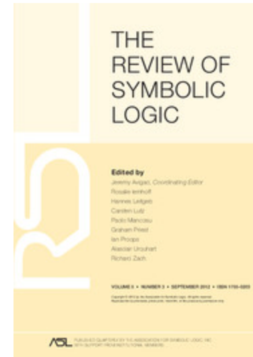

A NOTE ON THEORIES FOR QUASI-INDUCTIVE DEFINITIONS

RICCARDO BRUNI

The Review of Symbolic Logic / Volume 2 / Issue 04 / December 2009, pp 684 - 699
DOI: 10.1017/S175502030909025X, Published online: 04 November 2009

THE REVIEW OF SYMBOLIC LOGIC
Volume 2, Number 4, December 2009

A NOTE ON THEORIES FOR QUASI-INDUCTIVE DEFINITIONS

RICCARDO BRUNI
Department of Philosophy, University of Florence

Abstract. This paper introduces theories for arithmetical *quasi-inductive definitions* (Burgess, 1986) as it has been done for first-order monotone and nonmonotone inductive ones. After displaying the basic axiomatic framework, we provide some initial result in the proof theoretic bounds line of research (the upper one being given in terms of a theory of sets extending Kripke–Platek set theory).

§1. Introduction. Inductive definitions play a crucial role in logic, mathematics, and computer science. Definitions of this sort which have first undergone a metamathematical investigation are those based on monotone operators (Moschovakis, 1974; Buchholz *et al.*, 1981). Studies in generalized recursion theory and on the theory of admissible sets have made it natural to drop the monotonicity constraint and start considering inductive definitions in a generalized form (Richter, 1971; Aczel & Richter, 1974).

In this latter sense, an inductive definition of sets of natural numbers for a given set theoretic operator $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, is the one satisfying the clause

$$I_\Phi^\alpha := I_\Phi^{<\alpha} \cup \Phi(I_\Phi^{<\alpha}) \quad \text{where} \quad I_\Phi^{<\alpha} := \bigcup\{I_\Phi^\beta \mid \beta < \alpha\}$$

for every α ordinal number.

Then, it is further indicated by

$$I_\Phi^\infty := \bigcup\{I_\Phi^\alpha \mid \alpha \text{ ordinal}\}$$

the set of natural numbers which is inductively defined by Φ.

The proof theory for inductive definitions in this generalized form is given by Jäger (2001), and further by Jäger & Studer (2002) where a connection with Feferman's theory T_0 for explicit mathematics is disclosed.

In this paper, we plan to study in a similar way a form of transfinite definition of sets of natural numbers which is known as *quasi-inductive*. The introduction of this sort of constructions goes back to a seminal paper by Burgess (1986) on formal theories of truth, as a result of his analysis of the Herzberger–Gupta–Belnap *Revision Theory of Truth* (Gupta & Belnap, 1993).

To allow an immediate comparison with the inductive case, for a given operator $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, a quasi-inductive sequence of sets $\langle H_\Phi^\alpha \mid \alpha \text{ ordinal}\rangle$ is the one defined by

$$\begin{aligned} H_\Phi^0 &= \emptyset \\ H_\Phi^{\alpha+1} &= \Phi(H_\Phi^\alpha) \\ H_\Phi^\lambda &= \liminf_{\beta<\lambda} H_\Phi^\beta = \bigcup_{\alpha<\lambda} \bigcap_{\alpha\leq\beta<\lambda} H_\Phi^\beta, \quad \lambda \text{ limit.} \end{aligned}$$

Received: May 4, 2009
Research partially supported by the Consiglio Nazionale delle Ricerche (CNR) and the Swiss National Science Foundation (SNSF).

doi:10.1017/S175502030909025X

It should be clear that none of the usual argument showing inductive constructions to have fixed points (namely, levels γ's such that $I^{\gamma}_{\Phi} = I^{\gamma+1}_{\Phi}$) apply here in general. However, it can be shown that similar results hold for a natural modification of the notions involved therein.

In particular, one can show that the whole sequence enters into a 'cycle' as there are limit ordinals σ's such that $H^{\sigma}_{\Phi} = H^{+\infty}_{\Phi} := \liminf_{\beta<\infty} H^{\beta}_{\Phi}$, and $H^{-\sigma}_{\Phi} := \liminf_{\beta<\sigma}(\omega \setminus H^{\beta}_{\Phi}) = \liminf_{\beta<\infty}(\omega \setminus H^{\beta}_{\Phi}) =: H^{-\infty}_{\Phi}$. In fact, it is possible to show that there is a closed and unbound class of *countable* ordinals enjoying this property.

Since the paper by Burgess we referred to above, revision theoretic constructions (of which quasi-inductive definitions are a specific instance) proved to have a tight relationship with the independently discovered paradigm of Infinite Time Turing Machines (Hamkins & Lewis, 2000—the connections between the two notions were first made by Löwe, 2001[1]). By sometimes exploiting this very same connection, Welch (2007) has, in very recent time, attempted at characterizing levels in the hierarchy of game determinacy by means of quasi-inductively defined sets (this work being more in the line of the proof theoretic character of our research).

Methodologically speaking, we stick here to axiomatic and proof theoretic means of analysis.

We first devise an axiomatic framework which comprises the properties of *arithmetical* quasi-inductive definitions (something we do in a similar fashion as it has been done for the inductive cases). Then, we will focus on the problem of setting the proof theoretic strength of the outcoming system of axioms. The problem of finding a lower bound is solved in such a way to give support to the claim that quasi-inductive definitions are a natural generalization of the inductive sort of constructions. As for the upper bound, this is calculated in terms of a theory of sets extending the usual Kripke–Platek system of axioms. None of the results in question allow us to fix *sharp* bounds, hence the whole paper serves mainly the purpose of providing some definite technical problem, on which some comment is made in the closing section.

§2. The family QID(K) of formal theories. We start from a language $\mathcal{L}_{Ar}$ which is the one of Peano arithmetic. Hence, $\mathcal{L}_{Ar}$ has countably many individual variables for natural numbers $x, y, z, \ldots$ (possibly with subscripts), and symbols for all primitive recursive functions and relations.

The language $\mathcal{L}_0$ we now define extends $\mathcal{L}_{Ar}$ with a second sort of denumerably many variables $\alpha, \beta, \gamma, \ldots$ (with possible subscripts) for ordinal numbers, individual constants $0_\Omega, \omega$, function symbols $succ_\Omega, +_\Omega, \times_\Omega$, as well as predicate constants $<_\Omega, =_\Omega$.[2]

The collection $TERM^N$ of *arithmetical terms* is defined as usual. The collection of *ordinal terms* $TERM^\Omega$, instead, is the least containing $0_\Omega, \omega$, individual variables for ordinals, and which is closed under the primitive ordinal functions. Formulas are built up as usual by closing the collection of *atomic formulas*, which feature arithmetical atomic

1 Löwe's result actually applied to a proper initial subclass of Herzberger revision sequences. The full connection, as Löwe himself acknowledged, was established by Welch (2001) (or Welch, 2003).

2 The subscript Ω is conceived in such a way to distinguish symbols which will be applied only to ordinal terms—to be defined below—from the corresponding ones applying to arithmetical terms. The subscript will be dropped whenever no confusion can possibly arise.

formulas as well as formulas $s =_\Omega t$, $s <_\Omega t$ for $s, t \in TERM^\Omega$, under logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and quantification $\forall, \exists$ on *both sorts* of variables.

Symbols $m, n, \ldots$ will be used in the following to indicate arithmetical terms, while lowercase greek letters $\alpha, \beta, \ldots$ are the chosen metavariables for *arbitrary* ordinal terms of our language.

The next step is devoted to encapsulating in the language the tools for representing set theoretical operators which are to be iterated along the ordinals. As it is customary, this is done via *operator forms*.

Let $\mathcal{L}(X)$ indicate $\mathcal{L}_{Ar} \cup \{X\}$, X being a fresh unary predicate variable. An operator form is a formula $A(x, X)$ of $\mathcal{L}(X)$ with displayed free variables. Notice that there is no constraint as to how the higher order variable occurs in A. Operator forms are then grouped into complexity classes $\mathbf{K}$ according to their logical complexity (then $\mathbf{K}$ is Π_n, Σ_n, or Δ_n, for some $n \in \mathbb{N}$, which have their usual definitions).

Finally, the language $\mathcal{L}(\mathbf{K})$ of our theory $\mathsf{QID}(\mathbf{K})$ is obtained from $\mathcal{L}_0$ by adding binary predicate constants $\mathcal{H}_A$, for each operator form $A(x, X)$ in $\mathbf{K}$. Terms and formulas of the expanded language are thus as before, except that a clause for atoms of the form $\mathcal{H}_A(n, \alpha)$ with $n \in TERM^N$, $\alpha \in TERM^\Omega$, must now be added.

Bounded quantification on both sort of variables is introduced by definition as expected. Additional notational conventions are

$$
\begin{aligned}
x \in \mathcal{H}_A^\alpha &:= \mathcal{H}_A(x, \alpha) \\
x \in \mathcal{H}_A^{+\alpha} &:= (\exists \beta < \alpha)(\forall \gamma < \alpha)(\beta \leq \gamma \rightarrow x \in \mathcal{H}_A^\gamma) \\
x \in \mathcal{H}_A^{-\alpha} &:= (\exists \beta < \alpha)(\forall \gamma < \alpha)(\beta \leq \gamma \rightarrow x \notin \mathcal{H}_A^\gamma) \\
x \in \mathcal{H}_A^{+\infty} &:= \exists \beta \forall \gamma\, (\beta \leq \gamma \rightarrow x \in \mathcal{H}_A^\gamma) \\
x \in \mathcal{H}_A^{-\infty} &:= \exists \beta \forall \gamma\, (\beta \leq \gamma \rightarrow x \notin \mathcal{H}_A^\gamma) \\
\mathcal{H}_A^\alpha \equiv \mathcal{H}_A^\beta &:= \forall x (x \in \mathcal{H}_A^\alpha \leftrightarrow x \in \mathcal{H}_A^\beta)
\end{aligned}
$$

(where obviously $\alpha \leq \beta := (\alpha < \beta \vee \alpha = \beta)$).

The theory $\mathsf{QID}(\mathbf{K})$ for $\mathbf{K}$ complex quasi-inductive definitions is the one based on the language $\mathcal{L}(\mathbf{K})$, whose axioms are those grouped as follows:

I. Logical axioms. The axioms for first-order predicate logic with identity.

II. Number theoretic axioms. The usual axioms of Peano arithmetic, except the schema of complete induction.

III. Ordinal theoretic axioms. The (universal closure of the) following list of assumptions:[3]

(Ω.1) $\alpha = \beta \vee \alpha < \beta \vee \beta < \alpha$
(Ω.2) $\neg(\alpha < \alpha)$
(Ω.3) $\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma$
(Ω.4) $0 \leq \alpha$
(Ω.5) $\alpha < \alpha'$ [with $\alpha' = succ_\Omega(\alpha)$]
(Ω.6) $\alpha < \beta \rightarrow \alpha' \leq \beta$

[3] Here and in the rest of the paper we abbreviate $\alpha \times_\Omega \beta$ by $\alpha\beta$ for the sake of readability.

(Ω.7) $0 < \omega \wedge (\forall\alpha < \omega)\alpha' < \omega$
(Ω.8) $Lim(\lambda) \to \omega \leq \lambda \;\; [Lim(\alpha) := (0 < \alpha \wedge (\forall\beta < \alpha)\beta' < \alpha)]$
(Ω.9) $\alpha + 0 = \alpha$
(Ω.10) $\alpha + \beta' = (\alpha + \beta)'$
(Ω.11) $\alpha < \beta \to \gamma + \alpha < \gamma + \beta$
(Ω.12) $\alpha \leq \beta \to \alpha + \gamma \leq \beta + \gamma$
(Ω.13) $\alpha 0 = 0\alpha = 0$
(Ω.14) $\alpha\beta' = \alpha\beta + \alpha$
(Ω.15) $0 < \gamma \wedge \alpha < \beta \to \gamma\alpha < \gamma\beta$
(Ω.16) $\alpha \leq \beta \to \alpha\gamma \leq \beta\gamma$
(Ω.17) $\alpha < \beta \to (\exists\gamma \leq \beta)(\alpha + \gamma = \beta)$
(Ω.18) $0 < \beta \to (\exists\gamma \leq \alpha)(\exists\delta < \beta)(\alpha = \beta\gamma + \delta)$.

IV. QID axioms. For all operator forms $A(x, X)$ in **K**, the (universal closure of the) following schemas:

(QID.1) $x \in \mathcal{H}_A^0 \to x \neq x$
(QID.2) $x \in \mathcal{H}_A^{\alpha'} \leftrightarrow A(x, \mathcal{H}_A^\alpha)$
(QID.3) $Lim(\lambda) \to (x \in \mathcal{H}_A^\lambda \leftrightarrow (\exists\alpha < \lambda)(\forall\beta < \lambda)(\alpha \leq \beta \to x \in \mathcal{H}_A^\beta))$
(QID.4) $\forall\alpha\exists\lambda[Lim(\lambda) \wedge \alpha < \lambda \wedge (\mathcal{H}_A^{+\lambda} \equiv \mathcal{H}_A^{+\infty}) \wedge (\mathcal{H}_A^{-\lambda} \equiv \mathcal{H}_A^{-\infty})]$

[notice that we have $\mathcal{H}_A^\lambda \equiv \mathcal{H}_A^{+\lambda}$, for λ limit].

V. Induction principles. Finally, this group contains the schema of complete induction on the natural numbers ($\mathcal{L}(\mathbf{K}) - \mathrm{I}_N$), and the one of transfinite induction on the ordinals ($\mathcal{L}(\mathbf{K}) - \mathrm{I}_\Omega$) for all formulas $A(x)$ and $B(\alpha)$ of $\mathcal{L}(\mathbf{K})$:

($\mathcal{L}(\mathbf{K}) - \mathrm{I}_N$) $A(0) \wedge \forall x(A(x) \to A(x')) \to \forall x A(x)$

($\mathcal{L}(\mathbf{K}) - \mathrm{I}_\Omega$) $\forall\alpha((\forall\beta < \alpha)(B(\beta) \to B(\alpha))) \to \forall\alpha B(\alpha)$.

The axioms from the QID group are clearly conceived in order to capture the properties of quasi-inductive constructions in a similar manner as the corresponding group of 'operator' axioms do in the inductive cases (see §3 below). The fourth of them embodies the stabilization property we have spoken of in the introduction in a strong form. This formulation of the principle is nonetheless needed in the proof of a theorem to follow stating that levels corresponding to 'stable' ordinals in the sense of (QID.4) occur in the sequence according to a period (and consequently admit a characterization in terms of ordinal arithmetic).

Before going into that, we need a couple of auxiliary lemmas. The first of them contains some basic facts on ordinal arithmetic, the proof of which is straightforward and therefore omitted.

LEMMA 2.1. *QID(**K**) proves (the universal closure of):*

(i) $Lim(\alpha) \wedge \delta < \beta + \alpha \to (\exists\gamma < \alpha)(\delta < \beta + \gamma)$
(ii) $\alpha + 0 = 0 + \alpha = \alpha$
(iii) $Lim(\alpha) \wedge \delta < \beta\alpha \to (\exists\gamma < \alpha)(\delta < \beta\gamma)$
(iv) $Lim(\alpha) \to Lim(\beta + \alpha)$
(v) $Lim(\alpha) \wedge 0 < \beta \to Lim(\beta\alpha)$
(vi) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

Then, one proves some further properties so to clarify the meaning of ordinal addition in the quasi-inductive iteration of a given operator form.

LEMMA 2.2.

(i) *For every* $\boldsymbol{K}$ *operator form* $A(x, X)$, *QID(K) proves:*

(i.1) $x \in \mathcal{H}_A^{\alpha+0} \leftrightarrow x \in \mathcal{H}_A^{\alpha}$

(i.2) $x \in \mathcal{H}_A^{\alpha+\beta'} \leftrightarrow x \in \mathcal{H}_A^{(\alpha+\beta)'}$

(i.3) $Lim(\lambda) \rightarrow \forall x(x \in \mathcal{H}_A^{\alpha+\lambda} \leftrightarrow (\exists\beta < \lambda)(\forall\delta < \lambda)(\beta \leq \delta \rightarrow x \in \mathcal{H}_A^{\alpha+\delta})$

(ii) *For every* $\boldsymbol{K}$ *operator form* $A(x, X)$, *QID(K) proves*

$$\forall\alpha\beta\gamma\,(\mathcal{H}_A^{\alpha} \equiv \mathcal{H}_A^{\beta} \rightarrow \mathcal{H}_A^{\alpha+\gamma} \equiv \mathcal{H}_A^{\beta+\gamma}).$$

Proof. (i.1) and (i.2) are trivial application of the sum equations (**Ω**.9, 10) and the identity axioms.

The proof of (i.3) is, for the most part, a matter of ordinal arithmetic. By Lemma 2.1. (iv) we have $Lim(\alpha + \lambda)$ from the assumption $Lim(\alpha)$ and hence

$$x \in \mathcal{H}_A^{\alpha+\lambda} \leftrightarrow (\exists\beta < \alpha + \lambda)(\forall\delta < \alpha + \lambda)(\beta \leq \delta \rightarrow x \in \mathcal{H}_A^{\delta}).$$

Then, it suffices to prove

$$(\exists\beta < \alpha + \lambda)(\forall\delta < \alpha + \lambda)(\beta \leq \delta \rightarrow x \in \mathcal{H}_A^{\delta}) \leftrightarrow$$
$$\leftrightarrow (\exists\beta < \lambda)(\forall\delta < \lambda)(\beta \leq \delta \rightarrow x \in \mathcal{H}_A^{\alpha+\delta})$$

which is a consequence of the defining properties of ordinal addition.

Having proved that, (ii) then comes from an easy induction on γ, using (i.1–i.3). □

Lemmas 2.1. (vi) and 2.2. (ii) justify our getting rid of parentheses in the ordinal terms indexing levels in a quasi-inductive construction.

PROPOSITION 2.3. *Let* σ *be any limit ordinal such that* $\mathcal{H}_A^{\sigma} \equiv \mathcal{H}_A^{+\infty}$ *and* $\mathcal{H}_A^{-\sigma} \equiv \mathcal{H}_A^{-\infty}$, *for a* $\boldsymbol{K}$ *operator form* $A(x, X)$. *Then QID(K) proves that there exists a unique ordinal* $p(\sigma) > 0$, *the period of* σ, *such that:*

(i) *for every ordinal* γ, $\mathcal{H}_A^{\sigma} \equiv \mathcal{H}_A^{\sigma+p(\sigma)\gamma}$

(ii) *for every ordinal* $\alpha > \sigma$ *there exists an ordinal* $0 \leq \nu < p(\sigma)$ *such that* $\mathcal{H}_A^{\alpha} \equiv \mathcal{H}_A^{\sigma+\nu}$.

Proof. We argue informally in QID(**K**). Let σ be as in the statement of the proposition. Then, by applying axioms (QID.4) and ($\mathcal{L}$(**K**) $-$ I$_\Omega$) we can find an ordinal δ such that

$$\delta = \min\xi.\sigma < \xi \wedge \mathcal{H}_A^{\xi} \equiv \mathcal{H}_A^{+\infty} \wedge \mathcal{H}_A^{-\xi} \equiv \mathcal{H}_A^{-\infty}$$

Let $p(\sigma) = \delta - \sigma$, namely *the* ordinal ξ such that $\sigma + \xi = \delta$. Existence of $p(\sigma)$ is then a consequence of (**Ω**.17), while uniqueness follows from (**Ω**.11).

It is easy thus to prove by induction on γ (using Lemma 2.2. (i–ii)) that

$$\forall\gamma\,(\mathcal{H}_A^{\sigma} \equiv \mathcal{H}_A^{\sigma+p(\sigma)\gamma})$$

holds.

If $\gamma = \beta + 1$ then:

$$\begin{aligned} \mathcal{H}_A^{\sigma+p(\sigma)\gamma} &\equiv \mathcal{H}_A^{\sigma+p(\sigma)\beta+p(\sigma)} \equiv \\ &\overset{\text{IH, 2.2.(ii)}}{\equiv} \mathcal{H}_A^{\sigma+p(\sigma)} \equiv \\ &\equiv \mathcal{H}_A^{\sigma}. \end{aligned}$$

If γ is a limit ordinal and by IH $\mathcal{H}_A^{\sigma} \equiv \mathcal{H}_A^{\sigma+p(\sigma)\beta}$ for all $\beta < \gamma$, then Lemma 2.2. (i.3) gives $\forall x(x \in \mathcal{H}_A^{\sigma} \rightarrow x \in \mathcal{H}_A^{\sigma+p(\sigma)\gamma})$. Conversely, Lemma 2.2. (i.3) and $x \in \mathcal{H}_A^{\sigma+p(\sigma)\gamma}$ yield $x \in \mathcal{H}_A^{\sigma+p(\sigma)\beta}$ for some $\beta < \gamma$, from which one gets $x \in \mathcal{H}_A^{\sigma}$ by IH.

Turning now to the proof of (ii), if $\sigma < \alpha$ then axioms (Ω.17,18) entail $\alpha = \sigma + p(\sigma)\beta + \nu$ for some $\nu < p(\sigma)$. Hence Lemma 2.2. (ii) yields $\mathcal{H}_A^{\alpha} \equiv \mathcal{H}_A^{\sigma+\nu}$. □

This last result clearly implies that all and only the ordinals enjoying the stabilization property admit this characterization in terms of the least of them and its 'period'.

§3. Lower bound. In this section we will show that theories for first-order nonmonotone inductive definitions can be embedded into our formalism for arithmetical quasi-inductive constructions. This will be shown to hold both 'globally', for theories of both sorts comprising all arithmetical operator forms, and 'locally', between **K** complex nonmonotone inductive definitions and the corresponding instance of the QID formalism.

Theories for nonmonotone inductive definitions were introduced and studied in Jäger (2001) and Jäger & Studer (2002). As it was said, we have chosen to define our formalism so to closely resemble what was done for the inductive case before. The reader will notice that similarity as the definition of the latter goes on below.[4]

The language $\mathcal{L}_{FID}^{K}$ for **K** complex nonmonotone inductive definitions is as follows: it contains number theoretic variables $x, y, z, , \ldots$, ordinal variables $\alpha, \beta, \gamma, \ldots$, symbols for all arithmetical primitive recursive functions and relations, a binary relation symbol $<$ between ordinals, and binary relation symbols $\mathcal{P}_A$ for every operator form $A(x, X)$ from $\mathcal{L}_{Ar} \cup \{X\}$ (where, as in the previous section, we put no constraint as to how the predicate variable X should occur). Terms include number terms, which have the usual definition, and ordinal variables. Atomic formulas have then the form $R(s_1, \ldots, s_n)$ where R is primitive recursive and all of the s_i are number terms, $\alpha < \beta$, or $\mathcal{P}_A(\alpha, s)$ (abbreviated $\mathcal{P}_A^{\alpha}(s)$) where s is a number term. Formulas are then built by closing the collection of atomic formulas under Boolean connectives, and quantification on both sorts of variables. Bounded quantification is introduced by definition as expected.

Additional abbreviations which are made use of in the axioms are:

$$\mathcal{P}_A^{<\alpha}(s) := (\exists \beta < \alpha)\mathcal{P}_A^{\beta}(s) \qquad \mathcal{P}_A^{\infty}(s) := \exists \beta \mathcal{P}_A^{\beta}(s).$$

The theory $\mathsf{FID}(\mathbf{K})$ has the following groups of nonlogical axioms:

I. Number theoretic axioms. The axioms of Peano arithmetic with the exception of complete induction (see Group IV of axioms below).

II. Linearity axiom.

$$\neg(\alpha < \alpha) \wedge (\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma) \wedge (\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha).$$

III. Operator axioms. For all operator forms $A(x, X)$:

(OP.1) $\quad \mathcal{P}_A^{\alpha}(s) \leftrightarrow \mathcal{P}_A^{<\alpha}(s) \vee A(s, \mathcal{P}_A^{<\alpha})$

(OP.2) $\quad A(s, \mathcal{P}_A^{\infty}) \rightarrow \mathcal{P}_A^{\infty}(s).$

[4] Due to uniformity matters, the following description of the collection of theories for first-order nonmonotone inductive definitions may differ, on certain superficial typesetting aspects, from the one as it was originally given in the two papers we have referred to above.

IV. Induction principles. For all formulas $A(x)$ and $B(\alpha)$ of $\mathcal{L}^{K}_{FID}$, we have:

(CI) $A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x)$

(TI) $\forall\alpha((\forall\beta < \alpha)(B(\beta) \rightarrow B(\alpha))) \rightarrow \forall\alpha B(\alpha)$.

Systems $\mathsf{FID}(\mathbf{K})$ which have been taken into consideration in Jäger (2001), Jäger & Studer (2002), come from both restricting the attention to special classes $\mathbf{K}$ of operator forms (in particular, those defined according to Richter's method for producing nonmonotone operators), and weakening the induction principles.

The embedding we are going to describe is based on a straightforward idea. Namely, it is obtained by restricting the attention to *inflationary* operator forms, namely formulas $B(x, X) := (X(x) \vee A(x, X))$ where $A(x, X)$ is an operator form whatsoever. Notice that if $A(x, X)$ is of complexity $\mathbf{K}$, then also $B(x, X)$ is so. Operator forms of this sort are trivially *inclusive*, in the sense that, axiomatically speaking, $C(s) \rightarrow B(s, C)$ holds for every formula $C(x)$. Then, first one proves that, without loss of generality, we can assume operator forms that get used for defining the corresponding theories FID to have always this form. Then, the result basically reduces to exploiting the fact that the quasi-inductive iteration of inflationary operator forms turns out to be in fact a monotone inductive construction due to the inclusivity property.

LEMMA 3.1. *Let $A(x, X)$ be a $\mathbf{K}$ complex operator form whatsoever, and take $B(x, X) := (X(x) \vee A(x, X))$ to be the inflationary version of it. Then*

$$\vdash_{\mathsf{FID}(\mathbf{K})} \forall\alpha(\mathcal{P}^{\alpha}_{A}(s) \leftrightarrow \mathcal{P}^{\alpha}_{B}(s)).$$

Proof. By induction on α. Hence, assume (IH) that, for all $\beta < \alpha$

$$\mathcal{P}^{\beta}_{A}(s) \leftrightarrow \mathcal{P}^{\beta}_{B}(s)$$

is the case, which, by obvious considerations entails then

$$\mathcal{P}^{<\beta}_{A}(s) \leftrightarrow \mathcal{P}^{<\beta}_{B}(s)$$

for every $\beta < \alpha$. Hence:

$$\begin{aligned}
\mathcal{P}^{\alpha}_{B}(s) &\overset{(OP.1)}{\leftrightarrow} \mathcal{P}^{<\beta}_{B}(s) \vee B(s, \mathcal{P}^{<\beta}_{B}) \\
&\leftrightarrow \mathcal{P}^{<\beta}_{B}(s) \vee A(s, \mathcal{P}^{<\beta}_{B}) \\
&\overset{IH}{\leftrightarrow} \mathcal{P}^{<\beta}_{A}(s) \vee A(s, \mathcal{P}^{<\beta}_{A}) \\
&\overset{(OP.1)}{\leftrightarrow} \mathcal{P}^{\alpha}_{A}(s).
\end{aligned}$$

□

For the next lemma, we assume, without loss of generality, that operator forms are in negative normal form (which is obtained by pushing negations deep inside the formula using De Morgan laws insofar as atomic subformulas are reached). Further, given an operator form $A(x, X)$ whatsoever, we make use for convenience of the notation $A(s, C(a), \neg C(a))$, for $C(x)$ formula, just to indicate the result of the following substitution:

$$A(s, X)[X^{+} := C(a), X^{-} := C(a)]$$

where X^{+}/X^{-} indicate the positive/negative occurrences of the variable X inside A.

LEMMA 3.2. *For every* $\boldsymbol{K}$ *complex inflationary operator form* $B(x, X)$ *and* $\alpha \neq 0_\Omega$ *ordinal, we have*

$$\vdash_{\mathsf{QID}(\mathbf{K})} s \in \mathcal{H}^\alpha_B \leftrightarrow (\exists\beta < \alpha)(s \in \mathcal{H}^\beta_B) \vee B(s, (\exists\beta < \alpha)\mathcal{H}^\beta_B).$$

Proof. The proof is, again, by induction on α.
Then, assume $\alpha = \beta + 1$ and we have:

$$\begin{aligned} s \in \mathcal{H}^\alpha_B &\overset{(\mathrm{QID}.2)}{\leftrightarrow} B(s, \mathcal{H}^\beta_B) \\ &\leftrightarrow s \in \mathcal{H}^\beta_B \vee A(s, \mathcal{H}^\beta_B) \\ &\overset{\mathrm{IH}}{\leftrightarrow} (\exists\gamma < \beta)(s \in \mathcal{H}^\gamma_B) \vee \\ &\qquad \vee B(s, (\exists\gamma < \beta)\mathcal{H}^\gamma_B) \vee \\ &\qquad \vee A(s, \mathcal{H}^\beta_B). \end{aligned}$$

Now:

1. $(\exists\gamma < \beta) s \in \mathcal{H}^\gamma_B \overset{\beta < \alpha}{\rightarrow} (\exists\gamma < \alpha) s \in \mathcal{H}^\gamma_B$.
2. $B(s, (\exists\gamma < \beta)\mathcal{H}^\gamma_B) \overset{\mathrm{IH}}{\rightarrow} s \in \mathcal{H}^\beta_B \overset{\beta < \alpha}{\rightarrow} (\exists\gamma < \alpha) s \in \mathcal{H}^\gamma_B$.

From both the desired conclusion is reached by logic. We further show, which concludes the direction from left to right of the theorem, that $A(s, \mathcal{H}^\beta_B)$ implies it as well. But, since:

$$t \in \mathcal{H}^\beta_B \rightarrow (\exists\gamma < \alpha)(t \in \mathcal{H}^\gamma_B)$$

and, by IH

$$t \notin \mathcal{H}^\beta_B \rightarrow (\forall\gamma < \alpha)(t \notin \mathcal{H}^\gamma_B)$$

we have:

$$A(s, \mathcal{H}^\beta_B, \neg\mathcal{H}^\beta_B) \rightarrow A(s, (\exists\gamma < \alpha)\mathcal{H}^\gamma_B, (\forall\gamma < \alpha)\neg\mathcal{H}^\gamma_B) =: A(s, (\exists\gamma < \alpha)\mathcal{H}^\gamma_B).$$

This, by logic again, ensures the left-to-right direction of the desired conclusion.
Vice versa, we first observe that:

$$\begin{aligned} (\exists\gamma < \alpha)(s \in \mathcal{H}^\gamma_B) &\rightarrow (\exists\gamma \leq \beta)(s \in \mathcal{H}^\gamma_B) \\ &\overset{\mathrm{IH}}{\rightarrow} s \in \mathcal{H}^\beta_B \end{aligned}$$

which yields, by inclusivity and (QID.2), $s \in \mathcal{H}^\alpha_B$.
It remains then to prove

$$A(s, (\exists\gamma < \alpha)\mathcal{H}^\gamma_B) \rightarrow s \in \mathcal{H}^\alpha_B.$$

But, again

$$\begin{aligned} (\exists\gamma < \alpha)(t \in \mathcal{H}^\gamma_B) &\rightarrow (\exists\gamma \leq \beta)(t \in \mathcal{H}^\gamma_B) \\ &\overset{\mathrm{IH}}{\rightarrow} t \in \mathcal{H}^\beta_B \end{aligned}$$

and

$$(\forall\gamma < \alpha)(t \notin \mathcal{H}^\gamma_B) \rightarrow t \notin \mathcal{H}^\beta_B.$$

Hence:

$$A(s, (\exists\gamma < \alpha)\mathcal{H}_B^{\gamma}, (\forall\gamma < \alpha)\neg\mathcal{H}_B^{\gamma}) \rightarrow A(s, \mathcal{H}_B^{\beta}, \neg\mathcal{H}_B^{\beta}) =: A(s, \mathcal{H}_B^{\beta})$$

which, by logic and axiom (QID.2), entails the conclusion.

Assume, instead, that for all $\beta < \alpha$ the theorem holds and α is a limit ordinal.

First, notice that by the inclusivity property one gets

$$\begin{aligned} s \in \mathcal{H}_B^{\alpha} &\overset{\text{(QID.3)}}{\leftrightarrow} (\exists\beta < \alpha)(\forall\gamma < \alpha)(\beta \leq \gamma \rightarrow s \in \mathcal{H}_B^{\gamma}) \\ &\leftrightarrow (\exists\beta < \alpha)(s \in \mathcal{H}_B^{\beta}). \end{aligned} \tag{1}$$

It follows that the direction from left to right of the theorem is straightforward in this case.

For the other direction, observe that, since α is a limit ordinal we have

$$\begin{aligned} &(\exists\beta < \alpha)(s \in \mathcal{H}_B^{\beta}) \vee B(s, (\exists\beta < \alpha)\mathcal{H}_B^{\beta}) \rightarrow \\ &\quad \rightarrow (\exists\gamma < \alpha)[(\exists\beta < \gamma)(s \in \mathcal{H}_B^{\beta}) \vee B(s, (\exists\beta < \gamma)\mathcal{H}_B^{\beta})]. \end{aligned}$$

Now, the induction hypothesis allows to conclude $s \in \mathcal{H}_B^{\gamma}$ for such a $\gamma < \alpha$, which yields $(\exists\beta < \alpha)(s \in \mathcal{H}_B^{\beta})$.

By (1), this ends the proof. □

Notice that we also have

$$\begin{aligned} s \in \mathcal{H}_B^{+\infty} &\leftrightarrow \exists\beta\forall\gamma\,(\beta \leq \gamma \rightarrow s \in \mathcal{H}_B^{\gamma}) \\ &\leftrightarrow \exists\beta(s \in \mathcal{H}_B^{\beta}). \end{aligned} \tag{2}$$

This fact will be required for the proposed translation, together with the next lemma on the fixed point axiom:

COROLLARY 3.3. *QID(**K**) proves* $B(s, \mathcal{H}_B^{+\infty}) \rightarrow s \in \mathcal{H}_B^{+\infty}$.

Proof. Using the stabilization property we have, for every ordinal σ such that $\mathcal{H}_B^{\sigma} \equiv \mathcal{H}_B^{+\infty}$ is the case:

$$\begin{aligned} B(s, \mathcal{H}_B^{+\infty}) &\leftrightarrow B(s, \mathcal{H}_B^{\sigma}) \\ &\overset{(1)}{\rightarrow} (\exists\beta < \sigma)(s \in \mathcal{H}_B^{\beta}) \vee B(s, (\exists\beta < \sigma)\mathcal{H}_B^{\beta}) \\ &\overset{\text{L.3.2.}}{\leftrightarrow} s \in \mathcal{H}_B^{\sigma} \\ &\leftrightarrow s \in \mathcal{H}_B^{+\infty}. \end{aligned}$$

□

The details of the promised embedding result modulo a translation $(\,)^{\bullet}$ of formulas of $\mathcal{L}_{FID}^{K}$ into formulas of $\mathcal{L}(\mathbf{K})$, should now be obvious. We let in fact ordinal terms of $\mathcal{L}(\mathbf{K})$ correspond to ordinal variables from $\mathcal{L}_{FID}^{K}$ in a fashion that agrees with Lemma 3.2., and a similar correspondence is then established between the order relation from the two languages. Finally, to atomic formulas $\mathcal{P}_B(s, \alpha)$ we let correspond formulas $\mathcal{H}_B(s, \alpha)$,[5]

[5] Where, by abuse of notation, we have used the same symbols in order to indicate terms which correspond to each others under the translation we are describing.

and the translation is further set so to commute with respect to Boolean connectives and quantifiers.

The following theorem is then easily established using Lemma 3.2. and its Corollary 3.3. above:

THEOREM 3.4. *For every formula A of* $\mathcal{L}^{K}_{FID}$ *we have:*

$$\vdash_{\mathsf{FID}(\mathbf{K})} A \Rightarrow \ \vdash_{\mathsf{QID}(\mathbf{K})} A^{\bullet}$$

§4. Embedding QID(Π_∞) in a theory of sets. The idea underlying the upper bound result we present here is straightforward. It consists in finding the least theory extending KP within which it is possible to reproduce the arguments needed for proving the stabilization property for arithmetical quasi-inductive definitions, as the latter is embodied by our axiom (QID.4) (see, e.g., Cantini, 1996).

We start by describing the theory we shall make use of here, which is the standard Kripke–Platek set theory KP where two of its main assumptions, the schema of collection and the schema of separation, are extended to formulas of appropriately chosen complexity classes.

Then, the language $\mathcal{L}^*$ of our theory T is a standard first-order language comprising two binary relation constants, $\in$ for membership and equality $=$. Terms and formulas, as well as collections Δ_0, Σ, Π, Π_n, Σ_n of formulas of $\mathcal{L}^*$ ($n \in \mathbb{N}$), have their usual definitions.

We make use of standard abbreviations $\mathsf{Tran}(x)$ and $\mathsf{Ord}(x)$ ('x is a transitive set' and 'x is an ordinal number', respectively) for the following bounded formulas

$$\mathsf{Tran}(x) := (\forall y \in x)(\forall z \in y)(z \in x)$$

$$\mathsf{Ord}(x) := \mathsf{Tran}(x) \wedge (\forall y \in x)\mathsf{Tran}(y).$$

The nonlogical axioms of T are:

(EXT) $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$
(PAIR) $\forall x \forall y \exists z(x \in z \wedge y \in z)$
(UNION) $\forall z \exists x \forall y(y \in x \leftrightarrow (\exists w \in z)(y \in w))$
(SEP) $\forall z \exists x \forall y[y \in x \leftrightarrow y \in z \wedge A(y)]$
(COLL) $\forall z[(\forall x \in z)\exists y B(x, y) \rightarrow \exists w(\forall x \in z)(\exists y \in w)B(x, y)]$
(INF) $\exists x[\mathsf{Ord}(x) \wedge \exists y(y \in x) \wedge (\forall z \in x)(\exists w \in x)(z \in w)]$
(IND$_\in$) $\forall x((\forall y \in x)(A(y) \rightarrow A(x)) \rightarrow \forall x A(x)$

[where in (SEP) and (COLL), $A(x)$ and $B(x, y)$ are Δ_2 and Δ_3 formulas, respectively[6]].

For the sake of readability below, we further introduce the following abbreviation

$$x \in \mathsf{N} := (x = \emptyset \vee (\exists y \in x)(x = y \cup \{y\}))$$

and, for an arithmetical formula A of $\mathcal{L}_{Ar}$ (or of $\mathcal{L}(X) := \mathcal{L}_{Ar} \cup \{X\}$), we indicate by A^{N} the translation of it (in the expected manner) into a bounded formula of $\mathcal{L}^*$.

In the following we use, as before, lowercase greek letters $\alpha, \beta, \gamma, \dots$ (possibly with subscripts) to denote ordinals in our theory of sets, and, as usual, we write $\alpha < \beta$ for $\alpha \in \beta$. Similarly, we will use letters $m, n, \dots$ (possibly with subscripts) for natural numbers.

6 By correcting an oversight of the author, P. Welch pointed out that a slight modification of the argument that was contained in a previous version of the paper led in fact to this improved result. The author wishes to thank him for his suggestions in this sense.

Let now $A(x, X)$ be an operator form. As a first step in our proof we define a Δ_0 formula $\mathsf{QH}_A(\alpha, f)$ to express that f is a function describing the iteration of $A(x, X)$ up to α according to the quasi-inductive clauses. Thus we set:

$$\mathsf{QH}_A(\alpha, f) := \begin{cases} Fun(f) \wedge dom(f) = \alpha \wedge \\ \wedge(\forall \beta < \alpha)[(\beta = 0 \wedge f(\beta) = 0) \vee \\ \vee(\exists \gamma < \alpha)(\beta = \gamma + 1 \wedge f(\beta) = \{z \in \mathsf{N} \mid A^{\mathsf{N}}(z, f(\gamma))\}) \vee \\ \vee(Lim(\beta) \wedge f(\beta) = \bigcup_{\gamma < \beta} \bigcap_{\gamma \leq \delta < \beta} f(\delta))]. \end{cases}$$

Further, we put:

$$x \in H_A^{\alpha} := \exists f[\mathsf{QH}_A(\alpha + 1, f) \wedge x \in f(\alpha)]$$

$$x \in H_A^{-\alpha} := (\exists \beta < \alpha)(\forall \gamma < \alpha)(\beta \leq \gamma \rightarrow x \notin H_A^{\gamma})$$

$$x \in H_A^{+\infty} := \exists \beta \forall \gamma\, (\beta \leq \gamma \rightarrow x \in H_A^{\gamma})$$

$$x \in H_A^{-\infty} := \exists \beta \forall \gamma\, (\beta \leq \gamma \rightarrow x \notin H_A^{\gamma}).$$

The following lemma is an immediate consequence of our definitions, and the properties of our theory. Proofs are standard, hence left to the reader.

LEMMA 4.1. *For all operator forms $A(x, X)$, T proves:*

1. $\forall \alpha \exists f \mathsf{QH}_A(\alpha, f)$.
2. $\mathsf{QH}_A(\alpha, f) \wedge \beta < \alpha \rightarrow \mathsf{QH}_A(\beta, f)$.
3. $\mathsf{QH}_A(\alpha, f) \wedge \mathsf{QH}_A(\beta, g) \wedge \alpha \leq \beta \rightarrow (\forall \gamma < \alpha)(f(\gamma) = g(\gamma))$.
4. $n \in \mathsf{N} \rightarrow (n \in H_A^{\alpha+1} \leftrightarrow A^{\mathsf{N}}(n, H_A^{\alpha}))$.
5. $n \in \mathsf{N} \wedge Lim(\lambda) \rightarrow (n \in H_A^{\lambda} \leftrightarrow (\exists \alpha < \lambda)(\forall \beta < \lambda)(\alpha \leq \beta \rightarrow n \in H_A^{\beta}))$.

The unicity condition that one can prove by that, allows one to equivalently describe stages H_A^{α}'s by the Π_1 formula

$$x \in H_A^{\alpha} \leftrightarrow \forall f[\mathsf{QH}_A(f, \alpha + 1) \rightarrow x \in f(\alpha)].$$

This means that formulas of the form $x \in H_A^{\alpha}$ are Δ_1^{T}, while formulas $x \in H_A^{+\infty}$, $x \in H_A^{-\infty}$ are both of complexity Σ_2^{T}.

The first step toward the embedding result is then the following lemma:

LEMMA 4.2. (COVERING) *In T it is provable that, for every ordinal α, there exists a limit ordinal $\delta > \alpha$ such that $H_A^{+\infty} \subseteq H_A^{\delta}$, $H_A^{-\infty} \subseteq H_A^{-\delta}$, $H_A^{\delta} \cap H_A^{-\infty} = \emptyset$ and $H_A^{-\delta} \cap H_A^{+\infty} = \emptyset$.*

Proof. Since

$$(\forall x \in \mathsf{N})\exists \beta(x \in H_A^{+\infty} \rightarrow (\forall \gamma \geq \beta)(x \in H_A^{\gamma}))$$

is a simple consequence of the definition given above, (COLL) ensures then that

$$\exists b(\forall x \in \mathsf{N})(\exists \beta \in b)(x \in H_A^{+\infty} \rightarrow (\forall \gamma \geq \beta)(x \in H_A^{\gamma})). \tag{3}$$

Now, take b' to be the set, which exists by (SEP), such that $b' = \{\beta \in b \mid (\exists n \in \mathsf{N})(\forall \gamma \geq \beta)(n \in H_A^{\gamma})\}$.

By a completely similar argument one finds sets c, c' playing for $H_A^{-\infty}$ the role b and b' play for $H_A^{+\infty}$ (with $\notin$ substituting $\in$ in the consequent of (3)).

So let α be any ordinal. Take δ to be the least limit ordinal such that $\xi < \delta$ where $\xi = \alpha \cup b' \cup c'$. By the choice of δ we have $x \in H_A^{+\infty} \to x \in H_A^{\delta}$ and $x \in H_A^{-\infty} \to x \in H_A^{-\delta}$. But also $x \in H_A^{\delta} \to x \notin H_A^{-\infty}$ and $x \in H_A^{-\delta} \to x \notin H_A^{+\infty}$ which completes the proof of the lemma. □

Notice that if δ is an ordinal given by covering, we have that

$$x \in H_A^{+\infty} \leftrightarrow \forall\beta(\delta \leq \beta \to x \in H_A^{\beta})$$

$$x \in H_A^{-\infty} \leftrightarrow \forall\beta(\delta \leq \beta \to x \notin H_A^{\beta})$$

are satisfied. This motivates the following definition which in turn will allow us to refer to 'unstable' elements by means of a Σ_1 formula:

DEFINITION 4.3. *For every $n \in \mathsf{N}$ and δ arbitrary but fixed ordinal given by covering, we say that n* is unstable (relatively to δ) *(abbreviated: $U_\delta(n)$) if*

$$U_\delta(n) := \exists\beta(\delta \leq \beta \wedge n \in H_A^{\beta}) \wedge \exists\gamma\,(\delta \leq \gamma\ \wedge n \notin H_A^{\gamma}).$$

A lemma which (implicitly) gets used below is the following. Its proof is standard, hence left out for space consideration.[7] It should be noticed, however, that the argument for proving this makes essential use of (SEP) and (COLL).

LEMMA 4.4.

(i) *If A, B are Σ_n^{T} and Π_n^{T} formulas respectively, then so are* $(\exists x \in z)A$ *and* $(\forall x \in z)B$.
(ii) *If A is a Σ_n^{T} formula ($n \in \{1, 2\}$), then so is* $(\forall x \in a)A$.
(iii) *If A is a Π_n^{T} formula ($n \in \{1, 2\}$), then so is* $(\exists x \in a)A$.

The next proposition contains the core result for the proposed embedding, namely that our theory T proves the existence of arbitrarily many, and arbitrarily large ordinals stabilizing the quasi-inductive sequence of sets that is produced by iterating any given operator form $A(x, X)$. The basic idea of it is to show that, for an ordinal δ given by covering, it is possible to filter out all of the unstable elements possibly occurring into H_A^{δ} and $H_A^{-\delta}$.

PROPOSITION 4.5. (STABILITY) *In T it is provable that, for every arithmetical operator form $A(x, X)$, $\forall\alpha\exists\lambda(\alpha < \lambda \wedge H_A^{\lambda} \equiv H_A^{+\infty} \wedge H_A^{-\lambda} \equiv H_A^{-\infty})$.*

Proof. We informally work in T as follows. Assume δ is an arbitrary but fixed ordinal given by the covering lemma. Assume also that $(\exists z \in \mathsf{N})U_\delta(z)$. We first need a function enumerating the set $W = \{n \in \mathsf{N} \mid U_\delta(n)\}$ of the unstable elements relatively to the chosen δ. This is done by listing them in their given order as finite ordinals. That is we set

$$F(0) := \min_N z \in \mathsf{N}.U_\delta(z)$$

$$F(\alpha) := \begin{cases} \min_N z \in \mathsf{N}.U_\delta(z) \wedge (\forall\beta < \alpha)(F(\beta) < z), & \text{if it exists} \\ F(0), & \text{otherwise} \end{cases}$$

(where $\min_N$ is written as such so to make it clear that one chooses the minimum with respect to the ordering $<_N$ of the natural numbers).

[7] A reference for that is Devlin (1974) (or Devlin, 1984).

Having noticed that W is a Σ_1 set, the totality of F simply follows by the Σ_2 recursion theorem. [8]

Clearly, F will have exhausted W after ω steps, and it will keep exhausting it after every ξ number of steps afterward, for ξ limit ordinal. Hence, this function will enable us to have elements of W occurring *infinitely often* in a list provided by F[9] if we set $dom(F) = \gamma$, where γ is such that $Lim^+(\gamma) := 0 < \gamma \wedge (\forall \alpha < \gamma)(\exists \beta < \gamma)(\alpha < \beta \wedge Lim(\beta))$ is the case.[10]

Let then λ be an ordinal such that $Lim^+(\lambda)$ holds. We define a function G with $dom(G) = \lambda$ by transfinite recursion in the F we have considered above, according to the clauses:

$$G(0) = \delta$$
$$G(\alpha+1) = \begin{cases} \min \mu . G(\alpha) < \mu \wedge F(\alpha) \in H_A^{\mu}, & \text{if } F(\alpha) \notin H_A^{G(\alpha)} \\ \min \mu . G(\alpha) < \mu \wedge F(\alpha) \notin H_A^{\mu}, & \text{otherwise} \end{cases}$$
$$G(\xi) = \min \mu . \sup(\{G(\beta) \mid \beta < \xi\}) < \mu, \ \xi \text{ limit}$$

As before, one must show in T that this function always yields a value. This is done exactly in the same way as for the function F above: it comes in fact from the Σ_2 recursion theorem, being G parametric in F (which is a Δ_2^{T} function[11]) and in the ordinal δ given by covering which, owing to the proof of that result, is of complexity Π_1^{T}.

Hence, G is itself a provably total Δ_2^{T} function. By (COLL), (SEP), and ($\mathsf{IND}_\in$), one can find an ordinal μ_0 such that

$$\mu_0 = \min \xi . \forall \gamma \, ((\exists \beta < \lambda)(\gamma = G(\beta)) \rightarrow \gamma < \xi).$$

It should then turn out quite clearly that, since G is strictly increasing below λ, μ_0 is a limit ordinal and that it further satisfies the property given by the covering lemma. Moreover, it is also clear from the definition of G that at stages H_A^{α}'s, $\alpha < \mu_0$, the members of W behave as unstable elements, hence they are not retained at $H_A^{\mu_0}$, $H_A^{-\mu_0}$. It is then an easy task to verify, by further exploiting the fact that elements of W occur infinitely often in the F enumeration of it, that for *no* n such that $n \in H_A^{\mu_0} \cup H_A^{-\mu_0}$, $U_\delta(n)$ can be the case as well.

Hence the theorem. □

Given this result, we are now ready for stating the promised embedding in detail. As usual, this is based on a translation of the language $\mathcal{L}(\Pi_\infty)$ of theories for arithmetical quasi-inductive definitions into the language $\mathcal{L}^*$ of our theory T of sets. The terms of this translation should not come as a surprise: both languages extend the language $\mathcal{L}_{Ar}$ of Peano arithmetic; the additional sort of variables for ordinals which is present among the $\mathcal{L}(\Pi_\infty)$ alphabet will be interpreted by ordinals in $\mathcal{L}^*$, and the 'less than' relation of the former

[8] A reference for Σ_2 recursion to be provable in Σ_2–KP:= KP + Δ_2–SEP + Σ_2–COLL, could be again Devlin (1974).

[9] Formally: $(\forall x \in W)(\forall \beta < dom(F))(\exists \eta < dom(F))(F(\beta) = x \rightarrow F(\eta) = x \wedge \beta < \eta)$.

[10] Equivalently, one could choose γ to be a 'principal additive ordinal number' greater than ω (namely, such that: $\omega < \gamma \wedge (\forall \beta < \gamma)(\forall \xi < \gamma)(\beta + \xi < \gamma)$).

[11] By a well-known trick, one could in fact set

$$y = F(x) \leftrightarrow (x < \lambda \wedge \forall z (z = F(x) \rightarrow z = y))$$

showing F to be Π_2^{T} as well.

by the elementhood one of the latter; finally, to atomic formulas of the form $\mathcal{H}_A(t, \alpha)$ in $\mathcal{L}(\Pi_\infty)$ we will make correspond formulas $t \in H_A^\alpha$ in $\mathcal{L}^*$.

To be more precise: arithmetical and ordinal variables from $\mathcal{L}(\Pi_\infty)$ are mapped into individual variables of $\mathcal{L}^*$ in such a way that they are kept separate. We indicate by $\dot{x}$ a variable in $\mathcal{L}^*$ corresponding to the number theoretic variable x in $\mathcal{L}(\Pi_\infty)$, and we do the same for ordinal variables.

Then, the translation A' in $\mathcal{L}^*$ of a formula A in $\mathcal{L}(\Pi_\infty)$ is obtained by applying the following steps: (i) replace all variables occurring in A by their translation in $\mathcal{L}^*$; (ii) replace formulas of the form $\alpha < \beta$ in A by $\dot{\alpha} < \dot{\beta}$; (iii) replace formulas of the form $\mathcal{H}_A(s, \alpha)$ by formulas $\dot{s} \in H_A^{\dot{\alpha}}$;[12] (iv) replace numerical quantifiers $\forall x(\dots), \exists x(\dots)$ by $(\forall \dot{x} \in \mathsf{N})(\dots), (\exists \dot{x} \in \mathsf{N})(\dots)$ respectively; replace ordinal quantifiers $\forall \alpha(\dots), \exists \alpha(\dots)$ by $\forall \dot{\alpha}(\mathsf{Ord}(\dot{\alpha}) \to \dots)$ and $\exists \dot{\alpha}(\mathsf{Ord}(\dot{\alpha}) \wedge \dots)$ respectively. To A formula of $\mathcal{L}(\Pi_\infty)$ with arithmetical variables $x_1, \dots, x_n$ and ordinal variables $\alpha_1, \dots, \alpha_m$ it is then associated the formula A° of $\mathcal{L}^*$ given by

$$A^\circ := (\dot{x_1} \in \mathsf{N} \wedge \dots \wedge \dot{x_n} \in \mathsf{N} \wedge \mathsf{Ord}(\dot{\alpha_1}) \wedge \dots \wedge \mathsf{Ord}(\dot{\alpha_m}) \to A').$$

In order to establish the embedding, it is then sufficient to prove the following theorem:

THEOREM 4.6. *Let A be an axiom of* $\mathsf{QID}(\Pi_\infty)$. *Then* T *proves* A°.

Proof. The argument is given by distinguishing various cases, using Lemma 4.1. and Proposition 4.5. for the QID group of axioms. □

§5. Concluding remarks. As it was hinted at above, the results that are provided here should be regarded as a first scratching of the surface of a proof theoretical kind of approach to quasi-inductive definitions. The main reason for pursuing the research resides in the fact that, as far as we know, substantially nothing in this sense has been attempted so far. In Welch (2007) (which was still unpublished at the time this note was submitted for publication), he independently studied the strength of an axiom that he called **AQI**, which he formulated in the language of second-order arithmetic, in the hierarchy of the axioms of determinacy for games. This assumption comprises the stabilization property we have also been referring to, in that it states that the quasi-inductive iteration of any given arithmetic operator Φ along a well ordering has a repeat pair, namely levels γ, δ such that $\Phi_\gamma = \Phi_\delta$.

Welch (2007) states that **AQI** has KP+(Σ_2-SEP)+(Σ_2-COLL) as an upper bound. The result that has been presented here, could then be viewed as just a step toward the verification of that statement.

This might be relevant for the work that we plan to pursue to refine the proposed investigation. The main goal to attain at in this direction is the sharpening of the results that have been obtained so far, possibly so to achieve an exact bound for the theory of arithmetical quasi-inductive definitions. It is not altogether impossible then, that some new interesting information in this sense could be found by exploting the interplay between Welch's approach and ours.

§6. Acknowledgments. The author would like to thank A. Cantini for first directing his attention to the topic of this research, as well as for all the useful comments he provided on the intermediate stages in the revision procedure to get to the actual state of the paper.

[12] In this and the previous clause, we implicitly assume to have extended the correspondence $(\dot{\ })$ to arbitrary terms in the expected manner.

I am also grateful to the Swiss National Science Foundation and the Consiglio Nazionale delle Ricerche for providing me a grant for a short stay in Bern while the research was at an early stage of development. In this respect, the author is deeply indebted with G. Jäger and all the people of the *Theoretische Informatik und Logik* group at the *Institut für Informatik und angewandte Mathematik* in Bern, for attending my seminars there and providing comments and criticisms which proved to be crucial for achieving the results that have been presented here. Finally, I owe to P. Welch some corrections and suggestions that substantially improved a previous version of the paper by giving a definite shape to the main result of it, and by eliminating redundancies. Of course, I am the only responsible for mistakes that might still be present in this final version of the paper.

BIBLIOGRAPHY

Aczel, P., & Richter, W. (1974). Inductive definitions and reflecting properties of admissible ordinals. In Fenstad, J. E., and Hinman, P. G., editors. *Generalized Recursion Theory: Proceedings of the 1972 Oslo Symposium*, 1974, Amsterdam. North-Holland, pp. 301–381.

Buchholz, W., Feferman, S., Pohlers, W., & Sieg, W. (1981). *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*. Lecture Notes in Mathematics 897. Berlin: Springer-Verlag.

Burgess, J. (1986). The truth is never simple. *Journal of Symbolic Logic*, **51**, 663–681.

Cantini, A. (1996). *Logical Frameworks for Truth and Abstraction. An Axiomatic Study*. Studies in Logic and the Foundation of Mathematics 135. Amsterdam: Elsevier.

Devlin, K. (1974). An introduction to the fine structure of the constructible hierarchy (results by Ronald Jensen). In Fenstad, J. E., and Hinman, P. G., editors. *Generalized Recursion Theory: Proceedings of the 1972 Oslo Symposium*, 1974, Amsterdam. North-Holland, pp. 123–163.

Devlin, K. (1984). *Constructibility*. Berlin: Springer-Verlag.

Gupta, A., & Belnap, N. (1993). *The Revision Theory of Truth*. Cambridge, MA: MIT Press.

Hamkins, J. D., & Lewis, A. (2000). Infinite time turing machines. *Journal of Symbolic Logic*, **65**, 567–604.

Jäger, G. (2001). First-order theories for non-monotone inductive definitions: Recursively inaccessible and Mahlo. *Journal of Symbolic Logic*, **66**, 1073–1089.

Jäger, G., & Studer, T. (2002). Extending the system T_0 of explicit mathematics: The limit and Mahlo axioms. *Annals of Pure and Applied Logic*, **114**, 79–101.

Löwe, B. (2001). Revision sequences and computers with an infinite amount of time. *Journal of Logic and Computation*, **11**, 25–40.

Moschovakis, Y. (1974). *Elementary Induction on Abstract Structures*. Studies in Logic and the Foundation of Mathematics 77. Amsterdam: North-Holland.

Richter, W. (1971). Recursively Mahlo ordinals and inductive definitions. In Gandy, R. O., and Yates, C. M. E., editors. *Logic Colloquium '69*. Amsterdam: North-Holland. pp. 273–288.

Welch, P. (2001). On Gupta–Belnap revision theories of truth, Kripkean fixed points and the next stable set. *Bulletin of Symbolic Logic*, **7**, 345–360.

Welch, P. (2003). On revision operators. *Journal of Symbolic Logic*, **68**, 689–711.

Welch, P. (2007). *Weak Systems of Determinacy and Arithmetical Quasi–Inductive Definitions*. Preprint.

DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF FLORENCE
VIA BOLOGNESE 52, FIRENZE 50139, ITALY
E-mail: riccardobruni@hotmail.com
URL: http://www.philos.unifi.it/CMpro-v-p-88.html