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A COEXISTENCE PROBLEM FOR NONOSCILLATORY SOLUTIONS TO EMDEN-FOWLER TYPE DIFFERENTIAL EQUATIONS

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Abstract

We consider a coexistence problem for nonoscillatory solutions to the Emden-Fowler type differential equation

$$(a(t)|x'|^{\alpha} \operatorname{sgn} x')' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \; t \geq 1. \qquad (*)$$

For the special case

$$x'' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \; t \geq 1, \qquad (**)$$

this problem has been posed by Moore and Nehari when $1 < \beta$ [24] and by Belohorec when $0 < \beta < 1$ [2]. Nonoscillatory solutions to $(**)$ can be classified into three types, according their asymptotic behavior as $t \to \infty$, and in [2, 24] it is shown that these three types of nonoscillatory solutions cannot simultaneously coexist for $(**)$. When the sublinear case $\alpha > \beta$ occurs, this

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Dedicated to the memory of Professor George Sell

result has been recently extended to (*) in [17, 25]. In the superlinear case $\alpha < \beta$, a partial answer has been given in [9]. Here we complete this study, by showing that in any case this triple coexistence for nonoscillatory solutions is impossible also for (*).

1. Introduction

In this paper, we consider the second order nonlinear differential equation

$$(a(t)\,|\,x'\,|^{\alpha}\,\mathrm{sgn}\,x')' + b(t)\,|\,x\,|^{\beta}\,\mathrm{sgn}\,x = 0, \tag{1}$$

where α, β are positive constants such that $\alpha < \beta$, the functions a, b are continuous on $[t_0, \infty)$ and

$$a(t) > 0,\ b(t) \geq 0,\ \sup\{\,b(t) : t \geq \tau\,\} > 0 \text{ for any } \tau > t_0. \tag{2}$$

A prototype of (1) is the equation

$$(t^2|\,x'\,|^{\alpha}\,\mathrm{sgn}\,x')' + b(t)\,|\,x\,|^{\beta}\,\mathrm{sgn}\,x = 0,$$

arising in the study of radially symmetric solutions of partial differential equations with Laplacian operator in $\mathbb{R}^3$, [7, 16].

Since $0 < \alpha < \beta$, the initial value problem associated to (1) has a unique local solution, that is, a solution x such that $x(\bar{t}) = x_0,\ x'(\bar{t}) = x_1$ for arbitrary numbers x_0, x_1 and any $\bar{t} \leq t_0$. Moreover, if a, b satisfy suitable smoothness conditions and $b(t) > 0$, then any local solution of (1) is continuable to infinity, see, [28, Appendix A]. On the other hand, under the weaker assumption (2), equations of type (1) with uncontinuable solutions may exist, see, [27].

Here, by a solution of (1) we mean a function x defined on some ray $[\tau_x, \infty)$, $\tau_x \geq t_0$, such that it's quasiderivative $x^{[1]}$, i.e. the function

$$x^{[1]}(t) = a(t)\,|\,x'(t)\,|^{\alpha}\,\mathrm{sgn}\,x'(t), \tag{3}$$

is continuously differentiable, and satisfies (1) for any $t \leq \tau_x$.

As usual, a solution x of (1) is said to be nonoscillatory if $x(t) \neq 0$ for large t and oscillatory otherwise. Equation (1) is said to be nonoscillatory if any solution is nonoscillatory. Notice that, if $\alpha \neq \beta$, nonoscillatory solutions of (1) may coexist with oscillatory ones, while if $\alpha = \beta$ this fact is impossible, [23, Chapter III, Section 10].

If both integrals

$$I_a = \int_{t_0}^{\infty} a^{-1/\alpha}(s)\, ds, \quad I_b = \int_{t_0}^{\infty} b(s)\, ds$$

are divergent, then all solutions of (1) are oscillatory, while, if both integrals I_a and I_b are convergent, then all nonoscillatory solutions and their quasiderivatives are bounded, [21, 23]. Thus, the interesting case is when only one of integrals I_a, I_b is divergent, that is either

$$I_a < \infty,\ I_b = \infty \tag{P_1}$$

or

$$I_a = \infty,\ I_b < \infty. \tag{P_2}$$

Our aim here is to consider the superlinear case $0 < \alpha < \beta$ when (P_1) holds, that is the case

$$0 < \alpha < \beta, \quad I_a < \infty, \quad I_b = \infty. \tag{4}$$

Observe that the remaining cases have been already treated in the literature and are discussed below in Section 4.

If x is a solution of (1), then $-x$ is a solution, too. Thus, let $\mathbb{P}$ be the class of all eventually positive solutions x of (1). In view of (4), the class $\mathbb{P}$ can be divided into three subclasses, according to the asymptotic behavior of x as $t \to \infty$, [19]. More precisely, any solution $x \in \mathbb{P}$ satisfies one of the following asymptotic properties:

$$\lim_{t\to\infty} x(t) = \ell_x, \quad 0 < \ell_x < \infty, \tag{5}$$

$$\lim_{t\to\infty} \frac{x(t)}{A(t)} = \infty, \tag{6}$$

$$\lim_{t\to\infty} \frac{x(t)}{A(t)} = \ell_x, \quad 0 < \ell_x < \infty, \tag{7}$$

where ℓ_x is a positive constant depending on x and

$$A(t) = \int_t^\infty a^{-1/\alpha}(s)\, ds.$$

Let $x, y \in \mathbb{P}$ satisfy (6), (7) respectively. Then x, y tend to zero as $t \to \infty$ and $0 < y(t) < x(t)$ for large t. Hence, solutions of (1) satisfying (6) are called slowly decaying solutions, and solutions satisfying (7) strongly decaying solutions, [17]. Solutions satisfying (5), (6), (7) are referred also as dominant solutions, intermediate solutions and subdominant solutions, respectively, [4, 15, 16, 20].

The interesting problem which arises is whether all three types of nonoscillatory solutions can simultaneously exist. Since there are known necessary and sufficient conditions for the existence of subdominant and dominant solutions, the coexistence problem leads to the problem on the nonexistence of intermediate solutions.

Observe that for the linear equation

$$(a(t)x') + b(t)\, x = 0$$

this triple coexistence is impossible. This question has a long history, which started sixty years ago by Belohorec [2] and Moore and Nehari [24], for the special case

$$x'' + b(t)\,|\,x\,|^{\beta} \operatorname{sgn} x = 0, \quad \beta \neq 1. \tag{8}$$

Equation (8) is the well-known Emden-Fowler equation and it is widely studied in the literature, [18, 23, 27] and references therein. Moreover, it is easy to show that nonoscillatory solutions of (8) can be divided into three types, according to their behavior as $t \to \infty$. which cannot simultaneously coexist, [2, 24].

Later on, for the limit case $\alpha = \beta$, that is for the half-linear equation

$$(a(t)\,|\,x'\,|^{\alpha} \operatorname{sgn} x')' + b(t)\,|\,x\,|^{\alpha} \operatorname{sgn} x = 0, \tag{9}$$

the same coexistence problem has been proposed by Kusano et al. in [16, page 213]. A negative answer has been given in [3], by using an extension of the wronskian and a Minkowsky type inequality. It is worth noting that results in [3] are proved assuming the positivity of the function b, but they continue to hold also when (2) holds, with minor modification.

When (P_1) holds, in [17] the sublinear case has been considered and the above coexistence problem has been solved in a negative way for the more general equation (1). Here we complete this study, by considering (1) when (4) is valid.

The main result is given in Section 3, jointly with some consequences and examples. In Section 4 a summary of the coexistence problem, jointly with some comments and suggestions for future researches, are presented.

2. Auxiliary Results

In view of (4), any eventually positive solution x of (1) is decreasing and $x^{[1]}$ is negative nonincreasing. Hence, the class $\mathbb{P}$ can be divided into the subclasses:

$$M^{-}_{\ell, -\infty} = \{x \in \mathbb{P} : x(\infty) = \ell_x,\ x^{[1]}(\infty) = -\infty,\ 0 < \ell_x < \infty\},$$

$$M^{-}_{0, -\infty} = \{x \in \mathbb{P} : x(\infty) = 0,\ x^{[1]}(\infty) = -\infty\}, \tag{10}$$

$$M^{-}_{0, -\ell} = \{x \in \mathbb{P} : x(\infty) = 0,\ x^{[1]}(\infty) = -\ell_x,\ 0 < \ell_x < \infty\},$$

The superscript symbol "_" means that solutions $x \in \mathbb{P}$ satisfy $x(t)x^{[1]}(t) < 0$ for any large t. Clearly, classes $M^{-}_{\ell, -\infty}$, $M^{-}_{0, -\infty}$ and $M^{-}_{0, -\ell}$ coincide with the ones given by (5), (6) and (7), and are called dominant solutions, intermediate solutions and subdominant solutions, respectively.

Define

$$J = \int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}} \left(\int_{t_0}^{s} b(r)\, dr \right)^{1/\alpha} ds,$$

$$Y = \int_{t_0}^{\infty} b(s) \left(\int_{s}^{\infty} \frac{1}{a^{1/\alpha}(r)}\, dr \right)^{\beta} ds,$$

$$Z = \int_{t_0}^{\infty} b(s)\left(\int_s^{\infty} a^{1/\alpha}(r)\,dr\right)^{\alpha} ds.$$

Here, we recall some results, which are needed in the proof of our main result. The following holds.

Proposition 1. *Assume* (P_1). *Then*:

(i_1) *For* (1), *including* (9), *the class* $M^-_{\ell, -\infty}$ *is nonempty if and only if* $J < \infty$. *Moreover, for any* ℓ, $0 < \ell < \infty$, *there exists* $x \in M^-_{\ell, -\infty}$ *such that* $\lim_{t\to\infty} x(t) = \ell$.

(i_2) *For* (1) [9] *the class* $M^-_{0, -\ell}$ *is nonempty if and only if* $Y < \infty$ $[\, Z < \infty \,]$. *Moreover, for any* ℓ, $0 < \ell < \infty$, *there exists* $x \in M^-_{0, -\ell}$ *such that* $\lim_{t\to\infty} x^{[1]}(t) = -\ell$.

(i_3) *Any solution of* (1) *is oscillatory if and only if* $Y = \infty$.

Proposition 1 follows from [15, 19], see also [1, Theorems 3.13.11, 3.13.12]. Concerning the half-linear case $\alpha = \beta$. we refer to [3, Theorems 6 and 7]. Observe that these results are proved by assuming the positivity of the function b. Nevertheless, it is easy to verify that they continue to hold also in case when (2) is valid.

As already claimed, the problem of coexistence of the above types of nonoscillatory solutions is completely solved for the half-linear equation (9) in [3, Corollary 1]. Using this result, we obtain the following.

Proposition 2. *For equation* (9) *at most two of the subclasses* $M^-_{0, -\ell}$, $M^-_{0, -\infty}$ $M^-_{\ell, -\infty}$ *are nonempty.*

The relations between the convergence of the integrals J, Y and Z are completely described in [11, Lemma 3] if $\alpha \neq \beta$ and in [8, Lemma 2] if $\alpha = \beta$. Here we report some of these results, which will be useful.

Lemma 1. (i_1) *If* $\alpha \geq 1$ *and* $Z = \infty$, *then* $J = \infty$.

(i_2) *If* $0 < \alpha < \beta$ *and* $J < \infty$, *then* $Y < \infty$.

We close this section with two inequalities, which are needed in the sequel. The first one is a Hölder-type inequality and it is proved in [9, Lemma 1].

Lemma 2. *Let* λ, μ *be such that* $\mu > 1, \lambda\mu > 1$ *and let* f, g *be nonnegative continuous functions for* $t \geq T$. *Then*

$$\left(\int_T^t g(s)\left(\int_s^t f(\tau)\,dt\right)^{\lambda} ds\right)^{\mu}$$

$$\leq \lambda^{\mu}\left(\frac{\mu-1}{\lambda\mu-1}\right)^{\mu-1}\left(\int_T^t f(\tau)\left(\int_T^{\tau} g(s)\,ds\right)^{\mu} d\tau\right)^{\lambda\mu-1}.$$

Lemma 3. *Let* $\lambda > 0$, *and let* X, Y *be two positive numbers. Then*

$$(X+Y)^{\lambda} \leq \sigma_{\lambda}(X^{\lambda}+Y^{\lambda}),$$

where

$$\sigma_{\lambda} = 1 \text{ if } \lambda < 1 \text{ and } \sigma_{\lambda} = 2^{\lambda-1} \text{ if } \lambda \geq 1. \tag{11}$$

Proof. If $\lambda \geq 1$, the assertion follows from the convexity of the function $\theta^{\lambda}\,(\theta \geq 0)$. Let $0 < \lambda < 1$ and consider for $\theta \geq 0$ the function $G(\theta) = 1 + \theta^{\lambda} - (1+\theta)^{\lambda}$. Since $G(0) = 0$ and G is increasing, putting $\theta = Y/X$ the assertion follows.

3. The Coexistence Problem

The main result is the following.

Theorem 1. *Assume* (4). *If* $J < \infty$, *then for equation* (1) *the subclass* $M^-_{0,\,-\infty}$, *is empty.*

This result will be proved by using some integral inequalities, a half linearization technique and a sharp asymptotic estimate for nonoscillatory solutions of (1). We start by stating the decay rate of intermediate solutions of (1). The following holds.

Lemma 4. *If* (4) *holds,* $J < \infty$, *and* $0 < \alpha < 1$, *then for any* $x \in M^-_{0,\,-\infty}$ *we have*

$$\liminf_{t\to\infty}\left|\frac{x^{\beta}(t)}{x^{[1]}(t)}\right|\int_{t_0}^{t} b(r)\,dr > 0.$$

Proof. Without loss of generality, suppose for $t \geq t_1 \geq t_0$

$$0 < x(t) < 1, \quad x^{[1]}(t) < 0. \tag{12}$$

Integrating (1), we get for $t \geq t_1$

$$x^{[1]}(t) = x^{[1]}(t_1) - \int_{t_1}^{t} b(s)x^{\beta}(s)\,ds. \tag{13}$$

In view of Lemma 3 we have

$$x^{\beta}(t) = \left(\int_t^{\infty} |\,x'(r)\,|\,dr\right)^{\beta} \leq \sigma_{\beta}\left(\int_t^{\bar{t}} |\,x'(r)\,|\,dr\right)^{\beta} + \sigma_{\beta}\left(\int_{\bar{t}}^{\infty} |\,x'(r)\,|\,dr\right)^{\beta},$$

where $\bar{t} \geq t$ and σ_{β} is defined in (11). Hence, from (13) we obtain

$$x^{[1]}(\bar{t}) \geq x^{[1]}(t_1) - \sigma_{\beta}\int_{t_1}^{\bar{t}} b(r)\left[\left(\int_r^{\bar{t}} |\,x'(s)\,|\,ds\right)^{\beta} + \left(\int_{\bar{t}}^{\infty} |\,x'(s)\,|\,ds\right)^{\beta}\right] dr \tag{14}$$

$$x^{[1]}(t_1) - \sigma_{\beta}\int_{t_1}^{\bar{t}} b(r)\left(\int_r^{\bar{t}} |\,x'(s)\,|\,ds\right)^{\beta} dr - \sigma_b\left(\int_{\bar{t}}^{\infty} |\,x'(s)\,|\,ds\right)^{\beta}\int_{t_1}^{\bar{t}} b(r)\,dr.$$

Setting

$$M = \beta\left(\frac{1-\alpha}{\beta-\alpha}\right)^{1-\alpha},$$

in view of Lemma 2 with $\lambda = \beta$, $\mu = \alpha^{-1}$, $f(\tau) = |\,x'(\tau)\,|$ and $g(r) = b(r)$, we obtain

$$\int_{t_1}^{\bar{t}} b(r)\left(\int_r^{\bar{t}} |\,x'(s)\,|\,ds\right)^{\beta} dr$$

$$\leq M\left(\int_{t_1}^{\bar{t}} |\,x'(\tau)\,|\left(\int_{t_1}^{\tau} b(r)\,dr\right)^{1/\alpha} d\tau\right)^{\alpha}\left(\int_{t_1}^{\bar{t}} |\,x'(\tau)\,|\,d\tau\right)^{\beta-\alpha}.$$

Since, in view of (12), we have

$$0 \le \int_{t_1}^{\bar{t}} |\, x'(\tau)\,|\, d\tau = x(t_1) - x(\bar{t}) < 1,$$

taking into account that $|x^{[1]}|$ is nondecreasing for $t \ge t_1$, we get

$$\int_{t_1}^{\bar{t}} b(r) \left(\int_r^{\bar{t}} |\, x'(s)\,|\, ds \right)^{\beta} dr$$

$$\le M \left(\int_{t_1}^{\bar{t}} \left(\frac{1}{a(\tau)} \right)^{1/\alpha} |x^{[1]}(\tau)|^{1/\alpha} \left(\int_{t_1}^{\tau} b(r)\, dr \right)^{1/\alpha} d\tau \right)^{\alpha}$$

$$M\, |x^{[1]}(\bar{t})| \left(\int_{t_1}^{\bar{t}} \left(\frac{1}{a(\tau)} \int_{t_1}^{\tau} b(r)\, dr \right) d\tau \right)^{1/\alpha}.$$

Thus, from (14) we have

$$x^{[1]}(\bar{t}) - x^{[1]}(t_1)$$

$$\ge -\sigma_\beta M\, |x^{[1]}(\bar{t})| \left(\int_{t_1}^{\bar{t}} \left(\frac{1}{a(\tau)} \int_{t_1}^{\tau} b(r)\, dr \right)^{1/\alpha} d\tau \right)^{\alpha}$$

$$-\sigma_\beta \left(\int_{\bar{t}}^{\infty} |\, x'(s)\,|\, ds \right)^{\beta} \int_{t_1}^{\bar{t}} b(r)\, dr.$$

Since $x^{[1]}(\bar{t}) < 0$ and

$$\left(\int_{\bar{t}}^{\infty} |\, x'(s)\,|\, ds \right)^{\beta} = x^{\beta}(\bar{t}),$$

we obtain

$$1 - \frac{x^{[1]}(t_1)}{x^{[1]}(\bar{t})} \le \sigma_\beta M \left(\int_{t_1}^{\bar{t}} \left(\frac{1}{a(\tau)} \int_{t_1}^{\tau} b(r)\, dr \right)^{1/\alpha} d\tau \right)^{\alpha} - \sigma_\beta \frac{x^{\beta}(\bar{t})}{x^{[1]}(\bar{t})} \int_{t_1}^{\bar{t}} b(r)\, dr$$

or

$$\sigma_\beta \frac{x^{\beta}(\bar{t})}{\left| x^{[1]}(\bar{t}) \right|} \int_{t_1}^{\bar{t}} b(r)\, dr \ge 1 - \sigma_\beta M \left(\int_{t_1}^{\bar{t}} \left(\frac{1}{a(\tau)} \int_{t_1}^{\tau} b(r)\, dr \right)^{1/\alpha} d\tau \right)^{\alpha} - \frac{x^{[1]}(t_1)}{x^{[1]}(\bar{t})}.$$

Since $J < \infty$ and $\lim_{\bar{t} \to \infty} x^{[1]}(\bar{t}) = -\infty$, the assertion follows.

Now we are in position to prove Theorem 1.

Proof of Theorem 1. By contradiction, suppose that there exists $x \in M^{-}_{0, -\infty}$ such that $0 < x(t) \le 1, x^{[1]}(t) < 0$ on $[t_1, \infty), t_1 \ge t_0$ and $\lim_{t\to\infty} x(t) = 0$.

First, suppose $0 < \alpha < 1$. In view of Lemma 4, there exist $\varepsilon > 0$ such that for $t \ge \bar{t} \ge t_1$

$$x^{\beta}(t)\int_{t_0}^{t} b(r)\,dr \ge \varepsilon |x^{[1]}(t)|. \tag{15}$$

Without loss of generality, suppose

$$\int_{\bar{t}}^{\infty}\left(\frac{1}{a(s)}\int_{t_0}^{s} b(r)\,dr\right)^{1/\alpha} ds < \varepsilon^{1/\alpha}\Big/2 \tag{16}$$

From the equality

$$x(\bar{t}) - x(t) = -\int_{\bar{t}}^{t} x'(r)\,dr = \int_{\bar{t}}^{t}\left(\frac{|x^{[1]}(r)|}{a(r)}\right)^{1/\alpha} dr,$$

in view of (15), we have for $t \ge \bar{t}$

$$x(\bar{t}) - x(t) \le \left(\frac{1}{\varepsilon}\right)^{1/\alpha}\int_{\bar{t}}^{t}\left(\frac{1}{a(s)}\right)^{1/\alpha} x^{\beta/\alpha}(s)\left(\int_{t_0}^{s} b(r)\,dr\right)^{1/\alpha} ds.$$

Since x is positive decreasing, $x(s) \le 1$, and $\beta > \alpha$, in view of (16) we obtain

$$x(\bar{t}) - x(t) \le \left(\frac{1}{\varepsilon}\right)^{1/\alpha} x(\bar{t})\int_{\bar{t}}^{t}\left(\frac{1}{a(s)}\right)^{1/\alpha}\left(\int_{t_0}^{s} b(r)\,dr\right)^{1/\alpha} ds \le \frac{1}{2}x(\bar{t}).$$

Thus

$$\frac{1}{2}x(\bar{t}) \le x(t),$$

which gives a contradiction as $t \to \infty$, because $\lim_{t\to\infty} x(t) = 0$.

Now, assume $1 \le \alpha$. Consider on $[t_1, \infty)$ the half-linear equation

$$(a(t)|z'(t)|^{\alpha}\operatorname{sgn} z'(t))' + b_x(t)|z(t)|^{\alpha}\operatorname{sgn} z(t) = 0, \tag{17}$$

where

$$b_x(t) = b(t) x^{\beta-\alpha}(t).$$

Clearly, $z = x$ is a solution of (17) in the class $M^-_{0,\,-\infty}$. We claim that

$$\int_{t_1}^{\infty} b_x(t)\, dt = \infty. \tag{18}$$

Indeed, integrating (17) on $(t_1, \bar{t})$, we get

$$x^{[1]}(t_1) - x^{[t]}(\bar{t}) = z^{[1]}(t_1) - z^{[1]}(\bar{t}) = \int_{t_1}^{\bar{t}} b_x(s) z^{\alpha}(s)\, ds$$

or, since z is decreasing for $t \geq t_1$,

$$x^{[1]}(t_1) - x^{[1]}(\bar{t}) \leq z^{\alpha}(t_1) \int_{t_1}^{\bar{t}} b_x(s)\, ds,$$

which gives (18), because $\lim_{\bar{t}\to\infty} x^{[1]}(\bar{t}) = -\infty$.

Since $x(t) \leq 1$ on $[t_1, \infty)$, we have $b_x(t) \leq b(t)$. Hence, from $J < \infty$ we obtain

$$\int_{t_1}^{\infty} \left(\frac{1}{a(s)} \int_{t_1}^{s} b_x(r)\, dr \right)^{1/\alpha} ds < \infty.$$

Applying Proposition 1, we get that (17) has solutions also in the class $\mathbb{M}^-_{0,\,-\ell}$. In view of Proposition 2, the three types of nonoscillatory solutions cannot coexist in the half-linear case. Thus, the class $\mathbb{M}^-_{\ell,\,-\infty}$ is empty for (17) and, again from Proposition 1, we obtain

$$\int_{t_1}^{\infty} b_x(s) \left(\int_{s}^{\infty} a^{-1/\alpha}(r)\, dr \right)^{\alpha} ds = \infty,$$

which implies $Z = \infty$. Since $\alpha \geq 1$, applying Lemma 1 we get $J = \infty$, which is a contradiction.

From Theorem 1 we get the following coexistence result.

Corollary 1. *Assume* (4). *For* (1) *solutions in the class* $\mathbb{M}^-_{\ell,\,-\infty}$ *cannot coexist with solutions in the class* $\mathbb{M}^-_{0,\,-\infty}$. *Consequently, at most two of the* *subclasses* $\mathbb{M}^-_{0,\,-\ell}$, $\mathbb{M}^-_{0,\,-\infty}$, $\mathbb{M}^-_{\ell,\,-\infty}$ *are nonempty for* (1).

Proof. Let $z \in \mathbb{M}^-_{\ell,\,-\infty}$. From Proposition 1 we have $J < \infty$. Hence, Theorem 1 gives the assertion.

A necessary condition for the existence of solutions in the class $\mathbb{M}^-_{0,\,-\infty}$ is given by the following.

Corollary 2. *If* (4) *holds and* (1) *has solutions in the class* $\mathbb{M}^-_{0,\,-\infty}$, *then* $J = \infty$, $Y < \infty$. *Moreover*, (1) *has also infinitely many solutions in the class* $\mathbb{M}^-_{0,\,-\ell}$ *and every nonoscillatory solution of* (1) *tends to zero as* $t \to \infty$, i.e. $\mathbb{M}^-_{\ell,\,-\infty} = \varnothing$.

Proof. Let us show that $J = \infty$. By contradiction, assume $J < \infty$. Now, from Lemma 1-(i_2) we get $Y < \infty$ and so, in virtue of Proposition 1, both classes $\mathbb{M}^-_{0,\,-\ell}$, $\mathbb{M}^-_{\ell,\,-\infty}$ are nonempty. Since (1) has also solutions in the class $\mathbb{M}^-_{0,\,-\infty}$. we obtain a contradiction with Corollary 1. Thus $J = \infty$ and $\mathbb{M}^-_{\ell,\,-\infty} = \varnothing$. Moreover, since (1) has nonoscillatory solutions, again from Proposition 1, we get $Y < \infty$ and the assertion follows.

The following examples illustrate Corollary 1.

Example 1. Consider for $t \geq e$ the equation

$$(t \log t \,|\, x' \,|^{\alpha} \operatorname{sgn} x')' + \frac{|\sin t|}{t} |\, x' \,|^{\beta} \operatorname{sgn} x' = 0$$

where $0 < \alpha < 1$ and $\beta > \alpha$. Clearly, (2) and (4) hold. From

$$\int_e^T \frac{1}{a^{1/\alpha}(s)} \left(\int_e^s b(r)\, dr \right)^{1/\alpha} ds \leq \int_e^T \left(\frac{\log t - 1}{t \log t} \right)^{1/\alpha} dt$$

we obtain $J < \infty$. Thus, in view of Theorem 1, we have $\mathbb{M}^-_{0,\,-\infty} = \varnothing$. Moreover, from Lemma 1 we get $Y < \infty$ and so, in virtue of Proposition 1, both classes $\mathbb{M}^-_{0,\,-\ell}$, $\mathbb{M}^-_{\ell,\,-\infty}$ are nonempty.

Example 2. Consider for $t \geq 1$ the equation

$$(t|x'|^{1/5}\operatorname{sgn} x')' + c_0 t^{4/15}|x|^{1/3}\operatorname{sgn} x = 0, \tag{19}$$

where

$$c_0 = \frac{1}{10}\left(\frac{7}{2}\right)^{1/5}.$$

Clearly, (2) and (4) holds. It is easy to shows that $x(t) = t^{-7/2}$ is a solution of (19) and $x^{[1]}(t) = -(7/2)^{1/5}t^{1/10}$. Since $\lim_{t\to\infty} x^{[1]}(t) = -\infty$, we have $x \in \mathbb{M}^-_{0,\,-\infty}$. Moreover, a standard calculation shows that $Y < \infty$. Hence, from Proposition 1 and Corollary 1, for equation (19) we have $\mathbb{M}^-_{0,\,-\ell} \neq \varnothing$ and $\mathbb{M}^-_{\ell,\,-\infty} = \varnothing$.

4. Concluding Remarks

1. A summary. For the reader's convenience, we briefly summarize the situation when (P_2) holds. In this case, any solution $x \in \mathbb{P}$ is nondecreasing for large t and the class $\mathbb{P}$ can be divided into the three subclasses, [13]:

$$\begin{aligned} &\mathbb{M}^+_{\infty,\,\ell} = \{x \in \mathbb{P} : x(\infty) = \infty,\, x^{[1]}(\infty) = \ell_x,\, 0 < \ell_x < \infty\},\\ &\mathbb{M}^+_{\infty,\,0} = \{x \in \mathbb{P} : x(\infty) = \infty,\, x^{[1]}(\infty) = 0\}, \\ &\mathbb{M}^+_{\ell,\,0} = \{x \in \mathbb{P} : x(\infty) = \ell_x,\, x^{[1]}(\infty) = 0,\, 0 < \ell_x < \infty\}, \end{aligned} \tag{20}$$

The superscript symbol + means that solutions are eventually positive increasing. Solutions in $\mathbb{M}^+_{\infty,\,\ell}$, $\mathbb{M}^+_{\infty,\,0}$, and $\mathbb{M}^+_{\ell,\,0}$ continue to be referred as dominant solutions, intermediate solutions and subdominant solutions, respectively, since if $x \in \mathbb{M}^+_{\infty,\,\ell}$, $y \in \mathbb{M}^+_{\infty,\,0}$, $z \in \mathbb{M}^+_{\ell,\,0}$ we have for large t

$$z(t) \leq y(t) \leq z(t).$$

The problem concerning the coexistence of these three types of nonoscillatory solutions has been solved in a negative way in [25] in the sublinear case, and in [9] in the superlinear case. Summarizing these results with ones in [17] and using Corollary 1, we get the following complete answer to the question posed in [2, 24] on the triple coexistence of nonoscillatory solutions of (1).

Using the terminology of subdominant solutions, intermediate solutions and dominant solutions for classes $\mathbb{M}^-_{0,-\ell}$, $\mathbb{M}^-_{0,-\infty}$ and $\mathbb{M}^-_{\ell,-\infty}$ defined in case (P_1) by (10) and for classes $\mathbb{M}^+_{\ell,0}$, $\mathbb{M}^+_{\infty,\ell}$ and $\mathbb{M}^+_{\infty,0}$ defined in case (P_2) by (20) respectively, the following holds.

Corollary 3. *Assume* (2) *and either* (P_1) *or* (P_2). *Then* (1) *does not have simultaneously subdominant solutions*, *intermediate solutions and dominant solutions*.

In other words, the triple coexistence for nonoscillatory solutions of (1) is impossible for any positive value of α, β independently on the convergence of the integrals I_a, and I_b.

2. The sublinear case. In case (P_1), necessary and suficient conditions for the existence of subdominant solutions and dominant solutions are given in Proposition 1. Observe that Proposition 1 can be applied also when $\alpha > \beta > 0$.

In the sublinear case, the intermediate solutions can be also characterized by means of the convergence of integrals J and Y, as the following result shows.

Theorem 2. [17] *Let* $\alpha > \beta > 0$ *and assume that* (2) *and the case* (P_1) *hold*. *Then* (1) *has solutions in the class* $\mathbb{M}^-_{0,-\infty}$ if and only if $J < \infty$ and $Y = \infty$.

We recall that in the sublinear case, the asymptotic growth of intermediate solutions of (1) can be found in [14, 20], where the study is accomplished in the framework of regular variation. This approach is motivated by the monograph of Marić [22] which provides a powerful tool for obtaining a precise asymptotic analysis of various kinds of nonoscillatory solutions. The papers [20, 14] require additional assumptions on the function

b, in particular the positiveness of b. Since Theorem 2 is valid when b has a sequence of zeros, clustering at infinity, it should be interesting to obtain asymptotics for intermediate solutions also in this case.

3. The superlinear case. In the superlinear case, in spite of many examples of equations of type (1) having solutions in the class $\mathbb{M}^{-}_{0,\,-\infty}$ which can be easily produced, until now no necessary and sufficient conditions for their existence are known. This difficulty is due to the problem of finding sharp upper and lower bounds, [1, page 241], [20, page 2].

When (4) holds, sufficient conditions for existence of intermediate solutions can be found in [9, 10] where monotonicity properties of a suitable energy function are used. Nevertheless, we point out that [9, 10] require additional assumptions on the function b, like the positiveness. Now, the following question arises. In the superlinear case, does equation (1) have intermediate solutions when b has a sequence of zeros, clustering at infinity? Recall that in the superlinear case equation (1) can have non-continuable solutions, if b vanishes at some $t_1 > t_0$, [27]. For this reason, the above question seems very difficult.

4. A change of variable. For any solution x of (1), consider the change of variable $z = x^{[1]}$. Then z is a solution of the dual equation to (1)

$$(b^{-1/\beta}(t)|z'|^{1/\beta}\operatorname{sgn} z')' + a^{-1/\alpha}(t)|z'|^{1/\alpha}\operatorname{sgn} z = 0 \tag{21}$$

and $z^{[1]}(t) = b^{-1/\beta}(t)|z'|^{1/\beta}\operatorname{sgn} z' = -x(t)$. Equation (21) is obtained from (1), when a is replaced by $b^{-1/\beta}$ and b by $a^{-1/\alpha}$. Moreover β^{-1} plays the role of α and vice versa. Thus, this transformation maps superlinear [sublinear] equations of type (1) into superlinear [sublinear] equations of the same type. It is easy to shows that the condition $(P_1)\,[(P_2)]$ for (1) becomes the condition $(P_2)\,[(P_1)]$ for (21). Consequently, it is often used in the literature for deriving the corresponding results for the case (P_1) from known results for the case (P_2) or vice-versa. Since this transformation requires the positivity of b, this transformation does not work in the case considered in this paper. We refer to [26, page 94] for more details on this topic.

5. Extensions

In the discrete case, for the half-linear difference equation

$$\Delta(a_n \mid \Delta x_n|^{\alpha} \operatorname{sgn} \Delta x_n) + b_n| x_{n+1}|^{\alpha} \operatorname{sgn} x_{n+1} = 0,$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, the coexistence problem between nonoscillatory solutions with a different growth at infinity has been completely solved in [5, Theorem 3.1.], for any positive value of α, independently of the convergence of the series

$$S_a = \sum_{n=0}^{\infty}\left(\frac{1}{a_n}\right)^{1/\alpha}, \quad S_b = \sum_{n=0}^{\infty} b_n.$$

Later on, a partial answer to the same problem for the Emden-Fowler type discrete equation

$$\Delta(a_n \mid \Delta x_n|^{\alpha} \operatorname{sgn} \Delta x_n) + b_n| x_{n+1}|^{\beta} \operatorname{sgn} x_{n+1} = 0,$$

has been given in [6, Theorems 2.4 and 3.3]. It would be interesting to complete the study on the coexistence both for Emden-Fowler difference equations and dynamic equations on time-scale.

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