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Some inverse limit approaches to the Riordan group

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ABSTRACT

Using the inverse limit tool, we obtain the Riordan group in its infinite and bi-infinite representations from groups of finite Riordan matrices. We define two different reflections for bi-infinite Riordan matrices. Employing these definitions, we give answers to the problems $D^{\perp} = D$ and $D^{\diamond} = D$ in the Riordan group. So, we describe the self-complementary and self-dual Riordan matrices.

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1. Introduction

The main tool used in this paper to relate finite, infinite and bi-infinite Riordan matrices is the inverse limit concept. This concept has been and is still being widely used in practically all branches of mathematics, sometimes under the name of projective limit. Usually it is a way to approximate objects by better behaved or widely known ones. The concept of inverse system, from which the inverse limit is derived, can be defined in any category, where the related concept of pro-category appears. An introductory text for these topics is [8]. In [2] the authors developed, from the categorical point of view, the Algebraic Topology. There, it can be found a study on inverse limit in the categories of both groups with homomorphisms and topological spaces with continuous maps.

Because of our previous works, [4,5,11,3], we feel the necessity of a deeper understanding of different reflections in bi-infinite Riordan matrices. To do that, we introduce herein the natural concept of finite Riordan matrices. In these kind of matrices, we get an internal transformation reflecting across $y = x$. This transformation makes no sense in usual infinite Riordan matrices. Both previous concepts on finite framework have been independently used in [1] to obtain sloping constructions of Riordan arrays. Using inverse sequences of finite Riordan matrices and the inverse limit concept, see Chapter VIII of [2], we are able to define essentially two different reflections in bi-infinite Riordan matrices. This allows us to reformulate some questions left open in [5]. Finally, we give answers to those problems.

The horizontal and vertical constructions that we explored in [4] have enabled us to understand these reflections. In fact, the fundamental point has been the symmetry between the sequence g found in [6] and used there for a vertical construction of a Riordan matrix and the A-sequence of Rogers [10] for horizontal construction of such object. Actually, the A-sequence is the parameter g in the inverse matrix, equivalently g is the A-sequence of the inverse matrix, see Proposition 7 in [3, p. 3615]. Both sequences are the same in many important cases, for instance, in self-complementary, self-dual and involutory Riordan matrices. In some works in progress, we are treating these latest kind of matrices from this point of view. We are also developing the natural pro-Lie group structure of the Riordan group derived from the constructions made herein.

After the preliminary Section 2, in Section 3 we obtain the groups of finite Riordan matrices $\mathcal{R}_n$ of order $n+1$ for $n = 0, 1, \cdots$ by means of natural projections from the Riordan group $\mathcal{R}$. Later, we recuperate the Riordan group as an inverse limit of these groups of finite matrices with appropriate bonding maps. We also give an internal characterization of such finite matrices.

In Section 4 we describe first the complementary and the dual, see [5], as elements in the inverse limit described in the previous section. To do that, we apply a natural reflection on finite Riordan matrices $\mathcal{R}_n$ of order $n+1$ for $n = 0, 1, \cdots$. Later, employing again the inverse limit concept for a constant inverse sequence, we achieve the bi-infinite representation of Riordan matrices from the usual infinite one. We end this section reaching this bi-infinite representation as inverse limit of sequences of groups of finite

Riordan matrices by two different natural ways depending overall on the parity of the size of finite matrices in the inverse sequences.

Reflecting term by term the finite matrices involved in both sequences above, in Section 5 we get two different, but related, reflections on bi-infinite Riordan matrices and the two corresponding concepts of symmetric matrices where the dual and the complementary appears. This allows us to translate symmetries in bi-infinite matrices in terms of the problems self-complementarity and self-duality in Riordan matrices left open in [5]. We end the paper solving these problems.

The notation of a Riordan matrix used in this paper is taken from [6] but to make understanding easier for people not familiar with this notation we will include, in Section 2, the conversion formula to a more usual one. Along the paper some results will be written in both ways. The notation used for the complementary, the dual, etc., are taken from [4] and [5].

Always, in this paper, $\mathbb{K}$ is a field of characteristic 0. By $\mathbb{N}$ we denote the set $\{0, 1, 2, 3, \cdots\} \subset \mathbb{K}$ and by $\mathbb{Z}$ the set $\{0, \pm 1, \pm 2, \pm 3, \cdots\} \subset \mathbb{K}$.

2. Basic concepts and notations

As we said in the introduction, the inverse limit concept is a general construction used in Category Theory. We only need few things about this general construction to understand the material herein. Consequently, we will restrict our definitions to the Group Category and to inverse sequences which are particular cases of the more general concept of inverse system.

Definition 1. An inverse sequence of groups is a pair $\mathcal{S} = \{(G_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}}\}$ where G_n is a group and $\psi_n : G_{n+1} \to G_n$ is a group homomorphism for every $n \in \mathbb{N}$.

The homomorphisms ψ_n are called the *bonding morphisms* or the *bonding maps* of the inverse sequence.

Proposition 2. *Let $\mathcal{S} = \{(G_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}}\}$ be an inverse sequence of groups. Consider*

$$\varprojlim \mathcal{S} = \{(z_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} G_n \mid \psi_n(z_{n+1}) = z_n \ \forall\ n \in \mathbb{N}\}.$$

Then, $\varprojlim \mathcal{S}$ has a natural group structure given by the product $(a_n)_{n\in\mathbb{N}} \cdot (b_n)_{n\in\mathbb{N}} = (a_n \cdot b_n)_{n\in\mathbb{N}}$ for $(a_n)_{n\in\mathbb{N}},\ (b_n)_{n\in\mathbb{N}} \in \varprojlim \mathcal{S}$ and where the products inside the parentheses are those in each group G_n. The sequence $e = (e_n)_{n\in\mathbb{N}}$ formed by neutral elements in each group is always in $\varprojlim \mathcal{S}$. This is the neutral element in the inverse limit. In fact, the group $\varprojlim \mathcal{S}$ is a subgroup of the direct product group $\prod_{n\in\mathbb{N}} G_n$.

Other important devices to deal with inverse limits are projections to each group G_n in the inverse sequence. In the general definition of inverse limit in categories those projections are, in fact, part of the definition. In our case the projections can be obtained from above in the following way.

Given the set-theoretic Cartesian product we have a natural projection $\Psi_k : \prod_{n\in\mathbb{N}} G_n \to G_k$ defined by $\Psi_k((z_n)_{n\in\mathbb{N}}) = z_k$. In our context Ψ_k is a group homomorphism. The restrictions of Ψ_k to $\varprojlim \mathcal{S}$ are called the projections in the inverse limit. The following result is satisfied. For every $n \in \mathbb{N}$ the diagram below is commutative, that is, $\Psi_n = \psi_n \circ \Psi_{n+1}$:

$$\begin{tikzcd} \varprojlim \mathcal{S} \arrow[d, "\Psi_n"'] \arrow[dr, "\Psi_{n+1}"] & \\ G_n & G_{n+1} \arrow[l, "\psi_n"] \end{tikzcd}$$

Now, we are going to describe briefly the Riordan group and the two notations we will use for Riordan matrices.

The Riordan group is a subgroup of the group of invertible infinite lower triangular matrices with the usual product of matrices as the operation. The elements of the Riordan group are those matrices whose columns are the coefficients of successive terms of a geometric progression in $\mathbb{K}[[x]]$ where the initial term is a formal power series of order 0 and the common ratio is a formal power series of order 1. For example, in [6,7], the Riordan matrix $D = (d_{i,j})_{i,j\in\mathbb{N}}$ is represented as $T(f \mid g) = D$, with $f(0) \neq 0$ and $g(0) \neq 0$, so that $d_{i,j} = [x^i]\dfrac{x^j f(x)}{g^{j+1}(x)}$. Consequently, the first term is $\dfrac{f(x)}{g(x)}$ and the common ratio is $\dfrac{x}{g(x)}$. Another way to denote this is $D = \mathcal{R}(d(x), h(x))$ where $d(x)$ and $h(x)$ are formal power series with $d(0) \neq 0$, $h(0) = 0$ and $h'(0) \neq 0$, see [4,5]. For this representation $d_{i,j} = [x^i]d(x)h^j(x)$. Then, clearly, the relationship between both notations is

$$d(x) = \frac{f(x)}{g(x)}, \quad h(x) = \frac{x}{g(x)} \qquad \text{or equivalently} \qquad f(x) = \frac{xd(x)}{h(x)}, \quad g(x) = \frac{x}{h(x)}.$$

The product and the inverse become (see [6] Proposition 20, pp. 2629–2630).

$$T(f \mid g)T(l \mid m) = T\left(fl\left(\frac{x}{g}\right) \Big| gm\left(\frac{x}{g}\right)\right)$$

where $fl\left(\dfrac{x}{g}\right) \equiv f(x)\cdot l\left(\dfrac{x}{g(x)}\right)$, analogously for the second term.

For what concerns the inverse:

$$(T(f \mid g))^{-1} \equiv T^{-1}(f \mid g) = T\left(\frac{1}{f(\frac{x}{A})}\Big| A\right)$$

where $\left(\frac{x}{A}\right) \circ \left(\frac{x}{g}\right) = \left(\frac{x}{g}\right) \circ \left(\frac{x}{A}\right) = x$. The previous formal power series denoted by A is the so-called A-sequence introduced by Rogers [10]. The previous formula for the product and the inverse in the another notation can be found in almost all papers on Riordan matrices and one can get it directly using the conversion formula above.

3. General properties of finite Riordan matrices

For every $n \in \mathbb{N}$ consider the general linear group $GL(n+1, \mathbb{K})$ formed by all $(n+1) \times (n+1)$ invertible matrices with coefficients in $\mathbb{K}$. Let $\mathcal{R}$ be the Riordan group. Since every Riordan matrix is lower triangular, we can define a natural homomorphism $\Pi_n : \mathcal{R} \to GL(n+1, \mathbb{K})$ given by

$$\Pi_n((d_{i,j})_{i,j\in\mathbb{N}}) = (d_{i,j})_{i,j=0,\cdots,n}.$$

For obvious reasons, many times we will refer to this homomorphism as the projection of a certain infinite Riordan matrix. In fact, they are the projections defined in the previous section.

To describe the Riordan group as an inverse limit of an inverse sequence of groups of finite matrices, we consider first the subgroup of $GL(n+1, \mathbb{K})$ defined by $\mathcal{R}_n = \Pi_n(\mathcal{R})$ (recall that the image under a group homomorphism is a subgroup of the target group). The other needed part is the sequence of bonding maps.

Definition 3. Let $D = (d_{i,j})_{i,j=0,\cdots,n+1} \in \mathcal{R}_{n+1}$. We define $P_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$ by

$$P_n((d_{i,j})_{i,j=0,1,\cdots,n+1}) = (d_{i,j})_{i,j=0,\cdots,n}.$$

$P_n(D)$ is obtained from D by deleting its last row and its last column. P_n is a group homomorphism for every n because the matrices are lower triangular. Moreover the diagram below is commutative

$$\begin{tikzcd} \mathcal{R} \arrow[d, "\Pi_n"'] \arrow[dr, "\Pi_{n+1}"] & \\ \mathcal{R}_n & \mathcal{R}_{n+1} \arrow[l, "P_n"] \end{tikzcd}$$

Consequently we get

Proposition 4. *The Riordan group $\mathcal{R}$ is isomorphic to $\varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$.*

The above proposition means that a Riordan matrix can be uniquely described by a sequence of finite matrices $(D_n)_{n\in\mathbb{N}}$ with $D_n \in \mathcal{R}_n$ and such that $P_n(D_{n+1}) = D_n$

for every $n \in \mathbb{N}$. Furthermore, the product in the Riordan group corresponds to the component-wise products in the sequences.

Obviously if $n = 0$ then $\mathcal{R}_0 = \mathbb{K}^*$ with the usual product in $\mathbb{K}$ being $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Now, as consequences of known results about the horizontal and vertical constructions of Riordan matrices in the literature, see for example [10,6,4], we have the following internal characterization for finite dimensional Riordan matrices.

Theorem 5. *Let $D = (d_{i,j})_{i,j=0,\cdots,n}$ with $n \geq 1$ be a lower triangular matrix. Then*

(i) *$D \in \mathcal{R}_n$ if and only if $d_{0,0} \neq 0$ and there are $a_0, a_1, \cdots, a_{n-1}$ in $\mathbb{K}$ with $a_0 \neq 0$ such that*

$$d_{i,j} = \sum_{k=0}^{i-j} a_k d_{i-1,j-1+k} \qquad i,j = 1, \cdots, n. \tag{1}$$

(ii) *$D \in \mathcal{R}_n$ if and only if $d_{0,0} \neq 0$ and there are $g_0, g_1, \cdots, g_{n-1}$ in $\mathbb{K}$ with $g_0 \neq 0$ such that*

$$d_{i,j} = \sum_{k=0}^{i-j} g_k d_{i+1-k,j+1} \qquad i,j = 0, \cdots, n-1. \tag{2}$$

Moreover if $g(x) = \sum_{k=0}^{n-1} g_k x^k$ and $A(x) = \sum_{k=0}^{n-1} a_k x^k$, then

$$[x^{k-1}]\frac{1}{A(x)} = \frac{1}{k}[x^{k-1}]g^k(x) \qquad \textit{for} \qquad k = 1, \cdots, n. \tag{3}$$

Proof. (ii) Suppose that there are $g_0, g_1, \cdots, g_{n-1}$ in $\mathbb{K}$ with $g_0 \neq 0$ such that

$$d_{i,j} = \sum_{k=0}^{i-j} g_k d_{i+1-k,j+1} \qquad i,j = 0, \cdots, n-1.$$

If we consider the Riordan matrix $T(f \mid g)$ for

$$g(x) = \sum_{k=0}^{n-1} g_k x^k, \qquad f(x) = \left(\sum_{k=0}^{n-1} g_k x^k\right)\left(\sum_{k=0}^{n} d_{k,0} x^k\right)$$

and using Theorem 11 in [6] we have $D = \Pi_n(T(f \mid g))$.

On the other hand if $D \in \mathcal{R}_n$ then there is a Riordan matrix $T(f \mid g)$ such that $\Pi_n(T(f \mid g)) = D$ using again Theorem 11 in [6] we get that if $g_i = [x^i]g(x)$ for $i = 0, \cdots, n-1$ then (2) is satisfied.

The proof of part (i) is similar using the A-sequence construction of Riordan matrices, see for example [10,9].

Finally (3) is a consequence of Lagrange Inversion Formula and the fact given in Proposition 7(i) in [3]. $\square$

Example 6.

$$P_5 = \Pi_5(T(1 \mid 1-x)) = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}.$$

$P_5 \in \mathcal{R}_5$ because $d_{0,0} = 1$, $a_0 = 1$, $a_1 = 1$, $a_2 = a_3 = a_4 = 0$ satisfy (1). Equivalently $g_0 = 1$, $g_1 = -1$, $g_2 = g_3 = g_4 = 0$ satisfy (2). Moreover, (3) holds since $A(x) = 1 + x$ and $g(x) = 1 - x$.

Observe that:

$\mathcal{R}_1$ is the group of 2×2 lower triangular invertible matrices.

$\mathcal{R}_2$ is the group of 3×3 lower triangular invertible matrices where the main diagonal is formed by three consecutive terms of a geometric progression.

$\mathcal{R}_3$ is the group of 4×4 lower triangular invertible matrices where the main diagonal is formed by four consecutive terms of a geometric progression and the first sub-diagonal is formed by three consecutive terms of an arithmetic–geometric progression with the same ratio as the geometric one in the main diagonal.

The elements of any group $\mathcal{R}_n$ are called *finite Riordan matrices*. Note that the size of any element in $\mathcal{R}_n$ is $(n+1) \times (n+1)$.

It is obvious that we can have a finite Riordan matrix as the projection of different infinite Riordan matrices. In fact, there are infinite many different infinite Riordan matrices with the same n-projection. To clarify the general situation on equality of n-projection we have:

Proposition 7. *Given a finite Riordan matrix $D \in \mathcal{R}_n$ with $n \geq 1$, there are unique polynomials $\tilde{f}$ and $\tilde{g}$ with $\deg(\tilde{f}) \leq n$ and $\deg(\tilde{g}) \leq n-1$ such that*

$$\Pi_n(T(\tilde{f} \mid \tilde{g})) = D.$$

We call $T(\tilde{f} \mid \tilde{g})$ *the canonical Riordan representative* of the finite Riordan matrix D.

Proof. As in the proof of Theorem 5 using part (2), choose

$$\tilde{g} = \sum_{k=0}^{n-1} g_k x^k, \qquad \tilde{f} = \mathcal{T}_n\left(\left(\sum_{k=0}^{n-1} g_k x^k\right)\left(\sum_{k=0}^{n} d_{k,0} x^k\right)\right)$$

where $\mathcal{T}_n(h)$ represents the n-degree h's Taylor polynomial. $\square$

Example 8. In this case $f(x) = (1-x)(1+x+x^2+x^3+x^4+x^5) = 1-x^6$ then $\tilde{f} = \mathcal{T}_n(f(x)) = 1$, $\tilde{g(x)} = g(x) = 1-x$. So

$$P_5 = \Pi_5(T(1 \mid 1-x)) = \Pi_5(T(1-x^6 \mid 1-x)),$$

in the other notation we get

$$P_5 = \Pi_5\left(\mathcal{R}\left(\frac{1}{1-x}, \frac{x}{1-x}\right)\right) = \Pi_5\left(\mathcal{R}\left(\frac{1-x^6}{1-x}, \frac{x}{1-x}\right)\right).$$

Proposition 9. *Let $D = (d_{i,j})_{i,j=0,\cdots,n}$, with $n \geq 1$ be a finite Riordan matrix. If $T(f \mid g)$ is such that $\Pi_n(T(f \mid g)) = D$, then*

$$g = \tilde{g} + x^n m, \qquad f = \tilde{f} + m_0 d_{00} x^n + x^{n+1} l$$

with $l, m \in \mathbb{K}[[x]]$, $m(x) = \sum_{k\geq 0} m_k x^k$ and $\tilde{f}$ and $\tilde{g}$ the polynomials of the canonical representative of D.

Proof. It easy to see that if $f(x) = \sum_{k\geq 0} f_k x^k$ and $g(x) = \sum_{k\geq 0} g_k x^k$ and $\Pi_n(T(f \mid g)) = D$ then

$$g_k = \tilde{g}_k, \qquad f_k = \tilde{f}_k, \qquad \text{for} \qquad k = 0, \cdots, n-1.$$

Then $g = \tilde{g} + x^n m$. Moreover

$$[x^n]f = \sum_{k=0}^{n} g_k d_{n-k,0} = \sum_{k=0}^{n-1} g_k d_{n-k,0} + g_n d_{0,0} = [x^n]\tilde{f} + m_0 d_{0,0}$$

and the proof is finished. $\square$

Example 10.

$$P_5 = \Pi_5(T(1 + m_0 x^5 + x^6 l(x) \mid 1 - x + x^5 m(x)))$$

with l, m and m_0 as in the above proposition. In the other notation

$$P_5 = \Pi_5\left(\mathcal{R}\left(\frac{1 + m_0 x^5 + x^6 l(x)}{1 - x + x^5 m(x)}, \frac{x}{1 - x + x^5 m(x)}\right)\right).$$

In particular, by taking $m(x) = 3 + 15x^3$ and $l(x) = 7$, we have

$$\begin{aligned} P_5 &= \Pi_5(T(1 + 3x^5 + 7x^6 \mid 1 - x + 3x^5 + 15x^8)) \\ &= \Pi_5\left(\mathcal{R}\left(\frac{1 + 3x^5 + 7x^6}{1 - x + 3x^5 + 15x^8}, \frac{x}{1 - x + 3x^5 + 15x^8}\right)\right). \end{aligned}$$

Remark 11. For $n = 0$, we have

$$\Pi_0(T(f \mid g)) = \Pi_0(T(r \mid s)) = D = (d_{0,0}) \quad \Leftrightarrow \quad \frac{f_0}{g_0} = \frac{r_0}{s_0} = d_{0,0}$$

where the diagonal matrix $T(d_{0,0} \mid 1)$ is the canonical representative of $D \in \mathcal{R}_0$.

As consequence of the above results we obtain:

Proposition 12 *(The equality of banded submatrices). Given two Riordan matrices, $T(f \,|\, g) = (d_{i,j})_{i,j \in \mathbb{N}}$ and $T(r \,|\, s) = (c_{i,j})_{i,j \in \mathbb{N}}$ with $\Pi_n(T(f \,|\, g)) = \Pi_n(T(r \mid s))$ and $n \geq 1$, then*

$$\Pi_{n-1}(T(fg^m \mid g)) = \Pi_{n-1}(T(rs^m \mid s)) \qquad \forall\, m \in \mathbb{Z}.$$

In particular,

$$d_{i,j} = c_{i,j} \qquad \textit{for} \quad i - j = k, \quad \textit{with} \quad k = 0, \cdots, n-1.$$

Proof. Since $\Pi_n(T(f \mid g)) = \Pi_n(T(r \mid s))$ then, in particular, $g_k = s_k$ and $f_k = r_k$ for $k = 0, \cdots, n-1$. Note that $[x^l]g^m$ depends only on the coefficients $g_0, \cdots, g_l$. Consequently the coefficients $[x^l]fg^m$ depend only on the coefficients $f_0, \cdots, f_l$ and $g_0, \cdots, g_l$. So, we conclude that $s_l = g_l$ and $[x^l]fg^m = [x^l]rs^m$ for $0 \leq l \leq n-1$. Hence $\Pi_{n-1}(T(fg^m \,|\, g)) = \Pi_{n-1}(T(rs^m \,|\, s))$. In particular, for $i - j = k$ with $k = 0, \cdots, n-1$

$$d_{i,j} = d_{j+k,j} = [x^{j+k}]\frac{x^j f}{g^{j+1}} = [x^k]\frac{f}{g^{j+1}} = [x^k]\frac{r}{s^{j+1}} = [x^{k+j}]\frac{x^j r}{s^{j+1}} = c_{j+k,j} = c_{i,j}. \qquad \square$$

The reciprocal of the above proposition is not true, because if $\Pi_{n-1}(T(fg^m \mid g)) = \Pi_{n-1}(T(rs^m \mid s))$ $\forall\, m \in \mathbb{Z}$ then $d_{i,j} = c_{i,j}$ for $i - j \leq n-1$ but no conditions or relations about $d_{n,0}$ and $c_{n,0}$ are implied. So, in general $\Pi_n(T(f \mid g)) \neq \Pi_n(T(r \mid s))$.

Recall that $\Pi_n : \mathcal{R} \to \mathcal{R}_n$ is a group homomorphism. Then, using Proposition 9 we have that the kernel

$$\ker \Pi_n = \left\{ T(f \mid g) \mid g = 1 + x^n m, f = 1 + m_0 x^n + x^{n+1} l \quad \text{with} \quad l, m \in \mathbb{K}[[x]] \right\}$$

satisfies the following result:

Proposition 13. $\ker \Pi_n$ *is a normal subgroup of the Riordan group $\mathcal{R}$. Moreover*

(1)

$$\ker \Pi_n \supset \ker \Pi_{n+1} \quad \forall\, n \geq 0, \quad \textit{and} \quad \bigcap_{n \geq 0} \ker \Pi_n = T(1 \mid 1).$$

(2)

$$\Pi_n(T(f \mid g)) = \Pi_n(T(r \mid s)) \;\Leftrightarrow\; \exists\, l, m \in \mathbb{K}[[x]]$$

such that

$$T(r \mid s) = T(f \mid g)T(1 + m_0 x^n + x^{n+1} l \mid 1 + x^n m).$$

The proof is obvious and part (1) was first proved in [6, pp. 2633–2634] using an invariant complete ultrametic on the Riordan group $\mathcal{R}$.

From now on, we denote by $T_n(f \mid g) = \Pi_n(T(f \mid g))$, with $n \in \mathbb{N}$. If the subindex n does not appear, we are referring to the infinite Riordan matrix.

4. Reflections on matrices: from finite to bi-infinite Riordan matrices

4.1. The inverse limit description of the complementary and the dual

Note that matrix transposition can be viewed as a reflection on the matrix across $y = -x$. To stay inside the Riordan group transposition is not allowed. However, looking at the structure of finite Riordan matrices we realize that reflecting these matrices across $y = x$ we obtain another lower triangular matrix. In fact, we have:

Proposition 14 *(The reflected Riordan matrix). Let $D = (d_{i,j})_{i,j=0,\cdots,n}$ be a finite Riordan matrix and consider the matrix $D^R = (c_{i,j})_{i,j=0,\cdots,n}$ with $c_{i,j} = d_{n-j,n-i}$. Then D^R is a finite Riordan matrix that we call the reflected matrix of D.*

Proof. If $D = (d_{i,j})_{i,j=0,\cdots,n}$, then $D^R = (c_{i,j})_{i,j=0,\cdots,n}$ with $c_{i,j} = d_{n-j,n-i}$. D^R is a finite Riordan matrix because following Theorem 5

$$c_{0,0} = d_{n,n} = a_0 d_{n-1,n-1} = \frac{d_{n-1,n-1}}{g_0} \neq 0$$

and

$$c_{n-j,n-i} = d_{i,j} = \sum_{k=0}^{i-j} a_k d_{i-1,j-1+k} = \sum_{k=0}^{i-j} a_k c_{n-(j-1+k),n-(i-1)}.$$

Renaming the subscripts: $n - j = \tilde{i}$ and $n - i = \tilde{j}$

$$c_{\tilde{i},\tilde{j}} = \sum_{k=0}^{\tilde{i}-\tilde{j}} \tilde{g}_k c_{\tilde{i}+1-k,\tilde{j}+1} \qquad \Rightarrow \qquad \tilde{g}_k = a_k$$

then D^R is a finite Riordan matrix where the coefficients $\tilde{g}_k$ are equal to the coefficients a_k of D. In a similar way we can prove that the coefficients $\tilde{a}_k$ of D^R are equal to the coefficients g_k of D. □

Remark 15. Note that, in the proof above, we obtain that the finite sequence $g_0, g_1, \cdots, g_{n-1}$ used to construct the reflected matrix of D by columns in Theorem 5(ii) is equal to the finite sequence $a_0, a_1, \cdots, a_{n-1}$ used to construct D by rows in Theorem 5(i). Analogously, the constants used to construct the reflected matrix of D by rows are the same as those used to construct D by columns. Moreover, the first column of D^R can be calculated by the expression

$$c_{i,0} = d_{n,n-i} = [x^i]\frac{f}{g^{n+1-i}}. \tag{4}$$

Example 16.

$$P_5^R = \begin{pmatrix} 1 & & & & & \\ 5 & 1 & & & & \\ 10 & 4 & 1 & & & \\ 10 & 6 & 3 & 1 & & \\ 5 & 4 & 3 & 2 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$P_5^R = \Pi_5(T((1+x)^6 \mid 1+x)) = \Pi_5\left(\mathcal{R}\left((1+x)^5, \frac{x}{1+x}\right)\right).$$

Some properties that can be deduced from the above proposition are:

Corollary 17.

(i) *If $D \in \mathcal{R}_n$ is a Toeplitz matrix, then its reflected matrix is the same Toeplitz matrix.*
(ii) *If $D \in \mathcal{R}_n$, $(D^R)^R = D$.*
(iii) *If $D, C \in \mathcal{R}_n$, then $(DC)^R = C^R D^R$.*
(iv) $(D^R)^{-1} = (D^{-1})^R$.
(v) *If $D \in \mathcal{R}_n$ is such that its canonical representative is $D = T_n(f \mid g)$, then the representative of D^R is $T_n(l \mid A)$ where*

$$A = \sum_{k=0}^{n-1} a_k x^k, \qquad l = \sum_{k=0}^{n}\left(\sum_{j=0}^{k} d_{n,n-k-j}a_j\right)x^k.$$

(vi) *If $D \in \mathcal{R}_n$ then DD^R is a Toeplitz matrix, i.e., $DD^R = T_n(h \mid 1)$.*
(vii) $D^{-1} = D^R T_n\left(\frac{1}{h}\middle|1\right)$.

Proof. (i) If $D = (d_{i,j})_{i,j=0,\cdots,n}$ is a Toeplitz matrix then there are $f_0, f_1, \cdots, f_n \in \mathbb{K}$ with $d_{i,j} = f_{i-j}$. So, if $D^R = (c_{i,j})_{i,j=0,\cdots,n}$ then $c_{i,j} = d_{n-j,n-i} = f_{n-j-n+i} = f_{i-j}$.

(ii) It is obvious.

(iii) and (iv) are straightforward.

(v) Following the previous proposition we have that the second parameter is the A defined above. And, following as in the proof of Theorem 5 we obtain the first parameter l

$$l = \mathcal{T}_n\left(\left(\sum_{k=0}^{n} d_{n,n-k}x^k\right)\left(\sum_{k=0}^{n-1} a_k x^k\right)\right).$$

(vi) Let $D = T_n(f \mid g)$ and using (v) we get $D^R = T_n(l \mid A)$ where $g(x)A\left(\frac{x}{g(x)}\right) = 1$ by (3) in Theorem 5, then $DD^R = T_n(h \mid 1)$ which is a Toeplitz matrix.

(vii) Since $DD^R = T_n(h \mid 1)$ then $DD^R T_n\left(\frac{1}{h} \mid 1\right) = T_n(1|1)$ which is the corresponding identity. Consequently $D^{-1} = D^R T_n\left(\frac{1}{h}\middle|1\right)$. □

In Definition 3 we constructed $P_n(D)$ by deleting its last row and its last column, analogously, we can consider $Q_n(D)$ by deleting its first row and its first column.

Definition 18. Let $D = (d_{i,j})_{i,j=0,\cdots,n+1} \in \mathcal{R}_{n+1}$. We define $Q_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$ by

$$Q_n((d_{i,j})_{i,j=0,\cdots,n+1}) = (d_{i,j})_{i,j=1,\cdots,n+1} = (\tilde{d}_{i,j})_{i,j=0,\cdots,n} \quad \text{with} \quad \tilde{d}_{i,j} = d_{i+1,j+1}.$$

The fact that $Q_n(D) \in \mathcal{R}_n$, when $D \in \mathcal{R}_{n+1}$, is an immediate consequence of Theorem 5. Concretely, since the columns of any Riordan matrix $T(f \mid g)$ form a geometric progression in $\mathbb{K}[[x]]$ with common ratio $\frac{x}{g}$ and first term $\frac{f}{g}$, then

$$Q_n(T_{n+1}(f \mid g)) = T_n\left(\frac{f}{g}|g\right).$$

Moreover, Q_n is a group homomorphism because all matrices are lower triangular. Therefore, we get the following result:

Proposition 19. $(D_n)_{n\in\mathbb{N}} \in \varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$ *if and only if* $D_n^R = Q_n(D_{n+1}^R)$ *for all* $n \in \mathbb{N}$.

Theorem 20. *Let* $T(f \mid g)$ *be any Riordan matrix. Then*

(i) *The sequence* $(T_n^R(fg^{n+1} \mid g))_{n\in\mathbb{N}}$ *is a Riordan matrix. In fact, the above sequence, as an element in the inverse limit of* $\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$, *is the complementary Riordan matrix of* $T(f \mid g)$ *denoted in* [5] *by* $T^{\perp}(f \mid g)$.

(ii) *The sequence* $(T_n^R(fg^n \mid g))_{n\in\mathbb{N}}$ *is a Riordan matrix. In fact, the above sequence, as an element in the inverse limit of* $\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$, *is the dual Riordan matrix of* $T(f \mid g)$ *denoted in* [5] *by* $T^{\diamondsuit}(f \mid g)$.

Proof. (i) To see that $(T_n^R(fg^{n+1} \mid g))_{n\in\mathbb{N}}$ is a Riordan matrix, i.e. it is an element in the inverse limit $\varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$, we call $D_n = T_n^R(fg^{n+1} \mid g)$. So, $D_n^R = T_n(fg^{n+1} \mid g)$. By the description of Q_n we obtain

$$Q_n(T_{n+1}(fg^{n+2} \mid g)) = T_n\left(fg^{n+1}|g\right)$$

and by Proposition 19 we get the desired result. The proof of (ii) is analogous. □

Remark 21. In fact, any $[m]$-complementary in sense of [5] can be described as above.

The following corollary is now obvious

Corollary 22. *For every Riordan matrix* $T(f \mid g)$ *and for every* $n \in \mathbb{N}$ *we have*

$$T_n^R(f \mid g) = T_n^{\diamondsuit}\left(\frac{f}{g^n}|g\right),$$

or equivalently

$$T_n^{\diamondsuit}(f \mid g) = T_n^R\left(fg^n|g\right), \qquad T_n^{\perp}(f \mid g) = T_n^R\left(fg^{n+1}|g\right),$$
$$T_n^R(f \mid g) = T_n^{\perp}\left(\frac{f}{g^{n+1}}|g\right).$$

Example 23.

$$P_5 = \Pi_5(T(1 \mid 1-x)),$$
$$P_5^R = \Pi_5\left(T^{\diamondsuit}\left(\frac{1}{(1-x)^5}\Big|1-x\right)\right) = \Pi_5(T((1+x)^6 \mid 1+x)).$$

4.2. A special constant inverse sequence: from infinite to bi-infinite

As we showed the Riordan group is, in some sense, the asymptotic behavior of the inverse sequence $\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$. Similarly, we can study the asymptotic behavior of the sequence of homomorphisms $(Q_n)_{n\in\mathbb{N}}$. In this sense, using the definition of Q_n we easily have:

Proposition 24. *There is a unique isomorphism* $\Phi : \mathcal{R} \to \mathcal{R}$ *such that* $\Pi_n \circ \Phi = Q_n \circ \Pi_{n+1}$ *for every* $n \in \mathbb{N}$. *In fact it is defined by* $\Phi(T(f \mid g)) = T\left(\frac{f}{g} \mid g\right)$.

This isomorphism was first used in [5].

Obviously $\Phi^{-1}(T(f \mid g)) = T\left(fg|g\right)$. We are now going to use this result to get the bi-infinite representation of the Riordan group found in [4] by a different approach and using again the concept of inverse limit of an inverse sequence.

Proposition 25. *The Riordan group $\mathcal{R}$ is isomorphic to the $\varprojlim\{(G_n)_{n\in\mathbb{N}},(\Psi_n)_{n\in\mathbb{N}}\}$, where $G_n = \mathcal{R}$ and $\Psi_n = \Phi$ for every $n \in \mathbb{N}$.*

Proof. Let $\Gamma = \{(G_n)_{n\in\mathbb{N}},(\Psi_n)_{n\in\mathbb{N}}\}$ be the inverse sequence of groups where $G_n = \mathcal{R}$ for every $n \in \mathbb{N}$ and $\Psi_n : G_{n+1} \to G_n$ is such that $\Psi_n = \Phi$ for every $n \in \mathbb{N}$.

Consider $\tau_m : \varprojlim \Gamma \to G_m$ be the projection. Let us prove that $\tau_0 : \varprojlim \Gamma \to G_0$ is, in our case, a group isomorphism. Note first that $\alpha \in \varprojlim \Gamma$ if and only if there is a $T(f \mid g) \in \mathcal{R}$ such that

$$\alpha = \Big(\cdots, T(fg^n \mid g\Big), \cdots, T(fg^2 \mid g), T(fg \mid g), T(f \mid g)\Big).$$

It is now obvious that τ_0 is onto. To prove the injectivity, take $\alpha \in \ker \tau_0$ then $\tau_0(\alpha) = T(1 \mid 1)$. Consequently, $\alpha = \Big(\cdots, T(1 \mid 1), \cdots, T(1 \mid 1), T(1 \mid 1), T(1 \mid 1)\Big)$ which is the neutral element in $\varprojlim \Gamma$. $\square$

The above construction allows us to get the following representation of the elements in the Riordan group.

Let $\alpha \in \varprojlim \Gamma$ and $T(f \mid g) \in \mathcal{R} = G_0$ be such that $T(f \mid g) = \tau_0(\alpha)$ then

$$\begin{array}{cccccccccccc} \varprojlim \Gamma & \cdots \to & \mathcal{R} & \xrightarrow{\Phi} & \mathcal{R} & \cdots \to & \mathcal{R} & \xrightarrow{\Phi} & \mathcal{R} & \xrightarrow{\Phi} & \mathcal{R} = G_0 \\ \alpha = \Big(& \cdots & T(fg^{n+1} \mid g), & & T(fg^n \mid g), & \cdots & T(fg^2 \mid g), & & T(fg \mid g), & & T(f \mid g)\Big). \end{array}$$

The 0-approximation of α is $\Big(T(f \mid g)\Big)$. The 1-approximation is $\Big(T(fg \mid g), T(f \mid g)\Big)$ with $\Phi(T(fg \mid g)) = T(f \mid g)$ is obtained from the 0-approximation adding adequately a column (to the left) and a row. The fact that $\Phi(T(fg \mid g)) = T(f \mid g)$ is also encoded in the above matrix. This means that deleting the first row and the first column we obtain the 0-approximation.

In a similar way we can associate a unique matrix, just the matrix $T(fg^n \mid g)$, to the n-approximation which is $\Big(T(fg^n \,|\, g), T(fg^{n-1} \,|\, g), \cdots, T(f \,|\, g)\Big)$ with $\Phi(T(fg^k \,|\, g)) = T(fg^{k-1} \,|\, g)$ for $k = 1, \cdots, n$. Then now, it is very natural to identify α with the bi-infinite matrix $B(f \mid g) = (d_{n,k})_{n,k\in\mathbb{Z}}$ where

$$d_{n,k} = [x^{n-k}]fg^{-k-1} \quad n, k \in \mathbb{Z},\ n \geq k.$$

Moreover the product of two elements of $\varprojlim \Gamma$ converts to the usual row-by-column product of the corresponding bi-infinite representations.

The 0-column and the 0-row are fixed by the rule above and it only depends on the canonical embedding of $T(f|g)$ into $B(f|g)$. Note that the assignment $B(f|g) \to T(f|g)$ is a group isomorphism because is just the action of the projection τ_0.

$$B(f|g) = \begin{pmatrix} \ddots & & & & & & & \\ \cdots & d_{-3,-3} & & & & & & \\ \cdots & d_{-2,-3} & d_{-2,-2} & & & & & \\ \cdots & d_{-1,-3} & d_{-1,-2} & d_{-1,-1} & & & & \\ \cdots & d_{0,-3} & d_{0,-2} & d_{0,-1} & d_{0,0} & & & \\ \cdots & d_{1,-3} & d_{1,-2} & d_{1,-1} & d_{1,0} & d_{1,1} & & \\ \cdots & d_{2,-3} & d_{2,-2} & d_{2,-1} & d_{2,0} & d_{2,1} & d_{2,2} & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $d_{n,k}$, $n, k \in \mathbb{Z}$, are those described above.

Note that if we have a Toeplitz matrix $T(f \mid 1)$ with $f = \sum_{n\geq 0} f_n x^n$ then $B(f \mid 1) = (d_{n,k})_{n,k\in\mathbb{Z}}$ is such that $d_{n,k} = 0$ if $n < k$ and $d_{n,k} = f_{n-k}$ if $n \geq k$.

Related to the notation used in [4] we have the following relation:

$$B(f \mid g) = \chi\left(\frac{f(x)}{g(x)}, \frac{x}{g(x)}\right) \qquad \text{or} \qquad \chi(d(x), h(x)) = B\left(\frac{xd}{h}\middle|\frac{x}{h}\right).$$

Remark 26. It is clear that Φ induces an isomorphism, denoted again by Φ, in the bi-infinite Riordan group given by

$$\Phi(B(f \mid g)) = B\left(\frac{f}{g}|g\right),$$

or equivalently $\Phi((d_{n,k})_{n,k\in\mathbb{Z}}) = (d_{n+1,k+1})_{n,k\in\mathbb{Z}}$.

4.3. Approximating bi-infinite matrices by finite ones

At this time we are going to achieve the bi-infinite representation of the Riordan group in a new way, using only finite Riordan matrices.

Given $B(f \mid g)$ above, we define the sequence of finite matrices $\gamma_n(B(f \mid g)) = T_{2n}(fg^n \mid g)$ for $n \geq 0$. For example

$$\cdots,\ \gamma_1(B(f \mid g)) = \begin{pmatrix} d_{-1,-1} & & \\ d_{0,-1} & d_{0,0} & \\ d_{1,-1} & d_{1,0} & d_{1,1} \end{pmatrix},\ \gamma_0(B(f \mid g)) = \begin{pmatrix} d_{0,0} \end{pmatrix}. \tag{5}$$

Note that if $n \geq 0$

$$\gamma_n = Q_{2n} \circ P_{2n+1} \circ \gamma_{n+1} = P_{2n} \circ Q_{2n+1} \circ \gamma_{n+1}.$$

Analogously, given $B(f \mid g)$ we define the sequence of finite matrices $\delta_n(B(f \mid g)) = T_{2n+1}(fg^n \mid g))$ for $n \geq 0$. For example

$$\cdots,\ \delta_1(B(f \mid g)) = \begin{pmatrix} d_{-1,-1} & & & \\ d_{0,-1} & d_{0,0} & & \\ d_{1,-1} & d_{1,0} & d_{1,1} & \\ d_{2,-1} & d_{2,0} & d_{2,1} & d_{2,2} \end{pmatrix}, \quad \delta_0(B(f \mid g)) = \begin{pmatrix} d_{0,0} & \\ d_{1,0} & d_{1,1} \end{pmatrix}.$$

Moreover for $n \geq 0$

$$\delta_n = Q_{2n+1} \circ P_{2n+2} \circ \delta_{n+1} = P_{2n+1} \circ Q_{2n+2} \circ \delta_{n+1},$$

then we obtain

Proposition 27.

(i) *The bi-infinite representation of the Riordan group $\mathcal{R}$ jointly with the projections $(\gamma_n)_{n\geq 0}$ is the inverse limit of the inverse sequence $\{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$ where $s_n = Q_{2n} \circ P_{2n+1} = P_{2n} \circ Q_{2n+1}$.*

(ii) *The bi-infinite representation of the Riordan group $\mathcal{R}$ jointly with the projections $(\delta_n)_{n\geq 0}$ is the inverse limit of the inverse sequence $\{(\mathcal{R}_{2n+1})_{n\geq 0}, (r_n)_{n\geq 0}\}$ where $r_n = Q_{2n+1} \circ P_{2n+2} = P_{2n+1} \circ Q_{2n+2}$.*

Proof. To prove (i) in Proposition 27, suppose $(D_{2n})_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$. This means that $s_n(D_{2n+2}) = D_{2n}$. We have to find a Riordan matrix $T(f \mid g) = (C_n)_{n\geq 0} \in \varprojlim \{(\mathcal{R}_n)_{n\geq 0}, (P_n)_{n\geq 0}\}$ such that $(D_{2n})_{n\geq 0} = B(f \mid g)$. Choose $C_0 = D_0$. Since $s_0(D_2) = D_0$ and $s_0 = P_0 \circ Q_1$ taking $C_1 = Q_1(D_2)$ we get that $P_0(C_1) = C_0$. For the next step, let us consider the equalities $s_1(D_4) = D_2$, $s_1 = P_2 \circ Q_3$ and $C_1 = Q_1(D_2)$ then $Q_1(P_2 \circ Q_3)(D_4) = C_1$. As $Q_1 \circ P_2 = P_1 \circ Q_2$ then $P_1(Q_2(Q_3(D_4))) = C_1$ so we take $C_2 = Q_2(Q_3(D_4))$. Proceeding by induction, consider

$$C_{n+1} = Q_{n+1} \circ Q_{n+2} \circ \cdots \circ Q_{2n+1}(D_{2n+2}).$$

Since $s_n(D_{2n+2}) = D_{2n}$, $s_n = P_{2n} \circ Q_{2n+1}$, $Q_m \circ P_{m+1} = P_m \circ Q_{m+1}$ for every $m \in \mathbb{N}$ and

$$C_n = Q_n \circ Q_{n+1} \circ \cdots \circ Q_{2n-1}(D_{2n})$$

we get that $P_n(C_{n+1}) = C_n$. Now, (5) above, allows us to obtain any $B(f \mid g)$ as an element in $\varprojlim \{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$. Finally, by the construction, the assignment $B(f \mid g) \to (D_{2n})_{n\geq 0}$ is an isomorphism. Moreover, $D_{2n} = T_{2n}(fg^n \mid g)$.

To prove (ii) we proceed analogously starting at $C_0 = P_1(C_1)$ where $C_1 = D_1$. □

Remark 28. It is interesting to be aware of the action of s_n and r_n, which corresponds just to delete all the boundary coefficients in the corresponding finite matrices.

Note that (i) and (ii) above are each independent on the other, because the groups and the bonding maps in each one are different to the other.

5. Symmetries in bi-infinite Riordan matrices: some applications

In [5] we introduced, among other things, the concept of complementary and dual Riordan matrices. It is natural to look for some symmetries related to these concepts. In particular, we would like to clarify the following two problems

$$\begin{array}{lll} (P_1) & D = D^{\perp} & D \in \mathcal{R}. \\ (P_2) & D = D^{\diamondsuit} & D \in \mathcal{R}, \end{array}$$

that is, to characterize Riordan matrices which coincide with their complementary arrays (P_1) and their dual arrays (P_2). These problems will be solved in Theorem 33 and Theorem 37, respectively.

What we are going to do first is to reinterpret these problems in terms of two different reflections of bi-infinite Riordan matrices. To accomplish this task, we will use the odd reflection $B^{R_o}(f \mid g)$ and the even reflection $B^{R_e}(f \mid g)$ of a bi-infinite Riordan matrix $B(f \mid g)$. The existence of these reflections is explained in the following result:

Theorem 29.

(i) *Let* $B(f \mid g) = (D_{2n})_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$ *then*

$$(D_{2n}^R)_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$$

and it represents a bi-infinite Riordan matrix,

$$(D_{2n}^R)_{n\geq 0} = B^{R_o}(f \mid g).$$

Moreover

$$B^{R_o}(f \mid g) = B\left(A(A - xA')f\left(\frac{x}{A}\right) \mid A\right),$$

where A is the A-sequence of $B(f \mid g)$.

(ii) *Let* $B(f \mid g) = (D_{2n+1})_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n+1})_{n\geq 0}, (r_n)_{n\geq 0}\}$ *then*

$$(D_{2n+1}^R)_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n+1})_{n\geq 0}, (r_n)_{n\geq 0}\}$$

and it represents a bi-infinite Riordan matrix,

$$(D_{2n+1}^R)_{n\geq 0} = B^{R_e}(f \mid g).$$

Moreover

$$B^{R_e}(f \mid g) = B\left(A^2(A - xA')f\left(\frac{x}{A}\right) \mid A\right),$$

where A is the A-sequence of $B(f \mid g)$.

Proof. (i) Let us denote by $\rho_n : \mathcal{R}_n \to \mathcal{R}_n$ given by $\rho_n(D) = D^R$, then $\rho_{2n} \circ s_n = s_n \circ \rho_{2n+2}$ so $(D^R_{2n})_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$ and consequently it represents a new bi-infinite Riordan matrix $B(l \mid m)$. Using the representation given in Proposition 27 and the formula given for the dual of a Riordan matrix in p. 81 in [5] we get the values of l and m.

(ii) It is completely analogous to (i) because in this case $(D^R_{2n+1})_{n\geq 0} \in \varprojlim \{(\mathcal{R}_{2n+1})_{n\geq 0}, (r_n)_{n\geq 0}\}$ also represents $B(A^2(A - xA')f\left(\frac{x}{A}\right) \mid A)$. $\square$

Note that the words odd and even in the above definitions of reflections are related to the parity of the finite matrices used in each of the inverse sequences. Moreover, $\Phi(B^{R_e}(f \mid g)) = B^{R_o}(f \mid g)$. We call a bi-infinite matrix *even-symmetric* if $B^{R_e}(f \mid g) = B(f \mid g)$. Analogously we define the *odd-symmetric* matrices.

Remark 30. Observe that, using the notations analogous to that in [5], we have

$$B^{R_o}(f \mid g) = B^{\diamondsuit}(f \mid g), \qquad B^{R_e}(f \mid g) = B^{\perp}\left(\frac{f}{g^2} \mid g\right).$$

Moreover $B^{R_o}(f \mid g) = B^{R_e}(fg \mid g)$.

Proposition 31.

(i) *$T^{\perp}(f \mid g) = T(f \mid g)$ if and only if $B(fg \mid g)$ is an even-symmetric matrix.*
(ii) *$T^{\diamondsuit}(f \mid g) = T(f \mid g)$ if and only if $B(f \mid g)$ is an odd-symmetric matrix.*

Proof. (i) Consider the matrix $B(f \mid g)$. From this, we obtain $B(fg \mid g) = (D_{2n+1})_{n\in\mathbb{N}} \in \varprojlim \{(\mathcal{R}_{2n+1})_{n\geq 0}, (r_n)_{n\geq 0}\}$ starting at

$$D_1 = \begin{pmatrix} d_{-1,-1} & \\ d_{0,-1} & d_{0,0} \end{pmatrix}.$$

Obviously, if $B(fg \mid g)$ is even-symmetric, then $T^{\perp}(f \mid g) = T(f \mid g)$. To prove the converse, suppose $T^{\perp}(f \mid g) = T(f \mid g)$. It is clear that $D_1 = D_1^R$ because $T_0(f \mid g) = T_0^{\perp}(f \mid g)$. Consequently $g_0 = 1$. In a similar way

$$D_3 = \begin{pmatrix} d_{-2,-2} & & & \\ d_{-1,-2} & d_{-1,-1} & & \\ d_{0,-2} & d_{0,-1} & d_{0,0} & \\ d_{1,-2} & d_{1,-1} & d_{1,0} & d_{1,1} \end{pmatrix}$$

if $T_1(f \mid g) = T_1^{\perp}(f \mid g)$, and $r_0(D_3) = D_1$ therefore we only need to prove that $d_{0,-2} = d_{1,-1}$ to conclude that $D_3 = D_3^R$. Since

$$d_{0,-2} = g_0 d_{1,-1} + g_1 d_{0,-1} + g_2 d_{-1,-1}$$
$$d_{1,-1} = a_0 d_{0,-2} + a_1 d_{0,-1} + a_2 d_{0,0}$$

and $g = A$ because $T^{\perp}(f \mid g) = T(f \mid g)$ we obtain the result. Proceeding analogously by induction we finally get that $B(fg \mid g)$ is even-symmetric.

(ii) In this case $B(f \mid g) = (D_{2n})_{n\in\mathbb{N}} \in \varprojlim \{(\mathcal{R}_{2n})_{n\geq 0}, (s_n)_{n\geq 0}\}$ and we start at

$$D_0 = \begin{pmatrix} d_{0,0} \end{pmatrix}.$$

Of course, if $B(f \mid g)$ is odd-symmetric, then $T^{\diamondsuit}(f \mid g) = T(f \mid g)$. We sketch the proof of the converse. Suppose $T^{\diamondsuit}(f \mid g) = T(f \mid g)$, then $T_n^{\diamondsuit}(f \mid g) = T_n(f \mid g)$ for all n. From this we have $D_0 = D_0^R$ and $D_2 = D_2^R$. Using induction and similar arguments as in the previous case, we get the desired result. □

We also need the following result to solve (P_1) and (P_2).

Proposition 32. *Let $D_m = (d_{i,j}) \in \mathcal{R}_m$ be such that $D_m = D_m^R$ with $m \geq 1$.*

(a) *If m is odd then D_m is a Toeplitz matrix.*
(b) *If m is even and $d_{0,0} = d_{1,1}$, then D_m is a Toeplitz matrix.*

Proof. To prove (a) we proceed by induction. Let $m = 2n + 1$.

If $n = 0$ then

$$\begin{pmatrix} d_{0,0} & \\ d_{1,0} & d_{1,1} \end{pmatrix} = \begin{pmatrix} d_{1,1} & \\ d_{1,0} & d_{0,0} \end{pmatrix}, \ \Rightarrow \ d_{0,0} = d_{1,1}, \ \Rightarrow \ g_0 = 1.$$

If $n = 1$, take

$$D = \begin{pmatrix} d_{0,0} & & & \\ d_{1,0} & d_{1,1} & & \\ d_{2,0} & d_{2,1} & d_{2,2} & \\ d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} \end{pmatrix} = \begin{pmatrix} d_{3,3} & & & \\ d_{3,2} & d_{2,2} & & \\ d_{3,1} & d_{2,1} & d_{1,1} & \\ d_{3,0} & d_{2,0} & d_{1,0} & d_{0,0} \end{pmatrix} = D^R.$$

So

$$r_0(D) = \begin{pmatrix} d_{1,1} & \\ d_{2,1} & d_{2,2} \end{pmatrix} = r_0(D)^R$$

and consequently $g_0 = 1$. Since $d_{1,0} = d_{3,2}$, $d_{1,0} = d_{2,1} + g_1 d_{2,2}$, $d_{2,1} = d_{3,2} + g_1 d_{2,2}$ and $d_{1,1} = d_{2,2}$ we get $d_{1,0} = d_{2,1} = d_{3,2}$ and $g_1 = 0$. Proceeding by the same way from the equality $d_{2,0} = d_{3,1}$ we get $g_2 = 0$. Suppose now that this is true for n and consider $D \in \mathcal{R}_{2n+3}$ with $D = D^R$, we know that $r_n(D) \in \mathcal{R}_{2n+1}$ satisfies $r_n(D) = r_n(D)^R$. Hence $g_0 = 1$ and $g_i = 0$ for all $i = 1, \cdots, 2n$. Since $D = D^R$, $d_{2n+1,0} = d_{2n+3,2}$ and $D \in \mathcal{R}_{2n+3}$. Consequently $d_{2n+1,0} = d_{2n+2,1} + g_{2n+1} d_{0,0}$ and $d_{2n+2,1} = d_{2n+3,2} + g_{2n+1} d_{0,0}$. This implies that $g_{2n+1} = 0$. Moreover, since $d_{2n+2,0} = d_{2n+3,1}$ and $d_{2n+2,0} = d_{2n+3,1} + g_{2n+2} d_{0,0}$, we obtain $g_{2n+2} = 0$ and D is a Toeplitz matrix. Finally, putting all above together we have $g_0 = 1$, $g_n = 0$ for all $n \geq 1$ and then $T(f \mid g)$ is a Toeplitz matrix. To prove (b), we note that the condition $d_{0,0} = d_{1,1}$, implies that $g_0 = 1$ and we proceed in a similar way as in (a). □

Now, the solution of the (P_1) problem is given by

Theorem 33. *$T^{\perp}(f \mid g) = T(f \mid g)$ if and only if $T(f \mid g)$ is a Toeplitz matrix.*

Proof. We only need to prove that if $T^{\perp}(f \mid g) = T(f \mid g)$ then $T(f \mid g)$ is a Toeplitz matrix. From Proposition 31 we get that $B(fg \mid g)$ is even symmetric. According to Proposition 27, if $\delta_n(B(fg \mid g)) = D_{2n+1}$ then $D_{2n+1} = D_{2n+1}^R$. Now using Proposition 32(a) we obtain that D_{2n+1} is a Toeplitz matrix for all n. This means that $B(fg \mid g)$ is a bi-infinite Toeplitz matrix. So, in particular, $T(f \mid g)$ is a Toeplitz matrix. □

The previous theorem allows us to prove the following result concerning the resolution of functional differential equations.

Corollary 34. *Let $f, \omega \in \mathbb{K}[[x]]$ with $f(0) \neq 0$, $\omega(0) = 0$ and $\omega'(0) \neq 0$ where $\omega'(x)$ is the usual derivative of ω in $\mathbb{K}[[x]]$. The solutions of*

$$\begin{cases} x^2 f(\omega(x))\omega'(x) = f(x)\omega^2(x), \\ \omega(\omega(x)) = x \end{cases} \tag{6}$$

in $\mathbb{K}[[x]]$ are just $\omega(x) = x$ and f arbitrary with $f(0) \neq 0$.

Proof. It is clear that $\omega(x) = x$ and f arbitrary with $f(0) \neq 0$ is a solution of (6). Suppose now that the pair (ω, f) is any solution of (6), then there is a series $g \in \mathbb{K}[[x]]$

with $g(0) \neq 0$ such that $\omega = \frac{x}{g}$. Consider the Riordan matrix $T(f \mid g)$. The first equation in (6) means that $A = g$, being A the corresponding A-sequence. The second equation converts to $(A - xA')f\left(\frac{x}{A}\right) = f(x)$. This implies $T^{\perp}(f \mid g) = T(f \mid g)$ and then it is a Toeplitz matrix. Consequently $g \equiv 1$, equivalently $\omega(x) = x$, and f arbitrary with $f(0) \neq 0$. □

Corollary 35. *Let $T(f \mid g)$ be a Riordan matrix. Then, $T(f \mid g)$ is Toeplitz if and only if $T_n^R(f \mid g) = T_n(f \mid g)$ for all $n \in \mathbb{N}$.*

Proof. If $T(f \mid g)$ is a Toeplitz then, obviously, $T_n^R(f \mid g) = T_n(f \mid g)$ for all $n \in \mathbb{N}$. The reciprocal is also obvious from Proposition 32. □

Remark 36. Note that, in general, $(T_n^R(f \mid g))_{n \in \mathbb{N}}$ does not represent any Riordan matrix.

We will finish this paper treating (P_2) whose answer is very different to that of (P_1). Our result is

Theorem 37. *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the solutions of (P_2) are the Riordan matrices $T(f \mid g)$ such that*

$$A(x) = g(x), \qquad f(x) = \lambda\sqrt{g(x)(g(x) - xg'(x))}e^{\phi\left(x, \frac{x}{g(x)}\right)}$$

with $\lambda \in \mathbb{K}^*$ and $\phi(x, z)$ is a symmetric bivariate power series with $\phi(0, 0) = 0$. If in addition $g(0) = 1$, then $T(f \mid g)$ is a Toeplitz matrix.

In other words, the Riordan array $\mathcal{R}(d(x), h(x))$ is self-dual if and only if h is self inverse for the composition operation and

$$d(x) = \lambda\sqrt{x\frac{h'(x)}{h(x)}}e^{\phi(x, h(x))}$$

for λ, and ϕ as above. Moreover, if $h'(0) = 1$ then $h(x) = x$.

Proof. $T^{\diamond}(f \mid g) = T(f \mid g)$ if and only if

$$g(x) = A(x) \quad \text{and} \quad A(x)(A(x) - xA'(x))f\left(\frac{x}{A(x)}\right) = f(x)$$

if and only if

$$\omega(\omega(x)) = x, \quad \text{and} \quad f(x) = \left(\frac{x}{\omega(x)}\right)^3 \omega'(x) f(\omega(x))$$

where $\omega(x) = \dfrac{x}{g(x)}$. This is equivalent to

$$\frac{f(x)}{f(\omega(x))} = \left(\frac{x}{\omega(x)}\right)^3 \omega'(x) \Leftrightarrow \log(f(x)) - \log(f(\omega(x))) = \log\left(\left(\frac{x}{\omega(x)}\right)^3 \omega'(x)\right)$$

where the last equality has full meaning for formal power series f with $f(0) > 0$ in real case and $f(0) \neq 0$ in the complex case.

Consider the linear equation on $y(x)$

$$y(x) - y(\omega(x)) = h(x), \quad \text{with} \quad h(x) = \log\left(\left(\frac{x}{\omega(x)}\right)^3 \omega'(x)\right). \tag{7}$$

Since $h(x) = -h(\omega(x))$, then $\frac{1}{2}h(x)$ is a particular solution. To get the general solution, note that the corresponding homogeneous equation has as general solution $y_H(x) = \phi(x, \omega(x))$ where $\phi(x, z)$ is a symmetric bivariate formal power series. It is clear that such a $\phi(x, \omega(x))$ is a solution. On the other hand if $y_H(x)$ is a solution, then

$$\phi(x, z) = \frac{y_H(x) + y_H(z)}{2}$$

satisfies all needed conditions. So the general solution of (7) is

$$y(x) = \frac{1}{2}h(x) + \phi(x, \omega(x)) \quad \text{with } \phi \text{ symmetric.}$$

Note also that for preciseness reasons in the final formula, we can suppose $\phi(0, 0) = 0$. Since $\log(f(x))$ is a solution of (7) then

$$f(x) = \lambda\sqrt{g(x)(g(x) - xg'(x))}\, e^{\phi\left(x, \frac{x}{g(x)}\right)}$$

with $\lambda \in \mathbb{K}^*$.

The remaining case to prove is $\mathbb{K} = \mathbb{R}$ and $f(0) < 0$. We proceed analogously changing f by $-f$ and we get the same result.

Note that for solution of (P_2), $(g(0))^2 = 1$. If $g(0) = 1$, using Proposition 31(ii) and Proposition 32(b) we obtain that $T(f \mid g)$ is a Toeplitz matrix. $\square$

A consequence of the above result is a way to construct odd-symmetric bi-infinite Riordan matrices

Example 38. Take $g(x) = 2x - 1$ and $\phi(x, z) = 0$. $B(\sqrt{1 - 2x} \mid 2x - 1)$ is then odd-symmetric. Below we write

$$\gamma_3(B(\sqrt{1-2x} \mid 2x-1)) = B_3(\sqrt{1-2x} \mid 2x-1) = T_6(\sqrt{1-2x}(2x-1)^3 \mid 2x-1)$$

$$\begin{pmatrix} 1 & & & & & & \\ -5 & -1 & & & & & \\ 15/2 & 3 & 1 & & & & \\ -5/2 & -3/2 & -1 & -1 & & & \\ -5/8 & -1/2 & -1/2 & -1 & 1 & & \\ -3/8 & -3/8 & -1/2 & -3/2 & 3 & -1 & \\ -5/16 & -3/8 & -5/8 & -5/2 & 15/2 & -5 & 1 \end{pmatrix}.$$

Example 39. In general, if we take $g(x) = \alpha x - 1$, see [7], $\phi(x, z) = 0$ and we proceed as in the previous example we get that for any $\lambda \in \mathbb{K}^*$

$$B(\lambda\sqrt{1-\alpha x} \mid \alpha x - 1) \qquad \alpha \in \mathbb{K}$$

is a family of odd-symmetric bi-infinite matrices and then

$$T(\lambda\sqrt{1-\alpha x} \mid \alpha x - 1) \qquad \alpha \in \mathbb{K}$$

or equivalently, taking $\mu = -\lambda$,

$$\mathcal{R}\left(\frac{\mu}{\sqrt{1-\alpha x}}, \frac{x}{\alpha x - 1}\right)$$

is a family of self-dual matrices. The case $\mu = 1$ and $\alpha = 4$ was first detected as self-dual in [5, p. 82].

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