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A Dirichlet problem on the half-line for nonlinear equations with indefinite weight

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Abstract. We study the existence of positive solutions on the half-line $[0, \infty)$ for the nonlinear second order differential equation

$$\big(a(t)x'\big)' + b(t)F(x) = 0, \quad t \geq 0,$$

satisfying Dirichlet type conditions, say $x(0) = 0$, $\lim_{t\to\infty} x(t) = 0$. The function b is allowed to change sign and the nonlinearity F is assumed to be asymptotically linear in a neighborhood of zero and infinity. Our results cover also the cases in which b is a periodic function for large t or it is unbounded from below.

Keywords. Second order nonlinear differential equation, boundary value problem on the half line, Dirichlet conditions, globally positive solution, disconjugacy, principal solution.

MSC 2010: Primary 34B40, Secondary 34B18.

1 Introduction

Consider the boundary value problem (BVP) on the half-line $[0, \infty)$

$$\big(a(t)x'\big)' + b(t)F(x) = 0, \tag{1}$$

$$x(0) = 0, \quad x(t) > 0 \ \text{ on } (0, \infty), \ \lim_{t\to\infty} x(t) = 0, \tag{2}$$

where we assume the following:

(i) The function a is continuous on $[0,\infty)$, $a(t)>0$, and

$$\int_0^\infty \frac{1}{a(t)}\,dt<\infty. \tag{3}$$

(ii) The function b is continuous on $[0,\infty)$, nonnegative and not identically zero on $[0,1]$, and is allowed to change sign for $t>1$. Moreover, b is bounded from above, that is, there exists a positive constant B such that

$$b(t)\le B \quad \text{on } [1,\infty). \tag{4}$$

(iii) The function F is continuous on $\mathbb{R}$, $F(u)u>0$ for $u\ne 0$, F is differentiable on $[0,\infty)$ with bounded nonnegative derivative:

$$0\le \frac{dF(u)}{du}\le 1 \quad \text{for } u\ge 0, \tag{5}$$

and satisfies

$$\lim_{u\to 0^+}\frac{F(u)}{u}=k_0, \qquad \lim_{u\to\infty}\frac{F(u)}{u}=k_\infty, \tag{6}$$

where

$$0\le k_0\ne k_\infty.$$

Observe that (5) implies that $k_0,k_\infty\le 1$.

The BVP (1)-(2) is a Dirichlet-type BVP on an unbounded domain. Recently, there has been a growing interest in studying infinite interval problems associated to second order nonlinear differential equations, under various points of view. For a wide bibliography, we refer the reader to [1, 2, 21] and the references therein. When the weight b is of fixed sign or it is sign-indefinite, we refer to [7, 14, 26] or [15, 16, 23, 24], respectively. The BVP (1)-(2) arises in the investigation of positive radial solutions for elliptic equations, when the nonlinearity is asymptotically linear, see, e.g., [3].

Our main aim is to continue this study when the function b is allowed to change its sign and the nonlinearity F can be, roughly speaking, close to a linear function. The investigated problem can be viewed as an extension to the half-line of recent results on nonlinear BVPs with a sign-indefinite weight on a compact interval, see, e.g., [4, 5], and reference therein for a brief survey on this topic.

Denote by $|\cdot|_L$ the norm in $L^1[0,1]$ and set

$$A(t)=\int_0^t \frac{1}{a(s)}ds. \tag{7}$$

Our main result is the following, in which the disconjugacy of a suitable auxiliary differential linear equation plays a key role, see Section 3 below.

Theorem 1. *Assume that the linear differential equation*

$$v'' + \frac{B}{a(t)} v = 0 \tag{8}$$

is disconjugate on $[1, \infty)$*, where the constant* B *is defined in (4).*

If there exist $t_1, t_2 \in (0, 1)$*,* $t_1 < t_2$ *such that* $\int_{t_1}^{t_2} b(t)\, dt > 0$*, and*

$$0 \leq \min\{k_0, k_\infty\} A(1)\ |b|_L < 1, \tag{9}$$

$$\max\{k_0, k_\infty\} \int_{t_1}^{t_2} b(t)\, dt > \frac{A(1)}{A(t_1)(A(1) - A(t_2))}, \tag{10}$$

then the BVP (1)-(2) has a solution.

Moreover, the solution x *has a local maximum in the interval* $(0, 1]$*, is decreasing in* $[1, \infty)$ *and satisfies*

$$\int_1^\infty \frac{1}{a(t) x^2(t)} dt = \infty. \tag{11}$$

Theorem 1 covers also the cases in which the weight b is a periodic function for large t or it is unbounded from below.

Our approach is based on a shooting method and a continuity result. More precisely, Theorem 1 is proved by considering two auxiliary BVPs, the first one on the compact interval $[0, 1]$, where b is nonnegative, and the second one on the half-line $[1, \infty)$, where b is allowed to change its sign. The problem of the existence of solutions for (1), emanating from zero, positive in the interval $(0, 1)$, and satisfying additional assumptions at $t = 1$, is considered in Section 2 and is solved by using some results from [22], with minor changes. The BVP on $[1, \infty)$ is examined in Section 4. It deals with positive decreasing solutions on $[1, \infty)$ for (1) which tend to zero as $t \to \infty$. This second problem is solved by using a fixed point theorem for operators defined in a Fréchet space by a Schauder's linearization device, see [11, Theorem 1.3]. This method does not require the explicit form of the fixed point operator, but only some *a-priori* bounds. These estimations are obtained using some properties of principal solutions of disconjugate second order linear equations, see [20, Chapter 11]. Finally, roughly speaking, the solvability of (1)-(2) is obtained by using a shooting method on $[0,1]$ and, by means some continuity arguments, pasting a solution of (1) on $[0,1]$ with a solution of the BVP on $[1,\infty)$. This last argument can be viewed as a generalization to non compact intervals of some ideas in [19].

Notice that our approach allows us to obtain also an estimation of the decay to zero of solutions of (1)-(2). Some examples complete the paper.

2 Two auxiliary BVPs on $[0,1]$

In this section, we recall some results about the existence of solutions of (1) on $[0,1]$, which belong either to Δ_1 or Δ_2, where

$$\Delta_1 = \{u \in C[0,1] : u(0) = u(1) = 0, u(t) > 0 \text{ on } (0,1)\}$$
$$\Delta_2 = \{u \in C[0,1] : u(0) = u'(1) = 0, u(t) > 0 \text{ on } (0,1]\}\,.$$

These results can be obtained from [22], with minor changes.

BVPs on a compact interval, associated to equations of the form

$$z'' + g(t)F(z) = 0, \tag{12}$$

where g is a continuous nonnegative function on $[0,1]$, have been widely investigated in the literature, under many different points of view. We refer to [6, Introduction] and references therein for a brief survey.

In particular, the existence of solutions of (12), which satisfy either $z \in \Delta_1$ or $z \in \Delta_2$, has been considered in [18], where the key conditions on the nonlinearity are either that F is superlinear, that is, $k_0 = 0, k_\infty = \infty$ or F is sublinear, that is, $k_0 = \infty$, $k_\infty = 0$. When the nonlinearity F is not necessarily superlinear nor sublinear, these results have been extended in several ways in [22].

Using [22, Corollaries 3.1 and 3.5] and the continuity of g, we obtain the following result.

Lemma 1. *Assume that there exist $t_1, t_2 \in (0,1)$, $t_1 < t_2$, such that $\int_{t_1}^{t_2} g(t)\,dt > 0$ and*

$$0 \le \min\{k_0, k_\infty\}\,|g|_L < 1, \quad \max\{k_0, k_\infty\} \int_{t_1}^{t_2} g(t)\,dt > \frac{1}{t_1(1-t_2)}\,. \tag{13}$$

Then (12) has both solutions $z_1 \in \Delta_1$ and $z_2 \in \Delta_2$.

Proof. In virtue of the continuity of g, every nonnegative solution z of (12), $z \not\equiv 0$, satisfies $z(t) > 0$ on $(0,1)$, since z' is nonincreasing. Hence, the assertion follows from [22, Corollaries 3.1 and 3.5]. $\square$

When g does not have zeros on $[0,1]$, from Lemma 1 we obtain the following.

Lemma 2. *Let g be positive on $[0,1]$. If*

$$0 \leq \min\{k_0, k_\infty\}|g|_L < 1, \quad \max\{k_0, k_\infty\} \min_{t\in[0,1]} g(t) > 27, \tag{14}$$

then (12) has both solutions $z_1 \in \Delta_1$ and $z_2 \in \Delta_2$.

Proof. Fixed $t_1, t_2 \in (0,1)$, $t_1 < t_2$, we have

$$\int_{t_1}^{t_2} g(\tau)\, d\tau \geq (t_2 - t_1) \min_{t\in[0,1]} g(t).$$

Thus, the second condition in (13) is satisfied if

$$\max\{k_0, k_\infty\} \min_{t\in[0,1]} g(t) \geq \frac{1}{t_1(1-t_2)(t_2-t_1)}$$

for a suitable choice of t_1, t_2. Put $\rho = \varrho(t_1,t_2) = t_1(1-t_2)(t_2-t_1)$, it is easily checked that ϱ takes its maximum $1/27$ on the region $0 \leq t_1 < t_2 \leq 1$ when $t_1 = 1/3, t_2 = 2/3$. Therefore, the second inequality in (14) follows. $\square$

Define for $t \in [0,1]$

$$\tau(t) = \frac{A(t)}{A(1)}, \tag{15}$$

where A is given in (7). Thus, τ maps the interval $[0,1]$ into itself. Let x be a solution of (1) on $[0,1]$ and put $z(\tau) = x(t(\tau))$, where $t(\tau)$ is the inverse function of $\tau(t)$. Then, z is a solution on $[0,1]$ of

$$\frac{d^2 z}{d\tau^2} + \tilde{b}(\tau)F(z) = 0, \tag{16}$$

where $\tilde{b}(\tau) = A^2(1)a(t(\tau))b(t(\tau))$. Vice versa, if z is a solution of (16) on $[0,1]$, then $x(t) = z(\tau(t))$ is a solution of (1) on the same interval. Moreover, it is easy to show that x belongs to Δ_i if and only if $z \in \Delta_i, i = 1,2$. Hence, Lemmas 1 and 2 read for (1) as follow.

Proposition 1. *Assume that one of the following conditions is satisfied.*

(i) There exist $t_1, t_2 \in (0,1)$, $t_1 < t_2$ such that $\int_{t_1}^{t_2} b(t)\,dt > 0$, and

$$\begin{aligned} &0 \le \min\{k_0, k_\infty\}\, A(1)\, |b|_L < 1, \\ &\max\{k_0, k_\infty\} \int_{t_1}^{t_2} b(t)\,dt > \frac{A(1)}{A(t_1)(A(1) - A(t_2))}. \end{aligned}$$

(ii) $b(t) > 0$ on $[0,1]$ and

$$\begin{aligned} &0 \le \min\{k_0, k_\infty\}\, A(1)\, |b|_L < 1, \\ &27 < \max\{k_0, k_\infty\}\, A(1) \min_{t\in[0,1]} b(t). \end{aligned}$$

Then (1) has both solutions $x_1 \in \Delta_1$ and $x_2 \in \Delta_2$.

Proof. Since

$$\int_0^1 \tilde{b}(\tau)\,d\tau = A^2(1) \int_0^1 b(t(\tau))a(t(\tau))\,d\tau = A(1) \int_0^1 b(t)\,dt = A(1)\,|b|_L$$

and

$$\int_{t_1}^{t_2} b(t)\,dt = \frac{1}{A(1)} \int_{\tau_1}^{\tau_2} \tilde{b}(\tau)\,d\tau,$$

where $\tau_i = \tau(t_i) = A(t_i)/A(1), i = 1, 2$, the assertion follows from Lemmas 1 and 2.

$\square$

Other sufficient conditions for the existence of solutions of (1) in the sets Δ_1 and Δ_2, can be obtained in a similar way from other results in [22].

3 Principal solutions and disconjugacy

Consider the linear equation

$$\big(a(t)y'\big)' + \beta(t)y = 0, \tag{17}$$

where β is a continuous function for $t \ge T \ge 0$. In our study, an important role is played by the disconjugacy property and the notion of principal solutions for (17).

We recall that (17) is said to be *disconjugate* on an interval $I \subset [T, \infty)$ if any nontrivial solution of (17) has at most one zero on I. We refer to [13, 20] and references therein for basic properties of disconjugacy. In particular, the following results will be useful in the sequel.

Lemma 3. *Let $T_1 \geq T$. The following statements are equivalent.*
(i_1) Equation (17) is disconjugate on $[T_1, \infty)$;
(i_2) Equation (17) is disconjugate on (T_1, ∞);
(i_3) Equation (17) has a solution without zeros on (T_1, ∞).

Proof. (i_1)$\Longleftrightarrow$(i_2). If (17) is disconjugate on $[T_1, \infty)$, then it is disconjugate on (T_1, ∞). The vice versa follows from [13, Theorem 2, Chapt.1], with minor changes. Finally, (i_2)$\Longleftrightarrow$(i_3) follows from [20, Corollary 6.1]. □

The concept of principal solution was introduced in 1936 by W. Leighton and M. Morse and, later on, analyzed by P. Hartman and A. Wintner, see, e.g., [20, Chapter 11]. If (17) is nonoscillatory, then there exists a solution u_0 of (17), which is uniquely determined up to a constant factor by one of the following conditions (in which u denotes an arbitrary solution of (17), linearly independent of u_0):

$$\lim_{t\to\infty} \frac{u_0(t)}{u(t)} = 0, \tag{18}$$

$$\frac{u_0'(t)}{u_0(t)} < \frac{u'(t)}{u(t)} \quad \text{for large } t,$$

$$\int_{t_u}^{\infty} \frac{dt}{a(t)u_0^2(t)} = \infty, \tag{19}$$

where $t_u \geq T$ is such that $u_0(t) \neq 0$ on $[t_u, \infty)$. The solution u_0 is called *principal solution* of (17) and any solution u of (17), which is linearly independent of u_0, is called a *nonprincipal solution* of (17). Property (18) is the simplest and most typical property characterizing principal solutions, because, roughly speaking, it means that the principal solution is the smallest one in a neighborhood of infinity.

Remark 1. If (17) is disconjugate on $[T_1, \infty), T_1 \geq T$, then any principal solution of (17) does not have zeros on (T_1, ∞), see [20, Chapter XI, Exercise 6.6]. Thus, a necessary condition for positiveness of the principal solution on the open interval (T, ∞), is the disconjugacy of the equation. Nevertheless, disconjugacy cannot be sufficient for the positiveness of principal solution on the close half-line $[T, \infty)$, as the following example shows.

Example 1. Consider the equation

$$(a(t)y')' + y = 0, \quad t \geq 0, \tag{20}$$

where $a(1) = 1$ and

$$a(t) = \frac{1 + t - 2e^{t-1}}{1 - t} \text{ if } t \neq 1.$$

Hence, a is a positive continuous function on $[0, \infty)$ and (3) holds for a. Using (19), we get that $y_0(t) = te^{-t}$ is the principal solution of (20). Moreover, in view of Lemma 3, equation (20) is disconjugate on $[0, \infty)$.

Consider now the special case $\beta(t) \equiv M > 0$ in (17), i.e. the equation

$$\big(a(t)y'\big)' + My = 0. \tag{L}$$

In view of Example 1, the disconjugacy of (L) on $[T, \infty)$ does not guarantee the positiveness of principal solution at the initial point $t = T$. To obtain this additional property, consider the so-called *dual equation to (L),* that is the equation

$$v'' + \frac{M}{a(t)}v = 0, \tag{D}$$

which is obtained from (L) by the change of variable $v(t) = a(t)y'(t)$. The dual equation has been often used in the literature for studying oscillatory properties of second order self-adjoint linear equations, see, e.g., [8, 9, 25], and, for the half-linear case, [10, 17].

The following necessary and sufficient condition for the disconjugacy of (D) holds, see also [20, page 352].

Lemma 4. *Equation (D) is disconjugate on $[T, \infty)$ if and only if (D) has a solution v_0 such that $v_0(t) > 0$ on (T, ∞) and $v_0'(t) > 0$ on $[T, \infty)$.*

Proof. Assume that (D) is disconjugate on $[T, \infty)$. From Lemma 3 there exists a solution v_0 of (D) such that $v_0(t) > 0$ for $t > T$. Thus, v_0' is decreasing for $t > T$. We claim that $v_0'(t) > 0$ on the whole interval $[T, \infty)$. By contradiction, if v_0' has a zero on $[T, \infty)$, then there exists $t_1 > T$ such that $v_0'(t) \leq v_0'(t_1) < 0$ for $t \geq t_1$. Integrating this inequality we get

$v_0(t) \leq v_0(t_1) + v_0'(t_1)(t - t_1)$, which gives a contradiction with the positiveness of v_0 when t tends to infinity. The opposite statement follows again in virtue of Lemma 3. $\square$

From Lemma 4 we obtain the following.

Lemma 5. *If (D) is disconjugate on $[T, \infty)$, then (L) has a principal solution y_0 such that $y_0(t) > 0$ on $[T, \infty)$ and $y_0'(t) < 0$ on (T, ∞).*

Proof. In view of Lemma 4 and the change of variable $y(t) = v'(t)$, equation (L) has a solution y_0 which satisfies $y_0(t) > 0$ on $[T, \infty)$ and $y_0'(t) < 0$ on (T, ∞). Hence, the disconjugacy of (L) follows from Lemma 3. If y_0 is not principal solution, from [20, Corollary 6.3] the solution $\overline{y}$ given by

$$\overline{y}(t) = y_0(t) \int_t^\infty \frac{ds}{a(s) y_0^2(s)},$$

is the desired principal solution of (L). $\square$

Remark 2. Example 1 shows that the assumption on disconjugacy of (D) in Lemma 5 cannot by replaced by the disconjugacy of (L). Moreover, observe that the dual equation of (20) is

$$v'' + a^{-1}(t) v = 0, \tag{21}$$

where a is defined in Example 1. It is easy to verify that the function $v_0(t) = 2e^{-1} - (1+t)e^{-t}$ is a principal solution of (21). Since $v_0(1) = 0$, any principal solution of (21) has a zero at $t = 1$. Consequently, (21) is not disconjugate on $[0, \infty)$.

4 An auxiliary BVP on $[1, \infty)$

For any $c > 0$, consider for $t \geq 1$ the existence of solutions x of (1) which satisfy the boundary conditions

$$x(1) = c, \quad x'(1) \leq 0, \; x(t) > 0 \text{ on } [1, \infty), \; \lim_{t \to \infty} x(t) = 0. \tag{22}$$

The solvability of this BVP is based on a general fixed point theorem for operators defined in a Fréchet space, see [11, Theorem 1.3]. In particular, this result reduces the existence of solutions of a BVP for differential equations on noncompact intervals to the existence of suitable *a-priori* bounds and it is mainly useful when the associated fixed point operator is not known in an explicit form. We recall this result in the form that will be used.

Theorem 2. *Consider the BVP on* $[T,\infty), T \geq 0$,

$$(a(t)x')' + b(t)F(x) = 0, \quad x \in S, \tag{23}$$

where S *is a nonempty subset of the Fréchet space* $C[T,\infty)$. *Let* G *be a continuous function on* $\mathbb{R}^2$, *such that* $F(d) = G(d,d)$ *for any* $d \in \mathbb{R}$ *and assume that there exist a nonempty, closed, convex and bounded subset* $\Omega \subset C[T,\infty)$ *such that for any* $u \in \Omega$ *the BVP on* $[T,\infty)$

$$(a(t)x')' + b(t)G(u(t),x(t)) = 0, \quad x \in S$$

admits a unique solution x_u. *Let* Ψ *be the operator* $\Omega \to C[T,\infty)$, *such that* $\Psi(u) = x_u$. *Assume*

(i_1) $\Psi(\Omega) \subset \Omega$;

(i_2) if $\{u_n\} \subset \Omega$ *is a sequence converging in* Ω *and* $\Psi(u_n) \to x$, *then* $x \in S$.

Then Ψ *has a fixed point in* Ω, *which is a solution of the BVP (23).*

Let $\widetilde{F}$ be the function

$$\widetilde{F}(v) = \frac{F(v)}{v} \text{ if } v > 0, \quad \widetilde{F}(0) = k_0, \tag{24}$$

where k_0 is defined in (6) and set $b_+(t) = \max\{b(t),0\}, b_-(t) = -\min\{b(t),0\}$. Thus $b(t) = b_+(t) - b_-(t)$. The following holds.

Theorem 3. *Assume that equation (8) is disconjugate on* $[1,\infty)$. *Then, for any* $c > 0$, *equation (1) has a unique globally positive decreasing solution* x *on* $[1,\infty)$ *satisfying (22) and (11).*

Proof. Fixed $c > 0$, consider the equations

$$\big(a(t)y'\big)' + By = 0, \tag{25}$$

$$\big(a(t)w'\big)' - b_-(t)w = 0. \tag{26}$$

From Lemma 5, equation (25) is disconjugate on $[1,\infty)$ and has a principal solution y_0 such that $y_0(1) = c$, $y_0(t) > 0$ on $[1,\infty)$, $y_0'(t) < 0$ on $(1,\infty)$. Moreover, from [9, Theorem 1] we obtain $\lim_{t\to\infty} y_0(t) = 0$.

Since $-b_-(t) \leq 0$, equation (25) is a Sturm majorant for (26). Thus (26) has a positive principal solution w_0 such that $w_0(1) = c$, $w_0'(t) \leq 0$ for $t \geq 1$, see, e.g., [20, Corollary 6.4]. Using the comparison result for the principal solutions, see e.g. [20, Corollary 6.5], we get on $(1,\infty)$

$$\frac{w_0'(t)}{w_0(t)} \le \frac{y_0'(t)}{y_0(t)}$$

and so $0 < w_0(t) \le y_0(t)$ for $t \ge 1$.

Let Ω and S be the subsets of the Fréchet space $C[1,\infty)$ given by

$$\Omega = \left\{u \in C[1,\infty), \frac{1}{2}w_0(t) \le u(t) \le y_0(t)\right\},$$
$$S = \left\{x \in C[1,\infty), x(1) = c,\ x(t) > 0,\ \int_1^\infty \frac{1}{a(t)x^2(t)}dt = \infty\right\},$$

respectively.

For any $u \in \Omega$ consider the linear equation

$$(a(t)x')' + b(t)\widetilde{F}(u(t))x(t) = 0, \tag{27}$$

where $\widetilde{F}$ is given in (24). In view of (5), we have $\sup_{v\ge 0}\widetilde{F}(v) \le 1$. Hence, (25) is a majorant for (27). Thus, using again the comparison result [20, Corollary 6.5], equation (27) has a unique positive principal solution x_u, such that $x_u(1) = c$, and for $t > 1$

$$\frac{x_u'(t)}{x_u(t)} \le \frac{y_0'(t)}{y_0(t)}.$$

Hence, taking into account that y_0 is decreasing to zero as t tends to infinity, we get

$$\begin{array}{ll} 0 < x_u(t) \le y_0(t) & \text{on } [1,\infty), \\ \lim_{t\to\infty} x_u(t) = 0, & x_u'(t) < 0 \text{ on } (1,\infty) \end{array}. \tag{28}$$

Thus, for any $u \in \Omega$, equation (27) has a solution $x_u \in S$, which is unique in view of (19).

Denote by $\Psi : \Omega \to C[1,\infty)$ the operator

$$\Psi(u) = x_u.$$

Using again the comparison result [20, Corollary 6.5] for equations (27) and (26), we obtain for any $u \in \Omega$ and $t \ge 1$

$$\frac{w_0'(t)}{w_0(t)} \le \frac{x_u'(t)}{x_u(t)}. \tag{29}$$

Then, in view of (28) we get for any $u \in \Omega$ and $t \geq 1$

$$w_0(t) \leq x_u(t) \leq y_0(t),$$

i.e., the operator Ψ maps Ω into itself.

Now, let $\{u_n\} \subset \Omega$ be a sequence converging in Ω and $x_{u_n} = \Psi(u_n) \to x$. Clearly $x(1) = c$. Since $\overline{\Psi(\Omega)} \subset \Omega$, we get $x(t) > 0$. Moreover, since y_0 is a principal solution of (25), from (28) we obtain

$$\int_1^\infty \frac{1}{a(t)x^2(t)} dt \geq \int_1^\infty \frac{1}{a(t)y_0^2(t)} dt = \infty.$$

Thus, $x \in S$ and, by Theorem 2, there exists a fixed point $\overline{x}$ of Ψ in Ω. Clearly, $\overline{x}$ is a solution of (1) on $[1, \infty)$ and $\overline{x}(1) = c$. Since $\overline{x}$ is also a principal solution of (27) with $u = \overline{x}$, from (28) we get $\overline{x}(t) > 0$, $\overline{x}'(t) < 0$ for $t > 1, \overline{x}'(1) \leq 0$ and $\lim_{t\to\infty} \overline{x}(t) = 0$. Thus $\overline{x}$ is positive decreasing on $(1, \infty)$ and satisfies (22) and (11).

Finally, it remains to verify that (1) has a unique solution which satisfies (22). Let x, v be two positive solutions of (1) defined on $[1, \infty)$ and satisfying (22). In view of the first part of the proof, we can suppose also that

$$\int_1^\infty \frac{dt}{a(t)x^2(t)} = \infty. \tag{30}$$

Denote by $\Phi(u, v)$ the function $(u \geq 0, v \geq 0)$

$$\Phi(u,v) = \begin{cases} (F(u) - F(v))/(u - v) & \text{if} \quad u \neq v \\ \\ dF(u)/du & \text{if} \quad u = v \end{cases}$$

and set $z(t) = x(t) - v(t)$. Thus, z is a solution of the equation

$$\big(a(t)z'\big)' + b(t)\overline{\Phi}(t)z = 0, \tag{31}$$

where $\overline{\Phi}(t) = \Phi(x(t), v(t))$. In virtue of (5), we have

$$b(t)\overline{\Phi}(t) \leq B.$$

Since, from Lemma 5, equation (25) is disconjugate on $[1, \infty)$, the equation (31) is disconjugate on $[1, \infty)$ too. Since $z(1) = 0$, the solution z does not have zeros for $t > 1$ and so, without loss of generality, we can suppose $z(t) > 0$

for $t > 1$. Because $\lim_{t\to\infty} z(t) = 0$, there exists $t_1 > 1$ such that $z'(t_1) = 0$. Moreover, taking into account that x satisfies (30) and $z(t) < x(t)$, we get that z is a principal solution of (31). Using again the comparison result [20, Corollary 6.5] for equations (25) and (31), we obtain for $t > 1$

$$\frac{z'(t)}{z(t)} \leq \frac{y_0'(t)}{y_0(t)}, \tag{32}$$

where y_0 is the positive decreasing principal solution of (25) defined in the first part of the proof. Thus, the inequality (32) gives a contradiction at $t = t_1$, because

$$\frac{y_0'(t_1)}{y_0(t_1)} < 0.$$

□

We conclude this section with the following continuity result for starting points of solutions of (1) which satisfy (22).

Theorem 4. *Assume that equation (8) is disconjugate on* $[1, \infty)$. *Let* $\{c_n\}$ *be a positive sequence converging to zero and denote by* x_n *the unique solution of (1) which satisfies (22) with* $c_n = c$. *Then the sequence* $\{x_n'(1)\}$ *converges to zero.*

Proof. In virtue of Theorem 3, for any $c_n > 0$, equation (1) has a unique solution x_n which satisfies (22) with $c_n = c$. Denote by w_n the principal solution of (26) such that $w_n(1) = c_n$. From (29) and (22) we get

$$w_n'(1) \leq x_n'(1) \leq 0. \tag{33}$$

Since principal solutions are determined up to a constant factor, we have

$$w_n(t) = \frac{c_n}{c_1} w_1(t).$$

Hence $w_n'(1) = c_n w_1'(1)/c_1$ and from (33) the assertion follows. □

5 Proof of the main result

In this section we prove Theorem 1 and we show some its consequences. To this aim, the following generalization of the well known Kneser's theorem (see for instance [12, Section 1.3]), plays a key role.

Proposition 2. *Consider the system*

$$z' = F(t,z), \quad (t,z) \in [T_1, T_2] \times \mathbb{R}^n$$

where F is continuous and bounded, and let K_0 be a continuum (i.e., a compact and connected subset) of $\{(T_1, w) : w \in \mathbb{R}^n\}$. Let $\mathcal{Z}(K_0)$ be the family of all the solutions emanating from K_0. If any solution $z \in \mathcal{Z}(K_0)$ is defined on the whole interval $[T_1, T_2]$, then the cross-section $\mathcal{Z}(T_2; K_0) = \{z(T_2) : z \in \mathcal{Z}(K_0)\}$ is a continuum in $\mathbb{R}^n$.

Proof of Theorem 1. Let $x_1 \in \Delta_1$ and $x_2 \in \Delta_2$ be the solutions on $[0,1]$ of (1), whose existence is guaranteed by Proposition 1, and let $\alpha = \max\{x_1'(0), x_2'(0)\} > 0$, $\beta = \min\{x_1'(0), x_2'(0)\} > 0$. Put

$$L = \alpha\, a(0) A(1) \tag{34}$$

and let $\widehat{F}$ be a Lipschitz function on $\mathbb{R}$ such that

$$\widehat{F}(u) = \begin{cases} 0, & u < 0 \\ F(u), & 0 \le u \le L \\ F(L), & u > L \end{cases} .$$

For $\ell \in (0, \alpha]$, consider the Cauchy problem

$$\begin{cases} (a(t)x')' + b(t)\widehat{F}(x) = 0, \quad t \in [0,1] \\ x(0) = 0,\ x'(0) = \ell, \end{cases} \tag{35}$$

and denote by x_ℓ the unique solution of (35). Let us show that x_ℓ is defined on the whole interval $[0,1]$. For any solution x of the equation in (35), the function $a(\cdot)x'(\cdot)$ is nonincreasing, so $a(t)x_\ell'(t) \le a(0)x_\ell'(0) = a(0)\ell$. Integrating this inequality, in view of (34) we get for $t \in [0,1]$

$$x_\ell(t) \le a(0)\ \ell \int_0^t \frac{1}{a(s)}\, ds \le a(0)\ \ell\ A(1) \le L.$$

Assume now that $x_\ell(t) > 0$ on $(0, t_1), 0 < t_1 \le 1$, and $x_\ell(t_1) = 0$. Then, in virtue of the uniqueness of the Cauchy problem (35), we obtain $x_\ell'(t_1) < 0$. If $t_1 < 1$, then $x_\ell(t) < 0$ in a right neighborhood of t_1 and satisfies $(a(t)x_\ell')' = 0$, which gives $x_\ell(t) < 0$ for every $t \geq t_1$ for which this solution exists. Since $x'_\ell(t_1) < 0$, by integration we obtain for $t > t_1$

$$x_\ell(t) = a(t_1)x'_\ell(t_1)\int_{t_1}^{t}\frac{1}{a(s)}\,ds > a(t_1)x'_\ell(t_1)A(1),$$

that is, x_ℓ is bounded from below. Therefore the solution x_ℓ of (35) is defined on the whole interval $[0,1]$.

Let x be any solution of (1), nonnegative on $[0,1]$ and satisfying $x(0) = 0$, $x'(0) = \ell \in (0, \alpha]$. Then x is also a solution of (35) for $0 \leq t \leq 1$, and vice versa. Indeed, reasoning as above, we obtain $x(t) \leq L$ on $[0,1]$ and therefore $F(x(t)) = \widehat{F}(x(t))$ for all $t \in [0,1]$.

Put $K_0 = \{(x(1), x'(1)) : x$ is solution of (35) with $\ell \in [\beta, \alpha]\}$. Since any solution of (35) is defined on the whole $[0,1]$, by Proposition 2 the set K_0 is a continuum in $\mathbb{R}^2$, containing the points $(0, x'_1(1))$, $(x_2(1), 0)$, with $x'_1(1) < 0$, $x_2(1) > 0$. Further, K_0 does not contain any point $(0, c)$ with $c \geq 0$. Therefore a continuum $K_1 \subseteq K_0$ exists, $K_1 \subseteq \overline{\pi} = \{(u,v) : u \geq 0, v \leq 0\}$, $(0,0) \notin K_1$, and there exist two points $P, Q \in K_1$, $P = (p, 0)$, $Q = (0, -q)$, $p > 0, q > 0$.

In order to complete the proof, we use a similar argument to the one given in [23, Theorem 1.1], with minor changes. Consider equation (1) for $t \geq 1$. By Theorem 3, for every $c > 0$, (1) has a unique positive decreasing solution x satisfying (22) and (11). Then, the set S_1 of the initial data of the solutions of (1) on $[1, \infty)$ satisfying (11) and (22) is connected, $S_1 \subset \bar{\pi}$, and its projection on the first component is the half-line $(0, \infty)$. Further, from Theorem 4, $(0,0) \in \bar{S}_1$. Therefore we have

$$K_1 \cap S_1 \neq \emptyset.$$

Let us show that each point $(c,d) \in K_1 \cap S_1$ corresponds to a solution of the BVP (1)-(2). Let $(c,d) \in K_1 \cap S_1$. Then $c > 0, d \leq 0$. Since $(c,d) \in K_1$, there exists a solution u of (35), for a suitable $\ell \in [\beta, \alpha]$, such that $u(1) = c > 0$ and $u'(1) = d \leq 0$. Since $u(1) > 0$ we have $u(t) > 0$ on $(0,1]$. Therefore u is also a solution of (1) in $[0,1]$, with $u(0) = 0$, $u(t) > 0$ for $t \in (0,1]$. As $(c,d) \in S_1$, a positive decreasing solution v of (1) exists on $[1, \infty)$, which satisfies (22) and $v(1) = c = u(1)$, $v'(1) = d = v'(1)$. Hence, the function

$$x(t) = \begin{cases} u(t), & t \in [0,1], \\ v(t), & t > 1. \end{cases}$$

is a solution of the BVP (1)-(2) and the proof is complete. □

From Theorem 1 and Proposition 1, we get the following.

Corollary 1. *Let assumptions of Proposition 1-(ii) be satisfied and equation (8) be disconjugate on* $[1, \infty)$*. Then the BVP (1)-(2) has a solution.*

We close this section with the solvability of our BVP for the perturbed equation

$$\big(a(t)z'\big)' + (b(t) + b_1(t))F(z) = 0, \tag{36}$$

where b_1 is a continuous function for $t \geq 0$ such that $b_1(t) \equiv 0$ on $[0, 1]$ and $b_1(t) \leq 0$ for $t > 1$.

Corollary 2. *If assumptions of Theorem 1 are satisfied, then equation (36) has a solution* z *satisfying boundary conditions (2).*

6 Examples and concluding remarks

Theorem 1 is illustrated by the following example.

Example 2. Consider the equation

$$\big(a(t)x'\big)' + b(t)\ F(x) = 0, \tag{37}$$

where

$$a(t) = (1+t)^2, \quad b(t) = \frac{1}{5e} \exp\left(\frac{16}{1+16t^4}\right) \cos\left(\frac{\pi t}{2}\right) \quad \text{for } t \geq 0. \tag{38}$$

and F satisfies (5) and (6) with

$$k_0 = \frac{9}{e^{15}}, \quad k_\infty = 1.$$

Since b is decreasing on $[0, 1]$, we get

$$\int_{1/3}^{1/2} b(\tau) d\tau \geq \frac{1}{6} b(1/2) = \frac{\sqrt{2}}{60} e^7.$$

For equation (37), the function A in (7) becomes

$$A(t) = \frac{t}{1+t},$$

so assumptions (9), (10) are verified for $t_1 = 1/3$ and $t_2 = 1/2$, because

$$A(1)|b|_L \geq \frac{1}{2}b(0) = \frac{e^{15}}{10},$$
$$\frac{A(1)}{A(t_1)(A(1) - A(t_2))} = 12 < \frac{\sqrt{2}}{60}e^7.$$

Finally, for $t \in [1, \infty)$ we have

$$b(t) \leq \frac{1}{5e} < \frac{1}{4}$$

and the equation (8) becomes the Euler equation

$$v'' + \frac{1}{4(1+t)^2}v = 0,$$

which is disconjugate on $[1, \infty)$, see, e.g., [25, Chapter 2.1]. Hence, in view of Theorem 1, equation (37) has solutions x which satisfy the boundary conditions (2) and

$$\int_1^\infty \frac{1}{(1+t)^2 x^2(t)} dt = \infty.$$

Remark 3. Example 2 can be slightly modified for the nonlinearity

$$F(u) = \frac{u^2}{1+u}$$

or the nonlinearity

$$F(u) = \frac{u}{1+\sqrt{u}}.$$

Remark 4. Consider the equation

$$\big(a(t)x'\big)' + (b(t) \ + b_1(t))F(x) = 0, \tag{39}$$

where the functions a, b are given in (38), b_1 is the function

$$b_1(t) = (e - e^t)(|\cos t| - \cos t) \text{ for } t \geq 1, \quad b_1(t) \equiv 0 \text{ for } t \in [0, 1),$$

and F is as in Example 2. Since $b_1(t) \leq 0$, in view of Corollary 2, equation (39) has solutions x which satisfy the boundary conditions (2).

Remark 5. Theorem 1 and Corollaries 1, 2 continue to hold if the assumption (5) is replaced by the more general condition

$$\exists K > 0 : 0 \leq \frac{dF(u)}{du} \leq K \quad \text{for } u \geq 0$$

and the disconjugacy of (8) is substituted by the disconjugacy on $[1, \infty)$ of the linear equation

$$v'' + \frac{B\,K}{a(t)} v = 0.$$

Remark 6. The assumption $k_0 \neq k_\infty$ implies that F cannot be a linear function on $[0, \infty)$. If the linear equation

$$\big(a(t)x'\big)' + b(t)x = 0 \tag{40}$$

has a solution x satisfying (2), then in virtue of Lemma 3, (40) is disconjugate on $[0, \infty)$. However, x is not necessarily the principal solution of (40). The following example illustrates this case.

Example 3. Consider the equation

$$\big(e^{2t}x'\big)' + e^{2t}x = 0, \quad t \geq 0 \tag{41}$$

A standard calculation shows that $x_0(t) = e^{-t}$, $x_1(t) = te^{-t}$ are solutions of (41). Obviously, x_1 satisfies (2). Observe that x_1 is a nonprincipal solution and x_0 is the principal solution.

In a forthcoming paper we will consider this kind of BVPs for nonlinear equations for which $k_0 = k_\infty$.

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