and the Alfvénic Mach number normal to the z axis, namely $\rho=\rho_0\mathscr{F}(\xi)$ and $M_A=M_A(\xi)$. Thus, the two flux functions A and Ψ are given by

$$\dot{A}=-A_0\xi\mathscr{M}(\xi), \quad \dot{\Psi}=-\Psi_0\xi\sqrt{\mathscr{F}(\xi)}\mathscr{N}(\xi), \tag{27}$$

where

$$\mathscr{M}(\xi)=\left|\frac{1-M_0^2}{1-M_A^2(\xi)}\right|^{1/2}, \quad \mathscr{N}(\xi)=\frac{M_A(\xi)}{M_0}\mathscr{M}(\xi), \tag{28}$$

and $M_0 \equiv M_A(0)=\Psi_0/\sqrt{4\pi\rho_0}A_0$, $h_0=A_0^2-\Psi_0^2/4\pi\rho_0=A_0^2(1-M_0^2)$. In the limit $M_A=M_0$ the two functions $\mathscr{M}$ and $\mathscr{N}$ equal unity. The magnetic and velocity fields are respectively

$$B_r \equiv \frac{\dot{A}}{r}\frac{\partial\xi}{\partial\varphi}=-\frac{A_0}{R_0^2}r\beta\sin 2\varphi\mathscr{M}(\xi),$$

$$B_\varphi \equiv -\dot{A}\frac{\partial\xi}{\partial r}=\frac{A_0}{R_0^2}r(1-\beta\cos 2\varphi)\mathscr{M}(\xi),$$

and

$$V_r \equiv \frac{\dot{\Psi}}{4\pi\rho}\frac{1}{r}\frac{\partial\xi}{\partial\varphi}=-\frac{\Psi_0}{4\pi\rho_0R_0^2}r\beta\sin 2\varphi\frac{\mathscr{N}(\xi)}{\sqrt{\mathscr{F}(\xi)}},$$

$$V_\varphi \equiv -\frac{\dot{\Psi}}{4\pi\rho}\frac{\partial\xi}{\partial r}=\frac{\Psi_0}{4\pi\rho_0R_0^2}r(1-\beta\cos 2\varphi)\frac{\mathscr{N}(\xi)}{\sqrt{\mathscr{F}(\xi)}},$$

Note that on the magnetic axis only the z components of the two vectors can be non-null. The analytical form of the *total* pressure may be found from Eqs. (22) and (24); in the simple case M_A = const it reads

$$P+\frac{B_z^2}{8\pi}=\tilde{\Pi}_0-\frac{A_0^2}{4\pi R_0^2}\xi^2\left(1-\frac{1-\beta^2}{1-\beta\cos 2\varphi}\frac{M_0^2}{2}\right),$$

which is a surface function only when $M_0=0$, as expected. It is interesting to notice that the presence of a flow in the $x-y$ plane allows for a larger pressure, while $V_z(\xi)$ does not enter at all in this balance relation. These properties may have a great importance for the study of the stability of steady flows inside magnetic flux tubes and our solution can be used as a non-trivial unperturbed configuration.

Finally, we want to point out that this solution could also be found using the standard mathematical methods, but only in the particular case $\boldsymbol{V}=\Omega(\xi)\,\mathbf{e}_z \Leftrightarrow \dot{\Psi}=0$; in this situation the magnetic flux function A can be used directly instead of ξ, so the equation to solve is simply:

$$\nabla^2 A+4\pi\frac{\mathrm{d}\tilde{\Pi}}{\mathrm{d}A}=0.$$

In order to linearize it, the generalized pressure can be chosen to be $\tilde{\Pi}(A)=\tilde{\Pi}_0-(k/4\pi)(A-\overline{A})$, where k and $\overline{A}$ are arbitrary constants, so the equation to solve becomes (in polar coordinates):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 A}{\partial\varphi^2}=k.$$

The general solution of this Poisson equation is

$$A(r,\varphi)=\frac{k}{4}r^2+\sum_{n=0}^{+\infty}a_n r^n\cos(n\varphi-\varphi_n),$$

where all the constants a_n and φ_n are arbitrary and where all the terms diverging for $r \rightarrow 0$ have been rejected. Choosing $a_0=\overline{A}, a_2=\beta A_0/2R_0, k=-2A_0/R_0^2$ and setting all the other constants to zero, the magnetic flux function becomes

$$A=\overline{A}-\frac{A_0}{2}\left(\frac{r}{R_0}\sqrt{1-\beta\cos 2\varphi}\right)^2,$$

that is the known solution, since the term within brackets is just the function $\xi=r/R(\varphi)$, satisfying the second relation in (27). Note that our corresponding dynamical solution could have been also derived from this relation making use of the method described by Gebhardt & Kiessling (1992), with the same choice for the function $h(\xi)$.

3.2. Translational symmetry: plasma in a uniform gravitational field

As a second application of our method of solution, we consider the situation in which there are two mutually orthogonal preferential directions: that of a uniform gravitational field and the direction of invariance. Using cartesian coordinates, the assumptions $U = gz$ and $\partial/\partial y \equiv 0$ can be freely chosen, so that the unknown function $\xi(x^1, x^2)$ will be separated this time as

$$\xi = z/Z(x),$$

where $x^1 = z, x^2 = x$ and $x^3 = y$ in order to follow the previous notation.

This time $g_{33} = \sqrt{g} = 1$, so that the expressions for the differential operators and field components are trivial. The equation to solve is again Eq. (23), but now we consider the gravity term with $U = g\xi Z$. After some rearranging, the equation reduces to

$$\frac{Z''}{Z} - \left[2 + \frac{\xi}{2h}\frac{\mathrm{d}h}{\mathrm{d}\xi}\right]\frac{Z'^2}{Z^2} =$$

$$\left[\frac{1}{2\xi h}\frac{dh}{d\xi}\right]\frac{1}{Z^2} + \left[\frac{4\pi}{h\xi}\frac{d\tilde{\Pi}}{d\xi}\right] + \left[-\frac{4\pi g}{h}\frac{d\rho}{d\xi}\right] Z.$$

By imposing the constancy of the coefficients within the square brackets, as required by our method, we get the compatibility conditions

$$h = h_0, \quad \tilde{\Pi} = \tilde{\Pi}_0 + \frac{\mu h_0}{8\pi}\xi^2, \quad \rho = \rho_0 - \frac{\lambda h_0}{4\pi g}\xi. \tag{29}$$

The last condition holds only for $g \neq 0$. With these assumptions the equation to solve becomes

$$\frac{Z''}{Z} - 2\frac{Z'^2}{Z^2} = \lambda Z + \mu$$

with the first integral

$$Z'^2 = CZ^4 + DZ^3 + EZ^2, \tag{30}$$

where C is a new arbitrary constant, while $D = -2\lambda$ and $E = -\mu$.

Assuming $Z \geq 0$ for every x, in order to describe, for example, the atmosphere just above a stellar surface (coincident with $\xi = 0$), three different cases are considered depending on the values of E:

1. $E = 0$. In this case $\tilde{\Pi} \equiv \tilde{\Pi}_0 =$ const and the solution is (see Fig. 2)
$$Z(x) = \frac{Z_0}{1 + \beta x^2}, \tag{31}$$
with $Z_0 = D/|C|, \beta = D^2/4|C|, D > 0$ and $C < 0$.
2. $E < 0$. In this case a periodic solution is found (see Fig. 3):
$$Z(x) = \frac{Z_0}{1 - \beta \cos \alpha x}, \tag{32}$$
where $Z_0 = 2|E|/D, \beta = \sqrt{1 - 4|E||C|/D^2}, \alpha = \sqrt{E}$, with $D > 0$ and $-D^2/4|E| \leq C < 0$.

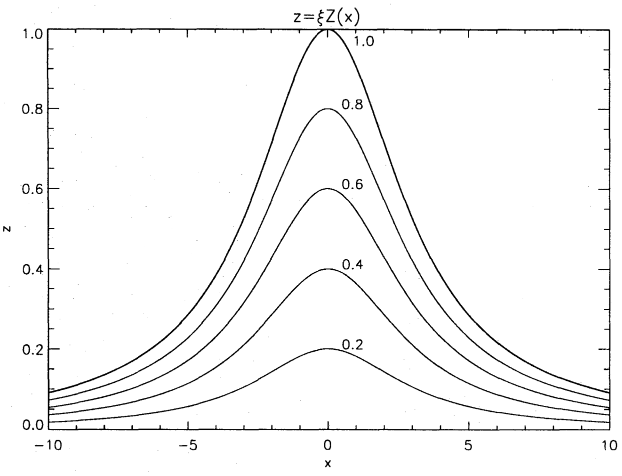

Fig. 2. Projection of field lines in the $x - z$ plane [from Eq. (31) with $Z_0 = 1, \beta = 0.1$].

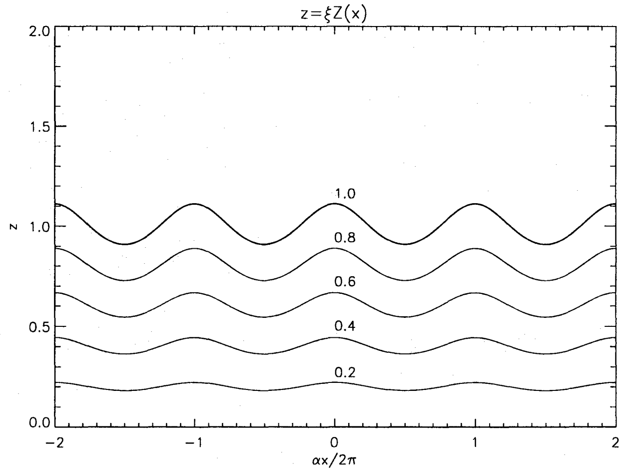

Fig. 3. Projection of field lines in the $x - z$ plane [from Eq. (32) with $Z_0 = 1, \beta = 0.1$].

3. $E > 0$. This last case is the only one that allows realistic solutions in absence of gravity, that is when $D = 0$. Here there are again three different cases:
   (a) $D < 0$; the solution is (see Fig. 4)
   $$Z(x) = \frac{Z_0}{1 + \beta \cosh \alpha x}, \tag{33}$$
   where $Z_0 = 2E/|D|, \alpha = \sqrt{E}$ and $\beta = \sqrt{1 - 4EC/D^2}$, with $C \leq D^2/4E$.
   (b) $D > 0$; the solution is
   $$Z(x) = \frac{Z_0}{-1 + \beta \cosh \alpha x}, \tag{34}$$
   where $Z_0 = 2E/D, \alpha = \sqrt{E}$ and $\beta = \sqrt{1 + 4E|C|/D^2}$, with $C < 0$ in order to avoid divergencies ($\beta > 1$).
   (c) $D = 0$; the solution is
   $$Z(x) = \frac{Z_0}{\cosh \alpha x}, \tag{35}$$
   where $Z_0 = \sqrt{E/|C|}$ and $\alpha = \sqrt{E}$, with $C < 0$.

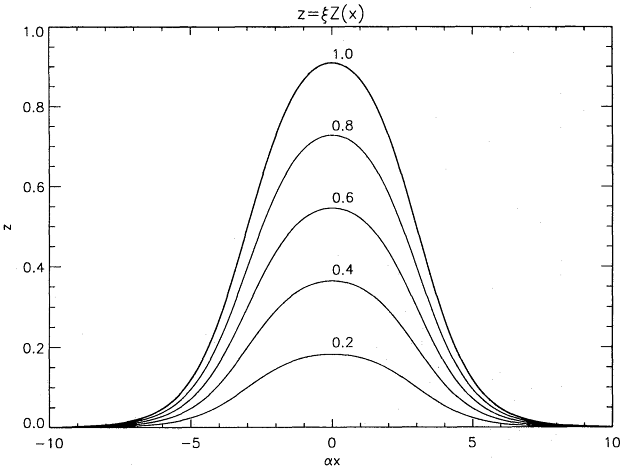

Fig. 4. Projection of field lines in the $x - z$ plane [from Eq. (33) with $Z_0 = 1, \beta = 0.1$].

The most interesting case for astrophysical applications is the solution with $E > 0$ and $D < 0$ (Eq. (33) and Fig. 4), which is also that resembling more closely a magnetic arcade in the solar corona. For sub-Alfvénic flows, as expected in the low-beta corona, both the density and the total pressure surface functions are decreasing with height. Taking

$$\rho = \rho_0 \mathscr{F}(\xi), \; \dot{A} = -A_0 \mathscr{M}(\xi), \; \dot{\Psi} = -\Psi_0 \sqrt{\mathscr{F}(\xi)} \mathscr{N}(\xi), \quad (36)$$

where all the symbols retain the same meaning as in the previous sub-section, the density reads

$$\mathscr{F}(\xi) = 1 - \delta\xi, \quad \delta = \frac{\alpha^2 A_0^2}{4\pi\rho_0 g Z_0}(1 - M_0^2)$$

(when $g = 0$ the density is still a free function of ξ). The magnetic field components are

$$B_z \equiv -\dot{A}\xi\frac{Z'}{Z} = \frac{A_0}{Z_0}\alpha z \beta \sinh \alpha x \mathscr{M}(\xi),$$

$$B_x \equiv -\dot{A}\frac{1}{Z} = \frac{A_0}{Z_0}(1 + \beta\cosh \alpha x)\mathscr{M}(\xi),$$

while the velocity components are

$$V_z \equiv -\frac{\dot{\Psi}}{4\pi\rho}\xi\frac{Z'}{Z} = \frac{\Psi_0}{4\pi\rho_0 Z_0}\alpha z\beta \sinh \alpha x \frac{\mathscr{N}(\xi)}{\sqrt{\mathscr{F}(\xi)}},$$

$$V_x \equiv -\frac{\dot{\Psi}}{4\pi\rho}\frac{1}{Z} = \frac{\Psi_0}{4\pi\rho_0 Z_0}(1 + \beta\cosh \alpha x)\frac{\mathscr{N}(\xi)}{\sqrt{\mathscr{F}(\xi)}},$$

while the expression for P may be derived from Eqs. (22) and (29).

3.3. The axisymmetric case

In this section the validity of the solution found in BC, for a system with azimuthal invariance and an incompressible flow parallel to the direction of the magnetic field, will be extended by removing the assumption that $\boldsymbol{V}$ is parallel to $\boldsymbol{B}$. In our formulation this means $G \neq 0$. The coordinate system suitable for the axisymmetric case is obviously the cylindrical system with $x^1 = z, x^2 = r, x^3 = \varphi$ and the condition $\partial/\partial\varphi \equiv 0$. The function $\xi(z, r)$ is assumed to be separable in the form

$$\xi = r/R(z),$$

so that the magnetic magnetic and flow surfaces are given by the relation $r = \xi R(z)$, with $R > 0$, for every positive value of ξ.

In axisymmetry the differential operators in Eq. (19) are ($g_{33} = r^2$, $\sqrt{g} = r$):

$$\nabla \cdot \left(\frac{\nabla\xi}{g_{33}}\right) = \frac{1}{r^2}\frac{\partial^2\xi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\xi}{\partial r}\right),$$

$$|\nabla\xi|^2 = \frac{1}{r^2}\left[\left(\frac{\partial\xi}{\partial z}\right)^2 + \left(\frac{\partial\xi}{\partial r}\right)^2\right],$$

hence the equation becomes

$$\frac{R''}{R} - \left[2 + \frac{\xi}{2h}\frac{\mathrm{d}h}{\mathrm{d}\xi}\right]\frac{R'^2}{R^2} = \left[\frac{\xi^3}{2h}\frac{\mathrm{d}}{\mathrm{d}\xi}(4\pi\rho h G^2)\right]R^4 +$$

$$\left[4\pi\frac{\xi}{h}\frac{\mathrm{d}\tilde{\Pi}}{\mathrm{d}\xi}\right]R^2 + \left[\frac{1}{2h\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}(hH^2)\right] + \left[\frac{1}{2h\xi}\frac{\mathrm{d}h}{\mathrm{d}\xi} - \frac{1}{\xi^2}\right]\frac{1}{R^2}.$$

The compatibility conditions give

$$\rho = \rho_0\mathscr{F}(\xi), \quad h = h_0\xi^2, \quad \Pi = \Pi_0 + \frac{\mu h_0}{8\pi}\xi^2,$$

$$G = G_0/(\xi\sqrt{\mathscr{F}(\xi)}), \quad H = \sqrt{\lambda/2}\xi, \quad (37)$$

where ρ_0, h_0, Π_0, μ and λ are arbitrary constants ($\rho_0 > 0, \lambda \geq 0$), $\mathscr{F}(\xi)$ is a free surface function. The last two relations are not the most general, but these forms have been chosen in order to avoid singularities in the physical quantities as $\xi \to 0$. Now, the equation for R reduces simply to

$$\frac{R''}{R} - 3\frac{R'^2}{R^2} = \mu R^2 + \lambda,$$

with its first integral given by

$$R'^2 = -CR^6 + DR^4 - ER^2, \quad (38)$$

where $E = \lambda/2 > 0, D = -\mu$ and C is an integration constant. Eq. (38) is analogous to the one found in Sect. 3.1 in the only analytically integrable case, that is when $\nu = 1$. Since $E > 0$, the conditions $D > 0$ and $0 < C \leq D^2/4E$ must be imposed

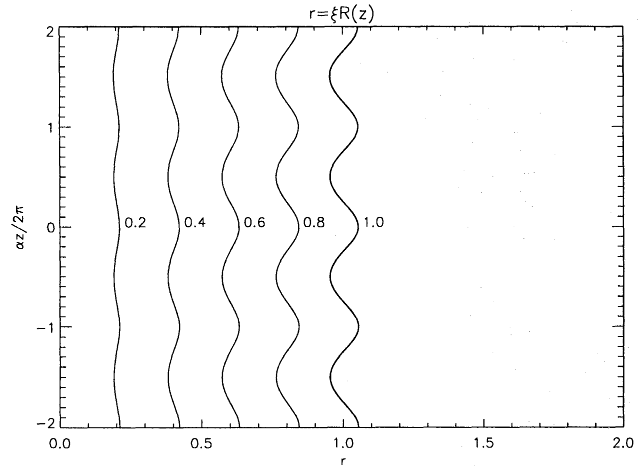

Fig. 5. Projection of field lines in the $r-z$ plane [from Eq. (39) with $R_0=1, \beta=0.1$].

in order to find *well behaved* solutions. The shape function is then

$$R(z)=\frac{R_0}{\sqrt{1-\beta\cos\alpha z}}, \tag{39}$$

being $R_0=\sqrt{2E/D}, \beta=\sqrt{1-4EC/D^2}, \alpha=2\sqrt{E}$ (see Fig. 5). As previously anticipated, the periodic solution (39) is the same found in BC, but now $G(\xi)\neq 0$, that is to say that the two vectors $\boldsymbol{V}$ and $\boldsymbol{B}$ are not parallel.

The usual choice of $M_A=M_A(\xi)$ as free surface functions, together with the assumptions

$$\dot{A}=A_0\xi\mathscr{M}(\xi), \qquad \dot{\Psi}=\Psi_0\xi\sqrt{\mathscr{F}(\xi)}\mathscr{N}(\xi), \tag{40}$$

leads to the following expressions for the magnetic field components:

$$B_z\equiv\dot{A}\frac{\partial\xi}{\partial r}=\frac{A_0}{R_0^2}(1-\beta\cos\alpha z)\mathscr{M}(\xi),$$

$$B_r\equiv-\dot{A}\frac{\partial\xi}{\partial z}=-\frac{A_0}{R_0^2}\frac{\alpha}{2}r\beta\sin\alpha z\mathscr{M}(\xi),$$

$$B_\varphi\equiv\frac{H\dot{A}}{r}+rG\dot{\Psi}=$$

$$\frac{A_0}{R_0^2}\frac{\alpha}{2}r(1-\beta\cos\alpha z)\mathscr{M}(\xi)+\Psi_0G_0r\mathscr{N}(\xi),$$

and for the velocity field components:

$$V_z\equiv\frac{\dot{\Psi}}{4\pi\rho}\frac{\partial\xi}{\partial r}=\frac{\Psi_0}{4\pi\rho_0R_0^2}(1-\beta\cos\alpha z)\frac{\mathscr{N}(\xi)}{\sqrt{\mathscr{F}(\xi)}},$$

$$V_r\equiv-\frac{\dot{\Psi}}{4\pi\rho}\frac{\partial\xi}{\partial z}=-\frac{\Psi_0}{4\pi\rho_0R_0^2}\frac{\alpha}{2}r\beta\sin\alpha z\frac{\mathscr{N}(\xi)}{\sqrt{\mathscr{F}(\xi)}},$$

$$V_\varphi\equiv\frac{H\dot{\Psi}/4\pi\rho}{r}+rG\dot{A}=$$

$$\frac{\Psi_0}{4\pi\rho_0R_0^2}\frac{\alpha}{2}r(1-\beta\cos\alpha z)\frac{\mathscr{N}(\xi)}{\sqrt{\mathscr{F}(\xi)}}+A_0G_0r\frac{\mathscr{M}(\xi)}{\sqrt{\mathscr{F}(\xi)}}.$$

Finally the pressure can be derived from Eq. (21). It is interesting to notice that in the particular case $M_A=$ const the choice of the free surface functions is exactly the same as in Villata & Ferrari (1994b). Therefore, despite the different methods adopted to solve the equations (self-similarity in the present work, linearization in the other), the same solutions may be derived. In fact, when $M_A=M_0$ our solutions reduce to one of their classes of axisymmetric solutions.

The obvious astrophysical application of this class of solutions is the modeling of the *knotty* jet-type structures in the outflows from both proto-stellar object and extra-galactic nuclei. Although our solution refers to the incompressible case the basic structure of the jet could be modeled in these simple terms. An example of this approach is given in Villata & Ferrari (1995), where the M 87 jet is modeled matching the synchrotron emissivity that results from their incompressible solutions (taken as proportional to the plasma density times the magnetic field squared) with the observed radio contours.

4. Conclusions

In this paper the problem of finding exact solutions to the set of steady ideal MHD equations has been treated. Assuming general symmetry with one ignorable spatial coordinate, a set of reduced equations and integrals have been derived in general curvilinear coordinates. Discussing the equations we have seen that the major problem is the coupling between the generalized Grad-Shafranov and Bernoulli equations due to the equation of state relating P and ρ, even for simple thermodynamical situations such as the adiabatic, isothermal or barotropic cases. In order to avoid this difficulty incompressibility has been assumed and a single elliptical equation for the non-dimensional flux function ξ has been derived.

A new, general, *completely analytical* and *non-linear* method of solution has been proposed and applied to three kinds of geometries and several classes of solutions have been obtained. Due to the self-similarity assumption the magnetic and flow surfaces are regularly nested around the magnetic axis and so physical discontinuities are avoided a-priori. All our classes of solutions are general and flexible, since they contain a minimum of three free surface functions. In translational symmetry we found flows in a magnetic flux tube with elliptical section. This solution might represent a starting point to the study of its stability properties, since usually only cylindrical flux tubes with circular section are considered. Moreover, elliptical flows are known to be unstable (Lifschitz & Hameiri 1991) but it is not clear whether the presence of the magnetic field can stabilize them. Solutions in translational symmetry are also found for a plasma in a uniform gravity field. Some of these resemble arcade-type structures above the solar surface. Our incompressible solutions may be considered a first step towards more accurate modeling of the flows in this kind of structures. Finally, an interesting class of jet-type, axisymmetric solutions has been found. Again, we must keep in mind that these are only incompressible solutions and that for realistic modeling of jets from proto-steallar objects or active galaxies more complex energy equations have to be considered.

However, we want to underline once again that apart from the direct application of our solutions for models of astrophysical structures, the importance of having non-linear, exact solutions to the stationary, ideal, MHD equations is universally recognized. For example, these can provide a valuable basis for stability calculations or may be used as a test for numerical codes.

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