PONTE ALLA VERGINE.
STRUCTURAL ANALYSIS OF A MEDIEVAL MASONRY FOOTBRIDGE

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Summary

The paper is concerned with the application of a numerical method, available to the analysis of voussoir arches and/or any type of masonry structure, in the structural investigation of the medieval masonry footbridge called *Ponte alla Vergine*, close Pistoia.
The masonry structure is modelled as a discrete system of rigid blocks connected by unilateral elastic contact constraints.
The contact device which links the blocks, through which both a mortar joint and a dry joint could be simulated, consists of a set of elastic links, orthogonal to the contact surface between two adjacent blocks, and an additional link, parallel to the interface, through which the shear forces can be transmitted.
In accordance with the assumption of no tensile strength in the joint, only compressive forces can be transmitted from one element to another. Reasonable hypotheses can be assumed for the link parallel to the contact surface in order to calibrate both the shear behaviour and the influence of the friction between the blocks.
Through the results of the numerical procedure it is possible both to define the cracking failure pattern, highlighting the actual reacting structure within the apparent one, and to evaluate the width of the cracks located in the mortar joints.

Keywords: Masonry, no tension behaviour, non linear analysis, generalized inverse.

1. Introduction

It is well known that the main problem in the analysis of masonry structures is the different compressive and tensile strength that characterizes the material behaviour. Such a circumstance makes it impossible to understand which is, in a pre-assigned structural configuration subject to any external action (loads and/or inelastic displacements), the actual reacting structure. Such a reacting structure, which is the real unknown of the problem, does not necessarily correspond to the apparent one, but depends, from time to time, on the external actions.
The assumption of no tensile strength for the masonry, although in some cases may be unrealistic, can be considered an appropriate hypothesis or at least a *safe* assumption. Nevertheless, a no tension assumption can be considered almost exactly true if, for instance, we deal with arches, or any masonry structure, built with stone blocks assembled dry or with very weak mortar joints. Generally the analysis of structural problems involving unilateral constraints, expressed through systems of equations and inequalities, requires the use of Q.P. techniques. Otherwise, as an alternative, it is possible to obtain the solution by using a step by step procedure according to which the solution relative to the *standard material* (linear elastic and bilateral) is assumed as starting point and is subsequently corrected according to the actual material skills. Such a method has already been practiced by Castigliano in 1879.
The procedure presented here performs the solution through the introduction of suitable *distortion terms* capable to generate internal coactions such as to give back the compatibility in the sign conditions where tensile stresses are not admissible. The numerical procedure is based on the use of Moore-Penrose *generalized inverse* and reduces the problem to the solution of systems of linear equations.

2. General formulation and numerical procedure

Following a typically eighteenth century idea, let us consider the general problem of a masonry structure consisting of rigid blocks (elastically undeformable) linked through elastic mortar layers.
In such a model the no-tension behaviour of the material is totally supposed concentrated in the mortar joint located in between two adjacent blocks. Such a joint can therefore be assumed as an unilateral elastic contact constraint. In particular, the mortar joint can be idealized, in a Drucker's way, through an *interface device* consisting of a set of elastic links, orthogonal to the contact surface, capable of transmitting only compressive forces between the blocks, and additional links, parallel to the interface, through which the shear forces can be transmitted. The behaviour of the orthogonal links is assumed unilateral linear elastic, whereas for the parallel ones further hypotheses can be added in order to specify either the shear strength and to calibrate, for instance, the influence of the friction between the blocks, or a bilateral rigid behaviour totally capable to prevent the sliding. In practice a reasonably low number of orthogonal bars is enough to describe with significant expressiveness the behaviour of the joint and to appraise clearly the location ad depth of possible cracks.
Let us consider, therefore, a masonry structure consisting of *n* three dimensional rigid elements linked through *m* unilateral elastic contact interfaces.
Assuming the structure subjected to the action of external loads and inelastic displacements represented respectively by the vectors $F \in \Re^{6n}$ and $\Omega_1 \in \Re^{km}$ (where the value k depends on the number of contact constraints chosen to characterize the interface device and defines the degree of statically indeterminacy of the structure), the problem can be expressed through a system of equilibrium and elastic-kinematical equations, with some variables, those which correspond to the unilateral links in the interface model, subjected to inequalities which express sign conditions:

$$\begin{cases} AX = F \\ A^T x + KX = \Omega_1 + \overline{\Omega}_2 \end{cases} \quad \text{sub} \quad \begin{cases} X \le 0 \\ \overline{\Omega}_2 \ge 0 \end{cases} \qquad (1)$$

In the previous form (1) $A \in \Re^{6n \times km}$ is the geometrical configuration matrix; $X \in \Re^{km}$ indicates the unknown vector of internal forces located on the interface joints; the components of the $x \in \Re^{6n}$ represents the unknown vector of displacement and rotation components of the centroids of the elements; $K \in \Re^{km \times km}$ is the diagonal stiffness matrix of the contact constraints; $\Omega_1 \in \Re^{km}$ is the vector of possible external inelastic displacements; $\overline{\Omega}_2 \in \Re^{km}$ indicates the unknown vector whose components are *internal distorsions* which need for obtaining a solution capable of satisfying both the equilibrium equations, while respecting the sign conditions, and the elastic-kinematical compatibility of the actual reacting structure. On this subject, it is convenient to distinguish, within the vector $\overline{\Omega}_2$, two types of entities, assuming for the former, related to the equilibrium aspects, the notation $\overline{\Omega}_2^*$ and for the latter, related to the compatibility ones, the notation $\overline{\Omega}_2^{**}$.
Of course the system of equations (1), subject to the first sign conditions, could also have no solution; in such a case it means that the structure cannot be equilibrated under the given system of the external actions. In this case there is no vector $X \in \Re^{km}$ which satisfies the *6n* equations and the *km* inequalities simultaneously.
However let us suppose that the system (1) in consistent. In such a case the general solution $X = X_0 + X_N$, that is able to satisfy the equilibrium problem and the first of the two inequalities, can be obtained assuming, as initial solution X_0, that is relative to the bilateral linear elastic behaviour of the contact constraints:

$$X_0 = K^{-1}A^T(AK^{-1}A^T)^{-1}F + K^{-1}(I - A^T(AK^{-1}A^T)^{-1}AK^{-1})\Omega_1 \qquad (2)$$

The initial vector X_0 can be suitably arranged in two sub-vectors: X_{0t}, whose components do not satisfy the sign conditions, and X_{0c} whose components satisfy the sign conditions:

$$X_0 = \begin{bmatrix} X_{0t} \\ X_{0c} \end{bmatrix} \qquad (3)$$

Note that is $X_{0t} \in \Re^{tm}$, where *t* is the number of the contact constraints that, in the initial solution, come out stretched. In any case *t* can be greater than the degree of statically indeterminacy of the structure.
According to the Colonnetti's theorem, the maximum number of imposed linear independent terms of distortion - that is the maximum number of the iterations in the procedure - are, at most, equal to the degree of statically indeterminacy of the structure. Such an initial solution is then modified through the vector:

$$X_N = (I - K^{-1}A^T(AK^{-1}A^T)^{-1}A)\overline{\Omega}_2^* \qquad (4)$$

which, added to X_0, satisfies the first of the (1) while respecting the sign conditions.
The properties of the orthogonal projection matrix $C = (I - K^{-1}A^T(AK^{-1}A^T)^{-1}A)$ (see [4]) and the appropriate choice of the unknown vector $\overline{\Omega}_2^*$, are the keys to understanding the meaning of the procedure. In its turn also the matrix *C* can be suitably partitioned in four sub-matrices C_t, C_1, C_1^T, C_c:

$$C = \begin{bmatrix} C_t & C_1 \\ C_1^T & C_c \end{bmatrix} \qquad (5)$$

where the sub-matrix $\overline{C}_t = [C_t \quad C_1] \in \mathcal{R}^{t \times km}$ has to be chosen as a full row rank matrix. On this subject the elimination of any linearly dependent row of the matrix $\overline{C}_t$, plays a key role in ascertaining the number of strictly necessary *internal distorsions* to give back the compatibility in the sign conditions. Computing the Moore-Penrose generalized inverse of $\overline{C}_t$, it is easily possible to evaluate the vector $\overline{\Omega}_2^*$:

$$\overline{\Omega}_2^* = \overline{C}_t^{-1} X_{0t} \qquad (6)$$

If the solution of the unilateral problem exists, the vector solution which satisfies simultaneously the equilibrium equations and the first of the two inequalities (1), assumes the form :

$$X = \begin{bmatrix} 0 \\ X_c \end{bmatrix} \quad \text{with } X_c < 0 \qquad (7)$$

Since the final vector *X* is different from the first elastic vector solution X_0, it cannot satisfy, of course, the kinematical compatibility expressed through the second set of equations in the system (1).
A very easy way to build up again such a compatibility is to consider the second set of equations in the system (1) in the form $A^T\overline{x} + KX = 0$. Partitioning both the general matrix A^T in two sub-matrices A_t^T, A_c^T, and the constitutive matrix K in K_t, K_c, we obtain the solution:

$$\overline{x} = -(A_c A_c^T)^{-1} A_c K_c X_c \qquad (8)$$

which represents the vector of the displacements of the centroids of the elements only due to the actual reacting structure. Finally the vector $\overline{\Omega}_2^{**}$ can be determined, so that the compatibility of the second of the (1) is already reached : $\overline{\Omega}_{2t}^{**} = A_t^T \overline{x}$.
The components of the vector $\overline{\Omega}_2^{**} \neq 0$ give the position and width of the cracks located in the mortar joints:

$$\overline{\Omega}_2^{**} = \begin{bmatrix} \overline{\Omega}_{2t}^{**} \\ 0 \end{bmatrix} \qquad (9)$$

The procedure can be illustrated with an elementary example shown in figure 1. Between two rigid elements is supposed the presence of a strip - joint made with no-tension material characterized by an elastic-cracking behaviour.
The figure 1a) gives an idea of the *contact device* consisting of a set of elastic links and shows the location of the external load. The case 1b) corresponds to the linear elastic solution, whereas the case 1c) shows the exact solution

relative to the unilateral elastic contact constraint. In figure 1d) the joint behaviour becomes clearer through the enlargement of the deformed configuration of the structure. In particular the crack width in the joint can be highlighted.

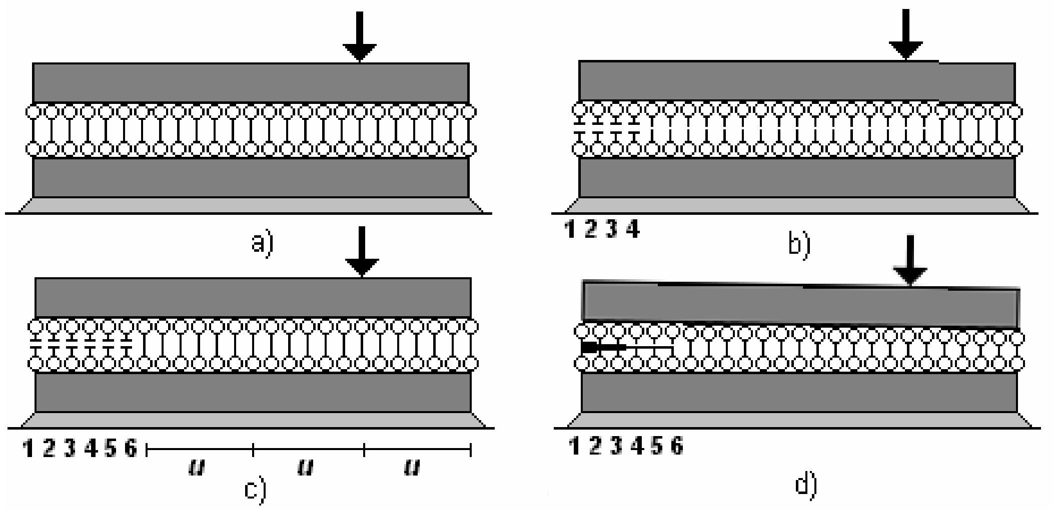

Fig. 1 - Behaviour of strip - joint between two rigid elements under the assumption of no tensile strength.

3. The case of voussoir arches

The masonry arch is widely known as a conventionally monodimentional structure consisting of stone or brick elements assembled dry or with mortar joints. Since the difference between the mechanical parameters of stone and mortar deformability is so substantial, the latter case is suitable to be analyzed using the proposed model of rigid blocks linked through elastic mortar layers.

Even if three contact constraints would be strictly sufficient (Fig.2a), an *interface device*, consisting of four orthogonal bars (two of which located at the edges of block and the other two in the middle third position) and a parallel one, describes the behaviour of joint better (Fig.2b). The general behaviour of the structure depends on the ratio between the stiffness value assumed for the contact constraints which are orthogonal to the interface surface, and the stiffness value of the contact constraint parallel to the interface surface. Moreover in addition to the no-tension behaviour of the orthogonal contact constraints, we can assume that a limited strength exists also for the shear forces. It is convenient to assume that the appropriate limit value of the tangential forces depends on the compressive action transmitted between the blocks. Nonetheless, because of it is very improbable that any sliding of one block upon another occurs under statical load conditions, the actual behaviour of the arch is marked by the presence of opening hinges.

The results obtained allow us both to locate the actual line of thrust and the cracked joints. In addition the corresponding width and depth of the crack, measured in its radial direction, can be also evaluated.

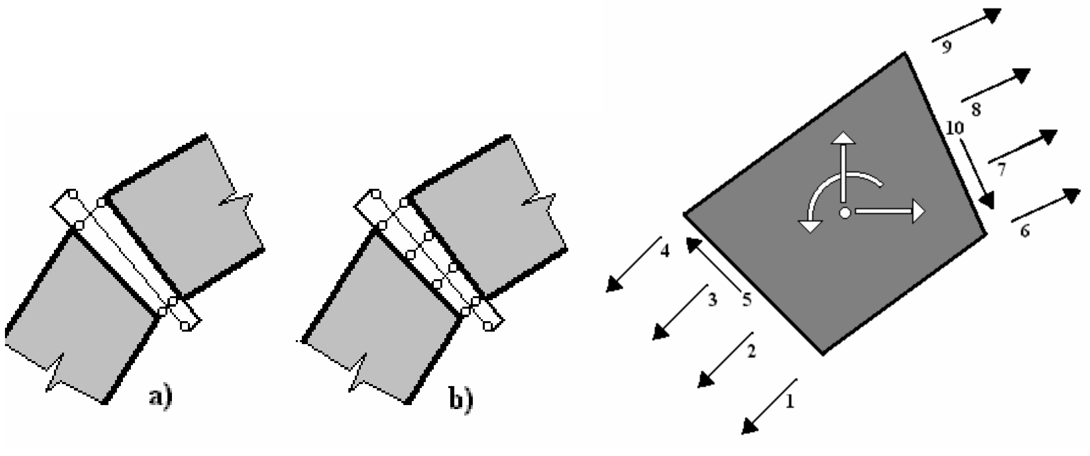

Fig. 2 - Two feasible interface devices between adjacent blocks and reading order of the interactions acting on the block corresponding to the case b).

4. The analysis of masonry arch bridge Ponte alla Vergine

The Ponte alla Vergine, crossing the river Vincio di Montagnana (PT), is a footbridge medieval masonry bridge with a width of 2 meters, including the parapets, provided with three arch rings and a total development in length of 37 meters.
In particular the central arch has a span of 7 meters, the larger lateral one, which crosses the river, ha a span of 13 meters, the other one a span of 6.3 meters.
The arch rings are made of masonry bricks whereas the piers, the abutments and the spandrel faces present a brickwork of masonry stones.

Fig. 3 - Ponte alla Vergine. Complete front view.

Fig. 4 - Ponte alla Vergine. Crossino estradox of the brigde

Fig. 5 - Ponte alla Vergine. Spandrel face

The numerical procedure has been applied to the analysis of each of the the three arch rings of the bridge. No-tension hypotheses for the mortar joints have been considered.
The numerical output, in terms of internal stresses, takes into account the strength characteristics evaluated in the actual partialized interfaces of mortar joints.

The load conditions considered are the followings:

1. Self – weight of the arch ring + filling self - weight
2. Self – weight of the arch ring + filling self - weight + uniform load of 400 dN/mq
3. Collapse load performed by a moving pointed load applied on any single voussoir
4. Collapse seismic acceleration applied to the most vulnerable arch ring.

With reference to point 1 the following values have been used in the analysis :

- Brick stone masonry 1800 dN/mc
- Filling materials 1000 dN/mc

With reference to point 3, the maximum moving load has been considered in the location in which a four hinges mechanism is reached.

The figures 6 and the next show the results in terms of location of the actual line of thrust. The table 1 shows the numerical values obtained in terms of stresses for each case.

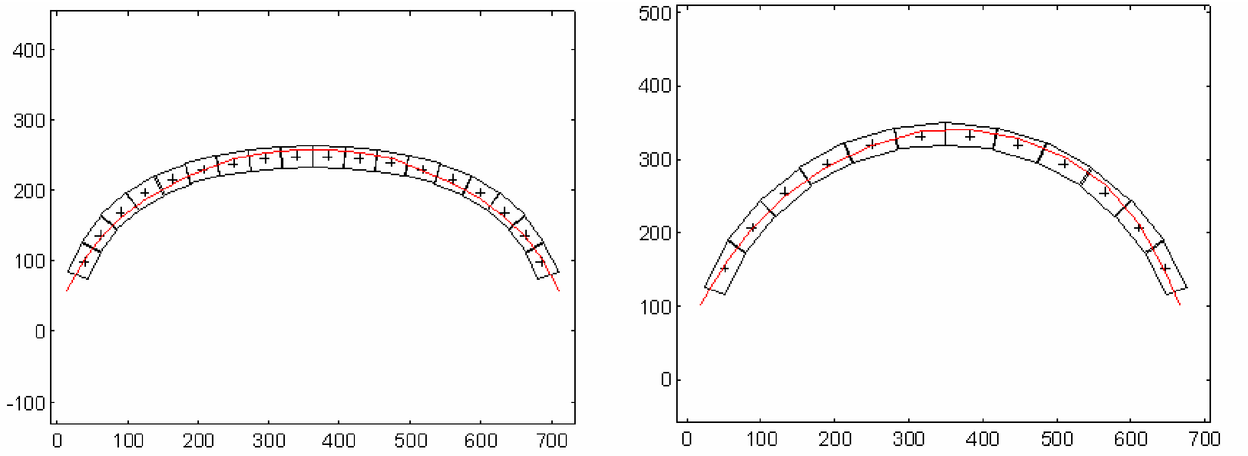

Fig. 6 - Self weight of the structure + uniform load of 400 dN/mq. Actual line of thrust. On the left side the lesser lateral arch; on the right side the central one.

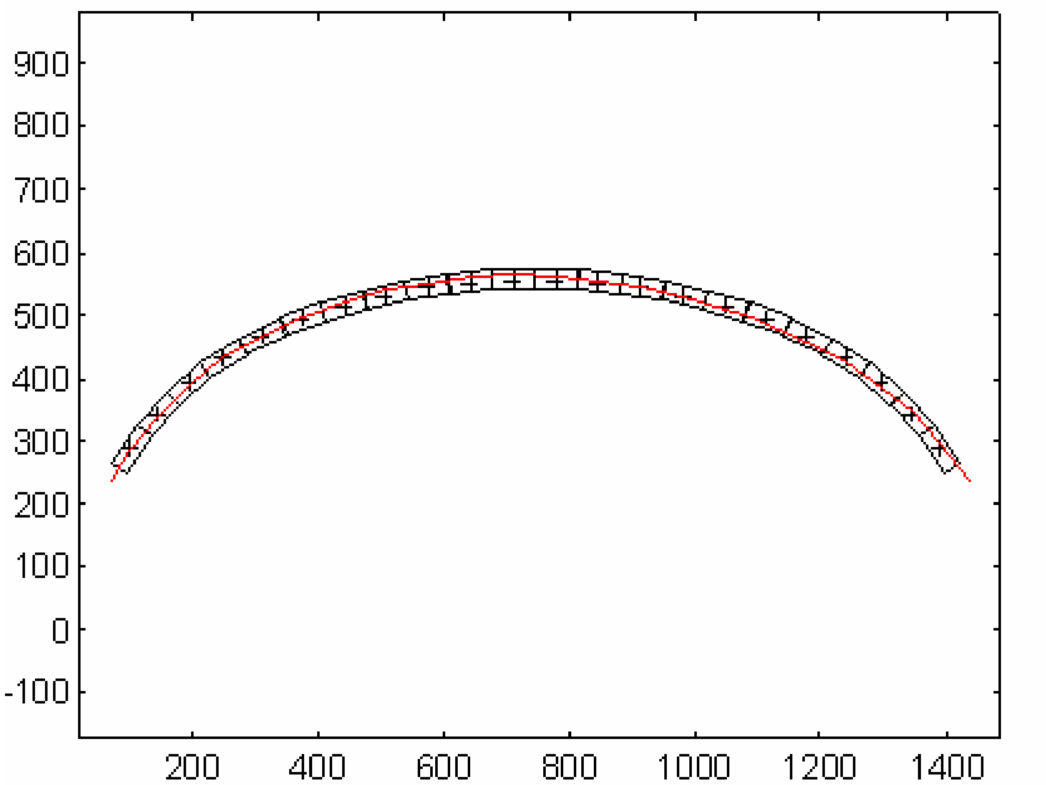

Fig. 6 - Self weight of the structure + crowd load of 400 dN/mq. Actual line of thrust. Larger lateral arch

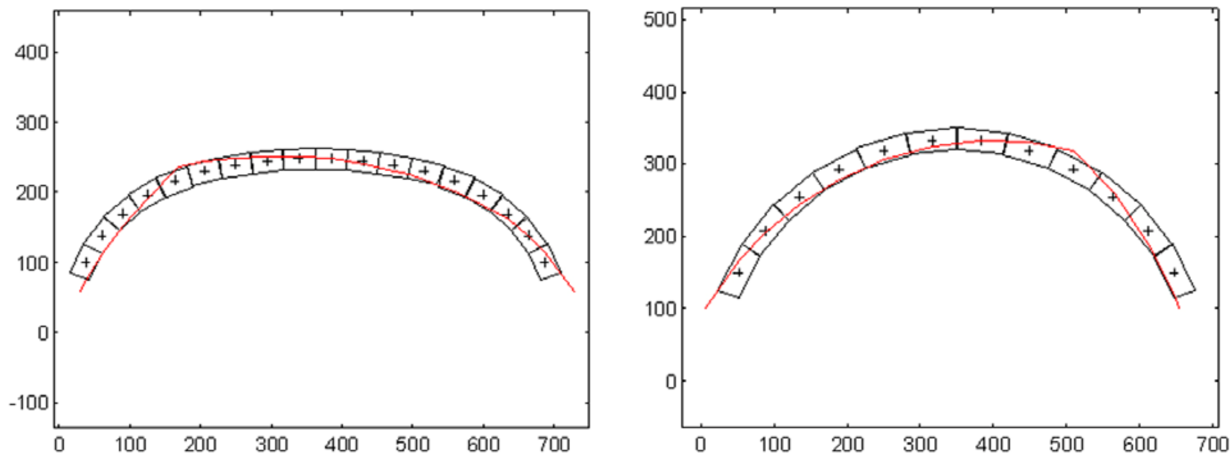

Fig. 6 - Collapse load performed by a moving pointed load applied on single voussoir. Actual line of thrust. On the left side the lesser lateral arch: 2400 dN on n° 4; on the right side the central one: 3600 dN on n° 5.

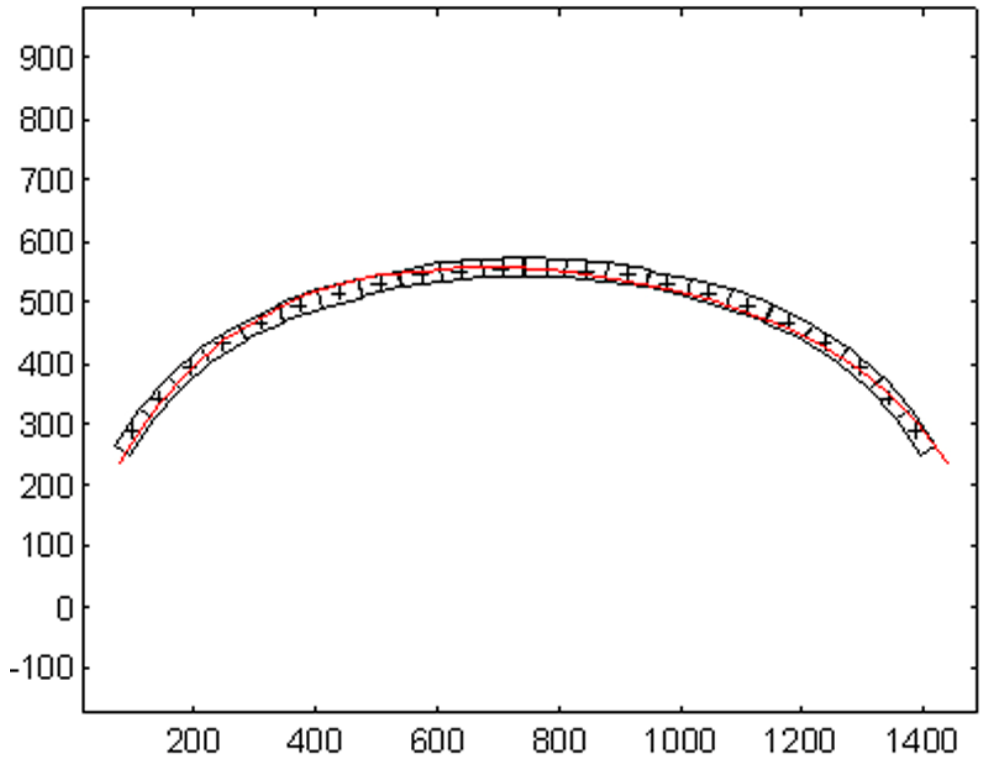

Fig. 6 - Collapse load performed by a moving pointed load applied on single voussoir. Actual line of thrust. Larger lateral arch: 1600 dN on n° 7.

Table 3

	Lateral arch ring with smaller span $\sigma_{c\,max}$ (dN/cmq)	Central arch ring $\sigma_{c\,max}$ (dN/cmq)	Lateral arch ring with larger span $\sigma_{c\,max}$ (dN/cmq)
Self weight of the arch ring + filling self weight	2.42	3.00	10.17
Self weight of the arch ring + filling self weight + uniform load of 400 dN/mq	3.54	4.73	12.45
Collapse moving pointed load	2400 dN voussoir .n°4 13.11	3600 dN voussoir n°5 12.19	1600 dN voussoir n°7 16.06
Collapse seismic acceleration > 0.25g	12.07	11.96	16.86

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