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Transverse invariants from deformations of Khovanov $\mathfrak{sl}_2$- and $\mathfrak{sl}_3$-homologies

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MY soul is an entangled knot,
Upon a liquid vortex wrought
By Intellect, in the Unseen residing,
And thine cloth like a convict sit,
With marlinspike untwisting it,
Only to find its knottiness abiding;
Since all the tools for its untying
In four-dimensioned space are lying

Extract from “A Paradoxical Ode“
by James Clerk Maxwell

Contents

Introduction

The aim of this thesis is the study of transverse link invariants coming from Khovanov $\mathfrak{sl}_2$- and $\mathfrak{sl}_3$-homologies and from their deformations. As a by-product of our work we get computable estimates on some concordance invariants coming from Khovanov $\mathfrak{sl}_2$-homologies. In order to explain the meaning of the the previous sentences it is necessary to give some definitions.

1. Links in contact manifolds and effective invariants

A contact manifold (Definition 3.1) is an odd-dimensional manifold $\mathcal{M}$ endowed with a totally non-integrable hyperplane field ξ – intuitively speaking, a plane field which twists too much to allow the existence of a co-dimension one local sub-manifold of $\mathcal{M}$ admitting ξ as tangent space. The simplest contact manifold is probably $\mathbb{R}^3$ endowed with the symmetric contact structure ξ_{sym} (see Chapter 3, Section 1, for the definition). Despite its simplicity, the study of the contact manifold $(\mathbb{R}^3, \xi_{sym})$ is fundamental to understand the "zoology" of the contact structures over any 3-manifold. In fact, on the one hand $(\mathbb{R}^3, \xi_{sym})$ provides a local model for all contact 3-manifolds (cf. Darboux's theorem [**16**, Theorem 2.5.1] and [**14**, Example 2.1]). On the other hand, any contact closed 3-manifold can be obtained from the one point compactification of $(\mathbb{R}^3, \xi_{sym})$, namely $(\mathbb{S}^3, \xi_{sym})$, via surgical constructions along links which are compatible, in some sense, with the contact structure (cf. [**16**, Theorems 6.4.4 & 7.3.5]).

There are two natural ways for a link λ in a 3-manifold $\mathcal{M}$ to be compatible with a contact structure ξ on $\mathcal{M}$:

- ▷ λ is everywhere tangent to the contact structure;
- ▷ λ is be everywhere transverse to the contact structure;

the first type of links are called Legendrian, while the second type of links are called transverse. Two members of each of these families of links are considered equivalent if they are ambient isotopic through links in the same family.

Legendrian and transverse links have been extensively studied, and they have some classical invariants: the link type, the Thurston-Bennequin number and the rotation number for Legendrian links, and the link type and the self-linking number for transverse links. These invariants are complete invariants for some families of links (e.g. the unknot, the torus links, the figure eight knot etc.) but in general they are not good at distinguishing transverse or Legendrian links. There are infinite families of links (or even knots) whose members share the same classical invariants even though they are distinct as transverse or Legendrian links (e.g. some cablings of torus knots). An invariant of Legendrian, or transverse, links which can tell apart two different Legendrian, respectively transverse, links with the same classical invariants is called **effective**.

The study of Legendrian and transverse links gives also information on the topology of the contact three manifold. For instance, Rudolph in [**49**] used an inequality to relate the classical invariants of Legendrian and transverse links with the slice genus of the underlying link. Other similar inequalities, called **Bennequin-type inequalities**, have been produced through the years.

For Legendrian links there are lots of well known effective invariants, for example the Chekanov-Eliashberg DGA [**12**], the LOSS invariants coming from knot Floer Homology [**36**], the invariants $\lambda_{\pm}$ coming from the grid version of knot Floer Homology [**43**] etc. and some of them give rise to effective transverse invariants, e.g. the θ invariant coming from grid Floer homology [**43**].

The work of various authors (for example [**36, 43, 45, 34, 60, 59**]) has established that link homologies, such as HFK and Khovanov-Rozansky homologies, contain not only topological but also Legendrian and transverse information. The Legendrian and transverse information contained in Heeegaard Floer type theories is, in some sense, more explicit. Perhaps this is due to the more direct connection of the Heegaard Floer theories with the topology of three manifolds. On the other hand, the Legendrian and transverse information in the Khovanov-Rozansky homologies is less understood. So, in the hope of shedding new light on this subject we investigated the presence of transverse invariants in (the deformations of) Khovanov $\mathfrak{sl}_2$- and $\mathfrak{sl}_3$-homologies.

2. Transverse invariants in Khovanov-Rozansky homologies

Let R be the ring $\mathbb{F}[U_1, ..., U_s]$, where $\mathbb{F}$ is a field and $deg(U_i) = 2k_i$. A **potential** ω for the (equivariant) Khovanov-Rozansky $\mathfrak{sl}(N)$-homology is an homogeneous polynomial in the ring $R[X]$, where $deg(X) = 2$, of the form

$$\omega(X, U_1, ..., U_s) = X^{N+1} + XF(X, U_1, ..., U_s), \qquad F(X, 0, ..., 0) = 0.$$

Given a oriented link diagram L, one can associate to L a bi-graded chain complex $(C_{\omega}^{\bullet,\bullet}(L), d_{\chi})$ over the ring R. Where the first grading is the homological grading and the second grading is induced by the polynomial grading on $R[X]$. The differential d_{χ} is homogeneous with respect to both gradings. The homology of this complex is called the **equivariant Khovanov-Rozansky $\mathfrak{sl}(N)$-homology** (over R with respect to ω).

The chain module $C_{\omega}^{\bullet,\bullet}(L)$ is itself obtained as the homology of another complex, here denoted $(M_{\omega}(L), \partial)$. This complex is obtained from the diagram D by associating to each MOY-resolution[1] of L, say Γ, a matrix factorization $(M(\Gamma), \partial_{\Gamma})$, and to the diagram L the direct sum of these complexes. The complex $M_{\omega}(L)$ is

[1]A web resolution with markers. For the definition of web resolution the reader may consult Chapter 5 Section 1.

triply graded: it has a $\mathbb{Z}/2\mathbb{Z}$-degree coming from the degree of the matrix factorizations, an homological degree coming from the type of resolution performed, and a polynomial grading.

The whole construction is technically quite complex. The complexity of the theory makes it difficult to work with the chain complex. This is the main reason why in this thesis we work with a potential ω of degree either 3 or 4. In these cases it is possible to bypass some of the technical difficulties using different approaches. The general case will be the subject of a future work by the author.

When $deg_X(\omega) = 3$ we will use an approach similar to the original definition of Khovanov homology (cf. [**25, 27**]). When $deg_X(\omega) = 4$ we use the approach introduced by Khovanov ([**26**]) for $\omega = X^4$, and generalized by Mackaay and Vaz ([**40**]) to arbitrary potentials of degree 4, which uses foams. The construction via matrix factorizations described above will be hidden in both these approaches. Even the choice of the potential will be partially hidden; both the constructions we are going to describe in this thesis will depend on the choice of a polynomial, which corresponds to the derivative (with respect to X) of the potential.

The idea of defining transverse invariants using Khovanov-Rozansky homologies is not new. In 2006 Olga Plamenevskaya defined a transverse invariant ψ which is an homology class in the Khovanov homology of a transverse link (presented as a closed braid). The Plamenevskaya invariant ψ was generalized in 2008, by Hao Wu (see [**59**]). Wu defined an invariant ψ_n which is an homology class in the non-deformed Khovanov-Rozansky $\mathfrak{sl}_N$-homology (that is the theory with potential $\omega = X^N$).

In 2013, Robert Lipshitz, Lenhard Ng and Sucharit Sarkar ([**34**]) defined two trasverse link invariants, denoted by $\psi^{\pm}$, which are classes in (a twisted version of) Lee's deformation of Khovanov Homology (i.e. the theory with potential $\frac{1}{3}(X^3 - 3X)$). Moreover, Lipshitz, Ng and Sarkar defined a family of transverse link invariants $\psi_{p,q}$ living into an appropriate quotient of the graded object (with respect to a natural filtration) associated to Lee Homology.

All these invariant are suspected to be non-effective. However, their effectiveness is still unknown, also because of the lack of examples of trasverse links with the same classical invariants and a high number of crossings. These invariants have been proved to be non-effective in distinguishing flype equivalent, transversally nonequivalent, links, which were first studied by Joan Birman and William Menasco ([**8**]).

3. Outline of contents and main results

Our work is divided into two main parts and some appendices. The first part is concerned with the study of transverse invariants in the deformations of Khovanov's $\mathfrak{sl}_2$-theory. The second part deals with the study of transverse invariants in the deformations of Khovanov's $\mathfrak{sl}_3$-theory.

Part I: Transverse invariants in Khovanov-type $\mathfrak{sl}_2$-theories. In the first part of this thesis we deal with the deformations of Khovanov $\mathfrak{sl}_2$-homology. We introduce two transverse braid invariants in the Bar-Natan deformation of Khovanov homology, called β-invariants, and study their properties. In particular, we prove that they are equivalent to the invariants $\psi^{\pm}$, and that they recover the Plamenevskaya invariant. Furthermore, using the peculiar structure of Bar-Natan homology we are able to extract from the β-invariants two numerical invariants, which we call c-invariants. We also provide computable bounds on the value of the c-invariants. As a by-product of these estimates we provide a new computable bound for Rasmussen's s-invariant for links. Finally, we investigate the presence of similar transverse invariants in a general deformation of Khovanov $\mathfrak{sl}_2$-homology. We manage to give some sufficient conditions for the existence of transverse invariants and to study some of their properties. We obtain a correspondence between these invariants and polynomial invariants for transverse braids.

The first part is divided into four chapters.

The first chapter reviews the construction of a Khovanov-type link homology theory (i.e. a deformation of the Khovanov $\mathfrak{sl}_2$-homology) starting from a Frobenius algebra. We start by introducing the notion of Frobenius algebra, including its graded and filtered versions, and of morphism of Forbenius algebras. Then, we describe some constructions that can be used to modify a Frobenius algebra (namely twist, base change and dual). Finally, we define Khovanov-type homology theories, describe how to construct a bi-graded (resp. filtered) complex from a link diagram and a graded (resp. filtered) Frobenius algebra, and analyse the relationship between the mirror image and the complex associated to the dual Frobenius algebra.

The second chapter is an overview of the Bar-Natan homology theory (which is a particular Khovanov-type homology theory). In particular, we focus on the bi-graded structure of Bar-Natan homology and its structure of $\mathbb{F}[U]$-module. We provide a description of the structure of bi-graded $\mathbb{F}[U]$-module of Bar-Natan homology in terms of known invariants. We prove that "the free part"[2] of Bar-Natan homology is completely determined by the linking matrix and John Pardon's concordance invariants. Finally, we briefly review the definition of some concordance invariants defined in Lee theory.

In the third chapter we deal with some transverse invariants coming from Bar-Natan theory. After a brief review of some basic contact topology, which occupies the first section, in the second section we define the β-cycles. These are two distinguished elements in the Bar-Natan chain complex of a link diagram L, and they are denoted by $\beta(L)$ and $\bar{\beta}(L)$. We prove that the β-invariants are also cycles and that their homology classes generate a rank 2 $\mathbb{F}[U]$-sub-module of

[2]The free part of a module over a PID is not a canonically defined sub-module. However, the isomorphism class (as a graded module) of the free part is unique.

Bar-Natan homology. In particular, the homology classes of the β-cycles are non-trivial. Moreover, we prove the following statement which combines Propositions 3.8, 3.12, 3.16, 3.18 and Corollary 3.17.

Theorem. *Let L be a oriented link diagram. Then, the following assertions hold:*

(A) if L' is obtained by performing a positive first Reidemeister move, then

$$\Phi(\beta(L)) = \beta(L') \quad \textit{and} \quad \Psi(\beta(L')) = \beta(L);$$

(B) if L' is obtained by performing a negative first Reidemeister move, then

$$\pm U\Phi_*([\beta(L)]) = [\beta(L')];$$

(C) if L' is obtained by performing a braid-like second or third Reidemeister move, then

$$\Phi(\beta(L)) = \beta(L') \quad \textit{and} \quad \Psi(\beta(L')) = \beta(L);$$

where

$$\Phi : C^{\bullet}_{BN}(L) \longrightarrow C^{\bullet}_{BN}(L') \quad \textit{and} \quad \Psi : C^{\bullet}_{BN}(L') \longrightarrow C^{\bullet}_{BN}(L)$$

are maps associated to a Reidemeister move and to its inverse, respectively. In particular, if B is the closure of a braid diagram, the $\beta(B)$ is a transverse braid invariant. Moreover, $\beta(B)$ is is flype invariant.

The previous result can be re-stated without changes replacing $\beta(L)$ with $\bar{\beta}(L)$.

Furthermore, we prove the equivalence between the β-invariants and the $\psi^{\pm}$-invariants and establish the relationship between the β-invariants and the $\psi_{p,q}$ invariants. In the third section, we use the $\mathbb{F}[U]$-module structure of Bar-Natan homology to define two numerical invariants for transverse braids, namely the c-invariants. The c-invariants $c(L)$ and $\bar{c}(L)$ are defined as the maximal power of U which divides the homology classes of $\beta(L)$ and $\bar{\beta}(L)$, respectively. We manage to give some estimates from below on their values. More precisely, the following result holds.

Theorem (Corollaries 3.35, 4.4& 3.40). *Let L be a link diagram. Then, the following inequality holds*

$$o_-(L_-) - \ell_-(L_-) \geq max\{c(L), \bar{c}(L)\} \geq min\{c(L), \bar{c}(L)\} \geq o_-(L) - \ell_-(L) + \delta_-(L),$$

where $o_-(L)$, $\ell_-(L)$ and $\delta_-(L)$ are combinatorial quantities which can be easily computed from the diagram L. In particular, if L is negative the bound is sharp.

Then, we study the behaviour of the β-invariants under crossing changes, obtaining the following result.

Proposition (Corollary 3.39). *If L_+ and L_- are two oriented link diagrams, which differ only in a crossing which is positive in L_+, and negative in L_-. Then,*

$$\Phi_{+\to-}(\beta(L_+, R)) = \beta(L_-, R), \quad \Phi_{-\to+}(\beta(L_-, R)) = \pm U^2\beta(L_+, R),$$

where $\Phi_{+\to-}$ and $\Phi_{-\to+}$ are maps between the Bar-Natan complexes associated to the crossing changes. In particular, $c_R(L_-) \geq c_R(L_+)$. □

The same result holds replacing β with $\bar{\beta}$ and c with $\bar{c}$.

We conclude the chapter with a fourth section. This section is dedicated to the generalizations of the both the β-invariants and the c-invariants to other deformations of Khovanov homology. In particular, we give a sufficient condition for the existence of transverse braid invariants in a large class of Khovanov-type chain complexes. Furthermore, we investigate the presence of other transverse invariants similar to the β-invariants in Bar-Natan homology. As a result we prove that there is a bijection between the set of such invariants and the set of polynomial invariants of transverse braids. Finally,we conclude by proving the non-effectiveness of the c-invariants for all prime knots with less than 12 crossings.

The final chapter of this part of this thesis is dedicated to a Bennequin-type inequality arising from the estimates on the values of the c-invariants. More precisely, we prove the following result.

Theorem (Theorem 4.2). *Let L be an oriented link diagram representing the link type λ, then*

$$(s\text{-ineq})\qquad w + o - 2o_{+} + 2\ell_{+} - 2\delta^{+} + 1 \geq s(\lambda) \geq w - o + 2o_{-} - 2\ell_{-} + 2\delta^{-} + 1$$

where $o_{-}(L)$, $o_{+}(L)$, $\ell_{-}(L)$, $\ell_{+}(L)$, $\delta_{-}(L)$ and $\delta_{+}(L)$ are combinatorial quantities which can be easily computed from the diagram L.

We provide an ample set of examples in which this bound is sufficient to compute Rasmussen's s-invariant for links. Moreover, we prove that (s-ineq) is independent of any known bound.

Part II: Transverse invariants in Khovanov-type $\mathfrak{sl}_3$-theories. The second part of this thesis has a structure which is similar to the first part. We use Khovanov-Mackaay-Vaz foam technology to find transverse invariants in (the deformations of) Khovanov $\mathfrak{sl}_3$ homology theory. We give necessary and sufficient conditions for a deformation Khovanov $\mathfrak{sl}_3$-homology to contain such an invariant. Then, we specialize to two particular deformations, and use their structure of $\mathbb{F}[U]$-modules to recover numerical transverse braid invariants. We also prove that these two specialized invariants recover Wu's invariant ψ_3. Finally, we provide some bounds on the values of the numerical invariants.

This second part of this thesis is divided into two chapters.

The first chapter is a review of the foam technology and the universal $\mathfrak{sl}_3$-link homology theory. We start by revising some basic material regarding foams. Then, we describe how to obtain a link homology theory, whose chain complex is denoted by $C^{\bullet}_{p}(L, R)$ from a monic polynomial $p(x)$ of degree 3 in x using foams.

In the second chapter we define a set of transverse braid invariants in $C^{\bullet}_{p}(L, R)$, called β_3-invariants. These invariants are elements of $C^{\bullet}_{p}(L, R)$, and they are in bijection with the set of distinct roots of $p(x)$. In particular, we recover Wu's ψ_3-invariant.

Theorem (Corollary 6.7). *Given a polynomial $p(x) \in R[x]$, such that $deg_x(p) = 3$, and a root of $p(x)$, say x_1, there is an element $\beta_3(L; p, x_1) \in C_p^\bullet(L, R)$ such that $\beta_3(\overline{B}; p, x_1)$ is a transverse braid invariant.*

Afterward, we specialize our construction to the case $p(x) = x^3 - U^3$ and $R = \mathbb{C}[U]$ and use the structure of $\mathbb{C}[U]$-module of the theory to define three numerical invariants, called c_3-invariants, and give some bounds on their values. More precisely we prove the following result.

Theorem (Corollary 6.13). *Let L be an oriented link diagram. Then,*

$$c_{U\xi_3^i}(L, \mathbb{C}) \geq 2(o_+(L) - \ell_+(L))), \quad \forall i \in \{0, 1, 2\},$$

where $c_{U\xi_3}(L, \mathbb{C})$, $c_{U\xi_3^2}(L, \mathbb{C})$ and $c_U(L, \mathbb{C})$ are the c_3-invariants of L, and o_+ and ℓ_+ are combinatorial quantities easily computable form the diagram L. In particular, if B' is the negative stabilization of a braid B, then $c_{U\xi_3^i}(\overline{B'}, \mathbb{C}) \geq 2$. □

Similarly to the $\mathfrak{sl}_2$-theories, the bound on the value of the c_3-invariants yields a bound on the s_3-concordance invariant. However, this result will be omitted form this thesis, due to lack of time, and will be included in a future paper by the author.

Appendices. We conclude this thesis with three appendices. The first appendix deals with some basic algebra needed throughout the thesis, and also contains some (fairly general) propositions which are essential for our work. The second appendix contains various computations. More precisely, it includes the computation of Bar-Natan homology for a few families of links, and some details on the computations made in Chapter 3 of Part 1. To conclude, in the third and last appendix we prove the self-duality of Bar-Natan theory.

Part 1

Transverse invariants in Khovanov-type $\mathfrak{sl}_2$-theories

CHAPTER 1

Frobenius algebras and Khovanov-type homologies

In this chapter we will briefly review the construction of a link homology theory starting from a Frobenius algebra of rank 2. This construction is originally due to Khovanov (cf. [**25, 28**]). The construction presented here is slightly different in the notation, but the approach is very similar to the original one. We will assume some familiarity with the machinery of homological algebra, in particular with filtered and bi-graded complexes. However, the reader can find the basic definitions, the notation, and the properties needed in Appendix A.

1. Frobenius algebras

1.1. Definitions. A **Frobenius algebra** $\mathcal{F}$, over the ring $R_{\mathcal{F}}$, is a commutative unitary $R_{\mathcal{F}}$-algebra $A_{\mathcal{F}}$, together with two maps

$$\Delta_{\mathcal{F}} : A_{\mathcal{F}} \to A_{\mathcal{F}} \otimes_{R_{\mathcal{F}}} A_{\mathcal{F}}, \quad \varepsilon_{\mathcal{F}} : A_{\mathcal{F}} \to R_{\mathcal{F}},$$

satisfying the following requirements

(a) $A_{\mathcal{F}}$ is a finitely generated, and projective $R_{\mathcal{F}}$-module;
(b) $\Delta_{\mathcal{F}}$ is an $A_{\mathcal{F}}$-bi-module isomorphism (i.e. commutes with the left and right action[1] of $A_{\mathcal{F}}$ over $A_{\mathcal{F}} \otimes A_{\mathcal{F}}$);
(c) $\varepsilon_{\mathcal{F}}$ is $R_{\mathcal{F}}$-linear;
(d) $\Delta_{\mathcal{F}}$ is co-associative[2] and co-commutative[3];
(e) $(id_{A_{\mathcal{F}}} \otimes \varepsilon_{\mathcal{F}}) \circ \Delta_{\mathcal{F}} = id_{A_{\mathcal{F}}} = (\varepsilon_{\mathcal{F}} \otimes id_{A_{\mathcal{F}}}) \circ \Delta_{\mathcal{F}}$.

The map $\Delta_{\mathcal{F}}$ is called **co-multiplication**, while $\varepsilon_{\mathcal{F}}$ is the **co-unit** relative to $\Delta_{\mathcal{F}}$.

As we are going to deal with more than a Frobenius algebra, we will usually keep the subscript indicating to which Frobenius algebra the maps Δ, ε, the algebra A, and the ring R belong to. Sometimes, it will be necessary to specify the multiplicative structure on $A_{\mathcal{F}}$, so we will denote $m_{\mathcal{F}}$ the ($R_{\mathcal{F}}$-linear) multiplication map from $A_{\mathcal{F}} \otimes_{R_{\mathcal{F}}} A_{\mathcal{F}}$ to $A_{\mathcal{F}}$. Finally, we will denote by $\iota_{\mathcal{F}}$ the map that specifies the $R_{\mathcal{F}}$-algebra structure over $A_{\mathcal{F}}$, that is: the $R_{\mathcal{F}}$-linear map from $R_{\mathcal{F}}$ to $A_{\mathcal{F}}$ sending $1_{R_{\mathcal{F}}}$ to $1_{A_{\mathcal{F}}}$.

Now let us give an equivalent condition for an algebra to be a Frobenius algebra.

[1] The left and right actions of A on $A \otimes_R A$ are given by

$$a.(x \otimes y) = ax \otimes y, \quad (x \otimes y).a = x \otimes ay.$$

[2] $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$.

[3] $\tau \circ \Delta = \Delta$, where $\tau(a \otimes b) = b \otimes a$.

Proposition 1.1. *Given a commutative R-algebra* (R, A, m, ι)*, and an R-linear map*

$$\varepsilon : A \to R,$$

the following conditions are equivalent

(1) *there exists a unique map* Δ*, such that* $(R, A, m, \iota, \Delta, \varepsilon)$ *is a Frobenius algebra;*
(2) *the (R-)bilinear pairing* $(\cdot,\cdot) = \varepsilon(m(\cdot,\cdot))$ *is non-degenerate, and A is a finitely generated R-module;*

$\square$

For a complete proof of the previous statement the interested reader may consult [**22**, Chapter 1]. However, it will be needed in the follow up to see how the co-multiplication can be defined from the bilinear pairing $(\cdot,\cdot)$ (see also [**55**, Chapter 2]). Given a commutative R-algebra (R, A, m, ι), and an R-linear map

$$\varepsilon : A \to R,$$

such that $\varepsilon(m(\cdot,\cdot)) = (\cdot,\cdot)$ is non degenerate, for each $x \in A$ set

$$\Delta(x) = \sum_i x_i' \otimes x_i'',$$

to be the unique element such that:

$$m(x, y) = \sum_i (x_i'', y) x_i', \quad \forall y \in A,$$

and this defines the desired co-multiplication.

For our purposes, will be useful to introduce also the graded and filtered versions of Frobenius algebras.

Definition 1.1. A **graded Frobenius algebra** is a Frobenius algebra $\mathcal{F}$, satisfying the following properties

(a) $R = \bigoplus_k R_k$ is a graded ring;
(b) $A = \bigoplus_i A_i$ is a graded R-module;
(c) m, ι are graded maps;
(d) Δ, ε are graded maps (where $A \otimes A$ is given the usual tensor grading);

where m is the multiplication on A, and $\iota : R \to A$ is the unique ring homomorphism, called **unit**, such that

$$\iota(r) \cdot \alpha = r.\alpha, \quad \forall r \in R,\ \alpha \in A.$$

Definition 1.2. A **filtered Frobenius algebra** is a Frobenius algebra $\mathcal{F}$ over a (possibly trivially) graded ring R together with a filtration $\mathscr{F}_\circ$ of A as an R-module, for which there exists an integer d such that:

$$\mathscr{F}_i \mathscr{F}_j \subseteq \mathscr{F}_{j+i+d}$$

for each i and each j, and

$$\Delta(\mathscr{F}_n) \subseteq \sum_k \mathscr{F}_k \otimes \mathscr{F}_{n-d-k} \subseteq A \otimes A,$$

for each n.

Definition 1.3. Let $\mathcal{F} = (R, A, m, \iota, \Delta, \varepsilon)$, and $\mathcal{G} = (S, B, n, \jmath, \Gamma, \eta)$ be two (graded) Frobenius Algebras. A **Frobenius algebra morphism** is a couple of ring homomorphisms

$$\psi : R \to S, \quad \varphi : A \to B,$$

such that

$$\varphi \circ \iota = \jmath \circ \psi, \quad \eta \circ \varphi = \psi \circ \varepsilon.$$

and

$$\varphi \otimes \varphi \circ \Delta = \Gamma \circ \varphi.$$

Given two Frobenius algebra morphisms, say (φ, ψ) and (γ, ω), their **composition** $(\varphi, \psi) \circ (\gamma, \omega)$ is defined as $(\varphi \circ \gamma, \psi \circ \omega)$. An **isomorphism of Frobenius algebras** is a morphism (φ, ψ) such that both φ and ψ are ring isomorphisms.

It is not difficult to see that Definition 1.3 allows us to define a category of Frobenius algebras. In this category the role of the identity morphism for an object $\mathcal{F}$ is played by $(Id_{A_{\mathcal{F}}}, Id_{R_{\mathcal{F}}})$.

The definition of morphism easily extends to graded and filtered Frobenius algebras: it is sufficient to require the map on the base ring to be graded, and the morphism of algebras to be either graded or filtered, depending on which type of Frobenius algebra one is dealing with. Hence, we can define the categories of graded Frobenius algebras, and the category of Filtered Frobenius algebras.

1.2. Examples. Now let us turn to some examples. These examples are crucial from our view point, as we will work only with this family of Frobenius algebras. However, this choice is not as restrictive as one may imagine. In fact, these algebras codify all the information we are interested in (cf. **[28, 41]**).

Let R be a ring. Define A_{BIG} to be the (graded) free $R[U, T]$-algebra

$$A_{BIG} = \frac{R[U, T][X]}{(X^2 - UX + T)},$$

where $x_- := X$, and $x_+ := 1$, have degrees, respectively, -1 and $+1$. In order to define the structure of Frobenius algebra, define a co-multiplication $\Delta = \Delta_{BIG}$, as follows

$$\Delta(x_+) = x_+ \otimes_R x_- + x_- \otimes_R x_+ - Ux_+ \otimes x_+,$$
$$\Delta(x_-) = x_- \otimes_R x_- - Tx_+ \otimes x_+.$$

Finally, the co-unit map is defined by

$$\varepsilon : A_{BIG} \to R[U, T] : \alpha(U, T)x_+ + \beta(U, T)x_- \mapsto \beta(U, T).$$

All the other theories, are obtained by specifying U, T or both, in elements u or t of R (that is, applying the functor $\cdot \otimes_{R[U,T]} R[U, T]/(U - u, T - t)$). In particular, we define

(1) Khovanov theory Kh, by setting $U = 0$, $T = 0$;
(2) the original Lee theory, denoted by $OLee$, is obtained by setting $T = 1$ and $U = 0$;
(3) the twisted Lee theory (also known as filtered Bar-Natan theory), denoted by $TLee$, is obtained by setting $T = 0$ and $U = 1$;

(4) the Bar-Natan theory, denoted by BN, is obtained by setting $T = 0$;
(5) the T-theory, denoted by TT, is obtained by setting $U = 0$.

By setting

$$deg(U) = -2, \quad \text{and} \quad deg(T) = -4,$$

BIG becomes a graded Frobenius algebra, and hence BN, TT, and Kh, inherit this structure; while $TLee$ and $OLee$ become filtered Frobenius algebras.

Another, more general example which will be of use in the second part of this thesis, is the following. Let R be a Noetherian domain, and set

$$R_k = R[U_1, ..., U_k],$$

and fix a monic polynomial of degree $N \geq k$, say $p \in R_k[X]$. The Frobenius algebra $\mathcal{K}(p,k)$, which will be called in this thesis **Krasner algebra** (with respect to p), is defined as follows:

$$R_{\mathcal{K}(p,k)} = R_k, \quad A_{\mathcal{K}(p,k)} = \frac{R_k[X]}{(p)},$$

and the co-unit is defined on the generators of $A_{\mathcal{K}(p,k)}$ as follows

$$\varepsilon([X^i]) = \delta_{i,N-1}, \quad \forall i \leq N-1.$$

Notice that BIG, and all the other theories, are specifications of Krasner theory with $N = 2$, for a suitable choice of the number of variables k and of the polynomial p.

1.3. Base change and twisting. In **[28]** Khovanov describes two constructions to define a new Frobenius algebra from a given one. The first one of these constructions takes a Frobenius algebra $\mathcal{F}$ over a ring R, and a ring homomorphism

$$\psi : R \to S,$$

and gives back a Frobenius algebra $\mathcal{F}_\psi = (R_\psi, A_\psi, m_\psi, \iota_\psi, \Delta_\psi, \varepsilon_\psi)$. This is defined as follows:

$$R_\psi = R \otimes_R S, \quad A_\psi = A \otimes_R S,$$

where the algebra structure on A_ψ is the natural one[4] and

$$\varepsilon_\psi = \varepsilon \otimes_R Id_S, \quad \Delta_\psi = \Delta \otimes Id_S.$$

The algebra $\mathcal{F}_\psi$ is said to be a **base change of** $\mathcal{F}$. Given a graded Frobenius algebra $\mathcal{F}$, one can define the **graded** (resp. **filtered**) **base change of** $\mathcal{F}$ exactly as above, provided that S is a graded ring, ψ is a graded map, and all tensor products are endowed with the induced graded (resp. filtered) structure.

Remark 1. Any (graded, resp. filtered) base change ψ induces a natural (graded, resp. filtered) morphism of Frobenius algebras $\Psi = (\psi, Id_A \otimes \psi(1))$. Moreover, if the morphism ψ is an isomorphism, then Ψ is also an isomorphism.

[4] For the definition of the tensor product of algebras the reader may consult [**2**, Chapter 2], or [**31**, Chapter XIV].

Given a Frobenius algebra $\mathcal{F}$, and an invertible element $u \in A$, we may define a new Frobenius algebra $\mathcal{F}_u$, by twisting the co-multiplication and co-unit as follows

$$\varepsilon_u(\cdot) = \varepsilon(m(u, \cdot)), \quad \Delta_u(\cdot) = \Delta(m(u^{-1}, \cdot)),$$

where m is the multiplication on A. The result $\mathcal{F}_u = (R, A, m, \iota, \varepsilon_u, \Delta_u)$ is still a Frobenius algebra, and is called **twist of** $\mathcal{F}$ (**by** u). Twisting is the only way to modify co-multiplication and co-unit on a Frobenius algebra.

Proposition 1.2 ([**22**], Theorem 1.6). *Given a Frobenius algebra $(R, A, m, \iota, \Delta, \varepsilon)$, all the other structures of Frobenius algebra on (R, A, m, ι) are obtained from $\mathcal{F}$ via a twist by an invertible element.* $\square$

Twisting by an invertible element, gives an isomorphic Frobenius algebra.

Proposition 1.3. *Let $\mathcal{F}$ be a Frobenius algebra, and let $u \in A$ be an invertible element. Then there exists a natural Frobenius algebra isomorphism (φ_u, ψ_u) between $\mathcal{F}$ and $\mathcal{F}_u$. Moreover, if u is an homogeneous element, and if $\mathcal{F}$ and $\mathcal{F}_u$ are graded, then they are graded-isomorphic (up to shift). Finally, if u is an homogeneous element of $R_{\mathcal{F}} \subseteq A_{\mathcal{F}}$, and $\mathcal{F}$, $\mathcal{F}_u$ are filtered, then they are filtered isomorphic (up to shift).*

Proof. Because in this case the underlying structure of algebra is the same for both $A_{\mathcal{F}}$ and $A_{\mathcal{F}_u}$, throughout the proof we will denote both of them simply by A. Define $\psi : A \to A$ as the (left) multiplication by u^{-1}. This is clearly a isomorphism of R-algebras. Moreover, ψ commutes with Δ. In fact, let $x \in A$

$$\psi \otimes \psi \circ \Delta(x) = u^{-1}.\Delta(x).u^{-1} =$$

by definition of Frobenius algebra Δ is an (A, A)-bi-module map from A to $A \otimes A$,

$$= \Delta(m(u^{-1}, m(x, u^{-1}))) = \Delta_u \circ \psi(x),$$

where the last equality is due to the commutativity of (A, m). Finally, a simple computation shows that

$$\varepsilon_u \circ \psi = \varepsilon.$$

Thus, by setting $\Psi = (Id_R, \psi)$ one gets the desired isomorphism. The graded version of the proposition follows immediately once one notice that, if u is homogeneous, then ψ is a graded map (of degree $deg(u)$).

Similarly, if $\mathcal{F}$ is filtered, the multiplication by u^{-1} respects the filtration on A (shifting it by $-deg(u)$, while its inverse shifts the filtration by $deg(u)$). So, the map ψ is a filtered isomorphism between $A_{\mathcal{F}}$ and $A_{\mathcal{F}_u}$ (the latter shifted by $deg(u)$).

Q.E.D.

1.4. Dual Frobenius algebras. Let $\mathcal{F}$ be a Frobenius algebra, and set

$$R_{\mathcal{F}^*} = R_{\mathcal{F}}, \quad A_{\mathcal{F}^*} = A^*_{\mathcal{F}} \left(= Hom_R(A_{\mathcal{F}}, R_{\mathcal{F}})\right).$$

Because A is a projective R-module, and R is a Noetherian domain, there is a canonical isomorphism $(A \otimes A)^* \simeq A^* \otimes A^*$ (cf. Section 1 in Appendix A). Hence, is it possible to define the maps

$$\Delta^*_{\mathcal{F}} : A^*_{\mathcal{F}} \otimes_{R_{\mathcal{F}}} A^*_{\mathcal{F}} \to A^*_{\mathcal{F}}, \quad \varepsilon^*_{\mathcal{F}} : R^*_{\mathcal{F}}(= R_{\mathcal{F}}) \to A^*_{\mathcal{F}},$$

and

$$m^*_{\mathcal{F}} : A^*_{\mathcal{F}} \to A^*_{\mathcal{F}} \otimes_{R_{\mathcal{F}}} A^*_{\mathcal{F}}, \quad \iota^*_{\mathcal{F}} : A^*_{\mathcal{F}} \to R^*_{\mathcal{F}}(= R_{\mathcal{F}}),$$

as the duals of the corresponding maps in $\mathcal{F}$. Finally, set

$$\Delta_{\mathcal{F}^*} = m^*_{\mathcal{F}},\ m_{\mathcal{F}^*} = \Delta^*_{\mathcal{F}},\ \varepsilon_{\mathcal{F}^*} = \iota^*_{\mathcal{F}},\ \iota_{\mathcal{F}^*} = \varepsilon^*_{\mathcal{F}}.$$

Direct computations show that $\mathcal{F}^*$ is a Frobenius algebra, and is called **dual Frobenius algebra** of $\mathcal{F}$ (cf. [**28**, Subsection: The dual system]).

By Proposition 1.1, we have a natural, non degenerate pairing given by the Frobenius algebra structure, defined as follows

$$(\cdot,\cdot)_{\mathcal{F}} = \varepsilon_{\mathcal{F}}(m_{\mathcal{F}}(\cdot,\cdot)).$$

This pairing defines an isomorphism of $R_{\mathcal{F}}$-modules

$$\Phi : A_{\mathcal{F}} \longrightarrow A^*_{\mathcal{F}} :\ \alpha \mapsto (\alpha,\cdot).$$

Definition 1.4. A Frobenius algebra $\mathcal{F}$ is called **strongly self-dual** if the canonical isomorphism[5] is an isomorphism of Frobenius algebra between $\mathcal{F}$ and $\mathcal{F}^*$. A Frobenius algebra is **self-dual** if (id,Φ) is an isomorphism of Frobenius algebras. Finally, $\mathcal{F}$ is **weakly self-dual** if is isomorphic to its dual.

Remark 2. Direct computations show that the Frobenius algebras Kh, $OLee$ and TT are strongly self dual. On the other hand, BN and $TLee$ are not strongly self-dual (unless, $2 = 0$ in $R_{\mathcal{F}}$), even though they are self-dual (cf. Appendix C).

Remark 3. Any base change preserve strong self-duality, while twisting in general does not (e.g. $OLee$ and $TLee$ are twist-equivalent if $R_{\mathcal{F}}$ is a ring where 2 is invertible, see [**39**, Proposition 3.1]).

The dual Frobenius algebra of a graded (resp. filtered) Frobenius algebra $\mathcal{F}$, is still graded (resp. filtered) in a natural way. The algebra $A_{\mathcal{F}}$ is a projective $R_{\mathcal{F}}$-module of finite type, hence has a dual basis $\{(x_i,\varphi_i)\}$, set $deg(\varphi_i)$ as their degree as maps from $A_{\mathcal{F}}$ to $R_{\mathcal{F}}$ (which means that $deg(\varphi_i) = -deg(x_i)$). Being the co-multiplication, the multiplication, unit and co-unit of $A_{\mathcal{F}}$ graded (resp. filtered) maps, the same holds for the multiplication, co-multiplication, co-unit and unit of $A_{\mathcal{F}^*}$. Self-duality, of any type, in the graded (resp. filtered) case will be assumed to be graded (resp. filtered) (i.e. the isomorphism is an isomorphism of graded (resp. filtered) Frobenius algebras).

[5]The isomorphism sending a basis of $A_{\mathcal{F}}$ to its dual (in the sense of Appendix A, Section 1)

2. Link homology theories

2.1. Khovanov-type theories. Let L be an oriented link diagram. A **local resolution** of L in a crossing is its replacement with either , a 0-resolution, or with , a 1-resolution.

Definition 1.5. A **resolution of** L is the set of circles, embedded in $\mathbb{R}^2$, obtained from L by performing a local resolution at each crossing. The total number of 1-resolutions performed in order to obtain a resolution $\underline{s}$ will be called **weight** of $\underline{s}$, and will be denoted by $|\underline{s}|$.

Let $\mathcal{R}_L$ the set of all the possible resolutions of L. It is possible to define an elementary relation on $\mathcal{R}_L$ as follows

$$\underline{r} \prec \underline{s} \iff |\underline{r}| < |\underline{s}|, \text{ and } \underline{r},\ \underline{s} \text{ differ by a single local resolution.}$$

A **square** $[s_0, s_1, s_2, s_3]$ is a collection of four, different, resolutions such that:

$$\underline{s}_0 \prec \underline{s}_1, \quad \underline{s}_0 \prec \underline{s}_2, \quad \underline{s}_1 \prec \underline{s}_3, \quad \text{and} \quad \underline{s}_2 \prec \underline{s}_3.$$

Definition 1.6. A **sign function** is a map

$$S : \mathcal{R}_L \times \mathcal{R}_L \to \{0, 1, -1\},$$

satisfying the following two properties:

(1) $S(\underline{r}, \underline{s}) = 0$ if, and only if, $\underline{r} \not\prec \underline{s}$;
(2) for each square $[s_0, s_1, s_2, s_3]$, we have

$$S(\underline{s}_0, \underline{s}_1)S(\underline{s}_1, \underline{s}_3) = -S(\underline{s}_0, \underline{s}_2)S(\underline{s}_2, \underline{s}_3).$$

Given a Frobenius algebra, say $\mathcal{F} = (R, A, \iota, \Delta, \varepsilon)$, define

$$C^i_{\mathcal{F}}(L, R) = \bigoplus_{|\underline{r}| - n_- = i} A_{\underline{r}}, \quad A_{\underline{r}} = \bigotimes_{\gamma \in \underline{r}} A_\gamma,$$

where A_γ is just an indexed copy of A, and $\underline{r}$ ranges in $\mathcal{R}_L$. These are the R-modules that are going to play the role of (co)chain groups. In order to define a (co)chain complex, all that is left to do is to define a differential. This will be done in two steps. Start by defining

$$d^{\underline{s}}_{\underline{r}} : A_{\underline{r}} \to A_{\underline{s}}, \quad \underline{r} \prec \underline{s}.$$

Consider $x = \bigotimes_{\gamma \in \underline{r}} \alpha_\gamma$, where $\underline{r} \prec \underline{s}$. By definition of $\prec$, the two resolutions $\underline{r}$ and $\underline{s}$ differ by a single local resolution. Hence there is an identification of all the circles in the two resolutions, except the ones involved in the change of local resolution. There are only two cases to consider: (a) two circles of $\underline{r}$, say $\gamma_1,\ \gamma_2$ are merged in a single circle γ_1' in $\underline{s}$, or (b) a circle γ_1 belonging to $\underline{r}$ is split in into two circles, say γ_1' and γ_2', in $\underline{s}$. Our map is defined as follows

$$d^{\underline{s}}_{\underline{r}}(x) = \begin{cases} \bigotimes_{\gamma \in \underline{r} \cap \underline{s}} \alpha_\gamma \otimes m(\alpha_{\gamma_1}, \alpha_{\gamma_2}) & \text{in case (a)} \\ \bigotimes_{\gamma \in \underline{r} \cap \underline{s}} \alpha_\gamma \otimes \Delta(\alpha_{\gamma_1}) & \text{in case (b)} \end{cases}$$

Finally, fix a sign function S and define

$$d^i_{\mathcal{F}} : C^i_{\mathcal{F}}(L, R) \to C^{i+1}_{\mathcal{F}}(L, R) : x \in A_{\underline{r}} \mapsto \sum_{\underline{r} \prec \underline{s}} S(\underline{r}, \underline{s}) d^{\underline{s}}_{\underline{r}}(x).$$

Remark 4. Notice that $d_{\underline{r}}^{S}$ is well defined because of the commutativity of m, and of the co-commutativity of Δ. On the other hand, $d_{\mathcal{F}}$ depends on the choice sign function S. In particular, the existence of $d_{\mathcal{F}}$ depends on the existence of such a function.

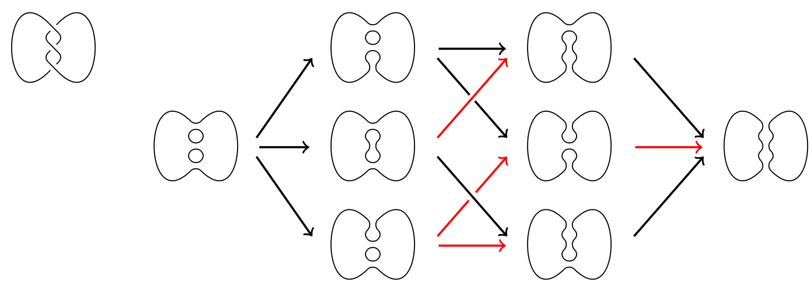


FIGURE 1. A diagram of the trefoil knot, the poset of its resolutions, and a sign function. The arrows represent the relation $\prec$, the red arrows are those whose pair of start and end points has image -1 via the sign function, and all the pairs start point-end point of the black arrows have image 1.

Proposition 1.4 (Khovanov, **[25]**). *There exists a sign function S such that the complex $(C^{\bullet}_{\mathcal{F}}(L,R), d^{\bullet}_{\mathcal{F}})$ is a (co)chain complex. Moreover, the homology of this complex does not depend, up to isomophism, on the choice of the sign function S, or on the order of the circles in each resolution.* □

Remark 5. It is immediate from the definition of Khovanov-type homology that

$$C^{\bullet}_{\mathcal{F}}(L \sqcup L', R) \simeq C^{\bullet}_{\mathcal{F}}(L,R) \otimes_R C^{\bullet}_{\mathcal{F}}(L',R),$$

as complexes of R-modules. Moreover, if $\mathcal{F}$ is a graded (resp. filtered) Frobenius algebra the above isomorphism respects the quantum grading (resp. the filtration) defined in the next subsection.

2.2. Invariance and gradings. Which are the conditions on $\mathcal{F}$ under which the homology of the complex $(C^{\bullet}_{\mathcal{F}}(L,R), d^{\bullet}_{\mathcal{F}})$ is a link invariant? Or, better, which are the ones needed to get a link homology theory (i.e. functorial in some sense)? A simple, necessary, condition can be found by noticing that: if the homology of $(C^{\bullet}_{\mathcal{F}}(L,R), d^{\bullet}_{\mathcal{F}})$ is a link invariant, then its Euler characteristic should also be independent of the diagram L. In particular, the chain complexes associated to the two diagrams of the unknot drawn below must have the same Euler characteristic.

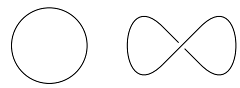


FIGURE 2. Two diagrams of the unknot.

The associated chain complexes are, respectively,

$$0 \longrightarrow A \longrightarrow 0, \quad \text{and} \quad 0 \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$$

where the first non-trivial group is the one in (homological) degree 0, and A indicates $A_{\mathcal{F}}$. Thus, by imposing the equality of the two characteristics, one gets the following equation

$$rank(A) = -rank(A) + rank(A \otimes A) = rank(A)^2 - rank(A),$$

which, once excluded the trivial case, implies

$$rank(A) = 2.$$

It turns out that the rank of A being 2 is the only property $\mathcal{F}$ must satisfy in order to have the invariance of the homology (cf. [**28**, Prop. 3, Thm. 5 & 6]). So, as we are concerned only with link invariant theories, from now on all Frobenius algebras will be supposed of rank 2. Once a basis of A is fixed, say $x_+,\ x_-$, the elements of $C^i_{\mathcal{F}}(L, R)$ of the form $\bigotimes_{\gamma\in\underline{r}} \alpha_\gamma$, with $\alpha_\gamma \in \{x_+,\ x_-\}$ and $\underline{r} \in \mathcal{R}_L$, will be called **states**; while those of the form $\bigotimes_{\gamma\in\underline{r}} \alpha_\gamma$, with $\alpha_\gamma \in A$, will be called **enhanced states**. Notice that the states are an R-basis of $C^\bullet_{\mathcal{F}}(L, R)$, while the enhanced states are a system of generators.

Remark 6. If $\mathcal{F}$ is a graded (resp. filtered) Frobenius algebra, then the basis $\{x_+, x_-\}$ will be taken to be composed of homogeneous elements (resp. to be a filtered basis, see Appendix A). Under these conditions, it is possible to define another grading (resp. filtration) over the complex $(C^\bullet_{\mathcal{F}}(L, R), d^\bullet_{\mathcal{F}})$, as follows

$$qdeg(\bigotimes_{\gamma\in\underline{r}} \alpha_\gamma) = \sum_{\gamma\in\underline{r}} deg_A(\alpha_\gamma) - 2n_- + n_+ + |\underline{r}|,$$

for each state $\bigotimes_{\gamma\in\underline{r}} \alpha_\gamma$. (Then the filtration is given by considering all the elements which can be written as combination of states of degree greater or lower than a fixed *qdeg*, depending on whether the multiplication is non-decreasing or non-increasing with respect to the *qdeg*.) Moreover, by definition of graded (resp. filtered) Frobenius algebra, the differential $d^\bullet_{\mathcal{F}}$ is homogeneous (resp. filtered, see Section 5 Appendix A) with respect to the *qdeg* degree (resp. induced filtration), and the resulting homology theory is hence doubly-graded (resp. filtered). Let $\mathcal{F}$ be a filtered Frobenius algebra, we will denote by $\mathscr{F}_\circ C^\bullet_{\mathcal{F}}(L, R)$ the filtration induced on the complex $C^\bullet_{\mathcal{F}}(L, R)$.

Theorem 1.5. *Let L be an oriented link diagram. If $\mathcal{F}$ and $\mathcal{G}$ are isomorphic (graded, resp. filtered) Frobenius algebras, then $(C^\bullet_{\mathcal{F}}(L, R_{\mathcal{F}}), d^\bullet_{\mathcal{F}})$ and $(C^\bullet_{\mathcal{G}}(L, R_{\mathcal{G}}), d^\bullet_{\mathcal{G}})$ are isomorphic as (doubly-graded, resp. filtered) complexes of both $R_{\mathcal{F}}$ and $R_{\mathcal{G}}$ modules.*

Proof. Let $\mathcal{F} = (R, A, m, \iota, \Delta, \varepsilon)$ and $\mathcal{G} = (S, B, n, \jmath, \Gamma, \eta)$ be two isomorphic (graded) Frobenius algebras, and let (φ, ψ) the (graded, resp. filtered) isomorphism between them. Then, for each resolution $\underline{r}$ we have the isomorphism[6] of (graded) R-modules

$$\bigotimes_{\gamma \in \underline{r}} \psi : A_{\underline{r}} \to B_{\underline{r}},$$

where B is seen as an R-module with the induced structure. This induces naturally an isomorphism of (bi-graded, resp. filtered) chain modules that commutes (by definition of morphism between Frobenius algebras) with the differentials. The same reasoning works replacing R with S.

Q.E.D.

Remark 7. Until now we required the diagrams to be oriented: this is essential for the invariance. As the reader may have noticed, the orientation comes up only in the degrees shift. The homological degree has been shifted by a the number of negative crossings. Without this shift the homology is not invariant as graded module (much less as bi-graded or filtered module).

2.3. Duality and mirror image. It has been shown by Khovanov (see [**25**, Section 7.3]) that there is a relationship between the complex of the mirror image and the dual complex of a link. More precisely, the following proposition holds.

Proposition 1.6 ([**25**], Proposition 32). *Let $\mathcal{F} = (A, R, m, \iota, \Delta, \varepsilon)$ be a Frobenius algebra, L be an oriented link diagram, and L^* be the mirror diagram*[7]*. Then, there is an isomorphism of chain complexes*

$$(C_{\mathcal{F}})^* (L, R) \simeq C_{\mathcal{F}^*}(L^*, R).$$

Where the dual complex of an R-complex $(C^\bullet, d^\bullet)$ is defined as

$$(C^*)^i = Hom_R(C^{-i}, R), \quad (d^*)^i = (d^{-i})^*.$$

□

In particular, if a theory is (strongly) self-dual there is an isomorphism

$$C_{\mathcal{F}}^{-\bullet}(L, R)(=) \simeq C_{\mathcal{F}}^{\bullet}(L^*, R).$$

At this point a natural question is whether or not this isomorphism, in the case $\mathcal{F}$ is a graded Frobenius algebra, preserves the quantum degree (i.e. the second degree). The answer is not positive, in general we cannot say anything about this grading. Nonetheless, in some special cases something can be said. Before stating the result, we wish to recall some notation. Let R be a graded ring and $M^{\bullet,\bullet}$ be a bi-graded R-module (cf. Appendix A). Fix two integers, say a and b. The (a, b)**-shift** of $M^{\bullet,\bullet}$ is the graded R-module $M(a, b)$, whose underlying

[6]This is injective because A, B are both flat R-modules, and is obviously surjective.

[7]That is the diagram whose positive crossing are replaced with negative ones, and *vice-versa*.

module structure is the same as M, but whose (i,j)-th homogeneous component is the $(i+a, j+b)$-th homogeneous component of M. More explicitly,

$$M(a,b) = \bigoplus_{i,j} (M(a,b))^{i,j}, \qquad (M(a,b))^{i,j} = M^{i+a,j+b}.$$

More generally, if M is a C-graded R-module, with C a commutative monoid and R a C-graded ring (see Appendix A), and $c' \in C$, we will denote by $M(c')$ the **c'-shift of** M, that is the C-graded R-module, whose c-th homogeneous component corresponds to the $(c+c')$-th homogeneous component of M, and whose underlying module structure is the same as M.

Proposition 1.7. *Let $\mathcal{F} = (R, A, m, \iota, \Delta, \varepsilon)$ be a graded Frobenius algebra, such that:*

(a) R is a principal ideal domain;
(b) R contains only homogeneous elements of non-negative (non-positive) degree;

Suppose that

$$H^{\bullet,\bullet}_{\mathcal{F}}(L,R) \simeq \bigoplus_{i=1}^{s} \frac{R}{(a_i)}(h_i, q_i) \oplus \bigoplus_{j=1}^{r} R(k_j, p_j), \tag{1}$$

as bi-graded R-modules, then

$$H^{\bullet,\bullet}_{\mathcal{F}^*}(L^*,R) \simeq \bigoplus_{i=1}^{s} \frac{R}{(a_i)}(-h_i, -q_i) \oplus \bigoplus_{j=1}^{r} R(-k_j, -p_j). \tag{2}$$

Where the grade structures of $\mathcal{F}^*$ and $C^{\bullet,\bullet}_{\mathcal{F}^*}(L,R)$ are the natural ones (cf. Subsection 1.4).

Proof. Proposition 1.6 implies that $H(C^*_{\mathcal{F}}(L))$ and $H_{\mathcal{F}^*}(L^*)$ are isomorphic as graded R-module with respect to the homological degree; more precisely, we have

$$H^{\bullet,\bullet}(C^*_{\mathcal{F}}(L,R)) = H^{\bullet,\bullet}_{\mathcal{F}^*}(L^*,R) \simeq \bigoplus_{i=1}^{s} \frac{R}{(a_i)}(-h_i, q^*_i) \oplus \bigoplus_{j=1}^{r} R(-k_j, p^*_j).$$

Hence it is sufficient to show that

$$p^*_j = -p_j, \quad q^*_j = -q_j.$$

First, notice that the isomorphism Φ used in Proposition 1.6 (cf. [**25**, Proposition 32]), and the definition of degree in the dual (graded) Frobenius algebra, are such that: given a state $\underline{s} \in C_{\mathcal{F}}(L,R)$, with respect to any fixed homogeneous basis of A, we have

$$qdeg(\underline{s}) = -qdeg(\Phi(\underline{s})).$$

A homology class $[x] \in H_{\mathcal{F}}(L^*,R)$ is called **primitive** if, and only if, do not exist $[y] \in H_{\mathcal{F}}(L^*,R)$ and $\alpha \in R_i$, with α non-invertible and $i \neq 0$, such that

$$[x] = \alpha[y].$$

Remark 8. Primitive elements do not necessary exist in general, even in the case of a Khovanov-type homology over an arbitrary Frobenius algebra. However, the hypotheses (a) and (b) made ensure us that, not only we have primitive elements, but also that any minimal set of homogeneous generators of $H_{\mathcal{F}}(L,R)$ is composed of primitive elements. In fact, pick an homogeneous class $[x]$ generating one of the summands in (1). If $[x]$ is not primitive there is a $\alpha \in R_+$ (resp. R_-, see below for the definitions), such that:

$$[x] = \alpha[y].$$

which is absurd, because $[x]$ should have minimal (resp. maximal) degree (↯).

We claim that if $[x]$ is primitive, then any decomposition as linear combination of states of any representative of $[x]$ has at least one (non-trivial) term with degree 0 coefficient. In fact, suppose not. Then, as R contains only homogeneous elements of non-negative (non-positive) degree

$$R_+ = \bigoplus_{i=1}^{\infty} R_i, \quad \left(\text{resp. } R_- = \bigoplus_{i=-1}^{-\infty} R_i, \right)$$

is an ideal. Moreover, as R is a principal ideal domain, R_+ (resp. R_-) is generated by a single element, say r_+ (resp. r_-). If a representative $\tilde{x}$ of $[x]$, decomposes as

$$\tilde{x} = \sum_{n=1}^{m} \alpha_n \underline{s}_n,$$

with $\alpha_n \in R_+$ (resp. $\alpha_n \in R_-$) for each n, then

$$\tilde{x} = r_+ \sum_{n=1}^{m} \alpha'_n \underline{s}_n, \quad \left(\text{resp. } \tilde{x} = r_- \sum_{n=1}^{m} \alpha'_n \underline{s}_n, \right)$$

for some $\alpha'_n \in R$, which is absurd because $[x]$ was supposed to be primitive (↯). Being the isomorphism Φ an homogeneous map (i.e. sends homogeneous elements to homogeneous elements, see also Section 2 of Appendix A), and seen that it changes the sign in the (quantum) degree of the states, for each primitive homology class we have

$$qdeg([x]) = -qdeg(\Phi_*([x])).$$

Since the generator of each summand in the decomposition in (1) is necessarily a primitive class (see the remark above), and the homology of the mirror image is generated by the duals of the generators of $H_{\mathcal{F}}(L,R)$, the claim follows.

Q.E.D.

Corollary 1.8. *Let $\mathbb{F}$ be a field, and λ be an oriented link. If*

$$H^{\bullet,\bullet}_{BN}(\lambda, \mathbb{F}[U]) = \bigoplus_{i=1}^{n} \frac{\mathbb{F}[U]}{(U^{t_i})}(h_i, q_i) \oplus \bigoplus_{j=1}^{m} \mathbb{F}[U](k_j, p_j)$$

then

$$H^{\bullet,\bullet}_{BN}(\lambda^*, \mathbb{F}[U]) = \bigoplus_{i=1}^{n} \frac{\mathbb{F}[U]}{(U^{t_i})}(-h_i, -q_i) \oplus \bigoplus_{j=1}^{m} \mathbb{F}[U](-k_j, -p_j),$$

where λ^* is the mirror image of λ.

Proof. It follows from the previous proposition, and from the self-duality of Bar-Natan theory (Appendix C).

Q.E.D.

2.4. The Saddle move, functoriality and some notation. A **traced resolution** is a set of circles, and (coloured) segments between them, obtained by replacing a crossing  with either  or . A coloured segment keeps trace of both the position of a crossing split, and the type of splitting performed on that crossing (which is encoded in the colour: red for the 1-resolution, and blue for the 0-resolution).

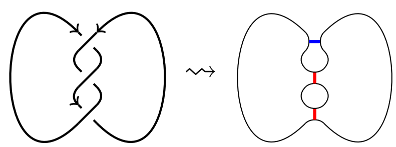


FIGURE 3. A traced resolution of a trefoil knot diagram

A **traced state** (TS) is a traced resolution with the circles labeled with either $+$ or $-$. The traced states are, once an R-basis for the algebra A is fixed, naturally identified with the states. In a similar way, an **enhanced traced state** (ETS) is traced resolution with the circles labeled with elements of A, and each ETS could be naturally identified with an enhanced state (notice that no choice of a basis is needed).

Let L be an oriented link diagram. A **surgery arc** on L is an embedded copy of $[0,1]$ in $\mathbb{R}^2$, such that:

(a) the ending points of γ belong to L;
(b) the interior of γ does not meet L;

Once we pick a (traced) resolution $\underline{r}$ of L, γ joins the circle, or circles, of $\underline{r}$ to which its ending points belongs to. As the interior of γ does not meet L, then interior of γ does not meet neither the circles, nor the traces, in $\underline{r}$.

Consider a link diagram L, together with a surgery arc γ. The **saddle morphism** (along γ) assigns to each ETS x, with underlying resolution $\underline{r}$, a linear combination of ETSs, say $S(x,\gamma)$, whose circles, and traces are the same as in x, except near the surgery arc where the local picture on the left hand side of Figure 4 is replaced with the right hand side of the same figure. The labels of the circles not meeting γ are left invariant, while those involved are replaced as follows;

(a) $m(a,b)$ if the arcs meeting γ belong to different circles;
(b) $\Delta(a)$ if the arcs meeting γ belong to the same circle (in particular $a = b$);

this assignment will be denoted, more compactly, by specifying an arc between the two new arcs and a label $s(a,b)$ (see the right hand side of Figure 4). Then this map (which, as things are now, is only defined on the ETSs) extends, by R-linearity, to $C^{\bullet}_{\mathcal{F}}$.

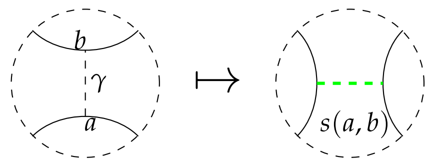


FIGURE 4. The saddle morphism.

Remark 9. Even if in our notation $S(x,\gamma)$ is represented as a single ETS, there is no need for $S(x,\gamma)$ to be such. For example, suppose $\mathcal{F} = Kh$, $a = b = x_+$, and that the two arc joined by the surgery arc belong to the same circle, then $S(a,\gamma)$ is the sum of two ETSs (see Figure 5).

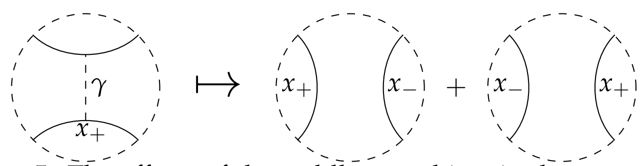


FIGURE 5. The effects of the saddle morphism in the case where $\mathcal{F} = Kh$, the two arcs in the left hand side belong to the same circle, and the circle is labelled x_+.

The saddle move is a part of a more general construction that, to each generic (oriented) cobordism Σ, properly embedded in $\mathbb{R}^3 \times [0,1]$ or $\mathbb{S}^3 \times [0,1]$, assigns a morphism between the chain complexes of any Khovanov-type theory of the two (oriented) links $\Sigma \cap \mathbb{R}^3 \times \{0\}$, and $\Sigma \cap \mathbb{R}^3 \times \{1\}$ (the latter considered with the opposite of the induced orientation). More precisely, given a movie description of a generic cobordism Σ, there is a well defined R-linear map

$$\Phi_\Sigma : C^{\bullet}_{\mathcal{F}}(L_0, R) \to C^{\bullet}_{\mathcal{F}}(L_1, R),$$

where L_0 and L_1 are the first and the last clips of the movie (see [**4**, Section 8] for the details).

Theorem 1.9 (Jacobsson, [**21**], Bar-Natan, [**4**]). *If Σ and Σ' are two isotopic cobordisms, via an isotopy which leaves the boundary fixed, the induced map, in any Khovanov-type homology theory, are equal up to sign. Furthermore, if the theory is graded (resp. filtered) then the map induced by Σ is graded (resp. filtered) of degree $\chi(\Sigma)$.* □

2.5. Reduction and co-reduction. Given a link diagram L, and chosen a number of base points, say $p_1, ..., p_k$, in distinct components, denote by C_i an unknotted circle near p_i. There is an action of

$$A_k = \underbrace{A_{\mathcal{F}} \otimes_R ... \otimes_R A_{\mathcal{F}}}_{k \text{ times}},$$

on the complex $C^\bullet_{\mathcal{F}}(L, R)$, given by merging the circle C_i with the marked point p_i, for each marked point. This gives an R-linear map

$$F : C^\bullet_{\mathcal{F}}(L \sqcup C_1 \sqcup ... \sqcup C_k, R) = C^\bullet_{\mathcal{F}}(L, R) \otimes_R A_k \longrightarrow C^\bullet_{\mathcal{F}}(L, R),$$

which describes the above-mentioned action. The canopoleis structure described in [**4**], which is valid for any Khovanov type theory, tells us that the action is well defined and that it is an invariant under isotopies of the link fixing the base points.

Fix $x \in A_{\mathcal{F}}$, and consider the polynomial ring $R[X_1, ..., X_k]$, this could be made to act on $C^\bullet_{\mathcal{F}}(L, R)$ in the following way: the variable X_i acts on a chain element α as $F(\alpha \otimes x_i)$, that is

$$X_i.\alpha = F(\alpha \otimes x_i),$$

where

$$x_i = \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes x \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-i-1},$$

and by R-linearity the action is extended to the whole $R[X_1, ..., X_k]$. Furthermore, this action commutes with the differential of $C^\bullet_{\mathcal{F}}(L, R)$; this is due to the associativity of the multiplication, and to the fact that

$$\Delta_{\mathcal{F}} \circ m_{\mathcal{F}} = Id_A \otimes m_{\mathcal{F}} \circ (\Delta_{\mathcal{F}} \otimes Id_A),$$

in any Frobenius algebra (cf. Relation (F) in [**1**, Section 5]).

Remark 10. In the case of Khovanov homology with $R = \mathbb{F}_2$, the structure of $R[X_1, ..., X_k]/Ann(A_k)$-module (where X_i acts on A_k as the multiplication by x_i) on $H^\bullet_{\mathcal{F}}(L, R)$ is independent of the chosen diagram, as shown by Hedden and Ni, using an argument due to Sarkar ([**19**, Proposition 2.2]).

So, the image of the action of each variable defines a sub-complex. To be precise, chosen a (non-empty) subset $S \subseteq \{1, ..., k\}$ and an element x, is possible to define the (S, x)**-co-reduced complex** $\widehat{C}^\bullet_{\mathcal{F}}(L, R)_{(S,x)}$ as the sub-complex which is annihilated by the ideal $(X_s | s \in S) \subseteq R[X_1, ..., X_k]$ under the action described above. The quotient complex of $C^\bullet_{\mathcal{F}}(L, R)$ by $\widehat{C}^\bullet_{\mathcal{F}}(L, R)_{(S,x)}$ is called the (S, x)-**reduced complex**, and is denoted by $\widetilde{C}^\bullet_{\mathcal{F}}(L, R)_{(S,x)}$. These complexes naturally fit in the following short exact sequence

$$0 \to \widehat{C}^\bullet_{\mathcal{F}}(L, R)_{(S,x)} \longrightarrow C^\bullet_{\mathcal{F}}(L, R) \longrightarrow \widetilde{C}^\bullet_{\mathcal{F}}(L, R)_{(S,x)} \to 0 \tag{3}$$

Whenever clear from the context we will omit x from the notation, and if there is no ambiguity also S will be removed from the notation. If $S = \{1, ..., k\}$ we will call it **full** (**co**-)**reduction**.

When there is only one marked point (in particular, $k = 1$), one can also consider the image of X_1, and this is again sub-complex. Of course, it is isomorphic (as an R-complex) to the reduced complex. The x**-simply co-reduced complex** is the quotient of $C^\bullet_{\mathcal{F}}(L, R)$ by the x-reduced complex, and is denoted by $\underline{C}^\bullet_{\mathcal{F}}(L, R)_x$. There is a natural short exact sequence of complexes

$$0 \to \widetilde{C}^\bullet_{\mathcal{F}}(L, R)_x \longrightarrow C^\bullet_{\mathcal{F}}(L, R) \longrightarrow \underline{C}^\bullet_{\mathcal{F}}(L, R)_x \to 0 \tag{4}$$

As said before, all the reduced an co-reduced homologies are invariants of links with marked points (i.e. invariants of links up to isotopies fixing the base points). However, if we are dealing with knots all isotopies of a diagram can be taken to happen far from a given point in the diagram. This implies the following

Proposition 1.10 (Khovanov, **[26]**). *The isomorphism class of the (co)reduced homologies is an invariant of links with a fixed component (i.e they do not depend on the chosen diagram, or the chosen marked point on the fixed component). In particular, they are knot invariants.* □

Remark 11. If the Frobenius algebra is graded (resp. filtered), and the element x is homogeneous (resp. is an element of a filtered basis) then the resulting (co)reduced theory will be graded (resp. filtered).

CHAPTER 2

Bar-Natan Theory

The aim of this chapter is to provide structure theorems for the Bar-Natan homology, in a sense that will be specified in the next section. The idea is to recover "the shape" of the Bar-Natan homology in terms of the Khovanov homology, and (twisted) Lee homology, similarly to what has been done by Turner ([**54**]) over $\mathbb{F}_2$. Most of the material contained here is either known or folklore; however, proofs are supplied either if (to the author's knowledge) no proof of the statement can be found in literature, or if the proof of a certain fact is useful to understand the subsequent material. *Vice-versa* the proof of a statement will be omitted if it is a trivial consequence of previously stated propositions, or does not provide any insight, or the result could be found in literature. In this last two cases the result will be attributed to one or more authors (if possible), and a reference where to find the proof will be given.

The chapter is divided into two sections. In the first section we will study the free pat of Bar-Natan homology, determining its rank, and how this rank is distributed in the quantum and homological degrees. The second second section of this chapter will deal with the torsion sub-module of the Bar-Natan homology, and its relationship with Khovanov homology.

1. The structure of Bar-Natan homology I: the free part

The Bar-Natan homology of an oriented link-type λ is a bi-graded $R[U]$-module (up to bi-graded isomorphism). In order to have a structure theorem for modules over $R[U]$, some conditions on the base ring R are necessary. The simplest (and, probably, the only possible) requirement on R is to be a field. In this case, there is a (well-known) structure theorem for graded modules, whose proof can be found in various sources (see, for example, [**61**, Theorem 3.19]).

Theorem 2.1 (Structure Theorem for Graded Modules over a PID). *If A is a graded principal ideal domain, and M is a finitely generated graded module over A, then M is graded isomorphic to the module*

$$\bigoplus_{j\in J} \frac{A}{(d_j)}(k_j) \oplus \bigoplus_{i\in I} A(h_i)$$

where the d_js are non-trivial homogeneous elements of A, $h_i,\ k_i \in \mathbb{Z}$, and $(\cdot)$ indicates the degree-shift (see page 11). Moreover, this decomposition is unique up to permutations of the summands. □

In the case of Bar-Natan theory, the variable U has bi-degree $(0,-2)$, hence each $H^{i,\bullet}_{BN}(L,R[U])$ is both a sub-module and a direct summand of $H^{\bullet,\bullet}_{BN}(L,R[U])$. This fact, combined with the previous theorem, implies the following (rough) description of the Bar-Natan homology of a link.

Corollary 2.2. *Given an oriented link-type λ, there exist $h_i,\ k_j,\ q_i,\ p_j \in \mathbb{Z}$, and $t_i \in \mathbb{N}\setminus\{0\}$, such that*

$$H^{\bullet,\bullet}_{BN}(\lambda,\mathbb{F}[U]) = \bigoplus_{i=1}^{m} \frac{\mathbb{F}[U]}{(U^{t_i})}(h_i,q_i) \oplus \bigoplus_{j=1}^{n} \mathbb{F}[U](k_j,p_j),$$

for some $m,\ n \in \mathbb{N}$, where (a,b) indicates a bi-degree shift of a in the homological degree[1] *and b in the quantum degree.* □

Now it is possible to give a rigorous meaning to the sentence ”determine the structure (or ”the shape“) of Bar-Natan homology“. This will simply mean to determine the integers m, n, t_i, h_i, q_i, k_j and p_j (or relations between them) in terms of known link invariants. Notice that the free part of a module M over a PID R is not a well defined sub-module of M, while the torsion sub-module (or torsion part, see [**31**, Chapter III, §7]) is canonically defined. So the sentence: ”determine the free part of Bar-Natan homology“ is to be read as ”determine the integers n, k_j and p_j”, and similarly the sentence ”determine the torsion part of Bar-Natan homology“ will stand for ”determine integers m, t_i, h_i, and q_i“.

Let us start by determining the rank (that is n) of Bar-Natan homology, as well as the “distribution” of the rank in each homological degree (the k_js). It turns out that these numbers are determined by the rank of (Twisted) Lee theory, and hence by the linking matrix (cf. the next sub-section).

Proposition 2.3 (Turner [**54**]). *Let $\mathbb{F}$ be a field, and L an oriented link diagram with ℓ components. Then, we have:*

$$rank_{\mathbb{F}[U]}(H^{\bullet,\bullet}_{BN}(L;\mathbb{F}[U])) = 2^{\ell},$$

where all the generators lie in even (homological) degree, and if L represents a knot the (homological) degree of the generators is 0. More precisely, we have

$$rank_{\mathbb{F}[U]}(H^{i,\bullet}_{BN}(L;\mathbb{F}[U])) = dim_{\mathbb{F}}(H^{i}_{TLee}(L;\mathbb{F})) =$$

$$= 2\ card\left\{ E \subseteq \{1,2,\cdots,\ell\} \,\middle|\, \sum_{l\in E, m\notin E} 2lk(L_l,L_m) = i \right\},$$

where $L_1,...,L_\ell$ are the components of L.

Proof. Since $\mathbb{F}[U]$ is a PID, it is possible to apply the Künneth formula to the complexes $C^{\bullet,\bullet}_{BN}(L;\mathbb{F}[U])$, and $\mathbb{F}(U)$ (the latter considered with trivial differential). Thus, one obtains the following isomorphism

$$H^{\bullet,\bullet}_{BN}(L;\mathbb{F}[U]) \otimes_{\mathbb{F}[U]} \mathbb{F}(U) \simeq H^{\bullet}(C_{BN}(L,\mathbb{F}[U]) \otimes \mathbb{F}(U)),$$

[1] All elements of A_{BN} are taken to be homogeneous of homological degree 0.

and the first part of the Proposition follows immediately from [**39**, Theorem 2.3]. Regarding the second assertion, consider $\mathbb{F}$ as an $\mathbb{F}[U]$-module where U acts as 1. Again by the Künneth formula there is an isomorphism

$$H^{i}_{TLee}(L,\mathbb{F}) \simeq H^{i,\bullet}_{BN}(L;\mathbb{F}[U]) \otimes_{\mathbb{F}[U]} \mathbb{F} \oplus Tor_{\mathbb{F}[U]}(H^{i+1,\bullet}_{BN}(L,\mathbb{F}[U]),\mathbb{F}).$$

As the torsion of $H^{\bullet,\bullet}_{BN}(L;\mathbb{F}[U])$ is only of the type $\mathbb{F}[U]/(U^k)$ (see Corollary 2.2), the second direct summand in the previous formula vanishes. Finally, the computation of the rank in terms of the linking numbers follows from the rank considerations above, and from a simple computations of the (homological) degree of the canonical generators in $H^{\bullet}_{TLee}(L,\mathbb{F})$ (see [**51**, Theorem 4.1.A], [**32**, Theorem 4.2], [**54**, Theorem 3.1], and also loc. cit. Propositions 2.6 and 2.7).

Q.E.D.

For the sake of completeness, we restate a version of the previous theorem for the TT-homology.

Proposition 2.4 (Khovanov [**28**]). *Let $\mathbb{F}$ be a field, and L an oriented link diagram with ℓ components. Then, we have:*

$$rank_{\mathbb{F}[T]}(H^{i,\bullet}_{TT}(L;\mathbb{F}[T])) = dim_{\mathbb{F}}(H^{i}_{OLee}(L;\mathbb{F})).$$

In particular, if $char(\mathbb{F}) = 2$, the following holds

$$rank_{\mathbb{F}[T]}(H^{\bullet,\bullet}_{TT}(L;\mathbb{F}[T])) = dim_{\mathbb{F}}(H^{\bullet,\bullet}_{Kh}(L;\mathbb{F})).$$

Furthermore, if $char(\mathbb{F}) \neq 2$, we have

$$rank_{\mathbb{F}[T]}(H^{i,\bullet}_{TT}(L;\mathbb{F}[T])) = 2\ card \left\{ E \subseteq \{1,2,\cdots,\ell\} \,\middle|\, \sum_{l\in E, m\notin E} 2lk(L_l,L_m) = i \right\},$$

where $L_1,\ldots,L_\ell$ are the components of L. In particular all the generators lie in even (homological) degree, if L represents a knot the (homological) degree of the generators is 0, and the overall rank is 2^ℓ.

Proof. The Theorem [**39**, Theorem 2.2] ensures that, if $char(\mathbb{F}) \neq 2$, there is a twist equivalence

$$H^{\bullet}_{OLee}(L;\mathbb{F}(T)) \simeq H^{\bullet}(C_{TT}(L,\mathbb{F}[T]) \otimes \mathbb{F}(T));$$

in particular, we get:

$$H^{i}_{OLee}(L;\mathbb{F}(T)) \simeq H^{i}(C_{TT}(L,\mathbb{F}[T]) \otimes \mathbb{F}(T)),$$

as $\mathbb{F}(U)$-modules, for each i. And the claims follow as in Proposition 2.3. If $char(\mathbb{F}) = 2$, by (i) of [**39**, Theorem 2.2] there exists a twist equivalence

$$H^{\bullet,\bullet}_{Kh}(L;\mathbb{F}(T)) \simeq H^{\bullet,\bullet}_{TT}(L;\mathbb{F}(T)),$$

and the statement follows.

Q.E.D.

Remark 12. Proposition 2.4 is due to Khovanov in the case $\mathbb{F} = \mathbb{Q}$, and its proof generalizes easily to any field of odd characteristic. Similarly, Turner gives the proof of Proposition 2.3 only in the case $\mathbb{F} = \mathbb{F}_2$, but the result extends to any field without changing the proof.

In order to completely determine the free part of Bar-Natan theory, all that is left is to compute the p_js. This represent the most difficult part of our task, and relies on properties of the filtered Lee theory.

1.1. Lee theory and the canonical generators. In order to understand the structure of Bar-Natan theory it is fundamental to understand Lee theory. This theory, introduced in the original version by Eun Soo Lee (cf. [**32**]), has been thoroughly studied in the last 11 years in both its original and its twisted version. As stated in the previous chapter, Lee theories are filtered theories. The essential information contained in Lee theories is stored in their filtrations. In fact, as we will see in a moment, disregarding the filtration the Lee homology of an oriented link depends only on the linking matrix (i.e. on the number of components and on the linking numbers between them).

Let L be an oriented link diagram, and denote by $\widetilde{L}$ the underlying unoriented diagram. Consider the set $\mathbb{O}(\widetilde{L})$ of all possible orientations over $\widetilde{L}$. This set contains exactly 2^{ℓ} elements, where ℓ is the number of components of L.

Given an orientation $\mathfrak{o} \in \mathbb{O}(\widetilde{L})$, consider the corresponding oriented diagram $L_{\mathfrak{o}}$, and denote by $n_+(\mathfrak{o})$ and $n_-(\mathfrak{o})$ the number of positive and negative crossings of $L_{\mathfrak{o}}$, respectively. For each choice of an orientation $\mathfrak{o}$, the oriented resolution[2] of $L_{\mathfrak{o}}$, say $\underline{r}_{\mathfrak{o}}$, can be identified with a resolution of L.

Bearing in mind this fact, is possible to define, for each orientation $\mathfrak{o} \in \mathbb{O}(\widetilde{L})$, an enhanced state $\mathbf{v}_{\circ}(\mathfrak{o}) \in C^{\bullet}_{\circ}(L;\mathbb{F})$, where $\circ \in \{OLee, TLee\}$, as follows

(1) mark a point p_γ for each circle γ in $\underline{r}_{\mathfrak{o}}$;
(2) let q_γ be the point in $\mathbb{S}^2$ obtained by pushing p_γ slightly to the left[3] with respect to the orientation induced on $\underline{r}_{\mathfrak{o}}$ by $\mathfrak{o}$;
(3) define the **nesting number** of γ, denoted by N_γ, as the number of intersection points between the circles in $\underline{r}_{\mathfrak{o}}$ and a generic segment between q_γ and the point at the infinity in $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$, modulo 2;
(4) label γ as follows

$$v^{\circ}_{\gamma} = \begin{cases} x_- & \text{if } N(\gamma) \equiv 0 \; mod\, 2 \text{ and } \circ = TLee \\ x_+ + x_- & \text{if } N(\gamma) \equiv 0 \; mod\, 2 \text{ and } \circ = OLee \\ x_+ - x_- & \text{if } N(\gamma) \equiv 1 \; mod\, 2 \end{cases}$$

Finally, set $\mathbf{v}_{\circ}(\mathfrak{o}) = \bigotimes_{\gamma \in \underline{r}_{\mathfrak{o}}} v^{\circ}_{\gamma}$. We will call $\mathbf{v}_{\circ}(L)$, and $\overline{\mathbf{v}}_{\circ}(L) = \mathbf{v}_{\mathfrak{o}}(-L)$ (i.e. the $\mathbf{v}_{\circ}(\mathfrak{o})$ corresponding to the orientation of L and to the opposite orientation,

[2]The only resolution that inherits the orientation from the diagram. More precisely, its the resolution where all negative crossing are replaced by a 1-resolution, and all the positive crossing are replaced by a 0-resolution.

[3]That is, consider the normal vector field $\mathbf{n}$ on γ, and consider a point q_γ in the half-line $p_\gamma + t\,\mathbf{n}(p_\gamma)$, $t \geq 0$, such that the segment between q_γ and p_γ do **not** intersect any circle except γ in p_γ.

respectively) **canonical cycles**. Notice that the homological and quantum degrees of $\mathbf{v}_\circ(\mathfrak{o})$ are, respectively,

$$hdeg(\mathbf{v}_\circ(\mathfrak{o})) = n_-(\mathfrak{o}) - n_-(L)$$

and

$$qdeg(\mathbf{v}_\circ(\mathfrak{o})) = -o(\mathfrak{o}) + n_-(\mathfrak{o}) + n_+(L) - 2n_-(L).$$

where $o(\mathfrak{o})$ is the number of circles in $r_{\mathfrak{o}}$, and we extended the $qdeg$ as the minimum quantum degree of the homogeneous components. In particular,

$$hdeg(\mathbf{v}_\circ(L)) = 0, \qquad qdeg(\mathbf{v}_\circ(L)) = o(L) + w(L),$$

where $o(L)$ denotes the number of circles in the oriented resolution of L. By changing the orientation of a single component, say λ_i, the number of negative crossing in the new orientation is changed by adding twice[4] the sum of the linking numbers of λ_i with all the other components (all positive crossings with the other components become negative, hence should be added, while the previously negative crossings become positive, and so they should be removed from the count). From this simple consideration, and from the fact that $\mathbf{v}_\circ(\mathfrak{o})$ and $\mathbf{v}_\circ(\overline{\mathfrak{o}})$ have the same homological degree, it follows that:

$$card\{\mathfrak{o} \in \mathbb{O}(\widetilde{\lambda}) \,|\, hdeg(\mathbf{v}_\circ(\mathfrak{o})) = i\} = 2card\left\{ E \subseteq \{1,2,\cdots,\ell\} \,\middle|\, \sum_{l\in E, m\notin E} 2lk(\lambda_l,\lambda_m) = i \right\} \tag{5}$$

Proposition 2.5 ([**32**], [**47**]). *The enhanced state $\mathbf{v}_\circ(\mathfrak{o}) \in C^\bullet_\circ(L;\mathbb{F})$ is a cycle, for each $\mathfrak{o} \in \mathbb{O}(\widetilde{L})$ and each $\circ \in \{OLee, TLee\}$.*

Proof. The statement follows directly from the fact that the set of circles of any oriented resolution is bipartite (cf. [**47**, Lemma 2.4 & Corollary 2.5]), and from the definitions of *OLee* and *TLee*.

Q.E.D.

Proposition 2.6 ([**47**], Proposition 2.3). *Let L be a link diagram. The homology class*

$$\mathfrak{s}_\circ(\mathfrak{o},\mathbb{F}) = [\mathbf{v}_\circ(\mathfrak{o})] \in H^\bullet_\circ(L;\mathbb{F}), \quad \mathfrak{o} \in \mathbb{O}(L),$$

is left invariant (up to sign) by the maps induced in homology by the Reidemeister moves, for each $\circ \in \{TLee, OLee\}$, and $\mathbb{F}$ such that $char(\mathbb{F}) \neq 2$ if $\circ = OLee$. □

Proposition 2.7 ([**32**], Theorem 4.2). *Let L be a link diagram. The homology class*

$$\mathfrak{s}^\circ_{\mathfrak{o}} = [\mathbf{v}_\circ(\mathfrak{o})] \in H^\bullet_\circ(L;\mathbb{F}), \quad \mathfrak{o} \in \mathbb{O}(L),$$

are a basis for $H^\bullet_\circ(L;\mathbb{F})$, for each $\circ \in \{OLee, TLee\}$, and for each field $\mathbb{F}$ such that $char(\mathbb{F}) \neq 2$ if $\circ = OLee$. In particular,

$$rank(H^i_\circ(L;\mathbb{F})) = card\{\mathfrak{o} \in \mathbb{O}(\widetilde{L}) \,|\, hdeg(\mathbf{v}_\circ(\mathfrak{o})) = i\}.$$

□

[4] Recall that the linking number is *half* the signed sum of the crossings between two components.

Proofs of the Propositions 2.6 and 2.7 are split in various sources: see [**51**, Theorem 4.1.A] for the case $\circ = OLee$ and $char(\mathbb{F})$ odd or zero, and [**54**, Theorem 3.1], for the case $char(\mathbb{F}) = 2$ and $\circ = TLee$. Notice that, in the case $char(\mathbb{F}) \neq 2$, $OLee$ and $TLee$ are twist equivalent, and the equivalence (cf. [**39**, Proposition 3.1]) sends $\mathbf{v}_{OLee}(\mathfrak{o})$ to (a non zero multiple of) $\mathbf{v}_{TLee}(\mathfrak{o})$; hence, if $\circ = TLee$ the statement holds true for every field $\mathbb{F}$. All the above-mentioned proofs are suitable adaptations of the original proof by Lee (cf. [**32**, Theorem 4.2]). Because of Proposition 2.6 and Proposition 2.7, the $\mathfrak{s}_\circ(\mathfrak{o}, \mathbb{F})$ are also called **canonical generators**.

1.2. Concordance invariants from Lee homologies. Let λ_0 and λ_1 be two oriented links in $\mathbb{R}^3$. A **cobordism** between λ_0 and λ_1 is a compact oriented surface Σ, properly embedded in $\mathbb{R}^3 \times [0,1]$, such that

$$\Sigma \cap \mathbb{R}^3 \times \{i\} = \lambda_i, \qquad i \in \{0,1\},$$

with the induced orientation on λ_0 and the opposite of the induced orientation on λ_1.

Given a cobordism Σ between two links, say λ and λ', it is useful to distinguish between its connected components; a component of Σ is of **first type** if it bounds a component of λ, and is of **second type** otherwise. Let $\widetilde{\lambda}$ be the unoriented link underlying λ. Two orientations $\mathfrak{o}$ and $\mathfrak{o}'$, on $\widetilde{\lambda}$ and $\widetilde{\lambda}'$, are **compatible via** Σ if there exists an orientation of Σ bounding the oriented links $(\widetilde{\lambda}, \mathfrak{o})$ and $(\widetilde{\lambda}', \overline{\mathfrak{o}'})$. The set of pair of compatible orientations will be denoted by $\mathbb{O}_\Sigma(\widetilde{\lambda}, \widetilde{\lambda}')$. Finally, the set of pair $(\mathfrak{o}, \mathfrak{o}') \in \mathbb{O}_\Sigma(\widetilde{\lambda}, \widetilde{\lambda}')$, for a fixed $\mathfrak{o}$, will be denoted by $\mathbb{O}_\Sigma(\mathfrak{o}, \widetilde{\lambda}')$.

A cobordism Σ is a **weak cobordism** (or weakly connected in the language of [**47**]) if all its components are of first type. While Σ is a **strong cobordism** if each component of Σ bounds exactly one component of λ and one component of λ'.

Remark 13. Every strong cobordism is also a weak cobordism. For each weak cobordims between λ and λ', there is a unique orientation of $\widetilde{\lambda}'$ compatible with the orientation of λ.

Definition 2.1. A link λ is **strongly concordant** (resp. **weakly concordant**) to a link λ' if there exists a strong (resp. weak) cobordism of genus 0 between λ and λ'. Any link which is strongly (resp. weakly) concordant to an unlink is called **strongly** (resp. **weakly**) **slice**

There is a theorem due to Rasmussen ([**47**, Proposition 4.1]), describing the behaviour of the canonical generators under cobordism. Below is a slight restatement of the theorem, whose proof is identical to the one given by Rasmussen.

Proposition 2.8 (Rasmussen, [**47**]). *Given a cobordism Σ with no closed components between λ and λ'*

$$(\Phi_\Sigma)_*(\mathfrak{s}_\circ(\mathfrak{o}, \mathbb{F})) = \pm \sum_{\mathfrak{o}' \in \mathbb{O}_\Sigma(\mathfrak{o}, \widetilde{\lambda}')} \alpha_{\mathfrak{o}'} \mathfrak{s}_\circ(\mathfrak{o}', \mathbb{F}),$$

where $\alpha_{\mathfrak{o}'}$ is different from zero for each $\mathfrak{o}'$, $\circ \in \{TLee, OLee\}$, and $char(\mathbb{F}) \neq 2$ if $\circ = OLee$. In particular, if Σ is a strong cobordism, then $(\Phi_\Sigma)_*$ is a filtered isomorphism of degree $-\chi(\Sigma)$. □

Corollary 2.9. *The filtered isomorphism class of $H^\bullet_\circ(\lambda, \mathbb{F})$ is a strong concordance invariant.* □

Suppose $char(\mathbb{F}) \neq 2$. Following Rasmussen ([**47**]), Beliakova and Wehrli ([**6**]) introduced a family of numerical invariants for links, which are defined as follows

$$s_\circ(\mathfrak{o}, \mathbb{F}) = \frac{Fdeg\left(\mathfrak{s}_\circ(\mathfrak{o}, \mathbb{F}) + \mathfrak{s}_\circ(\bar{\mathfrak{o}}, \mathbb{F})\right) + Fdeg\left(\mathfrak{s}_\circ(\mathfrak{o}, \mathbb{F}) - \mathfrak{s}_\circ(\bar{\mathfrak{o}}, \mathbb{F})\right)}{2}.$$

As the two filtered degrees (see Appendix A) in the above formula differ exactly by 2 (see [**47**], or [**6**, Section 7.1]), one can also give an alternative, equivalent, definition

$$s_\circ(\mathfrak{o}, \mathbb{F}) = min\left\{Fdeg\left(\mathfrak{s}_\circ(\mathfrak{o}, \mathbb{F}) + \mathfrak{s}_\circ(\bar{\mathfrak{o}}, \mathbb{F})\right), Fdeg\left(\mathfrak{s}_\circ(\mathfrak{o}, \mathbb{F}) - \mathfrak{s}_\circ(\bar{\mathfrak{o}}, \mathbb{F})\right)\right\} + 1.$$

However, the definition of the s-invariant should be modified to include the case $char(\mathbb{F}) = 2$ and $\circ = TLee$. Following Mackaay, Turner and Vaz ([**39**]) define

$$s_{TLee}(\mathfrak{o}, \mathbb{F}) = min\left\{Fdeg(x) \mid x \in \langle \mathfrak{s}_{TLee}(\mathfrak{o}, \mathbb{F}), \mathfrak{s}_{TLee}(\bar{\mathfrak{o}}, \mathbb{F})\rangle_{\mathbb{F}} \subseteq H^\bullet_{TLee}(L, \mathbb{F})\right\} + 1.$$

From Corollary A.6 in Appendix A follows that this definition is equivalent to the other definitions if $char(\mathbb{F}) \neq 2$.

From the basic properties of self-dual Frobenius algebras, and their relationships with the homology of the mirror link, and from Proposition 2.8 the next result follows immediately.

Proposition 2.10 (Rasmussen, [**47**], Beliakova-Wehrli [**6**]). *Let λ and λ' be two oriented links in $\mathbb{R}^3$, and let Σ be a weak cobordism between them. Then the following inequality holds*

$$|s_\circ(\mathfrak{o}, \mathbb{F})(\lambda) - s_\circ(\mathfrak{o}', \mathbb{F})(\lambda')| \leq -\chi(\Sigma), \quad \forall (\mathfrak{o}, \mathfrak{o}') \in \mathbb{O}_\Sigma(\widetilde{\lambda}, \widetilde{\lambda}').$$

In particular, since $\chi(\Sigma) = 0$ for any strong concordance, the $s_\circ(\mathfrak{o}, \mathbb{F})$s are strong concordance invariants. □

These invariants are more than just strong concordance invariants, they also provide an obstruction to weak sliceness (as follows almost immediately from the previous proposition).

Corollary 2.11 (Beliakova-Wehrli, [**6**]). *If λ is weakly slice, then*

$$|s_\circ(\lambda, \mathbb{F})| \leq \ell - 1,$$

where ℓ is the number of components in λ, and $s_\circ(\lambda, \mathbb{F})$ indicates $s_\circ(\mathfrak{o}_0, \mathbb{F})$, where $\mathfrak{o}_0$ is the orientation of λ. □

Finally, let us recall the other properties of the s-invariant (i.e the Rasmussen-Beliakov-Wehrli invariant associated to the orientation of L).

Theorem 2.12 (Rasmussen [**47**], Beliakova and Wehrli [**6**]). *Let λ and λ' be two oriented links, and let ℓ be the number of components of λ, then*

$$s(\lambda \sqcup \lambda', \mathbb{F}) = s(\lambda, \mathbb{F}) + s(\lambda', \mathbb{F}) - 1 \tag{6}$$

$$s(\lambda, \mathbb{F}) + s(\lambda', \mathbb{F}) - 2 \leq s(\lambda \sharp \lambda', \mathbb{F}) \leq s(\lambda, \mathbb{F}) + s(\lambda', \mathbb{F}) \tag{7}$$

$$2 - 2\ell \leq s(\lambda, \mathbb{F}) + s(\lambda^*, \mathbb{F}) \leq 2 \tag{8}$$

where $\sharp$ stands for the connected sum, and * stands for the mirror link.

Another generalization of the Rasmussen invariants for knots are **Pardon's invariants** (cf. [**44**]). These invariants were introduced by John Pardon, and are strong concordance invariants generalizing the Rasmussen invariant for knots to links. They are defined as follows

$$d^{\circ}_{h,q}(\lambda, \mathbb{F}) = dim_{\mathbb{F}} \left(\frac{\mathscr{F}^q H^h_{\circ}(\lambda; \mathbb{F})}{\mathscr{F}^{q+1} H^h_{\circ}(\lambda; \mathbb{F})} \right), \quad \circ \in \{OLee, TLee\};$$

where

$$\mathscr{F}^q H^h_{\circ}(\lambda; \mathbb{F}) = (\tilde{\iota}_k)_* \left(H^h(\mathscr{F}^q C_{\circ}(\lambda; \mathbb{F})) \right),$$

and $\tilde{\iota}_q$ is the inclusion of chain complexes $\mathscr{F}^q C_{\circ}(L, R)) \hookrightarrow C_{\circ}(L, R)$ (see Chapter 1 Subsection 2.2 for the definition of filtrations in Lee theories, and Appendix A Section 5 for some general definitions regarding filtered chain complexes).

Thanks to [**39**, Theorem 2.3], there is a twist equivalence, respecting the filtration, between $TLee$ and $OLee$ if $char(\mathbb{F}) \neq 2$. This implies

$$d^{TLee}_{h,q}(\lambda, \mathbb{F}) = d^{OLee}_{h,q}(\lambda, \mathbb{F}), \text{ and } s_{TLee}(\mathfrak{o}, \mathbb{F}) = s_{OLee}(\mathfrak{o}, \mathbb{F}) \quad \text{if } char(\mathbb{F}) \neq 2.$$

So we are going to omit the reference to the theory, taking $s(\mathfrak{o}, \mathbb{F})(\lambda)$ and $d_{h,q}(\lambda, \mathbb{F})$ to be the ones defined from twisted Lee theory.

Remark 14. The field will always appear in the notation, in fact there is a difference between these invariants in characteristic 0 and in characteristic 2 (cf. [**35**]). Whether or not there is a difference among fields with odd characteristic, and characteristic 0 is still unknown (at least to the author's knowledge, cf. [**38**]).

The original Pardon's invariants, defined in [**44**], in our notation correspond to $d^{OLee}_{h,q}(\mathbb{Q})$. However, the strong concordance invariance, as well as the other properties (cf. [**44**, Theorem 1.2], loc. cit. Corollary 2.9), of $d_{h,q}$ are still valid, for both $d^{TLee}_{h,q}(\mathbb{F})$ and $d^{OLee}_{h,q}(\mathbb{F})$, over any field $\mathbb{F}$ of zero or odd characteristic, or in case $char(\mathbb{F}) = 2$ for $TLee$ (in fact, the properties listed in [**44**, Theorem 2.1] still hold in these contexts).

1.3. The quantum grading in the Free part. In order to complete the task of determine quantum degree of the free part (i.e. the p_js of Corollary 2.2), some preliminary definition is needed. Let L be an oriented link diagram, and λ the oriented link-type represented by L. Set

$$s_i(\lambda, \mathbb{F}) = max\left\{k \mid dim_{\mathbb{F}}\left((\tilde{\iota}_k)_*\left(H(\mathscr{F}^k_{TLee}(L, \mathbb{F}))\right)\right) \geq i\right\},$$

where $\mathscr{F}^j_{TLee}(L, \mathbb{F})$ denotes the filtration in the twisted Lee complex (previously denoted by $\mathscr{F}^j C^\bullet_{TLee}(L, \mathbb{F})$, see Chapter 1 Subsection 2.2 for the definition), and $\tilde{\iota}_k$ is the natural inclusion map $\mathscr{F}^k_{TLee}(L, R) \hookrightarrow C^\bullet_{TLee}(L, R)$. In the case κ is a knot, we have

$$s_1(\kappa, \mathbb{F}) = s(\kappa, \mathbb{F}) + 1, \quad s_2(\kappa, \mathbb{F}) = s(\kappa, \mathbb{F}) - 1;$$

the above equalities can be deduced from two facts: first, the Lee homology of a knot is bi-dimensional, and second the homology classes of the canonical generators have filtered degree $s(\kappa, \mathbb{F}) \pm 1$. Now it is necessary to relate the filtered Twisted Lee theory with (a modified version of) Bar-Natan theory.

In order to "build a bridge" between the two theories is necessary to modify slightly the complex $C^{\bullet,\bullet}_{BN}$. Consider $\mathbb{F}[U^{1/2}]$, with its natural structure of $\mathbb{F}[U]$-module, as a bi-graded $\mathbb{F}[U]$-complex with trivial differential, whose generators 1 and $U^{1/2}$ lie in bi-degree $(0,0)$ and $(0,-1)$, respectively. Finally, define the **total Bar-Natan complex** $\overline{C}^{\bullet,\bullet}_{BN}(\lambda, \mathbb{F})$ to be the tensor complex $C^{\bullet,\bullet}_{BN}(\lambda, \mathbb{F}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U^{1/2}]$. Notice that the total Bar-Natan complex is also a bi-graded chain complex over $\mathbb{F}[U^{1/2}]$.

As a simple application of the Künneth formula, one can compute the homology of the total Bar-Natan complex from the homology of the (usual) Bar-Natan complex. More precisely,

$$\overline{H}^{i,\bullet}_{BN}(\lambda, \mathbb{F}) \simeq H^{i,\bullet}_{BN}(\lambda, \mathbb{F}) \otimes 1_{\mathbb{F}[U^{1/2}]} \oplus H^{i,\bullet+1}_{BN}(\lambda, \mathbb{F}) \otimes U^{1/2},$$

as graded $\mathbb{F}[U]$-modules, where $H^{i,\bullet}_{BN}(\lambda, \mathbb{F}) \otimes a$ denotes the $\mathbb{F}[U]$-sub-module of $H^{i,\bullet}_{BN}(\lambda, \mathbb{F}) \otimes \mathbb{F}[U^{1/2}]$ generated by all elementary tensors of the form $x \otimes a$ with $x \in H^{i,\bullet}_{BN}(\lambda, \mathbb{F})$.

Proposition 2.13. *Let $\mathbb{F}$ be a field, and L be an oriented link diagram. Then, the map*

$$\pi_j : \overline{C}^{\bullet,j}_{BN}(L, \mathbb{F}) \to \mathscr{F}^j_{TLee}(L, \mathbb{F}) : \sum_r x_r \otimes P_r(U^{\frac{1}{2}}) \mapsto \sum_r P_r(1) x_r,$$

is an isomorphism of ($\mathbb{F}$-)complexes. Moreover, the following diagram commutes

$$\begin{tikzcd}
\overline{C}^{\bullet,j}_{BN}(L, \mathbb{F}[U]) \arrow[r, "\pi_j"] \arrow[d, "U^{\frac{1}{2}}"'] & \mathscr{F}^{j,\bullet}_{TLee}(L, \mathbb{F}) \arrow[d, "\iota_j"] \\
\overline{C}^{\bullet,j-1}_{BN}(L, \mathbb{F}[U]) \arrow[r, "\pi_{j-1}"'] & \mathscr{F}^{j-1}_{TLee}(L, \mathbb{F})
\end{tikzcd}$$

Proof. It is sufficient to observe that the maps π_j are just restrictions of the quotient map

$$\overline{\pi} : \overline{C}^{\bullet,\bullet}_{BN}(L, \mathbb{F}) \to C^{\bullet}_{TLee}(L, \mathbb{F}) = \frac{\overline{C}^{\bullet,\bullet}_{BN}(L, \mathbb{F})}{(U^{\frac{1}{2}} - 1)\overline{C}^{\bullet,\bullet}_{BN}(L, \mathbb{F})}.$$

Then, the existence of the unique homogeneous lift proven in Appendix A (loc. cit. Lemma A.10), ensures the bijectivity of π_j for each j. The fact that π_j is a chain map is an immediate consequence of the definition of π_j, and of the fact that the identification of A_{TLee} with $A_{BN} \otimes \mathbb{F}[U]/(U-1)$ commutes with multiplication and co-multiplication (that is: this identification induces[5] a morphism of Frobenius Algebras). The commutativity of the diagram is trivial.

Q.E.D.

Lemma 2.14. *Let M be a graded $\mathbb{F}[U]$-module (deg$(U) = -2$). Denote by $T(M)$ the torsion sub-module of M (see* [**31**, Ch. III §7]*). Then*

$$T(M) \subseteq (U-1)M,$$

and

$$M_k \cap T(M) = M_k \cap (U-1)M.$$

Proof. Consider $x \in T(M)$, by theorem 2.1, then there exists a $r > 0$ such that $U^r x = 0$, hence

$$x = -U^r x + x = -(U^r - 1)x = -(U-1)(U^{r-1} + ... + 1)x \in (U-1)M.$$

In particular, we have

$$M_k \cap T(M) \subseteq M_k \cap (U-1)M.$$

Consider $x_k \in M_k \cap (U-1)M$, by definition, there exists $y \in M$ such that

$$x_k = (U-1)y = (U-1)\left(\sum_{i=m}^{n} y_i\right), \quad y_i \in M_i.$$

Hence,

$$\delta_{i,k} x_k = (U y_{i+2} - y_i),$$

where we set $y_j = 0$, for each $j > n$ and $j < m$. As a consequence y_j is torsion for each $k \geq j \geq m$. While, y_j is trivial for each $n \geq j \geq k+2$. Thus, x_k is either trivial or torsion, completing the proof.

Q.E.D.

[5]Recall that a morphism of Frobenius algebras is a pair (φ, ψ), in this case it is sufficient to take $\psi = id_{\mathbb{F}}$ and as φ the above mentioned identification.

Lemma 2.15. *Let L be an oriented link diagram. Consider the commutative diagram*

$$\begin{tikzcd}
 & & \frac{H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}{T\left((H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])\right)} \\
H^{\bullet,k}_{BN}(L,\mathbb{F}[U]) \arrow[r] \arrow[urr, "\jmath'_k"] \arrow[drr, "\jmath_k"'] & H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U]) \arrow[ur, "\pi'"'] \arrow[dr, "\pi"] & \\
 & & \frac{H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}{(U-1)H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}
\end{tikzcd}$$

where the horizontal map is the inclusion in the direct sum, while π, π' are projection onto the quotient; then

$$dim_{\mathbb{F}}(Im(\jmath_k)) = dim_{\mathbb{F}}(Im(\jmath'_k))$$

Proof. Notice that, thanks to the previous lemma, there is a well defined map π_∞ that makes the diagram below commute

$$\begin{tikzcd}
 & & \frac{H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}{T(H^{\bullet,\bullet}_{BN})(L,\mathbb{F}[U])} \arrow[dd, "\pi_\infty"] \\
H^{\bullet,k}_{BN}(L,\mathbb{F}[U]) \arrow[r] \arrow[urr, "\jmath'_k"] \arrow[drr, "\jmath_k"'] & H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U]) \arrow[ur, "\pi'"'] \arrow[dr, "\pi"] & \\
 & & \frac{H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}{(U-1)H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}
\end{tikzcd}$$

This map, seen as a map from $Im(\jmath'_k)$ to $Im(\jmath_k)$, is surjective. Moreover, because $\jmath'_k$ and $\jmath_k$ share the same kernel (see Lemma 2.14), it is also injective, and the claims follows.

Q.E.D.

Theorem 2.16. *Let λ be a link type, and ℓ the number of components in λ. The Bar-Natan homology of λ is isomorphic, as bi-graded $\mathbb{F}[U]$-module, to*

$$\bigoplus_{r=1}^{n} \frac{\mathbb{F}[U]}{(U^{t_r})}(h_r, q_r) \oplus \bigoplus_{i=1}^{2^\ell} \mathbb{F}[U](k_i, s_i).$$

In particular, the following holds

$$s_i \equiv \ell \quad mod\ 2, \qquad \forall i \in \{1, ..., 2^\ell\}.$$

Proof. Using the notation of Corollary 2.2, fix an order on the p_is in such a way that $p_i \geq p_{i-1}$. The first part of the theorem may be re-stated as follows:

$$s_i = p_i, \qquad \forall i \in \{1, ..., 2^\ell\}.$$

Consider the commutative square

$$\begin{tikzcd}
\overline{H}^{\bullet,k}_{BN}(L,\mathbb{F}[U]) \arrow[r, "\jmath_k+\jmath_{k+1}"] \arrow[d, "\pi_k"', "\simeq"] & \frac{H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}{(U-1)H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])} \arrow[d, "\simeq"', "\pi_\infty"] \\
H^i(\mathscr{F}^{k,\bullet}_{TLee}(L,\mathbb{F})) \arrow[r, "(\tilde{\imath}_k)_*"'] & H^{\bullet}_{TLee}(L,\mathbb{F})
\end{tikzcd}$$

where we have identified $\overline{H}^{\bullet,k}_{BN}(L,\mathbb{F}[U])$ and $H^{\bullet,k}_{BN}(L,\mathbb{F}[U]) \oplus H^{\bullet,k+1}_{BN}(L,\mathbb{F}[U])$, and π_∞ is the natural quotient map (cf. proof of Lemma 2.15). Being the vertical arrows in the diagram above isomorphisms, the dimension (over $\mathbb{F}$) of the image of $(\tilde{\imath}_k)_*$ is the same as the dimension of the image of the map $\jmath_k + \jmath_{k+1}$. Thus we get the following characterization of s_i.

$$s_i(\lambda,\mathbb{F}) = max\left\{ k \mid dim_{\mathbb{F}}\left((\jmath_k + \jmath_{k+1})\left(\overline{H}^{\bullet,k}_{BN}(L,\mathbb{F}[U])\right)\right) \geq i \right\}. \tag{9}$$

In order to conclude it is necessary to find a characterization of the p_is in terms of the maps $\jmath_k$. Our claim is

$$p_i = max\{k \mid \jmath_k(H^{\bullet,k}_{BN}(L,\mathbb{F}[U])) \text{ has dimension } \geq i\}. \tag{10}$$

Assuming (10), from (9) follows that

$$s_i \geq p_i.$$

The first part of Corollary 2.24 (which does not rely on any result in this section), asserts that if $r \equiv \ell + 1$ modulo 2, then

$$H^{\bullet,r}_{BN}(L;\mathbb{F}[U]) \equiv 0.$$

It follows that one among $\jmath_k$ and $\jmath_{k+1}$ is trivial. If $\jmath_{s_i+1}$ is trivial then it follows that

$$i \leq dim_{\mathbb{F}}\left((\jmath_{s_i} + \jmath_{s_i+1})(\overline{H}^{\bullet,s_i}_{BN}(L,\mathbb{F}[U^{1/2}]))\right) = dim_{\mathbb{F}}\left(\jmath_{s_i}(H^{\bullet,s_i}_{BN}(L,\mathbb{F}[U]))\right),$$

which implies, always assuming (10),

$$p_i \geq s_i,$$

and the statement follows. Consider the following (commutative) diagram

$$\begin{tikzcd}
H^{\bullet,s_i}_{BN}(L,\mathbb{F}[U]) \oplus H^{\bullet,s_i+1}_{BN}(L,\mathbb{F}[U]) \otimes U^{\frac{1}{2}} \arrow[dr] & \\
 & \frac{H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}{(U-1)H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])} \\
H^{\bullet,s_i+1}_{BN}(L,\mathbb{F}[U]) \oplus H^{\bullet,s_i+2}_{BN}(L,\mathbb{F}[U]) \otimes U^{\frac{1}{2}} \arrow[ur] \arrow[uu, "U^{\frac{1}{2}}"] &
\end{tikzcd}$$

and assume, by contradiction, that the modules in gray (and hence $\jmath_{s_i}$) are trivial. Because the multiplication by $U^{\frac{1}{2}}$ maps $H^{\bullet,s_i+1}_{BN}(L,\mathbb{F}[U]) \otimes 1$ surjectively onto

$H^{\bullet,s_i+1}_{BN}(L,\mathbb{F}[U])\otimes U^{\frac{1}{2}}$, also the image of $\jmath_{s_i+1}+\jmath_{s_i+2}$ would have dimension at least i, which is absurd by (9) (↯).

So all is left is to prove (10). Fix an isomorphism

$$H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])\simeq\bigoplus_{i=1}^{m}\frac{\mathbb{F}[U]}{(U^{t_i})}(h_i,q_i)\oplus\bigoplus_{j=1}^{n}\mathbb{F}[U](k_j,p_j),\tag{11}$$

which exists by Corollary 2.2. Consider the natural generators of the module on the right hand side of (11), that is

$$e_i=(0,...,0,\overset{i-th\text{ place}}{\overset{\downarrow}{[1]}},0,...,0),\quad\text{and}\quad f_j=(0,...,0,\overset{(m+j)-th\text{ place}}{\overset{\downarrow}{1}},0,...,0),$$

where $i\in\{1,...,m\}$ and $j\in\{1,...,n\}$. Denote by $\tilde{e}_i$ and $\tilde{f}_j$ the pull-back under the isomorphism in (11) of e_i and f_j, respectively. It is immediate that $H^{\bullet,k}_{BN}(L,\mathbb{F}[U])$, as an $\mathbb{F}$-vector space, has a basis composed by all elements of the form $U^{n_i}\tilde{e}_i$ and $U^{m_i}\tilde{f}_j$ such that

$$qdeg(\tilde{e}_i)-2n_i=k=qdeg(\tilde{f}_j)-2n_j.$$

The homogeneous generators of $H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])$ have quantum degree which is congruent to ℓ modulo 2. This follows from the first part of Corollary 2.24 which is independent from the results in this section. From Lemma 2.15 it follows that the dimension of $\jmath_k(H^{\bullet,k}_{BN}(L,\mathbb{F}[U]))$ is the dimension of the ($\mathbb{F}$-)vector subspace of $H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])$ generated by all homogeneous non-torsion elements of degree k. It follows that the dimension of the image of $\jmath_k$ is greater than or equal to i if, and only if, at least i among the $\tilde{f}_j$'s have degree at least k.

Q.E.D.

Corollary 2.17. *Let κ be a knot type. The Bar-Natan homology of κ is isomorphic, as bi-graded $\mathbb{F}[U]$-module, to*

$$\bigoplus_{i=1}^{n}\frac{\mathbb{F}[U]}{(U^{t_i})}(h_i,q_i)\oplus\mathbb{F}[U](0,s(\kappa,\mathbb{F})-1)\oplus\mathbb{F}[U](0,s(\kappa,\mathbb{F})+1).$$

□

By definition of the total Bar-Natan complex, one has the following isomorphism of bi-graded $\mathbb{F}[U]$-modules:

$$\overline{H}^{h,q}_{BN}(L,\mathbb{F}[U^{1/2}])\simeq H^{h,q}_{BN}(L,\mathbb{F}[U])\oplus H^{h,q+1}_{BN}(L,\mathbb{F}[U])=H^{h,k(\ell,q)}_{BN}(L,\mathbb{F}[U]),$$

where $k(\ell,q)$ is the lowest number, which is greater than, or equal to q, such that $k(\ell,q)\equiv\ell\ mod\,2$. The isomorphisms (of $\mathbb{F}$-complexes)

$$\pi_q^{-1}:\mathscr{F}^q_{TLee}(L,\mathbb{F})\longrightarrow\overline{H}^{\bullet,q}_{BN}(L,\mathbb{F}[U^{1/2}]),$$

induce an isomorphism between the direct limits (which is the natural map used also in the proof of Proposition 2.13)

$$\pi_\infty^{-1}:\underset{\leftarrow}{lim}\,\mathscr{F}^q_{TLee}(L,\mathbb{F})(\simeq H^{\bullet}_{TLee}(L,\mathbb{F}))\longrightarrow\underset{\leftarrow}{lim}\,\overline{H}^{\bullet,q}_{BN}(L,\mathbb{F}[U^{1/2}])\left(\simeq\frac{H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}{(U-1)H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U])}\right).$$

Consider the commutative diagram

$$\begin{tikzcd} \overline{H}^{\bullet,q}_{BN}(L,\mathbb{F}[U^{1/2}]) \arrow[r, "J_q = \jmath_q + \jmath_{q+1}"] \arrow[d, "\pi_q^{-1}"', "\simeq"] & \frac{H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U^{1/2}])}{(U-1)H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U^{1/2}])} \arrow[d, "\pi_\infty^{-1}", "\simeq"'] \\ H^i(\mathscr{F}^q_{TLee}(L,\mathbb{F})) \arrow[r, "(\imath_q)_*"'] & H^\bullet_{TLee}(L,\mathbb{F}) \end{tikzcd}$$

Since the inclusion in the direct limit J_q, is such that

$$J_q\left(\overline{H}^{h,q}_{BN}(L,\mathbb{F}[U^{1/2}])\right) = \frac{H^{h,k(\ell,q)}_{BN}(L,\mathbb{F}[U])}{[(U-1)H^{h,\bullet}_{BN}(L,\mathbb{F}[U])] \cap H^{h,k(\ell,q)}_{BN}(L,\mathbb{F}[U])}$$

there is an isomorphism

$$\mathscr{F}^q H^h_{TLee}(L,\mathbb{F}) \simeq \frac{H^{h,k(\ell,q)}_{BN}(L,\mathbb{F}[U])}{[(U-1)H^{h,\bullet}_{BN}(L,\mathbb{F}[U])] \cap H^{h,k(\ell,q)}_{BN}(L,\mathbb{F}[U])} \simeq$$

and the latter can be identified, thanks to Lemma 2.14, with

$$\simeq \frac{H^{h,k(\ell,q)}_{BN}(L,\mathbb{F}[U])}{Tor\ H^{h,\bullet}_{BN}(L,\mathbb{F}[U]) \cap H^{h,k(\ell,q)}_{BN}(L,\mathbb{F}[U])}.$$

Finally, because the multiplication by $U^{1/2}$ from $\overline{H}^{\bullet,q+1}_{BN}$ to $\overline{H}^{\bullet,q}_{BN}$ is surjective if $q \equiv \ell + 1\ mod\, 2$, and corresponds to the multiplication by U from $H^{\bullet,q+2}_{BN}$ to $H^{\bullet,q}_{BN}$, if $q \equiv \ell\ mod\, 2$ (cf. Proposition 2.13), it follows that

$$\frac{\mathscr{F}^q H^h_{TLee}(L,\mathbb{F})}{\mathscr{F}^{q+1} H^h_{TLee}(L,\mathbb{F})} \simeq \frac{\frac{H^{h,k(\ell,q)}_{BN}(L,\mathbb{F}[U])}{Tor\ H^{h,\bullet}_{BN}(L,\mathbb{F}[U]) \cap H^{h,q}_{BN}(L,\mathbb{F}[U])}}{U^{\frac{k(\ell,q)-k(\ell,q+1)}{2}} \frac{H^{h,k(\ell,q+1)}_{BN}(L,\mathbb{F}[U])}{Tor\ H^{h,\bullet}_{BN}(L,\mathbb{F}[U]) \cap H^{h,k(\ell,q+1)}_{BN}(L,\mathbb{F}[U])}}.$$

We observe that using the $\mathbb{F}[U]$-module structure of Bar-Natan homology, the above isomorphism can be interpreted as saying that the Pardon's invariant $d^{TLee}_{h,q}$ counts how many "generators of the free part" there are in bi-degree (h,q). More formally,

Proposition 2.18. *If the Bar-Natan homology of an oriented link λ is*

$$\bigoplus_{r=1}^{n} \frac{\mathbb{F}[U]}{(U^{t_r})}(h_r,q_r) \oplus \bigoplus_{i=1}^{2^\ell} \mathbb{F}[U](k_i,s_i),$$

then

$$d_{h,q}(\lambda,\mathbb{F}) = card(\{i \mid (k_i,s_i) = (h,q)\}).$$

□

From which follows immediately

Corollary 2.19. *All the s_is are strong concordance invariants.* □

Remark 15. In general, for any knot κ we have

$$d_{h,q}(\kappa) \neq 0 \iff (h,q) \in \{(0, s(\kappa) \pm 1)\},$$

re-proving Corollary 2.17. In the case of links, the points where $d_{h,q}$ is non-trivial do not determine the Rasmussen-Beliakova-Wehrli invariants (shifted by ± 1). For example, consider the n-components unlink $\mathcal{U}_n$ and $\mathbb{F} = \mathbb{Q}$. Direct computations show that

$$d_{h,q}(\mathcal{U}_n) = \begin{cases} \binom{k}{n} & \text{if } (h,q) \in \{(0, 2k-n)\}_{k=0,\dots,n} \\ 0 & \text{otherwise} \end{cases},$$

while, for $n = 2$

$$s_{\mathfrak{o}}(\mathcal{U}_2) = -1, \quad \forall \mathfrak{o} \in \mathbb{O}(\mathcal{U}_2).$$

2. The structure of Bar-Natan homology II: torsion

To complete our description of Bar-Natan homology all that is left is to determine the torsion sub-module of $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$. In a similar way to how the free part of Bar-Natan homology could be recovered from (twisted) Lee theory, the torsion of $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$ could be recovered from Khovanov homology. From now on by "U-torsion" we will mean the elements of $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$ which are annihilated by the multiplication for some power of U.

2.1. Torsion and Khovanov homology.

Proposition 2.20. *Let L be an oriented link diagram, and $\mathbb{F}$ be a field. Suppose that the $\mathbb{F}$-vector space $H^{i-1,j-2}_{Kh}(L, \mathbb{F})$ is trivial. Then, the map*

$$U\cdot \; : H^{i,j}_{BN}(L, \mathbb{F}[U]) \to H^{i,j-2}_{BN}(L, \mathbb{F}[U]),$$

is injective. In particular, if $H^{i-1,\bullet}_{Kh}(L, \mathbb{F})$ is trivial, then $H^{i,\bullet}_{BN}(L, \mathbb{F})$ is either trivial, or a free $\mathbb{F}[U]$-module.

Proof. Consider the short exact sequence of complexes, which descends directly from the definitions,

$$0 \to C^{\bullet,j}_{BN}(L, \mathbb{F}[U]) \xrightarrow{U\cdot} C^{\bullet,j-2}_{BN}(L, \mathbb{F}[U]) \xrightarrow{\pi_{Kh}} C^{\bullet,j-2}_{Kh}(L, \mathbb{F}) \to 0.$$

This sequence induces a long exact sequence in homology

$$\cdots \to H^{i-1,j-2}_{Kh}(L, \mathbb{F}) \xrightarrow{\partial_*} H^{i,j}_{BN}(L, \mathbb{F}[U]) \xrightarrow{U_*} H^{i,j-2}_{BN}(L, \mathbb{F}[U]) \to \cdots$$

Thus, by exactness we have

$$Ker(U_* |_{H^{i,j}_{BN}}) = \partial_*(H^{i-1,j-2}_{Kh}(L, \mathbb{F})),$$

and the first part of the claim follows. The second part of the Proposition is a simple consequence of the first part, and of Corollary 2.2.

Q.E.D.

Remark 16. The previous proposition holds for any ring R in place of $\mathbb{F}$. This because all we have used are the definitions of BN and Kh. In the following we will rely on the Künneth formula, as well as on the structure theorem for graded modules (i.e. Theorem 2.1), and both of them require $R[U]$ to be a PID (to be precise, the Künneth formula requires $R[U]$ to be a hereditary domain, plus some more hypotheses regarding the complex, see **[20]**). This is the main reason why our theorems and considerations are limited to the case $R = \mathbb{F}$.

Now we can state a precise relationship between Khovanov homology and Bar-Natan homology.

Proposition 2.21. *Let L be an oriented link diagram. Given a field $\mathbb{F}$, there is a short exact sequence of $\mathbb{F}[U]$-modules*

$$0 \to H^{i,\bullet}_{BN}(L, \mathbb{F}[U]) \otimes \mathbb{F} \longrightarrow H^{i,\bullet}_{Kh}(L, \mathbb{F}) \longrightarrow Tor_{\mathbb{F}[U]}(H^{i+1,\bullet}_{BN}(L, \mathbb{F}[U]), \mathbb{F}) \to 0,$$

where $\mathbb{F}$ has been identified with the $\mathbb{F}[U]$-module $\mathbb{F}[U]/(U)$. Moreover, the first map is graded of degree 0, and this sequence splits (not naturally).

Proof. The proof of the statement, descends directly from a careful inspection of the proof of the Künneth formula. Some care should be put into checking the degrees of the maps involved in our case. Reporting the whole proof here would not give any insight, so we refer the reader to, for example, **[20**, Chapter V], or to any basic text of Algebraic Topology.

Q.E.D.

Corollary 2.22. *If L is an oriented link diagram, and $\mathbb{F}$ is a field, then*

$$H^{i,\bullet}_{Kh}(L, \mathbb{F}) = 0$$

implies

$$H^{i,\bullet}_{BN}(L, \mathbb{F}[U]) = 0, \quad Tor_{\mathbb{F}[U]}(H^{i+1,\bullet}_{BN}(L, \mathbb{F}[U]), \mathbb{F}) = 0.$$

□

Corollary 2.23. *If L is an oriented knot diagram with $n_- > 0$ negative crossings, then*

$$H^{-n_-,\bullet}_{Kh}(L, \mathbb{F}) \simeq Tor_{\mathbb{F}[U]}(H^{-n_-+1,\bullet}_{BN}(L, \mathbb{F}[U]), \mathbb{F});$$

as $\mathbb{F}[U]$-modules, for any field $\mathbb{F}$. □

Another consequence of Proposition 2.21, which describes the shape of Bar-Natan homology, is the following one.

Corollary 2.24. *Let $\mathbb{F}$ be any field, and let L be an ℓ-components link. If $j \equiv \ell + 1$ modulo 2, then*

$$H^{\bullet,j}_{BN}(L, \mathbb{F}[U]) \equiv 0.$$

In particular, if L is a knot, its Bar-Natan homology is isomorphic (as bi-graded $\mathbb{F}[U]$-module) to

$$\bigoplus_{i=1}^{n} \frac{\mathbb{F}[U]}{(U^{t_i})}(h_i, 2k_i + 1) \oplus \mathbb{F}[U](0, s(\kappa) - 1) \oplus \mathbb{F}[U](0, s(\kappa) + 1).$$

Proof. The statement holds true for Khovanov homology, see [**25**, Proposition 24]. If $H^{i,j}_{BN}(L;\mathbb{F}[U])$ is non-zero for some i, then so is

$$H^{i,j+2k}_{BN}(L,\mathbb{F}[U])\otimes\mathbb{F}\hookrightarrow H^{i,j+2k}_{Kh}(L,\mathbb{F}),$$

for some $k\in\mathbb{N}$, which is absurd ($\lightning$).

Q.E.D.

2.2. Hooks, Pawns and Torsion. In order to recover also the (quantum) graded structure of Khovanov's homology from Bar-Natan's, one has to investigate the graded structure of $Tor_{\mathbb{F}[U]}(H^{i,\bullet}_{BN}(L,\mathbb{F}[U]),\mathbb{F})$.

Let $\mathcal{F}$ be a graded Frobenius algebra, and M a graded $R_{\mathcal{F}}$ module. There is a graded structure on

$$Tor_{R_{\mathcal{F}}}(H^{\bullet,\bullet}_{\mathcal{F}}(L,R),M),$$

coming from the projective resolution

$$0\to B^{\bullet,\bullet}_{\mathcal{F}}(L,R)\longrightarrow Z^{\bullet,\bullet}_{\mathcal{F}}(L,R)\longrightarrow H^{\bullet,\bullet}_{\mathcal{F}}(L,R)\to 0,$$

where in general $Z^{\bullet}_{\mathcal{F}}(L,R)$ indicates the set of cycles in $C^{\bullet}_{\mathcal{F}}(L,R)$, $B^{\bullet}_{\mathcal{F}}(L,R)$ indicates the set of boundaries.

First, we need to show that this structure is well-defined, that is: under a suitable set of hypotheses the graded structure should not depend on the choice of projective resolution (cf. [**18**, Chapter 3]).

Lemma 2.25. *Let $M^{\bullet}$, $N^{\bullet}$ be two graded R-modules. Consider two graded projective presentations of $M^{\bullet}$, say*

$$\mathcal{F}:0\to F^{\bullet}_1\xrightarrow{\varphi}F^{\bullet}_2\to M^{\bullet}\to 0,\quad \mathcal{G}:0\to G^{\bullet}_1\xrightarrow{\gamma}G^{\bullet}_2\to M^{\bullet}\to 0.$$

If γ and φ are graded maps of degree 0 *and if the projections onto $M^{\bullet}$ are graded of degree* 0, *then*

$$Tor_{\mathcal{F}}(M^{\bullet},N^{\bullet})\simeq Tor_{\mathcal{G}}(M^{\bullet},N^{\bullet}),$$

as graded R-modules.

Proof. The proof is just the proof of the invariance of *Tor* with respect to the change of projective resolutions (see, for example, [**20**]). If the hypothesis on the gradings of φ, γ and the projections are satisfied, then the up-to-homotopy isomorphism between the two complexes are also graded of degree 0 by construction.

Q.E.D.

So, coming back to the relationship between Bar-Natan and Khovanov theory, we notice that with respect to graded structure of *Tor* described above the map

$$H^{i,\bullet}_{Kh}(L;\mathbb{F})\longrightarrow Tor_{\mathbb{F}[U]}(H^{i+1,\bullet}_{BN}(L,\mathbb{F}[U]),\mathbb{F}),$$

appearing in Proposition 2.21 is graded of degree 0 (this is due to the fact that that all the maps appearing in the proof of Proposition 2.21 are bi-graded with bi-degree 0). Now, we are ready to prove the main result of this section.

Proposition 2.26. *Given a link L with ℓ connected components and Bar-Natan homology of the form*

$$\bigoplus_{i=1}^{n} \frac{\mathbb{F}[U]}{(U^{t_i})}(h_i, q_i) \oplus \bigoplus_{j=1}^{2^\ell} \mathbb{F}[U](2k_j, s_j),$$

there is an isomorphism of doubly graded $\mathbb{F}[U]$-modules between $H^{\bullet,\bullet}_{Kh}(L;\mathbb{F})$ and

$$\bigoplus_{i=1}^{n} (\mathbb{F}(h_i, q_i) \oplus \mathbb{F}(h_i - 1, q_i - 2t_i)) \oplus \bigoplus_{j=1}^{2^\ell} \mathbb{F}(2k_j, s_j),$$

Proof. Consider the projective resolution of $\mathbb{F}[U]$-modules

$$0 \to U^k(\mathbb{F}[U](a)) \to \mathbb{F}[U](a) \to \frac{\mathbb{F}[U]}{(U^k)}(a) \to 0,$$

this tells us that

$$Tor_{\mathbb{F}[U]}\left(\frac{\mathbb{F}[U]}{(U^k)}(a), \mathbb{F}\right) = \mathbb{F}(a - 2k).$$

So, we have the following chain of isomorphisms of bi-graded $\mathbb{F}[U]$-modules

$$Tor_{\mathbb{F}[U]}(H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U]), \mathbb{F}) \simeq Tor_{\mathbb{F}[U]}(T\left(H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])\right), \mathbb{F}) \simeq$$

thanks to Lemma 2.25

$$\simeq Tor_{\mathbb{F}[U]}\left(\bigoplus_{j=1}^{n} \frac{\mathbb{F}[U]}{(U^{t_j})}(h_j, q_j), \mathbb{F}\right) \simeq$$

$$\simeq \bigoplus_{j=1}^{n} Tor_{\mathbb{F}[U]}\left(\frac{\mathbb{F}[U]}{(U^{t_j})}(h_j, q_j), \mathbb{F}\right) \simeq$$

finally,

$$\simeq \bigoplus_{j=1}^{n} \mathbb{F}(h_j, q_j - 2t_j).$$

Hence, each torsion-tower in Bar-Natan's homology contributes a **hook move**, i.e. $\mathbb{F}(i-1, j-2k) \oplus \mathbb{F}(i, j)$, in Khovanov homology. In particular, each $\mathbb{F}[U]/(U^2)$ contributes with a knight move. On the other hand, by Propositions 2.20 and 2.21, each free-tower (i.e. each copy of $\mathbb{F}[U]$) contributes a single copy of $\mathbb{F}$ in the same bi-degree of its generator.

Q.E.D.

Corollary 2.27. *If the torsion in Bar-Natan's homology of a knot κ is only of the form $\mathbb{F}[U]/(U^2)$, then there are only a single pawn move (in bi-degree $(0, s(\kappa)+1)$), and some knight moves (i.e. hook moves with $k = 2$) in Khovanov homology.* $\square$

2.3. T-theory and knight moves. The propositions in Subsections 2.1 and 2.2 all of formal nature, hence they hold, with minor modifications, for the TT-homology. Moreover, the proofs are the same. For the sake of completeness, we restate below the main propositions.

Proposition 2.28. *Let L be an oriented link diagram, and $\mathbb{F}$ be a field. Suppose that the $\mathbb{F}$-vector space $H^{i-1,j-4}_{Kh}(L,\mathbb{F})$ is trivial. Then the map*

$$T\cdot\ : H^{i,j}_{TT}(L,\mathbb{F}[T]) \to H^{i,j-4}_{TT}(L,\mathbb{F}[T]),$$

is injective. In particular, if $H^{i-1,\bullet}_{Kh}(L,\mathbb{F})$ is trivial, then $H^{i,\bullet}_{TT}(L,\mathbb{F})$ is either trivial or a free $\mathbb{F}[T]$-module. □

Proposition 2.29. *Let L be an oriented link diagram. Given a field $\mathbb{F}$, there is a short exact sequence of $\mathbb{F}[T]$-modules*

$$0 \to H^{i,\bullet}_{TT}(L,\mathbb{F}[T])\otimes\mathbb{F} \longrightarrow H^{i,\bullet}_{Kh}(L,\mathbb{F}) \longrightarrow Tor_{\mathbb{F}[T]}\left(H^{i+1,\bullet}_{TT}(L,\mathbb{F}[T]),\mathbb{F}\right) \to 0,$$

where $\mathbb{F} = \mathbb{F}[T]/T$. Moreover, the first map is graded of degree 0, and this sequence splits (not naturally). □

Proposition 2.30. *Given a link L with ℓ connected components and TT-homology of the form*

$$\bigoplus_{i=1}^{n} \frac{\mathbb{F}[T]}{(T^{t_i})}(h_i,q_i) \oplus \bigoplus_{j=1}^{m} \mathbb{F}[T](2k_j,s_j),$$

there is an isomorphism of doubly graded $\mathbb{F}[T]$-modules between $H^{\bullet,\bullet}_{Kh}(L,\mathbb{F})$ and

$$\bigoplus_{i=1}^{n} (\mathbb{F}(h_i,q_i)\oplus\mathbb{F}(h_i-1,q_i-4t_i)) \oplus \bigoplus_{j=1}^{2^\ell} \mathbb{F}(2k_j,s_j).$$

□

Proposition 2.30 gives stronger constraints on the shape of Khovanov homology than the ones given by Proposition 2.26; in fact, Proposition 2.30 asserts that there are hooks of length multiple of 4. In order to extract information on Bar-Natan homology from this result one should prove that the (h_i,q_i)s appearing in Proposition 2.26 are the same as the ones in Proposition 2.30. In order to prove this, one has to relate TT-homology and Bar-Natan homology. We are going to do this by using Lee theory.

Similarly to what was done for Bar-Natan theory, we have to modify slightly the complex $C^{\bullet,\bullet}_{TT}(L,\mathbb{F}[T])$. In particular, we need to introduce a fourth root of T.

Consider $\mathbb{F}[T^{1/4}]$, with is natural structure of $\mathbb{F}[T]$-module, as a bi-graded $\mathbb{F}[T]$-complex (with trivial differential) whose generators 1, and $T^{1/4}$ lie in bi-degree $(0,0)$ and $(0,-1)$, respectively. Finally, define the **total** TT**-complex** as the tensor complex $C^{\bullet,\bullet}_{TT}(L,\mathbb{F}[T])\otimes_{\mathbb{F}[T]}\mathbb{F}[T^{1/4}]$, and denote it by $\overline{C}^{\bullet,\bullet}_{TT}(L,\mathbb{F}[T^{1/2}])$. The total TT-complex is a bi-graded chain complex over $\mathbb{F}[T^{1/2}]$.

By Künneth formula,

$$\begin{aligned}\overline{H}^{i,\bullet}_{TT}(L,\mathbb{F}[U]) \simeq H^{i,\bullet}_{TT}(L,\mathbb{F}[U])\otimes\langle 1_{\mathbb{F}[T^{1/4}]}\rangle_{\mathbb{F}[T]} \oplus H^{i,\bullet+1}_{TT}(L,\mathbb{F}[U])\otimes\langle T^{1/4}\rangle_{\mathbb{F}[T]}\oplus\\ \oplus\; H^{i,\bullet+2}_{TT}(L,\mathbb{F}[U])\otimes\langle T^{1/2}\rangle_{\mathbb{F}[T]} \oplus H^{i,\bullet+3}_{TT}(L,\mathbb{F}[U])\otimes\langle T^{3/4}\rangle_{\mathbb{F}[T]},\end{aligned}$$

as graded $\mathbb{F}[T]$-modules.

Proposition 2.31. *Let $\mathbb{F}$ be a field, and L be an oriented link diagram. Then, the map*

$$\Phi_j : \overline{C}^{\bullet,j}_{TT}(L,\mathbb{F}[T^{1/4}]) \to \mathscr{F}^{j,\bullet}_{OLee}(L,\mathbb{F}) :\ \sum_r x_r\otimes P_r(T^{\frac{1}{4}}) \mapsto \sum_r P_r(1)x_r,$$

is an isomorphism of ($\mathbb{F}$-)complexes, where $\mathscr{F}_{OLee}$ is the quantum filtration in the original Lee theory. Moreover, the following diagram commutes

$$\begin{tikzcd} \overline{C}^{\bullet,j}_{TT}(L,\mathbb{F}[T^{1/4}]) \arrow[r, "\Phi_j"] \arrow[d, "T^{\frac{1}{4}}"'] & \mathscr{F}^{j,\bullet}_{OLee}(L,\mathbb{F}) \arrow[d, "\imath_j"] \\ \overline{C}^{\bullet,j-1}_{TT}(L,\mathbb{F}[T^{1/4}]) \arrow[r, "\Phi_{j-1}"'] & \mathscr{F}^{j-1,\bullet}_{OLee}(L,\mathbb{F}) \end{tikzcd}$$

Proof. The proof is the same as the proof of Proposition 2.13.

Q.E.D.

Theorem 2.32. *Let $\mathbb{F}$ be a field such that $char(\mathbb{F})\neq 2$, and let L be an oriented link diagram. Then,*

$$H^{\bullet,\bullet}_{TT}(L,\mathbb{F}[T]) \simeq \bigoplus_{i=1}^{n}\frac{\mathbb{F}[T]}{(T^{t_i})}(h_i,q_i)\oplus\bigoplus_{j=1}^{m}\mathbb{F}[T](2k_j,s_j),$$

as bi graded $\mathbb{F}[T]$-modules, if and only if

$$H^{\bullet,\bullet}_{BN}(L,\mathbb{F}[U]) \simeq \bigoplus_{i=1}^{n}\frac{\mathbb{F}[U]}{(U^{2t_i})}(h_i,q_i)\oplus\bigoplus_{j=1}^{m}\mathbb{F}[U](2k_j,s_j),$$

as bi graded $\mathbb{F}[U]$-modules.

Proof. One can relate Bar-Natan and TT-theory as follows

$$\overline{C}^{\bullet,j}_{TT}(L,\mathbb{F}[X]) \xrightarrow{\Phi_j} \mathscr{F}^{j,\bullet}_{OLee}(L,\mathbb{F}) \overset{\simeq}{\longleftrightarrow} \mathscr{F}^{j,\bullet}_{TLee}(L,\mathbb{F}) \xleftarrow{\pi_j} \overline{C}^{\bullet,j}_{BN}(L,\mathbb{F}[X]),$$

where $char(\mathbb{F})\neq 2$, and $\mathbb{F}[T^{\frac{1}{4}}]$, $\mathbb{F}[U^{\frac{1}{2}}]$ are (naturally) identified with $\mathbb{F}[X]$ as graded rings. Moreover, we have the following commutative diagram

$$\begin{tikzcd} \overline{C}^{\bullet,j}_{TT}(L,\mathbb{F}[X]) \arrow[r, "\Phi_j"] \arrow[d, "X\cdot"] & \mathscr{F}^{j,\bullet}_{OLee}(L,\mathbb{F}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "\imath_j"] & \mathscr{F}^{j,\bullet}_{TLee}(L,\mathbb{F}) \arrow[d, "\iota_j"] & \overline{C}^{\bullet,j}_{BN}(L,\mathbb{F}[X]) \arrow[l, "\pi_j"'] \arrow[d, "X\cdot"] \\ \overline{C}^{\bullet,j-1}_{TT}(L,\mathbb{F}[X]) \arrow[r, "\Phi_{j-1}"] & \mathscr{F}^{j-1,\bullet}_{OLee}(L,\mathbb{F}) \arrow[r, leftrightarrow, "\simeq"] & \mathscr{F}^{j-1,\bullet}_{TLee}(L,\mathbb{F}) & \overline{C}^{\bullet,j-1}_{BN}(L,\mathbb{F}[X]) \arrow[l, "\pi_j"'] \end{tikzcd}$$

Hence, there is an isomorphism of bi-graded $\mathbb{F}[X]$-modules (to lighten the notation the link and the ring will be omitted for the rest of the proof)

$$H^{\bullet,\bullet}_{BN} \oplus H^{\bullet,\bullet+1}_{BN} \otimes X \simeq H^{\bullet,\bullet}_{TT} \oplus H^{\bullet,\bullet+1}_{TT} \otimes X \oplus H^{\bullet,\bullet+2}_{TT} \otimes X^2 \oplus H^{\bullet,\bullet+3}_{TT} \otimes X^3.$$

If we consider the natural $\mathbb{F}[X^2]$-module structure, both the right hand side and the left hand side of the equation above, split into the direct sum of two sub-modules: one supported in the quantum gradings which are equal to ℓ modulo 2, and the other supported in $qdeg \equiv \ell + 1$ modulo 2. Thus, thanks to this splitting, we get the following isomorphisms of bi-graded $\mathbb{F}[X^2]$-modules

$$H^{\bullet,\bullet}_{BN} \simeq H^{\bullet,\bullet}_{TT} \oplus H^{\bullet,\bullet+2}_{TT} \otimes X^2, \quad H^{\bullet,\bullet+1}_{BN} \otimes X \simeq H^{\bullet,\bullet+1}_{TT} \otimes X \oplus H^{\bullet,\bullet+3}_{TT} \otimes X^3.$$

Notice that the torsion sub-module of $H^{\bullet,\bullet}_{BN}$ is isomorphic to the module

$$T(H^{\bullet,\bullet}_{TT}) \oplus T(H^{\bullet,\bullet}_{TT}) \otimes X^2;$$

In particular, we have

$$\bigoplus_{i=1}^{n} \left(\frac{\mathbb{F}[T]}{(T^{t_i})}(h_i, q_i) \oplus \frac{\mathbb{F}[T]}{(T^{t_i})}(h_i, q_i) \otimes X^2 \right) \simeq \bigoplus_{j=1}^{m} \frac{\mathbb{F}[U]}{(U^{r_j})}(k_j, p_j).$$

By the uniqueness of the decomposition of a graded module over a PID (loc. cit. Theorem 2.1), we have that $n = m$ and, for each i exists j such that

$$\frac{\mathbb{F}[T]}{(T^{t_i})}(h_i, q_i) \oplus \frac{\mathbb{F}[T]}{(T^{t_i})}(h_i, q_i - 2) \simeq \frac{\mathbb{F}[U]}{(U^{r_j})}(k_j, p_j),$$

as graded $\mathbb{F}[X^2]$-modules. In other words, a torsion tower in $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$ corresponds to two copies of a torsion tower in $H^{\bullet,\bullet}_{TT}(L, \mathbb{F}[T])$, with one of these copies shifted by $(0, -2)$ (see Figure 1 for a visual representation).

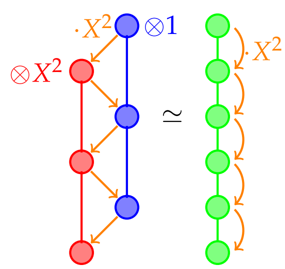


FIGURE 1. Torsion towers

This implies that $(h_i, q_i) = (k_j, p_j)$ and $2t_i = r_j$. The same reasoning applies for the free part, and the claim follows.

Q.E.D.

The previous proposition is not entirely unexpected. In fact, [**41**, Theorems 3 & 4] assert that the TT-theory is the universal theory over any field where 2 is invertible, while BN is universal over $\mathbb{Z}$; hence, the two must hold the same amount of information if $char(\mathbb{F}) \neq 2$.

Corollary 2.33. *If $char(\mathbb{F}) \neq 2$, then the torsion in $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$ is of even order. More precisely, the torsion sub-module of $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$ is isomorphic to a direct sum of $\mathbb{F}[U]$-modules of the form $\mathbb{F}[U]/(U^{2k})$.* $\square$

The "knight move conjecture", known also as the Bar-Natan-Garoufalidis conjecture (cf. [**3**] and [**15**], see also [**26**, Section 3.2]), which was proved by Lee, in [**32**], for the homologically-thin knots (cf. [**26**]), could be stated as follows.

Conjecture (Bar-Natan, Garoufalidis). *If $\mathbb{F} = \mathbb{Q}$, then for any link λ*

$$H^{\bullet,\bullet}_{Kh}(\lambda; \mathbb{F}) = \bigoplus_{i=1}^{n} (\mathbb{F}(h_i, q_i) \oplus \mathbb{F}(h_i - 1, q_i - 2)) \oplus \bigoplus_{j=1}^{2^{\ell}} \mathbb{F}(2k_j, s_j),$$

as bi-graded $\mathbb{F}$-module.

In other words, there are only hooks of length 2 in the rational Khovanov homology. It is our opinion that this kind of (conjectural) phenomenon is the shadow of a (conjectural) phenomenon in Bar-Natan homology, which can be stated as follows.

Conjecture. *If $char(\mathbb{F}) \neq 2$, then the torsion in $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$ is of order 2. More precisely, the torsion sub-module of $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$ is isomorphic to a direct sum of $\mathbb{F}[U]$-modules of the form $\mathbb{F}[U]/(U^2)$.*

This conjecture implies the Bar-Natan-Garoufalidis conjecture thanks to Corollary2.27.

Remark 17. It is not difficult to see that the Bar-Natan–Garoufalidis conjecture is false over $\mathbb{F}$ with $char(\mathbb{F}) = 2$.

CHAPTER 3

Transverse invariants in the $\mathfrak{sl}_2$-theory

In this chapter we introduce two transverse braid invariants coming from Bar-Natan theory, and study their properties. These invariants consist of a pair of chains in $C_{BN}^{\bullet,\bullet}(B,\mathbb{F})$ (the β-invariants), and a pair of non-negative integers (the c-invariants). Both the β-invariants and the c-invariants are proved to be flype invariants, and closely related to another family of invariants, namely the NLS-invariants. Moreover, the c-invariants are proved to be non-effective – in a technical sense – on all transverse knots whose knot type is a prime knot with crossing number lower than 12.

The chapter is structured as follows: in the first section we introduce some basic material regarding transverse and Legendrian links, as well as some material regarding Bennequin-type inequalities. The second section is devoted to the study of the β-invariants and their relationship with other transverse invariants coming from Khovanov and Lee theories. The third section is dedicated to the c-invariants. After having defined the c-invariants we will provide some bounds to compute them. Finally, we investigate the presence of other transverse invariants in the Bar-Natan complex, and in the more general case of a Khovanov-type homology theory.

1. Transverse knots and braiding

Let us start by reviewing some well-known facts on transverse links in the contact manifold $(\mathbb{R}^3,\xi_{sym})$. The material contained in this section is the bare minimum needed to understand the subsequent material. The interested reader may refer to [**14**] for the general background on transverse and Legendrian link theories.

1.1. Transverse links and braids. As was said before, in this thesis we are interested only in transverse links in $\mathbb{R}^3$ endowed with the symmetric contact structure. In particular, we use a combinatorial description of these links given by the Transverse Markov Theorem. Nonetheless, we introduce some more general material on Legendrian and transverse links. This material will be necessary to understand some aspects of the problem, and to state some results. That said, let us start by giving the general definition of contact structure on a 3-manifold.

Definition 3.1. A **contact structure** ξ on a 3-manifold $\mathcal{M}$, is a totally non-integrable distribution of planes over $\mathcal{M}$ – i.e. for each point p, and each neighbourhood $\mathcal{U}$ of p, does not exists a surface tangent to the distribution ξ in $\mathcal{U}$.

The **symmetric contact structure** on $\mathbb{R}^3$, is the distribution of planes given as the kernel of the 1-form

$$\xi_{sym} = dz - ydx + xdy.$$

This distribution is easily proved to be totally non-integrable by the Frobenius criterion for integrability of distributions (see, for example, [**30**, Chapter IV]), and hence defines a true contact structure.

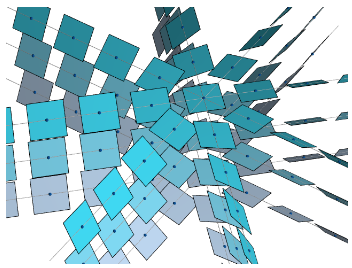


FIGURE 1. The kernel of ξ_{sym} in $\mathbb{R}^3$, near the z-axis.

A **transverse link** (respectively, **Legendrian link**) in $(\mathbb{R}^3, \xi_{sym})$ is a smooth embedding of ℓ disjoint copies of $\mathbb{S}^1$ into $\mathbb{R}^3$, in such a way that at each point the tangent space is transverse (respectively, tangent) to $Ker(\xi_{sym})$. A transverse (respectively, Legendrian) knot is a one-component transverse (respectively, Legendrian) link. Two transverse (resp. Legendrian) links, say λ and λ' are said to be **transversely** (resp. **Legendrian**) **equivalent** if there exists a smooth ambient isotopy

$$H : \mathbb{R}^3 \times [0,1] \longrightarrow \mathbb{R}^3,$$

such that: $H(\lambda, 1) = \lambda'$ and $H(\lambda, t)$ is a transverse (resp. Legendrian) link for each t. If two links are equivalent are said to be of the same type.

A classic theorem by Bennequin relates transverse links and closed braids, but first let us recall the definition of a braid.

Definition 3.2. The **braid group** on i strands, denoted by B_i, is the group generated by σ_j, with $j \in \{1, ..., i-1\}$, subject to the following relations

$$\sigma_k \sigma_j = \sigma_j \sigma_k, \quad |k - j| > 1,$$

$$\sigma_{k+1}\sigma_k\sigma_{k+1} = \sigma_k\sigma_{k+1}\sigma_k.$$

A **braid** B is an element of B_n, for a certain n. The integer n is called **braid index of** B. The braid group admits a (more geometrical) diagrammatic representation as shown in Figure 2.

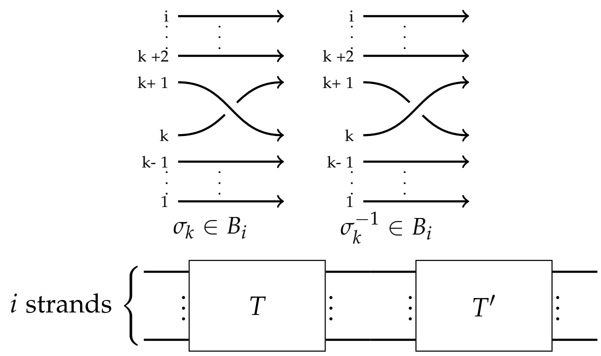


FIGURE 2. Diagrammatic representation of a generator of B_i, and its inverse, together with the pictorial representation of the product in the braid group.

The Alexander closure (see Figure 3) of a braid B represents a link embedded in a thickened annulus $A \times (-\varepsilon, \varepsilon)$. This thickened annulus can be naturally seen as an embedded sub manifold of $\mathbb{R}^3$: it is sufficient to identify A with the annulus on the place between the unit circle and the circle of radius 2. Up to perturbations of B by small isotopies near the crossings, this embedding represents a transverse link. The above-mentioned theorem by Bennequin states that the converse is also true.

Theorem 3.1. *(Bennequin '83, [7]) Any transverse link in $(\mathbb{R}^3, \xi_{sym})$ is transversely isotopic to a the Alexander closure braid (embedded in $\mathbb{R}^3$ as described above).* □

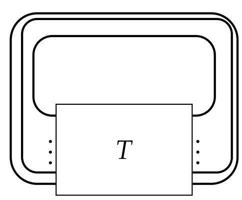


FIGURE 3. The Alexander closure of a braid T

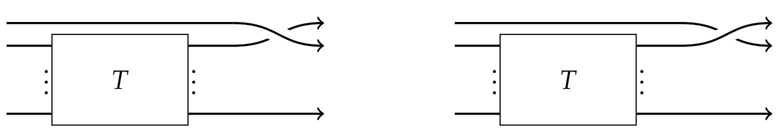


FIGURE 4. Positive (left) and negative (right) stabilizations of a braid T.

A refinement of Bennequin's theorem, called the **Transverse Markov Theorem** (due to Orevkov, Shevchishin and, independently, Wrinkle) allows one to identify (closed) braids, modulo a certain finite set of combinatorial moves, with transverse links, up to transverse isotopy.

Theorem 3.2. *(Orevkov and Shevchishin,* **[42]***, Wrinkle,* **[58]***) Two braids represent the same transverse link type if and only if they are related by a finite sequence of braid conjugations, positive stabilizations, and destabilizations*[1]. □

1.2. Classical and effective invariants. There are two classical invariants for transverse links: the topological link-type, and the **self-linking number** sl. The latter could be defined, in the case of a transverse braid B, as the difference between the writhe[2] of the braid, and the braid index. More explicitly

$$sl(B) = n_+(B) - n_-(B) - b(B) = w(B) - b(B).$$

Remark 18. A negative stabilization decreases the self-linking number by 2. In particular, a negative stabilization changes the transverse type of a braid.

Both the self linking number and the link-type are not very powerful as transverse invariants; there are ample families of transverse links (or even knots) which could not be distinguished by the means of these invariants. Any invariant which can distinguish two transverse links with the same self-linking number, and the same link-type is called **effective**. A family of transverse links whose elements are told apart one from the other by the two classical invariants is called (**transversely**) **simple** – e.g. positive torus links.

In **[8]**, Joan Birman and William Menasco, introduced a construction to produce a family of non-simple closed 3-braids. This construction, called **flype**, had been generalized to n-strands braids by the same authors (see **[9]**). Here, as in **[34]**, the world flype will refer to a special case of those in **[9]**. Consider two braids, say $A,\ B \in B_n$, and define

$$F_1^{\pm}(A, B; k) = A\sigma_n^k B\sigma_n^{\mp 1}, \quad F_2^{\pm}(A, B; k) = A\sigma_n^{\mp 1} B\sigma_n^k.$$

Then, we will say that $F_1^{+}(A, B; k)$ (resp. $F_1^{-}(A, B; k)$) is obtained from $F_2^{+}(A, B; k)$ (resp. $F_1^{-}(A, B; k)$) by a **positive** (resp. **negative**) **flype**. It is immediate that the two braids $F_1^{\pm}(A, B; k)$, and $F_2^{\pm}(A, B; k)$, have the same self linking number. Less immediate, but nonetheless true, is that the closure of $F_1^{\pm}(A, B; k)$ and $F_2^{\pm}(A, B; k)$, represent the same link type (see the end of Section 2). While two links which are obtained one from the other by a positive flype represent the same transverse link type, there are examples of negative flype pairs – i.e. two links related by a negative flype – whose elements represent non-transversely isotopic link types.

Legendrian and transverse links are related. We do not wish to enter into the details of their relationship, but there are some facts we need to state. Legendrian links have (at least) two diagrammatic theories, both of them being similar to the usual diagrammatic knot theory. Of course, in this case some care is needed: not any diagram can be seen as diagram of a Legendrian link. The two main

[1]Let $B \in B_{m-1}$, the **positive** (resp. **negative**) **stabilization** of $B \in B_{m-1}$ is the braid $B\sigma_m \in B_m$ (resp. $B\sigma_m^{-1} \in B_m$). The destabilization is just the inverse process: if one considers a braid of the form $A\sigma_m^{\pm 1}B$, where $A,\ B \in B_{m-1}$, then its destabilization is the braid AB. See also Figure 4.

[2]The number of positive crossings minus the number of negative crossings in an oriented link diagram L, usually denoted by $w(L)$.

ways to produce diagrams for Legendrian links are: through front projection, and through a Lagrangian projection.

Given a front projection of a Legendrian link (which looks like a link diagram with cusps, see [**14**]), there exist two ways to produce the diagram (a front projection to be precise) of a transverse link with the same link type. The two transverse links corresponding to the diagram obtained are called **transverse push-offs** of the Legendrian link (cf. [**14**]). There is a positive push-off and a negative push-off and in general these are non-isotopic as transverse links.

There are also three **classical Legendrian invariants**: the topological link type, the **Thurston-Bennequin number** tb and the **rotation number** r. The Thurston-Bennequin and rotation numbers of a Legendrian link are related to the self-linking number of its transverse push-offs by the equations

$$tb(\lambda) \mp r(\lambda) = sl(\lambda_{\pm}),$$

where λ is a Legendrian link and λ_+ (resp. λ_-) its positive (resp. negative) push-off. In particular, given a transverse link λ and a Legendrian link λ' such that λ is a transverse push-off of λ', then

$$tb(\lambda') - |r(\lambda')| \leq sl(\lambda).$$

Hence, it follows that any bound from above on the value of the self linking number, is also a bound on the value of difference between the Turston-Bennequin number and the absolute value of the rotation number. Finally, notice that, for any link λ, the previous equation implies

$$tb_{max}(\lambda) \leq sl_{max}(\lambda),$$

where sl_{max} and tb_{max} are the maximal self-linking number, and the maximal Thurston-Bennequin number, respectively.

1.3. Bennequin-type inequalities. Finding upper bounds for the Thurston-Bennequin, rotation and self-linking numbers of links from link invariants originated as a problem in contact topology, but now has become more a subject of knot and braid theories. This is in some sense due to the fact that, while is easy to compute the self-linking number, the rotation number, or the Thurston-Bennequin number of a (suitable) diagram, it is often difficult to compute the invariants estimating them. These bounds are given by numerical three dimensional (e.g. the Seifert genus), four dimensional (e.g. the slice genus) or combinatorial link invariants (e.g. the un-knotting number), or also by quantities related to polynomial link invariants (such as the Kauffman and HOMFLY-PT polynomials) and, more recently, link homologies (such as Khovanov, Khovanov-Rozansky and knot Floer homologies). This collection of bounds has been named, generically, **Bennequin-type inequalities**.

Among the earlier Bennequin-type inequalities there are: the inequality proven by Rudolph ([**49**]), in terms of the slice genus, and the inequality proved by Eliashberg ([**13**]), in terms of the Euler characteristic of a surface bounding a transverse or a Legendrian link in a tight contact manifold. Then, a few years later, Rasmussen ([**47**]), Plamenevskaya ([**45**]), and Shumakovitch ([**50**]), introduced similar inequalities for the s-invariant of knots (we call them **Bennequin** s**-inequalities**). These inequalities where subsequently sharpened by Kawamura ([**23**]), and generalized by Wu ([**59**]) to the sl_N Khovanov-Rozansky homology. Finally, Kawamura ([**24**]) and Lobb ([**37**]), independently, sharpened these inequalities further in the case of Khovanov homology. A slight improvement of Lobb's inequalities for a particular class of links (namely, the pseudo-thin links) is due to Cavallo ([**10**]).

Let L be an oriented link diagram, and λ the link-type represented by L. Denote by $o_+(L)$, $o_-(L)$ and $o_0(L)$ the number of circles in the oriented resolution of L which are adjacent only to positive crossing, negative crossings and both type of crossings, respectively. These types of circles will be called, respectively, **positive circles**, **negative circles** and **neutral circles**. A circle in the oriented resolution is called **strongly negative** if it is a negative circle and is touched by at least two crossings. The number of strongly negative circles in L will be denoted by $o'_-(L)$. Let $\mathbb{F}$ be a field. Now that the notation is in place, we can state two of the above-mentioned Bennequin s-inequalities.

[**45, 50**] Let λ be an oriented knot, and L an oriented diagram representing it, then

$$s(\lambda, \mathbb{F}) \geq w(L) - o(L) + 1. \tag{P-S04}$$

[**23**] Let λ be an oriented non splittable link, and L an oriented diagram representing it. If $o_+(L) + o_0(L) > 0$, then

$$s(\lambda, \mathbb{F}) \geq w(L) - o(L) + 2o'_-(L) + 1. \tag{Kw11}$$

To state the others Bennequin s-inequalities is necessary to fix some more notation. Denote by $\Gamma(L)$ the **simplified Tait graph**, that is a graph with marked edges which has the circles of the oriented resolution as vertices, and there is an edge between two vertices if they share at least a crossing. The markings on the edges are either $+$, $-$, or 0, depending on the fact that two circle share only positive crossings, only negative crossings, or both type of crossings. $\Gamma_\pm(L)$ is the sub-graph of $\Gamma(L)$ spanned by positive (resp. negative) circles. Finally, denote by $s_+(L)$ (resp. $s_-(L)$) the number of connected components of the graph obtained from $\Gamma(L)$ by removing the negative (resp. positive) edges. Finally, we can state Lobb's, Kawamura's and Cavallo's bounds on the value of the s-invariant.

[**37**] Let λ be an oriented link, and L an oriented diagram representing λ, then

$$w(L) + o(L) - 2s_-(L) + 1 \geq s(\lambda, \mathbb{F}) \geq w(L) - o(L) + 2s_+(L) - 2\ell + 1. \tag{Lb12}$$

[**24**] Let λ be a non-splittable oriented link, and L an oriented diagram representing it, then

$$s(\lambda, \mathbb{F}) \geq w(L) - o(L) + 2s_+(L) - 1. \tag{Kw15}$$

[10] Let λ be an oriented pseudo-thin link, L an oriented diagram representing λ, and ℓ_s the number of its split components (i.e. connected components of L seen as a four-valent graph), then

$$s(\lambda, \mathbb{F}) \geq w(L) - o(L) + 2s_+(L) - 2\ell_s(L) + 1. \tag{Cv15}$$

Remark 19. The inequalities stated in this section are slight reformulations of the original ones. In fact, we stated them for any field $\mathbb{F}$. The proof of the inequality (Kw15) uses only formal properties of the s-invariant, so (Kw15) holds true for any field (cf. [**24**, Theorem 4.4]). Because the proof of (Cv15) uses only a formal property of the s-invariants and (Lb12), the inequality (Cv15) holds true for any field if, and only if, (Lb12) holds true for any field. Finally, the proof of (Lb12) can be adapted without changing a single word to all fields of characteristic different than 2. While, the case $char(\mathbb{F}) = 2$ could be dealt with by replacing original Lee theory with twisted Lee theory in the proof of (Lb12). The inequalities (P-S04) and (Kw11) hold for any field as all the other inequalities are strictly stronger.

Remark 20. Although Kawamura's inequality is limited to non-splittable links, it can be extended (thanks to property (6) in Chapter 2, Theorem 2.12) to link diagrams whose split components are diagrams of non-splittable links. Given such a link diagram L, Kawamura's inequality can be re-written as follows

$$s(\lambda, \mathbb{F}) \geq w(L) - o(L) + 2s_+(L) - 2\ell_s(L) + 1. \tag{Kw15}$$

When L is (the Alexander closure of) a braid diagram, $o(L)$ becomes the braid index. Thus, the above inequalities provide an upper bound for the self-linking number of L. Of course, these inequalities can be read in the other way, and used to estimate the value of the s-invariant (and hence the slice genus). For example, each of the above inequalities (except (Lb12)) can be used to compute the slice-genus of the positive torus link $T(p, q)$, as shown in the following proposition.

Proposition 3.3 (Generalized Milnor conjecture, Cavallo [**10**]). *The slice genus of the positive torus link $T(p, q)$ is*

$$g_*(p, q) = \frac{(p-1)(q-1) - GCD(p, q) + 1}{2}$$

and

$$s(T(p, q), \mathbb{F}) = (p-1)(q-1).$$

Proof. Let $B_{p,q}$ be the standard braid representing the positive torus link $T(p, q)$ – i.e. the braid given by the word $(\sigma_1\sigma_2 \cdots \sigma_{p-1})^q$. Its self-linking number is

$$sl(B_{p,q}) = (p-1)q - p.$$

Any of (P-S04), (Kw11), (Kw15), (Cv15) – since the diagram $B_{p,q}$ is positive, the link is pseudo-thin – implies,

$$s(T(p, q), \mathbb{F}) \geq sl(B_{p,q}) - 1 = (p-1)(q-1).$$

As $s(T(p,q),\mathbb{F})$ is a lower bound for minus the Euler characteristic of any surface Σ in $\mathbb{D}^4$ bounding $T(p,q)$ and the unknot, it follows that

$$-\chi(\Sigma) = 2g(\Sigma) - 2 + GCD(p,q) + 1 \geq s(\lambda,\mathbb{F}) \geq (p-1)(q-1), \tag{12}$$

consequently,

$$2g_*(p,q) \geq (p-1)(q-1) - GCD(p,q) + 1.$$

The other inequality follows from the computation of the genus of the surface obtained, via the Seifert algorithm, from (the Alexander closure of) $B_{p,q}$ – see **[48]** or **[33]**, or any other book on knot theory, for a description of the Seifert algorithm. The statement on the s-invariant follows directly from (12).

Q.E.D.

2. Transverse invariants from Bar-Natan Homology I: the β-invariant

Let L be an oriented link diagram. In 2006, Olga Plamenevskaya introduced the first transverse invariant in Khovanov homology.

Definition 3.3. The **Plamenevskaya cycle** $\psi(L,\mathbb{F})$ of an oriented link diagram L is the homogeneous element in $C^{0,\bullet}_{Kh}(L,\mathbb{F})$ defined as the state whose underlying resolution is the oriented resolution, and whose circles are all labeled with x_-.

Given a braid diagram B, denote by $\psi(B,\mathbb{F})$ the Plamenevskaya cycle of the Alexander closure of B, and call it, in view of the next proposition, the **Plamenevskaya invariant** of B.

Proposition 3.4 (Plamenevskaya, **[45]**). *The Plamenevskaya invariant is a transverse braid invariant. That is, if B and B' are two braids related by a transverse braid move, then the map*[3] *between the Khovanov chain complexes associated to the move sends $\psi(B,\mathbb{F})$ to $\pm\psi(B',\mathbb{F})$. Moreover, $qdeg(\psi(B,\mathbb{F})) = sl(B)$, and the homology class of ψ detects negative stabilization – i.e. $[\psi(B,\mathbb{F})] = 0$ if B is the negative stabilization of another braid.*

In 2013, Lenhard Ng, Robert Lipshitz, and Sucharit Sarkar, noticed that the canonical cycles $\psi^+(L,\mathbb{F}) = \mathbf{v}_{TLee}(L,\mathbb{F})$ and $\psi^-(L,\mathbb{F}) = \mathbf{v}_{TLee}(L,\mathbb{F})$, shared similar properties:

Proposition 3.5 (Ng-Lipshitz-Sarkar, **[34]**). *The canonical cycles are transverse braid invariants. TThat is, if B and B' are two braids related by a transverse braid move, then the map between the twisted Lee chain complexes associated to the move sends $\psi^{\pm}(B,\mathbb{F})$ to $\psi^{\pm}(B',\mathbb{F})$, up to sign. While the image of $\psi^{\pm}(B,\mathbb{F})$ under the map induced by a negative (de)stabilization differs from $\psi^{\pm}(B',\mathbb{F})$ by a boundary term whose quantum degree is $sl(B,\mathbb{F})$. Moreover, $qdeg(\psi^{\pm}(B,\mathbb{F})) = sl(B)$, and their projection in to the associated graded object to the twisted Lee complex (considered with the filtration introduced in Chapter 1, Subsection 2.2) can be identified with $\psi(B,\mathbb{F})$.*

[3] These maps form a very specific set of morphisms which is used throughout the literature. This matter will be discussed at the beginning of the next subsection.

We will call $\psi^{\pm}(B,\mathbb{F})$ the **Ng-Lipshitz-Sarkar invariants** (or NLS **invariants**) of the transverse braid B. There are other transverse invariants which were introduced in the same paper, namely $\psi^{\pm}_{p,q}$ and ψ^{diff}. For the instant we will leave these invariants aside. We will come back to them later.

Let L be an oriented link diagram, and R be a ring. The β**-cycles** (associated to L over the ring R) are homogeneous cycles in $C^{\bullet,\bullet}_{BN}(L,R[U])$, defined similarly to the canonical cycles of Lee theory. Suppose that an orientation $\mathfrak{o}\in\mathbb{O}(\widetilde{L})$ is fixed and define $\beta(\mathfrak{o},R)$ to be the enhanced state with underlying resolution the oriented resolution $\underline{r}_{\mathfrak{o}}$, where each circle γ has label b_γ, defined as follows

$$b_\gamma=\begin{cases}x_- & \text{if } N(\gamma)\equiv 0\ mod\,2\\ x_\bullet=x_--Ux_+ & \text{if } N(\gamma)\equiv 1\ mod\,2\end{cases}$$

where $N(\gamma)$ denotes the nesting number of γ (see page 20 for the definition). With the exact same computations done in the case of the canonical cycles, one obtains the homological and quantum degrees of $\beta(\mathfrak{o},\mathbb{F})$, which are

$$hdeg(\beta(\mathfrak{o},R))=n_-(\mathfrak{o})-n_-(L),$$

and

$$qdeg(\beta(\mathfrak{o},R))=-o(\mathfrak{o})+n_-(\mathfrak{o})+n_+(L)-2n_-(L),$$

where $o(\mathfrak{o})$ is the number of circles in $\underline{r}_{\mathfrak{o}}$.

Denote by $\beta(L,R)$ the β-cycle corresponding to the orientation of L, and by $\overline{\beta}(L,R)$ the β-cycle corresponding to the opposite orientation. Then,

$$hdeg(\beta(L,R))=hdeg(\overline{\beta}(L,R))=0$$

and

$$qdeg(\beta(L,R))=qdeg(\overline{\beta}(L,R))=-o(L)+w(L). \tag{13}$$

In particular, when the diagram is the Alexander closure of a braid B, then

$$qdeg\left(\beta(B,R)\right)=sl(B).$$

Directly from the definitions it follows that

$$\pi_{Kh}(\beta(L,\mathbb{F}))=\psi(L,\mathbb{F})=\pi_{Kh}(\overline{\beta}(L,\mathbb{F}))$$

and

$$\pi_{TLee}(\beta(L,\mathbb{F}))=\psi^+(L,\mathbb{F}),\ \pi_{TLee}(\overline{\beta}(L,\mathbb{F}))=\psi^-(L,\mathbb{F}),$$

where

$$\pi_\circ: C^{\bullet,\bullet}_{BN}(L,R[U])\longrightarrow C^{\bullet}_{\circ}(L,R),\quad \circ\in\{Kh,TLee\},$$

are the quotient maps.

In order to study the properties of $\beta(\mathfrak{o},R)$, it is convenient to analyse the behaviour of x_- and $x_\bullet$ with respect to the operations of Frobenius algebra. For this purpose, it is useful to introduce an involution on the set $\{x_-,x_\bullet\}$ called **conjugation**, and defined as follows:

$$\overline{x_-}=x_\bullet,\quad \overline{x_\bullet}=x_-.$$

Simple computations show that

$$m(x,\overline{x}) = 0, \text{ and } m(x,x) = \varepsilon_x U x, \tag{14}$$

for each $x \in \{x_-, x_\bullet\}$, where $\varepsilon_{x_-} = 1$, and $\varepsilon_{x_\bullet} = -1$. With this notation we have

$$\overline{x} = x - \varepsilon_x U x_+ \tag{15}$$

Moreover, this involution behaves well with respect to the co-multiplication, more precisely

$$\Delta(\overline{x}) = \overline{\Delta(x)} = \overline{x}, \quad \forall\, x \in \{x_-, x_\bullet\}. \tag{16}$$

Remark 21. Seen the formal properties shared by x_- and $x_\bullet$, all theorem concerning β are valid also for $\bar{\beta}$.

Proposition 3.6. *Let L be an oriented link diagram, and let $\mathfrak{o} \in \mathbb{O}(L)$. Then, the enhanced state $\beta(\mathfrak{o}, R) \in C^{\bullet,\bullet}_{BN}(L, R[U])$ is a cycle.*

Proof. It follows directly from the fact that the set of circles in the oriented resolution is bipartite (see [**47**, Lemma 2.4 & Corollary 2.5]), and from (14).

Q.E.D.

Given an oriented link diagram L, the set

$$B_{\mathbb{F}}(L) = \{[\beta(\mathfrak{o}, \mathbb{F})]\}_{\mathfrak{o} \in \mathbb{O}(\widetilde{L})} \subseteq H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$$

is a free set (i.e. all the elements are linearly independent over $\mathbb{F}[U]$) as shown in the following proposition.

Proposition 3.7. *Let L be an oriented link diagram. The set $B_{\mathbb{F}}(L)$ generates a free sub-module of rank 2^ℓ in $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$, where ℓ is the number of components of L. In particular, each $[\beta(\mathfrak{o}, \mathbb{F})]$ is always non trivial.*

Proof. Let us suppose that

$$\sum_{\mathfrak{o} \in \mathbb{O}_h(\widetilde{L})} P_{\mathfrak{o}}(U)[\beta(\mathfrak{o}, \mathbb{F})] = 0, \quad P_{\mathfrak{o}} \in \mathbb{F}[U]. \tag{17}$$

where $\mathbb{O}_h(\widetilde{L})$ is the set of all orientations such that $hdeg(\beta(\mathfrak{o}, \mathbb{F})) = h$. Being $[\beta(\mathfrak{o}, \mathbb{F})]$ bi-homogeneous, we may assume that

$$P_{\mathfrak{o}}(U) = a_{\mathfrak{o}} U^{k_{\mathfrak{o}}}, \quad a_{\mathfrak{o}} \in \mathbb{F}.$$

Thus, applying π_{TLee} to both sides of the equation (17), one gets

$$\sum_{\mathfrak{o} \in \mathbb{O}_h(\widetilde{L})} \alpha_{\mathfrak{o}} \mathfrak{s}_{\mathfrak{o}} = 0,$$

which implies $\alpha_{\mathfrak{o}} = 0$, for each $\mathfrak{o} \in \mathbb{O}(\widetilde{L})$.

Q.E.D.

Remark 22. Seen that the β-cycles are defined in the same way as the canonical cycles in $TLee$, and that they form a free set, one may expect their homology classes to generate $H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])/T(H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U]))$. However this is generally false. In fact, it may happen that the homology classes of some of the β-cycles are multiples of other homology classes (cf. Proposition 3.8).

Remark 23. Proposition 3.7 holds true also when $R = \mathbb{Z}$. It is sufficient to notice that the set $B_{\mathbb{Q}}(L) \subseteq H^{\bullet,\bullet}_{BN}(L, \mathbb{Q}[U]) = H^{\bullet,\bullet}_{BN}(L, \mathbb{Z}[U]) \otimes \mathbb{Q}$ generates a $\mathbb{Q}[U]$-sub-module of rank 2^{ℓ}.

2.1. Invariance of β. We now turn to the analysis of the behaviour of $\beta(L, R)$ under the maps induced by the Reidemeister moves. These maps are not canonical, but there is a set of maps which is conventionally used throughout the literature. These were originally introduced by Khovanov (see [**25**]), and translated to a more abstract setting by Bar-Natan (cf. [**4**]), allowing one to work with an arbitrary Frobenius algebra.

At the beginning of each sub-section we will recall the general (i.e. for an arbitrary Frobenius algebra $\mathcal{F}$) definition of the map associated to the move we are dealing with. In the subsequent propositions we are going to use the specialization of these maps to the case $R_{\mathcal{F}} = \mathbb{F}[U]$ and $\mathcal{F} = BN$. So, for the rest of the section we fix a Frobenius algebra $\mathcal{F} = (R, A, m, \iota, \Delta, \varepsilon)$.

2.2. First Reidemeister move. Let L be an oriented link diagram. Denote by L'_+ the oriented link diagram obtained from L via a positive first Reidemeister move (i.e. the addition of a positive curl, see Figure 5) on an arc **a**. Finally, denote by c_+ the crossing appearing only in L'_+.

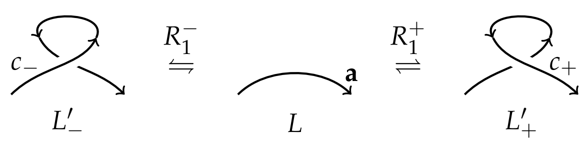


FIGURE 5. The first Reidemeister move.

The complex $C_{\mathcal{F}}(L'_+)$ can be identified (as a graded R-module) with the complex

$$C^{\bullet}_{\mathcal{F}}(L \cup \bigcirc) \oplus C^{\bullet}_{\mathcal{F}}(L)(-1) \simeq (C^{\bullet}_{\mathcal{F}}(L) \otimes_R A) \oplus C^{\bullet}_{\mathcal{F}}(L)(-1), \tag{18}$$

where $(\cdot)$ denotes the (homological) degree shift. In fact, each resolution of L'_+ obtained by performing a 0-resolution in c_+ can be identified with a resolution of $L \cup \bigcirc$, while each of the remaining resolutions can be identified with a resolution of L. To turn this identification into an isomorphism of R-complexes it is sufficient to endow the graded R-module on the left-hand side of (18) with the differential

$$d'_{\mathcal{F}} = \begin{pmatrix} d_{\mathcal{F}} \otimes_R id_A & 0 \\ D_{\mathcal{F}} & d_{\mathcal{F}} \end{pmatrix},$$

where $D_{\mathcal{F}}$ is the map associated to a saddle move merging the unknotted component with the circle γ' containing **a**. More explicitly,

$$D_{\mathcal{F}} : C^{\bullet}_{\mathcal{F}}(L) \otimes_R A \to C^{\bullet}_{\mathcal{F}}(L) : \left(\bigotimes_{\gamma \in \underline{r}} \alpha_\gamma \right) \otimes \alpha \mapsto \left(\bigotimes_{\gamma \in \underline{r} \setminus \{\gamma'\}} \alpha_\gamma \right) \otimes m(\alpha_{\gamma'}, \alpha).$$

To use a more "algebraic" turn of phrase: the complex $C^{\bullet}_{\mathcal{F}}(L')$ is isomorphic to the mapping cone of the map D. Now, we are ready to define the map associated to the addition of the (positive) curl. This map, denoted by $\Phi_1^+(\mathcal{F})$, is defined as follows

$$\begin{aligned}\Phi_1^+(\mathcal{F}) : C_{\mathcal{F}}(L) &\longrightarrow (C_{\mathcal{F}}(L) \otimes_R A) \oplus C_{\mathcal{F}}(L)(-1,0)\\ \bigotimes_{\gamma \in \underline{r}} \alpha_\gamma &\mapsto \left(\left(\bigotimes_{\gamma \in \underline{r} \setminus \{\gamma'\}} \alpha_\gamma \right) \otimes \left(\alpha_{\gamma'} \otimes T_{\mathcal{F}}(1_R) - \Delta(\alpha_{\gamma'}) \right) \right) \oplus 0\end{aligned}$$

where $T_{\mathcal{F}}$ is the "de-cupped torus map"

$$T_{\mathcal{F}} : R \to A : x \mapsto m(\Delta(\iota(x))). \tag{19}$$

To conclude the positive version of the first Reidemeister move, we need to define the map associated to the removal of a positive curl. This map, denoted by $\Psi_1^+(\mathcal{F})$ is given by

$$\begin{aligned}\Psi_1^+(\mathcal{F}) : (C^{\bullet}_{\mathcal{F}}(L) \otimes_R A) \oplus C^{\bullet}_{\mathcal{F}}(L)(-1) &\longrightarrow C^{\bullet}_{\mathcal{F}}(L)\\ \left(\left(\bigotimes_{\gamma \in \underline{r}} \alpha_\gamma \right) \otimes a \right) \oplus \bigotimes_{\gamma \in \underline{s}} \delta_\gamma &\mapsto \varepsilon(a) \bigotimes_{\gamma \in \underline{r}} \alpha_\gamma.\end{aligned}$$

Remark 24. It is not difficult to check that $\Psi_1^+(\mathcal{F})$ is an up-to-chain-homotopy inverse of the map $\Phi_1^+(\mathcal{F})$, and that both of them are chain maps.

Now, let us turn to the negative version of the first Reidemeister move. For our scopes it is sufficient to define only the map associated to the creation of a negative curl. Let us denote by L'_- the diagram obtained from L by adding a negative curl on the arc **a** (see Figure 5). Denote by c_- the crossing of L'_- created by the addition of the curl. Similarly to the case of the positive Reidemeister move, there is an identification of the resolutions of L'_- where c_- is replaced with is 0-resolution and the resolutions of L. All the remaining resolutions of L'_- can be identified with the resolutions of $L \cup \bigcirc$. These identifications induce the following isomorphisms of (graded) R-modules

$$C^{\bullet}_{\mathcal{F}}(L'_-) \simeq C^{\bullet}_{\mathcal{F}}(L)(-1) \oplus C^{\bullet}_{\mathcal{F}}(L \cup \bigcirc) \simeq C^{\bullet}_{\mathcal{F}}(L)(-1) \oplus (C^{\bullet}_{\mathcal{F}}(L) \otimes_R A). \tag{20}$$

Remark 25. Suppose $\mathcal{F}$ is a graded Frobenius algebra. Then the complex $C^{\bullet}_{\mathcal{F}}(L, R)$ can be endowed with a second grading (see Chapter 1 Subsection 2.2). To turn the isomorphisms in (20) into isomorphisms of *bi*-graded R-modules it is necessary to introduce an appropriate (quantum) grade shift (cf. [**4**, Section 6]). This shift is not necessary in the case of the positive version of the first Reidemeister move.

As in the case of R_1^+, we wish to turn the isomorphisms in (20) into isomorphisms of chain complexes. In order to do so it is sufficient to endow the rightmost R-module in (20) with the differential

$$d'_{\mathcal{F}} = \begin{pmatrix} d_{\mathcal{F}} & 0 \\ D'_{\mathcal{F}} & d_{\mathcal{F}} \otimes_R id_A \end{pmatrix};$$

where $D'_{\mathcal{F}}$ is the map associated to a saddle move splitting the circle γ' containing the arc **a**. More explicitly,

$$D'_{\mathcal{F}} : C^{\bullet}_{\mathcal{F}}(L) \to C^{\bullet}_{\mathcal{F}}(L) \otimes_R A : \bigotimes_{\gamma \in \underline{r}} \alpha_\gamma \mapsto \left(\bigotimes_{\gamma \in \underline{r} \setminus \{\gamma'\}} \alpha_\gamma \right) \otimes \Delta(\alpha_{\gamma'}).$$

Remark 26. There is no ambiguity in the labels given by $\Delta(\alpha_{\gamma'})$ because of the co-commutativity of Δ (cf. page 1).

Finally, we can define the map associated to the addition of a negative curl, denoted by Φ_1^-, as follows

$$\begin{aligned} \Phi_1^- (= \Phi_1^-(\mathcal{F})) : C^{\bullet}_{\mathcal{F}}(L) &\longrightarrow C^{\bullet}_{\mathcal{F}}(L)(-1) \oplus (C^{\bullet}_{\mathcal{F}}(L) \otimes_R A) \\ \bigotimes_{\gamma \in \underline{r}} \alpha_\gamma &\longmapsto 0 \oplus \left(\bigotimes_{\gamma \in \underline{r}} \alpha_\gamma \right) \otimes \varepsilon(1_R) \end{aligned}$$

Now we are finally ready to state (and prove) a result describing the behaviour of $\beta(L, R)$ with respect to the maps associated to the first Reidemeister move(s).

Proposition 3.8. *Let L be an oriented link diagram, and R be any ring. With the above notation (cf. Figure 5) we have*

$$\text{(R1p)} \qquad \Psi_1^+(\beta(L,R)) = \beta(L'_+, R) \quad \textit{and} \quad \Phi_1^+(\beta(L'_+, R)) = \beta(L, R),$$

and

$$\text{(R1n)} \qquad -\varepsilon_x U(\Phi_1^-)_*([\beta(L,R)]) = [\beta(L'_-, R)];$$

where x is the label, in $\beta(L, R)$, of the circle γ' containing the arc **a**.

Proof. Let us start from the addition of a positive curl. A simple computation shows that

$$T_{BN}(1_{R[U]}) = x_- + x_\bullet. \tag{21}$$

Suppose $\alpha_{\gamma'} = x \in \{x_-, x_\bullet\}$. It follows

$$\alpha_{\gamma'} \otimes T_{BN}(1_{R[U]}) - \Delta_{BN}(\alpha_{\gamma'}) = x \otimes (x + \overline{x}) - x \otimes x = x \otimes \overline{x},$$

where $\bar{\cdot}$ denotes the conjugation on the set $\{x_-, x_\bullet\}$ (i.e. the involution exchanging x_- and $x_\bullet$). Identify the oriented resolution of L'_+ with the oriented resolution of $L \cup \bigcirc$ as in the definition of Φ_1^+. From the previous considerations it follows that the label of the un-knotted component which does not belong to L in $\Phi_1^+(\beta(L,R))$ is $\overline{x}$, the label of γ' is x, and all the other labels remain unchanged. Thus, it is immediate that

$$\Phi_1^+(\beta(L,R)) = \beta(L'_+, R).$$

To conclude the case of the positive R_1 move, we must verify that $\beta(L, R)$ is preserved by Ψ_1^+. This is immediate from the following considerations:

(a) if $a = b_{\gamma'}$ then $\varepsilon(a) = 1$;

(b) the direct summand in $C^{\bullet,\bullet}_{BN}(L'_+, R[U])$ corresponding to the oriented resolution of L' is mapped onto the direct summand in $C^{\bullet,\bullet}_{BN}(L, R[U])$ corresponding to the oriented resolution;
(c) the labels on the circles that are not involved in the move and in the circle γ' are left invariant by Ψ^+_1.

Now, let us turn to the behaviour of β with respect to the map associated to the addition of a negative curl. Immediately from the definition it follows that

$$\Phi^-_1(\beta(L,R)) = \left(\bigotimes_{\gamma\in\underline{r}} b_\gamma\right) \otimes x_+, \tag{22}$$

where $\underline{r}$ denotes the oriented resolution of L, and the oriented resolution of L'_- is identified with the oriented resolution of $L \cup \bigcirc$. Consider the chain

$$\eta \in C^{\bullet,\bullet}_{BN}(L'_-, R[U]) \simeq C^{\bullet,\bullet}_{BN}(L, R[U])(-1,-2) \oplus (C^{\bullet,\bullet}_{BN}(L, R[U]) \otimes_R A)(0,-1),$$

defined as follows

$$\eta = 0 \oplus \left(\left(\bigotimes_{\gamma\in\underline{r}} b_\gamma\right) \otimes x\right) = 0 \oplus (\beta(L,R) \otimes x),$$

where $x = b_{\gamma'}$. By (15) we have

$$\beta(L'_-, R) = 0 \oplus \left(\left(\bigotimes_{\gamma\in\underline{r}} b_\gamma\right) \otimes \bar{x}\right) = \eta - \varepsilon_x U \Phi^-_1(\beta(L)).$$

Thus, if η is a boundary in $C^{\bullet,\bullet}_{BN}(L'_-, R[U])$, then

$$[\beta(L'_-, R)] = [\eta - \varepsilon_x U \Phi^-_1(\beta(L,R))] = -\varepsilon_x U (\Phi^-_1)_*([\beta(L,R)]),$$

and the claim follows. So, consider the chain

$$\beta(L,R) \oplus 0 \in C_{\mathcal{F}}(L, R[U])(-1) \oplus (C_{\mathcal{F}}(L, R[U]) \otimes_R A),$$

and compute its boundary

$$d'_{BN}(\beta(L,R) \oplus 0) = d_{BN}(\beta(L,R)) \oplus \left(D'_{BN}(\beta(L,R))\right) =$$
$$= 0 \oplus \left(\bigotimes_{\gamma\in\underline{r}\setminus\{\gamma'\}} b_\gamma \otimes \Delta_{BN}(b_{\gamma'})\right) = 0 \oplus \left(\bigotimes_{\gamma\in\underline{r}\setminus\{\gamma'\}} b_\gamma \otimes x \otimes x\right) =$$
$$= 0 \oplus (\beta(L,R) \otimes x) = \eta,$$

and this concludes the proof.

Q.E.D.

Thus, $\beta(B,R)$ is invariant under positive stabilizations, and changes in a somewhat controlled way under negative stabilizations.

Corollary 3.9. *If B is an oriented braid, then $[\beta(B,R)] \in H^{0,sl(B)}_{BN}(B, R[U])$ is invariant under positive stabilizations. More formally,*

$$(\Psi^+_1)_*([\beta(B,R)]) = [\beta(B',R)],$$

where B' is the positive stabilization of B. Moreover, if B'' is an oriented braid obtained from B via a negative stabilization, then

$$U(\Phi_1^-)_*([\beta(B,R)]) = \pm[\beta(B'',R)].$$

□

2.3. Second Reidemeister move. Let L be an oriented link diagram. Let **a** and **b** be two (un-knotted) arcs of L lying in a small ball. Performing a second Reidemeister move on these arcs inserts two adjacent crossings, say c_1 and c_2, of opposite type (see Figure 6).

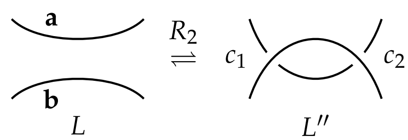


FIGURE 6. The (un-oriented) second Reidemeister move.

Denote by L'' the oriented link diagram obtained from L by performing a second Reidemeister move on the arcs **a** and **b**. There are four possible resolutions of the pair of crossing c_1, c_2. Let L''_{ij}, with $i, j \in \{0,1\}$, be the link obtained from L'' by performing a i-resolution on c_1 and a j-resolution on c_2 (see Figure 7). Notice that there is a natural identification of the link L''_{10} with L.

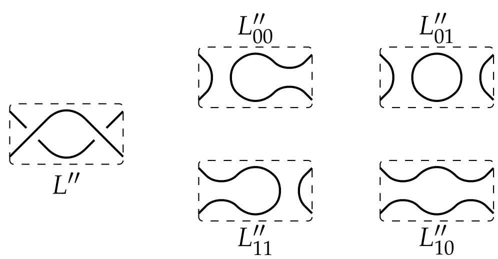


FIGURE 7. The possible resolutions of c_1 and c_2.

Remark 27. Only one among the links L''_{00}, L''_{10}, L''_{01} and L''_{11} inherits the orientation from L'', and this is either L''_{10} or L''_{01}.

Similarly to the case of the first Reidemeister move, there is an isomorphism of graded R-modules

$$C^\bullet_{\mathcal{F}}(L'') \simeq C^\bullet_{\mathcal{F}}(L''_{00}) \oplus C^\bullet_{\mathcal{F}}(L''_{10})(-1) \oplus C^\bullet_{\mathcal{F}}(L''_{01})(-1) \oplus C^\bullet_{\mathcal{F}}(L''_{11})(-2). \tag{23}$$

given by the identification of each resolution of L'' with a resolution of L''_{ij} (for a suitable choice of i and j).

Remark 28. Assume $\mathcal{F}$ to be a graded Frobenius algebra. To turn the isomorphism in (23) into an isomorphism of bi-graded R-modules a suitable shift of the quantum degree has to be taken into account.

The isomorphism in (23) is not an isomorphism of R-complexes. To obtain such an isomorphism it is necessary to modify the differential of the complex on the right-hand-side of (23). This modified differential can be (roughly) defined as follows

$$d''_{\mathcal{F}} = \begin{pmatrix} d^{00}_{\mathcal{F}} & 0 & 0 & 0 \\ D''_{00,10} & d^{10}_{\mathcal{F}} & 0 & 0 \\ D''_{00,01} & 0 & d^{01}_{\mathcal{F}} & 0 \\ 0 & D''_{10,11} & D''_{01,11} & d^{11}_{\mathcal{F}} \end{pmatrix}$$

where $d^{ij}_{\mathcal{F}}$ is the differential of the complex $C^{\bullet}_{\mathcal{F}}(L''_{ij})$, and

$$D''_{ij,hk} : C^{\bullet}_{\mathcal{F}}(L''_{ij}) \longrightarrow C^{\bullet}_{\mathcal{F}}(L''_{hk})$$

is the map corresponding to a saddle move from L''_{ij} to L''_{hk}. This description, even if it is imprecise, is more than sufficient for our scopes. The interested reader may consult [**25**, Section 5] or [**4**, Section 4] for a more detailed description of $d''_{\mathcal{F}}$.

Now, consider the link L''_{01}. Denote by **c** and **d** the two arcs appearing in the local picture in Figure 7 (see also Figure 8). Fix an arc **g**, meeting L''_{01} only at the endpoints, joining **c** and **d**. Finally, fix an arc **e**, meeting L only at the endpoints, joining the arcs **a** and **b**.

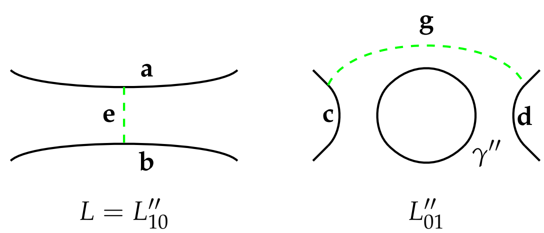


FIGURE 8.

Now, with the notation defined above, and using the notation introduced in Chapter 1 Subsection 2.4, we can finally define the map associated to the second Reidemeister move

$$\Psi_2 : C^{\bullet}_{\mathcal{F}}(L) \longrightarrow C^{\bullet}_{\mathcal{F}}(L''_{00}) \oplus C^{\bullet}_{\mathcal{F}}(L''_{10})(-1) \oplus C^{\bullet}_{\mathcal{F}}(L''_{01})(-1) \oplus C^{\bullet}_{\mathcal{F}}(L''_{11})(-2)$$

as follows

$$\Psi_2(x) = 0 \oplus x \oplus (S(x, \mathbf{e}) \otimes \iota(1_R)) \oplus 0,$$

where x is an enhanced state, we have identified L and L''_{10}, and $\iota(1_R)$ is the label of γ'' (cf. Figure 8). Similarly, the up-to-chain-homotopy inverse of Ψ_2

$$\Phi_2 : C^{\bullet}_{\mathcal{F}}(L''_{00}) \oplus C^{\bullet}_{\mathcal{F}}(L''_{10})(-1) \oplus C^{\bullet}_{\mathcal{F}}(L''_{01})(-1) \oplus C^{\bullet}_{\mathcal{F}}(L''_{11})(-2) \longrightarrow C^{\bullet}_{\mathcal{F}}(L)$$

is given by

$$\Phi_2(x_{00} \oplus x_{10} \oplus x_{01} \oplus x_{11}) = x_{10} + \varepsilon(x_{\gamma''})S(x_{01}, \mathbf{g}),$$

where x_{ij} denotes a (possibly trivial) enhanced state in $C^{\bullet}_{\mathcal{F}}(L''_{ij})$, and $x_{\gamma''}$ denotes the label of γ'' in x_{01} (cf. Figure 8).

Before stating the results concerning $\beta(L, R)$ it is necessary to distinguish between two versions of the R_2 move. This distinction is made according to the relative orientation of the arcs **a** and **b**; a R_2 move is called **coherent** if **a** and **b** are as in Figure 9, and **non-coherent** otherwise.

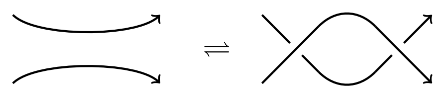


FIGURE 9. A coherent version of the second Reidemeister move. All other coherent second Reidemeister moves can be obtained by rotating or taking the mirror image of the one in figure.

Remark 29. Braid-like second Reidemeister moves are coherent.

Now, we can state the proposition concerning the behaviour of β under coherent second Reidemeister moves.

Proposition 3.10. *Let L be an oriented link diagram. Let L'' be the oriented link diagram obtained from L via a coherent second Reidemeister move. Then*

$$\Psi_2(\beta(L, R)) = \beta(L'', R) \quad \textit{and} \quad \Phi_2(\beta(L'', R)) = \beta(L, R). \tag{R2c}$$

Proof. Throughout this proof we will keep the notation shown in Figure 8. Let $\underline{r}$ be the oriented resolution of L. First, let us consider the behaviour of β with respect to the map Ψ_2. It is easy[4] to see that if the move is coherent then **a** and **b** do not belong to the same circle in $\underline{r}$. Let $\gamma_{\mathbf{a}}$ and $\gamma_{\mathbf{b}}$ be the circles to which **a** and **b**, respectively, belong to. It follows directly from the definition of S that

$$\Psi_2(\beta(L, R)) = 0 \oplus \beta(L, R) \oplus \left(\left(\bigotimes_{\gamma \in \underline{r} \setminus \{\gamma_{\mathbf{a}}, \gamma_{\mathbf{b}}\}} b_\gamma \right) \otimes m(b_{\gamma_{\mathbf{a}}}, b_{\gamma_{\mathbf{b}}}) \otimes x_+ \right) \oplus 0.$$

Because the move is coherent the labels in $\beta(L, R)$ of $\gamma_{\mathbf{a}}$ and of $\gamma_{\mathbf{b}}$ are conjugate. Thus, by (14) we have

$$m(b_{\gamma_{\mathbf{a}}}, b_{\gamma_{\mathbf{b}}}) = m(b_{\gamma_{\mathbf{a}}}, \overline{b_{\gamma_{\mathbf{a}}}}) = 0.$$

Finally, again because of the move is coherent, the oriented resolution of L'' is identified (via the isomorphism in (23)) with the oriented resolution of $L''_{10} = L$. Thus, it follows

$$\Psi_2(\beta(L, R)) = \beta(L'', R).$$

[4]It is a simple consequence of the Jordan curve theorem.

Now, let us turn to the behaviour of β under the map Φ_2. This is similar to the previous case. In fact, as we argued before, the isomorphism in (23) sends $\beta(L'', R)$ to

$$0 \oplus \beta(L, R) \oplus 0 \oplus 0.$$

With the same reasoning as above, from the coherence of the move it follows

$$\gamma_{\mathbf{a}} \neq \gamma_{\mathbf{b}} \quad \text{and} \quad b_{\gamma_{\mathbf{b}}} = \overline{b_{\gamma_{\mathbf{a}}}}.$$

From (14), and from the considerations we just made, we obtain

$$S(\beta(L, R), \mathbf{g}) = 0.$$

Since

$$\Phi_2(0 \oplus \beta(L, R) \oplus 0 \oplus 0) = \beta(L, R) + S(\beta(L, R), \mathbf{g})$$

the claim follows.

Q.E.D.

Corollary 3.11. *The cycles $\beta(B, R)$ and $\overline{\beta}(B, R)$ are invariant under braid-like R_2 moves.* □

Now let us turn to the case of non-coherent versions of the second Reidemeister move (see Figure 10).

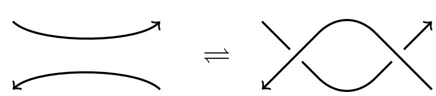


FIGURE 10. A non-coherent version of the second Reidemeister move. All other coherent second Reidemeister moves can be obtained by rotating or taking the mirror image of the one in figure.

Proposition 3.12. *Let L be an oriented link diagram and let L'' be obtained from L by a non-coherent second Reidemeister move along the arcs **a** and **b**. Then, either*

(R2n1) $$\Phi_2(\beta(L'', R)) = \beta(L, R)$$

or

(R2n2) $$\Phi_2(\beta(L'', R)) = \pm U\beta(L, R),$$

*depending whether **a** and **b** belong to different circles or to the same circle in the oriented resolution of L. In the first case* (R2n1) *holds, while* (R2n2) *holds in the latter case. Moreover, in neither case the map Ψ_2 does preserve the β-cycles.*

Proof. First, let us fix some notation. Denote by $\underline{r}$ the oriented resolution of L and by $\underline{s}$ the oriented resolution of L''. Since the move is non-coherent the oriented resolution of L'' can be identified with the oriented resolution of L''_{01} (Figure 11). Let **a**, **b**, **c**, **d**, **e**, **g** and γ'' be as in Figure 8. Finally, let **x** be one among the arcs **a**, **b**, **c** and **d**, and denote by $\gamma_{\mathbf{x}}$ the circle in the appropriate oriented resolution to which the arc **x** belongs to.

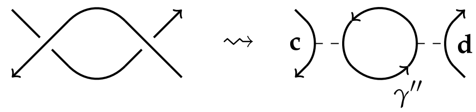


FIGURE 11. The oriented resolution of L''.

Before going into the details of the proof, it is useful to notice two things. First, it is easy to see that the circles $\gamma_{\mathbf{a}}$ in $\underline{r}$ and $\gamma_{\mathbf{c}}$ in $\underline{s}$ have the same nesting number. In particular, the label of $\gamma_{\mathbf{a}}$ in $\beta(L,R)$ and the label of $\gamma_{\mathbf{c}}$ in $\beta(L'',R)$ are equal. Second, from an easy check of the nesting numbers it follows

$$b_{\gamma_{\mathbf{a}}} = b_{\gamma_{\mathbf{b}}} \quad \text{and} \quad b_{\gamma_{\mathbf{c}}} = b_{\gamma_{\mathbf{d}}}.$$

We are now ready to prove the proposition. Let us start with the map Φ_2. With the identifications made above, we can write

$$\Phi_2(0 \oplus 0 \oplus \beta(L''_{01},R) \oplus 0) = \varepsilon(\gamma'')S(\beta(L''_{01},R),\mathbf{g}).$$

There are two cases:

(1) $\gamma_{\mathbf{c}} \neq \gamma_{\mathbf{d}}$
(2) $\gamma_{\mathbf{c}} = \gamma_{\mathbf{d}}$

and, since the move is not coherent, both of them are realized (see Figure 13 for an example). In the first case, $\gamma_{\mathbf{a}} = \gamma_{\mathbf{b}}$ and

$$S(\beta(L''_{01},R),\mathbf{g}) = \left(\bigotimes_{\gamma\in\underline{r}\setminus\gamma_{\mathbf{a}}} b_\gamma\right) \otimes m(b_{\gamma_{\mathbf{c}}}, b_{\gamma_{\mathbf{d}}}) =$$

$$= \left(\bigotimes_{\gamma\in\underline{r}\setminus\gamma_{\mathbf{a}}} b_\gamma\right) \otimes Ub_{\gamma_{\mathbf{d}}} = \left(\bigotimes_{\gamma\in\underline{r}\setminus\gamma_{\mathbf{a}}} b_\gamma\right) \otimes Ub_{\gamma_{\mathbf{a}}} =$$
$$= U\beta(L,R).$$

Thus, since $\varepsilon(b_{\gamma''}) = 1$, (R2n2) follows. Now, let us consider case (2). In this case $\gamma_{\mathbf{a}} \neq \gamma_{\mathbf{b}}$ and the map S behaves as follows

$$S(\beta(L''_{01},R),\mathbf{g}) = \left(\bigotimes_{\gamma\in\underline{r}\setminus\{\gamma_{\mathbf{a}},\gamma_{\mathbf{b}}\}} b_\gamma\right) \otimes \Delta(b_{\gamma_{\mathbf{c}}}) =$$

thanks to (16), it follows

$$= \left(\bigotimes_{\gamma\in\underline{r}\setminus\{\gamma_{\mathbf{a}},\gamma_{\mathbf{b}}\}} b_\gamma\right) \otimes b_{\gamma_{\mathbf{c}}} \otimes b_{\gamma_{\mathbf{c}}} = \beta(L,R),$$

where the last equality follows from the considerations on the labels made at the beginning of the proof. This concludes the proof of (R2n1).

Finally, the assertion about the map Ψ_2 is almost immediate in case (1); in fact, Ψ_2 is the up-to-homotopy inverse of Φ_2, and hence it does not preserve the homology class of the β-cycles (much less the β-cycles). In case (2) the assertion follows from two simple observations. On one hand, $\beta(L'',R)$ belongs to the direct summand $C_{\mathcal{F}}(L''_{01}, R)$ appearing in the decomposition (23). On the other hand, the image of $\beta(L, R)$ via Ψ_2 has a non-trivial component in the summand $C_{\mathcal{F}}(L''_{10}, R)$ appearing in the same decomposition.

Q.E.D.

There is also another way to represent transverse links: through front projections. These are ordinary link diagrams where none of the local configurations in Figure 12 appears (see [**14**, Section 2]). As in the case of transverse braid diagrams, there is a set of combinatorial moves which encodes transverse isotopy between front projections. This set is given by all second and third Reidemeister moves such that the condition of being a front projection is preserved on both sides.

Proposition 3.13. *The β-cycles are not invariants of transverse front projections. More precisely, there exists a Reidemeister move of the second type which preserves the transverse type of front projections but whose induced map does not preserve the β-cycles.*

Proof. First, notice that there is a non-coherent version of the second Reidemeister move which preserves the transverse type: the mirror of the one in Figure 10 (see also Figure 13). In Proposition 3.12 it has been shown that the β-cycles are not invariant under (the chain maps induced by) non-coherent second Reidemeister moves, and the claim follows.

Q.E.D.

Two examples of pairs of transverse front projections which are related by a non-coherent transverse second Reidemeister move are shown in Figure 13.

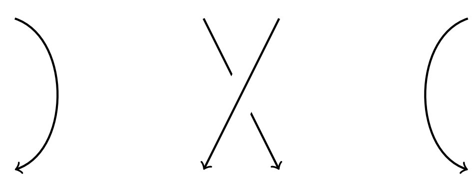


FIGURE 12. Local configurations excluded from front projections of transverse links.

2.4. Third Reidemeister move. Finally, we arrived to the case of the third Reidemeister move. This move is the hardest to deal with mainly because it involves the highest number of crossings among the Reidemeister moves. This fact has two main consequences. First, there are several versions of the third Reidemeister move. Second, the higher the number of crossings is the more complicated the splittings of the chain complexes are. Finally, the number of crossings is equal on both sides of the move, and hence there is no loss of complexity from one side to the other.

This complexity reflects into the chain homotopies associated to R_3. For this reason,we will avoid to give an explicit description of these maps. We will resort to the **categorified Kauffman trick**, which is due to Bar-Natan, to deduce the

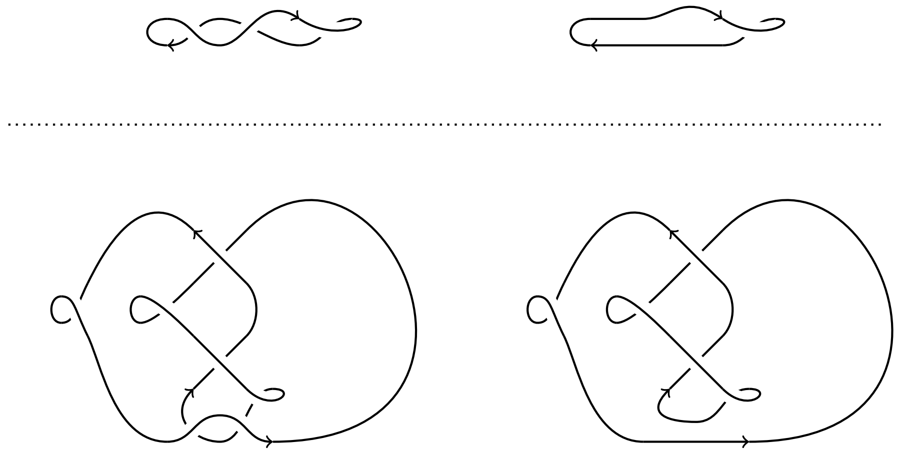


FIGURE 13. The two cases in Proposition 3.12 realized by front projections.

maps associated to the various versions of the third Reidmeister move. Thanks to this trick we are able to prove the invariance of the β-invariants without needing the explicit computation of these maps.

The original Kauffman trick is a smart way to deduce the invariance of the Jones polynomial with respect to the third Reidemeister move from the invariance with respect to the second move. Similarly, the categorified Kauffman trick allows one to deduce the invariance of Khovanov homology with respect to the third Reidemeister move from its invariance with respect to the second Reidemeister move. More precisely, it allows one to define the chain homotopy equivalences associated to the third Reidemeister moves from the chain homotopy equivalences associated to the second Reidemeister move. Of course, everything comes with a price: in our case we need to use some well-known constructions and results from homological algebra.

Definition 3.4. Let $(C_1^\bullet, d_1)$ and $(C_2^\bullet, d_2)$ be two chain complexes. Given a chain map

$$F : (C_1^\bullet, d_1) \longrightarrow (C_2^\bullet, d_2)$$

the (**mapping**) **cone** of F is the chain group

$$C^\bullet = C_1^{\bullet+1} \oplus C_2^\bullet$$

endowed with the differential

$$d = \begin{pmatrix} -d_1 & 0 \\ F & d_2 \end{pmatrix}$$

The mapping cone of F will be denoted by $\Gamma(F)$.

Definition 3.5. Let $(C_1^\bullet, d_1)$ and $(C_2^\bullet, d_2)$ be two chain complexes a chain map

$$\psi : (C_1^\bullet, d_1) \longrightarrow (C_2^\bullet, d_2)$$

is an **inclusion in a deformation retract** if there exists

$$\varphi : (C_2^\bullet, d_2) \longrightarrow (C_1^\bullet, d_1)$$

such that

$$\varphi \circ \psi = Id_{C_1^\bullet}, \qquad \psi \circ \varphi = Id_{C_2^\bullet} + d_1 \circ h \pm h \circ d_2,$$

and

$$h \circ \varphi = 0, \qquad \psi \circ h = 0,$$

for some chain-homotopy h. The map ψ is called **strong deformation retract** (associated to φ).

Notice that both inclusion in deformation retracts and strong deformation retracts are homotopy equivalence. Moreover, a strong deformation retract is an up-to-homotopy inverse of an inclusion in a deformation retract (the converse is not true in general).

Remark 30. It is easy to see that the maps Ψ_2 and Φ_2 are, respectively, an inclusion in a deformation retract and a strong deformation retract.

The categorified Kauffman trick is based on the behaviour of cones under retract. Let $C_1^\bullet$, $C_2^\bullet$, $D_1^\bullet$ and $D_2^\bullet$ be chain complexes and let

$$F : C_1^\bullet \longrightarrow C_2^\bullet$$

be a chain map. Consider the following diagram of complexes and chain maps

$$\begin{tikzcd}
C_1^\bullet \arrow[r, shift left, "\psi_1"] \arrow[d, "F"'] & D_1^\bullet \arrow[l, shift left, "\varphi_1"] \\
C_2^\bullet \arrow[r, shift right, "\varphi_2"'] & D_2^\bullet \arrow[l, shift right, "\psi_2"']
\end{tikzcd}$$

where φ_1 and φ_2 are inclusions in a deformation retract and ψ_1 and ψ_2 are the corresponding strong deformation retracts. Finally, let h_1 and h_2 be the two chain-homotopies such that

$$\varphi_i \circ \psi_i = id_{D_i} + \partial \circ h_i \pm h_i \circ \partial, \qquad i \in \{1, 2\}.$$

It turns out that the three cones $\Gamma(F)$, $\Gamma(F \circ \varphi_1)$ and $\Gamma(\psi_2 \circ F)$ are homotopy equivalent. Moreover, it is possible to explicitly write the homotopy equivalences. This is precisely the content of the following lemma.

Lemma 3.14 ([**4**], Lemma 4.5). *With the above notation, define the maps appearing in the following (not-necessarily-commutative) diagram*

$$\begin{tikzcd}
\Gamma(F\circ\psi_1): & D_1^{\bullet+1}\oplus C_2^{\bullet} \arrow[rr, "\partial_{\Gamma(F\circ\psi_1)}"] \arrow[d, shift left, "\widetilde{\Phi}_1"] & & D_1^{\bullet+2}\oplus C_2^{\bullet+1} \arrow[d, shift right, "\widetilde{\Phi}_1"'] \\
\Gamma(F)\quad : & C_1^{\bullet+1}\oplus C_2^{\bullet} \arrow[u, shift left, "\widetilde{\Psi}_1"] \arrow[d, shift right, "\widetilde{\Psi}_2"'] \arrow[rr, shift left, "\partial_{\Gamma(F)}"] & & C_1^{\bullet+2}\oplus C_2^{\bullet+1} \arrow[u, shift right, "\widetilde{\Psi}_1"'] \arrow[ll, shift left, "{H_1,\, H_2}"] \arrow[d, shift left, "\widetilde{\Psi}_2"] \\
\Gamma(\psi_2\circ F): & C_1^{\bullet+1}\oplus D_2^{\bullet} \arrow[u, shift right, "\widetilde{\Phi}_2"'] \arrow[rr, "\partial_{\Gamma(\psi_2\circ F)}"] & & C_1^{\bullet+2}\oplus D_2^{\bullet+1} \arrow[u, shift left, "\widetilde{\Phi}_2"]
\end{tikzcd}$$

as follows

$$\widetilde{\Phi}_1 = \begin{pmatrix} \psi_1 & 0 \\ F\circ h_1 & Id_{C_2} \end{pmatrix}, \quad \widetilde{\Psi}_1 = \begin{pmatrix} \varphi_1 & 0 \\ 0 & id_{C_2} \end{pmatrix},$$
$$\widetilde{\Phi}_2 = \begin{pmatrix} Id_{C_1} & 0 \\ h_2\circ F & \varphi_2 \end{pmatrix}, \quad \widetilde{\Psi}_2 = \begin{pmatrix} id_{C_2} & 0 \\ 0 & \psi_2 \end{pmatrix}$$
$$H_1 = \begin{pmatrix} -h_1 & 0 \\ 0 & 0 \end{pmatrix} \quad H_2 = \begin{pmatrix} 0 & 0 \\ 0 & -h_2 \end{pmatrix}.$$

Then, we have

$$\widetilde{\Psi}_i\circ\widetilde{\Phi}_i = Id,\ \widetilde{\Phi}_i\circ\widetilde{\Psi}_i = Id + \partial\circ H_i \pm H_i\circ\partial, \quad \widetilde{\Psi}_i\circ H_i = 0,\ H_i\circ\widetilde{\Phi}_i = 0,$$

for each $i\in\{1,2\}$. In particular, the cones $\Gamma(F)$, $\Gamma(F\circ\varphi_1)$ and $\Gamma(\varphi_2\circ F)$ are homotopy equivalent. □

Remark 31. Lemma 3.14 still holds if the maps ψ_1 and ψ_2 are inclusion in strong deformation retract and the maps φ_1 and φ_2 the corresponding retracts.

The following proposition is now immediate.

Proposition 3.15. *With the same notation as in Lemma 3.14, let $x_i \in C_i$ and $y_i\in D_i$, $i\in\{1,2\}$, be such that:*

$$\psi_1(x_1) = y_1, \quad \varphi_1(y_1) = x_1, \quad \psi_2(y_2) = x_2, \quad \varphi_2(x_2) = y_2.$$

Then

$$\widetilde{\Psi}_1(x_1\oplus 0) = y_1\oplus 0, \quad \widetilde{\Phi}_1(y_1\oplus 0) = x_1\oplus 0,$$

and

$$\widetilde{\Psi}_2(0\oplus y_2) = 0\oplus x_2, \quad \widetilde{\Phi}_2(0\oplus x_2) = 0\oplus y_2.$$

□

Now we are ready to start the description of the categorified Kauffman trick. Let L and L' be the links on the LHS and RHS of a third Reidemeister move as in Figure 14. The categorified Kauffman trick consist of three main steps.

Step 1 Write the complexes associated to L and L' as cones, say $\Gamma(F_1)$ and $\Gamma(F_2)$, where
$$F_1 : C_1 \to C_2, \quad F_2 : D_1 \to D_2,$$
and either $C_1 = D_1$ or $C_2 = D_2$.

Step 2 Show that there exists a complex C and either ψ_1, ψ_2, strong deformation retracts, such that

$$\psi_1 : D_2 \to C, \quad \psi_2 : C_2 \to C, \text{ and } \psi_1 \circ F_1 = \psi_2 \circ F_2,$$

if $C_1 = D_1$, or φ_1 and φ_2, inclusions in deformation retracts, such that

$$\varphi_1 : C \to D_1, \quad \varphi_2 : C \to C_1 \text{ and } F_1 \circ \varphi_1 = F_2 \circ \varphi_2,$$

if $C_2 = D_2$.

Step 3 Use Lemma 3.14 to write the maps

$$\widetilde{\Psi} : C_{\mathcal{F}}(L, R) = \Gamma(F_1) \to \Gamma(\psi_1 \circ F_1) \quad \widetilde{\Phi} : \Gamma(\psi_2 \circ F_2) \to C_{\mathcal{F}}(L', R) = \Gamma(F_2)$$

in the case $C_1 = D_1$, or the maps

$$\widetilde{\Psi} : C_{\mathcal{F}}(L, R) = \Gamma(F_1) \to \Gamma(F_1 \circ \varphi_1) \quad \widetilde{\Phi} : \Gamma(F_2 \circ \varphi_2) \to C_{\mathcal{F}}(L', R) = \Gamma(F_2)$$

in the case $C_2 = D_2$. Finally, define the chain map Ψ_3 associated to the third Reidemeister move as the composition $\widetilde{\Phi} \circ \widetilde{\Psi}$.

The same reasoning can be adapted to find the map Φ_3, which is the map associated to the other direction of the third Reidemeister move.

We can now proceed to the first step. Denote by c_1, c_2 and c_3 the crossings of L involved into the move, and denote by c_1', c_2' and c_3' the corresponding crossings in L', as in Figure 14.

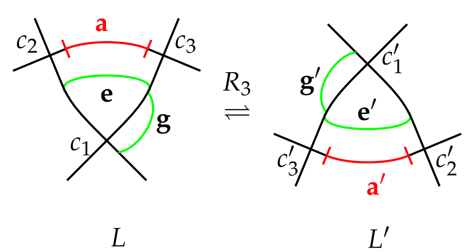


FIGURE 14. Naming arcs and crossings on both sides of a third Reidemeister move. The type of crossings is omitted as we will make use of the same notation for all cases. However, it is implicitly assumed that the strand of the link containing the arc **a** (resp. **a**′) either underpasses in both c_2 and c_3 (resp. c_2' and c_3'), or overpasses in both c_2 and c_3 (resp. c_2' and c_3').

Let L_1, L_1', L_2 and L_2' be oriented link diagram which differ only in a small ball as shown in Figure 15. There is a natural identification of the resolutions of L (resp. L') where c_1 (resp. c_1') is replaced by the resolution with the resolutions of L_1 (resp. L_1'). Similarly, there is an identification of the resolutions of L_2 (resp. L_2') with the resolution of L (resp. L') where c_1 (resp. c_1') has been replaced with the resolution . These identifications allow us to see the complex associated to L (resp. L') as the cone over a saddle map.

To be precise there are two cases depending on whether the crossing c_1 (and hence c_1') is or is . In the first case, the complexes associated to L and L' are, respectively, the cones over the maps

$$S(\cdot,\mathbf{g}) : C^{\bullet}_{\mathcal{F}}(L_1, R) \to C^{\bullet}_{\mathcal{F}}(L_2, R), \quad S(\cdot,\mathbf{g}') : C^{\bullet}_{\mathcal{F}}(L_1', R) \to C^{\bullet}_{\mathcal{F}}(L_2', R), \tag{24}$$

which are the maps associated to the saddle moves along $\mathbf{g}$ and $\mathbf{g}'$, respectively. Similarly, in the case c_1 is the crossing the complexes associated to L and L' are the cones over the maps

$$S(\cdot,\mathbf{e}) : C^{\bullet}_{\mathcal{F}}(L_2, R) \to C^{\bullet}_{\mathcal{F}}(L_1, R), \quad S(\cdot,\mathbf{e}') : C^{\bullet}_{\mathcal{F}}(L_2', R) \to C^{\bullet}_{\mathcal{F}}(L_1', R). \tag{25}$$

Thus, having completed Step 1 we can now proceed to Step 2.

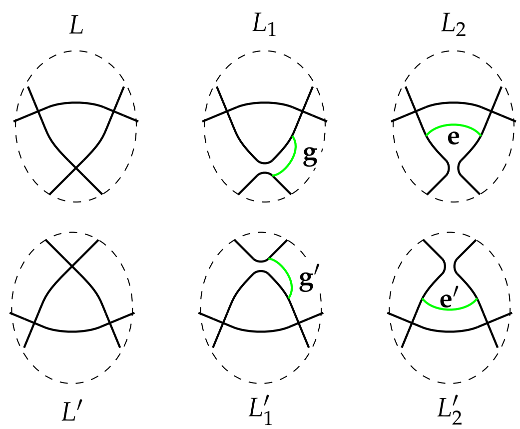


FIGURE 15. Resolution of a crossing on both sides of a third Reidemeister move.

Let us start by noticing that the diagrams L_2 and L_2' can be identified. Thus $C^{\bullet}_{\mathcal{F}}(L_1, R) = C^{\bullet}_{\mathcal{F}}(L_1', R)$. Recall that the crossings c_2 and c_3 have opposite type, and that c_2' and c_3' are of the same type as c_3 and c_2, respectively. So, the links L_1 and L_1' are related by two second Reidemeister moves. Denote by L_1'' the link which differs from L_1 and L_1' as shown in Figure 16, then the maps

$$\Phi_2 : C^{\bullet}_{\mathcal{F}}(L_1) \longrightarrow C^{\bullet}_{\mathcal{F}}(L_1''), \qquad \Phi_2' : C^{\bullet}_{\mathcal{F}}(L_1') \longrightarrow C^{\bullet}_{\mathcal{F}}(L_1'')$$

induced by the second Reidemeister moves, are strong deformation retracts, and the maps

$$\Psi_2 : C^{\bullet}_{\mathcal{F}}(L_1'') \longrightarrow C^{\bullet}_{\mathcal{F}}(L_1), \qquad \Psi_2' : C^{\bullet}_{\mathcal{F}}(L_1'') \longrightarrow C^{\bullet}_{\mathcal{F}}(L_1')$$

induced by the inverse move, are the corresponding inclusions.

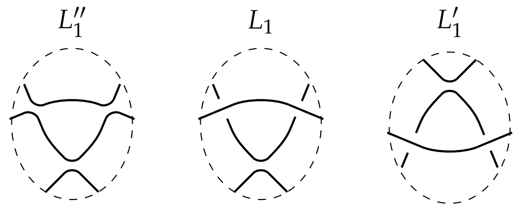


FIGURE 16. The links L_1'', L_1 and L_1'.

Then, to complete Step 2 all that is left is to verify that either

$$S(\cdot,\mathbf{g})\circ\Psi_2 = S(\cdot,\mathbf{g}')\circ\Psi_2'$$

or

$$\Phi_2\circ S(\cdot,\mathbf{e}) = \Phi_2'\circ S(\cdot,\mathbf{e}'),$$

depending on whether c_1 is  or , respectively. However, this is a simple verification and hence left to the reader.

Now, we could proceed to Step 3 and write down explicitly the maps associated to the third Reidemeister moves. However, we are going to take a shortcut, and turn directly to the proof of the invariance of the β-invariants. To do so we must restrict ourselves to a sub-set of all possible (oriented) third Reidmeister moves: the set of **coherent third Reidemeister moves**. This set is composed by all the versions in Figure 17 and their mirror images.

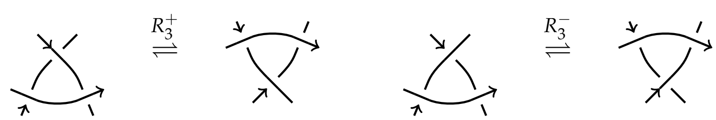


FIGURE 17. Basic versions of the coherent third Reidemeister moves.

Denote by Ψ_3 the chain map associated to the third Reidemeister move (the ones going left to right in Figure 17) and by Φ_3 the map associated to the inverse move. In particular,

$$\Psi_3 : C^{\bullet}_{\mathcal{F}}(L,R)\longrightarrow C^{\bullet}_{\mathcal{F}}(L',R),$$

and

$$\Phi_3 : C^{\bullet}_{\mathcal{F}}(L',R)\longrightarrow C^{\bullet}_{\mathcal{F}}(L,R).$$

With this notation we can state the main result of this section.

Proposition 3.16. *Let L and L' be two oriented link diagrams related by a coherent third Reidemeister move as in Figure 14. Then*

$$\text{(R3c)}\qquad \Psi_3(\beta(L,R)) = \beta(L',R)\quad\textit{and}\quad \Phi_3(\beta(L',R)) = \beta(L,R).$$

Proof. Because the move is coherent the second Reidemeister move used in Step 2 of the categorified Kauffman trick is coherent. Hence the claim is an immediate consequence of Propositions 3.15 and 3.10.

Q.E.D.

Corollary 3.17. *The cycles $\beta(B,R)$ and $\overline{\beta}(B,R)$ are transverse braid invariants. In particular, the homology classes $[\beta(B,R)]$ are invariants for transverse braids. Finally, the above mentioned invariants are sensible to negative stabilizations and destabilizations.* □

Given a braid diagram B, the cycles $\beta(B,R)$ are called **β-invariants of** B. Notice that all the proofs in this section do not rely on the specific base ring, and hence they are valid in general.

2.5. Flype invariance and relationship with the $\psi^{\pm}$ and $\psi^{\pm}_{p,q}$ invariants. To conclude the section, we wish to spend some words on the flype invariance of $\beta(B,R)$ and $\overline{\beta}(B,R)$, and on their relation ship with the $\psi^{\pm}$ and $\psi^{\pm}_{p,q}$ invariants. Consider the sequence of moves codifying a flype given in [**34**, Section 2.1]. which we report below for the sake of completeness.

$$\begin{gathered}
A\sigma_m^k B\sigma_m^{-1} \overset{\mathrm{II}}{\leftrightarrows} A\sigma_m^{-1}\sigma_m^{k+1}B\sigma_m^{-1} \overset{\mathrm{I}_-}{\leftrightarrows} A\sigma_m^{-1}\sigma_{m+1}^{-1}\sigma_m^{k+1}B\sigma_m^{-1} \overset{\mathrm{III}}{\leftrightarrows} \\
\overset{\mathrm{III}}{\leftrightarrows} A\sigma_{m+1}^{k+1}\sigma_m^{-1}\sigma_{m+1}^{-1}B\sigma_m^{-1} \overset{\mathrm{BR}}{\leftrightarrows} \sigma_{m+1}^{k+1}A\sigma_m^{-1}B\sigma_{m+1}^{-1}\sigma_m^{-1} \overset{\mathrm{BR}}{\leftrightarrows} \\
\overset{\mathrm{BR}}{\leftrightarrows} A\sigma_m^{-1}B\sigma_{m+1}^{-1}\sigma_m^{-1}\sigma_{m+1}^{k+1} \overset{\mathrm{III}}{\leftrightarrows} A\sigma_m^{-1}B\sigma_m^{k+1}\sigma_{m+1}^{-1}\sigma_m^{-1} \overset{\mathrm{I}_-^{-1}}{\leftrightarrows} \\
\overset{\mathrm{I}_-^{-1}}{\leftrightarrows} A\sigma_m^{-1}B\sigma_m^{k+1}\sigma_m^{-1} \overset{\mathrm{II}}{\leftrightarrows} A\sigma_m^{-1}B\sigma_m^{k}.
\end{gathered}$$

It could be easily checked directly, once completed Step 3 of the categorified Kauffman trick, that $\beta(B,R)$ and $\overline{\beta}(B,R)$ are invariant up to sign under this move. Moreover, can be noticed that the change in sign is the same for $\beta(B,R)$ and $\overline{\beta}(B,R)$. Thus any linear combination of the two is left invariant by the flype move. In particular, so does their difference, and hence $\psi^{\mathrm{diff}}(B,R) = \psi^{+}(B,R) - \psi^{-}(B,R)$. That is, the following result holds.

Proposition 3.18. *The β-invariants are flype invariants. More precisely, let L and L' be two links related by a negative flype. Then, there is a sequence of Reidemeister moves Σ from L to L' such that*

$$\Phi_\Sigma(\beta(L,R)) = \varepsilon\beta(L',R) \qquad \Phi_\Sigma(\bar{\beta}(L,R)) = \varepsilon\bar{\beta}(L',R),$$

and that

$$\Psi_\Sigma(\beta(L',R)) = \varepsilon'\beta(L,R) \qquad \Psi_\Sigma(\bar{\beta}(L',R)) = \varepsilon'\bar{\beta}(L,R),$$

where $\varepsilon,\varepsilon' \in \{\pm 1\}$, Φ_Σ is the map induced by the sequence Σ, and Ψ_Σ is the map induced by the inverse sequence[5]. □

There is an immediate corollary to Proposition 3.18.

Corollary 3.19. *All the $\mathbb{F}[U]$-linear combinations of the β-invariants are flype invariant in the sense of Proposition 3.18.* □

[5]By inverse sequence we mean the sequence of Reidemeister moves from L' to L obtained by reading Σ backwards.

Taking into account that the projection to the quotient

$$\pi_{TLee} : C^{\bullet}_{BN}(L, R[U]) \longrightarrow C^{\bullet}_{TLee}(L, R) = \frac{C^{\bullet}_{BN}(L, R[U])}{(U-1)C^{\bullet}_{BN}(L, R[U])}$$

maps $\beta(L, R)$ to $\psi^{+}(L, R)$ and $\bar{\beta}(L, R)$ to $\psi^{-}(L, R)$, the following result is immediate.

Corollary 3.20. *All the $\mathbb{F}$-linear combinations of the NLS-invariants are flype invariant in the sense of Proposition 3.18. In particular, ψ^{diff} is flype invariant.* □

However, flypes pairs are pretty difficult to distinguish, even with other effective invariants, so the question about the efficiency of the β-invariants is still open. The lack of examples of transverse braids with same classical invariants and an high number of crossing does not help any attempt to prove the efficiency of the β-invariants (cf. [**34**, Section 4]).

A less direct proof of both the transverse and the flype invariance of the β-invariants can be obtained by studying the relationship between the β and the NLS-invariants. To achieve this goal we resort to the following result. Let $R = \mathbb{F}[U]$, $\alpha \in R \setminus \{U\}$ and M be an R-module. Denote by $\overline{M}_{\alpha}$ the quotient module $M/(U-\alpha)$.

Proposition. A.11. *Let M and M' be two graded, free R-modules, and let $\alpha \in R$ be such that $deg(\alpha) \neq deg(U)$, and $\alpha \notin Div_0(R)$. Given two $R[U]$-linear maps*

$$\Phi : M \longrightarrow M', \qquad \Phi_{\alpha} : \overline{M}_{\alpha} \longrightarrow \overline{M}'_{\alpha},$$

which commute with the projections (i.e. $\pi'_{\alpha} \circ \Phi = \Phi_{\alpha} \circ \pi_{\alpha}$), with Φ homogeneous of degree 0. If $x \in M$, $x' \in M'$ are two homogeneous lifts of, respectively, $y \in \overline{M}_{\alpha}$, $y' \in \overline{M}'_{\alpha}$, such that $deg(x) = deg(x')$. Then,

$$\Phi_{\alpha}(y) = y' \quad \Longleftrightarrow \quad \Phi(x) = x'.$$

□

The proof of Proposition A.11 can be found in Appendix A. Let B and B' be two braids, and Σ a sequence of Reidemeister moves from B to B'. Let $R = \mathbb{F}[U]$, and consider the two free R-modules

$$M = C^{0,\bullet}_{BN}(B, \mathbb{F}[U]) \qquad M' = C^{0,\bullet}_{BN}(B', \mathbb{F}[U]).$$

Taking $\alpha = 1$ it follows

$$\overline{M}_{\alpha} = C^{0}_{TLee}(B, \mathbb{F}[U]) \qquad \overline{M}'_{\alpha} = C^{0}_{TLee}(B', \mathbb{F}[U]).$$

The sequence Σ induce two maps

$$\Phi : M \longrightarrow M', \qquad \Phi_{\alpha} : \overline{M}_{\alpha} \longrightarrow \overline{M}'_{\alpha},$$

which commute with the projections. Moreover, the map Φ is graded of degree 0. The β-cycles are nothing but a homogeneous lifts (with respect to the map π_{TLee}) of the NLS-invariants. Moreover, the degree of the β invariants is the self-linking number. So, if the self-linking number of B and B' is the same, then

$$\Phi(\beta(B, \mathbb{F})) = \beta(B', \mathbb{F}) \Longleftrightarrow \Phi_{\alpha}(\psi^{+}(B, \mathbb{F})) = \psi^{+}(B', \mathbb{F}),$$

and

$$\Phi(\bar{\beta}(B,\mathbb{F})) = \bar{\beta}(B',\mathbb{F}) \Longleftrightarrow \Phi_\alpha(\psi^-(B,\mathbb{F})) = \psi^-(B',\mathbb{F}).$$

So, from Proposition A.11 we get the following result.

Theorem 3.21. *The β-invariants are equivalent to the NLS-invariants.* □

From the results of **[34]** it is immediate the following corollary.

Corollary 3.22. *The β-invariants are transverse braid invariants. Moreover, the β-invariants are flype invariants.* □

The reader may wonder why we went to great length to prove directly the invariance of the β-invariants instead of proving Corollary 3.22. The main reason is that Corollary 3.22 works only in the case of Bar-Natan theory, whereas the proof of the invariance presented generalise to other Khovanov type homologies (as we will see in Section 4).

Remark 32. Notice that the information regarding the flype invariance provided by Corollary 3.22 is slightly less than the information provided by Proposition 3.18.

As we just observed, the information given by the β-invariants is exactly the same as the one given by the NLS-invariants. However, there is another family of transverse invariants introduced by Ng, Lipshitz, and Sarkar in **[34]**, namely the $\psi_{p,q}$-invariants.

Definition 3.6. The invariant $\psi_{p,q}$**-invariants of** B are defined as the image of $\psi^\pm(B,R)$ under the quotient map

$$\pi_{p,q} : \mathscr{F}_{sl(B)} C^0_{TLee}(B,R) \longrightarrow \frac{\mathscr{F}_{sl(B)-2p}(C^0_{TLee}(B,R))}{\mathscr{F}_{sl(B)+2q}(C^0_{TLee}(B,R))},$$

where $p+1, q > 0$.

Let us fix a field $\mathbb{F}$, and consider the (non-exact) sequence of chain complexes

$$\mathscr{F}_{sl(B)}(C^\bullet_{TLee}(B)) \longrightarrow \mathscr{F}_{sl(B)-2p}(C^\bullet_{TLee}(B)) \longrightarrow F^\bullet_{p,q}(B) := \frac{\mathscr{F}_{sl(B)-2p}(C^\bullet_{TLee}(B))}{\mathscr{F}_{sl(B)+2q}(C^\bullet_{TLee}(B))},$$

where the first map is an inclusion (hence injective) and the second map is the projection onto the quotient (hence surjective). Set

$$\overline{C}^{\bullet,\bullet}_{p,q}(B) = \frac{\overline{C}^{\bullet,\bullet-2p}_{BN}(B)}{U^{p+q}\overline{C}^{\bullet,\bullet+2q}_{BN}(B)},$$

We have the following commutative diagram,

$$\begin{tikzcd}
\mathscr{F}_{sl(B)}(C^\bullet_{TLee}(B)) \arrow[r, hook] & \mathscr{F}_{sl(B)-2p}(C^\bullet_{TLee}(B)) \arrow[r, two heads] & F^\bullet_{p,q}(B) \\
\overline{C}^{\bullet,sl(B)}_{BN}(B) \arrow[u, "\pi_{sl(B)}"] \arrow[r, hook, "U^p"'] & \overline{C}^{\bullet,sl(B)-2p}_{BN}(B) \arrow[u, "\pi_{sl(B)-2p}"] \arrow[r, two heads, "f"'] & \overline{C}^{\bullet,sl(B)}_{p,q}(B) \arrow[u, "\widetilde{\Pi}"]
\end{tikzcd}$$

where the vertical arrows are isomorphisms of $\mathbb{F}$-chain complexes, and

$$\pi_{sl(B)}(\beta(B,\mathbb{F})\otimes 1) = \psi^+(B,\mathbb{F}), \quad \pi_{sl(B)}(\overline{\beta}(B,\mathbb{F})\otimes 1) = \psi^-(B,\mathbb{F}).$$

Since these isomorphisms are functorial – i.e. they commute with the maps induced by a sequence of Reidemeister moves) – we obtain the following theorem

Theorem 3.23. *For each transverse braid B the invariants $\psi_{p,q}(B,\mathbb{F})$ are totally determined (and totally determine) by the image of $\beta(B,\mathbb{F})$ under the map $U^p f$ above.* □

Corollary 3.24. *The homology class $[\psi_{p,q}(B,\mathbb{F})] \in H^0(F^{\bullet}_{p,q}(B))$ is trivial if, and only if U^{p+q} divides $U^p[\beta(B,\mathbb{F})]$ – i.e. $\exists\, \delta \in H^{0,\bullet}_{BN}(B,\mathbb{F}[U])$ such that $U^{p+q}\delta = U^p[\beta(B,\mathbb{F})]$).* □

3. Transverse invariants from Bar-Natan Homology II: the invariant c

Both the β-invariants and the NLS-invariants are not practical to distinguish transverse links: one should verify that no map induced by a sequence of Reidemeister moves keeps the β-invariants (or the NLS-invariants) fixed. The idea we want to explore is to use the $\mathbb{F}[U]$-module structure of Bar-Natan homology to distinguish transverse links.

3.1. Definition and first properties. Let L be an oriented link diagram, and $x \in H^{\bullet,\bullet}_{BN}(L, R[U])$ be a non-zero homogeneous homology class. Then, the number

$$c_R(x, L) = max\left\{ k \in \mathbb{N} \;\middle|\; \exists y \in H^{\bullet,\bullet}_{BN}(L,R) \;:\; U^k y = x \right\},$$

is well-defined, and finite. The following proposition is immediate

Proposition 3.25. *Denote by $c_R(L)$ (resp. $\bar{c}_R(L)$) the number $c_R([\beta(L)], L)$ (resp. $c_R([\overline{\beta}(L)], L)$), and let B be a braid diagram. Then, $c_R(B)$ and $\bar{c}_R(B)$ are transverse braid invariants.* □

The numbers $c_R(B)$ and $\bar{c}_R(B)$ are called c**-invariants** of B. Now, let us state some propositions regarding the c-invariants. The first one is trivial consequence of the flype invariance of the β-invariants.

Proposition 3.26. *The c-invariants are flype invariant.* □

The second proposition relates the c-invariants of a braid with the c-invariants of its negative stabilizations, and is a trivial consquence of Proposition 3.8.

Proposition 3.27. *Let B be an oriented braid, and R a commutative integral domain. If B', with the inherited orientation, is obtained from B by k negative stabilizations, then*

$$0 \leq k \leq c_R(B') - c_R(B), \qquad 0 \leq k \leq \bar{c}_R(B') - \bar{c}_R(B).$$

□

Finally, to conclude this section let us show the relationship between the c-invariants and the Plamenevskaya invariant ψ.

Proposition 3.28. *Let B be an oriented braid. The following conditions are equivalent*

(1) $[\psi(B,R)]$ *is trivial;*
(2) $c_R(B) > 0$*;*
(3) $\overline{c}_R(B) > 0$*.*

Proof. Consider the short exact sequence of complexes

$$0 \to C^{i,\bullet}_{BN}(B, R[U]) \xrightarrow{U\cdot} C^{i,\bullet-2}_{BN}(B, R[U]) \xrightarrow{\pi_{Kh}} C^{i,\bullet-2}_{Kh}(B,R) \to 0;$$

this induces a long exact sequence

$$\cdots \xrightarrow{\partial_*} H^{i,j+2}_{BN}(B, R[U]) \xrightarrow{U_*} H^{i,j}_{BN}(B, R[U]) \xrightarrow{\pi_*} H^{i,j}_{Kh}(B,R) \to \cdots$$

Since $\pi_{Kh}(\beta(B,R)) = \pi_{Kh}(\overline{\beta}(B,R)) = \psi(B,R)$, then $[\psi(B,R)]$ is trivial if and only if

$$(\pi_{Kh})_*([\beta(B,R)]) = 0,$$

which is equivalent to

$$(\pi_{Kh})_*([\overline{\beta}(B,R)]) = 0.$$

By the exactness of the long sequence, the previous equalities are equivalent to $[\beta(B,R)] \in Im(U^*)$ and $[\overline{\beta}(B,R)] \in Im(U^*)$, respectively.

Q.E.D.

From a non-vanishing result for the Plamenevskaya invariant of quasi-positive braids and from the previous proposition it follows

Corollary 3.29. *If B is a quasi-positive braid – i.e. the product of conjugates of positive braids – then* $c_R(B) = 0$*.*

Proof. It follows from the fact that $[\psi(B)] \neq 0$ for quasi-positive braids (see [**45**, Corollary 1]).

Q.E.D.

In light of the previous proposition, and of the fact that $\psi_{0,1}(B,R) = \psi(B,R)$, one does wonder if the vanishing of the homology class of the $\psi_{p,q}$-invariants is somehow related to the c-invariants (or some deformation of them). It turns out that this fact is an immediate consequence of Theorem 3.23. In fact, denote by $c_R(B,p)$ the integer

$$c_R(B,p) = max\,\{k \mid U^k\delta = U^p[\beta]\,\delta \in H^{0,\bullet}_{BN}(B, R[U])\},$$

and call it the ***p*-th *c*-invariant**. The following result describes the relationship between the $\psi_{p,q}$ and the c-invariants.

Proposition 3.30. *Let B be an oriented braid and R be any ring. Then*

$$c_R(B,p) = max\{q \mid [\psi^+_{p,q}(B,R)] = 0\} = min\{q \mid [\psi^+_{p,q}(B,R)] \neq 0\}.$$

In particular,

$$c_R(B) = c_R(B,0) = max\{q \mid [\psi^+_{0,q}(B,R)] = 0\} = min\{q \mid [\psi^+_{0,q}(B,R)] \neq 0\}.$$

□

3.2. Estimates on the values of c-invariants. In the previous subsection we introduced the c-invariants and showed some relationships with other transverse invariants. Now, we wish to give some bounds on the value of the c-invariants. In order to do so we make an heavy use of the notation in Subsection 1.3.

Let L be an oriented link diagram, and let Γ' be a full sub-graph of $\Gamma_- = \Gamma_-(L)$ – i.e. if there is an edge in Γ_- between two vertices of Γ', then there is a edge between those vertices also in Γ'. Define $\alpha(\Gamma')$ to be the enhanced state whose underlying resolution is the oriented resolution, and whose labels coincide with those in $\beta(L,R)$ except on the circles corresponding to the vertices of $\Gamma_- \setminus \Gamma'$, where the label is x_+. Let v be a vertex of Γ', set $\alpha(\Gamma', v)$ the enhanced cycle which is identical to $\alpha(\Gamma')$ except on the circle corresponding to v, where the label is the conjugate of the corresponding label in $\beta(L,R)$ – i.e. the label is x_- if the corresponding label in $\beta(L,R)$ is $x_\bullet$, and *vice versa*.

Lemma 3.31. *Let L be an oriented link diagram. If Γ' is a (non-empty) full sub-graph of $\Gamma_-(L)$, and $v \in V(\Gamma')$, then*

(1) $\varepsilon_v U\alpha(\Gamma' \setminus \{v\}) = \alpha(\Gamma') - \alpha(\Gamma', v)$;
(2) if $v \in V(\Gamma')$ is a non-pure[6] *or a non-isolated (in Γ') vertex, then $\alpha(\Gamma', v)$ is a boundary.*

Proof. The claim (1) is immediate from the definitions (cf. the equation (16), at page 48).

Now let us turn to claim (2). Since $v \in V(\Gamma')$ is a non-isolated (in Γ') or non-pure vertex there exists v', which is either in Γ' or neutral, connected to v. Moreover, since $\Gamma' \subseteq \Gamma_-$, it follows that v' is connected to v by a negative edge.

Choose a negative crossing, say c, joining the circles γ_v and $\gamma_{v'}$ representing v and v', respectively. Then, consider the resolution $\underline{s}$ with all crossings but c resolved as in the oriented resolution. Finally, denote by γ the circle in $\underline{s}$ obtained by merging the circles associated to v and v' along c (see Figure 18).

With this notation, define the enhanced state $\theta(\Gamma', v)$ with underlying resolution $\underline{s}$ as follows: label the circle γ by $b_{v'}$, and label all the other circles as in[7] $\alpha(\Gamma')$.

Let us compute $\partial\theta(\Gamma', v)$. By definition, $\partial\theta(\Gamma', v)$ can be written as R-linear combination of enhanced states whose underlying resolution differ from $\underline{s}$ by a local resolution which is a 0-resolution in $\underline{s}$. Since in the oriented resolution the local resolution of each negative crossing is a 1-resolution, it follows that all contributions come from enhanced states obtained by either (Type A) merging two circles of $\underline{s}$ along a positive crossing or (Type B) splitting γ along c. Notice that we can take the expression of $\partial\theta(\Gamma', v)$ as sum of enhanced states in such a way that there is a unique enhanced state of Type B.

[6] A vertex in the simplified Tait graph is called **pure** if it is connected only with vertices of the same type.

[7] Here we are using the natural identification of all circles of the oriented resolution which do not meet c with the circles in $\underline{s}$ different from γ.

All the contributions coming from Type A enhanced states are trivial. In fact, either they merge two circles which have the same labels as in $\beta(L,R)$, or they merge γ with a circle which was connected by a positive crossing with $\gamma_{v'}$ (because v is negative, there are no positive crossing connecting γ_v to any other circle). In this latter case the contribute is trivial. In fact, γ has the same label as $\gamma_{v'}$ in $\beta(L,R)$ and all non negative circles in $\alpha(\Gamma')$ are labeled as in $\beta(L,R)$.

So, the only non trivial contribution to $\partial\theta(\Gamma',v)$ comes from the unique Type B enhanced state. It is immediate from the definition that the underlying resolution of the Type B enhanced state is the oriented resolution of L. Moreover, the labels of this enhanced state are easily computed: all circle different from γ_v are labeled as in $\alpha(\Gamma')$ and γ_v is labeled by $b_{v'} = \overline{b_v}$. So, by definition of $\alpha(\Gamma',v)$, the claim follows.

Q.E.D.

Proposition 3.32. *Let L be an oriented link diagram, and R be a ring. Denote by ℓ_- be the number of split components of L (i.e. connected components of L seen as a four-valent graph) which have only negative crossings. Then*

$$\beta(L) = U^{o_-(L)-\ell_-}\alpha(\{v_1',...,v_{\ell_-}'\}) + \partial\tau,$$

for some $\tau \in C_{BN}^{\bullet,\bullet}(L, R[U])$.

Proof. Our scope is to define a sequence of vertices, say $v_i \in \Gamma_-$, such that if we define (by induction)

$$\Gamma_i = \Gamma_{i-1} \setminus \{v_i\} \quad \text{and} \quad \Gamma_0 = \Gamma_-,$$

the vertex v_i is either non-isolated in Γ_{i-1} or non-pure.

In this way, by applying (1) of Lemma 3.31 several times, we obtain

$$\beta = \alpha(\Gamma_0) = \pm U\alpha(\Gamma_1) + \alpha(\Gamma_0, v_1) = \cdots =$$

$$= \pm U^{o_-(L)-\ell_-}\alpha(\{v_1',...,v_{\ell_-}'\}) + \sum_{i=1}^{o_-(L)-\ell_-} \varepsilon_i U^i \alpha(\Gamma_{i-1}, v_i).$$

The last summand is a boundary thanks to (2) of Lemma 3.31, and the claim follows.

To define the sequence of vertices $v_i \in V(\Gamma_-) \setminus \{v_1',...,v_{\ell_-}'\}$, one can start by fixing an order of the connected components of Γ_-, say $\Gamma_1,...,\Gamma_k$. For each component consider a spanning tree. There is way to define a total order on each spanning tree once one fixes a root: first consider the distance from the root, and then choose an arbitrary order on the nodes which are at the same distance from the root. The choice of the root can be almost arbitrary: one just have to pay attention to avoid pure vertices whenever possible. The only spanning trees having all pure vertices are those spanning connected components of Γ_- which are also connected components of Γ. These connected components of Γ are exactly those corresponding to negative split components of L. Now, fix a root on each spanning tree and an order on its vertices, as described above.

Denote by $v'_1, ..., v'_{\ell_-}$ the roots of the spanning trees spanning connected components of Γ_- which are also connected components of Γ. Order all the vertices of Γ_-, except $v'_1, ..., v'_{\ell_-}$, in the following way: first look at the order of the connected components to which they belong. Then, if they belong to the same component, use the order on the spanning tree. Finally, define v_i to be the i-th vertex with respect to this total order. Notice that each v_i is a non-isolated or non-pure vertex in Γ_{i-1} by definition, and this concludes the proof.

Q.E.D.

Corollary 3.33. *For each oriented link diagram L, we have*

$$c_R(L) \geq o_-(L) - \ell_-(L)$$

□

The estimate given in Corollary 3.33 can be sharpened a bit.

Proposition 3.34. *Let L be an oriented link diagram, and let $v'_1, ..., v'_{\ell_-}$ as in Proposition 3.32. If there is a negative edge between two neutral circles in $\Gamma(L)$, then*

$$\alpha(\{v'_1, ..., v'_{\ell_-}\}) = U\eta + \partial\theta,$$

for some η, $\theta \in C^{\bullet,\bullet}_{BN}(L, R[U])$.

Proof. Let v and v' two neutral vertices connected by a negative edge in $\Gamma(L)$. Denote by γ_v and $\gamma_{v'}$ the circles corresponding to v and v', respectively, in the oriented resolution $\underline{r} = \underline{r}(L)$. Choose a crossing c connecting the two circles γ_v and $\gamma_{v'}$. Denote by $\underline{s} = \underline{s}(c)$ the resolution in which every crossing, with the sole exception of c, is resolved as in the oriented resolution. Finally, denote by γ' the circle in $\underline{s}$ obtained by merging γ_v and $\gamma_{v'}$ along c (see Figure 18).

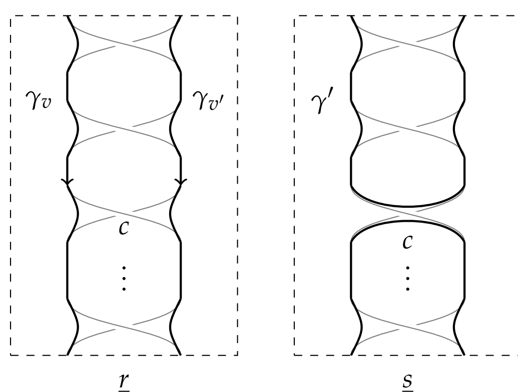


FIGURE 18. The circles γ_v and $\gamma_{v'}$, associated to v and v' respectively, in the oriented resolution $\underline{r}$, the crossing c and the circle γ' in the resolution $\underline{s} = \underline{s}(c)$.

Define θ to be the enhanced state whose underlying resolution in $\underline{s}$, and whose labels are as in $\alpha(\{v'_1, ..., v'_{\ell_-}\})$ for all the circles but γ', whose label is b_{γ_v}.

Remark 33. There is no special reason to use b_{γ_v} instead of $b_{\gamma_{v'}}$. In fact, the roles of v and v' can be exchanged and without affecting the proof.

The only 0-resolution which is not the resolution of a positive crossing in $\underline{s}$ is the resolution of c. Moreover, γ_v and $\gamma_{v'}$ are connected only by negative crossings. So, there exists an expression of $\partial\theta$ as sum of enhanced states whose underlying resolution is the oriented resolution (Type A), is by merging a circle with γ' along a positive crossing (Type B), is obtained by merging two circles (different from γ') along a positive crossing (Type C).

The contribution of Type A enhanced states amount to a single enhanced state α' whose labels are exactly those of $\alpha(\{v'_1, ..., v'_{\ell_-}\})$, except in $\gamma_{v'}$, where the label is $b_v = \overline{b_{v'}}$. Thus

$$\alpha(\{v'_1, ..., v'_{\ell_-}\}) - \alpha' = \pm U \bigotimes_{\gamma \in \underline{r} \setminus \gamma_{v'}} a_\gamma \otimes x_+,$$

where a_γ is the label of γ in $\alpha(\{v'_1, ..., v'_{\ell_-}\})$.

Each contribution coming from Type B enhanced states is either a multiple of U or trivial. In fact, consider a circle γ'' connected with a positive crossing to γ'. This circle, say γ'', in the oriented resolution either shares a crossing with γ_v or shares a crossing with $\gamma_{v'}$. In the first case the label of γ'' is $\overline{b_{\gamma_v}}$. In the second case the label of γ'' is b_{γ_v}. So, when we merge γ'' with γ' we get a circle whose label is either

$$m_{BN}(\overline{b_{\gamma_v}}, b_{\gamma_v}) = 0$$

or

$$m_{BN}(b_{\gamma_v}, b_{\gamma_v}) = \pm U b_{\gamma_v}.$$

Finally, contributions of Type C are trivial. In fact, all non-negative circles different from γ' in $\alpha(\{v'_1, ..., v'_{\ell_-}\})$ have the same labels as in $\beta(L, R)$. Hence, it follows

$$\alpha(\{v'_1, ..., v'_{\ell_-}\}) - \partial\theta = U\eta$$

Q.E.D.

Corollary 3.35. *For each oriented link diagram L, we have*

$$c(L) \geq o_-(L) - \ell_-(L) + \delta^-_L,$$

where $\delta^{\mp}_L$ is 1 *if there is a negative (positive) edge between two neutral circles in Γ_L, and* 0 *otherwise.* □

Remark 34. If L is a negative diagram, there are no non-negative vertices. So, the bound in Corollary 3.33 could be sharp for negative diagrams.

3.3. Crossing changes and an upper bound. Now we wish to give an upper bound to the value of the c-invariants. In order to obtain this estimate we shall analyse the behaviour of the β-invariants under crossing changes, obtaining a result of of independent interest.

First, we need to give a meaning to the expression ”crossing change“. To do so we need to define the crossing removal and the crossing creation moves.

Roughly speaking, we define **crossing removal** and **crossing creation** moves as the sequences of moves shown in Figure 19. More formally, let L be an oriented link diagram and let the crossing c and the arcs **a**, **b**, **e** and **g** be as in Figure 19. The removal of the crossing c is the composition of the saddle move along **g** with the first Reidemeister move which removes the curl created as a result of the saddle move. While the creation of a crossing c is the composition of a first Reidemeister move on the arc **b** with the saddle move along the arc **e**. A crossing creation (resp. removal) move is called **positive** if the crossing created (resp. removed) is positive, otherwise it will be called **negative**.

A **crossing change move** is a crossing removal move, followed by a crossing creation move of different type.

Remark 35. The crossing removal and the crossing creation moves are oriented moves, which means that one starts with an oriented link and ends up with an oriented link. This is basically because these moves represent a weak cobordism between two oriented links.

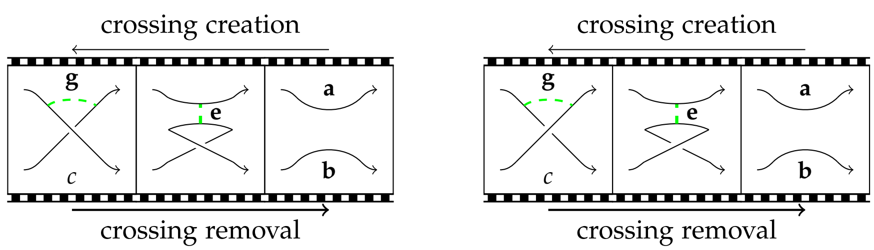


FIGURE 19. Positive (left) and negative (right) crossing removal and creation moves.

Given two links L and L' related by a crossing removal (resp. creation), and a Frobenius algebra $\mathcal{F}$, there is a natural induced map

$$\rho_\circ \text{ (resp. } \mu_\circ) : C^\bullet_{\mathcal{F}}(L, R) \longrightarrow C^\bullet_{\mathcal{F}}(L', R),$$

where $\circ$ is either $+$ or $-$, depending on the type of crossing removed (resp. created). These maps are the composition of a saddle map and a map associated to a first type Reidemeister move, in the order and of the type prescribed by the movie in Figure 19.

Proposition 3.36. *Let L be an oriented link diagram and let c be a positive crossing in L. Denote by L' the diagram obtained from L by removing c. Then*

$$\rho_+(\beta(L, R)) = \beta(L', R), \qquad \mu_+(\beta(L', R)) = \pm U\beta(L, R).$$

In particular, $c_R(L') \geq c_R(L)$.

Remark 36. Notice that we cannot extract any direct information on the c-invariants from

$$\mu_+(\beta(L', R)) = \pm U\beta(L, R).$$

In fact, from the above equality we get

$$c_R(U[\beta(L,R)]) \geq c_R(L');$$

however, the only obvious relation between $c_R(U\beta(L,R))$ and $c_R(L)$ is

$$c_R(U[\beta(L,R)]) \geq c_R(L) + 1.$$

Proof. Let $\underline{r}$ denote the oriented resolution of L. This can be identified with the oriented resolution $\underline{r}'$ of L', and the maps ρ_+ and μ_+ map $A_{\underline{r}}$ to $A_{\underline{r}'}$, and *vice versa*. In order to prove the proposition is useful to split the maps into the composition of a saddle map, and the map associated to a Reidemeister move, in the due order.

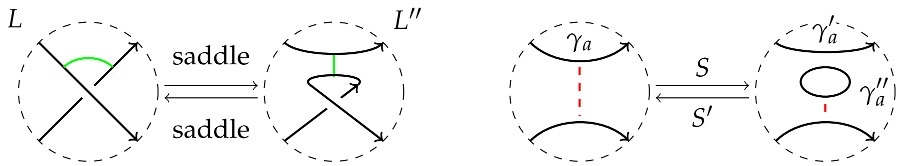


FIGURE 20. The saddle moves in the first clip of the crossing removal move, and in the last clip of the crossing creation move.

First, let us deal with the removal move. Let γ_a be the circle in $\underline{r}$ where the endpoints of the surgery arc $\mathbf{g}$ lie, this is split in two circles, say γ_a' and γ_a'', by the saddle move (see Figure 20). The saddle map S behaves as follows:

$$S(\beta(L,R)) = S\left(\bigotimes_{\gamma\in\underline{r}\setminus\gamma_a} b_\gamma \otimes b_{\gamma_a}\right) = \bigotimes_{\gamma\in\underline{r}\setminus\gamma_a} b_\gamma \otimes \Delta(b_{\gamma_a})$$

so, being b_{γ_a} either x_- or $x_\bullet$, the result is a single enhanced state, and (by (16)) labels of γ_a' and γ_a'' in this enhanced state are both b_{γ_a}. To complete the removal move, the saddle map should be composed with the map Ψ_1^+, and this acts as follows

$$\Psi_1^+\left(S(\beta(L,R))\right) = \Psi_1^+\left(\bigotimes_{\gamma\in\underline{r}\setminus\gamma_a} b_\gamma \otimes \Delta(b_{\gamma_a})\right) = \varepsilon(b_{\gamma_a''}) \bigotimes_{\gamma\in\underline{r}'\setminus\gamma_a'} b_\gamma \otimes b_{\gamma_a'},$$

which is exactly $\beta(L',R)$ (recall that $\varepsilon(x_\bullet) = \varepsilon(x_-) = 1$).

In the case of the positive crossing creation move, one has first to perform a positive first Reidemeister move, which has already been proved to preserve the β-cycles, so the labels of γ_a' and γ_a'' in $\Phi_1^+(\beta(L',R))$ are exactly $b_{\gamma_a'}$ and $b_{\gamma_a''}$. Then, to conclude, one must perform a saddle move. This has the effect of multiplying the labels of γ_a' and of γ_a''. From a simple consideration on the nesting numbers of γ_a' and γ_a'' it follows $b_{\gamma_a'} = b_{\gamma_a''}$. Now the claim follows immediately from Equation (14) at page 48.

Q.E.D.

Remark 37. Denote by L an oriented link diagram, and by L'' the oriented link diagram in the intermediate passage of the crossing removal move (i.e. the oriented link diagram obtained from L by a saddle move as in the right hand side of Figure 20). In the proof of the previous proposition we have shown that the saddle move S sends $\beta(L, R)$ to $\beta(L'', R)$.

In particular, we can re-prove the following

Corollary 3.37. *If B is a quasi-positive braid, then $c_R(B) = 0$.*

Proof. Since B is quasi-positive, it can be written as

$$B = \prod_{i=1}^{k} w_i B_i^+ w_i^{-1},$$

where B_i^+ are positive braids. By eliminating all the crossings in B_i^+, for each i, and performing a sequence of coherent Reidemeister moves, one ends up with the crossingless diagram of the l-components unlink U_l. By Proposition 3.36, it follows

$$0 = c_R(U_l) \geq c_R(B) \geq 0,$$

and this concludes the proof.

Q.E.D.

Proposition 3.38. *Let L be an oriented link diagram and let c be a negative crossing in L. Denote by L' the oriented link diagram obtained from L by removing c. Then*

$$\rho_-(\beta(L, R)) = \pm U\beta(L', R), \qquad \mu_-(\beta(L', R)) = \beta(L, R).$$

In particular, $c_R(L) \geq c_R(L')$.

Proof. The claim about the crossing removal map follows immediately from Remark 37 and from the behaviour of the β-cycles under a negative Reidemeister move. Let us turn to the case of the creation of a negative crossing. It follows immediately from the definition of Φ_1^- that

$$\Phi_1^-(\beta(L', R)) = \bigotimes_{\gamma \in \underline{r}' \setminus \gamma_a''} b_\gamma \otimes x_+, \tag{22}$$

where we are using the same notation used in the proof of Proposition 3.36 (cf. the proof of Proposition 3.8 Equation (22)). The saddle map assigns to $\Phi_1^-(\beta(L', R))$ the state whose underlying resolution is the oriented resolution of L, and all the labels are an in $\Phi_1^-(\beta(L', R))$, except the label of γ_a which is the product of the labels of γ_a' and of γ_a''. From a simple check on the nesting numbers it follows that $b_{\gamma_a'} = b_{\gamma_a}$. Taking into account that the label of γ_a'' is x_+ the claim follows.

Q.E.D.

Corollary 3.39. *If L_+ and L_- are two oriented link diagrams, which differ only in a crossing c. Suppose that c is positive in L_+, and negative in L_-, then*

$$\mu_- \circ \rho_+(\beta(L_+, R)) = \beta(L_-, R), \quad \mu_+ \circ \rho_-(\beta(L_-, R)) = \pm U^2 \beta(L_+, R)$$

In particular, $c_R(L_-) \geq c_R(L_+)$. □

The following will be an immediate consequence of Proposition 3.36 and of Corollary 4.4.

Corollary 3.40. *Let L be an oriented link diagram, and let L_- be the diagram obtained from L by removing all positive crossings. Then,*

$$o_-(L_-) - \ell_-(L_-) \geq c_R(L).$$

$\square$

4. Some philosophical remarks

Thus far we have introduced two (pairs of) transverse braid invariants, namely the β-invariants and the c-invariants, proved some of their properties. Moreover, we proved in Theorem 3.23 that the β-invariants are equivalent to the NLS-invariants. As a consequence the c-invariants can be recovered from the $\psi^{\pm}_{0,q}$ invariants (see Section 3). So one may wonder if there is some more transverse information coming from Khovanov-type theories.

4.1. The β-invariants. The first question we wish to address is whether or not is possible to find other transverse invariants in Bar-Natan theory.

Question 1. Let τ be a transverse link and let T be a diagram representing τ (either a front projection or a braid diagram). Is there an element of $C^{\bullet,\bullet}_{BN}(T, \mathbb{F}[U])$, different from $\beta(T)$ and $\bar{\beta}(T)$ in the case when T is a braid diagram, which is a both a cycle and a transverse invariant[8]?

This question, as stated, is quite general and hard to answer. Of course one may try to write a generic element of $C^{\bullet,\bullet}_{BN}(T, \mathbb{F}[U])$, and impose the invariance under the desired set of Reidemeister moves and the condition of being a cycle, getting some equations. Even though the transverse invariance can be reduced to some local relations, the structure of a generic element x of $C^{\bullet,\bullet}_{BN}(T, \mathbb{F}[U])$, and hence the number and the type of equations one gets by imposing $d_{BN}x = 0$, heavily depends on the diagram.

So, it is better to strengthen our requests in order to narrow the possible answers. For example, we can limit ourselves to the enhanced states with underlying resolution the oriented resolution.

Question 2. Let τ be a transverse link and let T be a diagram representing τ (either a front projection or a braid diagram). Is there $x \in C^{\bullet,\bullet}_{BN}(T, \mathbb{F}[U])$ such that x is an enhanced state with underlying resolution the oriented resolution, and x is a both a cycle and a transverse invariant?

Surprisingly enough we can give a complete answer to Question 2 in the case of transverse braids (cf. Corollary 3.44) and a partial answer in the case of transverse front projections (cf. Corollary 3.45).

---

[8]In the same sense as β, which means is preserved by the maps induced by a set of Reidedmeister moves codifying the transverse isotopy.

Suppose that we have a way to assign to each oriented link diagram L an enhanced state $x(L) \in C_{BN}^{\bullet,\bullet}(L, \mathbb{F}[U])$. Furthermore, suppose that the underlying resolution of $x(L)$ is the oriented resolution of L. The key to answer Question 2 is to analyse the behaviour of x under coherent second Reidemeister moves.

Lemma 3.41. *Let L be an oriented link diagram. Denote by $\boldsymbol{a}$ and $\boldsymbol{b}$ two unknotted arcs of L as*[9] *in Figure 21, and by L' the link diagram obtained by performing a coherent second Reidemeister move along $\boldsymbol{a}$ and $\boldsymbol{b}$. Finally, denote by $\gamma_{\boldsymbol{a}}$ and $\gamma_{\boldsymbol{b}}$ are the circles int the oriented resolution of L containing $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Suppose that*

$$\Phi_2(x(L)) = x(L') \quad \textit{and} \quad \Psi_2(x(L')) = x(L),$$

where Ψ_2 and Φ_2 are the maps associated to the second Reidemeister move and its inverse, respectively. Then the labels of $\gamma_{\boldsymbol{a}}$ and $\gamma_{\boldsymbol{b}}$ in $x(L)$ are, respectively, polynomial multiples of either $b_{\gamma_{\boldsymbol{a}}}$ and $b_{\gamma_{\boldsymbol{b}}}$, or $\bar{b}_{\gamma_{\boldsymbol{a}}}$ and $\bar{b}_{\gamma_{\boldsymbol{b}}}$.

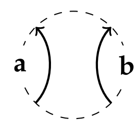


FIGURE 21. Two coherently oriented arcs.

Proof. Denote by $\underline{r}$ and $\underline{r}'$ the oriented resolutions of L and L', respectively. Finally, denote by $\underline{s}$ the oriented resolution of L' where all crossings but the two added by the second Reidemeister move are resolved as in the oriented resolution. First, let us recall the behaviour of the maps

$$\Phi_2 : C_{BN}(L, R) \longrightarrow C_{BN}(L', R) \quad \text{and} \quad \Psi_2 : C_{BN}(L', R) \longrightarrow C_{BN}(L, R),$$

associated to the second Reidemeister move, when restricted to $A_{\underline{r}}$[10] and $A_{\underline{r}'}$, respectively (cf. Subsection 2.3). The map Φ_2 sends $A_{\underline{r}}$ to $A_{\underline{r}'}$. By identifying the two resolutions $\underline{r}$ and $\underline{r}'$ we can identify $A_{\underline{r}}$ and $A_{\underline{r}'}$. With these identifications the map Φ_2 can be seen as the identity map. In particular, it preserves all states with underlying resolution the oriented resolution.

On the other hand, the map Ψ_2 sends $A_{\underline{r}}$ into $A_{\underline{r}'} \oplus A_{\underline{s}}$ and behaves on all the enhanced states in $A_{\underline{r}}$ as shown in Figure 22.

[9] More precisely, there exists a small ball intersecting L precisely in **a** and **b** which is ambient isotopic in $\mathbb{R}^2$ to the ball in Figure 21.

[10] Recall that, given a link diagram L and a resolution $\underline{t}$ of L, $A_{\underline{t}}$ is the $\mathbb{F}[U]$ sub-module of $C_{BN}(L; \mathbb{F}[U])$ generated by all states whose underlying resolution is $\underline{t}$.

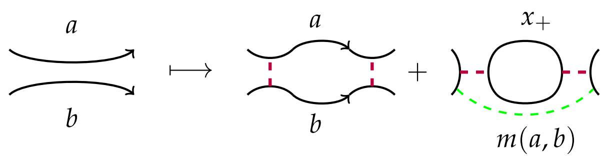


FIGURE 22. The behaviour of the map Ψ_2 on enhanced states with underlying resolution the oriented resolution.

Let a and b the labels of the circles $\gamma_{\mathbf{a}}$ and $\gamma_{\mathbf{b}}$ in $x(L)$. Since $\Psi_2(x(L)) = x(L') \in A_{\underline{r}'}$, it follows that $m(a,b) = 0$. Thus, a and b must be zero divisors in A_{BN}. It follows that a and b belong to either the ideal generated by x_- or to the ideal generated by $x_\bullet$ in A_{BN}. Moreover, the two labels should belong to different ideals. Since $b_{\gamma_{\mathbf{a}}}$ is either x_- or $x_\bullet$ and $b_{\gamma_{\mathbf{b}}} = \bar{b}_{\gamma_{\mathbf{a}}}$ the claim follows.

Q.E.D.

Lemma 3.42. *Let L be a non-split oriented link diagram (i.e. L is connected as a planar graph), and let $x \in C^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$ be an enhanced state. If x is invariant under coherent Reidemeister moves of the second type, then either $x = P(U)\beta(L,\mathbb{F})$ or $x = P(U)\bar{\beta}(L,\mathbb{F})$, for some $P \in \mathbb{F}[U]$.*

Proof. If two circles in the oriented resolution share a crossing it is possible to perform a coherent R_2 involving those circles. Thus, by Lemma 3.41 each pair of circles sharing a crossing should be labeled either as in β or $\bar{\beta}$, up to the multiplication by an element of $\mathbb{F}[U]$. Since L has only one split component, the Tait graph is connected. So, once the label of a single circle is chosen, all the other labels are determined up to multiplication by an element of $\mathbb{F}[U]$, and the claim follows.

Q.E.D.

Let L be an oriented link diagram, and let $L_1, ..., L_k$ be its split components. We will say that L_i and L_j have **compatible orientations** if there exists ball B intersecting L in two unknotted arcs **a** and **b**, with **a** belonging to L_i and **b** belonging to L_j, which is ambient isotopic in $\mathbb{R}^2$ to the ball in Figure 21.

The diagram L is said to be **coherently oriented** if for each pair of split components of L, say L_1 and L_2, there exists a sequence $L_1 = L_{i_1}, ..., L_{i_k} = L_2$ of split components of L such that the components L_{i_j} and $L_{i_{j+1}}$ have compatible orientations for each $j \in \{1, ..., k-1\}$.

Proposition 3.43. *Let L be a coherently oriented diagram, let $x \in C^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U])$ be an enhanced state whose underlying resolution is the oriented resolution, and let $\mathbb{F}$ be a field. If x is invariant under coherent Reidemeister moves of the second type, then x is a $\mathbb{F}[U]$-multiple of either $\beta(L,\mathbb{F})$ or $\bar{\beta}(L,\mathbb{F})$.*

Proof. Let L be a coherently oriented diagram and L_1,...,L_k be its split components. By Lemma 3.41 the labels of $x(L)$ on the components of the oriented resolution of a split component are exactly as in β or as in $\bar{\beta}$, up to multiplication by an element of $\mathbb{F}[U]$. By definition of coherently oriented link diagram, given two split component, say L_i and L_j, there exists a sequence $L_1 = L_{i_1}, ..., L_{i_k} = L_2$ of split components of L such that the components L_{i_j} and $L_{i_{j+1}}$ have compatible orientations for each $j \in \{1, ..., k-1\}$. By definition of compatible orientation it is possible to perform a second type coherent Reidemeister move using an arc of L_{i_j} and and arc of $L_{i_{j+1}}$. Thus, again by Lemma 3.41, if the labels of the circles corresponding to L_{i_j} in $x(L)$ are as in β (up to multiplication by an element of $\mathbb{F}[U]$), then also the labels of the circles corresponding to $L_{i_{j+1}}$ in $x(L)$ are as in β. Similarly, if the the labels of the circles corresponding to L_{i_j} in $x(L)$ are as in $\bar{\beta}$ then also the labels of the circles corresponding to $L_{i_{j+1}}$ in $x(L)$ are as in $\bar{\beta}$. So, once the label of a circle γ in $x(L)$ is a $\mathbb{F}[U]$-multiple of b_γ (resp. $\bar{b}_\gamma$), then $x(L)$ is an $\mathbb{F}[U]$-multiple of $\beta(L, \mathbb{F})$ (resp. $\bar{\beta}(L, \mathbb{F})$).

Q.E.D.

Remark 38. Link diagrams obtained as the Alexander closure of an oriented braid diagram are coherently oriented. Moreover, is always possible to perform a braid-like coherent second Reidemeister move between two coherently oriented the split components of the diagram.

From the previous remark and Proposition 3.43 it follows

Corollary 3.44. *All enhanced states, with underlying resolution equal to the oriented resolution, which are transverse braid invariants in $C_{BN}^{\bullet,\bullet}$ are multiples of one of the β-invariants. In particular, there is a bijection between such transverse invariants and polynomial transverse braid invariants.* □

The previous corollary settles the case of transverse braid invariants. Now, let us turn to the case of transverse front projections.

Suppose to have a way to assign to each transverse front projection T an element $x \in A_{\underline{r}(T)} \subseteq C_{BN}^{\bullet,\bullet}(T, \mathbb{F}[U])$, where $\underline{r}(T)$ denotes the oriented resolution of T. We say that x is a transverse invariant for front projections if given a sequence Σ of transverse Reidemeister moves from T to T' then the induced map

$$\Phi_\Sigma : C_{BN}^{\bullet,\bullet}(T, \mathbb{F}[U]) \longrightarrow C_{BN}^{\bullet,\bullet}(T', \mathbb{F}[U])$$

is such that

$$\Phi_\Sigma(x(T)) = x(T').$$

Corollary 3.45. *If x is an invariant for transverse front projections and T is a coherently oriented front projection, then $x(T)$ is an $\mathbb{F}[U]$-multiple of one of the β-invariants.* □

After having dealt with Question 2, another natural question comes into mind, that is: what happens if we change the Frobenius algebra? More formally, we pose the following question.

Question 3. Given a transverse knot τ, a diagram T for τ (either a front projection or a braid diagram), and a rank 2 Frobenius algebra $\mathcal{F}$, is there a chain in $C^{\bullet,\bullet}_{\mathcal{F}}(T,\mathbb{F}[U])$ which is a transverse invariant? Are there conditions on $\mathcal{F}$ which ensure the existence of such an element?

We would like to address Question 3 in the case
$$A_{\mathcal{F}} = \frac{\mathbb{F}[U,T][X]}{(X^2+PX+Q)},$$
with $P = P(U,T)$ and $Q = Q(U,T)$, where U and T are formal variables. The general case will be left for future work.

Remark 39. Our reasoning works in the slightly more general case
$$A_{\mathcal{F}} = \frac{R_{\mathcal{F}}[X]}{(X^2+PX+Q)} \quad R_{\mathcal{F}} = \frac{\mathbb{F}[U,T]}{(p(U,T),q(U,T))}$$
with p and q such that (p,q) is a (possibly trivial) prime ideal in $\mathbb{F}[U,T]$.

Up to twist equivalence (cf. Proposition 1.1 & Proposition 1.2) we may assume
$$\varepsilon_{\mathcal{F}}(X) = 1_{\mathbb{F}[U,T]}, \quad \varepsilon_{\mathcal{F}}(1_{A_{\mathcal{F}}}) = 0.$$
As in the case of Bar-Natan theory, we will avoid the full generality of the problem and limit ourselves to enhanced states with underlying resolution the oriented resolution. Again, as before, in order to get a cycle is necessary to have some zero divisors in $A_{\mathcal{F}}$, which implies that X^2+PX+Q factors over $\mathbb{F}[U,T]$. Thus, we may write
$$(X^2+PX+Q) = (X-x_1)(X-x_1),$$
where $x_1 = x_1(U,T)$, and $x_2 = x_2(U,T)$.

Remark 40. We are not excluding the case $x_1 = x_2$

Denote by
$$x_\circ = (X-x_1),$$
and
$$x_\bullet = (X-x_2).$$
These two elements have the same formal properties of x_- and $x_\bullet$. More precisely,
$$m_{\mathcal{F}}(x_\circ,x_\circ) = -(x_1-x_2)x_\circ, \quad m_{\mathcal{F}}(x_\bullet,x_\bullet) = (x_1-x_2)x_\bullet, \tag{26}$$
$$m_{\mathcal{F}}(x_\circ,x_\bullet) = 0, \tag{27}$$
and
$$\varepsilon_{\mathcal{F}}(x_\circ) = \varepsilon_{\mathcal{F}}(x_\bullet) = 1_{\mathbb{F}[U,T]}. \tag{28}$$
Furthermore, by setting
$$\varepsilon_{x_*} = \begin{cases} 1 & * = \circ \\ -1 & * = \bullet \end{cases},$$
and
$$\overline{x_\circ} = x_\bullet, \qquad \overline{x_\bullet} = x_\circ,$$

one can recover the analogue of Equation (15), more precisely

$$\overline{x} = x - \varepsilon_x(x_2 - x_1)1_{A_{\mathcal{F}}}, \quad x \in \{x_\circ, x_\bullet\}. \tag{29}$$

Moreover, one has to check also the behaviour of $x_\circ$ and $x_\bullet$ with respect to the co-multiplication $\Delta_{\mathcal{F}}$. In order to do so we will make use of the relation at page 2 (just after Proposition 1.1, see also [**55**, Chapter 2]) which relates the co-unit, the multiplication and the co-multiplication in a Frobenius algebra. For the sake of clarity we will recall it here. Let $(A, m, \iota, \Delta, \varepsilon)$ be a Frobenius algebra. For each $x \in A$ we have

$$\Delta(x) = \sum_i x'_i \otimes x''_i, \tag{30}$$

where the elementary tensors $x'_i \otimes x''_i$ are totally determined by the equations:

$$m(x, y) = \sum_i (x''_i, y)x'_i, \quad \forall y \in A; \tag{31}$$

where $(\cdot, \cdot)$ indicates the (non-degenerate) bi-linear pairing $\varepsilon(m(\cdot, \cdot))$.

Using Equations (26) and (27) and the description of the co-multiplication in Equations (30) and (31), one gets

$$\Delta_{\mathcal{F}}(x_\bullet) = x_\bullet \otimes x_\bullet, \quad \Delta_{\mathcal{F}}(x_\circ) = x_\circ \otimes x_\circ. \tag{32}$$

We will verify Equation (32) only for $x_\circ$, the other is formally identical and hence left to the reader. First notice that:

$$m_{\mathcal{F}}(X, x_\circ) = m_{\mathcal{F}}(x_\bullet, x_\circ) + m_{\mathcal{F}}(x_2, x_\circ) = x_2 x_\circ,$$

so it follows that

$$(X, x_\circ)_{\mathcal{F}} = \varepsilon_{\mathcal{F}}(m_{\mathcal{F}}(X, x_\circ)) = x_2.$$

Set

$$\Delta_{\mathcal{F}}(x_\circ) = aX \otimes X + bX \otimes 1 + c1 \otimes X + d1 \otimes 1.$$

By the above-mentioned relation between the co-unit, the multiplication and the co-multiplication, we have that

$$(x_2 a + b)X + (x_2 c + d) = m_{\mathcal{F}}(x_\circ, x_\circ) = -(x_1 - x_2)x_\circ,$$

and

$$(x_1 a + b)X + (x_1 c + d) = m_{\mathcal{F}}(x_\circ, x_\bullet) = 0 = 0X + 0.$$

Thus, we get

$$a = 1, \quad b = -x_1, \quad c = -x_1, \quad d = x_1^2,$$

and the second part of Equation (32) follows.

Finally, to complete our set of formal properties, we need to check the effect of the "de-cupped torus" map (cf. Equation (19)). Simple computations show that

$$\Delta_{\mathcal{F}}(1) = x_\circ \otimes 1_{A_{\mathcal{F}}} + 1_{A_{\mathcal{F}}} \otimes x_\bullet = x_\bullet \otimes 1_{A_{\mathcal{F}}} + 1_{A_{\mathcal{F}}} \otimes x_\circ,$$

from which it follows

$$T_{\mathcal{F}}(1_{\mathbb{F}[U,T]}) = m_{\mathcal{F}}(\Delta_{\mathcal{F}}(1_{A_{\mathcal{F}}})) = x_\bullet + x_\circ.$$

Let L be an oriented link diagram. Define $\beta_{\mathcal{F}}$**-invariants** as follows: $\beta_{\mathcal{F}}(L) \in C^{\bullet,\bullet}_{\mathcal{F}}(L, \mathbb{F}[U,T])$ is the enhanced state with underlying resolution the oriented resolution, where each circle γ has label $b_{\mathcal{F}}(\gamma)$, defined as follows

$$b_{\mathcal{F}}(\gamma) = \begin{cases} x_\circ & \text{if } N(\gamma) \equiv 0 \ mod\, 2 \\ x_\bullet & \text{if } N(\gamma) \equiv 1 \ mod\, 2 \end{cases}$$

Let $\bar{\beta}_{\mathcal{F}}(L)$ be defined as $\beta_{\mathcal{F}}(L)$, but with the roles of $x_\circ$ and $x_\bullet$ exchanged.

Remark 41. If $x_1 = x_2$ then $\beta_{\mathcal{F}} = \bar{\beta}_{\mathcal{F}}$.

Taking into account that all the formal properties of $x_\circ$ and $x_\bullet$ one can repeat all the proofs in this chapter. In particular, we get the following results.

Proposition 3.46. *Both $\beta_{\mathcal{F}}(L, \mathbb{F}[U,T])$ and $\bar{\beta}_{\mathcal{F}}(L, \mathbb{F}[U,T])$ are (possibly non distinct) cycles in $C^{\bullet}_{\mathcal{F}}(L, \mathbb{F}[U,T])$.* □

Proposition 3.47. *Let L be an oriented link diagram. If L' is the diagram obtained from L by a first Reidemeister move (with the induced orientation), then*

(R1p) $$\Psi_1^+(\mathcal{F})(\beta_{\mathcal{F}}(L,R)) = \beta_{\mathcal{F}}(L',R), \text{ and } \Phi_1^+(\mathcal{F})(\beta_{\mathcal{F}}(L',R)) = \beta_{\mathcal{F}}(L,R),$$

and

(R1n) $$(x_1 - x_2)(\Psi_1^-)_*(\mathcal{F})([\beta_{\mathcal{F}}(L)]) = -\varepsilon_x[\beta_{\mathcal{F}}(L')];$$

where x is the label of the circle containing the arc where the first move is performed in $\beta_{\mathcal{F}}(L,R)$. □

Proposition 3.48. *Let L be an oriented link diagram. Let L'' be the oriented link diagram obtained from L via a coherent second Reidemeister move. Then*

(R2c) $$\Psi_2(\beta_{\mathcal{F}}(L)) = \beta_{\mathcal{F}}(L'') \quad \text{and} \quad \Phi_2(\beta_{\mathcal{F}}(L'')) = \beta_{\mathcal{F}}(L).$$

□

Proposition 3.49. *Let L be an oriented link diagram and let L'' be obtained from L by a non-coherent second Reidemeister move along the arcs **a** and **b**. Then, either*

(R2n1) $$\Phi_2(\beta_{\mathcal{F}}(L'')) = \beta_{\mathcal{F}}(L)$$

or

(R2n2) $$\Phi_2(\beta_{\mathcal{F}}(L'')) = \pm(x_1 - x_2)\beta_{\mathcal{F}}(L),$$

*depending whether **a** and **b** belong to the same or to different circles in the oriented resolution of L. In the first case* (R2n1) *holds, while* (R2n2) *holds in the latter case. Moreover, in neither case the map Ψ_2 does preserve the $\beta_{\mathcal{F}}$-cycles.* □

Proposition 3.50. *Let L and L' be two oriented link diagrams related by a coherent third Reidemeister move as in Figure 14. Then*

(R3c) $$\Psi_3(\beta_{\mathcal{F}}(L)) = \beta_{\mathcal{F}}(L') \quad \text{and} \quad \Phi_3(\beta_{\mathcal{F}}(L')) = \beta_{\mathcal{F}}(L).$$

□

Suppose to have a way to assign to each oriented link diagram L an element $x \in A_{\underline{r}(L)} \subseteq C^{\bullet}_{\mathcal{F}}(L, R_{\mathcal{F}})$, where $\underline{r}(L)$ denotes the oriented resolution of L. We say that x is a **invariant under a sequence Σ of Reidemeister moves** from L to L' if the induced map

$$\Phi_{\Sigma} : C^{\bullet}_{\mathcal{F}}(L, R_{\mathcal{F}}) \longrightarrow C^{\bullet}_{\mathcal{F}}(L', R_{\mathcal{F}})$$

is such that

$$\Phi_{\Sigma}(x(T)) = x(T').$$

Finally, x is a **transverse invariant** for front projection (resp. for transverse braids) if is invariant under sequence of transverse Reidemeister moves of front projections (resp. transverse braid moves).

Proposition 3.51. *Let L be a coherently oriented diagram. If x is invariant under coherent Reidemeister moves of the second type, then x is a $R_{\mathcal{F}}$-multiple of either $\beta_{\mathcal{F}}(L, \mathbb{F})$ or $\bar{\beta}_{\mathcal{F}}(L, \mathbb{F})$.* □

Corollary 3.52. *If x is a transverse braid invariant, then x is an $R_{\mathcal{F}}$-multiple of one of the $\beta_{\mathcal{F}}$-invariants.* □

Corollary 3.53. *If x is an invariant for transverse front projections and T is a coherently oriented front projection, then $x(T)$ is an $R_{\mathcal{F}}$-multiple of one of the $\beta_{\mathcal{F}}$-invariants.* □

Finally, we address the question regarding the flype invariance of the $\beta_{\mathcal{F}}$-invariants.

Question 4. Given a Frobenius algebra $\mathcal{F}$ such that

$$A_{\mathcal{F}} = \frac{\mathbb{F}[U, T][X]}{(X^2 + P(U, T)X + Q(U, T))}$$

are there conditions on $\mathcal{F}$ such that at least one among $\beta_{\mathcal{F}}$ and $\bar{\beta}_{\mathcal{F}}$ is not flype invariant?

The answer is easy in this case. In fact using the sequence of moves codifying a flype given in **[34]**, it is easy to see that both $\beta_{\mathcal{F}}$ and $\bar{\beta}_{\mathcal{F}}$ are flype invariant.

4.2. The *c*-invariants. The usage of the β-invariants to distinguish transverse links is quite far from being practical. In fact, one should verify that all the homotopy equivalences induced by a sequence of Reidemeister moves do not preserve the β-invariants. Since the β-invariants are elements in $C^{0,\bullet}_{BN}$, which is a (graded) free $\mathbb{F}[U]$-module, it is difficult to prove algebraically that they are not left invariant by such maps.

So, in order to overcame this difficulty, we made use of the structure of $\mathbb{F}[U]$-module of Bar-Natan homology and defined the c-invariants. The c-invariants are completely determined by the homology classes of the β-invariants. Thus, they provide the same or less amount of transverse information as the β-invariants. For example, since the β-invariants are flype invariant also the c-invariants cannot be used to distinguish flypes.

Definition 3.7. An oriented link type λ is called c**-simple** if each pair of distinct transverse braids representatives of λ having the same classical invariants have also the same c-invariants.

The non-effectiveness of the c-invariants is equivalent to all links being c-simple. Now we wish to address the following question

Question 5. Let λ be an oriented link type. Which are the homological conditions which λ should satisfy to be c-simple?

This question as stated is intentionally vague. For example we did not specified which homology one should consider, or which type of condition one should look for. However, we manage to give some sufficient conditions for a knot type to be c-simple.

First, we need to make some preliminary consideration on the β-invariants. Let K be an oriented knot diagram representing the knot type κ. Fix an isomorphism of bi-graded $\mathbb{F}[U]$-modules

$$\varphi : H^{\bullet,\bullet}_{BN}(L, \mathbb{F}[U]) \to \bigoplus_{i=1}^{m} \frac{\mathbb{F}[U]}{(U^{t_i})}(h_i, q_i) \oplus \mathbb{F}[U](0, s(\kappa)+1) \oplus \mathbb{F}[U](0, s(\kappa)-1), \tag{33}$$

which exists by Corollary 2.17. Consider the natural generators of the module on the right hand side of (33), that is

$$e_i = (0, ..., 0, \overset{i-th \text{ place}}{\overset{\downarrow}{[1]}}, 0, ..., 0) \quad f_2 = (0, ..., 1, 0) \quad \text{and} \quad f_2 = (0, ..., 0, 1),$$

where $i \in \{1, ..., m\}$, and set

$$\tilde{e}_i = \varphi^{-1}(e_i) \quad \text{and} \quad \tilde{f}_j = \varphi^{-1}(f_j).$$

Notice that for each i we have

$$(hdeg(\tilde{e}_i), qdeg(\tilde{e}_i)) = (h_i, q_i).$$

Denote by I_0 the set of all $i \in \{1, ..., m\}$ such that $h_i = 0$. From the definitions of the $\tilde{e}_i$'s, $\tilde{f}_1$, $\tilde{f}_2$ and $c_{\mathbb{F}}(K)$ it follows immediately that

$$[\beta(K, \mathbb{F})] = U^{c_{\mathbb{F}}(K)} \left(\alpha_1 U^{r_1} \tilde{f}_1 + \alpha_2 U^{r_2} \tilde{f}_2 + \sum_{i \in I_0} \gamma_i U^{k_i} \tilde{e}_i \right), \tag{34}$$

where at least one among r_1, r_2, and the k_i's such that $\gamma_i U^{k_i} \tilde{e}_i \neq 0$, is zero. Moreover, as the homology classes of the β-invariants generate a rank 2 $\mathbb{F}[U]$-sub-module of $H^{\bullet,\bullet}_{BN}(\kappa, \mathbb{F}[U])$, it follows that at least one among α_1 and α_2 is non trivial. Let B_κ be a braid representing κ. From the homogeneity of the β-invariants and from Equation (13) it follows that

$$q_i - 2k_i = s(\kappa) - 1 - 2r_1 = s(\kappa) + 1 - 2r_2 = sl(B_\kappa) + 2c_{\mathbb{F}}(B_\kappa).$$

In particular, we get that

$$r_1 = r_2 + 1.$$

If r_1 equals 0 then we get

$$s(\kappa) - 1 = sl(B_\kappa) + 2c_{\mathbb{F}}(B_\kappa).$$

Thus, $c_{\mathbb{F}}$ would be (half of) the difference between a knot invariant and the self linking, and hence non-effective. A similar reasoning applies to $\bar{c}_{\mathbb{F}}$. Making use of these considerations we can prove the following proposition.

Proposition 3.54. *Let κ be an oriented knot type. If q_i is greater than or equal to $s(\kappa) - 1$ for each $i \in I_0$, then κ is c-simple.*

Proof. If $q_i \geq s(\kappa) - 1$ for all $i \in I_0$, then the k_i's are greater than or equal to r_1. Thus, if $r_2 > 0$, then the k_i's are also strictly greater than 0. It follows that r_2 must be equal to 0 and the claim follows.

Q.E.D.

Remark 42. Proposition 3.54 holds also in the case κ is a link is such that $H^{0,\bullet}(\kappa, \mathbb{F}[U])/T\left(H^{0,\bullet}(\kappa, \mathbb{F}[U])\right)$ is supported in two quantum degrees.

The following corollary is an immediate consequence of Propositions 2.26 and 3.54.

Corollary 3.55. *Let κ be an oriented knot type. If κ satisfies one of the following conditions*

(1) κ is Kh-pseudo-thin (i.e. $H^{0,\bullet}_{Kh}(\kappa, \mathbb{F})$ is supported in two quantum degrees);
(2) $H^{-1,j}_{Kh}(\kappa, \mathbb{F}) \equiv 0$ for each j strictly lower than $s(\kappa) - 3$;

then κ is c-simple. In particular, all Kh-thin[11] *knots are c-simple.* □

Another immediate consequence of the Propositions 2.26 and 3.54 is the following.

Corollary 3.56. *Let κ be an oriented knot type. Suppose that the torsion sub-module of $H^{0,\bullet}_{BN}(\kappa, \mathbb{F}[U])$ is isomorphic to the $\mathbb{F}[U]$-module*

$$M = \bigoplus_{i=1}^{m} \frac{\mathbb{F}[U]}{(U^{2k_i})},$$

for some m, k_1, ..., $k_m \in \mathbb{N} \setminus \{0\}$. Then, if $H^{-1,j}_{Kh}(\kappa, \mathbb{F}) \equiv 0$ for each j strictly lower than $s(\kappa) - 5$, then κ is c-simple. □

Let $\mathbb{F}$ be a field such that $char(\mathbb{F}) \neq 2$. From the analysis of the Bar-Natan and Khovanov homologies of all prime knots with less than 12 crossings it follows

Corollary 3.57. *All prime knots with less than 12 crossings and their mirror images are c-simple over $\mathbb{F}$.*

[11] An oriented link-type λ is *Kh*-**thin** (over $\mathbb{F}$) if its Khovanov homology is supported in two diagonals, that is

$$H^{i,j}_{Kh}(\lambda, \mathbb{F}) \not\equiv 0 \quad \Rightarrow i + j = s(\lambda) \pm 1.$$

Proof. For the computation of integral Khovanov homology the reader may refer to the KnotAtlas ([**5**]). Since there is only 2-torsion in the integral Khovanov homology of the prime knots with less than 12 crossings, their Khovanov homology over $\mathbb{F}$ is concentrated in the same bi-degrees as their rational Khovanov homology.

A well-known theorem due to Lee ([**32**]) states that alternating knots are *Kh*-thin. As a consequence of Corollary 3.55, all alternating knots are *c*-simple. So we may restrict our attention to the non-alternating knots. According to KnotInfo ([**11**]), among the 249 prime knots with less than 11 crossings the only non-alternating knots are the following

8_{19}	8_{20}	8_{21}	9_{42}	9_{43}	9_{44}	9_{45}	9_{46}	9_{47}	9_{48}
9_{49}	10_{124}	10_{125}	10_{126}	10_{127}	10_{128}	10_{129}	10_{130}	10_{131}	10_{132}
10_{133}	10_{134}	10_{135}	10_{136}	10_{137}	10_{138}	10_{139}	10_{140}	10_{141}	10_{142}
10_{143}	10_{144}	10_{145}	10_{146}	10_{147}	10_{148}	10_{149}	10_{150}	10_{151}	10_{152}
10_{153}	10_{154}	10_{155}	10_{156}	10_{157}	10_{158}	10_{159}	10_{160}	10_{161}	10_{162}
10_{163}	10_{164}	10_{165}							

The ones marked in red are the *Kh*-thin knots, while those in blue are the non-*Kh*-thin but pseudo-thin knots. If a knot is *Kh*-thin or *Kh*-pseudo-thin, then also its mirror image is *Kh*-thin or *Kh*-pseudo-thin (cf. Proposition 1.6). Thus, by Corollary 3.55 all coloured prime knots in the list above, and also their mirrors, are *c*-simple. The only prime knots with less than 11 crossing left are 9_{42} and 10_{136} (and their mirrors). These knots satisfy condition (2) of Corollary 3.55 and hence they are *c*-simple.

Finally, among the non-alternating prime knots with 11 crossings and their mirrors the ones which are neither pseudo-thin nor satisfy the condition (2) of Corollary 3.55 are

$$\begin{array}{ccccc} m(11_{n12}) & m(11_{n24}) & 11_{n34} & m(11_{n34}) & 11_{n42} \\ m(11_{n42}) & m(11_{n70}) & m(11_{n79}) & 11_{n92} & m(11_{n96}) \end{array}$$

However, since by hypothesis $char(\mathbb{F}) \neq 2$ by Corollary 2.33 the torsion submodule of $H^{\bullet,\bullet}_{BN}(\kappa, \mathbb{F}[U])$ is isomorphic to the $\mathbb{F}[U]$-module

$$M = \bigoplus_{i=1}^{m} \frac{\mathbb{F}[U]}{(U^{2k_i})},$$

for some m, k_1, ..., $k_m \in \mathbb{N} \setminus \{0\}$. Moreover, the links listed above satisfy the hypotheses of Corollary 3.56. Hence they are *c*-simple and the claim follows.

Q.E.D.

However, the reader should take into account that knots with less than 13 crossings seem to have pretty a simple Khovanov homology. For example, the first knot known to have different values of $s(\cdot, \mathbb{Q})$ and $s(\cdot, \mathbb{F}_2)$ is the knot $14n192465$ (see [**35**, Section 5]), and the first prime knot to have Khovanov homology supported in more than three diagonals, which is also the first with thick torsion, is the knot $13n3663$ (see [**51**, Appendix A.4]). Nonetheless there is a lack of knowledge of examples of transverse non-simple knots with high crossing number, so the question about the effectiveness of the c-invariants remains open.

CHAPTER 4

A Bennequin s-inequality from Bar-Natan homology

Let $\mathbb{F}$ be a field. Notice that the bounds for $c_{\mathbb{F}}(L)$ given in the previous chapter (Sections 3.2 and 3.3) are independent of the characteristic of $\mathbb{F}$ (actually they are independent of the ring used). The aim of this chapter is to give a new Bennequin s-inequality, and discuss its relationship with the other Bennequin s-inequalities.

1. A Bennequin s-inequality from the c-invariants

Before stating our Benenquin s-inequality it will be useful to recall some notation introduced in Chapter 3 Section 1.

Let L be an oriented link diagram of an ℓ-component link λ. Denote by ℓ_s the number of split-components of L (i.e. the connected components of L as planar graph). Define $o_+(L)$, $o_-(L)$ and $o_0(L)$ to be the number of circles in the oriented resolution of L which are touched only by positive crossing, negative crossings and by both type of crossings, respectively. The corresponding types of circles will be called positive, negative and neutral circles, respectively. By $\Gamma(L)$ we denote the simplified Tait graph. That is the graph whose vertices are the circles of the oriented resolution and two vertices are connected by an edge if they share at least a crossing. The edges of the simplified Tait graph are marked with either $+$, $-$, or 0, depending on the fact that two circle share only positive crossings, only negative crossings, or both type of crossings. Finally, $\Gamma_+(L)$ (resp. $\Gamma_-(L)$) is the sub-graph of $\Gamma(L)$ spanned by all positive (resp. negative) circles and $s_+(L)$ (resp. $s_-(L)$) denotes the number of connected components of the graph obtained from $\Gamma(L)$ by removing the negative (resp. positive) edges. Finally, we recall the Bennequin s-inequalities we are going to compare with our bound. Let start with Lobb's inequalities

$$w(L) + o(L) - 2s_-(L) + 1 \geq s(\lambda, \mathbb{F}) \geq w(L) - o(L) + 2s_+(L) - 2\ell + 1. \tag{Lb12}$$

Another inequality we are going to use is Kawamura's inequality, which is the following

$$s(\lambda, \mathbb{F}) \geq w(L) - o(L) + 2s_+(L) - 2\ell_s(L) + 1. \tag{Kw15}$$

The inequality (Kw15) holds with the hypothesis that all the split components of L are non splittable. Finally, we recall Cavallo's inequality

$$s(\lambda, \mathbb{F}) \geq w(L) - o(L) + 2s_+(L) - 2\ell_s(L) + 1, \tag{Cv15}$$

which holds for pseudo-thin links. Now, we are ready to state the main lemma.

Lemma 4.1. *Let L be an oriented link diagram representing the link type λ. Then,*

$$s(\lambda,\mathbb{F}) \geq w(L) + o(L) + 2c_{\mathbb{F}}(L) + 1$$

Proof. Let us recall (cf. **[39]**) that the filtered degree can be also defined as

$$Fdeg([x]) = max\{qdeg(x') \mid x' \in [x]\}$$

where the $qdeg$ has been extended as the minimal degree of all the homogeneous components. Notice that, given $x \in C^{\bullet,\bullet}_{BN}(L,\mathbb{F})$, then $qdeg(x) \leq qdeg\left(\pi_{TLee}(x)\right)$. Thanks to Corollary A.6, $s(\lambda,\mathbb{F}) - 1$ can be seen as the minimum among filtered degrees of

$$[\mathbf{v}_{TLee}(L,\mathbb{F})] = (\pi_{TLee})_*([\beta(L,\mathbb{F})]) \text{ and } [\overline{\mathbf{v}}_{TLee}(L,\mathbb{F})] = (\pi_{TLee})_*([\bar{\beta}(L,\mathbb{F})]).$$

Since

$$\pi_{TLee}(U^k x) = \pi_{TLee}(x),$$

the result follows immediately from Equation (13).

Q.E.D.

Theorem 4.2 (Bennequin s-inequality). *Let L be an oriented link diagram representing the link type λ, then*

$$(s\text{-ineq})\quad w + o - 2o_+ + 2\ell_+ - 2\delta^+ + 1 \geq s(\lambda,\mathbb{F}) \geq w - o + 2o_- - 2\ell_- + 2\delta^- + 1$$

Proof. The lower bound is an immediate consequence of Corollary 3.35, and Lemma 4.1. The upper bound is obtained by duality, which means that it follows from the lower bound on the mirror diagram L^* and from Equation (8) in Theorem 2.12.

Q.E.D.

The following corollary is an immediate consequence of Theorem 4.2 and Lemma 4.1.

Corollary 4.3. *Let L be an oriented link diagram. Then,*

$$o(L) - o_+(L) + \ell_+(L) - \delta^+_L \geq c_{\mathbb{F}}(L).$$

□

To conclude this section we state a corollary which follows immediately from the upper bound in Equation (Lb12) and the lower bound in Equation (s-ineq).

Corollary 4.4. *Let L be an oriented link diagram with only negative (resp. positive) crossings, and λ be the link type of L. Then,*

$$s(\lambda,\mathbb{F}) = w(L) + o(L) - 2\ell_s(L) + 1 \quad (\textit{resp. } s(\lambda,\mathbb{F}) = w(L) - o(L) + 1),$$

where ℓ_s indicates the number of split components of the diagram. In particular, the bound on $c_{\mathbb{F}}(L)$ given in Corollary 3.33 is sharp. □

2. Comparisons and examples

Taking aside the case of negative and positive links, where the bound is sharp, one wishes to understand how good are the estimates given by (s-ineq) on the value of the s-invariant. In particular, one wishes to compare the bounds given in (s-ineq) with similar bounds.

The upper bound in (s-ineq) is not as good as the one in (Lb12). In fact, it is easy to see that for any oriented link diagram L the following inequality holds

$$s_-(L) \geq o_+(L).$$

Moreover, we have the equality if, and only if, the link is positive. In fact, if we remove the positive edges from the simplified Tait graph all the positive vertices become connected components, and there are no other connected components if and only if L is a positive diagram. Thus, if L is not a positive diagram we have

$$s_-(L) \geq o_+(L) + 1 \geq o_+(L) + \delta_L^+ \geq o_+(L) + \delta_L^+ - \ell_+(L).$$

If L is a positive diagram, then $\delta_L^+ = 0$. Thus, if L is a positive diagram we get

$$s_-(L) = o_+(L) = o_+(L) + \delta_L^+ > o_+(L) + \delta_L^+ - \ell_+(L).$$

It follows immediately that

$$w(L) + o(L) - 2o_+(L) + 2\ell_+(L) - 2\delta_L^+ + 1 \geq w(L) + o(L) - 2s_-(L) + 1,$$

So, we have proved that the upper bound provided by (Lb12) is at least as good as the upper bound provided by (s-ineq).

However, a similar reasoning cannot be applied to the lower bounds. The lower bound given by Lobb depends linearly on the number of components of the link. Hence, it becomes gradually less efficient as the number of (linked) components grows. Thus, for links with an high number of components and a low number of split components (i.e. the components of the diagram as a planar graph) one expects the lower bound given in (s-ineq) to be better than the corresponding bound in (Lb12).

The lower bound given by Kawamura – meaning (Kw15) – is strictly better than the one given by Lobb. Nevertheless, is not always usable. In order for (Kw15) to be true the split components of the diagram should be non-splittable. In general it is extremely difficult to check whether or not a link is split. Moreover, if one has a splittable link diagram, it can be arbitrarily difficult to turn it into a disjoint union of non-splittable link diagrams.

Finally, Cavallo's bound (Cv15) works in a very limited setting, and in in which it provides a good bound; in the case of non-split pseudo-thin links the bound is as good as Kawamura's. The main drawback is that to be pseudo-thin is still quite a restrictive condition, especially when one deals with multi-component links.

We will show that the bound given by (s-ineq) is not comparable with the one given by (Kw15). More precisely, we will give an infinite family of examples where the lower bound in (s-ineq) is sharp, while Kawamura's is not.

We can start testing the Bennequin s-inequality (s-ineq) in some examples. The first example will be a family of non-split pseudo thin link with a large number of components. In this case Lobb's lower bound is extremely inefficient, while Kawamura's and Cavallo's give the same bound as (s-ineq), which is sharp.

2.1. The links $\lambda(k)$. The first family of links, see Figure 1 for the diagram $L(k)$, is obtained from the negative Trefoil knot ($\lambda(0)$) by taking consecutive connected sums with the positive Hopf link. The bounds on the s-invariant derived from the diagram $L(K)$ by using (Lb12) and (s-ineq) are easily computed and the results are displayed in Table 1.

(Lb12) upper	(s-ineq) upper	(Lb12) lower	(s-ineq) lower	(Cv15)	(Kw15)
k-2	k	-k	k-2	k-2	k-2

TABLE 1. Lower and upper bounds on the value of $s(\lambda(k), \mathbb{F})$.

The (absolute value of the) difference between the lower bound provided by (s-ineq) and the one provided by (Lb12) increases linearly with the number of components. Notice that the link $\lambda(k)$ is pseudo-thin – the number of canonical cycles in homological degree 0 can be computed by means of Equation (5) (see Section 2 of Chapter 2) and it is exactly two. Moreover, $\lambda(k)$ is clearly non-splittable. Thus, we can also compute the bounds from Kawamura's and Cavallo's (which of course are equal). Putting together the upper bound given by (Lb12) and the lower bound given by (s-ineq) we obtain the following result.

Proposition 4.5. *The s-invariant $s(\lambda(k), \mathbb{F})$ of the link $\lambda(k)$ is $k-2$.* $\square$

FIGURE 1. The diagram $L(k)$ representing the link λ_k, and the corresponding simplified Tait graph $\Gamma(L(k))$. The red vertices and edges are the positive ones, the blue vertices and edges are the negative ones, and the green vertices and edges are the neutral ones.

2.2. The links $\lambda'(h, k, p, r, t)$. The second family of links we are going to test (s-ineq) on is the family $\lambda'(h, k, p, r, t)$. This family of links depends on five integer parameters. The diagram $L'(h, k, p, r, t)$ representing the link $\lambda'(h, k, p, r, t)$ is drawn in Figure 2.

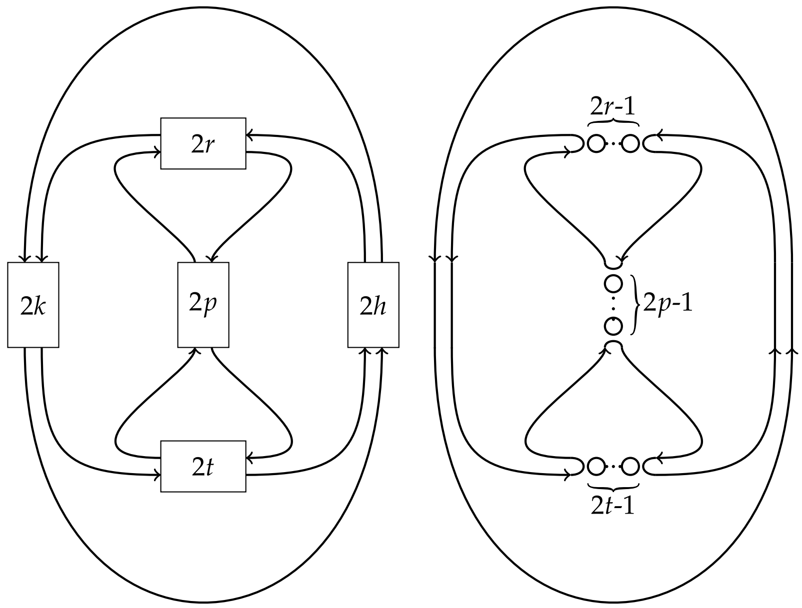


FIGURE 2. The diagram $L'(h, k, p, r, t)$ (on the left) representing the link $\lambda'(h, k, p, r, t)$, and its oriented resolution (on the right). Notice that the oriented resolution does not depend on the signs of the parameters.

During our computations we make use of symmetries to rule out some cases. For example, the cases $h < 0$, $k, t, r, p > 0$ and $k < 0$, $h, t, r, p > 0$ are symmetric (it is sufficient to rotate the diagram) so we will compute the bounds in one of the two cases. It is not difficult, albeit tedious, to compute in all the cases the bounds given by Inequalities (Lb12) and (s-ineq). The red entries in Table 1 are those for which the lower bound and the upper bound, given by either of the inequalities, disagree. The computations and all the simplified Tait graphs (whose unlabeled version is shown in Figure 3) in the various cases are reported in Section 5 of Appendix B. In all the cases were the two bounds disagree the links are not pseudo-thin, but it is more difficult to prove they are non-split. At any rate, the bound provided by Kawamura in these cases is equals to the one provided by (s-ineq). We can summarize the results of our computations in the following propositions. The values of the s-invariant listed in Proposition 4.6 are computed using the upper bound of Equation (Lb12) and the lower bound provided by (s-ineq).

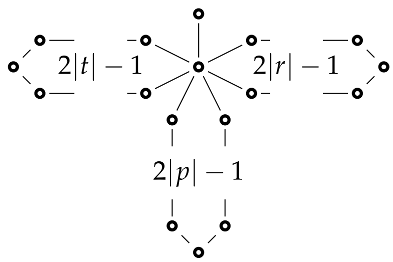


FIGURE 3. The simplified Tait graph for the diagram $L'(h,k,p,r,t)$.

Proposition 4.6. *The value of s-invariant of the link $\lambda'(h,k,p,r,t)$, for all cases, is listed in Table 2. The cases with two values, are those where the lower and the upper bounds given by Inequalities* (Lb12), (s-ineq) *and* (Kw15) *do not agree.* □

<0	>0	*value of s*	<0	>0	*value of s*
h,k,p,r,t	—	$2(h+k-1)$	h,k	p,r,t	$2(h+k+2)$
h,k,p,r	t	$2(h+k)$	t,k	h,p,r	$2(h+k)+1\pm 1$
h,k,p,t	r	$2(h+k)$	t,h	k,p,r	$2(h+k)+1\pm 1$
h,k,r,t	p	$2(h+k)$	r,t	h,k,p	$2(h+k-1)$
h,p,r,t	k	$2(h+k)-3\pm 1$	r,h	k,p,t	$2(h+k)+1\pm 1$
k,p,r,t	h	$2(h+k)-3\pm 1$	r,k	h,p,t	$2(h+k)+1\pm 1$
k,r,t	p,h	$2(h+k-1)$	p,t	h,k,r	$2(h+k-1)$
h,r,t	p,k	$2(h+k-1)$	p,r	h,k,t	$2(h+k-1)$
h,k,t	p,r	$2(h+k+1)$	p,k	h,r,t	$2(h+k)+1\pm 1$
h,k,r	p,t	$2(h+k+1)$	p,h	k,r,t	$2(h+k)+1\pm 1$
h,p,t	r,k	$2(h+k)-1\pm 1$	h	k,p,r,t	$2(h+k)+1\pm 1$
k,p,t	r,h	$2(h+k)-1\pm 1$	k	h,p,r,t	$2(h+k)+1\pm 1$
h,k,p	r,t	$2(h+k+1)$	p	h,k,r,t	$2(h+k)$
k,p,r	t,h	$2(h+k)-1\pm 1$	r	h,k,p,t	$2(h+k)$
h,p,r	t,k	$2(h+k)-1\pm 1$	t	h,k,p,r	$2(h+k)$
p,r,t	h,k	$2(h+k-2)$	—	h,k,p,r,t	$2(h+k+1)$

TABLE 2. The value of the s-invariant for the link $\lambda(h,k,p,r,t)$.

2.3. The links $\tau'(3,3k)$. In the previous example there were some cases in which the bound in (s-ineq) was not sharp. In this third example we wish to show how much the bounds given by (s-ineq) may depend on the chosen diagram.

Consider the 3-component oriented link $\tau'(3,3k)$ obtained from the positive $(3,3k)$-torus link by reversing the orientation of a single component. This is a thick non-split link which is not pseudo-thin. The diagrams in Figure 4 and in Figure 5 represent the link $T'(3,3k)$. In Figures 4 and 5 the orange strand goes in the opposite direction with respect to the black strands.

To see the equivalence of the diagrams T_k' and T_k'', one has to deal first with the case $k=1$. The sequence of Reidemeister moves in this case is shown in

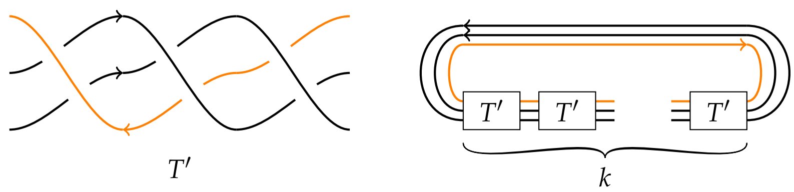


FIGURE 4. The tangle T' and the diagram T'_k representing the link $T'(3, 3k)$.

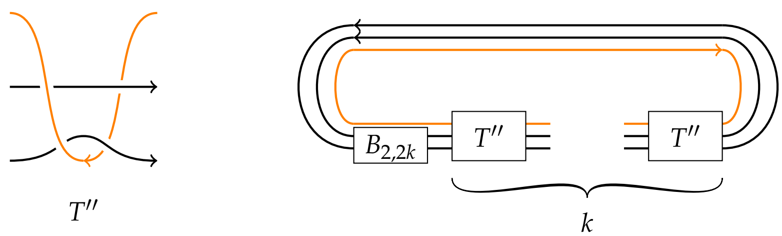


FIGURE 5. The tangle T'' and the diagram T''_k representing the link $T'(3, 3k)$.

Figure 6. When $k > 1$ it is possible to perform the sequence of Reidemeister moves in Figure 6 on each copy of the tangle T'. Then, it is easy to see that we can slide the copies of the tangle T'' past each copy of the tangle $B_{2,2}$, getting the desired diagram.

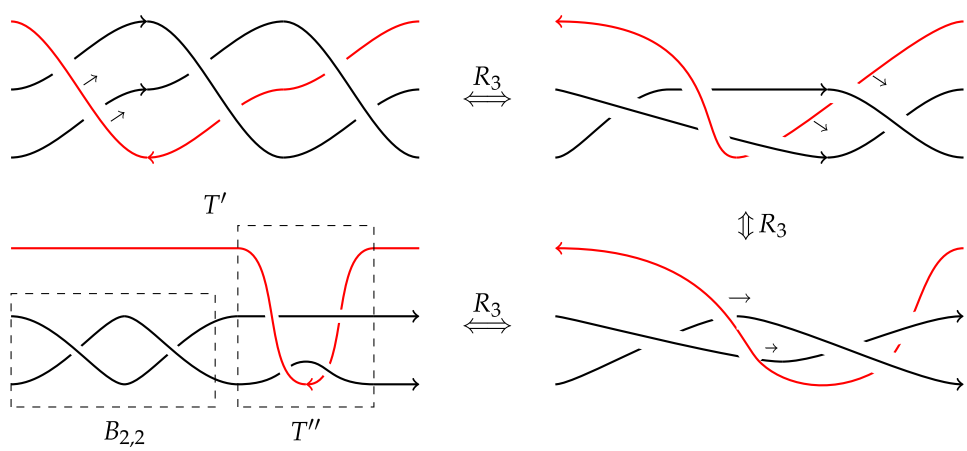


FIGURE 6. The equivalence between the tangle T'_1 and the tangle T''_1.

It is not difficult to compute the quantities needed to obtain the bounds given in Equations (Lb12), (Kw15) and (s-ineq). We listed them and the values of the bounds given by the above-mentioned equations in Table 3 below.

Quantities	T'_k	T''_k
w	$-2k$	$-2k$
o	$2k+1$	$2k+1$
o_-	0	$2k-1$
o_+	0	0
s_+	1	$2k$
s_-	1	1
ℓ_s	1	1
ℓ_-	0	0
δ_-	0	0
(Lb12) upper	0	0
(s-ineq) lower	$-2k$	-2
(Kw15) lower	$-2k$	-2

TABLE 3. Comparing the bounds on $s(T'(3,3k),\mathbb{F})$.

Notice that the difference between the upper bound given by (Lb12) and lower bound provided by either (Kw15) or (s-ineq) using the diagram T'_k increases linearly with k. On the other hand, the bounds obtained from the diagram T''_k are independent from k. As a result of our computations we obtain the following proposition.

Proposition 4.7. *For each field $\mathbb{F}$, the value s-invariant of the link $T'(3,3k)$ with respect to that field is either* 0 *or* -2. □

2.4. The links $\tau^{(h)}(m,mk)$. The fourth family of links is a generalization of the family of links in the previous subsection. Denote by $\tau^{(h)}(m,mk)$ the oriented link obtained from the (m,mk)-torus link by reversing the orientation of h strands. Notice that the link is independent of the chosen strands and that is non-splittable. Exactly as in the previous case, there are efficient and inefficient diagrams. We will compute the bound directly using the most efficient diagram we can provide, that is the diagram shown in Figure 7.

To compute the bounds provided by Inequalities (Lb12), (Kw15) and (s-ineq) it is first necessary to describe the oriented resolution of the tangle $T^{(h)}(m)$ in Figure 7. For this purpose, a simple analysis shows that if $h > m-h$ the oriented resolution of $T^{(h)}(m)$ is the one shown in Figure 8. If $h \leq m-h$ the oriented resolution of $T^{(h)}(m)$ is obtained by reflecting the picture in Figure 8 with respect to the dashed line and exchange the labels h and $m-h$. Our computations will be performed in the case $h \geq m-h$. The other case is dealt with similarly.

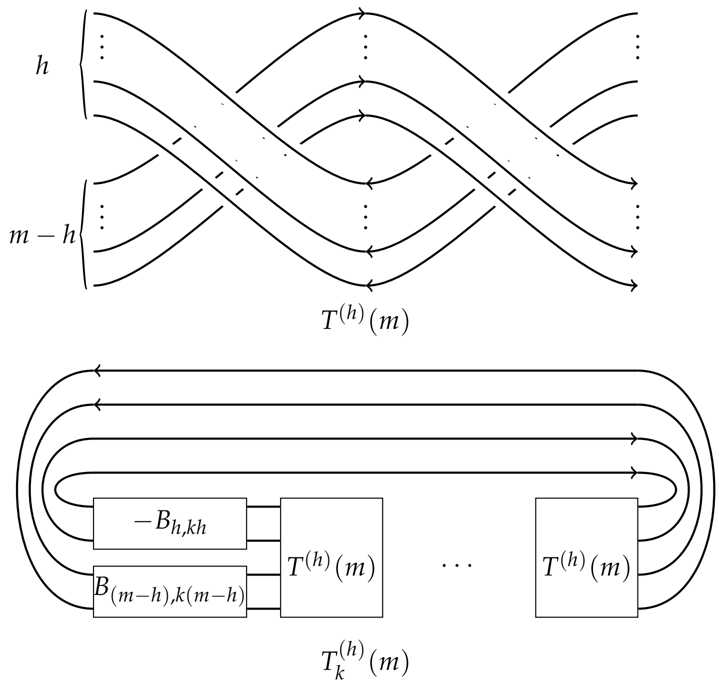


FIGURE 7. The tangle $T^{(h)}(m)$, and the diagram $T^{(h)}_k(m)$.

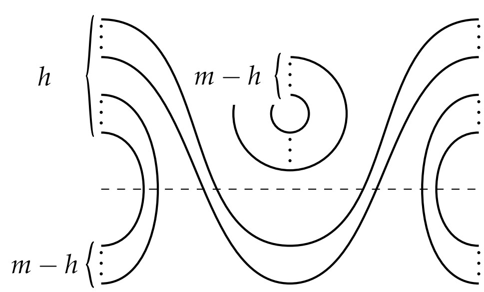


FIGURE 8. The oriented resolution of the tangle $T^{(h)}(m)$ in the case $2h \geq m$, all circles (and lines) in figure are connected by negative traces.

Now it is easy to count the circles in the oriented resolution of $T^{(h)}_k(m)$. The results of this computation, as well as the bounds given by (Lb12), (s-ineq), and (Kw15), are listed in Table 4.

Quantities	$T_k^{(h)}(m)$
w	$k[(m-2h)^2 - m]$
o	$2k(m-h) + 2h - m$
o_-	$2k(m-h) - m + h$
o_+	0
s_+	$2k(m-h) - m + h + 1$
s_-	1
ℓ_s	1
ℓ_-	0
δ_-	0
(Lb12) upper	$k(2h-m)^2 - k(2h-m) + 2h - m - 1$
(s-ineq) lower	$k(2h-m)^2 - k(2h-m) - m - 1$
(Kw15) lower	$k(2h-m)^2 - k(2h-m) - m - 1$

TABLE 4.

The difference between the upper bound given by Lobb's inequality, and the lower bound given by (s-ineq), is exactly $2h$. So the difference between the bounds does not depend on k or m, and is zero if, and only if, $h = 0$. The case $h = 0$ is precisely the case of positive torus links. The case $m - h \geq h$ yields a similar result, and the difference between the upper and the lower bounds is 0 if and only if $m = h$, which is the case of negative torus links. We can summarize our computations in the following proposition.

Proposition 4.8. *Let $\mathbb{F}$ be a field and let h, m, $k \in \mathbb{N}$. If $2h \geq m$, then*

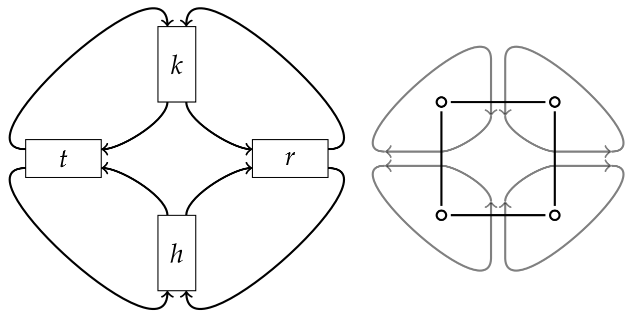
$$0 \leq s(\tau^{(h)}(m,km),\mathbb{F}) - \Big(k(2h-m)^2 - k(2h-m) - m - 1\Big) \leq 2h.$$

□

2.5. The links $\lambda''(h,k,t,r)$. The last example is a the family of oriented links $\lambda''(h,k,t,r)$ which depends on four parameters h, k, t, and r. As long as three out of the four parameters are non trivial the link is non split. For our purpose we will assume the value of each of them to be non-zero. The diagrams we are going to use for our computations are depicted, together with their simplified Tait graphs, in Figure 9.

Due to the simple nature of the Tait graph of the diagram $L''(k,h,t,r)$, it is not important which parameters are positive or negative. The only thing that matters as far as our computations are concerned is how many parameters are positive. In Table 5 we list the values of the quantities needed to compute the bounds given by (s-ineq) and (Lb12).

That said, it is pretty easy to compute the lower bounds given by (Kw15) and (s-ineq) and the upper bound given by (Lb12). Their values, for each case, are listed in Table 6.



$L''(k,h,t,r)$

FIGURE 9. The oriented diagram $L''(k,h,t,r)$ and its Tait graph.

Case	<0	>0	o	o_-	s_-	s_+	ℓ_s	ℓ_-	δ^-
a.	h,k,t,r	——	4	4	1	4	1	1	0
b.	h,k,t	r	4	2	1	3	1	0	1
c.	h,t,r	k	4	2	1	3	1	0	1
d.	h,k	t,r	4	0	2	2	1	0	1
e.	h,t	k,r	4	1	2	2	1	0	0
f.	r	h,k,t	4	0	3	1	1	0	0
g.	k	h,t,r	4	0	3	1	1	0	0
h.	——	h,k,t,r	4	0	4	1	1	0	0

TABLE 5.

$\sharp$ edges >0	$\sharp$ edges <0	(Lb12)	(Kw15)	(s-ineq)
4	0	$h+k+t+r+3$	$h+k+t+r+3$	$h+k+t+r+3$
3	1	$h+k+t+r+1$	$h+k+t+r+1$	$h+k+t+r+1$
2	2	$h+k+t+r-1$	$h+k+t+r-1$	$h+k+t+r-1$
1	3	$h+k+t+r-1$	$h+k+t+r-3$	$h+k+t+r-1$
0	4	$h+k+t+r-3$	$h+k+t+r-3$	$h+k+t+r-3$

TABLE 6.

The first consequence of our computations is the following proposition.

Proposition 4.9. *Let $\mathbb{F}$ be any field. Given $h,\ h,\ r,\ t \in \mathbb{Z} \setminus \{0\}$ we have*

$$s(\lambda''(k,h,t,r),\mathbb{F}) = \begin{cases} h+k+t+r+3 & \text{if } h,\ k,\ r,\ t > 0 \\ h+k+t+r-3 & \text{if } h,\ k,\ r,\ t < 0 \\ h+k+t+r+1 & \text{if only one among } h,\ h,\ r,\ t \text{ is lower than } 0 \\ h+k+t+r-1 & \text{otherwise} \end{cases}$$

$\square$

Another consequence of the results listed in Table 6 is independence of the bounds given by (s-ineq) and (Kw15).

Proposition 4.10. *The lower bounds for the value of the s-invariant given by* (Kw15) *and by* (s-ineq) *are independent.* □

3. Final remarks

To conclude this chapter we wish to investigate a bit more the sharpness of the lower bound provided by (s-ineq). The first thing to notice is that our bound is a consequence of an estimate of the value of the c-invariants. More precisely, we have

$$s(\lambda, \mathbb{F}) \geq qdeg(\beta(L, \mathbb{F})) + 2c_{\mathbb{F}}(L) + 1 \geq \underbrace{w - o + 2o_{-} - 2\ell_{-} + 2\delta^{-} + 1}_{lb(L)}$$

So, a necessary condition for $lb(L)$ to provide a sharp bound on the s-invariant is to have the equality

$$s(\lambda, \mathbb{F}) = qdeg(\beta(L, \mathbb{F})) + 2c_{\mathbb{F}}(L) + 1.$$

Thus for links which are not c-simple there are diagrams such that, even if the bound on the c-invariants given in Corollary 4.3 is sharp, $lb(L)$ is not sharp.

However, the previous statement does not exclude the possibility that for some diagram L the bound $lb(L)$ is sharp.

Question 6. For each link λ there always exists a diagram L such that either (s-ineq) is an equality?

Surprisingly enough, it is easy to answer this question, and the answer is negative. In fact, the bound $lb(L)$ does not depend on the characteristic of the field $\mathbb{F}$. On the other hand, the s depends on $char(\mathbb{F})$. For example $s_{\mathbb{F}_2}(14_{n1336}) < s_{\mathbb{Q}}(14_{n1336})$ (see [**51**]), thus for each diagram L of the knot 14_{n1336} the bound $lb(L)$ may be sharp over $\mathbb{F}_2 = \mathbb{Z}/(2)$ but is not sharp over $\mathbb{Q}$. This reasoning leads us to the following question, which still remains open.

Question 7. Is it possible to find a version of (s-ineq), depending on the characteristic of the field, such that each link admits a diagram for which the new inequality is sharp? More generally, fixed a field $\mathbb{F}$ is it possible to find a Bennequin s-inequality, depending only on combinatorial data of the diagram, such that each link λ has a diagram for which the bound on $s(\lambda, \mathbb{F})$ is sharp?

Now, let us turn to the last question. In the examples shown in the previous section the bound $lb(L)$ was not sharp for Kh-non-pseudo thin links. This is not a general behaviour. Consider the diagram L of the knot 9_{42} in Figure 10. This knot is not Kh-pseudo-thin. The bound $lb(L)$ is easily computed and is 0, which is also the value of $s(9_{42}, \mathbb{Q})$ (see [**5**]).

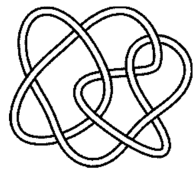


FIGURE 10. A diagram of the knot 9_{42}

However, the knot 9_{42} (being a knot) has rank 2 in $H^{0,\bullet}_{BN}(9_{42}, \mathbb{F}[U])$. Thus, the thickness of the 9_{42} is due to a torsion group in bi-degree $(0, 3)$. So, what happens if the thickness in Khovanov homology is not due to the presence of torsion in Bar-Natan homology? More formally, we call a link λ **strongly** BN**-thick** (over $\mathbb{F}$) if $H^{0,\bullet}_{BN}(\lambda, \mathbb{F}[U])/T(H^{0,\bullet}_{BN}(\lambda, \mathbb{F}[U]))$ is supported in more than two quantum degrees. Note that all the thick links used as examples in Section 2 are strongly BN-thick.

Question 8. Is there an oriented link diagram L, representing a strongly BN-thick link, such that (s-ineq) is sharp?

Our experiments seem to lead to a negative answers. However, links with low crossing number (i.e. less than 13 crossings) are far from being representative. Moreover, for each link we tested (s-ineq) on a finite number of diagrams, but it may exist another diagram on which (s-ineq) is sharp. Thus, Question 8 remains open.

Part 2

Transverse invariants in $\mathfrak{sl}_3$-homologies

CHAPTER 5

The universal $\mathfrak{sl}_3$-link homology via foams

Let L be an oriented link diagram corresponding to a link λ in $\mathbb{S}^3$, and let R be a ring. In his paper **[27]**, Mikhail Khovanov constructed a bi-graded complex $C^{\bullet,\bullet}_{x^3}(L, R)$, whose graded Euler-Poincaré characteristic is (a normalization of) the $\mathfrak{sl}_3$-Jones polynomial, which is an up-to-homotopy invariant of λ.

The chain complex $C^{\bullet,\bullet}_{x^3}(L, R)$ was defined starting from a "geometric complex" whose "chain groups" are (formal direct sums of) webs and whose "differential" is the (formal) sum of decorated "cobordisms" between webs, namely the foams. The algebraic chain complex is obtained from the geometric one via an appropriate functor.

A few years later, Marco Mackaay and Pedro Vaz, in **[40]**, generalized the geometric approach via webs and foams to encompass the deformations of Khovanov $\mathfrak{sl}_3$-theory. These deformations, analogues to the Lee deformation of the original Khovanov homology, were introduced by Gornik in the more general case of Khovanov and Rozansky's $\mathfrak{sl}_n$-homology, using different techniques.

Mackaay and Vaz defined, in the spirit of Bar-Natan's geometric construction (cf. **[4, 41]**), a "geometric complex" in an appropriate category. This category is (the category of complexes over matrices over the abelianized category of) the category of webs and foams, modulo a certain set of relations. These relations depend on the choice of a polynomial $p(x) \in R[x]$ whose degree in the variable x is 3. This geometric complex can be transformed into an (honest) algebraic complex using the so called tautological functors (cf. **[4, 40]**).

In this section we will revise briefly the construction of the universal $\mathfrak{sl}_3$-homology. Even though this thesis is meant to be as much self-contained as possible, some familiarity with the articles **[27, 4, 41, 40]** will be helpful.

1. Webs and Foams

Webs were originally introduced by Greg Kuperberg in **[29]** as a tool to study the representation theory of rank 2 Lie algebras.

Definition 5.1. A **web** Γ is a directed trivalent embedded planar graph with a finite number of components without vertices, called **loops**, satisfying the following properties:

(a) Γ has a finite number of vertices and a finite number of loops;
(b) there are two types of edges in Γ, called **thin** and **thick edges**, and for each vertex $v \in V(\Gamma)$ there is a unique thick edge incident in v;

(c) each vertex of Γ is either a source[1] or a sink[2].

For technical reasons, which will be clear afterwards, also the empty set is considered a web and called the **empty web**. A trivalent directed (abstract) graph satisfying (a) and (c) will be called **abstract web**.

Notice that for us the webs are closed, that is: there are no vertices with less than three incident edges. Moreover, note that from (c) it follows that all abstract webs are bi-partite graphs. Whenever it will be necessary we will mark the thick edges by making them thicker than the normal edges and by changing their colour.

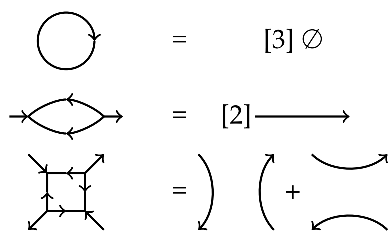


FIGURE 1. Kuperberg's local relations on webs.

To each closed web Γ is it possible to associate a Laurent polynomial $\langle\Gamma\rangle_3$ called the **Kuperberg bracket** of Γ. The Kuperberg bracket of a web is obtained from Γ via the Kuperberg relations[3] in Figure 1, where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-2} + \cdots + q^{-(n-1)} + q^{-(n-2)},$$

and the Kuperberg bracket of the empty web is defined to be $\langle\emptyset\rangle_3 = 1$.

Using the Kuperberg bracket it is possible to associate to an oriented link diagram L a Laurent polynomial $V_3(L) \in \mathbb{Z}[q, q^{-1}]$. Choose an ordering of the crossings in L, say c_1, ..., c_k. For each $v \in \{0,1\}^k$ define a web $\Gamma_v(L)$ by replacing the i-th crossing with its v_i-web resolution according to the rule in Figure 2.

[1]A vertex v of a directed graph is a **source** if all edges incident in v are directed outwards from v.

[2]A vertex v of a directed graph is a **sink** if all edges incident in v are directed towards v.

[3]For a proof of the consistency of the Kuperberg relations see [**29**].

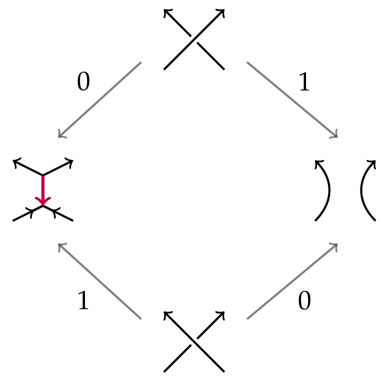


FIGURE 2. Web resolutions.

Finally, set

$$V_3(L) = \sum_{v \in \{0,1\}^k} (-1)^{|v|} q^{b_v} \langle \Gamma_v(L) \rangle_3,$$

where $|v|$ is the weight of v (i.e. the sum of its entries) and

$$b_v = \sum_{i=1}^{k} \frac{sign(c_i)5 + (-1)^{v_i}}{2}.$$

It is not difficult to prove that $V_3(L)$ is a link invariant. To be precise V_3 is a normalization of the Jones $\mathfrak{sl}_3$-polynomial. In the $\mathfrak{sl}_2$ setting, the Khovanov $\mathfrak{sl}_2$-homology categorifies (a normalization of) the Jones polynomial $V_2(L)$. Similarly, the (non-deformed) Khovanov $\mathfrak{sl}_3$-homology categorifies the polynomial $V_3(L)$.

To define the universal (geometric) $\mathfrak{sl}_3$-theory we need another ingredient: the foams. Roughly speaking, foams are decorated branched surfaces which are singular along a smooth 1-dimensional manifold of triple points. Let us put aside the decorations and let us start by defining the underlying topological structure of a foam.

A (**topological**) **pre-foam** Σ is a compact topological space such that each point has a neighbourhood homeomorphic to one of the four local models in Figure 3.

Remark 43. A topological pre-foam is a finite CW-complex.

A point $p \in \Sigma$ is **regular** if it has a neighbourhood which is homeomorphic to either (C) or (D). A non-regular point of Σ is called **singular** and the set of singular point is denoted by $Sing(\Sigma)$. The connected components of $\Sigma \setminus Sing(\Sigma) \subseteq \Sigma$ are called **regular regions** of Σ. Finally, a **boundary** point for Σ is a point which does not have a neighbourhood homeomorphic to either (A) or (C). A topological pre-foam with empty boundary is called **closed**.

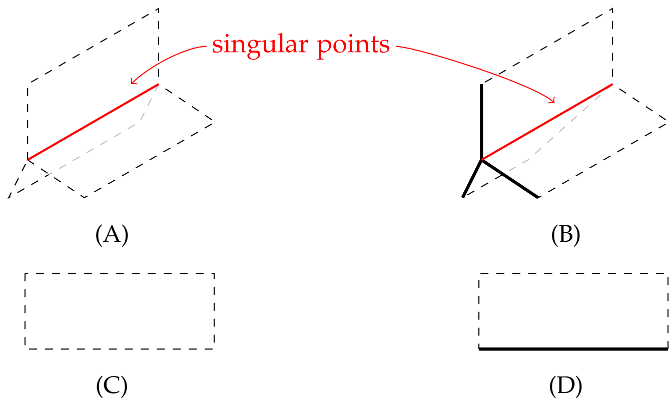


FIGURE 3. Local models for a pre-foam.

Remark 44. The singular locus of pre-foam is the disjoint union of circles and arcs, which are called **singular circles** and **singular arcs**. The singular boundary points correspond to the boundary points of the singular arcs.

The choice of a topological atlas[4] on a pre-foam determines a topological atlas on each regular region and also on $Sing(\Sigma)$. These atlases endow the closures of the regular regions and the singular locus (i.e. $Sing(\Sigma)$) with the structure of topological manifolds. A **smooth pre-foam** is a topological pre-foam with a chosen topological atlas such that the induced atlases on the regular regions and on the singular locus are smooth atlases.

Finally, an **orientable pre-foam** is a smooth pre-foam such that the closure of each regular region is an orientable (topological) manifold. Given an orientable pre-foam Σ, an **orientation** on Σ is the choice of an orientation of the closure of each regular region in such a way that the orientation induced in the intersection of two closed regions agrees.

Remark 45. If we choose an orientation on a orientable pre-foam this induces an orientation on the singular locus.

Remark 46. The boundary of a topological pre-foam is a (possibly empty) finite trivalent graph whose vertices correspond to singular boundary points. If the pre-foam is oriented its boundary is an directed graph. Moreover, each vertex of the boundary graph of an oriented pre-foam is either a sink or a source. In other words, the boundary of an oriented pre-foam is an abstract web.

Definition 5.2. A **decorated pre-foam** is an oriented pre-foam Σ together with the following data:

(a) a finite number (possibly zero) of marked points, called **dots**, in the interior of each regular region of Σ;

---

[4]By topological atlas on a pre-foam Σ we mean an open cover of Σ together with a homeomorphism of each element of the cover with one of the local models in Figure 3.

(b) a specified order on the regular regions incident on a singular arc or circle.

It is possible to define a category **PreFoam** whose objects are abstract webs and whose morphisms are formal R-linear combinations of the triples $(\Sigma, \partial_0\Sigma, \partial_1\Sigma)$ satisfying the following properties

- ▷ Σ is a decorated pre-foam;
- ▷ $\partial_0\Sigma$, $\partial_1\Sigma \in Obj(\mathbf{PreFoam})$, $\partial_0\Sigma$ is the source object and $\partial_1\Sigma$ the target object of the morphism;
- ▷ $\partial\Sigma = \partial_0\Sigma \sqcup -\partial_1\Sigma$, where the minus sign denotes the reversal of the orientation;
- ▷ the triple is seen up to boundary fixing isotopies which do not change the regular regions of the dots and preserve the ordering of the components near each singular arc.

Finally, the composition of two triples $(\Sigma, \partial_0\Sigma, \partial_1\Sigma)$ and $(\Sigma', \partial_1\Sigma, \partial_1\Sigma')$ is defined as the triple $(\Sigma'', \partial_0\Sigma, \partial_1\Sigma')$, where Σ'' is obtained by glueing Σ and Σ' along $\partial_1\Sigma$.

Definition 5.3. A **foam** is a decorated pre-foam properly and smoothly[5] embedded in $\mathbb{R}^2 \times I$. Moreover, we ask the cyclic order of the regular regions at each singular arc or circle to coincide with the cyclic order induced by rotating clockwise[6] around a singular arc or circle. Foams will be considered up to ambient isotopies of $\mathbb{R}^2 \times I$ which fix the boundary of the foam and does not change the regular regions of the dots.

Remark 47. The boundary of a foam is the disjoint union of two webs.

Given two webs W_0 and W_1, a **foam between** W_0 **and** W_1 is a foam Σ such that

$$\Sigma \cap \mathbb{R}^2 \times \{0\} = W_0 \quad \text{and} \quad \Sigma \cap \mathbb{R}^2 \times \{1\} = -W_1.$$

The category **Foam** is the category whose objects are webs, whose morphisms are a R-linear combinations of foams between two webs, and whose composition is defined as the glueing of two foams along the shared boundary. For technical reasons also the empty foam is included in **Foam** as an element of $Hom_{\mathbf{Foam}}(\emptyset, \emptyset)$.

Remark 48. There is a forgetful functor from **Foam** to **PreFoam**. Which sends the foam Σ to the triple $(\Sigma, \Sigma \cap \mathbb{R}^2 \times \{0\}, -\Sigma \cap \mathbb{R}^2 \times \{1\})$

The category **Foam** will be the geometric object underlying the construction of the universal $\mathfrak{sl}_3$-link homology theory. However, to obtain a true link invariant it is necessary to mod out **Foam** by a set of local relations.

[5]That is in such a way that the restriction of the embedding to each regular region and to the singular locus is smooth.

[6]We suppose fixed an orientation of $\mathbb{R}^2 \times I$.

2. Foams and Local relations

Local relations are equalities among (linear combinations of) foams which are identical except inside a small ball. The relations we are concerned with can be divided into two types:

reduction relations (c.f. Figure 4),
evaluation relations (c.f. Figure 5).

The reduction relations depend on the choice of a polynomial $p(x) \in R[x]$ of the form

$$p(x) = x^3 + a_2x^2 + a_1x + a_0.$$

Thus, we will hereby suppose p fixed. In the case R is a graded ring, one should require the coefficients a_2, a_1 and a_0 and the polynomial p to be homogeneous in order to obtain a graded theory. That said, let us leave aside the matter of the gradings, at least for the moment.

The reduction relations, as their name implies, allow us to "reduce" in some sense our foam. There are two types of reductions which can be performed: the "genus" reduction and the dot reduction. The **genus reduction relation** (GR) reduces the number of "handles" in a foam in exchange for adding dots and transforming a single foam into a linear combination of foams. While the **dot reduction relation** (DR) allows us to delete dots, if they are more than a certain number, in exchange to do so a single foam has to be traded with a linear combination of foams.

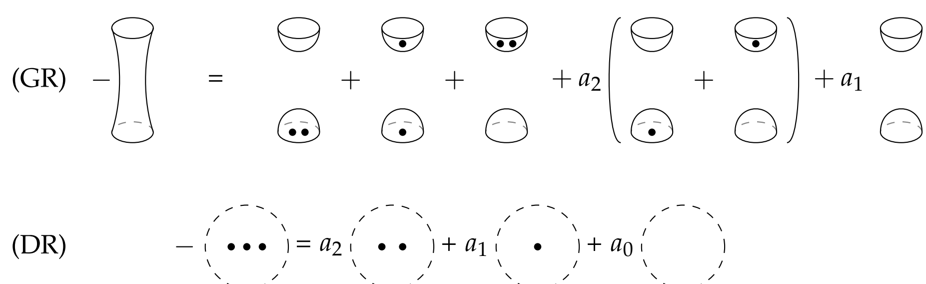


FIGURE 4. Reduction relations.

Given a closed foam Σ by using the genus reduction relation it is possible to express Σ as a linear combination of (disjoint unions of) spheres and theta foams (i.e. a sphere with a disk glued along the equator). Moreover, thanks to the dot reduction relations we can have less than three dots on each component.

The evaluation relations allow us to associate to each closed foam a multiple of the empty foam. In fact, we reduced our foam to a R-linear combination of spheres and theta foams with less than three dot each. The evaluation relations express the sphere and the theta foams with less than three dots as multiples of empty foam. There are two types of evaluation relations: the relations (S) (**sphere relations**) and the relations (Θ) (**theta foam relations**) and they are depicted in Figure 5.

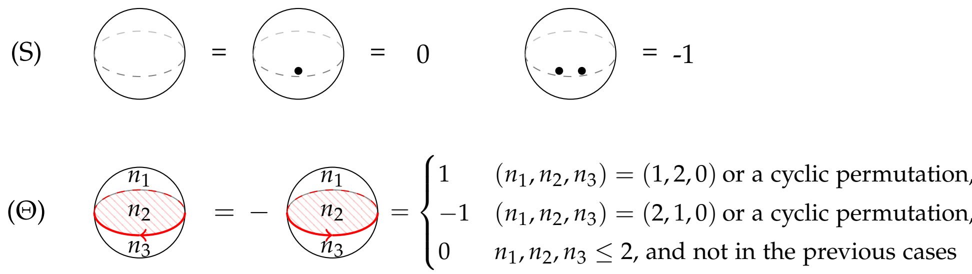


FIGURE 5. Evaluation relations. The numbers n_1, n_2 and n_3 on the theta foam indicate the number of dots in the corresponding region.

Since every closed foam evaluates to a polynomial (multiple of the empty foam), for each pair of webs W_0 and W_1, and each foam $F \in Hom_{\mathbf{Foam}}(W_0, W_1)$, there is a well defined R-bi-linear pairing

$$\langle \cdot | \cdot \rangle_F : Hom_{\mathbf{Foam}}(\emptyset, W_0) \otimes Hom_{\mathbf{Foam}}(W_1, \emptyset) \longrightarrow R$$

given by "capping off" F with an element of $Hom_{\mathbf{Foam}}(\emptyset, W_0)$, and closing the result by glueing an element of $Hom_{\mathbf{Foam}}(W_1, \emptyset)$.

The category $\mathbf{Foam}_{/\ell}$ has the same objects as foams, but the morphisms are considered up to the relation

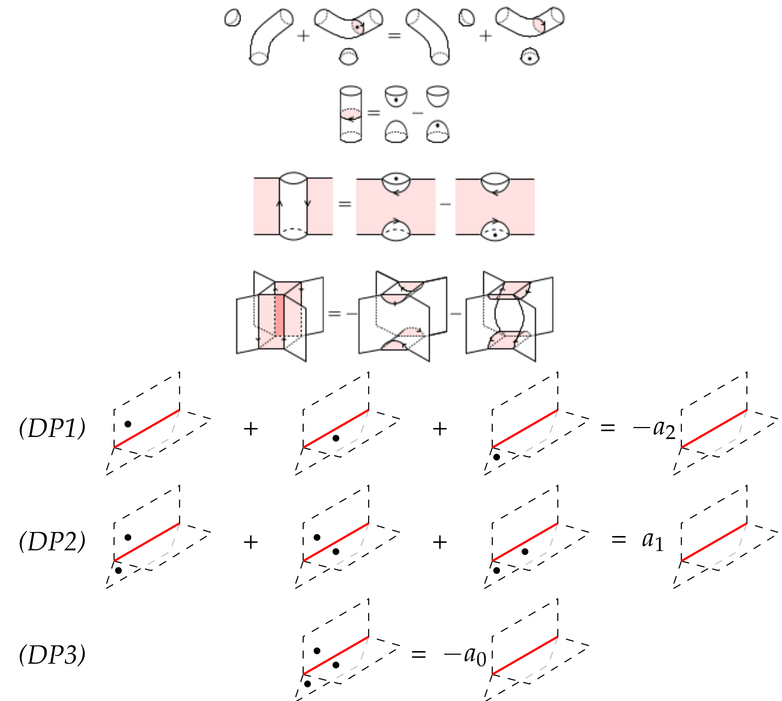
$$\sum_i P_i(U, V, W) F_i = 0, \quad F_i \in Hom_{\mathbf{Foam}}(W_0, W_1),$$

in the case

$$\sum_i P_i \langle F' | F'' \rangle_{F_i} = 0, \forall F' \in Hom_{\mathbf{Foam}}(\emptyset, W_0),\ F'' \in Hom_{\mathbf{Foam}}(W_1, \emptyset).$$

In other words, a R-linear combination of foams is trivial if the associated bi-linear form is trivial. Using the local relations it is possible to prove the following result, due to Khovanov in the case $p(x) = x^3$ and to Mackaay and Vaz when p is a general polynomial, and whose proof will be omitted. The reader may consult **[40]** for a proof.

Proposition 5.1 (Mackaay-Vaz, **[40]**). *The following local relations hold in* $\mathbf{Foam}_{/\ell}$



where (DP1), (DP2) and (DP3) are also called ***dot permutation relations****.* □

3. The $\mathfrak{sl}_3$-link homologies via Foams

In this section we will review the construction of a link homology theory via webs and foams. This construction consists of three steps. First we need some machinery coming from category theory, namely cube and abstract complexes. Then, we will describe how to build a cube from a link diagram, and apply the machinery developed to get a formal ”geometric“ complex of foams and webs. Finally, we make use of Bar-Natan's tautological functors to obtain an honest chain complex and an homology theory.

3.1. Cubes in categories and abstract complexes. Let us review the construction of the abstract complex associated to a link diagram. To define this complex, which is a ”complex of web and foams“ associated to an oriented link diagram, it is necessary a bit of abstract nonsense. We will try to keep the technicalities to a minimum while we try to be precise.

Denote by Q_n the standard n-dimensional cube $[0,1]^n$. Orient the edges of Q_n from the vertex with lower 1-norm to the vertex with greater 1-norm, where the 1-norm of a vector $v \in \mathbb{R}^n$ is defined as $\|v\|_1 = \sum_{i=1}^n |v_i|$.

A **square** in Q_n is an ordered collection of four distinct vertices (v_1, v_2, v_3, v_4) such that there are two edges from v_1 to v_2 and v_3, respectively, and two edges from v_2 and v_3 to v_4.

Definition 5.4. A n**-cube in a category C** is the assignment of an object O_v for each vertex v of Q_n and of a morphism $F(v,v') \in Hom_{\mathbf{C}}(O_v, O_{v'})$ for each oriented edge from v to v'.

Let R be a ring. An R**-linear category** is a (small) category **C** such that, for each pair of objects A, B, the set of morphisms $Hom_{\mathbf{C}}(A,B)$ has a structure of R-module and the composition is bilinear with respect to this structure. A $\mathbb{Z}$-linear category is often called a **pre-additive category**.

Definition 5.5. A n-cube in a category **C** is **commutative** if for each square (v_1, v_2, v_3, v_4) in Q_n we have

$$F(v_2, v_4) \circ F(v_1, v_2) = F(v_3, v_4) \circ F(v_1, v_3).$$

Similarly, a n-cube in a R-linear category **C** is **skew commutative** if for each square (v_1, v_2, v_3, v_4) in Q_n we have

$$F(v_2, v_4) \circ F(v_1, v_2) = -F(v_3, v_4) \circ F(v_1, v_3).$$

Is an easy exercise in combinatorics to prove that there always exists a sign assignment (cf. Section 2 of Chapter 1) that turns a commutative cube into a skew commutative cube. More precisely, the following result holds. The reader may consult, for example, **[25]** for a description of such a sign assignment.

Lemma 5.2. *Let* **C** *be an R-linear category. Then, there exists a map* ε *from the set of edges of* Q_n *to* $\{\pm 1\}$ *such that for each commutative n-cube* $Q = \{O_v, F(v,v')\}$ *in* **C** *the n-cube* $Q' = \{O_v, \varepsilon(v \to v')F(v,v')\}$ *is skew commutative.* □

Now, we wish to assign to a skew-commutative cube a formal chain complex. To do so one must first give a meaning to a complex over a category.

Definition 5.6. Let **C** be a R-linear category. The **category Com**$(\mathbf{C})$ **of complexes over C** is the category defined as follows

(1) the objects of $\mathbf{Com}(\mathbf{C})$ are ordered collections of pairs $(C_i, d_i)_{i\in\mathbb{Z}}$ where $C_i \in Obj(\mathbf{C})$ and $d_i \in Hom_{\mathbf{C}}(C_i, C_{i+1})$ such that

$$d_{i+1} \circ d_i = 0_{Hom_{\mathbf{C}}(C_i, C_{i+2})};$$

(2) the morphisms between two objects (C_i, d_i) and (D_i, ∂_i) of $\mathbf{Com}(\mathbf{C})$ are collection of maps $(F_i)_{i\in\mathbb{Z}}$ such that

$$\forall\, i \in \mathbb{Z} \;:\; F_i \in Hom_{\mathbf{C}}(C_i, D_{i+k}) \quad \text{and} \quad F_{i+1} \circ d_i = \partial_{i+k} \circ F_i,$$

for a fixed $k \in \mathbb{Z}$ called **degree of** $(F_i)_{i\in\mathbb{Z}}$;

(3) the composition of two morphisms $(F_i)_{i\in\mathbb{Z}}$ and $(G_j)_{j\in\mathbb{Z}}$ is the morphisms defined as $(G_{i+k} \circ F_i)_{i\in\mathbb{Z}}$, where k is the degree of $(F_i)_{i\in\mathbb{Z}}$.

In general an R-linear category does not have kernels and co-kernels. So, even though we can define chain complexes we cannot define the homology over an R-linear category. However, it is possible to define when two chain complexes over an R-linear category are homotopy equivalent.

Definition 5.7. Two morphisms F and G between two objects in $\mathbf{Com}(\mathbf{C})$, say $C_\bullet = (C_i, d_i)$ and $D_\bullet = (D_i, \partial_i)$, are **homotopy equivalent** if there exists a morphism $H \in Hom_{\mathbf{Com}(\mathbf{C})}(D_\bullet, C_\bullet)$ such that

$$F - G = d \circ H \pm H \circ \partial.$$

Two objects $C_\bullet,\ D_\bullet \in Obj(\mathbf{Com}(\mathbf{C}))$ are **homotopy equivalent** if there exists two morphisms

$$F \in Hom_{\mathbf{Com}(\mathbf{C})}(D_\bullet, C_\bullet) \quad \text{and} \quad G \in H_{\mathbf{Com}(\mathbf{C})}(C_\bullet, D_\bullet)$$

such that the compositions $F \circ G$ and $G \circ F$ are homotopy equivalent to the identity morphism of $D_\bullet$ and $C_\bullet$, respectively. We will denote by $\mathbf{Com}_{/h}(\mathbf{C})$ the category of complexes over $\mathbf{C}$ and morphisms of $\mathbf{Com}(\mathbf{C})$ up to homotopy equivalence.

To conclude the abstract construction of a complex from a skew commutative cube we need another definition.

Definition 5.8. Given an R-linear category $\mathbf{C}$, the **matrix category over C** is the category whose objects are formal direct sums of objects in $\mathbf{C}$ and whose morphisms are matrices with entries in the morphism of $\mathbf{C}$. The composition of two morphisms in the matrix category over $\mathbf{C}$ is given by the usual matrix multiplication rule.

Denote by $\mathbf{Kom}(\mathbf{C})$ the category of complexes over the matrix category over $\mathbf{C}$, where $\mathbf{C}$ is an arbitrary R-linear category. Given a skew commutative n-cube Q in $\mathbf{C}$, define

$$C_i^Q = \bigoplus_{|v|=i} Q_v \quad d_i^Q = \sum_{|v|=i} \bigoplus_{v'} F(v, v'),$$

where $F(v, v')$ is taken to be zero if there is no edge from v to v'. It is an easy verification that (C_i^Q, d_i^Q) is an object in $\mathbf{Kom}(\mathbf{C})$.

3.2. The Khovanov-Kuperberg bracket and the geometric complex. Now we are ready to define a complex from a oriented link diagram. Fix a monic polynomial $p(x) \in R[x]$ such that $deg_x(p) = 3$. We can define $\mathbf{Foam}_{/\ell}$. Let us assign a cube in $\mathbf{Foam}_{/\ell}$ to each oriented link diagram. Given an oriented link diagram L, fix an order of the crossings, say $\{c_1, ..., c_k\}$. Each crossing has two possible **web resolutions**, see Figure 2. These resolutions come with an integer depending on the crossing and the type of resolution performed. Define a natural bijection between the vertices of a k-dimensional cube $[0,1]^k$ and the web resolutions of the diagram[7] L as follows: to each $v \in \{0,1\}^n$ associate the web $W(L, v)$ obtained by replacing the crossing c_i with its v_i-web resolution.

To each oriented edge $v \to v'$ of Q_k is a associated a foam $F(v, v')$ between the webs $W(L, v)$ and $W(L, v')$. The foam $F(v, v')$ is everywhere a cylinder, except in a disk where the two webs differ, where the cobordism looks like one of the two elementary web cobordisms (see Figure 6).

[7] That is the web obtained by replacing each crossing with a web resolutions.

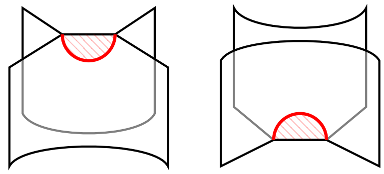


FIGURE 6. Elementary web cobordisms. The arc in red is a singular arc.

The cube we defined depends on the choice of an ordering of the crossings of L. Moreover, it is easy to see that, by a "Morse-theoretic" argument, the cube associated to an oriented link diagram is commutative. Thanks to Lemma 5.2 we can turn this commutative cube into a skew commutative cube $Q(L)$. Finally, using the abstract construction described in Subsection 3.1 we can associate to the skew commutative cube $Q(L)$ a complex $\langle L, p\rangle$ in $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$.

Remark 49. The dependence on p of $\langle L, p\rangle$ is quite subtle: what really depends on p is the category $\mathbf{Foam}_{/\ell}$. Of course complex could have defined a complex over the category **Foam** eliminating the dependence on p. However, the local relations will be necessary to have invariance under the Reidemeister moves.

The complex $\langle L, p\rangle$ will be called **Khovanov-Kuperberg bracket of** L (with respect to p). Whenever p is fixed or clear from the context we will remove it from the notation. With some standard machinery of homological algebra is easy to show the following proposition.

Proposition 5.3. *The Khovanov-Kuperberg bracket of an oriented link diagram L does not depend (up to homotopy equivalence) on the sign assignment and on the order of the crossings used to obtain the cube $Q(L)$.* □

The Khovanov-Kuperberg bracket is the analogue of the Khovanov bracket introduced by Bar-Natan in **[4]**. In terms of Knots polynomials it would be the Kauffman bracket. Exactly as in the case of the Kauffman bracket, to turn the Khovanov-Kuperberg bracket into an invariant we need a shift.

Definition 5.9. Given an R-linear category $\mathbf{C}$ and $C_\bullet = (C_i, d_i)_{i\in\mathbb{Z}} \in \mathbf{Com}(\mathbf{C})$, the **shift of** $C_\bullet$ **by** $k \in \mathbb{Z}$ is the object $C_\bullet(k)$ in $\mathbf{Com}(\mathbf{C})$ defined as follows:

$$C_\bullet(k) = (C_{i+k}, d_{i+k})_{i\in\mathbb{Z}}.$$

Finally, we can define the **the geometric $\mathfrak{sl}_3$-complex of** L (with respect to p) as follows

$$\widetilde{C}_p^\bullet(L, R) = \langle L, p\rangle(-n_+).$$

This is a link invariant in the sense of the following proposition.

Theorem 5.4. *(Mackaay-Vaz,* **[40]***) Let p be monic polynomial in $R[x]$ of degree 3 in x. If L and L' are oriented link diagram representing the same link, then $\widetilde{C}_p^\bullet(L, R)$ and $\widetilde{C}_p^\bullet(L', R)$ are homotopy equivalent.*

Furthermore, we have a functor

$$\widetilde{C}_p^\bullet : \mathbf{Link} \longrightarrow \mathbf{Kom}_{/\pm h}(\mathbf{Foam}_{/\ell}),$$

where $\mathbf{Link}$ *is the category of links in* $\mathbb{R}^3$ *and properly embedded surfaces in* $\mathbb{R}^3 \times [0,1]$ *(up-to-boundary fixing isotopies), and* $\mathbf{Kom}_{/\pm h}(\mathbf{Foam}_{/\ell})$ *is the category* $\mathbf{Kom}_h(\mathbf{Foam}_{/\ell})$ *whose morphism are considered up to sign.* □

3.3. Tautological functors and the $\mathfrak{sl}_3$-homology. The category $\mathbf{Mat}(\mathbf{Foam}_{/\ell})$ is not an Abelian category. Thus, it is not possible to define the homology of $\widetilde{C}_p^\bullet(L,R)$. There are different ways to turn the geometric complex into something more computable. Out of the different possibilities, following **[40]**, we chose the approach via tautological functors.

Definition 5.10. The **tautological functor** is the functor

$$T : \mathbf{Foam}_{/\ell} \longrightarrow R - \mathbf{Mod}$$

defined on an object $W' \in Obj(\mathbf{Foam}_{/\ell})$ by

$$T(W') = Hom_{\mathbf{Foam}_{/\ell}}(\emptyset, W')$$

and on morphisms by composition on the left, that is

$$\begin{array}{rccc} T(F): & T(W') & \longrightarrow & T(W'') \\ & G & \longmapsto & F \circ G \end{array}$$

for each $F \in Hom_{\mathbf{Foam}_{/\ell}}(W', W'')$ and W', $W'' \in Obj(\mathbf{Foam}_{/\ell})$.

Note that, if we have a disjoint union of the webs W' and W'', then

$$T(W' \sqcup W'') \simeq T(W') \otimes_R T(W'')$$

as R-modules. Before proceeding further let us show in an example how the functor T works. This example will be useful in the next chapter.

Example 1. *Let us compute* $T(\bigcirc)$*, i.e. find its isomorphism class as an* R*-module. We first wish to find a system of generators for* $T(\bigcirc)$ *as an* R*-module. Recall that* $T(W') = Hom_{\mathbf{Foam}_{/\ell}}(\emptyset, W')$ *is the* R*-module generated by all foams bounding* W' *modulo local relations.*

All closed components of a foam bounding the circle $\bigcirc$ *evaluates via local relations to elements of* R*. Thus,* $T(\bigcirc)$ *is generated (as* R*-module) by the connected foams bounding the circle* $\bigcirc$*. The latter is made of regular boundary points and hence admits a closed neighbourhood diffeomorphic to a cylinder* $\mathbb{S}^1 \times [0,1]$*. Thus we can use the genus reduction relation, and write any connected foam bounding* $\bigcirc$ *as an* R*-linear combinations of disks with at most two dots disjoint union a closed foam. Since closed foams evaluates to an element of* R*, it follows that* $T(\bigcirc)$ *is generated by dotted disks.*

Now, consider the epimorphism of R*-modules*

$$\Phi : R[x] \longrightarrow Hom_{\mathbf{Foam}_{/\ell}}(\emptyset, \bigcirc)$$

mapping x^k *to the disk with* k*-dots. The kernel of this epimorphism is the ideal generated by* $p(x)$*. In fact, the dot reduction relation tells us that* $p(x)$ *is in the kernel. On the* *other hand, the disk with two dots, the disk with a single dot and the disk with no dots are linearly independent over R. Thus there is no polynomial with degree less than three in the kernel of Φ. Consider $s(x) \in Ker(\Phi)$. Since $p(x)$ is monic it follows from the Euclidean division algorithm (see* [**31**, Theorem 1.1 Chapter IV]*) that there exists a unique pair of polynomials $q, r \in R[x]$ such that*

$$s(x) = q(x)p(x) + r(x),$$

and $deg_x(r) < deg_x(p)$. Thus, r is forced to be 0 by our previous considerations. It follows that

$$T(\bigcirc) \simeq \frac{R[x]}{(p(x))}$$

Moreover, there are analogues of the Kuperberg local relations on webs. More precisely, the following proposition holds.

Proposition 5.5. *(Khovanov-Kuperberg relations,* [**26, 40**]*) We have the following isomorphisms of R-modules.*

(circle removal)	$T(W' \sqcup \bigcirc) \simeq T(\bigcirc) \otimes T(W')$
(digon removal)	$T(W_1) \simeq T(W_2) \oplus T(W_2)$
(square removal)	$T(W_1') \simeq T(W_2') \oplus T(W_3')$

where W_1 and W_2 (resp. W_1', W_2' and W_3') are two (resp. three) webs which are identical but for a small ball where they look as depicted in Figure 7. □

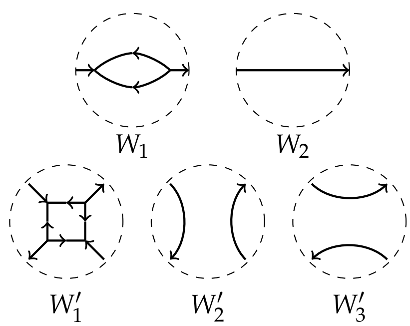


FIGURE 7. Webs involved in the Khovanov-Kuperberg relations.

For a proof of Proposition 5.5 the reader may consult [**27**, Subsection 3.4] and [**39**, Lemma 2.9]. Before, going back to the definition of the $\mathfrak{sl}_3$-homology theory, to make another example which will be useful in the follow up.

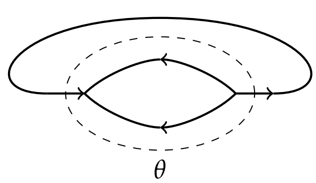


FIGURE 8. The θ-web as the closure of a digon.

Example 2. *Consider the θ-web depicted in Figure 8. It is easy to see that this web is a closed digon. Hence, by the digon removal relation and thanks to Example 1 we obtain*

$$T(\theta) \simeq \frac{R[x]}{(x)} \oplus \frac{R[x]}{(x)}$$

Now, let us get back to the definition of the $\mathfrak{sl}_3$-homology theory. There is a natural way to extend the tautological functor first to the category $\mathbf{Mat}(\mathbf{Foam}_{/\ell})$, and then to $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$. With an abuse of notation denote this extended functor by T.

Definition 5.11. The **$\mathfrak{sl}_3$-complex** (with respect to p) of an oriented link diagram L is

$$C_p^\bullet(L,R) = T\left(\widetilde{C}_p^\bullet(L,R)\right) \in Obj(\mathbf{Kom}(R - \mathbf{Mod})).$$

The homology of the $\mathfrak{sl}_3$-complex will be called **$\mathfrak{sl}_3$-homology of** L (with respect to p) and denoted by $H_p^\bullet(L,R)$.

The following proposition is an immediate consequence of Theorem 5.4.

Proposition 5.6. *The isomorphism class (as R-module) of $H_p^\bullet(L,R)$ is a link invariant. Moreover, $H_p^\bullet$ defines a functor between the category* **Link** *and the category R-$\mathbf{Mod}_{gr}$ of graded R-modules.* □

4. Grading and filtration

In this final section we wish to define a second grading or a filtration on the $\mathfrak{sl}_3$-complex. This can be done under some mild hypotheses on $p(x) \in R[x]$. But first one has to turn the category **Foam** into a graded category, in the following sense.

Definition 5.12. Let C be a commutative monoid and let R be a C-graded ring. A **C-graded category** is an R-linear category $\mathbf{C}$ together with

1. a structure of C-graded R-module on $Hom_{\mathbf{C}}(O,O')$, for each pair of objects O, $O' \in Obj(\mathbf{C})$;
2. a C-action on $Obj(\mathbf{C})$ called **shift** – i.e. an identification of C with a subset of $Hom_{\mathbf{Set}}(Obj(\mathbf{C}))$ such that: $c_1(c_2(O)) = (c_1+c_2)(O)$ and $0_C = Id_{Obj(\mathbf{C})}$;
3. a family of isomorphisms of graded R-modules
$$\Phi_{c_1,c_2}(O,O') : \ Hom_{\mathbf{C}}(O,O') \longrightarrow Hom_{\mathbf{C}}(c_1(O), c_2(O'))$$
of degree $c_1 - c_2$;

If the monoid C is not specified we implicitly assume $C = \mathbb{Z}$. An **n-graded category** is a $\mathbb{Z}^n$-graded category.

Remark 50. The category $\mathbf{Com}(\mathbf{C})$ is always graded. In particular, the category $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$ is graded.

Not every pre-additive category can be graded. However, there exists a "C-graded closure" $\mathbf{C}_{C\text{-}gr}$ of an R-linear category $\mathbf{C}$ satisfying Property (1) of the previous definition. This is the category is obtained by adding "artificial objects". More precisely, the objects of $\mathbf{C}_{C\text{-}gr}$ are pairs (O,c) where $O \in Obj(\mathbf{C})$ and $c \in C$. The morphisms between (O,c_1) and (O,c_2) are defined as follows

$$Hom_{\mathbf{C}}((O,c_1),(O',c_2)) = Hom_{\mathbf{C}}(O,O')(c_1 - c_2),$$

where $(\cdot)$ denotes the shift as a C-graded R-module (that is $M(k)_h = M_{h+k}$). Now, the shift in Property (2) and the identifications in Property (3) of Definition 5.12 can be defined as the natural ones.

We note that if $\mathbf{C}$ is a C-graded category, then also $\mathbf{Mat}(\mathbf{C})$ can be considered as a graded category; in fact, define a matrix to be homogeneous of degree d if all its entries are homogeneous of degree d. Similarly, the categories $\mathbf{Com}(\mathbf{C})$ and $\mathbf{Kom}(\mathbf{C})$ inherit a graded structure from $\mathbf{C}$ making them bi-graded categories.

Let R be a graded ring and $k \in \mathbb{Z} \setminus \{0\}$. Define a grading on $R[x]$ by setting $deg(x) = k$ (see Appendix A). Suppose that $p(x) \in R[x]$ is an homogeneous monic polynomial of degree 3 in x. We are ready to define a bi- grading on the geometric $\mathfrak{sl}_3$-complex.

First, we need to define a graded structure on $Hom_{\mathbf{Foam}}(W',W'')$. Given a foam Σ, define

$$deg(\Sigma) = -k\chi(\Sigma) + \chi(\partial\Sigma) + kd,$$

where d is the number of dots on Σ. It is easy to verify that deg is additive under composition of foams and defines a graded R-module structure on $Hom_{\mathbf{Foam}}(W',W'')$.

Since $p(x)$ is homogeneous also the reduction relation are homogeneous, while the evaluation relations are always homogeneous. Thus, the graded structure on **Foam** induces a graded structure on $\mathbf{Foam}_{/\ell}$. In turn, this structure induces a second graded structure over $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$ and hence a bi-grading on the Khovanov-Kuperberg bracket. Finally, given an oriented link diagram L define

$$\widetilde{C}_p^{\bullet,\bullet}(L,R) = \langle L,p\rangle(-n_+, 2w(L)).$$

It is possible to check that the differential is homogeneous of degree 0 with respect to the second grading. The first grading on $\widetilde{C}_p^{\bullet,\bullet}(L,R)$ is the homological grading (denoted by $hdeg$) and the second grading is called **quantum grading** and denoted by $qdeg$. With these definitions of grading the tautological functor

$$T : \mathbf{Foam}_{/\ell} \longrightarrow R\text{-}\mathbf{Mod}_{gr}$$

becomes a graded functor (i.e. it respects the gradings). Thus, the $\mathfrak{sl}_3$-complex becomes bi-graded. Moreover, we have the following theorem whose proof can be found in [**40**].

Theorem 5.7 (Mackaay-Vaz, [**40**]). *Let R be a graded ring and $p(x) \in R[x]$ be an homogeneous monic polynomial of degree $3k$, where $deg(x) = k$. We have a functor*

$$H_p^{\bullet,\bullet} : \mathbf{Link} \longrightarrow R\text{-}\mathbf{Mod}_{bi\text{-}gr}$$

such that

$$H_p^{\bullet,\bullet}(\lambda, R) \simeq H_p^{\bullet,\bullet}(\lambda', R)$$

if λ and λ' are isotopic, and given a surface Σ, properly embedded in $\mathbb{R}^3 \times [0,1]$, between two links $qdeg(H_p^{\bullet,\bullet}(\Sigma)) = -2\chi(\Sigma)$.

Finally, we wish to conclude this section with the case where $p(x)$ is not an homogeneous polynomial. In this case, suppose

$$deg(p(x) - x^3) < 3deg(x).$$

Then, we can still define a grading on **Foam** as above, although the category $\mathbf{Foam}_{/\ell}$ does not inherit a structure of graded category. This is due to the fact that the reduction relations are not homogeneous. However, $\mathbf{Foam}_{/\ell}$ becomes a category with a filtered structure in the following sense.

Definition 5.13. Let R be a graded ring. A **category with a filtered structure** is a R-linear category $\mathbf{C}$ together with

1. a structure of filtered R-module on $Hom_{\mathbf{C}}(O, O')$, for each pair of objects O, $O' \in Obj(\mathbf{C})$;
2. a $\mathbb{Z}$-action on $Obj(\mathbf{C})$ called **filtration shift**;
3. a family of isomorphism of filtered R-modules

$$\Phi_{c_1,c_2}(O, O') : \ Hom_{\mathbf{C}}(O, O') \longrightarrow Hom_{\mathbf{C}}(c_1(O), c_2(O'))$$

of filtered degree $c_1 - c_2$;

More explicitly, we can define the filtered degree of $[F] \in Hom_{\mathbf{Foam}_{/\ell}}(W', W'')$ (recall that $[F]$ is an equivalence class of linear combinations of foams) by setting

$$Fdeg([F]) = min\{deg(x) \mid x \in [F]\}$$

where the degree of a foam is defined as in the graded case, and

$$deg\left(\sum_{i=1}^{k} \alpha_i F_i\right) = \max_i \ (deg(\alpha_i) + deg(F_i))$$

where $\alpha_i \in R$ and F_i is a foam, for each $i \in \{1, ..., k\}$. One can define a increasing filtration on $Hom_{\mathbf{Foam}_{/\ell}}(W', W'')$ by setting

$$\mathscr{F}_i Hom_{\mathbf{Foam}_{/\ell}}(W', W'') = \langle [F] | Fdeg([F]) \leq i \rangle_{R_0}.$$

Similarly to the graded case we can extend the definition to the matrix category and then to the complex category. This filtered structure on $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$ induces a filtration $\widehat{\mathscr{F}}$ on the Khovanov-Kuperberg bracket. We can define a filtration on the geometric complex by setting

$$\mathscr{F}_i \widetilde{C}_p^{\bullet}(L, R) = \widehat{\mathscr{F}}_{i+2w(L)} \langle L, p \rangle.$$

By direct inspection of the differential of $\widetilde{C}_p^{\bullet}(L, R)$, it is immediate that the differential is a filtered map of filtered degree 0. Moreover, notice that the tautological funtor defines a functor from $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$ to the category of filtered

R-modules. Thus, the $\mathfrak{sl}_3$-complex becomes a filtered complex. Finally, by checking the chain homotopies used to prove Theorem 5.4 in **[40]** it is easy to prove the following theorem.

Theorem 5.8. *Let R be a graded ring and $p(x) \in R[x]$ be a monic polynomial of degree $3k$, where $deg(x) = k$. If*

$$deg(p(x) - x^3) < 3deg(x),$$

then the filtered homotopy type of $C_p^\bullet(L, R)$ is a link invariant. Moreover, the maps induced by a cobordism Σ between two links is filtered of filtered degree $-2\chi(\Sigma)$.

Theorem 5.8 has been proved, using different techniques, in the more general case of Krasner's deformations of the Khovanov-Rozansky $\mathfrak{sl}_n$-homologies, by Wu in **[60]**.

Now, we wish to discuss the graded (resp. filtered) version of the Khovanov-Kupeberg relations. We just state the result without proving it. The reader may consult **[27]** for a proof of the Khovanov-Kuperberg relations. The following result is then an immediate consequence of the definition of grading (resp. filtration) in the category $\mathbf{Foam}_{/\ell}$.

Proposition 5.9. *(Khovanov-Kuperberg relations,* **[26, 40]***) Let R be a graded ring. Set $deg(x) = k \in \mathbb{N} \setminus 0$ and let $p(x) \in R[x]$ be an homogeneous polynomial (resp. a polynomial satisfying $deg(p(x) - x^3) < 3deg(x)$ and) such that $deg_x(p) = 3$.*

Then, the tautological functor becomes a functor between the category $\mathbf{Foam}_{/\ell}$ and the category of graded (resp. filtered) R-modules. Moreover, we have the following isomorphisms of graded (resp. filtered) R-modules.

(circle removal)	$T(W' \sqcup \bigcirc) \simeq T(\bigcirc) \otimes T(W')$
(digon removal)	$T(W_1) \simeq T(W_2)(-1) \oplus T(W_2)(1)$
(square removal)	$T(W_1') \simeq T(W_2') \oplus T(W_3')$

where W_1 and W_2 (resp. W_1', W_2' and W_3') are two (resp. three) webs which are identical but for a small ball where they look as depicted in Figure 7, and $(\cdot)$ indicates the degree (resp. filtration) shift. $\square$

CHAPTER 6

Transverse invariants in the universal $\mathfrak{sl}_3$-theory

Let R be a ring. Fix a monic polynomial $p(x) \in R[x]$ of degree 3 in x. In this chapter we define a family of transverse braid invariants in $C^{\bullet}_{p}(\overline{B}, R)$, where B is a closed braid diagram. The elements of this family are in bijection with the (distinct) roots of p in R. From now on, unless otherwise stated, all tensor products are assumed to be taken over R and all the isomorphisms are assumed to be isomorphisms of R-modules.

1. The β-chains

The aim of this section is to define, given a root of $p(x)$, a cycle in $C^{\bullet}_{p}(L, R)$. This cycle will give raise to a transverse braid invariant, as we will see in the next section.

Assume that $p(x)$ has a root x_1 in R. Then, $p(x)$ is a multiple of $(x - x_1)$. More precisely, we have the following decomposition

$$p(x) = x^3 + a_2 x^2 + a_1 x + a_0 = (x - x_1)(x^2 + a_1' x + a_0'),$$

from which it follows

$$a_2 = a_1' - x_1 \qquad a_1 = a_0' - x_1 a_1' \qquad a_0 = -x_1 a_0'. \tag{35}$$

Let L be an oriented link diagram. Define the **oriented web resolution** $\underline{w}_L$ to be the web resolution where each positive crossing is replaced by its 1-web resolution and every negative crossing is resolved with its 0-resolution. In other words, the oriented web resolution is the web resolution where both and are replaced by . In particular, the oriented web resolution is a collection of oriented circles.

Definition 6.1. Let L be a oriented link diagram. The oriented web resolution $\underline{w}_L$ is a collection of circles in $\mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$. Consider a family of disjoint unknotted disks $\{\mathbb{D}_\gamma\}_{\gamma \in \underline{w}_L}$ properly embedded in $(\mathbb{R}^2 \times \{0\}) \times [0,1] \subseteq \mathbb{R}^3 \times [0,1]$, obtained by pushing the Jordan disks bounding the circles of $\underline{w}_L$ in $\mathbb{R}^2 \times \{0\}$. Denote by $\mathbb{D}^k_\gamma$ the disk $\mathbb{D}_\gamma$ with k dots on it. The **β_3-chain (with respect to p) associated to the root** x_1 is the element $\beta_3(L; p, x_1) \in T(\underline{w}_L)$ defined as follows

$$\sum_{S \subseteq \underline{w}_L} \sum_{S' \subseteq \underline{w}_L \setminus S} (a_1')^{card(S')} (a_0')^{card(S)} \left(\bigsqcup_{\gamma \in \underline{w}_L \setminus (S \cup S')} \mathbb{D}^2_\gamma \sqcup \bigsqcup_{\gamma \in S'} \mathbb{D}^1_\gamma \sqcup \bigsqcup_{\gamma \in S} \mathbb{D}^0_\gamma \right).$$

By definition of the $\mathfrak{sl}_3$-complex, to each web resolution $\underline{w}$ of L corresponds a direct summand $T(\underline{w})$ in $C_p^{\bullet}(L,R)$ in homological degree

$$-n_-(L) + |v(\underline{w})| = -n_-(L) + \sum_{i=1}^{n(L)} v_i$$

where $v(\underline{w})$ is the vertex of the standard n-cube associated to the web resolution $\underline{w}$. In particular, we have

$$T(\underline{w}_L) \subseteq C_p^0(L,R).$$

From the Khovanov-Kuperberg relations and from Example 1 in Chapter 5 it follows that

$$T(\underline{w}_L) \simeq \bigotimes_{\gamma \in \underline{w}_L} \frac{R[x_\gamma]}{(p(x_\gamma))} \simeq \frac{R[x_\gamma \mid \gamma \in \underline{w}_L]}{(p(x_\gamma))_{\gamma \in \underline{w}_L}} \tag{36}$$

where $\gamma \in \underline{w}_L$ should be read as "γ is a circle in $\underline{w}_L$". It is easy to see that the isomorphism in (36) maps $\beta_3(L;p,x_1)$ to

$$\prod_{\gamma \in \underline{w}_L} [x_\gamma^2 + a_1' x_\gamma + a_0'] \in T(\underline{w}_L) \subseteq C_p^0(L,R)$$

Moreover, it is also easy to notice that the multiplication of $\beta_3(L;p,x_1)$ by x_γ, which is an algebraic operation, corresponds "geometrically" to the addition on a dot on the disk $\mathbb{D}_\gamma$ in each summand of the "geometric expression" of $\beta_3(L;p,x_1)$.

This interplay between algebraic and geometric operations will be used throughout this chapter.

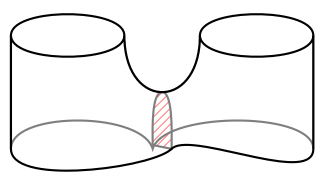


FIGURE 1. The foam F.

Lemma 6.1. *Let F be the foam in Figure 1, and consider the morphism of R-modules*

$$T(F) : T(\bigcirc \sqcup \bigcirc) \longrightarrow T(\theta).$$

If we consider

$$\beta = (x^2 + a_1'x + a_0')(y^2 + a_1'y + a_0') \in \frac{R[x,y]}{(p(y),p(x))} \simeq T(\bigcirc \sqcup \bigcirc),$$

then

$$T(F)(\beta) = 0.$$

Proof. Our aim to prove that $T(F)(\beta)$ is trivial. In order to do so, we will write $T(F)(\beta)$ as a linear combination of foams

$$T(F)(\beta) = \sum_{j=1}^{k} c_j(F_j + a_2F_j' + a_1F_j'' + a_0F_j'''), \quad c_j \in R$$

where F_j, F'_j, F''_j, F'''_j are identical except in a small region where they differ as shown in Figure 2.

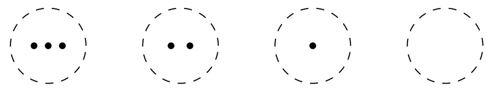


FIGURE 2. The local difference between the foams F_j, F'_j, F''_j, F'''_j.

Since $(F_j + a_2F'_j + a_1F''_j + a_0F'''_j)$ is trivial for each j by the dot reduction relation (DR), the claim will follow.

To avoid graphical calculus we use polynomials. So, let us denote by the monomial $A^rB^sC^t$ the foam (in $(\mathbb{R}^2 \times \{0\}) \times [0,1]$) shown in Figure 3, where s, r and t indicate the number of dots in the regions A, B and C respectively. By definition $T(F)(\beta)$ can be written as follows:

$$T(F)(\beta) = A^2B^2 + a'_1(A^2B + AB^2) + (a'_1 + 1)^2AB + a'_0(A^2 + B^2) + a'_1a'_0(A + B) + (a'_0)^2.$$

With this notation we can write the dot permutation relations (DP1), (DP2) and (DP3) described in Proposition 5.1 as follows:

$$A + B + C = -a_2 \tag{DP1}$$

$$AC + BC + AB = a_1 \tag{DP2}$$

$$ABC = -a_0 \tag{DP3}$$

Since all foam relations are local, and since we are allowed to move the dots inside regions, the formal products above satisfy associativity. Using Relations (DP1), (DP2) and (DP3) we obtain

$$T(F)(\beta) = B(A^3 + a_2A^2 + a_1A + a_0)(A - a'_1 + 1) =$$
$$= \Big((BA)A^3 + a_2(BA)A^2 + a_1(BA)A + a_0(BA)\Big) + (-a'_1 + 1)(A^3 + a_2A^2 + a_1A + a_0),$$

which is the desired decomposition of $T(F)(\beta)$.

Q.E.D.

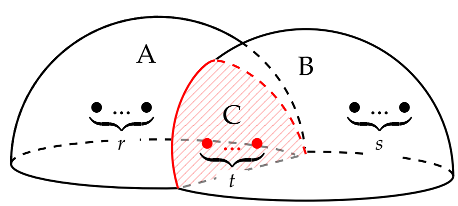


FIGURE 3. The foam $A^rB^sC^t$.

Now, we are ready to prove that each β-chain is a cycle.

Proposition 6.2. *If L is an oriented link diagram, then $\beta_{x_1}(L, R)$ is a cycle.*

Proof. The first thing to notice is that the oriented web resolution is bi-partite exactly as the oriented resolution; this means that if two arcs of a circle in $\underline{w}_L$ were connected by a crossing in L, then they belong to different circles in $\underline{w}_L$.

Let $\underline{w}$ be a web resolution which is obtained from $\underline{w}_L$ by replacing a 0-web resolution with a -1-resolution, and denote by W the set of such resolutions. Notice that each $\underline{w} \in W$ is the disjoint union of circles and a θ-web. In particular,

$$T(\underline{w}) \simeq \bigotimes_{\gamma \in \underline{w}_L \setminus \{\gamma_1, \gamma_2\}} \frac{R[x_\gamma]}{(p(x_\gamma))} \otimes \left(\frac{R[y]}{(p(y))} \oplus \frac{R[x]}{(p(x))} \right),$$

where γ_1 and γ_2 are the two circles which are merged into a θ-web, and the circles of γ different from γ_1 and γ_2 are identified with the circles of $\underline{w}$.

By definition, the differential d_{geo} of the geometric complex is such that

$$d_{geo \mid \underline{w}_L} = \sum_{\underline{w} \in W} F_i,$$

where the F_is are disjoint unions of cylinders and a copy of the foam F drawn in Figure 1. Applying the tautological functor T, we get

$$d(T(\underline{w}_L)) \subseteq \bigoplus_{w \in V} \left(\left(\frac{R[y]}{(p(y))} \oplus \frac{R[x]}{(p(x))} \right) \otimes \bigotimes_{\gamma \in \underline{w}_L \setminus \{\gamma_1(\underline{w}), \gamma_2(\underline{w})\}} \frac{R[x_\gamma]}{(p(x_\gamma))} \right),$$

and

$$d_{\mid \underline{w}_L} = \bigoplus_{\underline{w} \in W} \left(T(F) \otimes \bigotimes_{\gamma \in \underline{w}_L \setminus \{\gamma_1(\underline{w}), \gamma_2(\underline{w})\}} Id_\gamma \right).$$

Now, the statement follows immediately from Lemma 6.1.

Q.E.D.

2. The transverse invariance of the β-chains

In this section we define a family of transverse invariants from the β-chains. This section is devided into three sub-sections, one for each Reidemeister move. In each subsection we will describe the map associated to the corresponding Reidemeister move, and study its behaviour on β_{x_1}.

2.1. First negative Reidemeister move. Let L be an oriented link diagram. Denote by L'_- the oriented link diagram obtained from L via a positive (resp. negative) first Reidemeister move – i.e. the addition of a negative curl, see Figure 4 – on an arc **a**. Finally, denote by c_- the crossing created with the curl in L'_-.

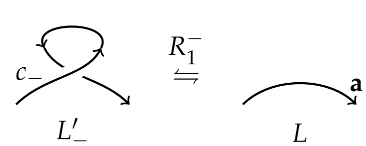


FIGURE 4. The negative version of the first Reidemeister move.

We wish to describe how the map associated to a negative Reidemeister move behaves at the level of geometric complexes. This is done in Figure 5. The picture should be read as follows: the foams are all embedded in $(\mathbb{R}^2 \times \{0\}) \times [0,1]$ and are cylinders except in a small region containing the arc $\mathbf{a}$, where they look like the ones depicted in Figure 5. Before proceeding we need the following lemma.

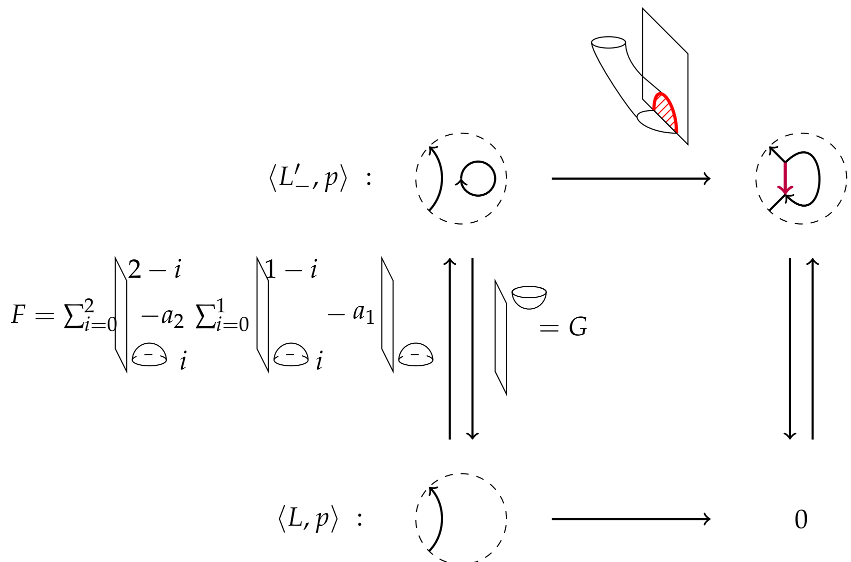


FIGURE 5. Schematic description of the maps encoding a negative first Reidemeister move. The numbers next to the foams (which are drawn in $(\mathbb{R}^2 \times \{0\}) \times [0,1]$ and should be read top to bottom) indicate the number of dots. The horizontal maps are the differential.

Lemma 6.3. *Let $p'(x) \in R[x]$ be the polynomial $x^2 + a'_1 x + a'_0$. Then,*

$$[x^2p'(x)] \otimes [1] + [xp'(x)] \otimes [y] + a_2[xp'(x)] \otimes [1] - x_1[p'(x)] \otimes [y] - x_1a_1[p'(x)] \otimes [1]$$

is trivial in

$$\frac{R[x]}{(p(x))} \otimes \frac{R[y]}{(p(y))}$$

Proof. First, let us point out that

$$xp'(x) = x_1p'(x) \quad mod\ p(x). \tag{37}$$

Using $a_2 = a'_1 - x_1$ we get

$$[x^2p'(x)] \otimes [1] + [xp'(x)] \otimes [y] + a_2[xp'(x)] \otimes [1] =$$
$$= [x^2p'(x)] \otimes [1] + [xp'(x)] \otimes [y] + a'_1[xp'(x)] \otimes [1] - x_1[xp'(x)] \otimes [1]$$

which, thanks to Equation (37), is equal to

$$\cancel{x_1[xp'(x)] \otimes [1]} + x_1[p'(x)] \otimes [y] + a'_1x_1[p'(x)] \otimes [1] \cancel{- x_1[xp'(x)] \otimes [1]}.$$

Q.E.D.

Denote by

$$\Phi_1 : C_p^\bullet(L, R) \longrightarrow C_p^\bullet(L'_-, R)$$

the map associated to the foam denoted by G in Figure 5, and by

$$\Psi_1 : C_p^\bullet(L'_-, R) \longrightarrow C_p^\bullet(L, R),$$

the map associated to the (linear combination of) foam(s) F.

Proposition 6.4. *Let L be an oriented link diagram. Then,*

$$\Phi_1(\beta_{x_1}(L, R)) = \beta_{x_1}(L'_-, R) \quad \Psi_1(\beta_{x_1}(L'_-, R)) = \beta_{x_1}(L, R).$$

Proof. Notice that $\underline{w}_L$ is mapped to $\underline{w}_{L'_-}$ by Φ_1 and that $\underline{w}_{L'_-}$ can be identified with $\underline{w}_L \sqcup \bigcirc$. Thus, by the Khovanov-Kuperberg circle removal relation, Example 1 and the sphere relation we can identify $\Phi_{1\,|\underline{w}_L}$ with the map

$$\frac{R[x_{\gamma'}]}{(p(x'_\gamma))} \otimes \bigotimes_{\gamma \in \underline{w}_L} \frac{R[x_\gamma]}{(p(x_\gamma))} \longrightarrow \bigotimes_{\gamma \in \underline{w}_L} \frac{R[x_\gamma]}{(p(x_\gamma))}$$

given by

$$q_{\gamma'}(x_{\gamma'}) \otimes \bigotimes_{\gamma \in \underline{w}_L} q_\gamma(x_\gamma) \longmapsto \varepsilon(q'(x_{\gamma'})) \bigotimes_{\gamma \in \underline{w}_L} q_\gamma(x_\gamma),$$

where γ' indicates the circle in $\underline{w}_{L'_-} \setminus \underline{w}_L$ and

$$\varepsilon : \frac{R[x]}{(p(x))} \longrightarrow R \; : \; [ax^2 + bx + c] \mapsto a.$$

Since in the case of β_{x_1} we have $q_\gamma(x) = q_{\gamma'}(x) = [p'(x)] = [x^2 + a'_1 x + a'_0]$ the first part of the statement follows.

Similarly to what has been done with Φ_1, we can identify Ψ_1 with the map

$$\bigotimes_{\gamma \in \underline{w}_L} \frac{R[x_\gamma]}{(p(x_\gamma))} \longrightarrow \frac{R[x_{\gamma'}]}{(p(x'_\gamma))} \otimes \bigotimes_{\gamma \in \underline{w}_L} \frac{R[x_\gamma]}{(p(x_\gamma))}$$

mapping $\bigotimes_{\gamma \in \underline{w}_L} q_\gamma(x_\gamma)$ to

$$-\left(\sum_{i=0}^{2} (x_{\gamma_{\mathbf{a}}}^{2-i} q_{\gamma_{\mathbf{a}}} \otimes x_{\gamma'}^{i}) + a_2 \sum_{i=0}^{1} (x_{\gamma_{\mathbf{a}}}^{1-i} q_{\gamma_{\mathbf{a}}} \otimes x_{\gamma'}^{i}) + a_1 (q_{\gamma_{\mathbf{a}}} \otimes 1) \right) \otimes \bigotimes_{\gamma \in \underline{w}_L \setminus \{\gamma_{\mathbf{a}}\}} q_\gamma(x_\gamma),$$

where $\gamma_{\mathbf{a}}$ is the circle in $\underline{w}_L$ containing the arc $\mathbf{a}$ of Figure 5 and $q_{\gamma_{\mathbf{a}}}$ is evaluated in $x_{\gamma_{\mathbf{a}}}$. It is easy to see that

$$\begin{aligned}\Psi_1(\beta_{x_1}(L, R)) = \Big(& [x_{\gamma_{\mathbf{a}}}^2 p'(x_{\gamma_{\mathbf{a}}})] \otimes [1] + [x_{\gamma_{\mathbf{a}}} p'(x_{\gamma_{\mathbf{a}}})] \otimes [x_{\gamma'}] + a_2 [x_{\gamma_{\mathbf{a}}} p'(x_{\gamma_{\mathbf{a}}})] \otimes [1] + \\ & - x_1 [p'(x_{\gamma_{\mathbf{a}}})] \otimes [x_{\gamma'}] - x_1 a_1 [p'(x_{\gamma_{\mathbf{a}}})] \otimes [1] \Big) \otimes \bigotimes_{\gamma \in \underline{w}_L \setminus \{\gamma_{\mathbf{a}}\}} p'_\gamma(x_\gamma) + \beta_{x_1}(L'_-, R) = \\ = & \beta_{x_1}(L'_-, R)\end{aligned}$$

where the last equality holds thanks to Lemma 6.3.

Q.E.D.

Remark 51. The reader may have noticed a similarity between the map ε in the proof of Proposition 6.4 and the co-unit of a Frobenius algebra defined in Chapter 1. Indeed ε is the co-unit of the Krasner Frobenius algebra relative to p defined in Subsection 1.2 of Chapter 1.

2.2. Second Reidemeister move. Now, let us turn to the second coherent Reidemeister move. Let L be an oriented link diagram. Let **a** and **b** be two (un-knotted) arcs of L lying in a small ball. Performing a second Reidemeister move on these arcs inserts two adjacent crossings, say c_1 and c_2, of opposite types.

Recall that a Reidemeister move is coherent if it can be obtained by rotating or taking the mirror image of the one in Figure 6. Denote by L'' the link obtained from L by performing a coherent second Reidemeister move. Finally, denote by $\underline{w}'_{L''}$. the web resolution of L'' where all crossings but c_1 and c_2 are resolved as in the oriented web resolution.

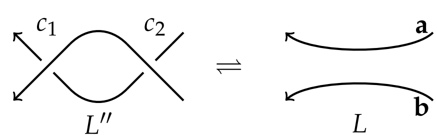


FIGURE 6. A coherent version of the second Reidemeister move. All other coherent second Reidemeister moves are obtained by rotating or taking the mirror image of the one in figure.

The map associated to the second Reidemeister move at the level of geometric complexes was defined by Mackaay and Vaz as in Figure 7. Denote by Φ_2 and Ψ_2 the two maps

$$\Phi_2 : C_p^\bullet(L, R) \longrightarrow C_p^\bullet(L'', R) \qquad \Psi_2 : C_p^\bullet(L'', R) \longrightarrow C_p^\bullet(L, R)$$

associated to the second coherent Reidemeister move.

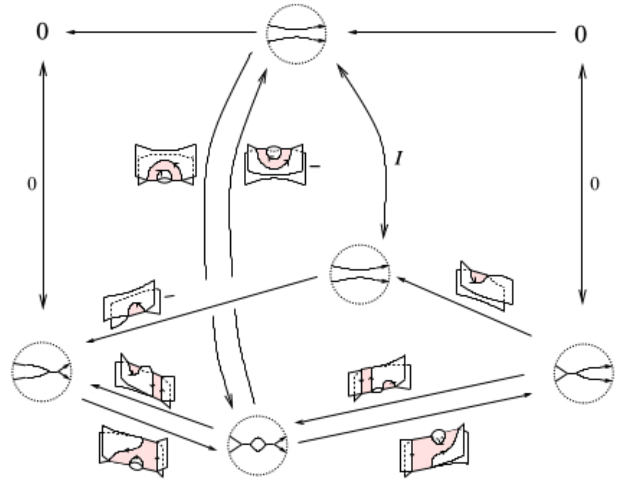


FIGURE 7.

Proposition 6.5. *Let L be an oriented link diagram and let L'' be the diagram obtained from L by performing a second coherent second Reidemeister move. Then,*

$$\Phi_2(\beta_3(L;p,x_1)) = \beta_3(L'';p,x_1) \qquad \Psi_2(\beta_3(L'';p,x_1)) = \beta_3(L;p,x_1)$$

Proof. First notice that $\underline{w}_L$ can be easily identified with $\underline{w}_{L''}$. With this identification we have that $\Psi_{2\,|T(\underline{w}_L)}$ behaves as the identity map (cf. Figure 7), and the second part of the statement follows.

Let us turn to the first part of our statement. The map $\Phi_{2\,|T(\underline{w}_L)}$ sends $T(\underline{w}_L)$ to $T(\underline{w}_L) \oplus T(\underline{w}'_{L''})$. More precisely, we have

$$\Phi_{2\,|\underline{w}_L} = Id_{\underline{w}_L} \oplus T(F'),$$

where F' is the foam drawn in Figure 8. To conclude is sufficient to prove that:

$$T(F')(\beta_3(L;p,x_1)) = 0.$$

This is immediate from Lemma 6.1 once one notices that the foam F' is the composition of the foam F in Figure 1 and a foam G (see Figure 8).

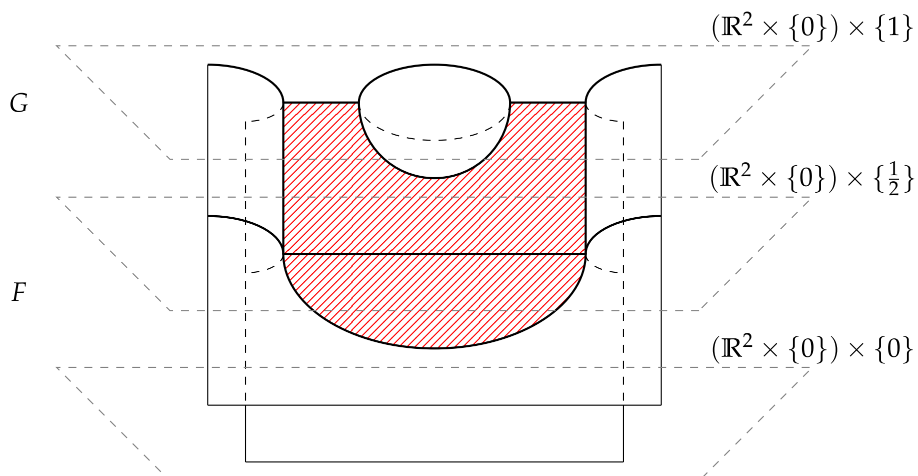


FIGURE 8. The cobordism F' as a composition of the cobordism F (on the bottom) and the cobordism G (on the top).

Q.E.D.

2.3. Third Reidemeister move. Finally, we have to prove the invariance of the β_3-chains under coherent (i.e. braid-like) versions of the third Reidemeister moves. Differently from the $\mathfrak{sl}_2-$case, it is not possible to apply the categorified Kauffman trick or some trivial modification of it. So, we will proceed in a more direct way.

Consider the version R_3° of the third Reidemeister move in Figure 9.

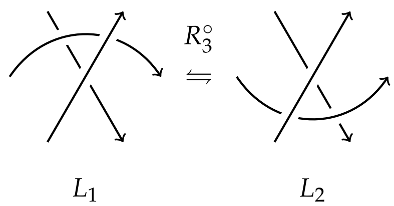


FIGURE 9. A version of the Third Reidemeister move.

All the coherent versions of the third Reidemeister move can be deduced, using two coherent second Reidemeister moves, from R_3° (cf. [**46**]). For example, R_3^+ (see Figure 17 in Chapter 3) can be deduced from the R_3° via a sequence of coherent versions of R_2 (as shown in Figure 10). The case of R_3^- is dealt with similarly (as for the other coherent versions of R_3 see [**46**, Lemma 2.6], and take into account that R_3° corresponds to Ω_{3b} in Polyak's notation).

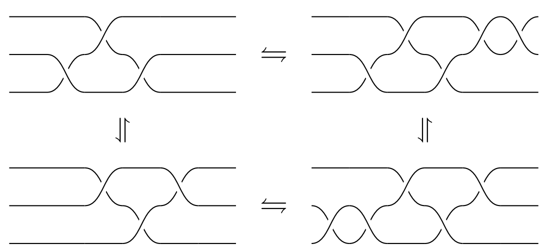


FIGURE 10. How to recover R_3^+ from R_3° via coherent second Reidemeister moves.

Chain maps between the geometric complexes associated to R_3° have been described explicitly by Khovanov (in the case $p(x) = x^3$) and by Mackaay and Vaz (in the general case). Each of these maps is defined as the composition of two maps. First, one defines an element $Q \in \mathbf{Kom}(\mathbf{Foam}_{/\ell})$ which is *not* the geometric complex associated to a link. Then, one defines chain maps

$$F_i : \langle L_i \rangle \longrightarrow Q \qquad G_i : Q \longrightarrow \langle L_i \rangle,$$

where $i \in \{1,2\}$, and L_1 and L_2 are the links on each side of the R_3° move (as in Figure 9), such that F_i is the up-to-homotopy inverse of G_i. For a description of such maps the reader may refer to [**40**] or to Figure 11.

The important thing that the reader should keep in mind is that for each web resolution $\underline{w}$ of L_i such that the crossings involved in R_3° are resolved as in the oriented resolution, there is the same direct summand in Q, and that the restriction of either G_i or F_i to $\underline{w}$ is minus the identity cobordism.

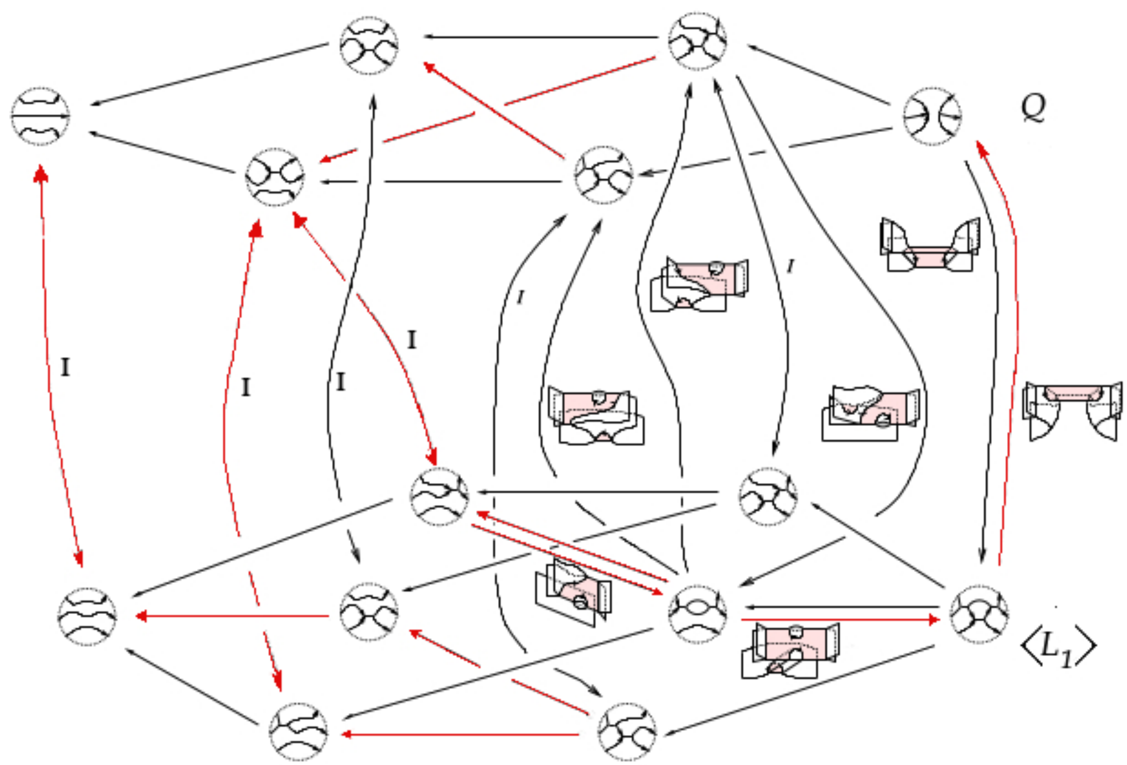


FIGURE 11. The maps G_1 and F_1. The maps G_2 and F_2 are obtained by reflecting the local figures along the horizontal axes, and the cobordisms accordingly. The red arrows indicate the maps which get a minus sign and the I's indicate the identity we cobordisms.

Finally, the maps associated to each direction of R_3° (between the Kuperberg brackets)

$$\widetilde{\Psi}_3 : \langle L_1, p\rangle \longrightarrow \langle L_2, p\rangle \qquad \widetilde{\Phi}_3 : \langle L_2, p\rangle \longrightarrow \langle L_1, p\rangle$$

are defined as follows

$$\widetilde{\Psi}_3 = G_2 \circ F_1 \qquad \widetilde{\Phi}_3 = G_1 \circ F_2.$$

It is immediate that these two maps, when restricted to the oriented web resolutions, are cylinders. Denote by Ψ_3 and Φ_3 the maps between $\mathfrak{sl}_3-$complexes associated to $\widetilde{\Psi}_3$ and $\widetilde{\Phi}_3$, respectively. Then, maps Ψ_3 and Φ_3 behave as the identity maps between the summands associated to the oriented web resolutions. So the following proposition is immediate.

Proposition 6.6. *Let L and L' be two oriented link diagrams related by a coherent third Reidemeister move. Then,*

$$\Psi_3(\beta_3(L; p, x_1)) = \beta_3(L'; p, x_1) \quad \textit{and} \quad \Phi_3(\beta_3(L'; p, x_1)) = \beta_3(L; p, x_1).$$

□

Directly from the behaviour of $\beta_3(L; p, x_1)$ under the maps induced by coherent Reidemeister moves, we get the following corollary.

Corollary 6.7. *Let B be a transverse braid. For each root x_1 of $p(x)$, the cycle $\beta_3(\overline{B}; p, x_1)$ is a transverse braid invariant, where the over-line indicates the mirror braid.* □

We will call $\beta_3(\overline{B}; p, x_1)$ the **β_3-invariant of** B **associated to** x_1.

Remark 52. The ψ_3-invariants introduced by Wu in **[59]** are a special case of our construction, more precisely the case $p(x) = x^3$ and $x_1 = 0$.

3. The c_3-invariants

In this final section we wish to introduce some numerical transverse invariants, similar in the spirit to the c-invariants introduced in the first part of this thesis. In order to do so we must fix a ring R and a polynomial p. From now on let us fix $R = \mathbb{C}[U]$, the univariate polynomial ring over the field of complex numbers, and $p(x) = x^3 - U^3$.

The theory $C^\bullet_{p(x)}(L, R)$ is bi-graded (cf. Chapter 5) once we set

$$deg(x) = 2 \quad \text{and} \quad deg(U) = 2.$$

In this case we have three roots of $p(x)$ in R. To be precise, the roots of $p(x)$ are $x_i = U\xi_3^i$, $i \in \{1, 2, 3\}$, where ξ_3 is a primitive third root of the unit. This theory is, in some sense, the analogue of the Bar-Natan theory in the $\mathfrak{sl}_3$-case.

Now, consider the theory $C^\bullet_{GLee}(L, \mathbb{C})$ defined as $C_{x^3-1}(L, \mathbb{C})$. This theory is no longer bi-graded but it is filtered, and is called the **Gornik-Lee theory**. The name itself indicates that this is the analogue of the Twisted Lee theory. It is immediate that there is an identification

$$C^{\bullet}_{GLee}(L,R) \simeq \frac{C^{\bullet,\bullet}_{x^3-U^3}(L,R)}{(U-1)C^{\bullet,\bullet}_{x^3-U^3}(L,R)}$$

and a surjective $\mathbb{C}$-linear map

$$\pi_1 : C^{\bullet,\bullet}_{x^3-U^3}(L,R) \longrightarrow C^{\bullet}_{GLee}(L,R)$$

Exactly as in the case of the Twisted Lee theory, in the case of knots the β-chains play the role of the canonical generators. These were introduced originally by Gornik and are called **Gornik generators** ([**17**]). The next proposition follows immediately from the definitions and the considerations above.

Proposition 6.8. *Let R be the ring $\mathbb{C}[U]$ and the polynomial $p(x)$ be $x^3 - U^3$. For each oriented link diagram L the homology classes of the β_3-chains generate a $\mathbb{F}[U]$-sub-module of $H_{p(x)}(L,R)$ of rank* 3*.*

Proof. Any linear combination of the homology classes of the β-chains is mapped by $(\pi_1)_*$ into a linear combination of the Gornik generators. Since the Gornik generators are linearly independent ([**17**]) the claim follows.

Q.E.D.

In particular, the homology classes of the β-chains are always non-torsion in the $\mathbb{C}[U]$-module $H_{x^3-U^3}(L,\mathbb{C}[U])$. Thus, it make sense to define

$$c_{U\xi_3^i}(L,\mathbb{C}) = max\{k \mid \exists \delta \in H_{x^3-U^3}(L,\mathbb{C}[U]) \,:\, U^k\delta = [\beta_3(L; x^3-U^3, U\xi_3^i)\}$$

The integers $c_{U\xi_3}(\overline{B},\mathbb{C})$, $c_{U\xi_3^2}(\overline{B},\mathbb{C})$ and $c_U(\overline{B},\mathbb{C})$ are called the **c_3-invariants** of the braid B. The name is motivated by the following corollary of the invariance of the β_3-invariants.

Proposition 6.9. *The c_3-invariants are transverse braid invariants.* □

We can now proceed exactly as in the case of the c-invariants and get some estimates on the value of the c-invariants. Let us recall first some notation.

Let L be an oriented link diagram of an ℓ-component link λ. Denote by ℓ_s the number of split-components of L (i.e. the connected components of L as planar graph). Define $o_+(L)$, $o_-(L)$ and $o_0(L)$ to be the number of circles in the oriented resolution of L which are touched only by positive crossing, negative crossings and by both type of crossings, respectively. The corresponding types of circles will be called positive, negative and neutral circles, respectively. By $\Gamma(L)$ we denote the simplified Tait graph. That is the graph whose vertices are the circles of the oriented resolution, with two vertices connected by an edge whenever the two corresponding circles share at least a crossing. The edges of the simplified Tait graph are marked with either $+$, $-$, or 0, depending on whether the two corresponding circles share only positive crossings, only negative crossings, or both type of crossings. A vertex of the simplified graph is called pure if it is connected with only vertices of the same type. Finally, $\Gamma_+(L)$ (resp. $\Gamma_-(L)$) is the sub-graph of $\Gamma(L)$ spanned by all positive (resp. negative) circles and $s_+(L)$ (resp. $s_-(L)$) be the number of connected components of the graph obtained from $\Gamma(L)$ by removing the negative (resp. positive) edges.

Consider the foam $A^rB^sC^t$ shown in Figure 12 the foam obtained by adding a dots in the region A, b dots in the region B and c dots in the region C.

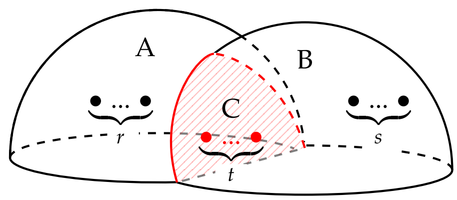


FIGURE 12. The foam $A^rB^sC^t$.

Then consider the foam F' obtained by reading the foam F in Figure 1 from the bottom to the top. We have the following result.

Lemma 6.10. *Consider the web $\bigcirc \sqcup \bigcirc$, and the map*

$$T(F') : T(\theta) \to T(\bigcirc \sqcup \bigcirc) \simeq \frac{R[x]}{(p(x))} \otimes \frac{R[y]}{(p(y))}$$

where the variable x is relative to the circle on the left. Then, we have

$$T(F')(A^2B + UAB + U^2B) = [x^2 + Ux + U^2] \otimes [y^2 + Uy]$$

Proof. The claim is a simple consequence of the fact that $[x^3] = U^3[1]$ in $\frac{R[x]}{(p(x))}$, and of the second relation from the top in Proposition 5.1.

Q.E.D.

Fix an oriented link diagram L. Consider $\Gamma' \subseteq \Gamma_+ = \Gamma_+(L)$. Denote by $\alpha(\Gamma')$ the element of $T(\underline{w}_L)$ defined as follows

$$\alpha(\Gamma') = \left(\prod_{\gamma \in \underline{w}_L \setminus V(\Gamma_+ \setminus \Gamma')} [x_\gamma^2 + Ux_\gamma + U^2] \right) \left(\prod_{\gamma \in V(\Gamma_+ \setminus \Gamma')} [1] \right).$$

Pick a non-isolated or non-pure vertex v in Γ', and consider a positive crossing c touching v. Denote by $\underline{w}(c)$ the web resolution obtained from the oriented resolution by replacing the 1-web resolution of c with its 0-web resolution. It is immediate that $\underline{w}(c)$ is the disjoint union of a θ-web, obtained merging the circle corresponding to v with the other circle touched by c, and some circles. Finally, define $\alpha(\Gamma', v, c) \in T(\underline{w}(c))$ as follows

$$(A^2B + UAB + U^2B) \otimes \left(\bigotimes_{\gamma \in \underline{w}_L \setminus V(\Gamma_+ \setminus \Gamma')} [x_\gamma^2 + Ux_\gamma + U^2] \right) \otimes \left(\bigotimes_{\gamma \in V(\Gamma_+ \setminus \Gamma') \cap \underline{w}(c)} [1] \right),$$

where we have identified the circles of $\underline{w}(c)$ with the corresponding circles in $\underline{w}_L$. The following lemma is an immediate consequence of Lemma 6.10 and of the definition of the differential, and it is left as an exercise to be done along the lines of the proof of Lemma 3.31.

Lemma 6.11. $d(\alpha(\Gamma', v, c)) = \alpha(\Gamma') + U^2\alpha(\Gamma' \setminus \{v\})$ □

Replacing Lemma 3.31 with Lemma 6.11 in the proof of Proposition 3.32 we obtain the following result.

Proposition 6.12. *Let L be an oriented link diagram, and R be a ring. Denote by ℓ_+ be the number of split components of L (i.e. the connected components of L seen as a four-valent graph) which have only positive crossings. Then*

$$\beta_3(L; x^3 - U^3, U) = U^{2(o_+(L)-\ell_+)}\alpha(\{v_1', \ldots, v_{\ell_+}'\}) + d\tau,$$

for some $\tau \in C^{-1,\bullet}_{x^3-U^3}(L, \mathbb{C}[U])$, where o_+ denotes the number of positive circles in Γ. □

A similar result can be produced for $\beta_3(L; x^3 - U^3, U\xi_3)$ and $\beta_3(L; x^3 - U^3, U\xi_3^2)$. An immediate consequence of Proposition 6.12 is the following corollary.

Corollary 6.13. *Let L be an oriented link diagram. Then,*

$$c_{U\xi_3^i}(L, \mathbb{C}) \geq 2(o_+(L) - \ell_+(L))), \quad \forall i \in \{0, 1, 2\}.$$

In particular, if B' is the negative stabilization of a braid B, then $c_{U\xi_3^i}(\overline{B'}, \mathbb{C}) \geq 2$. □

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Appendices

APPENDIX A

Algebra

In this section we collect all the basics of the algebraic objects we are going to use in this thesis. Throughout this appendix the word "ring" will stand for "commutative ring with unit element".

1. Duality and modules

Let R be a ring and M an R-module. In this section we will be concerned with some properties of the dual module, denoted by $M^* = Hom_R(M, R)$. Even without any hypotheses on either M or R, there is a natural map

$$\Phi : M^* \otimes_R M^* \longrightarrow (M \otimes_R M)^*,$$

defined by

$$\Phi\left(\sum_i \varphi_i \otimes_R \psi_i\right) : \; M \otimes_R M \to R : \; m \otimes n \mapsto \sum_i \varphi_i(m)\psi_i(n). \tag{38}$$

In general, this map is neither injective, nor surjective. Nonetheless, in some special cases something more can be said.

Proposition A.1 ([**31**], Corollary 5.6, Chapter XVI). *If M is a finitely generated free R-module, then Φ is an isomorphism.*

The hypothesis that M is finitely generated is necessary, as shown by the following example.

Example A.1. Let M be a non-finitely generated free R-module. Consider a basis $\{e_n\}_{n\in\mathbb{N}}$, for M, and set

$$\zeta : M \otimes M \to R : \; e_i \otimes e_j \mapsto \delta_{i,j} 1_R,$$

where $\delta_{i,j}$ is Kronecker's delta. If $\zeta = \Phi(\sum_{j=1}^{s} \varphi_j \otimes \psi_j)$, then the matrix $(\zeta(e_i \otimes e_j))_{i,j=0}^{k}$ has rank at most s for all k, which is absurd ($\lightning$).

For our scopes, is needed a somewhat weaker hypothesis: M should be projective of finite type. A finitely generated module M is said to be **projective of finite type** if there exists a **dual basis** for M, that is

$$(\varphi_i, x_i) \in M^* \times M, \quad i = 1, ..., k,$$

such that

$$\sum_{i=1}^{k} \varphi_i(m) x_i = m, \quad \forall\, m \in M.$$

Remark 53. From the definition of dual basis, it is immediate that the x_is form a system of generators for M. Similarly, the φ_is form a system of generators for M^*; in fact, given $\psi \in M^*$, we have

$$\sum_i \psi(x_i)\varphi_i = \psi.$$

Remark 54. Any projective module of finite type is projective (i.e. is a direct summand of a free module). Consider the free module F generated by $\tilde{x}_1, ..., \tilde{x}_k$, and the map

$$\pi : F \to M : \ \tilde{x}_i \mapsto x_i.$$

There is a map $s : M \to F$, defined as follows

$$s(x) = \sum_i \varphi_i(x)\tilde{x}_i,$$

such that

$$\pi \circ s(x) = \sum_i \varphi_i(x) x_i = x.$$

Hence, the exact sequence

$$0 \to ker(\pi) \longrightarrow F \xrightarrow{\pi} M \to 0$$

splits, and consequently $F = ker(\pi) \oplus M$. In particular, a projective module of finite type is torsion-free. Moreover, we have that

$$\varphi_i(x_j) = \delta_{i,j} 1_R \tag{39}$$

Remark 55. If $\mathcal{F}$ is a Frobenius algebra, then $A_{\mathcal{F}}$ is a projective $R_{\mathcal{F}}$-module of finite type.

Proposition A.2. *If M is a projective R-module of finite type, then*

$$\Psi : M^* \otimes N \to Hom_R(M, N) : \ \sum_j \psi_j \otimes y_j \mapsto \sum_j \psi_j(\cdot) y_j,$$

is an isomorphism, for each R-module N.

Proof. Consider $f \in Hom_R(M, N)$, and a dual basis for M, say $(\varphi_i, x_i)_{i=1,...,k}$, then set

$$y_j (= y_j(f)) = f(x_j).$$

With this notations

$$\begin{aligned} \Psi\left(\sum_j \varphi_j \otimes y_j\right)(m) &= \sum_j \varphi_j(m) y_j = \\ = \sum_j \varphi_j(m) f(x_j) &= \sum_j f(\varphi_j(m) x_j) = \\ &= f(\sum_j \varphi_j(m) x_j) = f(m). \end{aligned}$$

This proves the surjectivity of Ψ. Now, let us turn to the injectivity, and suppose

$$\Psi\left(\sum_j \psi_j \otimes y_j\right) = 0,$$

this implies

$$\Psi\left(\sum_j \psi_j \otimes y_j\right)(a) = 0, \quad \forall a \in M.$$

Being the φ_i generators of M^*, without loss of generality we may assume $\psi_j = \varphi_j$. Rewriting the previous equality one obtains

$$\sum_j \varphi_j(a) y_j = 0,$$

which implies,

$$0 = \sum_j \varphi_j(x_h) y_j = y_h, \quad \forall h = 1, ..., k.$$

And the claim follows by Equation (39).

Q.E.D.

Corollary A.3. *If M is a projective R-module of finite type, then the map Φ defined in Equation* (38) *is an isomorphism.*

Proof. Consider the (canonical) isomorphism (cf. [**2**, Chapter 2])

$$\Xi : Hom_R(M, Hom_R(N, P)) \longrightarrow Hom_R(M \otimes N, P),$$

defined as follows

$$\Xi(\psi)(m \otimes n) = \psi(m)(n).$$

Then by Proposition A.2, the composition

$$M^* \otimes_R M^* \xrightarrow{\Psi} Hom_R(M, M^*) = Hom_R(M, Hom_R(M, R)) \xrightarrow{\Xi} (M \otimes_R M)^*,$$

is an isomorphism. Moreover, we have that

$$\Xi(\Psi(\varphi \otimes \psi))(m \otimes n) = (\Psi(\varphi \otimes \psi)(m))(n) = \varphi(m)\psi(n),$$

which is precisely Φ.

Q.E.D.

2. Grading and Filtration

Definition A.1. Let R be a ring, and C be a commutative monoid. A C-**grading** on R is a decomposition of R as direct sum of Abelian subgroups

$$R = \bigoplus_{c \in C} R_c,$$

such that: $R_c R_{c'} \subseteq R_{c+c'}$. Whenever the monoid C is not specified will be either clear from the context or equals to $\mathbb{Z}$. Finally, a C-grading is **trivial** if $R = R_{0_C}$. The sub-group R_c is called **homogeneous component of degree** c. A element $x \in M$ **homogeneous** if $x \in R_c$, for some $c \in C$, and c is called the degree of x (usually, we will write $deg(x) = c$).

A C**-graded ring** is a ring R together with a fixed C-grading. Notice that every ring admits a trivial C-grading, and hence every ring can be considered trivially C-graded. So, whenever the grading is not specified we will assume our ring trivially graded, unless otherwise stated. Notice that R_{0_C} is a sub-ring of R, and every C-graded ring is an R_{0_C} (graded) algebra.

Definition A.2. A C**-graded module** M, over a C-graded ring R, is an R-module M together with a decomposition

$$M = \bigoplus_{c \in C} M_c,$$

into abelian subgroups, called **homogeneous components**, such that:

$$R_c M_{c'} \subseteq M_{c+c'}.$$

An element $x \in M$ is said to be **homogeneous of degree** c if $x \in M_c$.

Remark 56. There are few remarks concerning the previous definition

(1) if M is a C-graded module over a C-graded ring R, each homogeneous component M_c of M is an R_{0_C} sub-module of M.
(2) If R is G-graded, where G is an abelian group, and H is a sub-group of G, then R is also naturally G/H-graded

$$R = \bigoplus_{[g] \in G/H} R_{[g]}, \qquad R_{[g]} = \bigoplus_{h \in H} R_{g+h}$$

(3) If R is G-graded and we say that M is G/H-graded, we are considering R with the natural G/H-grading.
(4) The bi-grading (in general any multi-grading) is a particular case of the grading we defined: it is sufficient to take $C = \mathbb{Z}^2$ (in general, $C = C_1 \times .... \times C_k$ where the product of monoids is endowed with the natural operation).

Given a C-graded ring (possibly trivially graded) R, there exists a (natural) **polynomial grading** on $R[U_1, ..., U_k]$, for any choice of degrees $d_1, ..., d_k \in C$ for the variables $U_1, ..., U_k$, just set

$$deg(\alpha U_1^{t_1} \cdot \dots \cdot U_k^{t_k}) = deg_R(\alpha) + \sum_i t_i \cdot d_i,$$

where $\alpha \in R$ is homogeneous, and

$$t_i \cdot d_i = \underbrace{d_i + \dots + d_i}_{t_i},$$

and define $R[U_1, ..., U_k]_d$ to be the subgroup of $R[U_1, ..., U_k]$ generated by the homogeneous elements (with respect to deg) of degree d. Notice that in this way $R[U_1, ..., U_k]$ becomes both a C-graded R-module, and a C-graded ring.

Remark 57. The above construction extends naturally to $R[U_1^{\pm 1}, ..., U_k^{\pm 1}]$ if the monoid C is an abelian group.

Given two C-graded modules M and N, over a C-graded ring R, the modules

$$M \oplus N, \quad M \otimes_R N, \quad Hom_R(M,N),$$

are also naturally graded, and their homogeneous components are defined as follows

$$(M \oplus N)_i = M_i \oplus N_i,$$
$$(M \otimes N)_i = \bigoplus_{h+k=i} M_h \otimes_R N_k,$$
$$Hom_R(M,N)_i = \{f \in Hom_R(M,N) \mid f(M_k) \subseteq N_{k+i},\ \forall k\}.$$

In particular, a R-linear map $f : M \to N$ is called **homogeneous of degree** d if $f \in Hom_R(M,N)_d$.

Definition A.3. A **filtered module** M over a ($\mathbb{Z}$-)graded ring R, is a module M together with a family $\mathscr{F} = \{\mathscr{F}_j M\}_{j\in\mathbb{Z}}$ of sub-groups, called **ascending filtration** (resp. **descending filtration**), such that

$$\mathscr{F}_j M \subseteq \mathscr{F}_{j+1} M \text{ (resp. } \mathscr{F}_j M \subseteq \mathscr{F}_{j-1} M), \qquad \alpha \mathscr{F}_j M \subseteq \mathscr{F}_{j+d} M,\ \forall \alpha \in R_d,\ d \in \mathbb{Z}$$

for each $j \in \mathbb{Z}$. A filtration (either increasing or decreasing) is **convergent** if

$$\bigcup_j \mathscr{F}_j M = M, \qquad \bigcap_j \mathscr{F}_j M = (0).$$

Finally, a filtration is **bounded** if

$$\exists i_0,\ i_1, :\ \mathscr{F}_{i_0} M = (0), \quad \mathscr{F}_{i_1} M = M.$$

Remark 58. In particular, for each $j \in \mathbb{Z}$, $\mathscr{F}_j M$ is an R_0 sub-module of M.

Given a filtered module $(M, \mathscr{F})$ (over a graded ring R), and $x \in M$, the **filtered degree** of x, denoted by $Fdeg(x)$, is the integer k such that either $x \in \mathscr{F}_k M \setminus \mathscr{F}_{k+1} M$, if $\mathscr{F}$ is descending, or $x \in \mathscr{F}_k M \setminus \mathscr{F}_{k-1} M$, if $\mathscr{F}$ is ascending.

Let $(M, \mathscr{F})$ be a finitely generated, filtered R-module, and assume R to be a Noetherian domain. The **filtered dimension** of M is the graded dimension (over R) of its associated graded object $Gr^\bullet_{\mathscr{F}}(M)$, which is defined as follows

$$Gr^\bullet_{\mathscr{F}}(M) = \bigoplus_i Gr^i_{\mathscr{F}}(M), \qquad Gr^i_{\mathscr{F}}(M) = \frac{\mathscr{F}^i M}{\mathscr{F}^{i-1} M}.$$

More explicitly,

$$Fdim_R(M, \mathscr{F}) = \sum_i rank_R(Gr^i_{\mathscr{F}}(M)) q^i \in \mathbb{N}[[q, q^{-1}]],$$

where $Fdim$ indicates the filtered dimension and $rank_R(M)$ denotes the dimension of $M \otimes Q(R)$ as $Q(R)$-vector space, where $Q(R)$ is the field of quotients of R. Sometimes, we will be interested in the filtered dimension of M as an R_0-module, that is the reason why we will specify the ring with respect to which we are computing the filtered dimension.

Remark 59. If the filtration is bounded, then the filtered dimension is a polynomial.

Given two filtered modules M and N, over a graded ring R, the modules

$$M \oplus N, \quad M \otimes_R N, \quad Hom_R(M, N),$$

are also filtered with filtrations given by

$$\mathscr{F}_i(M \oplus N) = \mathscr{F}_i M \oplus \mathscr{F}_i N,$$

$$\mathscr{F}_i(M \otimes N) = \sum_{h+k=i} \mathscr{F}_h M \otimes_R \mathscr{F}_k N,$$

$$\mathscr{F}_i Hom_R(M, N) = \{f \in Hom_R(M, N) \mid f(\mathscr{F}_k M) \subseteq \mathscr{F}_{k+i} N, \forall k\}$$

In particular, the **filtered degree of a map** f between filtered modules, say M and N, is its filtered degree as an element of $Hom_R(M, N)$.

3. Filtered degree and bases

In this section we wish to study the behaviour of a basis for a free filtered module with respect to the filtration.

Definition A.4. A **filtered basis** B for a free filtered R-module, say M, is a basis for M (as an R-module), such that the canonical isomorphism

$$M \simeq \bigoplus_{b \in B} R \cdot b,$$

is an isomorphism of filtered modules with respect to the direct sum filtration, where each $R \cdot b$ is endowed with the filtration

$$\mathscr{F}_i(R \cdot b) = R \cdot b \cap \mathscr{F}_i M.$$

Not every basis is a filtered basis as shown by the following example.

Example. Consider the R module space $V_k = R[X]/(X^k + 1)$, with the filtration induced by the polynomial grading, namely

$$\mathscr{F}_i V = \begin{cases} (0) & i < 0, \\ \langle [1], ..., [X^i] \rangle & 0 \leq i \leq k-1, \\ V_k & k-1 < i \end{cases}$$

Then, the basis

$$[X+1], [X], [X^2], ..., [X^{k-1}],$$

is not a filtered basis. In fact, $[1]$ is sent, by the canonical isomorphism, to $(1, -1, 0, ..., 0)$. Thus, the filtered degree of $[1]$ with respect to the direct sum filtration is 1, while its filtered degree as an element of V_k is 0.

Filtered bases have a nice behaviour with respect to the filtered degree, and also with respect to direct sum and tensor product of filtered modules. The following proposition, whose proof is straightforward, summarize the main properties of filtered bases.

Proposition A.4. *Let R be a graded ring with degree bounded from above or below. Let M be a finitely generated, free, filtered R-module and let $\{m_1, ..., m_k\}$ be a filtered R-basis for M. Then, the following results hold*

(1) for any $i \in \mathbb{Z}$, and for any $\alpha_1, ..., \alpha_k \in R$,

$$\sum_{s=1}^{k} \alpha_s m_s \in \mathscr{F}_i M \quad \Longleftrightarrow \quad \alpha_s m_s \in \mathscr{F}_i M,\ \forall s$$

(2) for each $\varphi \in Hom_R(M, R)$

$$Fdeg(\varphi) = max\{Fdeg(\varphi(m_s)) - Fdeg(m_s) \mid s = 1, ..., k\};$$

(3) if N is also a finitely generated, free, filtered R-module, and $\{n_1, ..., n_h\}$ is a filtered basis for N, then $\{m_1, ..., m_k, n_1, ..., n_h\}$ is a filtered basis for $M \oplus N$, and the set $\{m_s \otimes n_t\}_{s,t}$ is a filtered basis for $M \otimes_R N$.

□

We conclude this section with a result concerning the filtered degree of the elements of an arbitrary basis of a filtered free module.

Proposition A.5. *Let R be a graded ring with only non-negative degrees. Let M be a finitely generated free R-module endowed with a descending filtration $\mathscr{F}$. Given a basis B for M define the integer $m(B)$ as follows*

$$m(B) = min\{Fdeg(b_i) | b_i \in B\}.$$

Then $m(B)$ does not depend on the choice of B.

Proof. Since R is supported in non-negative degrees, and since the filtration $\mathscr{F}$ is descending, $\mathscr{F}_i M$ is an R-sub-module of M for each $i \in \mathbb{Z}$. From the definitions of filtered degree and of $m(B)$ it follows that

$$b \in \mathscr{F}_{m(B)} M, \qquad \forall b \in B.$$

Let B' be another basis. Since B is a basis it follows that

$$b' = \sum_{b \in B} \alpha_b^{b'} b \in \mathscr{F}_{m(B)} M,$$

for each $b' \in B'$. Thus, it follows

$$Fdeg(b') \geq m(B),$$

which implies $m(B') \geq m(B)$. By exchanging the roles of B and B' we obtain the equality.

Q.E.D.

Corollary A.6. *Let $\mathbb{F}$ be a (trivially graded) field and let V be a filtered $\mathbb{F}$-vector space. If the filtration on V is descending, then the minimum filtered degree of the elements of a basis does not depend on the chosen basis.* □

4. Homogeneous lifts

Given a graded (commutative unital) ring R, consider the polynomial ring $R[U]$, where $deg(U) > 0$. Let M be a graded $R[U]$-module, and set

$$deg\,(\alpha) = max\{deg(\alpha_i)\},$$

where α_i are the homogeneous components of α, for each $\alpha \in M$. This is an extension of the degree which we are going to use in the follow up.

Lemma A.7. *Let M be a graded $R[U]$-module, $\alpha \in R$, and M_k be the k-th homogeneous component of M. If $U, \alpha \notin Ann(x)$ for each $x \in M$ and $deg(\alpha) \neq deg(U)$, then*

$$M_k \cap (U - \alpha)M = \{0\}.$$

Proof. Let $x \neq 0$ be an element of M, and x_1,..., x_k be its homogeneous components. Suppose that $deg(x_i) > deg(x_{i-1})$, and that $deg(\alpha) < deg(U)$ (the same proof works also for $deg(\alpha) > deg(U)$, up to exchanging the role of α and U). Then Ux_k and αx_1 are non trivial, and their degrees are, respectively, higher and lower than any other homogeneous component of $(U - \alpha)x$. Hence, $(U - \alpha)x$ cannot be homogeneous. □

Corollary A.8. *If R is an integral domain, and M a free graded $R[U]$-module, then*

$$(U - \alpha)M \cap M_k = \{0\}$$

for each $\alpha \in R$ such that $deg(\alpha) \neq deg(U)$ and for each $k \in \mathbb{Z}$. □

Assume that the degree in $R[U]$ is bounded from below. If α and U are homogeneous of different (positive) degrees, then the $R[U]$ quotient module $\overline{M}_\alpha = M/(U - \alpha)M$ is not (in general) graded, but it is filtered

$$\mathscr{F}_s\overline{M}_\alpha = \langle [x] \mid x \in M_k, k \geq s \rangle\,,$$

in this case, we may define in $\overline{M}_\alpha$ a **filtered degree** as follows

$$Fdeg([x]) = inf\{deg(y)\ :\ y \in [x]\} = sup\{s \mid [x] \in \mathscr{F}_s\overline{M}_\alpha\}.$$

Remark 60. The filtered degree defined above is the filtered degree induced by the filtration.

Remark 61. If $x \in M_k$, and $x \neq Uy$, then $Fdeg([x]) = k$. Moreover, the whole construction can be duplicated in case the degree of R is bounded from above: it is sufficient to reverse the inequality in the definition of the filtration and to exchange the infimum with the supremum (and vice-versa) in all the definitions above.

In the special case $M = R[U]$, where R is a graded ring satisfying some mild hypotheses, the quotient module $M/(U - \alpha)$ has a natural filtered basis. More formally the following proposition (whose proof is just a simple verification) holds.

Proposition A.9. *Let R be a graded Noetherian domain, with degrees bounded from either above or below, and let U be a formal variable graded consistently*[1]. *Given a non-homogeneous polynomial $P \in R[U]$, denote by $\mathscr{F}$ the induced filtration on the quotient $R[U]/(P)$. The basis $[1], ..., [U^{deg(P)}]$ is a filtered R-basis for $R[U]/P$.* □

Now, we wish to relate homogeneous elements of a graded module M with elements in a naturally filtered quotient of M. In particular, we wish to prove a uniqueness criteria for homogeneous lifts.

Lemma A.10. *Let M be a finitely generated, free, graded $R[U]$-module, with $R = R_0$. Suppose that $deg(U) = 1$, and consider an invertible element $\alpha \in R$.*

Given $y \in \overline{M}_\alpha$ there exists a unique homogeneous lift x_t (i.e. $[x_t] = y$), for each $t \in \mathbb{N}$, such that $deg(x_t) = Fdeg(y) + t$.

Proof. The uniqueness follows from Corollary A.8. More precisely, given two homogeneous lifts of the same degree, say y and y', their difference $y - y'$ is still homogeneous of the same degree. On the other hand, if $\pi_\alpha(y) = \pi_\alpha(y')$, then $y - y' \in (U - \alpha)M$. Which implies $y = y'$, by Corollary A.8.

For the existence of x_t it is sufficient, thanks to the above argument, to show the existence of $x = x_0$, and set $x_t = (\alpha^{-1}U)^t x$.

Let us start with $\tilde{x} = \sum_i \alpha_i U^{k_i} x_{j_i} \in y$ realizing the minimum degree (since M is finitely generated and $R[U]$ has only elements of non-negative degree, the minimum is attained). Then notice that

(1) $deg(\tilde{x}) = Fdeg(y)$;
(2) the leading term of $\tilde{x}$ is not a multiple of U (otherwise, evaluating U at α we get an element in y of lower degree);
(3) any other homogeneous term of $\tilde{x}$ has lower degree;

So, it is well defined

$$x = \sum_i \alpha_i \alpha^{k_i} (\alpha^{-1}U)^{deg(\tilde{x}) - deg(x_{j_i})} x_{j_i}$$

is an homogeneous element of degree $deg(\tilde{x}) = Fdeg(y)$ whose projection onto $\overline{M}_\alpha$ is y. □

The previous lemma generalizes to the case $deg(U) = n$, and $M = \bigoplus_{k \in \mathbb{Z}} M_{nk}$; the only change to make in the proof is the definition of x, where the exponent of U has to be replaced by:

$$t_i = \frac{deg(\tilde{x}) - deg(x_{j_i})}{n}.$$

Furthermore, we may also drop the hypothesis $R = R_0$. It is sufficient to assume the degree in R to be bounded by either above or below.

Remark 62. The uniqueness argument is unchanged even if we have no further hypothesis on the grading in R, or the degree of U, with the sole exception

[1]That is: the variable U has positive degree if R has degrees bounded from below, and negative degree if R has degrees bounded from above. In this way the degrees in the polynomial ring $R[U]$ are bounded.

of $deg(U) \neq deg(\alpha)$. Under these hypothesis (M being a free R-module, and α not a zero divisor for M) once the homogeneous lift exists is unique (in a fixed degree).

An immediate consequence of Lemma A.10 is the following proposition.

Proposition A.11. *Let M and M' be two graded, free R-modules, and let $\alpha \in R$ be such that $deg(\alpha) \neq deg(U)$, and $\alpha \notin Div_0(R)$. Given two $R[U]$-linear maps*

$$\Phi : M \longrightarrow M', \qquad \Phi_\alpha : \overline{M}_\alpha \longrightarrow \overline{M}'_\alpha,$$

which commute with the projections (i.e. $\pi'_\alpha \circ \Phi = \Phi_\alpha \circ \pi_\alpha$), with Φ homogeneous of degree 0. If $x \in M$, $x' \in M'$ are two homogeneous lifts of, respectively, $y \in \overline{M}_\alpha$, $y' \in \overline{M}'_\alpha$, such that $deg(x) = deg(x')$. Then,

$$\Phi_\alpha(y) = y' \quad \Longleftrightarrow \quad \Phi(x) = x'.$$

Proof. It is sufficient to notice that, for the assumptions on Φ and Φ_α, $\Phi(x)$ is an homogeneous lift of y' of degree $deg(x')$.

Q.E.D.

Lemma A.12. *Let M be a finitely generated, free, graded $R[U]$-modules, with R a graded integral domain with degrees bounded from above or below, and $deg(U) \neq 0$. Given a unit $\alpha \in R$, denote by $\overline{M}_\alpha$ the quotient of M by the sub-module $(U - \alpha)M$. Denoted by π the projection to the quotient, if x and y are homogeneous elements, and $x + y \in Ker(\pi)$, then*

$$deg(x) \equiv deg(y) \quad mod\, deg(U).$$

Proof. We will deal with the case $deg(U) > 0$, but the case $deg(U) < 0$ is dealt with in the same way. Because $x + y$ belongs to $Ker(\pi)$, there exists z such that $x + y = (U - \alpha)z$. Decompose z into homogeneous components

$$z = \sum_{i=m}^{M} z_i,$$

where

$$deg(z_m) < \cdots < deg(z_M).$$

Both Uz_M and αz_m do not vanish, and they have degrees different from all the other terms in the sum $\sum_{i=m+1}^{M-1}(U - \alpha)z_i - \alpha z_M + Uz_m$. Thus, either $deg(x) = deg(z_m)$, and $deg(y) = deg(z_M) + deg(U)$, or vice-versa. Up to exchange the roles of x and y, we may assume $deg(x) = deg(z_m)$. As $x + y$ is concentrated only in these degrees,

$$\sum_{i=m+1}^{M-1} (U - \alpha)z_i - \alpha z_M + Uz_m = 0.$$

Thus, we get

$$\alpha z_i = Uz_{i-1}, \qquad \forall i \in \{m + 1, ..., M\}.$$

In particular

$$deg(z_i) = deg(z_{i-1}) + deg(U), \qquad \forall i \in \{m + 1, ..., M\}.$$

And the claim follows. $\square$

Remark 63. Lemma A.10 works for a polynomial P in place of $(U - \alpha)$, in which case we have

$$deg(x) \equiv deg(y), \qquad mod\, GCD(deg(P_i) \mid P_i \neq 0),$$

where the P_i's are the homogeneous components of P.

Corollary A.13. *Under the hypotheses of Lemma A.12, the quotient module $\overline{M}_\alpha$ is $\mathbb{Z}/deg(U)\mathbb{Z}$-graded.* □

Proposition A.14. *Let M, M' be two finitely generated, free, graded $R[U]$-modules, with R an integral domain. Assume that $deg(\theta) \geq 0$, for each $\theta \in R$, and $deg(U) > 0$. Given an homogeneous unit $\alpha \in R$, denote by $\overline{M}_\alpha$, and $\overline{M}'_\alpha$ the quotients of M, and M' by the ideal $(U - \alpha)$, respectively. Then, given a degree* 0 *homogeneous map*

$$\Phi : M \longrightarrow M',$$

denote by $\overline{\Phi}$ the induced map on the quotients. Let $s \in \overline{M}_\alpha$. If

$$min\{Fdeg(s') \mid \overline{\Phi}(s') = s\} \leq k$$

and if $s \in Im(\overline{\Phi})$, then for each $\tilde{s} \in M'_k$ such that $\pi(\tilde{s}) = s$ we have

$$\tilde{s} \in Im(\Phi)$$

□

5. Spectral sequences

Let R be a Noetherian domain; during this section we will suppose all modules to be R-modules, and all maps to be R-linear, unless otherwise explicitly stated. The general references for this section are: [**52**, Chapter 9], [**31**, Chapter XX, §9], [**57**, Chapter 5] and [**20**, Chapter VIII].

Definition A.5. A **spectral sequence E** is a sequence $\{(E_r, d_r, \psi_r)\}_{r\in\mathbb{N}}$ of bi-graded R-modules E_r, and maps $d_r : E_r \to E_r$ satisfying the following properties:

(1) $d_r \circ d_r = 0$;
(2) d_r is (bi-)homogeneous of bi-degree $(r, -r+1)$, that is to say:

$$d_r(E_r^{p,q}) \subseteq E_r^{p+r,q-r+1},$$

for each $r \in \mathbb{N}$, and $p, q \in \mathbb{Z}$;
(3) for any $r \in \mathbb{N}$ is given an isomorphism of bi-graded modules

$$\psi_r : E_{r+1}^{\bullet,\bullet} \to H^{\bullet,\bullet}(E_r),$$

for each $r \in \mathbb{N}$.

The module E_r is the r-Th **page** , and the map d_r is the r-Th **differential** of the spectral sequence **E**.

Given two spectral sequences **E** and **E**$'$, a **morphism** (of spectral sequences) between them is a family of bi-homogeneous maps

$$\varphi_r : E_r \to E'_r,$$

which commute with the differentials (i.e. $\varphi_r \circ d_r = d'_r \circ \varphi_r$), and such that

$$\psi'_r \circ (\varphi_r)_* \circ \psi_r^{-1} = \varphi_{r+1}.$$

In a similar fashion one defines ephimorphisms, monomorphisms and isomorphisms.

In order to define the limit of a spectral sequence, consider the identification

$$E^{p,q}_{r+1} \simeq \frac{Z^{p,q}(E_r)}{B^{p,q}(E_r)} = \frac{Ker(d_r^{p,q} : E_r^{p,q} \to E_{r+1}^{p+r,q-r+1})}{Im(d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E^{p,q}_{r+1})},$$

performed by the means of ψ_r. Thanks to Noether isomorphism theorem (a.k.a. second isomorphism theorem for modules, see either **[52]**, or **[31]** for the statement), there exist[2] bi-graded sub-modules $Z^{p,q}_{r+1}$, and $B_r^{p,q}$ contained in $Z^{p,q}(E_r)$ and containing $B^{p,q}(E_r)$, such that

$$Z^{p,q}(E_{r+1}) = \frac{Z^{p,q}_{r+1}}{B^{p,q}(E_r)}, \quad B^{p,q}(E_{r+1}) = \frac{B^{p,q}_{r+1}}{B^{p,q}(E_r)}.$$

It follows that

$$B^{p,q}(E_r) \subseteq B^{p,q}_{r+1} \subseteq Z^{p,q}_{r+1} \subseteq Z^{p,q}(E_r).$$

By induction, one obtains a sequence

$$B^{p,q}(E_0) = B_0^{p,q} \subseteq ... \subseteq B_r^{p,q} \subseteq B^{p,q}_{r+1} \subseteq ... \subseteq Z^{p,q}_{r+1} \subseteq Z_r^{p,q} \subseteq ... \subseteq Z_0^{p,q} = Z^{p,q}(E_0)$$

for each $p, q \in \mathbb{Z}$. Moreover, we have

$$E_r^{p,q} = \frac{Z^{p,q}(E_r)}{B^{p,q}(E_r)} = \frac{\frac{Z_r^{p,q}}{B^{p,q}(E_{r-1})}}{\frac{B_r^{p,q}}{B^{p,q}(E_{r-1})}} = \frac{Z_r^{p,q}}{B_r^{p,q}}.$$

Finally, if we define $Z^{p,q}_\infty = \bigcap_r Z_r^{p,q}$ and $B^{p,q}_\infty = \bigcup_r B_r^{p,q}$, then the bi-graded module

$$E_\infty = \frac{Z_\infty^{\bullet,\bullet}}{B_\infty^{\bullet,\bullet}}$$

is called **limit** of the spectral sequence **E**.

Definition A.6. The spectral sequence **E** is said to **converge** if for every $p, q \in \mathbb{Z}$, there exists an integer $r_{p,q}$ such that $d_r^{p,q} \equiv 0$, for each $r > r_{p,q}$.

If a spectral sequence converges then $E^{p,q}_{r+1}$ is isomorphic to a quotient of $E_r^{p,q}$ (because $E_r^{p,q} = Ker(d_r^{p,q})$), for each $r > R_{p,q}$. Moreover, the (p,q)-component of the limit of the spectral sequence E_∞ is isomorphic to the direct limit of the system

$$\pi_r : E_r^{p,q} \to E^{p,q}_{r+1}, \quad r > R_{p,q},$$

where π_r is the projection onto the quotient module.

These considerations (and the properties of direct limits) imply that: if **E** and **E**$'$ are converging spectral sequences, then any morphism $\Phi = \{\varphi_r\}$ induces a morphism φ_∞ between the limits E_∞ and E'_∞. Furthermore, with a little bit of (basic) homological algebra one may deduce the following Proposition.

---

[2]Remember that $Z^{p,q}(E_{r+1})$ is a sub-module of E_{r+1}, and hence a sub-module of the quotient $Z^{p,q}(E_r)/B^{p,q}(E_r)$.

Proposition A.15. *Let $\Phi : \mathbf{E} \to \mathbf{E}'$ be a morphism of spectral sequences. If φ_r is an isomorphism for a certain r, then φ_s is an isomorphism for each $s > r$. Moreover, in this case also the map φ_∞ is an isomorphism.* □

In most of the cases, e.g. if p, q are bounded and $\mathbf{E}$ is a converging spectral sequence, there exists an $R \in \mathbb{Z}$ such that $d_r \equiv 0$ for each $r > R$; in this case, the sequence $\mathbf{E}$ is said to **converge at page** $R+1$. In such a situation the maps π_r are isomorphisms and $E_\infty \simeq E_r \simeq E_{R+1}$.

Given a (co)chain complex $(C^\bullet, d)$ over R, and a filtration $\mathscr{F}$ over $C^\bullet$, $\mathscr{F}$ is **compatible with the differential** if $d(\mathscr{F}_j C^\bullet) \subseteq \mathscr{F}_j C^\bullet$, for every j. In other words, a filtration is compatible with the differential if each level is a sub-complex of $(C^\bullet, d)$. Any time there is a filtered complex (i.e. a complex with a filtration), the filtration will be assumed, unless otherwise stated, to be compatible with the differential.

Given a filtered complex $(C^\bullet, d, \mathscr{F})$, the filtration $\mathscr{F}$ induces a filtration in homology. More precisely, the inclusions

$$\iota : \mathscr{F}_j C^\bullet \to C^\bullet = \mathscr{F}_\bullet C^\bullet,$$

induce a morphism in homology

$$\iota_* : H(\mathscr{F}_j C^\bullet) \to H(C^\bullet),$$

whose image defines a filtration $\mathscr{F}_j H(C^\bullet) = \iota_*(H(\mathscr{F}_j C^\bullet))$. The next theorem relates this filtration (to be precise its associated graded object), the filtration on the complex, and their respective homology via a spectral sequence. For a proof of the following result the reader may consult, for example, [**31**, Chapter XX, Propositions 9.1, 9.2 & 9.3].

Theorem A.16. *Let $\mathscr{F}$ be a convergent filtration, bounded from below, over a complex $(C^\bullet, d)$. There exists a convergent (graded) spectral sequence $\mathbf{E}$, such that*

$$E_0^{p,q} = Gr_{p,q}(C^\bullet) = \frac{\mathscr{F}_p C^{p+q}}{\mathscr{F}_{p+1} C^{p+q}}, \quad E_1^{p,q} \simeq H_{p+q}(E^0_{p,\bullet}) = H\left(\frac{\mathscr{F}_p C^{p+q}}{\mathscr{F}_{p+1} C^{p+q}}\right),$$

and

$$E_\infty^{p,q} \simeq Gr_{p,q}(H(C^\bullet)).$$

Moreover, the differential $d_1 : E^1_{p,\bullet} \to E^1_{p+1,\bullet}$ is, up to conjugation by the isomorphism $E_1^{p,q} \simeq H_{p+q}(E_0^{p,\bullet})$, the connecting morphism on the long exact sequence arising from the sequence

$$0 \to \frac{\mathscr{F}_{p+1} C^\bullet}{\mathscr{F}_{p+2} C^\bullet} \to \frac{\mathscr{F}_p C^\bullet}{\mathscr{F}_{p+2} C^\bullet} \to \frac{\mathscr{F}_p C^\bullet}{\mathscr{F}_{p+1} C^\bullet} \to 0$$

□

APPENDIX B

Computations

Corollary 2.33 can be used to compute Bar-Natan homology from Khovanov homology, if $char(\mathbb{F}) \neq 2$, in the case of knots. It suffices to start with the group in Khovanov homology that has the lowest quantum and homological degree, and to pair up the heads and the tails of the hooks (paying attention to the groups which are due to the free part of Bar-Natan homology, and which lay in bi degrees $(0, s-1)$, and $(0, s+1)$). There are some ambiguous configurations that may come up, as shown in Figure 1, but they are excluded by Corollary 2.33. Other ambiguous configurations (cf. Figure 1), that cannot be excluded, do not come up in the rational Khovanov homology of any knot with 12 crossings or less. Notice that the configuration in Khovanov homology shown in Figure 2, still satisfies the Bar-Natan-Goroufalidis conjecture.

Remark 64. The figures are meant to be read as follows: each row and each column correspond, respectively, to half of the quantum degree minus one, and to the homological degree; the 1 indicates the generator of a free summand, while U^k indicates the generator of a summand of the form $\mathbb{F}[U]/(U^k)$, in the corresponding bi-degree.

Kh			
			1
		1	
		1	
	1		

BN			
			U
		U	

BN			
			U^2
		U^2	

FIGURE 1. A possibly ambiguous configuration, and the corresponding possible configurations in Bar-Natan homology; the configuration in the middle is excluded by Corollary 2.33.

1. The knot 13_{n1336}

The knot 13_{n1336} is the first knot which is torsion rich (i.e. the torsion in the rational Khovanov homology is not contained in two diagonals, cf. [**51**, Section A.4]), is a non alternating, and is also the first knot whose Khovanov homology is supported in 4 diagonals. We report its Khovanov homology, over a field of characteristic different than 2, below (cf. [**51**, Section A.4]). As an example we compute the Bar-Natan homology of this knot. Start to pair up the groups in the

Kh			
			1
		1	
		1	
	1		

BN			
			U^2
		U^2	

BN			
			U^4
		U^4	

FIGURE 2. An ambiguous configuration, and the corresponding possible configurations in Bar-Natan homology; neither of them can be excluded using only Corollary 2.33.

lowest homological and quantum degrees. Whenever an apparently ambiguos configuration comes up, re-start with the groups of highest possible bi-degree. Moreover, take into account that we know exactly what is the rank of H_{BN} in each homological degree (cf. loc. cit. Proposition 2.3).

Kh	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
13														1
11														
9												1	1	
7										1	1			
5									1		1			
3									2	1				
1							2	1						
-1					1	1	1	1						
-3						1								
-5				1	1									
-7		1												
-9														
-11	1													

BN	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
13														U^2
11														
9												U^2		
7										U^2	U^2			
5									U^2					
3									U^2					
1							$1{+}U^2$	U^2						
-1					U^2	U^2	1							
-3														
-5														
-7		U^2												
-9														
-11														

2. Bar-Natan homology of $T(2,n)$ torus links

In [**25**, Section 6.2], Khovanov computed the integral Khovanov homology of the $T(2,n)$ torus links. The result, for a field $\mathbb{F}$ of characteristic different than 2, is

$$H_{Kh}^{-2j,4j-2+n}(T(2,n);\mathbb{F}) = \mathbb{F} = H_{Kh}^{-2j-1,4j+2+n}(T(2,n);\mathbb{F}),$$

with $1 \leq j \leq \lfloor (n-1)/2 \rfloor$,

$$H_{Kh}^{0,n-2}(T(2,n);\mathbb{F}) = \mathbb{F} = H_{Kh}^{0,n}(T(2,n);\mathbb{F}),$$

and all the other groups are trivial, for $n > 0$ odd and *char* $\mathbb{F} \neq 2$. Thus, since we have no ambiguity in identifying the hooks (knight moves, to be precise, cf. Figure 1), in this case

$$H_{BN}^{-2j-1,\bullet}(T(2,n);\mathbb{F}[U]) = \frac{\mathbb{F}[U]}{(U^2)}\{(0,4j+n)\},$$

with $1 \leq j \leq (n-1)/2$, and

$$H_{BN}^{0,\bullet}(T(2,n);\mathbb{F}[U]) = \mathbb{F}[U]\{(0,-n)\} \oplus \mathbb{F}[U]\{(0,-n+2)\}.$$

If n is even, then there are two more Khovanov homology groups which are non-trivial, more precisely

$$H_{Kh}^{-n,-3n}(T(2,n);\mathbb{F}) = \mathbb{F} = H_{Kh}^{-n,-3n+2}(T(2,n);\mathbb{F}),$$

also in this case no ambiguous configurations (cf. Figure 2) come up, hence

$$H_{BN}^{-2j,\bullet}(T(2,n);\mathbb{F}[U]) = \frac{\mathbb{F}[U]}{(U^2)}\{(0,-4j-n+2)\},$$

with $1 \leq j \leq \lfloor (n-1)/2 \rfloor$, and

$$H_{BN}^{0,\bullet}(T(2,n);\mathbb{F}[U]) = \mathbb{F}[U]\{(0,-n)\} \oplus \mathbb{F}[U]\{(0,-n+2)\},$$

$$H_{BN}^{-n,\bullet}(T(2,n);\mathbb{F}[U]) = \mathbb{F}[U]\{(0,-3n)\} \oplus \mathbb{F}[U]\{(0,-3n+2)\},$$

for n even. Thus, we may compute directly both the Ramussen-Beliakova-Wehrli invariants and Pardon's invariants. The result is the following:

$$d_{h,q}(T(2,n)) = \begin{cases} 1 & (h,q) \in \{(0,\text{-}n+2),(0,\text{-}n),(\text{-}n,\text{-}3n),(\text{-}n,\text{-}3n+2)\} \\ 0 & \text{otherwise} \end{cases},$$

$$RBW(T(2,n)) = \{\text{-}n+1, \text{-}3n+1\},$$

if n is even, and

$$RBW(T(2,n)) = \{n+1\},\ d_{h,q}(T(2,n)) = \begin{cases} 1 & (h,q) \in \{(0,\text{-}n+2),(0,\text{-}n)\} \\ 0 & \text{otherwise} \end{cases},$$

if n is odd; where $RBW(\lambda)$ denotes the set of the Ramussen-Beliakova-Wehrli invariants. These results are consistent with the known facts on the Rasmussen-Beliakova-Wehrli invariants for alternating links.

In the case $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, things do not change that much. In this case, it is immediate to isolate the two groups in bi-degrees $(0,\text{-}n{+}2), (0,\text{-}n)$. This amounts to the whole free part in the case n odd. Proposition 2.3 tells us that the remaining generators of the free part, in the case n even, are located in homological degree $-n$. Therefore, we obtain the same result as above. The remaining groups could be paired up into hooks as follows. Let us start from the lowest homological degree. Here we have a tetris piece (see Figure 3).

Kh	$-2j-1$	$-2j$		BN	$-2j-1$	$-2j$	
$\vdots$				$\vdots$			
$-4j+2-n$		1	$\cdots$	$-4j+2-n$		U	$\cdots$
$-4j-n$	1	1	$\cdots$	$-4j-n$		U	$\cdots$
$-4j-2-n$	1		$\cdots$	$-4j-2-n$			$\cdots$

FIGURE 3. A tetris piece and the corresponding figure in Bar-Natan homology.

If we couple the group in the lowest quantum degree with the ones in the highest quantum degree, then the two groups in the middle cannot be paired up, which is absurd. Hence, there is only a possible way to pair them up. Notice that there is no ambiguity because the dimension of $H^{h,\bullet}_{Kh}$ is either 2 or 0, for each homological degree h.

3. Bar-Natan homology of the links $T(3,k)$

Let us turn to a more interesting example, the case of $T(3,k)$. These links are neither alternating (in general), nor Kh-thin. Their rational Khovanov homology has been computed by M. Stošić (cf. **[53]**).

Theorem B.1 (Stošić). *The Khovanov polynomial, defined by*

$$P_{Kh}[\lambda](t,q) = \sum_{h,i} dim_{\mathbb{F}}(H^{h,i}_{BN})t^h q^i,$$

in the case $\lambda = T(3,k)$ *is given by*

$$P_{Kh}[T(3,k)](t,q) = q^{-2k}\Bigg(q^3 + q + t^{-2}q^{-1} + t^{-3}q^{-5} + \Big[t^{-4}(q^{-3}+q^{-5}) + t^{-5}(q^{-7}+q^{-9}) + t^{-6}q^{-7} + t^{-7}q^{-11}\Big]\sum_{i=0}^{n-2} t^{-4i}q^{-6i} + Q_k(t,q) \Bigg)$$

where $n = \lfloor k/3 \rfloor$, *and*

$$Q_k(t,q) = \begin{cases} t^{-4n}(q^{-6n+3} + 3q^{-6n+1} + 2q^{-6n-1}) & k \equiv 0 \ mod\, 3 \\ t^{-4n+2}q^{-6n+5} + t^{-4n+1}q^{-6n+1} & k \equiv 1 \ mod\, 3 \\ 0 & k \equiv 2 \ mod\, 3 \end{cases}$$

Now we have to isolate the terms which cannot be paired up in any hook.The terms of t-degree 0, cannot be paired up (as there are not other groups in neighbouring columns). Hence, two direct summands of the free part in $H^{\bullet,\bullet}_{BN}$ are in bi-degrees $(0,-2k+3)$, and $(0,-2k+1)$. In the cases $k \equiv 1$ and $k \equiv 2 \ mod\, 3$, this represent the whole free part. So, we get

$$d_{h,q}(T(3,k)) = \begin{cases} 1 & (h,q) \in \{(0,\text{-}2k+3),(0,\text{-}2k+1))\} \\ 0 & \text{otherwise} \end{cases}$$

and

$$RBW(T(3,k)) = \{-2k+2\}.$$

In accordance with the facts

$$d_{h,q}(\kappa) \neq 0 \iff d_{h,q}(\kappa) = 1 \iff (h,q) \in \{(0, s(\kappa) \pm 1)\}.$$

which hold for any knot κ (cf. **[44]**), and

$$s(T(p,q)) = -(p-1)(q-1),$$

for each $p,q \in \mathbb{N}$, with $gcd(p,q) = 1$, (cf. **[47]**). All the other terms can easily (and uniquely) be paired up into hooks.

So, it remains the case $k \equiv 0 \ mod\, 3$, or equivalently $k = 3n$. The total rank of $H^{\bullet,\bullet}_{BN}$ is, in this case, $2^3 = 8$. Two free summands are located, as in the previous cases, in bi-degrees $(0,-2k+3)$, and $(0,-2k+1)$. Consider the terms of the Khovanov polynomial representing the homology groups of the lowest homological degree, namely $t^{-4n}q^{-2k}(q^{-2k+3} + 3q^{-2k+1} + 2q^{-2k-1})$. Only one of the corresponding groups can be paired up with another group to get an hook, as $dim_{\mathbb{F}}(H^{-4n+1,\bullet}_{Kh}(T(3,3n)) = 1$. In the case one of the above mentioned terms is part of an hook, there would be five free summand in homological degree $-4n$, and one in another degree. But this is absurd (↯) because there is always an even number of canonical generator within the same homological degree. This implies that $H^{-4n,\bullet}_{BN}(T(3,k))$, is isomorphic to the bi-graded $\mathbb{Q}[U]$-module

$$\mathbb{Q}[U]\{(\text{-}4n,\text{-}4k-1)\} \oplus (\mathbb{Q}[U]\{(\text{-}4n,\text{-}4k{+}1)\})^{\oplus 3} \oplus (\mathbb{Q}[U]\{(\text{-}4n,\text{-}4k{+}3)\})^{\oplus 2}$$

Remark 65. We could have used Proposition 2.3. This proposition implies that the Bar-Natan homology of the link $T(3,3n)$ has rank 6 in homological degree $-12n$.

The remaining terms must be paired up into hooks, and there is only one possible way to do so. Thus, the Bar-Natan homology of $T(3,3n)$ is isomorphic to

$$\begin{aligned} H^{\bullet,\bullet}_{BN}(T(3,3n)) = \bigoplus_{i=0}^{n-2} \left(\frac{\mathbb{Q}[U]}{(U^2)}\{(-4i-4,-3)\} \oplus \frac{\mathbb{Q}[U]}{(U^2)}\{(-4i-4,-5)\} \oplus \right. \\ \left. \oplus \frac{\mathbb{Q}[U]}{(U^2)}\{(-4i-6,-7)\} \right) \oplus \frac{\mathbb{Q}[U]}{(U^2)}\{(-2,-6n+1)\} \oplus \\ \oplus \mathbb{Q}[U]\{(0,-6n+1)\} \oplus \mathbb{Q}[U]\{(0,-6n+3)\} \oplus \\ \oplus \mathbb{Q}[U]\{(\text{-}4n,\text{-}12n\text{+}3)\} \oplus (\mathbb{Q}[U]\{(\text{-}4n,\text{-}12n\text{+}1)\})^{\oplus 3} \oplus \\ \oplus (\mathbb{Q}[U]\{(\text{-}4n,\text{-}12n\text{-}1)\})^{\oplus 2}. \end{aligned}$$

As a corollary, we obtain

$$d_{h,q}(T(3,3n)) = \begin{cases} 1 & (h,q) \in \{(0,\text{-}6n\text{+}3),(0,\text{-}6n\text{+}1),(\text{-}4n,\text{-}12n\text{+}3)\} \\ 2 & (h,q) = (\text{-}4n,\text{-}12n\text{-}1) \\ 3 & (h,q) = (\text{-}4n,\text{-}12n\text{+}1) \\ 0 & \text{otherwise} \end{cases},$$

hence, we may conclude that

$$\{\text{-}6n\text{+}2,\text{-}12n\} \subseteq RBW(T(3,3n)) \subseteq \{\text{-}6n\text{+}2,\text{-}12n,\text{-}12n\text{+}2\}.$$

Since there are only two semi-orientations of $T(3,3n)$, it follows that there are only two Rassmussen-Beliakova-Wehrli invariants. Thus, we obtain either

$$RBW(T(3,3n)) = \{\text{-}6n\text{+}2,\text{-}12n\} \quad \text{or} \quad RBW(T(3,3n)) = \{\text{-}6n\text{+}2,\text{-}12n+2\}$$

Remark 66. Turner **[56]** states that the Khovanov homology of the $(3,k)$ torus links is the same in all the fields of characteristic which is not 2. Thus, our results extend to these fields as well.

From the computations above it follows that if $T'(3,3n)$ is the link obtained from $T(3,3n)$ by reversing the orientation of a single strand, then its Rasmussen Beliakova-Werhli invariants are either

$$RBW(T'(3,3n)) = \{0,6n+2\} \quad \text{or} \quad RBW(T'(3,3n)) = \{2,6n+2\}.$$

Which is in accordance with the computations made in Section 2. Moreover, we can deduce the Pardon's invariants of the link $T'(3,3n)$. These are easily computed and are

$$d_{h,q}(T(3,3n)) = \begin{cases} 1 & (h,q) \in \{(4n,3),(4n,6n\text{+}1),(0,\text{+}3)\} \\ 2 & (h,q) = (0,\text{-}1) \\ 3 & (h,q) = (0,1) \\ 0 & \text{otherwise} \end{cases},$$

4. The rational Bar-Natan homology of the prime knot with less than 12 crossing

In this section we list the Hilbert-Poincaré polynomials of the rational Bar-Natan homology of all prime knots with less than 12 crossings. THis polynomials have been computed as described at the beginning of the chapter. The Khovanov homologies of the knots are taken from KnotInfo [**?**]. In fact, we noticed that no-ambiguous configurations (cf. Figure 2) that could not be taken care of appear in these knots and paired up all the hooks with the help of a computer program (developed by Francesco di Baldassarre, to whom many thanks are due).

Before listing (the quite long) the sequence of all Hilbert-Poicaré polynomials, let us fix the notations. The polynomial BN should be read as follows: the term $cu^{2k}q^it^j$, with $k > 0$, corresponds to a copy of the $\mathbb{Q}[U]$-module $\left(\mathbb{Q}[U]/(U^{2k})\right)^c$ in homological degree j and quantum degree i, and the term cq^it^j corresponds to a copy of $\mathbb{Q}[U]^c$ in homological degree j and quantum degree i.

The reader may notice that only U^2-torsion appears and that all the knots satisfy the hypothesis of Proposition 3.54. In particular, they are c-simple over $\mathbb{Q}$ (cf. Corollary 3.57 in Chapter 3).

Name	BN
3_1	$q^{-1}+q^{-3}+u^2\big(q^{-5}t^{-2}\big)$
4_1	$q^{1}+q^{-1}+u^2\big(q^{-1}t^{-1}+q^{5}t^{2}\big)$
5_1	$q^{-3}+q^{-5}+u^2\big(q^{-11}t^{-4}+q^{-7}t^{-2}\big)$
5_2	$q^{-1}+q^{-3}+u^2\big(q^{-9}t^{-4}+q^{-5}t^{-2}+q^{-3}t^{-1}\big)$
6_1	$q^{1}+q^{-1}+u^2\big(q^{-5}t^{-3}+q^{-1}t^{-1}+q^{1}+q^{5}t^{2}\big)$
6_2	$q^{-1}+q^{-3}+u^2\big(q^{-7}t^{-3}+q^{-5}t^{-2}+q^{-3}t^{-1}+q^{-1}+q^{3}t^{2}\big)$
6_3	$q^{1}+q^{-1}+u^2\big(q^{-3}t^{-2}+q^{-1}t^{-1}+q^{1}+q^{3}t^{1}+q^{5}t^{2}+q^{7}t^{3}\big)$
7_1	$q^{-5}+q^{-7}+u^2\big(q^{-17}t^{-6}+q^{-13}t^{-4}+q^{-9}t^{-2}\big)$
7_2	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+q^{-9}t^{-4}+q^{-7}t^{-3}+q^{-5}t^{-2}+q^{-3}t^{-1}\big)$
7_3	$q^{5}+q^{3}+u^2\big(q^{9}t^{2}+q^{11}t^{3}+q^{13}t^{4}+2q^{15}t^{5}+q^{19}t^{7}\big)$
7_4	$q^{3}+q^{1}+u^2\big(2q^{7}t^{2}+q^{9}t^{3}+q^{11}t^{4}+2q^{13}t^{5}+q^{17}t^{7}\big)$
7_5	$q^{-3}+q^{-5}+u^2\big(q^{-15}t^{-6}+q^{-13}t^{-5}+2q^{-11}t^{-4}+$ $+q^{-9}t^{-3}+2q^{-7}t^{-2}+q^{-5}t^{-1}\big)$

Name	BN
7_6	$q^{-1}+q^{-3}+u^2(q^{-9}t^{-4}+q^{-7}t^{-3}+2q^{-5}t^{-2}+$ $+2q^{-3}t^{-1}+q^{-1}+q^1t^1+q^3t^2)$
7_7	$q^1+q^{-1}+u^2(q^{-3}t^{-2}+2q^{-1}t^{-1}+q^1+2q^3t^1+$ $+2q^5t^2+q^7t^3+q^9t^4)$
8_1	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+q^{-5}t^{-3}+q^{-3}t^{-2}+q^{-1}t^{-1}+q^1+q^5t^2)$
8_2	$q^{-3}+q^{-5}+u^2(q^{-13}t^{-5}+q^{-11}t^{-4}+q^{-9}t^{-3}+$ $+2q^{-7}t^{-2}+q^{-5}t^{-1}+q^{-3}+q^1t^2)$
8_3	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+2q^{-1}t^{-1}+q^1+q^3t^1+2q^5t^2+q^9t^4)$
8_4	$q^{-1}+q^{-3}+u^2(q^{-7}t^{-3}+q^{-5}t^{-2}+2q^{-3}t^{-1}+$ $+q^{-1}+q^1t^1+2q^3t^2+q^7t^4)$
8_5	$q^5+q^3+u^2(q^3t^{-1}+2q^7t^1+q^9t^2+2q^{11}t^3+$ $+2q^{13}t^4+q^{15}t^5+q^{17}t^6)$
8_6	$q^{-1}+q^{-3}+u^2(q^{-11}t^{-5}+q^{-9}t^{-4}+2q^{-7}t^{-3}+$ $+2q^{-5}t^{-2}+2q^{-3}t^{-1}+2q^{-1}+q^3t^2)$
8_7	$q^3+q^1+u^2(q^{-1}t^{-2}+q^1t^{-1}+q^3+2q^5t^1+$ $+2q^7t^2+2q^9t^3+q^{11}t^4+q^{13}t^5)$

Name	BN
8_8	$q^1+q^{-1}+u^2\big(q^{-3}t^{-2}+q^{-1}t^{-1}+2q^1+2q^3t^1+$ $+2q^5t^2+2q^7t^3+q^9t^4+q^{11}t^5\big)$
8_9	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+q^{-3}t^{-2}+2q^{-1}t^{-1}+2q^1+$ $+2q^3t^1+2q^5t^2+q^7t^3+q^9t^4\big)$
8_{10}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+q^1t^{-1}+2q^3+2q^5t^1+$ $+2q^7t^2+3q^9t^3+q^{11}t^4+q^{13}t^5\big)$
8_{11}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+q^{-9}t^{-4}+2q^{-7}t^{-3}+$ $+3q^{-5}t^{-2}+2q^{-3}t^{-1}+2q^{-1}+q^1t^1+q^3t^2\big)$
8_{12}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+q^{-3}t^{-2}+3q^{-1}t^{-1}+2q^1+$ $+2q^3t^1+3q^5t^2+q^7t^3+q^9t^4\big)$
8_{13}	$q^1+q^{-1}+u^2\big(q^{-3}t^{-2}+2q^{-1}t^{-1}+2q^1+2q^3t^1+$ $+3q^5t^2+2q^7t^3+q^9t^4+q^{11}t^5\big)$
8_{14}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+2q^{-7}t^{-3}+$ $+3q^{-5}t^{-2}+3q^{-3}t^{-1}+2q^{-1}+q^1t^1+q^3t^2\big)$
8_{15}	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+2q^{-15}t^{-6}+2q^{-13}t^{-5}+$ $+4q^{-11}t^{-4}+2q^{-9}t^{-3}+3q^{-7}t^{-2}+2q^{-5}t^{-1}\big)$
8_{16}	$q^{-1}+q^{-3}+u^2\big(q^{-9}t^{-4}+2q^{-7}t^{-3}+3q^{-5}t^{-2}+$ $+3q^{-3}t^{-1}+3q^{-1}+2q^1t^1+2q^3t^2+q^5t^3\big)$

Name	BN
8_{17}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+2q^{-3}t^{-2}+3q^{-1}t^{-1}+$ $+3q^1+3q^3t^1+3q^5t^2+2q^7t^3+q^9t^4\big)$
8_{18}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+3q^{-3}t^{-2}+3q^{-1}t^{-1}+$ $+4q^1+4q^3t^1+3q^5t^2+3q^7t^3+q^9t^4\big)$
8_{19}	$q^7+q^5+u^2\big(q^{13}t^3+q^{15}+q^{17}t^5\big)$
8_{20}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+q^{-3}t^{-2}+q^{-1}t^{-1}+q^3t^1\big)$
8_{21}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+q^{-9}t^{-4}+q^{-7}t^{-3}+$ $+2q^{-5}t^{-2}+q^{-3}t^{-1}+q^{-1}\big)$
9_1	$q^{-7}+q^{-9}+u^2\big(q^{-23}t^{-8}+q^{-19}t^{-6}+q^{-15}t^{-4}+q^{-11}t^{-2}\big)$
9_2	$q^{-1}+q^{-3}+u^2\big(q^{-17}t^{-8}+q^{-13}t^{-6}+q^{-11}t^{-5}+$ $+q^{-9}t^{-4}+q^{-7}t^{-3}+q^{-5}t^{-2}+q^{-3}t^{-1}\big)$
9_3	$q^7+q^5+u^2\big(q^{11}t^2+q^{13}t^3+q^{15}t^4+2q^{17}t^5+$ $+q^{19}t^6+2q^{21}t^7+q^{25}t^9\big)$
9_4	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+2q^{-15}t^{-6}+q^{-13}t^{-5}+$ $+2q^{-11}t^{-4}+2q^{-9}t^{-3}+q^{-7}t^{-2}+q^{-5}t^{-1}\big)$

Name	BN
9_5	$q^3+q^1+u^2\big(2q^7t^2+q^9t^3+2q^{11}t^4+2q^{13}t^5+ + q^{15}t^6+2q^{17}t^7+q^{21}t^9\big)$
9_6	$q^{-5}+q^{-7}+u^2\big(q^{-21}t^{-8}+q^{-19}t^{-7}+2q^{-17}t^{-6}+ + 2q^{-15}t^{-5}+3q^{-13}t^{-4}+q^{-11}t^{-3}+2q^{-9}t^{-2}+q^{-7}t^{-1}\big)$
9_7	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+q^{-17}t^{-7}+2q^{-15}t^{-6}+ + 2q^{-13}t^{-5}+3q^{-11}t^{-4}+2q^{-9}t^{-3}+2q^{-7}t^{-2}+q^{-5}t^{-1}\big)$
9_8	$q^{-1}+q^{-3}+u^2\big(q^{-9}t^{-4}+q^{-7}t^{-3}+2q^{-5}t^{-2}+ + 3q^{-3}t^{-1}+2q^{-1}+2q^1t^1+2q^3t^2+q^5t^3+q^7t^4\big)$
9_9	$q^{-5}+q^{-7}+u^2\big(q^{-21}t^{-8}+q^{-19}t^{-7}+3q^{-17}t^{-6}+ + 2q^{-15}t^{-5}+3q^{-13}t^{-4}+2q^{-11}t^{-3}+2q^{-9}t^{-2}+q^{-7}t^{-1}\big)$
9_{10}	$q^5+q^3+u^2\big(2q^9t^2+2q^{11}t^3+3q^{13}t^4+3q^{15}t^5+ + 2q^{17}t^6+3q^{19}t^7+q^{23}t^9\big)$
9_{11}	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+q^7t^1+3q^9t^2+ + 3q^{11}t^3+2q^{13}t^4+3q^{15}t^5+q^{17}t^6+q^{19}t^7\big)$
9_{12}	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+q^{-11}t^{-5}+2q^{-9}t^{-4}+ + 3q^{-7}t^{-3}+3q^{-5}t^{-2}+3q^{-3}t^{-1}+2q^{-1}+q^1t^1+q^3t^2\big)$
9_{13}	$q^5+q^3+u^2\big(2q^9t^2+2q^{11}t^3+3q^{13}t^4+4q^{15}t^5+ + 2q^{17}t^6+3q^{19}t^7+q^{21}t^8+q^{23}t^9\big)$

Name	BN
9_{14}	$q^1+q^{-1}+u^2\big(q^{-3}t^{-2}+2q^{-1}t^{-1}+2q^1+3q^3t^1+$ $+3q^5t^2+3q^7t^3+2q^9t^4+q^{11}t^5+q^{13}t^6\big)$
9_{15}	$q^3+q^1+u^2\big(q^1t^{-1}+q^3+2q^5t^1+4q^7t^2+$ $+3q^9t^3+3q^{11}t^4+3q^{13}t^5+q^{15}t^6+q^{17}t^7\big)$
9_{16}	$q^7+q^5+u^2\big(q^{11}t^2+3q^{13}t^3+2q^{15}t^4+4q^{17}t^5+$ $+3q^{19}t^6+3q^{21}t^7+2q^{23}t^8+q^{25}t^9\big)$
9_{17}	$q^{-1}+q^{-3}+u^2\big(q^{-9}t^{-4}+2q^{-7}t^{-3}+2q^{-5}t^{-2}+$ $+4q^{-3}t^{-1}+3q^{-1}+2q^1t^1+3q^3t^2+q^5t^3+q^7t^4\big)$
9_{18}	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+q^{-17}t^{-7}+3q^{-15}t^{-6}+$ $+3q^{-13}t^{-5}+4q^{-11}t^{-4}+3q^{-9}t^{-3}+3q^{-7}t^{-2}+2q^{-5}t^{-1}\big)$
9_{19}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+2q^{-3}t^{-2}+$ $+4q^{-1}t^{-1}+3q^1+3q^3t^1+3q^5t^2+q^7t^3+q^9t^4\big)$
9_{20}	$q^{-3}+q^{-5}+u^2\big(q^{-15}t^{-6}+2q^{-13}t^{-5}+3q^{-11}t^{-4}+$ $+3q^{-9}t^{-3}+4q^{-7}t^{-2}+3q^{-5}t^{-1}+2q^{-3}+q^{-1}t^1+q^1t^2\big)$
9_{21}	$q^3+q^1+u^2\big(q^1t^{-1}+2q^3+2q^5t^1+4q^7t^2+$ $+4q^9t^3+3q^{11}t^4+3q^{13}t^5+q^{15}t^6+q^{17}t^7\big)$
9_{22}	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+3q^1t^{-1}+3q^3+$ $+3q^5t^1+4q^7t^2+3q^9t^3+2q^{11}t^4+q^{13}t^5\big)$

Name	BN
9_{23}	$q^{-3}+q^{-5}+u^2(q^{-19}t^{-8}+2q^{-17}t^{-7}+3q^{-15}t^{-6}+$ $+3q^{-13}t^{-5}+5q^{-11}t^{-4}+3q^{-9}t^{-3}+3q^{-7}t^{-2}+2q^{-5}t^{-1})$
9_{24}	$q^{1}+q^{-1}+u^2(q^{-7}t^{-4}+q^{-5}t^{-3}+3q^{-3}t^{-2}+4q^{-1}t^{-1}+$ $+3q^{1}+4q^{3}t^{1}+3q^{5}t^{2}+2q^{7}t^{3}+q^{9}t^{4})$
9_{25}	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+2q^{-11}t^{-5}+3q^{-9}t^{-4}+$ $+4q^{-7}t^{-3}+4q^{-5}t^{-2}+4q^{-3}t^{-1}+3q^{-1}+q^{1}t^{1}+q^{3}t^{2})$
9_{26}	$q^{3}+q^{1}+u^2(q^{-1}t^{-2}+2q^{1}t^{-1}+2q^{3}+4q^{5}t^{1}+$ $+4q^{7}t^{2}+4q^{9}t^{3}+3q^{11}t^{4}+2q^{13}t^{5}+q^{15}t^{6})$
9_{27}	$q^{1}+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+3q^{-3}t^{-2}+$ $+4q^{-1}t^{-1}+4q^{1}+4q^{3}t^{1}+3q^{5}t^{2}+2q^{7}t^{3}+q^{9}t^{4})$
9_{28}	$q^{-1}+q^{-3}+u^2(q^{-11}t^{-5}+2q^{-9}t^{-4}+3q^{-7}t^{-3}+$ $+5q^{-5}t^{-2}+4q^{-3}t^{-1}+4q^{-1}+3q^{1}t^{1}+2q^{3}t^{2}+q^{5}t^{3})$
9_{29}	$q^{-1}+q^{-3}+u^2(q^{-9}t^{-4}+2q^{-7}t^{-3}+4q^{-5}t^{-2}+$ $+4q^{-3}t^{-1}+4q^{-1}+4q^{1}t^{1}+3q^{3}t^{2}+2q^{5}t^{3}+q^{7}t^{4})$
9_{30}	$q^{1}+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+3q^{-3}t^{-2}+$ $+5q^{-1}t^{-1}+4q^{1}+4q^{3}t^{1}+4q^{5}t^{2}+2q^{7}t^{3}+q^{9}t^{4})$
9_{31}	$q^{-1}+q^{-3}+u^2(q^{-11}t^{-5}+3q^{-9}t^{-4}+3q^{-7}t^{-3}+$ $+5q^{-5}t^{-2}+5q^{-3}t^{-1}+4q^{-1}+3q^{1}t^{1}+2q^{3}t^{2}+q^{5}t^{3})$

Name	BN
9_{32}	$q^3 + q^1 + u^2\big(q^{-1}t^{-2} + 3q^1t^{-1} + 3q^3 + 5q^5t^1 + {}$ $+ 5q^7t^2 + 5q^9t^3 + 4q^{11}t^4 + 2q^{13}t^5 + q^{15}t^6\big)$
9_{33}	$q^1 + q^{-1} + u^2\big(q^{-7}t^{-4} + 2q^{-5}t^{-3} + 4q^{-3}t^{-2} + {}$ $+ 5q^{-1}t^{-1} + 5q^1 + 5q^3t^1 + 4q^5t^2 + 3q^7t^3 + q^9t^4\big)$
9_{34}	$q^1 + q^{-1} + u^2\big(q^{-7}t^{-4} + 3q^{-5}t^{-3} + 4q^{-3}t^{-2} + {}$ $+ 6q^{-1}t^{-1} + 6q^1 + 5q^3t^1 + 5q^5t^2 + 3q^7t^3 + q^9t^4\big)$
9_{35}	$q^{-1} + q^{-3} + u^2\big(q^{-17}t^{-8} + 3q^{-13}t^{-6} + q^{-11}t^{-5} + {}$ $+ 2q^{-9}t^{-4} + 3q^{-7}t^{-3} + q^{-5}t^{-2} + 2q^{-3}t^{-1}\big)$
9_{36}	$q^5 + q^3 + u^2\big(q^3t^{-1} + q^5 + 2q^7t^1 + 3q^9t^2 + {}$ $+ 3q^{11}t^3 + 3q^{13}t^4 + 3q^{15}t^5 + q^{17}t^6 + q^{19}t^7\big)$
9_{37}	$q^1 + q^{-1} + u^2\big(q^{-7}t^{-4} + 2q^{-5}t^{-3} + 2q^{-3}t^{-2} + {}$ $+ 5q^{-1}t^{-1} + 3q^1 + 3q^3t^1 + 4q^5t^2 + q^7t^3 + q^9t^4\big)$
9_{38}	$q^{-3} + q^{-5} + u^2\big(q^{-19}t^{-8} + 2q^{-17}t^{-7} + 4q^{-15}t^{-6} + {}$ $+ 4q^{-13}t^{-5} + 6q^{-11}t^{-4} + 4q^{-9}t^{-3} + 4q^{-7}t^{-2} + 3q^{-5}t^{-1}\big)$
9_{39}	$q^3 + q^1 + u^2\big(q^1t^{-1} + 2q^3 + 3q^5t^1 + 5q^7t^2 + {}$ $+ 5q^9t^3 + 4q^{11}t^4 + 4q^{13}t^5 + 2q^{15}t^6 + q^{17}t^7\big)$
9_{40}	$q^{-1} + q^{-3} + u^2\big(q^{-11}t^{-5} + 3q^{-9}t^{-4} + 5q^{-7}t^{-3} + {}$ $+ 6q^{-5}t^{-2} + 7q^{-3}t^{-1} + 6q^{-1} + 4q^1t^1 + 4q^3t^2 + q^5t^3\big)$

Name	BN
9_{41}	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+3q^{-5}t^{-3}+ + 4q^{-3}t^{-2}+4q^{-1}t^{-1}+4q^1+3q^3t^1+2q^5t^2+q^7t^3\big)$
9_{42}	$q^1+q^{-1}+u^2\big(q^{-3}t^{-3}+q^1t^{-1}+q^3+q^7t^2\big)$
9_{43}	$q^5+q^3+u^2\big(q^3t^{-1}+q^7t^1+q^9t^2+q^{11}t^3+q^{13}t^4+q^{15}t^5\big)$
9_{44}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+q^{-5}t^{-3}+q^{-3}t^{-2}+2q^{-1}t^{-1}+ + q^1+q^3t^1+q^5t^2\big)$
9_{45}	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+q^{-11}t^{-5}+2q^{-9}t^{-4}+ + 2q^{-7}t^{-3}+2q^{-5}t^{-2}+2q^{-3}t^{-1}+q^{-1}\big)$
9_{46}	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-5}t^{-3}+q^{-3}t^{-2}+q^1\big)$
9_{47}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+2q^1t^{-1}+q^3+3q^5t^1+ + 2q^7t^2+2q^9t^3+2q^{11}t^4\big)$
9_{48}	$q^3+q^1+u^2\big(q^1t^{-1}+2q^3+q^5t^1+3q^7t^2+ + 3q^9t^3+q^{11}t^4+2q^{13}t^5\big)$
9_{49}	$q^5+q^3+u^2\big(2q^9t^2+2q^{11}t^3+2q^{13}t^4+3q^{15}t^5+ + q^{17}t^6+2q^{19}t^7\big)$

Name	BN
10_1	$q^1+q^{-1}+u^2\big(q^{-13}t^{-7}+q^{-9}t^{-5}+q^{-7}t^{-4}+q^{-5}t^{-3}+$ $+q^{-3}t^{-2}+q^{-1}t^{-1}+q^1+q^5t^2\big)$
10_2	$q^{-5}+q^{-7}+u^2\big(q^{-19}t^{-7}+q^{-17}t^{-6}+q^{-15}t^{-5}+$ $+2q^{-13}t^{-4}+q^{-11}t^{-3}+2q^{-9}t^{-2}+q^{-7}t^{-1}+q^{-5}+q^{-1}t^2\big)$
10_3	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-5}t^{-3}+q^{-3}t^{-2}+2q^{-1}t^{-1}+$ $+2q^1+q^3t^1+2q^5t^2+q^9t^4\big)$
10_4	$q^3+q^1+u^2\big(q^{-7}t^{-5}+2q^{-3}t^{-3}+q^{-1}t^{-2}+2q^1t^{-1}+$ $+2q^3+q^5t^1+2q^7t^2+q^9t^3+q^{11}t^4\big)$
10_5	$q^5+q^3+u^2\big(q^1t^{-2}+q^3t^{-1}+q^5+2q^7t^1+$ $+2q^9t^2+3q^{11}t^3+2q^{13}t^4+2q^{15}t^5+q^{17}t^6+q^{19}t^7\big)$
10_6	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+q^{-15}t^{-6}+2q^{-13}t^{-5}+$ $+3q^{-11}t^{-4}+3q^{-9}t^{-3}+3q^{-7}t^{-2}+2q^{-5}t^{-1}+2q^{-3}+q^1t^2\big)$
10_7	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+q^{-13}t^{-6}+2q^{-11}t^{-5}+$ $+3q^{-9}t^{-4}+3q^{-7}t^{-3}+4q^{-5}t^{-2}+3q^{-3}t^{-1}+2q^{-1}+q^1t^1+q^3t^2\big)$
10_8	$q^{-3}+q^{-5}+u^2\big(q^{-13}t^{-5}+q^{-11}t^{-4}+2q^{-9}t^{-3}+$ $+2q^{-7}t^{-2}+2q^{-5}t^{-1}+2q^{-3}+q^{-1}t^1+2q^1t^2+q^5t^4\big)$
10_9	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+2q^1t^{-1}+2q^3+$ $+3q^5t^1+3q^7t^2+3q^9t^3+2q^{11}t^4+q^{13}t^5+q^{15}t^6\big)$

Name	BN
10_{10}	$q^1+q^{-1}+u^2\big(q^{-3}t^{-2}+2q^{-1}t^{-1}+2q^1+3q^3t^1+$ $+4q^5t^2+3q^7t^3+3q^9t^4+2q^{11}t^5+q^{13}t^6+q^{15}t^7\big)$
10_{11}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+q^{-9}t^{-4}+3q^{-7}t^{-3}+$ $+3q^{-5}t^{-2}+4q^{-3}t^{-1}+3q^{-1}+2q^1t^1+3q^3t^2+q^7t^4\big)$
10_{12}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+q^1t^{-1}+2q^3+3q^5t^1+$ $+4q^7t^2+4q^9t^3+3q^{11}t^4+3q^{13}t^5+q^{15}t^6+q^{17}t^7\big)$
10_{13}	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+3q^{-5}t^{-3}+3q^{-3}t^{-2}+$ $+5q^{-1}t^{-1}+4q^1+3q^3t^1+4q^5t^2+q^7t^3+q^9t^4\big)$
10_{14}	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+2q^{-15}t^{-6}+3q^{-13}t^{-5}+$ $+5q^{-11}t^{-4}+4q^{-9}t^{-3}+5q^{-7}t^{-2}+4q^{-5}t^{-1}+2q^{-3}+q^{-1}t^1+q^1t^2\big)$
10_{15}	$q^3+q^1+u^2\big(q^{-5}t^{-4}+q^{-3}t^{-3}+2q^{-1}t^{-2}+3q^1t^{-1}+$ $+3q^3+3q^5t^1+3q^7t^2+3q^9t^3+q^{11}t^4+q^{13}t^5\big)$
10_{16}	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+3q^1t^{-1}+2q^3+$ $+4q^5t^1+4q^7t^2+3q^9t^3+3q^{11}t^4+q^{13}t^5+q^{15}t^6\big)$
10_{17}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+q^{-5}t^{-3}+2q^{-3}t^{-2}+3q^{-1}t^{-1}+$ $+3q^1+3q^3t^1+3q^5t^2+2q^7t^3+q^9t^4+q^{11}t^5\big)$
10_{18}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+3q^{-7}t^{-3}+$ $+4q^{-5}t^{-2}+5q^{-3}t^{-1}+4q^{-1}+3q^1t^1+3q^3t^2+q^5t^3+q^7t^4\big)$

Name	BN
10_{19}	$q^{-1}+q^{-3}+u^2(q^{-9}t^{-4}+2q^{-7}t^{-3}+3q^{-5}t^{-2}+ +4q^{-3}t^{-1}+4q^{-1}+3q^1t^1+4q^3t^2+2q^5t^3+q^7t^4+q^9t^5)$
10_{20}	$q^{-1}+q^{-3}+u^2(q^{-15}t^{-7}+q^{-13}t^{-6}+2q^{-11}t^{-5}+ +2q^{-9}t^{-4}+3q^{-7}t^{-3}+3q^{-5}t^{-2}+2q^{-3}t^{-1}+2q^{-1}+q^3t^2)$
10_{21}	$q^{-3}+q^{-5}+u^2(q^{-17}t^{-7}+q^{-15}t^{-6}+2q^{-13}t^{-5}+ +4q^{-11}t^{-4}+3q^{-9}t^{-3}+4q^{-7}t^{-2}+3q^{-5}t^{-1}+2q^{-3}+q^{-1}t^1+q^1t^2)$
10_{22}	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+q^{-3}t^{-2}+3q^{-1}t^{-1}+3q^1+ +4q^3t^1+4q^5t^2+3q^7t^3+3q^9t^4+q^{11}t^5+q^{13}t^6)$
10_{23}	$q^3+q^1+u^2(q^{-1}t^{-2}+2q^1t^{-1}+3q^3+4q^5t^1+ +5q^7t^2+5q^9t^3+4q^{11}t^4+3q^{13}t^5+q^{15}t^6+q^{17}t^7)$
10_{24}	$q^{-1}+q^{-3}+u^2(q^{-15}t^{-7}+q^{-13}t^{-6}+3q^{-11}t^{-5}+ +4q^{-9}t^{-4}+4q^{-7}t^{-3}+5q^{-5}t^{-2}+4q^{-3}t^{-1}+3q^{-1}+q^1t^1+q^3t^2)$
10_{25}	$q^{-3}+q^{-5}+u^2(q^{-17}t^{-7}+2q^{-15}t^{-6}+4q^{-13}t^{-5}+ +5q^{-11}t^{-4}+5q^{-9}t^{-3}+6q^{-7}t^{-2}+4q^{-5}t^{-1}+3q^{-3}+q^{-1}t^1+q^1t^2)$
10_{26}	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+2q^{-3}t^{-2}+4q^{-1}t^{-1}+ +4q^1+5q^3t^1+5q^5t^2+4q^7t^3+3q^9t^4+q^{11}t^5+q^{13}t^6)$
10_{27}	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+2q^{-11}t^{-5}+4q^{-9}t^{-4}+ +5q^{-7}t^{-3}+6q^{-5}t^{-2}+6q^{-3}t^{-1}+5q^{-1}+3q^1t^1+2q^3t^2+q^5t^3)$

Name	BN
10_{28}	$q^1 + q^{-1} + u^2\big(q^{-3}t^{-2} + 2q^{-1}t^{-1} + 3q^1 + 3q^3t^1 + + 5q^5t^2 + 4q^7t^3 + 3q^9t^4 + 3q^{11}t^5 + q^{13}t^6 + q^{15}t^7\big)$
10_{29}	$q^{-1} + q^{-3} + u^2\big(q^{-11}t^{-5} + 2q^{-9}t^{-4} + 4q^{-7}t^{-3} + + 4q^{-5}t^{-2} + 6q^{-3}t^{-1} + 5q^{-1} + 3q^1t^1 + 4q^3t^2 + q^5t^3 + q^7t^4\big)$
10_{30}	$q^{-1} + q^{-3} + u^2\big(q^{-15}t^{-7} + 2q^{-13}t^{-6} + 3q^{-11}t^{-5} + + 5q^{-9}t^{-4} + 5q^{-7}t^{-3} + 6q^{-5}t^{-2} + 5q^{-3}t^{-1} + 3q^{-1} + 2q^1t^1 + q^3t^2\big)$
10_{31}	$q^1 + q^{-1} + u^2\big(q^{-7}t^{-4} + 2q^{-5}t^{-3} + 3q^{-3}t^{-2} + + 4q^{-1}t^{-1} + 5q^1 + 4q^3t^1 + 4q^5t^2 + 3q^7t^3 + q^9t^4 + q^{11}t^5\big)$
10_{32}	$q^1 + q^{-1} + u^2\big(q^{-9}t^{-5} + 2q^{-7}t^{-4} + 3q^{-5}t^{-3} + + 5q^{-3}t^{-2} + 6q^{-1}t^{-1} + 5q^1 + 5q^3t^1 + 4q^5t^2 + 2q^7t^3 + q^9t^4\big)$
10_{33}	$q^1 + q^{-1} + u^2\big(q^{-7}t^{-4} + 2q^{-5}t^{-3} + 3q^{-3}t^{-2} + + 5q^{-1}t^{-1} + 5q^1 + 5q^3t^1 + 5q^5t^2 + 3q^7t^3 + 2q^9t^4 + q^{11}t^5\big)$
10_{34}	$q^1 + q^{-1} + u^2\big(q^{-3}t^{-2} + q^{-1}t^{-1} + 2q^1 + 2q^3t^1 + + 3q^5t^2 + 3q^7t^3 + 2q^9t^4 + 2q^{11}t^5 + q^{13}t^6 + q^{15}t^7\big)$
10_{35}	$q^1 + q^{-1} + u^2\big(q^{-5}t^{-3} + q^{-3}t^{-2} + 3q^{-1}t^{-1} + 3q^1 + + 4q^3t^1 + 4q^5t^2 + 3q^7t^3 + 3q^9t^4 + q^{11}t^5 + q^{13}t^6\big)$
10_{36}	$q^{-1} + q^{-3} + u^2\big(q^{-15}t^{-7} + 2q^{-13}t^{-6} + 2q^{-11}t^{-5} + + 4q^{-9}t^{-4} + 4q^{-7}t^{-3} + 4q^{-5}t^{-2} + 4q^{-3}t^{-1} + 2q^{-1} + q^1t^1 + q^3t^2\big)$

Name	BN
10_{37}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+q^{-5}t^{-3}+3q^{-3}t^{-2}+4q^{-1}t^{-1}+ +4q^1+4q^3t^1+4q^5t^2+3q^7t^3+q^9t^4+q^{11}t^5\big)$
10_{38}	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+3q^{-11}t^{-5}+ +4q^{-9}t^{-4}+5q^{-7}t^{-3}+5q^{-5}t^{-2}+4q^{-3}t^{-1}+3q^{-1}+q^1t^1+q^3t^2\big)$
10_{39}	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+2q^{-15}t^{-6}+3q^{-13}t^{-5}+ +5q^{-11}t^{-4}+5q^{-9}t^{-3}+5q^{-7}t^{-2}+4q^{-5}t^{-1}+3q^{-3}+q^{-1}t^1+q^1t^2\big)$
10_{40}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+2q^1t^{-1}+4q^3+5q^5t^1+ +6q^7t^2+7q^9t^3+5q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
10_{41}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+4q^{-7}t^{-3}+ +5q^{-5}t^{-2}+6q^{-3}t^{-1}+6q^{-1}+4q^1t^1+4q^3t^2+2q^5t^3+q^7t^4\big)$
10_{42}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+4q^{-3}t^{-2}+ +6q^{-1}t^{-1}+7q^1+6q^3t^1+6q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5\big)$
10_{43}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+4q^{-3}t^{-2}+ +5q^{-1}t^{-1}+6q^1+6q^3t^1+5q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5\big)$
10_{44}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+3q^{-9}t^{-4}+4q^{-7}t^{-3}+ +6q^{-5}t^{-2}+7q^{-3}t^{-1}+6q^{-1}+5q^1t^1+4q^3t^2+2q^5t^3+q^7t^4\big)$
10_{45}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+4q^{-3}t^{-2}+ +7q^{-1}t^{-1}+7q^1+7q^3t^1+7q^5t^2+4q^7t^3+3q^9t^4+q^{11}t^5\big)$

Name	BN
10_{46}	$q^7+q^5+u^2\big(q^5t^{-1}+2q^9t^1+q^{11}t^2+3q^{13}t^3+$ $+2q^{15}t^4+2q^{17}t^5+2q^{19}t^6+q^{21}t^7+q^{23}t^8\big)$
10_{47}	$q^5+q^3+u^2\big(q^1t^{-2}+q^3t^{-1}+2q^5+2q^7t^1+$ $+3q^9t^2+4q^{11}t^3+2q^{13}t^4+3q^{15}t^5+q^{17}t^6+q^{19}t^7\big)$
10_{48}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+q^{-5}t^{-3}+3q^{-3}t^{-2}+3q^{-1}t^{-1}+$ $+4q^1+4q^3t^1+3q^5t^2+3q^7t^3+q^9t^4+q^{11}t^5\big)$
10_{49}	$q^{-5}+q^{-7}+u^2\big(q^{-23}t^{-9}+2q^{-21}t^{-8}+3q^{-19}t^{-7}+$ $+5q^{-17}t^{-6}+4q^{-15}t^{-5}+6q^{-13}t^{-4}+3q^{-11}t^{-3}+3q^{-9}t^{-2}+2q^{-7}t^{-1}\big)$
10_{50}	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+3q^7t^1+3q^9t^2+$ $+5q^{11}t^3+4q^{13}t^4+4q^{15}t^5+3q^{17}t^6+q^{19}t^7+q^{21}t^8\big)$
10_{51}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+2q^1t^{-1}+4q^3+4q^5t^1+$ $+6q^7t^2+6q^9t^3+4q^{11}t^4+4q^{13}t^5+q^{15}t^6+q^{17}t^7\big)$
10_{52}	$q^3+q^1+u^2\big(q^{-5}t^{-4}+q^{-3}t^{-3}+3q^{-1}t^{-2}+4q^1t^{-1}+$ $+4q^3+5q^5t^1+4q^7t^2+4q^9t^3+2q^{11}t^4+q^{13}t^5\big)$
10_{53}	$q^{-3}+q^{-5}+u^2\big(q^{-21}t^{-9}+2q^{-19}t^{-8}+3q^{-17}t^{-7}+$ $+6q^{-15}t^{-6}+5q^{-13}t^{-5}+7q^{-11}t^{-4}+5q^{-9}t^{-3}+4q^{-7}t^{-2}+3q^{-5}t^{-1}\big)$
10_{54}	$q^3+q^1+u^2\big(q^{-5}t^{-4}+q^{-3}t^{-3}+3q^{-1}t^{-2}+3q^1t^{-1}+$ $+3q^3+4q^5t^1+3q^7t^2+3q^9t^3+q^{11}t^4+q^{13}t^5\big)$

Name	BN
10_{55}	$q^{-3}+q^{-5}+u^2\big(q^{-21}t^{-9}+2q^{-19}t^{-8}+3q^{-17}t^{-7}+$ $+5q^{-15}t^{-6}+4q^{-13}t^{-5}+6q^{-11}t^{-4}+4q^{-9}t^{-3}+3q^{-7}t^{-2}+2q^{-5}t^{-1}\big)$
10_{56}	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+3q^7t^1+4q^9t^2+$ $+6q^{11}t^3+5q^{13}t^4+5q^{15}t^5+4q^{17}t^6+2q^{19}t^7+q^{21}t^8\big)$
10_{57}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+2q^1t^{-1}+4q^3+5q^5t^1+$ $+7q^7t^2+7q^9t^3+5q^{11}t^4+5q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
10_{58}	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+4q^{-3}t^{-2}+6q^{-1}t^{-1}+5q^1+4q^3t^1+4q^5t^2+q^7t^3+q^9t^4\big)$
10_{59}	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+5q^3+$ $+6q^5t^1+6q^7t^2+6q^9t^3+4q^{11}t^4+2q^{13}t^5+q^{15}t^6\big)$
10_{60}	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+6q^{-3}t^{-2}+7q^{-1}t^{-1}+7q^1+6q^3t^1+5q^5t^2+3q^7t^3+q^9t^4\big)$
10_{61}	$q^5+q^3+u^2\big(q^{-1}t^{-3}+3q^3t^{-1}+q^5+2q^7t^1+$ $+3q^9t^2+2q^{11}t^3+2q^{13}t^4+q^{15}t^5+q^{17}t^6\big)$
10_{62}	$q^5+q^3+u^2\big(q^1t^{-2}+q^3t^{-1}+2q^5+3q^7t^1+$ $+3q^9t^2+4q^{11}t^3+3q^{13}t^4+3q^{15}t^5+q^{17}t^6+q^{19}t^7\big)$
10_{63}	$q^{-3}+q^{-5}+u^2\big(q^{-21}t^{-9}+2q^{-19}t^{-8}+2q^{-17}t^{-7}+$ $+5q^{-15}t^{-6}+4q^{-13}t^{-5}+5q^{-11}t^{-4}+4q^{-9}t^{-3}+3q^{-7}t^{-2}+2q^{-5}t^{-1}\big)$

Name	BN
10_{64}	$q^3+q^1+u^2(q^{-3}t^{-3}+q^{-1}t^{-2}+3q^1t^{-1}+3q^3+$ $+4q^5t^1+4q^7t^2+4q^9t^3+3q^{11}t^4+q^{13}t^5+q^{15}t^6)$
10_{65}	$q^3+q^1+u^2(q^{-1}t^{-2}+2q^1t^{-1}+3q^3+4q^5t^1+$ $+6q^7t^2+5q^9t^3+4q^{11}t^4+4q^{13}t^5+q^{15}t^6+q^{17}t^7)$
10_{66}	$q^{-5}+q^{-7}+u^2(q^{-23}t^{-9}+3q^{-21}t^{-8}+4q^{-19}t^{-7}+$ $+6q^{-17}t^{-6}+6q^{-15}t^{-5}+7q^{-13}t^{-4}+4q^{-11}t^{-3}+4q^{-9}t^{-2}+2q^{-7}t^{-1})$
10_{67}	$q^{-1}+q^{-3}+u^2(q^{-15}t^{-7}+2q^{-13}t^{-6}+3q^{-11}t^{-5}+$ $+5q^{-9}t^{-4}+5q^{-7}t^{-3}+5q^{-5}t^{-2}+5q^{-3}t^{-1}+3q^{-1}+q^1t^1+q^3t^2)$
10_{68}	$q^1+q^{-1}+u^2(q^{-11}t^{-6}+q^{-9}t^{-5}+3q^{-7}t^{-4}+$ $+4q^{-5}t^{-3}+4q^{-3}t^{-2}+5q^{-1}t^{-1}+4q^1+3q^3t^1+2q^5t^2+q^7t^3)$
10_{69}	$q^3+q^1+u^2(q^{-1}t^{-2}+3q^1t^{-1}+4q^3+6q^5t^1+$ $+8q^7t^2+7q^9t^3+6q^{11}t^4+5q^{13}t^5+2q^{15}t^6+q^{17}t^7)$
10_{70}	$q^3+q^1+u^2(q^{-3}t^{-3}+q^{-1}t^{-2}+4q^1t^{-1}+4q^3+$ $+5q^5t^1+6q^7t^2+5q^9t^3+4q^{11}t^4+2q^{13}t^5+q^{15}t^6)$
10_{71}	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+6q^{-1}t^{-1}+6q^1+6q^3t^1+6q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5)$
10_{72}	$q^5+q^3+u^2(q^3t^{-1}+q^5+3q^7t^1+5q^9t^2+$ $+6q^{11}t^3+6q^{13}t^4+6q^{15}t^5+4q^{17}t^6+3q^{19}t^7+q^{21}t^8)$

Name	BN
10_{73}	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+4q^{-9}t^{-4}+$ $+6q^{-7}t^{-3}+7q^{-5}t^{-2}+7q^{-3}t^{-1}+6q^{-1}+4q^{1}t^{1}+3q^{3}t^{2}+q^{5}t^{3}\big)$
10_{74}	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+q^{-13}t^{-6}+3q^{-11}t^{-5}+$ $+5q^{-9}t^{-4}+4q^{-7}t^{-3}+6q^{-5}t^{-2}+5q^{-3}t^{-1}+3q^{-1}+2q^{1}t^{1}+q^{3}t^{2}\big)$
10_{75}	$q^{1}+q^{-1}+u^2\big(q^{-5}t^{-3}+3q^{-3}t^{-2}+4q^{-1}t^{-1}+$ $+6q^{1}+7q^{3}t^{1}+6q^{5}t^{2}+6q^{7}t^{3}+4q^{9}t^{4}+2q^{11}t^{5}+q^{13}t^{6}\big)$
10_{76}	$q^{5}+q^{3}+u^2\big(q^{3}t^{-1}+3q^{7}t^{1}+3q^{9}t^{2}+5q^{11}t^{3}+$ $+5q^{13}t^{4}+4q^{15}t^{5}+4q^{17}t^{6}+2q^{19}t^{7}+q^{21}t^{8}\big)$
10_{77}	$q^{3}+q^{1}+u^2\big(q^{-1}t^{-2}+q^{1}t^{-1}+3q^{3}+4q^{5}t^{1}+$ $+5q^{7}t^{2}+6q^{9}t^{3}+4q^{11}t^{4}+4q^{13}t^{5}+2q^{15}t^{6}+q^{17}t^{7}\big)$
10_{78}	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+2q^{-15}t^{-6}+3q^{-13}t^{-5}+$ $+6q^{-11}t^{-4}+5q^{-9}t^{-3}+6q^{-7}t^{-2}+5q^{-5}t^{-1}+3q^{-3}+2q^{-1}t^{1}+q^{1}t^{2}\big)$
10_{79}	$q^{1}+q^{-1}+u^2\big(q^{-7}t^{-4}+q^{-5}t^{-3}+4q^{-3}t^{-2}+4q^{-1}t^{-1}+$ $+5q^{1}+5q^{3}t^{1}+4q^{5}t^{2}+4q^{7}t^{3}+q^{9}t^{4}+q^{11}t^{5}\big)$
10_{80}	$q^{-5}+q^{-7}+u^2\big(q^{-23}t^{-9}+2q^{-21}t^{-8}+4q^{-19}t^{-7}+$ $+6q^{-17}t^{-6}+5q^{-15}t^{-5}+7q^{-13}t^{-4}+4q^{-11}t^{-3}+4q^{-9}t^{-2}+2q^{-7}t^{-1}\big)$
10_{81}	$q^{1}+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+6q^{-1}t^{-1}+7q^{1}+7q^{3}t^{1}+6q^{5}t^{2}+5q^{7}t^{3}+2q^{9}t^{4}+q^{11}t^{5}\big)$

Name	BN
10_{82}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+3q^{-7}t^{-3}+$ $+5q^{-5}t^{-2}+5q^{-3}t^{-1}+5q^{-1}+4q^1t^1+3q^3t^2+2q^5t^3+q^7t^4\big)$
10_{83}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+3q^1t^{-1}+4q^3+6q^5t^1+$ $+7q^7t^2+7q^9t^3+6q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
10_{84}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+2q^1t^{-1}+4q^3+6q^5t^1+$ $+7q^7t^2+8q^9t^3+6q^{11}t^4+5q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$
10_{85}	$q^{-3}+q^{-5}+u^2\big(q^{-15}t^{-6}+2q^{-13}t^{-5}+3q^{-11}t^{-4}+$ $+4q^{-9}t^{-3}+5q^{-7}t^{-2}+4q^{-5}t^{-1}+4q^{-3}+2q^{-1}t^1+2q^1t^2+q^3t^3\big)$
10_{86}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+3q^{-3}t^{-2}+5q^{-1}t^{-1}+$ $+6q^1+7q^3t^1+7q^5t^2+6q^7t^3+4q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
10_{87}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+2q^{-3}t^{-2}+4q^{-1}t^{-1}+$ $+6q^1+6q^3t^1+7q^5t^2+6q^7t^3+4q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
10_{88}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+8q^{-1}t^{-1}+8q^1+8q^3t^1+8q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5\big)$
10_{89}	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+7q^{-7}t^{-3}+8q^{-5}t^{-2}+9q^{-3}t^{-1}+7q^{-1}+5q^1t^1+4q^3t^2+q^5t^3\big)$
10_{90}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+2q^{-3}t^{-2}+5q^{-1}t^{-1}+$ $+5q^1+6q^3t^1+7q^5t^2+5q^7t^3+4q^9t^4+2q^{11}t^5+q^{13}t^6\big)$

Name	BN
10_{91}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+5q^{-1}t^{-1}+6q^1+6q^3t^1+5q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5\big)$
10_{92}	$q^5+q^3+u^2\big(q^3t^{-1}+2q^5+4q^7t^1+6q^9t^2+$ $+8q^{11}t^3+7q^{13}t^4+7q^{15}t^5+5q^{17}t^6+3q^{19}t^7+q^{21}t^8\big)$
10_{93}	$q^{-1}+q^{-3}+u^2\big(q^{-9}t^{-4}+2q^{-7}t^{-3}+4q^{-5}t^{-2}+$ $+5q^{-3}t^{-1}+5q^{-1}+5q^1t^1+5q^3t^2+3q^5t^3+2q^7t^4+q^9t^5\big)$
10_{94}	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+4q^3+$ $+6q^5t^1+6q^7t^2+5q^9t^3+4q^{11}t^4+2q^{13}t^5+q^{15}t^6\big)$
10_{95}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+3q^1t^{-1}+5q^3+6q^5t^1+$ $+8q^7t^2+8q^9t^3+6q^{11}t^4+5q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
10_{96}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+3q^{-3}t^{-2}+6q^{-1}t^{-1}+$ $+6q^1+8q^3t^1+8q^5t^2+6q^7t^3+5q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
10_{97}	$q^3+q^1+u^2\big(q^1t^{-1}+2q^3+4q^5t^1+7q^7t^2+$ $+7q^9t^3+7q^{11}t^4+7q^{13}t^5+4q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
10_{98}	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+2q^{-15}t^{-6}+5q^{-13}t^{-5}+$ $+6q^{-11}t^{-4}+6q^{-9}t^{-3}+8q^{-7}t^{-2}+5q^{-5}t^{-1}+4q^{-3}+2q^{-1}t^1+q^1t^2\big)$
10_{99}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+5q^{-1}t^{-1}+7q^1+7q^3t^1+5q^5t^2+5q^7t^3+2q^9t^4+q^{11}t^5\big)$

Name	BN
10_{100}	$q^{-3}+q^{-5}+u^2\big(q^{-15}t^{-6}+2q^{-13}t^{-5}+4q^{-11}t^{-4}+$ $+4q^{-9}t^{-3}+6q^{-7}t^{-2}+5q^{-5}t^{-1}+4q^{-3}+3q^{-1}t^1+2q^1t^2+q^3t^3\big)$
10_{101}	$q^5+q^3+u^2\big(3q^9t^2+4q^{11}t^3+6q^{13}t^4+8q^{15}t^5+$ $+6q^{17}t^6+7q^{19}t^7+4q^{21}t^8+3q^{23}t^9+q^{25}t^{10}\big)$
10_{102}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+2q^{-3}t^{-2}+4q^{-1}t^{-1}+$ $+5q^1+6q^3t^1+6q^5t^2+5q^7t^3+4q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
10_{103}	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+4q^{-9}t^{-4}+$ $+5q^{-7}t^{-3}+7q^{-5}t^{-2}+6q^{-3}t^{-1}+5q^{-1}+4q^1t^1+2q^3t^2+q^5t^3\big)$
10_{104}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+6q^{-1}t^{-1}+6q^1+6q^3t^1+6q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5\big)$
10_{105}	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+6q^3+$ $+7q^5t^1+8q^7t^2+7q^9t^3+5q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
10_{106}	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+5q^3+$ $+6q^5t^1+6q^7t^2+6q^9t^3+4q^{11}t^4+2q^{13}t^5+q^{15}t^6\big)$
10_{107}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+7q^{-1}t^{-1}+8q^1+7q^3t^1+7q^5t^2+5q^7t^3+2q^9t^4+q^{11}t^5\big)$
10_{108}	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+3q^{-1}t^{-2}+5q^1t^{-1}+$ $+4q^3+5q^5t^1+5q^7t^2+3q^9t^3+2q^{11}t^4+q^{13}t^5\big)$

Name	BN
10_{109}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+6q^{-1}t^{-1}+7q^1+7q^3t^1+6q^5t^2+5q^7t^3+2q^9t^4+q^{11}t^5\big)$
10_{110}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+6q^{-5}t^{-2}+7q^{-3}t^{-1}+7q^{-1}+5q^1t^1+5q^3t^2+2q^5t^3+q^7t^4\big)$
10_{111}	$q^5+q^3+u^2\big(q^3t^{-1}+2q^5+4q^7t^1+5q^9t^2+$ $+7q^{11}t^3+6q^{13}t^4+6q^{15}t^5+4q^{17}t^6+2q^{19}t^7+q^{21}t^8\big)$
10_{112}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+3q^{-9}t^{-4}+4q^{-7}t^{-3}+$ $+7q^{-5}t^{-2}+7q^{-3}t^{-1}+7q^{-1}+6q^1t^1+4q^3t^2+3q^5t^3+q^7t^4\big)$
10_{113}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+3q^1t^{-1}+5q^3+8q^5t^1+$ $+9q^7t^2+10q^9t^3+8q^{11}t^4+6q^{13}t^5+4q^{15}t^6+q^{17}t^7\big)$
10_{114}	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+3q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+7q^{-3}t^{-2}+8q^{-1}t^{-1}+7q^1+7q^3t^1+5q^5t^2+3q^7t^3+q^9t^4\big)$
10_{115}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+8q^{-1}t^{-1}+9q^1+9q^3t^1+8q^5t^2+6q^7t^3+3q^9t^4+q^{11}t^5\big)$
10_{116}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+3q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+7q^{-5}t^{-2}+8q^{-3}t^{-1}+8q^{-1}+6q^1t^1+5q^3t^2+3q^5t^3+q^7t^4\big)$
10_{117}	$q^3+q^1+u^2\big(q^{-1}t^{-2}+3q^1t^{-1}+5q^3+7q^5t^1+$ $+9q^7t^2+9q^9t^3+7q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$

Name	BN
10_{118}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+7q^{-1}t^{-1}+8q^1+8q^3t^1+7q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5\big)$
10_{119}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+3q^{-3}t^{-2}+6q^{-1}t^{-1}+$ $+7q^1+8q^3t^1+9q^5t^2+7q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
10_{120}	$q^{-3}+q^{-5}+u^2\big(q^{-21}t^{-9}+3q^{-19}t^{-8}+5q^{-17}t^{-7}+$ $+8q^{-15}t^{-6}+8q^{-13}t^{-5}+10q^{-11}t^{-4}+7q^{-9}t^{-3}+6q^{-7}t^{-2}+4q^{-5}t^{-1}\big)$
10_{121}	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+3q^{-11}t^{-5}+6q^{-9}t^{-4}+$ $+8q^{-7}t^{-3}+10q^{-5}t^{-2}+10q^{-3}t^{-1}+8q^{-1}+6q^1t^1+4q^3t^2+q^5t^3\big)$
10_{122}	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+3q^{-3}t^{-2}+5q^{-1}t^{-1}+$ $+8q^1+8q^3t^1+9q^5t^2+8q^7t^3+5q^9t^4+4q^{11}t^5+q^{13}t^6\big)$
10_{123}	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+4q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+9q^{-1}t^{-1}+10q^1+10q^3t^1+9q^5t^2+6q^7t^3+4q^9t^4+q^{11}t^5\big)$
10_{124}	$q^9+q^7+u^2\big(q^{15}t^3+q^{17}+q^{19}t^5+q^{21}t^7\big)$
10_{125}	$q^3+q^1+u^2\big(q^{-5}t^{-4}+q^{-1}t^{-2}+q^1t^{-1}+q^5t^1+q^9t^3\big)$
10_{126}	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-9}t^{-4}+q^{-7}t^{-3}+$ $+2q^{-5}t^{-2}+2q^{-3}t^{-1}+q^1t^1\big)$

Name	BN
10_{127}	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+q^{-15}t^{-6}+2q^{-13}t^{-5}+$ $+3q^{-11}t^{-4}+2q^{-9}t^{-3}+3q^{-7}t^{-2}+q^{-5}t^{-1}+q^{-3}\big)$
10_{128}	$q^{7}+q^{5}+u^2\big(q^{11}t^{2}+q^{13}t^{3}+q^{15}t^{4}+q^{15}+2q^{17}t^{5}+q^{21}t^{7}\big)$
10_{129}	$q^{1}+q^{-1}+u^2\big(q^{-7}t^{-4}+q^{-5}t^{-3}+2q^{-3}t^{-2}+2q^{-1}t^{-1}+$ $+2q^{1}+2q^{3}t^{1}+q^{5}t^{2}+q^{7}t^{3}\big)$
10_{130}	$q^{1}+q^{-1}+u^2\big(q^{-11}t^{-6}+2q^{-7}t^{-4}+q^{-5}t^{-3}+$ $+q^{-3}t^{-2}+2q^{-1}t^{-1}+q^{3}t^{1}\big)$
10_{131}	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+q^{-13}t^{-6}+2q^{-11}t^{-5}+$ $+3q^{-9}t^{-4}+2q^{-7}t^{-3}+3q^{-5}t^{-2}+2q^{-3}t^{-1}+q^{-1}\big)$
10_{132}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-6}+q^{-7}t^{-4}+q^{-5}t^{-3}+q^{-5}t^{-2}+q^{-1}t^{-1}\big)$
10_{133}	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+q^{-13}t^{-6}+q^{-11}t^{-5}+$ $+2q^{-9}t^{-4}+q^{-7}t^{-3}+2q^{-5}t^{-2}+q^{-3}t^{-1}\big)$
10_{134}	$q^{7}+q^{5}+u^2\big(q^{11}t^{2}+2q^{13}t^{3}+q^{15}t^{4}+3q^{17}t^{5}+$ $+q^{19}t^{6}+2q^{21}t^{7}+q^{23}t^{8}\big)$
10_{135}	$q^{1}+q^{-1}+u^2\big(q^{-7}t^{-4}+q^{-5}t^{-3}+3q^{-3}t^{-2}+3q^{-1}t^{-1}+$ $+3q^{1}+3q^{3}t^{1}+2q^{5}t^{2}+2q^{7}t^{3}\big)$

Name	BN
10_{136}	$q^1+q^{-1}+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+q^1t^{-1}+2q^3+$ $+q^5t^1+q^7t^2+q^9t^3\big)$
10_{137}	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+2q^{-5}t^{-3}+2q^{-3}t^{-2}+$ $+2q^{-1}t^{-1}+2q^1+q^3t^1+q^5t^2\big)$
10_{138}	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+3q^1t^{-1}+2q^3+$ $+3q^5t^1+3q^7t^2+2q^9t^3+2q^{11}t^4\big)$
10_{139}	$q^9+q^7+u^2\big(q^{15}t^3+q^{17}+q^{19}t^5+q^{19}t^6+q^{21}t^7+q^{25}t^9\big)$
10_{140}	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+q^{-7}t^{-4}+q^{-5}t^{-3}+q^{-1}t^{-1}\big)$
10_{141}	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+q^{-5}t^{-3}+2q^{-3}t^{-2}+$ $+2q^{-1}t^{-1}+q^1+q^3t^1+q^5t^2\big)$
10_{142}	$q^7+q^5+u^2\big(q^{11}t^2+q^{13}t^3+q^{15}t^4+2q^{17}t^5+2q^{21}t^7\big)$
10_{143}	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+q^{-11}t^{-5}+2q^{-9}t^{-4}+$ $+2q^{-7}t^{-3}+3q^{-5}t^{-2}+2q^{-3}t^{-1}+q^{-1}+q^1t^1\big)$
10_{144}	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+3q^{-7}t^{-3}+$ $+3q^{-5}t^{-2}+4q^{-3}t^{-1}+3q^{-1}+q^1t^1+2q^3t^2\big)$

Name	BN
10_{145}	$q^{-3}+q^{-5}+u^2(q^{-17}t^{-8}+q^{-13}t^{-6}+q^{-11}t^{-5}+ +q^{-11}t^{-4}+q^{-7}t^{-3}+q^{-7}t^{-2})$
10_{146}	$q^{1}+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+2q^{-3}t^{-2}+ +3q^{-1}t^{-1}+3q^{1}+2q^{3}t^{1}+2q^{5}t^{2}+q^{7}t^{3})$
10_{147}	$q^{3}+q^{1}+u^2(q^{-3}t^{-3}+q^{-1}t^{-2}+2q^{1}t^{-1}+2q^{3}+ +2q^{5}t^{1}+2q^{7}t^{2}+2q^{9}t^{3}+q^{11}t^{4})$
10_{148}	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+q^{-11}t^{-5}+3q^{-9}t^{-4}+ +2q^{-7}t^{-3}+3q^{-5}t^{-2}+3q^{-3}t^{-1}+q^{-1}+q^{1}t^{1})$
10_{149}	$q^{-3}+q^{-5}+u^2(q^{-17}t^{-7}+2q^{-15}t^{-6}+3q^{-13}t^{-5}+ +4q^{-11}t^{-4}+3q^{-9}t^{-3}+4q^{-7}t^{-2}+2q^{-5}t^{-1}+q^{-3})$
10_{150}	$q^{5}+q^{3}+u^2(q^{3}t^{-1}+q^{5}+2q^{7}t^{1}+2q^{9}t^{2}+ +3q^{11}t^{3}+2q^{13}t^{4}+2q^{15}t^{5}+q^{17}t^{6})$
10_{151}	$q^{3}+q^{1}+u^2(q^{-1}t^{-2}+2q^{1}t^{-1}+3q^{3}+3q^{5}t^{1}+ +4q^{7}t^{2}+4q^{9}t^{3}+2q^{11}t^{4}+2q^{13}t^{5})$
10_{152}	$q^{-7}+q^{-9}+u^2(q^{-23}t^{-9}+q^{-21}t^{-8}+q^{-19}t^{-7}+ +2q^{-17}t^{-6}+q^{-15}t^{-5}+q^{-15}+2q^{-13}t^{-4}+q^{-11}t^{-2})$
10_{153}	$q^{1}+q^{-1}+u^2(q^{-7}t^{-4}+q^{-3}t^{-2}+q^{-1}t^{-1}+q^{-1}+ +q^{3}t^{1}+q^{3}t^{2}+q^{5}t^{3}+q^{9}t^{5})$

Name	BN
10_{154}	$q^7+q^5+u^2(q^{13}t^3+q^{13}t^4+2q^{15}+q^{17}t^5+$ $+q^{17}t^6+2q^{19}t^7+q^{21}t^8+q^{23}t^9+q^{25}t^{10})$
10_{155}	$q^1+q^{-1}+u^2(q^{-1}t^{-1}+q^1+2q^3t^1+2q^5t^2+$ $+2q^7t^3+2q^9t^4+q^{11}t^5+q^{13}t^6)$
10_{156}	$q^{-1}+q^{-3}+u^2(q^{-9}t^{-4}+2q^{-7}t^{-3}+3q^{-5}t^{-2}+$ $+3q^{-3}t^{-1}+3q^{-1}+2q^1t^1+2q^3t^2+q^5t^3)$
10_{157}	$q^5+q^3+u^2(q^7t^1+3q^9t^2+4q^{11}t^3+4q^{13}t^4+$ $+5q^{15}t^5+3q^{17}t^6+3q^{19}t^7+q^{21}t^8)$
10_{158}	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+2q^{-3}t^{-2}+4q^{-1}t^{-1}+$ $+3q^1+4q^3t^1+4q^5t^2+2q^7t^3+2q^9t^4)$
10_{159}	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+2q^{-11}t^{-5}+3q^{-9}t^{-4}+$ $+3q^{-7}t^{-3}+4q^{-5}t^{-2}+3q^{-3}t^{-1}+2q^{-1}+q^1t^1)$
10_{160}	$q^5+q^3+u^2(q^3t^{-1}+q^5+q^7t^1+2q^9t^2+$ $+2q^{11}t^3+q^{13}t^4+2q^{15}t^5)$
10_{161}	$q^{-5}+q^{-7}+u^2(q^{-19}t^{-8}+q^{-15}t^{-6}+q^{-13}t^{-5}+$ $+q^{-13}+q^{-11}t^{-4}+q^{-9}t^{-3}+q^{-9}t^{-2})$
10_{162}	$q^{-1}+q^{-3}+u^2(q^{-11}t^{-5}+q^{-9}t^{-4}+3q^{-7}t^{-3}+$ $+3q^{-5}t^{-2}+3q^{-3}t^{-1}+3q^{-1}+q^1t^1+2q^3t^2)$

Name	BN
10_{163}	$q^3+q^1+u^2(q^{-1}t^{-2}+3q^1t^{-1}+3q^3+4q^5t^1+$ $+5q^7t^2+4q^9t^3+3q^{11}t^4+2q^{13}t^5)$
10_{164}	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+3q^{-3}t^{-2}+$ $+4q^{-1}t^{-1}+4q^1+3q^3t^1+3q^5t^2+2q^7t^3)$
10_{165}	$q^3+q^1+u^2(q^5t^1+3q^7t^2+3q^9t^3+3q^{11}t^4+$ $+4q^{13}t^5+2q^{15}t^6+2q^{17}t^7+q^{19}t^8)$
$11a_1$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+7q^3+$ $+9q^5t^1+11q^7t^2+10q^9t^3+8q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7)$
$11a_2$	$q^5+q^3+u^2(q^3t^{-1}+2q^5+5q^7t^1+8q^9t^2+$ $+11q^{11}t^3+11q^{13}t^4+11q^{15}t^5+9q^{17}t^6+6q^{19}t^7+3q^{21}t^8+q^{23}t^9)$
$11a_3$	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+7q^{-7}t^{-3}+9q^{-5}t^{-2}+10q^{-3}t^{-1}+8q^{-1}+7q^1t^1+5q^3t^2+2q^5t^3+q^7t^4)$
$11a_4$	$q^1+q^{-1}+u^2(q^{-11}t^{-6}+2q^{-9}t^{-5}+4q^{-7}t^{-4}+$ $+6q^{-5}t^{-3}+7q^{-3}t^{-2}+8q^{-1}t^{-1}+7q^1+6q^3t^1+4q^5t^2+2q^7t^3+q^9t^4)$
$11a_5$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+2q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+7q^{-3}t^{-2}+10q^{-1}t^{-1}+10q^1+9q^3t^1+9q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5)$
$11a_6$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+7q^3+$ $+10q^5t^1+11q^7t^2+11q^9t^3+9q^{11}t^4+6q^{13}t^5+4q^{15}t^6+q^{17}t^7)$

Name	BN
$11a_7$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+4q^{-9}t^{-4}+$ $+6q^{-7}t^{-3}+7q^{-5}t^{-2}+8q^{-3}t^{-1}+7q^{-1}+5q^1t^1+4q^3t^2+2q^5t^3+q^7t^4\big)$
$11a_8$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+2q^{-9}t^{-5}+4q^{-7}t^{-4}+$ $+7q^{-5}t^{-3}+8q^{-3}t^{-2}+10q^{-1}t^{-1}+9q^1+7q^3t^1+6q^5t^2+3q^7t^3+q^9t^4\big)$
$11a_9$	$q^5+q^3+u^2\big(q^{-1}t^{-3}+q^1t^{-2}+3q^3t^{-1}+3q^5+$ $+4q^7t^1+5q^9t^2+5q^{11}t^3+4q^{13}t^4+3q^{15}t^5+2q^{17}t^6+q^{19}t^7\big)$
$11a_{10}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+6q^3+$ $+8q^5t^1+9q^7t^2+8q^9t^3+7q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11a_{11}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+7q^{-1}t^{-1}+8q^1+9q^3t^1+9q^5t^2+7q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{12}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+4q^1t^{-1}+5q^3+$ $+7q^5t^1+9q^7t^2+8q^9t^3+7q^{11}t^4+5q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$
$11a_{13}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+q^{-3}t^{-2}+3q^{-1}t^{-1}+3q^1+$ $+4q^3t^1+5q^5t^2+4q^7t^3+4q^9t^4+2q^{11}t^5+2q^{13}t^6+q^{15}t^7\big)$
$11a_{14}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+8q^{-1}t^{-1}+10q^1+11q^3t^1+10q^5t^2+9q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{15}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+7q^{-5}t^{-2}+8q^{-3}t^{-1}+9q^{-1}+7q^1t^1+7q^3t^2+4q^5t^3+2q^7t^4+q^9t^5\big)$

Name	BN
$11a_{16}$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+2q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+6q^{-3}t^{-2}+8q^{-1}t^{-1}+9q^1+7q^3t^1+7q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5)$
$11a_{17}$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+6q^1t^{-1}+7q^3+$ $+9q^5t^1+11q^7t^2+9q^9t^3+8q^{11}t^4+5q^{13}t^5+2q^{15}t^6+q^{17}t^7)$
$11a_{18}$	$q^3+q^1+u^2(q^{-1}t^{-2}+2q^1t^{-1}+5q^3+7q^5t^1+$ $+10q^7t^2+11q^9t^3+9q^{11}t^4+9q^{13}t^5+5q^{15}t^6+3q^{17}t^7+q^{19}t^8)$
$11a_{19}$	$q^3+q^1+u^2(q^{-3}t^{-3}+3q^{-1}t^{-2}+7q^1t^{-1}+9q^3+$ $+12q^5t^1+13q^7t^2+12q^9t^3+10q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7)$
$11a_{20}$	$q^5+q^3+u^2(q^3t^{-1}+q^5+4q^7t^1+6q^9t^2+$ $+9q^{11}t^3+9q^{13}t^4+9q^{15}t^5+8q^{17}t^6+5q^{19}t^7+3q^{21}t^8+q^{23}t^9)$
$11a_{21}$	$q^3+q^1+u^2(q^1t^{-1}+q^3+3q^5t^1+4q^7t^2+$ $+6q^9t^3+6q^{11}t^4+5q^{13}t^5+5q^{15}t^6+3q^{17}t^7+2q^{19}t^8+q^{21}t^9)$
$11a_{22}$	$q^5+q^3+u^2(q^1t^{-2}+2q^3t^{-1}+4q^5+6q^7t^1+$ $+7q^9t^2+9q^{11}t^3+7q^{13}t^4+7q^{15}t^5+4q^{17}t^6+2q^{19}t^7+q^{21}t^8)$
$11a_{23}$	$q^3+q^1+u^2(q^{-1}t^{-2}+2q^1t^{-1}+4q^3+6q^5t^1+$ $+8q^7t^2+9q^9t^3+7q^{11}t^4+7q^{13}t^5+4q^{15}t^6+2q^{17}t^7+q^{19}t^8)$
$11a_{24}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+3q^{-5}t^{-3}+7q^{-3}t^{-2}+$ $+10q^{-1}t^{-1}+12q^1+13q^3t^1+12q^5t^2+10q^7t^3+6q^9t^4+3q^{11}t^5+q^{13}t^6)$

Name	BN
$11a_{25}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+3q^{-1}t^{-2}+7q^1t^{-1}+9q^3+$ $+12q^5t^1+13q^7t^2+12q^9t^3+10q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$
$11a_{26}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+7q^{-3}t^{-2}+$ $+10q^{-1}t^{-1}+12q^1+13q^3t^1+12q^5t^2+10q^7t^3+6q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{27}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+3q^{-13}t^{-6}+6q^{-11}t^{-5}+$ $+9q^{-9}t^{-4}+11q^{-7}t^{-3}+12q^{-5}t^{-2}+11q^{-3}t^{-1}+9q^{-1}+5q^1t^1+3q^3t^2+q^5t^3\big)$
$11a_{28}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+7q^{-3}t^{-2}+9q^{-1}t^{-1}+10q^1+9q^3t^1+8q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{29}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+7q^{-5}t^{-2}+9q^{-3}t^{-1}+9q^{-1}+8q^1t^1+8q^3t^2+4q^5t^3+3q^7t^4+q^9t^5\big)$
$11a_{30}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+10q^{-1}t^{-1}+11q^1+12q^3t^1+12q^5t^2+9q^7t^3+6q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{31}$	$q^5+q^3+u^2\big(q^3t^{-1}+2q^5+5q^7t^1+7q^9t^2+$ $+10q^{11}t^3+10q^{13}t^4+10q^{15}t^5+8q^{17}t^6+5q^{19}t^7+3q^{21}t^8+q^{23}t^9\big)$
$11a_{32}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+3q^1t^{-1}+6q^3+8q^5t^1+$ $+11q^7t^2+12q^9t^3+10q^{11}t^4+9q^{13}t^5+5q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
$11a_{33}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+4q^{-7}t^{-3}+$ $+6q^{-5}t^{-2}+7q^{-3}t^{-1}+8q^{-1}+6q^1t^1+6q^3t^2+4q^5t^3+2q^7t^4+q^9t^5\big)$

Name	BN
$11a_{34}$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+6q^3+$ $+9q^5t^1+10q^7t^2+9q^9t^3+8q^{11}t^4+5q^{13}t^5+3q^{15}t^6+q^{17}t^7)$
$11a_{35}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+7q^{-1}t^{-1}+9q^1+10q^3t^1+9q^5t^2+8q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6)$
$11a_{36}$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+3q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+8q^{-3}t^{-2}+9q^{-1}t^{-1}+10q^1+9q^3t^1+7q^5t^2+5q^7t^3+2q^9t^4+q^{11}t^5)$
$11a_{37}$	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+2q^{-3}t^{-2}+5q^{-1}t^{-1}+$ $+5q^1+7q^3t^1+8q^5t^2+6q^7t^3+6q^9t^4+3q^{11}t^5+2q^{13}t^6+q^{15}t^7)$
$11a_{38}$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+2q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+7q^{-3}t^{-2}+9q^{-1}t^{-1}+10q^1+8q^3t^1+8q^5t^2+5q^7t^3+2q^9t^4+q^{11}t^5)$
$11a_{39}$	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+2q^{-3}t^{-2}+5q^{-1}t^{-1}+$ $+6q^1+7q^3t^1+9q^5t^2+7q^7t^3+6q^9t^4+4q^{11}t^5+2q^{13}t^6+q^{15}t^7)$
$11a_{40}$	$q^5+q^3+u^2(q^1t^{-2}+2q^3t^{-1}+3q^5+5q^7t^1+$ $+6q^9t^2+8q^{11}t^3+6q^{13}t^4+6q^{15}t^5+4q^{17}t^6+2q^{19}t^7+q^{21}t^8)$
$11a_{41}$	$q^3+q^1+u^2(q^{-1}t^{-2}+3q^1t^{-1}+5q^3+7q^5t^1+$ $+9q^7t^2+10q^9t^3+8q^{11}t^4+7q^{13}t^5+4q^{15}t^6+2q^{17}t^7+q^{19}t^8)$
$11a_{42}$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+6q^3+$ $+8q^5t^1+9q^7t^2+8q^9t^3+7q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7)$

Name	BN
$11a_{43}$	$q^7+q^5+u^2\big(3q^{11}t^2+6q^{13}t^3+7q^{15}t^4+12q^{17}t^5+$ $+10q^{19}t^6+11q^{21}t^7+9q^{23}t^8+5q^{25}t^9+3q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{44}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+7q^{-1}t^{-1}+9q^1+10q^3t^1+8q^5t^2+8q^7t^3+4q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
$11a_{45}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+4q^1t^{-1}+4q^3+$ $+6q^5t^1+8q^7t^2+6q^9t^3+6q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11a_{46}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+4q^{-1}t^{-2}+5q^1t^{-1}+$ $+6q^3+7q^5t^1+6q^7t^2+6q^9t^3+3q^{11}t^4+2q^{13}t^5+q^{15}t^6\big)$
$11a_{47}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+7q^{-1}t^{-1}+9q^1+10q^3t^1+8q^5t^2+8q^7t^3+4q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
$11a_{48}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+2q^{-17}t^{-7}+5q^{-15}t^{-6}+$ $+7q^{-13}t^{-5}+9q^{-11}t^{-4}+9q^{-9}t^{-3}+9q^{-7}t^{-2}+7q^{-5}t^{-1}+4q^{-3}+2q^{-1}t^1+q^1t^2\big)$
$11a_{49}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+4q^7t^1+6q^9t^2+$ $+8q^{11}t^3+9q^{13}t^4+8q^{15}t^5+7q^{17}t^6+5q^{19}t^7+2q^{21}t^8+q^{23}t^9\big)$
$11a_{50}$	$q^3+q^1+u^2\big(q^1t^{-1}+q^3+3q^5t^1+5q^7t^2+$ $+6q^9t^3+7q^{11}t^4+6q^{13}t^5+5q^{15}t^6+4q^{17}t^7+2q^{19}t^8+q^{21}t^9\big)$
$11a_{51}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+7q^{-7}t^{-3}+9q^{-5}t^{-2}+10q^{-3}t^{-1}+8q^{-1}+7q^1t^1+5q^3t^2+2q^5t^3+q^7t^4\big)$

Name	BN
$11a_{52}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+8q^{-1}t^{-1}+10q^1+11q^3t^1+11q^5t^2+9q^7t^3+6q^9t^4+4q^{11}t^5+q^{13}t^6\big)$
$11a_{53}$	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+2q^{-15}t^{-6}+4q^{-13}t^{-5}+$ $+6q^{-11}t^{-4}+7q^{-9}t^{-3}+8q^{-7}t^{-2}+7q^{-5}t^{-1}+6q^{-3}+3q^{-1}t^1+3q^1t^2+q^3t^3\big)$
$11a_{54}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+5q^{-11}t^{-5}+$ $+8q^{-9}t^{-4}+10q^{-7}t^{-3}+12q^{-5}t^{-2}+11q^{-3}t^{-1}+9q^{-1}+6q^1t^1+4q^3t^2+q^5t^3\big)$
$11a_{55}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+3q^{-7}t^{-3}+$ $+5q^{-5}t^{-2}+5q^{-3}t^{-1}+6q^{-1}+4q^1t^1+4q^3t^2+3q^5t^3+q^7t^4+q^9t^5\big)$
$11a_{56}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+7q^{-3}t^{-2}+8q^{-1}t^{-1}+9q^1+8q^3t^1+7q^5t^2+5q^7t^3+2q^9t^4+q^{11}t^5\big)$
$11a_{57}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+7q^{-5}t^{-2}+7q^{-3}t^{-1}+9q^{-1}+6q^1t^1+6q^3t^2+4q^5t^3+q^7t^4+q^9t^5\big)$
$11a_{58}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+5q^{-3}t^{-2}+6q^{-1}t^{-1}+7q^1+5q^3t^1+5q^5t^2+3q^7t^3+q^9t^4+q^{11}t^5\big)$
$11a_{59}$	$q^3+q^1+u^2\big(q^{-7}t^{-5}+q^{-5}t^{-4}+2q^{-3}t^{-3}+2q^{-1}t^{-2}+$ $+3q^1t^{-1}+3q^3+2q^5t^1+3q^7t^2+2q^9t^3+q^{11}t^4+q^{13}t^5\big)$
$11a_{60}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+3q^7t^1+5q^9t^2+$ $+6q^{11}t^3+7q^{13}t^4+7q^{15}t^5+5q^{17}t^6+4q^{19}t^7+2q^{21}t^8+q^{23}t^9\big)$

Name	BN
$11a_{61}$	$q^3+q^1+u^2\big(q^1t^{-1}+2q^3+4q^5t^1+7q^7t^2+$ $+8q^9t^3+8q^{11}t^4+8q^{13}t^5+6q^{15}t^6+4q^{17}t^7+2q^{19}t^8+q^{21}t^9\big)$
$11a_{62}$	$q^{-5}+q^{-7}+u^2\big(q^{-21}t^{-8}+q^{-19}t^{-7}+3q^{-17}t^{-6}+$ $+3q^{-15}t^{-5}+4q^{-13}t^{-4}+4q^{-11}t^{-3}+4q^{-9}t^{-2}+3q^{-7}t^{-1}+2q^{-5}+q^{-3}t^1+q^{-1}t^2\big)$
$11a_{63}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+q^{-17}t^{-7}+4q^{-15}t^{-6}+$ $+5q^{-13}t^{-5}+7q^{-11}t^{-4}+8q^{-9}t^{-3}+7q^{-7}t^{-2}+6q^{-5}t^{-1}+4q^{-3}+2q^{-1}t^1+q^1t^2\big)$
$11a_{64}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+2q^{-17}t^{-7}+5q^{-15}t^{-6}+$ $+6q^{-13}t^{-5}+8q^{-11}t^{-4}+8q^{-9}t^{-3}+7q^{-7}t^{-2}+6q^{-5}t^{-1}+3q^{-3}+q^{-1}t^1+q^1t^2\big)$
$11a_{65}$	$q^{-1}+q^{-3}+u^2\big(q^{-17}t^{-8}+q^{-15}t^{-7}+3q^{-13}t^{-6}+$ $+3q^{-11}t^{-5}+4q^{-9}t^{-4}+5q^{-7}t^{-3}+4q^{-5}t^{-2}+4q^{-3}t^{-1}+2q^{-1}+q^1t^1+q^3t^2\big)$
$11a_{66}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+7q^{-7}t^{-3}+9q^{-5}t^{-2}+10q^{-3}t^{-1}+9q^{-1}+7q^1t^1+5q^3t^2+3q^5t^3+q^7t^4\big)$
$11a_{67}$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+2q^{-9}t^{-5}+5q^{-7}t^{-4}+$ $+7q^{-5}t^{-3}+9q^{-3}t^{-2}+11q^{-1}t^{-1}+9q^1+8q^3t^1+6q^5t^2+3q^7t^3+q^9t^4\big)$
$11a_{68}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+4q^{-1}t^{-2}+6q^1t^{-1}+$ $+7q^3+8q^5t^1+8q^7t^2+7q^9t^3+4q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{69}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+9q^{-1}t^{-1}+11q^1+11q^3t^1+11q^5t^2+9q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6\big)$

Name	BN
$11a_{70}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+6q^1t^{-1}+8q^3+$ $+11q^5t^1+13q^7t^2+12q^9t^3+10q^{11}t^4+7q^{13}t^5+4q^{15}t^6+q^{17}t^7\big)$
$11a_{71}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+3q^{-1}t^{-2}+6q^1t^{-1}+9q^3+$ $+12q^5t^1+13q^7t^2+13q^9t^3+10q^{11}t^4+7q^{13}t^5+4q^{15}t^6+q^{17}t^7\big)$
$11a_{72}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+9q^{-1}t^{-1}+12q^1+12q^3t^1+12q^5t^2+10q^7t^3+6q^9t^4+4q^{11}t^5+q^{13}t^6\big)$
$11a_{73}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+4q^{-7}t^{-4}+7q^{-5}t^{-3}+$ $+11q^{-3}t^{-2}+14q^{-1}t^{-1}+14q^1+14q^3t^1+11q^5t^2+7q^7t^3+4q^9t^4+q^{11}t^5\big)$
$11a_{74}$	$q^5+q^3+u^2\big(q^{-1}t^{-3}+q^1t^{-2}+4q^3t^{-1}+3q^5+$ $+5q^7t^1+6q^9t^2+5q^{11}t^3+5q^{13}t^4+3q^{15}t^5+2q^{17}t^6+q^{19}t^7\big)$
$11a_{75}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+4q^1t^{-1}+4q^3+$ $+6q^5t^1+7q^7t^2+6q^9t^3+6q^{11}t^4+3q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11a_{76}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+3q^{-7}t^{-4}+6q^{-5}t^{-3}+$ $+9q^{-3}t^{-2}+11q^{-1}t^{-1}+12q^1+11q^3t^1+9q^5t^2+6q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{77}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+3q^{-13}t^{-6}+5q^{-11}t^{-5}+$ $+8q^{-9}t^{-4}+10q^{-7}t^{-3}+11q^{-5}t^{-2}+10q^{-3}t^{-1}+8q^{-1}+5q^1t^1+3q^3t^2+q^5t^3\big)$
$11a_{78}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+7q^3+$ $+9q^5t^1+10q^7t^2+10q^9t^3+8q^{11}t^4+5q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$

Name	BN
$11a_{79}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+3q^{-11}t^{-5}+6q^{-9}t^{-4}+$ $+9q^{-7}t^{-3}+11q^{-5}t^{-2}+12q^{-3}t^{-1}+11q^{-1}+8q^1t^1+6q^3t^2+3q^5t^3+q^7t^4\big)$
$11a_{80}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+3q^{-7}t^{-4}+6q^{-5}t^{-3}+$ $+8q^{-3}t^{-2}+11q^{-1}t^{-1}+11q^1+10q^3t^1+9q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{81}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+3q^{-3}t^{-3}+5q^{-1}t^{-2}+8q^1t^{-1}+$ $+9q^3+10q^5t^1+10q^7t^2+8q^9t^3+5q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{82}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+4q^{-7}t^{-3}+$ $+6q^{-5}t^{-2}+7q^{-3}t^{-1}+8q^{-1}+6q^1t^1+6q^3t^2+4q^5t^3+2q^7t^4+q^9t^5\big)$
$11a_{83}$	$q^5+q^3+u^2\big(q^1t^{-2}+2q^3t^{-1}+4q^5+6q^7t^1+$ $+8q^9t^2+10q^{11}t^3+8q^{13}t^4+8q^{15}t^5+5q^{17}t^6+3q^{19}t^7+q^{21}t^8\big)$
$11a_{84}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+6q^{-3}t^{-2}+8q^{-1}t^{-1}+8q^1+7q^3t^1+7q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5\big)$
$11a_{85}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+2q^1t^{-1}+4q^3+6q^5t^1+$ $+8q^7t^2+9q^9t^3+8q^{11}t^4+7q^{13}t^5+4q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
$11a_{86}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+3q^{-1}t^{-2}+6q^1t^{-1}+$ $+6q^3+7q^5t^1+7q^7t^2+6q^9t^3+4q^{11}t^4+2q^{13}t^5+q^{15}t^6\big)$
$11a_{87}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+8q^{-1}t^{-1}+9q^1+10q^3t^1+9q^5t^2+7q^7t^3+5q^9t^4+2q^{11}t^5+q^{13}t^6\big)$

Name	BN
$11a_{88}$	$q^1 + q^{-1} + u^2(q^{-9}t^{-5} + 2q^{-7}t^{-4} + 4q^{-5}t^{-3} + + 6q^{-3}t^{-2} + 8q^{-1}t^{-1} + 8q^1 + 7q^3t^1 + 7q^5t^2 + 4q^7t^3 + 2q^9t^4 + q^{11}t^5)$
$11a_{89}$	$q^3 + q^1 + u^2(q^{-3}t^{-3} + 2q^{-1}t^{-2} + 5q^1t^{-1} + 6q^3 + + 9q^5t^1 + 10q^7t^2 + 9q^9t^3 + 8q^{11}t^4 + 5q^{13}t^5 + 3q^{15}t^6 + q^{17}t^7)$
$11a_{90}$	$q^{-1} + q^{-3} + u^2(q^{-11}t^{-5} + 2q^{-9}t^{-4} + 4q^{-7}t^{-3} + + 5q^{-5}t^{-2} + 7q^{-3}t^{-1} + 7q^{-1} + 5q^1t^1 + 6q^3t^2 + 3q^5t^3 + 2q^7t^4 + q^9t^5)$
$11a_{91}$	$q^1 + q^{-1} + u^2(q^{-7}t^{-4} + 3q^{-5}t^{-3} + 5q^{-3}t^{-2} + + 8q^{-1}t^{-1} + 10q^1 + 10q^3t^1 + 10q^5t^2 + 8q^7t^3 + 5q^9t^4 + 3q^{11}t^5 + q^{13}t^6)$
$11a_{92}$	$q^{-1} + q^{-3} + u^2(q^{-13}t^{-6} + 2q^{-11}t^{-5} + 5q^{-9}t^{-4} + + 6q^{-7}t^{-3} + 8q^{-5}t^{-2} + 9q^{-3}t^{-1} + 7q^{-1} + 6q^1t^1 + 4q^3t^2 + 2q^5t^3 + q^7t^4)$
$11a_{93}$	$q^1 + q^{-1} + u^2(q^{-11}t^{-6} + 2q^{-9}t^{-5} + 4q^{-7}t^{-4} + + 5q^{-5}t^{-3} + 7q^{-3}t^{-2} + 8q^{-1}t^{-1} + 6q^1 + 6q^3t^1 + 4q^5t^2 + 2q^7t^3 + q^9t^4)$
$11a_{94}$	$q^7 + q^5 + u^2(2q^{11}t^2 + 4q^{13}t^3 + 5q^{15}t^4 + 9q^{17}t^5 + + 8q^{19}t^6 + 9q^{21}t^7 + 7q^{23}t^8 + 5q^{25}t^9 + 3q^{27}t^{10} + q^{29}t^{11})$
$11a_{95}$	$q^5 + q^3 + u^2(2q^9t^2 + 3q^{11}t^3 + 4q^{13}t^4 + 6q^{15}t^5 + + 5q^{17}t^6 + 6q^{19}t^7 + 4q^{21}t^8 + 3q^{23}t^9 + 2q^{25}t^{10} + q^{27}t^{11})$
$11a_{96}$	$q^1 + q^{-1} + u^2(q^{-9}t^{-5} + 2q^{-7}t^{-4} + 5q^{-5}t^{-3} + + 7q^{-3}t^{-2} + 9q^{-1}t^{-1} + 10q^1 + 9q^3t^1 + 8q^5t^2 + 5q^7t^3 + 3q^9t^4 + q^{11}t^5)$

Name	BN
$11a_{97}$	$q^3+q^1+u^2\big(q^{-7}t^{-5}+q^{-5}t^{-4}+3q^{-3}t^{-3}+4q^{-1}t^{-2}+$ $+5q^1t^{-1}+5q^3+5q^5t^1+5q^7t^2+3q^9t^3+2q^{11}t^4+q^{13}t^5\big)$
$11a_{98}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+3q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+6q^{-1}t^{-1}+6q^1+5q^3t^1+6q^5t^2+3q^7t^3+2q^9t^4+q^{11}t^5\big)$
$11a_{99}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+3q^{-3}t^{-3}+5q^{-1}t^{-2}+9q^1t^{-1}+$ $+9q^3+11q^5t^1+11q^7t^2+8q^9t^3+6q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{100}$	$q^5+q^3+u^2\big(q^3t^{-1}+2q^5+5q^7t^1+8q^9t^2+$ $+11q^{11}t^3+11q^{13}t^4+12q^{15}t^5+9q^{17}t^6+6q^{19}t^7+4q^{21}t^8+q^{23}t^9\big)$
$11a_{101}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+3q^1t^{-1}+6q^3+10q^5t^1+$ $+13q^7t^2+14q^9t^3+13q^{11}t^4+11q^{13}t^5+7q^{15}t^6+4q^{17}t^7+q^{19}t^8\big)$
$11a_{102}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+5q^1t^{-1}+5q^3+$ $+7q^5t^1+9q^7t^2+7q^9t^3+7q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11a_{103}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+q^{-3}t^{-2}+4q^{-1}t^{-1}+4q^1+$ $+6q^3t^1+7q^5t^2+5q^7t^3+6q^9t^4+3q^{11}t^5+2q^{13}t^6+q^{15}t^7\big)$
$11a_{104}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+8q^{-1}t^{-1}+9q^1+10q^3t^1+10q^5t^2+8q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{105}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+4q^7t^1+6q^9t^2+$ $+8q^{11}t^3+9q^{13}t^4+9q^{15}t^5+7q^{17}t^6+5q^{19}t^7+3q^{21}t^8+q^{23}t^9\big)$

Name	BN
$11a_{106}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+6q^{-1}t^{-1}+7q^1+7q^3t^1+7q^5t^2+6q^7t^3+3q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
$11a_{107}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+4q^{-1}t^{-2}+7q^1t^{-1}+$ $+7q^3+9q^5t^1+9q^7t^2+7q^9t^3+5q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{108}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+4q^{-7}t^{-3}+$ $+7q^{-5}t^{-2}+7q^{-3}t^{-1}+8q^{-1}+7q^1t^1+6q^3t^2+4q^5t^3+2q^7t^4+q^9t^5\big)$
$11a_{109}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+7q^{-1}t^{-1}+9q^1+9q^3t^1+9q^5t^2+8q^7t^3+4q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{110}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+6q^{-3}t^{-2}+7q^{-1}t^{-1}+8q^1+7q^3t^1+6q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5\big)$
$11a_{111}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+4q^{-1}t^{-2}+6q^1t^{-1}+$ $+7q^3+8q^5t^1+8q^7t^2+7q^9t^3+4q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{112}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+7q^{-3}t^{-2}+10q^{-1}t^{-1}+10q^1+9q^3t^1+9q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{113}$	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+2q^{-15}t^{-6}+4q^{-13}t^{-5}+$ $+7q^{-11}t^{-4}+8q^{-9}t^{-3}+9q^{-7}t^{-2}+8q^{-5}t^{-1}+7q^{-3}+4q^{-1}t^1+3q^1t^2+q^3t^3\big)$
$11a_{114}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+3q^1t^{-1}+6q^3+9q^5t^1+$ $+12q^7t^2+13q^9t^3+11q^{11}t^4+10q^{13}t^5+6q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$

Name	BN
$11a_{115}$	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+2q^{-3}t^{-2}+5q^{-1}t^{-1}+$ $+7q^1+9q^3t^1+10q^5t^2+9q^7t^3+8q^9t^4+5q^{11}t^5+3q^{13}t^6+q^{15}t^7)$
$11a_{116}$	$q^5+q^3+u^2(q^3t^{-1}+2q^5+5q^7t^1+8q^9t^2+$ $+11q^{11}t^3+11q^{13}t^4+11q^{15}t^5+9q^{17}t^6+6q^{19}t^7+3q^{21}t^8+q^{23}t^9)$
$11a_{117}$	$q^5+q^3+u^2(q^3t^{-1}+2q^5+4q^7t^1+7q^9t^2+$ $+9q^{11}t^3+9q^{13}t^4+10q^{15}t^5+7q^{17}t^6+5q^{19}t^7+3q^{21}t^8+q^{23}t^9)$
$11a_{118}$	$q^3+q^1+u^2(q^{-3}t^{-3}+q^{-1}t^{-2}+4q^1t^{-1}+4q^3+$ $+6q^5t^1+8q^7t^2+6q^9t^3+6q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7)$
$11a_{119}$	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+q^{-3}t^{-2}+4q^{-1}t^{-1}+4q^1+$ $+5q^3t^1+7q^5t^2+5q^7t^3+5q^9t^4+3q^{11}t^5+2q^{13}t^6+q^{15}t^7)$
$11a_{120}$	$q^{-3}+q^{-5}+u^2(q^{-19}t^{-8}+2q^{-17}t^{-7}+5q^{-15}t^{-6}+$ $+6q^{-13}t^{-5}+9q^{-11}t^{-4}+9q^{-9}t^{-3}+8q^{-7}t^{-2}+7q^{-5}t^{-1}+4q^{-3}+2q^{-1}t^1+q^1t^2)$
$11a_{121}$	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+7q^{-7}t^{-3}+9q^{-5}t^{-2}+10q^{-3}t^{-1}+9q^{-1}+7q^1t^1+5q^3t^2+3q^5t^3+q^7t^4)$
$11a_{122}$	$q^{-1}+q^{-3}+u^2(q^{-15}t^{-7}+2q^{-13}t^{-6}+5q^{-11}t^{-5}+$ $+8q^{-9}t^{-4}+9q^{-7}t^{-3}+11q^{-5}t^{-2}+10q^{-3}t^{-1}+8q^{-1}+5q^1t^1+3q^3t^2+q^5t^3)$
$11a_{123}$	$q^5+q^3+u^2(3q^9t^2+4q^{11}t^3+7q^{13}t^4+10q^{15}t^5+$ $+8q^{17}t^6+10q^{19}t^7+7q^{21}t^8+5q^{23}t^9+3q^{25}t^{10}+q^{27}t^{11})$

Name	BN
$11a_{124}$	$q^7+q^5+u^2(3q^{11}t^2+6q^{13}t^3+8q^{15}t^4+13q^{17}t^5+$ $+12q^{19}t^6+13q^{21}t^7+10q^{23}t^8+7q^{25}t^9+4q^{27}t^{10}+q^{29}t^{11})$
$11a_{125}$	$q^3+q^1+u^2(q^{-3}t^{-3}+3q^{-1}t^{-2}+7q^1t^{-1}+10q^3+$ $+13q^5t^1+15q^7t^2+14q^9t^3+11q^{11}t^4+8q^{13}t^5+4q^{15}t^6+q^{17}t^7)$
$11a_{126}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+8q^{-1}t^{-1}+11q^1+12q^3t^1+11q^5t^2+10q^7t^3+6q^9t^4+4q^{11}t^5+q^{13}t^6)$
$11a_{127}$	$q^5+q^3+u^2(q^1t^{-2}+3q^3t^{-1}+5q^5+8q^7t^1+$ $+10q^9t^2+12q^{11}t^3+10q^{13}t^4+9q^{15}t^5+6q^{17}t^6+3q^{19}t^7+q^{21}t^8)$
$11a_{128}$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+2q^{-7}t^{-4}+6q^{-5}t^{-3}+$ $+7q^{-3}t^{-2}+10q^{-1}t^{-1}+11q^1+9q^3t^1+9q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5)$
$11a_{129}$	$q^{-3}+q^{-5}+u^2(q^{-17}t^{-7}+2q^{-15}t^{-6}+5q^{-13}t^{-5}+$ $+7q^{-11}t^{-4}+8q^{-9}t^{-3}+10q^{-7}t^{-2}+8q^{-5}t^{-1}+7q^{-3}+4q^{-1}t^1+3q^1t^2+q^3t^3)$
$11a_{130}$	$q^{-1}+q^{-3}+u^2(q^{-15}t^{-7}+2q^{-13}t^{-6}+5q^{-11}t^{-5}+$ $+7q^{-9}t^{-4}+9q^{-7}t^{-3}+11q^{-5}t^{-2}+9q^{-3}t^{-1}+8q^{-1}+5q^1t^1+3q^3t^2+q^5t^3)$
$11a_{131}$	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+8q^{-7}t^{-3}+10q^{-5}t^{-2}+11q^{-3}t^{-1}+10q^{-1}+8q^1t^1+6q^3t^2+3q^5t^3+q^7t^4)$
$11a_{132}$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+6q^1t^{-1}+7q^3+$ $+10q^5t^1+12q^7t^2+10q^9t^3+9q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7)$

Name	BN
$11a_{133}$	$q^3+q^1+u^2\big(q^1t^{-1}+q^3+3q^5t^1+5q^7t^2+$ $+6q^9t^3+6q^{11}t^4+6q^{13}t^5+5q^{15}t^6+3q^{17}t^7+2q^{19}t^8+q^{21}t^9\big)$
$11a_{134}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+2q^1t^{-1}+5q^3+7q^5t^1+$ $+9q^7t^2+11q^9t^3+9q^{11}t^4+8q^{13}t^5+5q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
$11a_{135}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+10q^{-1}t^{-1}+11q^1+12q^3t^1+13q^5t^2+9q^7t^3+7q^9t^4+4q^{11}t^5+q^{13}t^6\big)$
$11a_{136}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+3q^{-13}t^{-6}+6q^{-11}t^{-5}+$ $+10q^{-9}t^{-4}+12q^{-7}t^{-3}+14q^{-5}t^{-2}+13q^{-3}t^{-1}+10q^{-1}+7q^1t^1+4q^3t^2+q^5t^3\big)$
$11a_{137}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+6q^{-5}t^{-2}+9q^{-3}t^{-1}+9q^{-1}+7q^1t^1+8q^3t^2+4q^5t^3+3q^7t^4+q^9t^5\big)$
$11a_{138}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+10q^{-1}t^{-1}+12q^1+13q^3t^1+13q^5t^2+10q^7t^3+7q^9t^4+4q^{11}t^5+q^{13}t^6\big)$
$11a_{139}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+6q^3+$ $+7q^5t^1+8q^7t^2+8q^9t^3+6q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11a_{140}$	$q^5+q^3+u^2\big(q^{-1}t^{-3}+q^1t^{-2}+3q^3t^{-1}+3q^5+$ $+4q^7t^1+5q^9t^2+5q^{11}t^3+4q^{13}t^4+3q^{15}t^5+2q^{17}t^6+q^{19}t^7\big)$
$11a_{141}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+4q^{-7}t^{-3}+$ $+6q^{-5}t^{-2}+8q^{-3}t^{-1}+8q^{-1}+7q^1t^1+7q^3t^2+4q^5t^3+3q^7t^4+q^9t^5\big)$

Name	BN
$11a_{142}$	$q^{-5}+q^{-7}+u^2(q^{-21}t^{-8}+q^{-19}t^{-7}+3q^{-17}t^{-6}+$ $+3q^{-15}t^{-5}+5q^{-13}t^{-4}+4q^{-11}t^{-3}+4q^{-9}t^{-2}+4q^{-7}t^{-1}+2q^{-5}+q^{-3}t^1+q^{-1}t^2)$
$11a_{143}$	$q^{-3}+q^{-5}+u^2(q^{-19}t^{-8}+q^{-17}t^{-7}+4q^{-15}t^{-6}+$ $+5q^{-13}t^{-5}+7q^{-11}t^{-4}+7q^{-9}t^{-3}+7q^{-7}t^{-2}+6q^{-5}t^{-1}+3q^{-3}+2q^{-1}t^1+q^1t^2)$
$11a_{144}$	$q^{-3}+q^{-5}+u^2(q^{-19}t^{-8}+q^{-17}t^{-7}+4q^{-15}t^{-6}+$ $+4q^{-13}t^{-5}+6q^{-11}t^{-4}+6q^{-9}t^{-3}+5q^{-7}t^{-2}+5q^{-5}t^{-1}+2q^{-3}+q^{-1}t^1+q^1t^2)$
$11a_{145}$	$q^{-1}+q^{-3}+u^2(q^{-17}t^{-8}+q^{-15}t^{-7}+4q^{-13}t^{-6}+$ $+4q^{-11}t^{-5}+6q^{-9}t^{-4}+7q^{-7}t^{-3}+6q^{-5}t^{-2}+6q^{-3}t^{-1}+3q^{-1}+2q^1t^1+q^3t^2)$
$11a_{146}$	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+7q^{-7}t^{-3}+10q^{-5}t^{-2}+10q^{-3}t^{-1}+9q^{-1}+8q^1t^1+5q^3t^2+3q^5t^3+q^7t^4)$
$11a_{147}$	$q^3+q^1+u^2(q^{-3}t^{-3}+3q^{-1}t^{-2}+6q^1t^{-1}+9q^3+$ $+11q^5t^1+13q^7t^2+12q^9t^3+9q^{11}t^4+7q^{13}t^5+3q^{15}t^6+q^{17}t^7)$
$11a_{148}$	$q^3+q^1+u^2(q^1t^{-1}+2q^3+4q^5t^1+7q^7t^2+$ $+9q^9t^3+9q^{11}t^4+9q^{13}t^5+7q^{15}t^6+5q^{17}t^7+3q^{19}t^8+q^{21}t^9)$
$11a_{149}$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+7q^3+$ $+9q^5t^1+11q^7t^2+10q^9t^3+8q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7)$
$11a_{150}$	$q^{-3}+q^{-5}+u^2(q^{-19}t^{-8}+2q^{-17}t^{-7}+5q^{-15}t^{-6}+$ $+7q^{-13}t^{-5}+10q^{-11}t^{-4}+10q^{-9}t^{-3}+10q^{-7}t^{-2}+8q^{-5}t^{-1}+5q^{-3}+3q^{-1}t^1+q^1t^2)$

Name	BN
$11a_{151}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+6q^{-9}t^{-4}+ +7q^{-7}t^{-3}+10q^{-5}t^{-2}+11q^{-3}t^{-1}+9q^{-1}+8q^{1}t^{1}+5q^{3}t^{2}+3q^{5}t^{3}+q^{7}t^{4}\big)$
$11a_{152}$	$q^{1}+q^{-1}+u^2\big(q^{-11}t^{-6}+2q^{-9}t^{-5}+5q^{-7}t^{-4}+ +6q^{-5}t^{-3}+9q^{-3}t^{-2}+10q^{-1}t^{-1}+8q^{1}+8q^{3}t^{1}+5q^{5}t^{2}+3q^{7}t^{3}+q^{9}t^{4}\big)$
$11a_{153}$	$q^{1}+q^{-1}+u^2\big(q^{-5}t^{-3}+2q^{-3}t^{-2}+4q^{-1}t^{-1}+ +6q^{1}+6q^{3}t^{1}+8q^{5}t^{2}+6q^{7}t^{3}+5q^{9}t^{4}+4q^{11}t^{5}+q^{13}t^{6}+q^{15}t^{7}\big)$
$11a_{154}$	$q^{3}+q^{1}+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+3q^{1}t^{-1}+4q^{3}+ +4q^{5}t^{1}+6q^{7}t^{2}+5q^{9}t^{3}+4q^{11}t^{4}+3q^{13}t^{5}+q^{15}t^{6}+q^{17}t^{7}\big)$
$11a_{155}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+3q^{-13}t^{-6}+6q^{-11}t^{-5}+ +11q^{-9}t^{-4}+12q^{-7}t^{-3}+15q^{-5}t^{-2}+14q^{-3}t^{-1}+10q^{-1}+8q^{1}t^{1}+4q^{3}t^{2}+q^{5}t^{3}\big)$
$11a_{156}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+q^{-11}t^{-5}+4q^{-9}t^{-4}+ +5q^{-7}t^{-3}+7q^{-5}t^{-2}+8q^{-3}t^{-1}+6q^{-1}+6q^{1}t^{1}+4q^{3}t^{2}+2q^{5}t^{3}+q^{7}t^{4}\big)$
$11a_{157}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+6q^{-9}t^{-4}+ +8q^{-7}t^{-3}+10q^{-5}t^{-2}+12q^{-3}t^{-1}+10q^{-1}+8q^{1}t^{1}+6q^{3}t^{2}+3q^{5}t^{3}+q^{7}t^{4}\big)$
$11a_{158}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+5q^{-7}t^{-3}+ +7q^{-5}t^{-2}+9q^{-3}t^{-1}+10q^{-1}+8q^{1}t^{1}+8q^{3}t^{2}+5q^{5}t^{3}+3q^{7}t^{4}+q^{9}t^{5}\big)$
$11a_{159}$	$q^{3}+q^{1}+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^{1}t^{-1}+6q^{3}+ +8q^{5}t^{1}+10q^{7}t^{2}+8q^{9}t^{3}+7q^{11}t^{4}+5q^{13}t^{5}+2q^{15}t^{6}+q^{17}t^{7}\big)$

Name	BN
$11a_{160}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+3q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+9q^{-1}t^{-1}+11q^1+12q^3t^1+11q^5t^2+9q^7t^3+6q^9t^4+3q^{11}t^5+q^{13}t^6)$
$11a_{161}$	$q^{-3}+q^{-5}+u^2(q^{-15}t^{-6}+q^{-13}t^{-5}+3q^{-11}t^{-4}+$ $+3q^{-9}t^{-3}+4q^{-7}t^{-2}+5q^{-5}t^{-1}+3q^{-3}+3q^{-1}t^1+3q^1t^2+q^3t^3+q^5t^4)$
$11a_{162}$	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+3q^{-11}t^{-5}+7q^{-9}t^{-4}+$ $+10q^{-7}t^{-3}+13q^{-5}t^{-2}+14q^{-3}t^{-1}+13q^{-1}+10q^1t^1+7q^3t^2+4q^5t^3+q^7t^4)$
$11a_{163}$	$q^3+q^1+u^2(q^{-5}t^{-4}+2q^{-3}t^{-3}+5q^{-1}t^{-2}+7q^1t^{-1}+$ $+8q^3+10q^5t^1+9q^7t^2+8q^9t^3+5q^{11}t^4+3q^{13}t^5+q^{15}t^6)$
$11a_{164}$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+3q^{-7}t^{-4}+7q^{-5}t^{-3}+$ $+10q^{-3}t^{-2}+13q^{-1}t^{-1}+14q^1+13q^3t^1+11q^5t^2+7q^7t^3+4q^9t^4+q^{11}t^5)$
$11a_{165}$	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+2q^{-3}t^{-2}+4q^{-1}t^{-1}+$ $+5q^1+6q^3t^1+7q^5t^2+5q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6+q^{15}t^7)$
$11a_{166}$	$q^3+q^1+u^2(q^{-3}t^{-3}+q^{-1}t^{-2}+3q^1t^{-1}+3q^3+$ $+4q^5t^1+5q^7t^2+4q^9t^3+4q^{11}t^4+2q^{13}t^5+q^{15}t^6+q^{17}t^7)$
$11a_{167}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+7q^{-1}t^{-1}+8q^1+9q^3t^1+9q^5t^2+7q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6)$
$11a_{168}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+8q^{-1}t^{-1}+9q^1+10q^3t^1+10q^5t^2+8q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6)$

Name	BN
$11a_{169}$	$q^1+q^{-1}+u^2(q^{-11}t^{-6}+2q^{-9}t^{-5}+5q^{-7}t^{-4}+$ $+7q^{-5}t^{-3}+9q^{-3}t^{-2}+10q^{-1}t^{-1}+9q^1+8q^3t^1+5q^5t^2+3q^7t^3+q^9t^4)$
$11a_{170}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+3q^{-5}t^{-3}+7q^{-3}t^{-2}+$ $+11q^{-1}t^{-1}+14q^1+15q^3t^1+15q^5t^2+12q^7t^3+8q^9t^4+5q^{11}t^5+q^{13}t^6)$
$11a_{171}$	$q^3+q^1+u^2(q^{-3}t^{-3}+4q^{-1}t^{-2}+7q^1t^{-1}+11q^3+$ $+14q^5t^1+15q^7t^2+15q^9t^3+11q^{11}t^4+8q^{13}t^5+4q^{15}t^6+q^{17}t^7)$
$11a_{172}$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+6q^1t^{-1}+8q^3+$ $+10q^5t^1+12q^7t^2+11q^9t^3+9q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7)$
$11a_{173}$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+8q^3+$ $+9q^5t^1+12q^7t^2+11q^9t^3+8q^{11}t^4+7q^{13}t^5+3q^{15}t^6+q^{17}t^7)$
$11a_{174}$	$q^{-1}+q^{-3}+u^2(q^{-11}t^{-5}+2q^{-9}t^{-4}+3q^{-7}t^{-3}+$ $+5q^{-5}t^{-2}+6q^{-3}t^{-1}+6q^{-1}+5q^1t^1+5q^3t^2+3q^5t^3+2q^7t^4+q^9t^5)$
$11a_{175}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+2q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+6q^{-1}t^{-1}+8q^1+8q^3t^1+8q^5t^2+7q^7t^3+4q^9t^4+3q^{11}t^5+q^{13}t^6)$
$11a_{176}$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+6q^3+$ $+8q^5t^1+9q^7t^2+9q^9t^3+7q^{11}t^4+5q^{13}t^5+3q^{15}t^6+q^{17}t^7)$
$11a_{177}$	$q^5+q^3+u^2(q^1t^{-2}+2q^3t^{-1}+3q^5+5q^7t^1+$ $+7q^9t^2+8q^{11}t^3+7q^{13}t^4+7q^{15}t^5+4q^{17}t^6+3q^{19}t^7+q^{21}t^8)$

Name	BN
$11a_{178}$	$q^3 + q^1 + u^2(q^{-1}t^{-2} + 3q^1t^{-1} + 5q^3 + 7q^5t^1 + + 10q^7t^2 + 10q^9t^3 + 9q^{11}t^4 + 8q^{13}t^5 + 4q^{15}t^6 + 3q^{17}t^7 + q^{19}t^8)$
$11a_{179}$	$q^5 + q^3 + u^2(q^{-1}t^{-3} + q^1t^{-2} + 3q^3t^{-1} + 2q^5 + + 4q^7t^1 + 4q^9t^2 + 4q^{11}t^3 + 4q^{13}t^4 + 2q^{15}t^5 + 2q^{17}t^6 + q^{19}t^7)$
$11a_{180}$	$q^1 + q^{-1} + u^2(q^{-9}t^{-5} + 2q^{-7}t^{-4} + 4q^{-5}t^{-3} + + 5q^{-3}t^{-2} + 7q^{-1}t^{-1} + 7q^1 + 6q^3t^1 + 6q^5t^2 + 3q^7t^3 + 2q^9t^4 + q^{11}t^5)$
$11a_{181}$	$q^3 + q^1 + u^2(q^{-3}t^{-3} + 2q^{-1}t^{-2} + 5q^1t^{-1} + 5q^3 + + 8q^5t^1 + 8q^7t^2 + 7q^9t^3 + 7q^{11}t^4 + 3q^{13}t^5 + 2q^{15}t^6 + q^{17}t^7)$
$11a_{182}$	$q^{-3} + q^{-5} + u^2(q^{-17}t^{-7} + 2q^{-15}t^{-6} + 3q^{-13}t^{-5} + + 5q^{-11}t^{-4} + 5q^{-9}t^{-3} + 6q^{-7}t^{-2} + 5q^{-5}t^{-1} + 4q^{-3} + 2q^{-1}t^1 + 2q^1t^2 + q^3t^3)$
$11a_{183}$	$q^{-1} + q^{-3} + u^2(q^{-15}t^{-7} + 2q^{-13}t^{-6} + 4q^{-11}t^{-5} + + 7q^{-9}t^{-4} + 8q^{-7}t^{-3} + 10q^{-5}t^{-2} + 9q^{-3}t^{-1} + 7q^{-1} + 5q^1t^1 + 3q^3t^2 + q^5t^3)$
$11a_{184}$	$q^{-1} + q^{-3} + u^2(q^{-13}t^{-6} + 2q^{-11}t^{-5} + 4q^{-9}t^{-4} + + 5q^{-7}t^{-3} + 7q^{-5}t^{-2} + 7q^{-3}t^{-1} + 6q^{-1} + 5q^1t^1 + 3q^3t^2 + 2q^5t^3 + q^7t^4)$
$11a_{185}$	$q^1 + q^{-1} + u^2(q^{-11}t^{-6} + 2q^{-9}t^{-5} + 4q^{-7}t^{-4} + + 6q^{-5}t^{-3} + 8q^{-3}t^{-2} + 9q^{-1}t^{-1} + 8q^1 + 7q^3t^1 + 5q^5t^2 + 3q^7t^3 + q^9t^4)$
$11a_{186}$	$q^7 + q^5 + u^2(2q^{11}t^2 + 4q^{13}t^3 + 5q^{15}t^4 + 8q^{17}t^5 + + 7q^{19}t^6 + 8q^{21}t^7 + 6q^{23}t^8 + 4q^{25}t^9 + 2q^{27}t^{10} + q^{29}t^{11})$

Name	BN
$11a_{187}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+ +7q^{-3}t^{-2}+9q^{-1}t^{-1}+9q^1+9q^3t^1+8q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{188}$	$q^3+q^1+u^2\big(q^{-7}t^{-5}+q^{-5}t^{-4}+3q^{-3}t^{-3}+3q^{-1}t^{-2}+ +5q^1t^{-1}+5q^3+4q^5t^1+5q^7t^2+3q^9t^3+2q^{11}t^4+q^{13}t^5\big)$
$11a_{189}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+3q^{-7}t^{-4}+6q^{-5}t^{-3}+ +9q^{-3}t^{-2}+12q^{-1}t^{-1}+12q^1+11q^3t^1+10q^5t^2+6q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{190}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+2q^{-3}t^{-2}+4q^{-1}t^{-1}+ +5q^1+6q^3t^1+7q^5t^2+6q^7t^3+5q^9t^4+3q^{11}t^5+2q^{13}t^6+q^{15}t^7\big)$
$11a_{191}$	$q^7+q^5+u^2\big(2q^{11}t^2+3q^{13}t^3+4q^{15}t^4+7q^{17}t^5+ +6q^{19}t^6+7q^{21}t^7+5q^{23}t^8+4q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{192}$	$q^5+q^3+u^2\big(3q^9t^2+4q^{11}t^3+6q^{13}t^4+8q^{15}t^5+ +7q^{17}t^6+8q^{19}t^7+5q^{21}t^8+4q^{23}t^9+2q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{193}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+4q^{-11}t^{-5}+ +6q^{-9}t^{-4}+7q^{-7}t^{-3}+8q^{-5}t^{-2}+7q^{-3}t^{-1}+6q^{-1}+3q^1t^1+2q^3t^2+q^5t^3\big)$
$11a_{194}$	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+2q^{-15}t^{-6}+4q^{-13}t^{-5}+ +6q^{-11}t^{-4}+7q^{-9}t^{-3}+8q^{-7}t^{-2}+6q^{-5}t^{-1}+6q^{-3}+3q^{-1}t^1+2q^1t^2+q^3t^3\big)$
$11a_{195}$	$q^1+q^{-1}+u^2\big(q^{-13}t^{-7}+q^{-11}t^{-6}+2q^{-9}t^{-5}+ +3q^{-7}t^{-4}+3q^{-5}t^{-3}+4q^{-3}t^{-2}+4q^{-1}t^{-1}+3q^1+2q^3t^1+2q^5t^2+q^7t^3\big)$

Name	BN
$11a_{196}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+3q^{-11}t^{-5}+6q^{-9}t^{-4}+$ $+9q^{-7}t^{-3}+12q^{-5}t^{-2}+12q^{-3}t^{-1}+11q^{-1}+9q^{1}t^{1}+6q^{3}t^{2}+3q^{5}t^{3}+q^{7}t^{4}\big)$
$11a_{197}$	$q^{3}+q^{1}+u^2\big(q^{-1}t^{-2}+2q^{1}t^{-1}+5q^{3}+8q^{5}t^{1}+$ $+11q^{7}t^{2}+12q^{9}t^{3}+11q^{11}t^{4}+10q^{13}t^{5}+6q^{15}t^{6}+4q^{17}t^{7}+q^{19}t^{8}\big)$
$11a_{198}$	$q^{3}+q^{1}+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+4q^{-1}t^{-2}+7q^{1}t^{-1}+$ $+8q^{3}+9q^{5}t^{1}+9q^{7}t^{2}+8q^{9}t^{3}+5q^{11}t^{4}+3q^{13}t^{5}+q^{15}t^{6}\big)$
$11a_{199}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+6q^{-5}t^{-2}+8q^{-3}t^{-1}+8q^{-1}+6q^{1}t^{1}+7q^{3}t^{2}+3q^{5}t^{3}+2q^{7}t^{4}+q^{9}t^{5}\big)$
$11a_{200}$	$q^{5}+q^{3}+u^2\big(2q^{9}t^{2}+3q^{11}t^{3}+5q^{13}t^{4}+7q^{15}t^{5}+$ $+6q^{17}t^{6}+7q^{19}t^{7}+5q^{21}t^{8}+4q^{23}t^{9}+2q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{201}$	$q^{1}+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+4q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+6q^{-1}t^{-1}+7q^{1}+5q^{3}t^{1}+6q^{5}t^{2}+3q^{7}t^{3}+2q^{9}t^{4}+q^{11}t^{5}\big)$
$11a_{202}$	$q^{3}+q^{1}+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+5q^{1}t^{-1}+5q^{3}+$ $+8q^{5}t^{1}+10q^{7}t^{2}+8q^{9}t^{3}+8q^{11}t^{4}+5q^{13}t^{5}+3q^{15}t^{6}+q^{17}t^{7}\big)$
$11a_{203}$	$q^{7}+q^{5}+u^2\big(q^{5}t^{-1}+q^{7}+2q^{9}t^{1}+3q^{11}t^{2}+$ $+5q^{13}t^{3}+4q^{15}t^{4}+5q^{17}t^{5}+4q^{19}t^{6}+3q^{21}t^{7}+2q^{23}t^{8}+q^{25}t^{9}\big)$
$11a_{204}$	$q^{5}+q^{3}+u^2\big(q^{3}t^{-1}+2q^{5}+4q^{7}t^{1}+6q^{9}t^{2}+$ $+8q^{11}t^{3}+8q^{13}t^{4}+8q^{15}t^{5}+6q^{17}t^{6}+4q^{19}t^{7}+2q^{21}t^{8}+q^{23}t^{9}\big)$

Name	BN
$11a_{205}$	$q^{-1}+q^{-3}+u^2(q^{-11}t^{-5}+2q^{-9}t^{-4}+4q^{-7}t^{-3}+$ $+6q^{-5}t^{-2}+7q^{-3}t^{-1}+7q^{-1}+6q^1t^1+6q^3t^2+3q^5t^3+2q^7t^4+q^9t^5)$
$11a_{206}$	$q^{-5}+q^{-7}+u^2(q^{-21}t^{-8}+q^{-19}t^{-7}+3q^{-17}t^{-6}+$ $+2q^{-15}t^{-5}+4q^{-13}t^{-4}+3q^{-11}t^{-3}+3q^{-9}t^{-2}+3q^{-7}t^{-1}+q^{-5}+q^{-3}t^1+q^{-1}t^2)$
$11a_{207}$	$q^{-3}+q^{-5}+u^2(q^{-19}t^{-8}+q^{-17}t^{-7}+4q^{-15}t^{-6}+$ $+4q^{-13}t^{-5}+7q^{-11}t^{-4}+7q^{-9}t^{-3}+6q^{-7}t^{-2}+6q^{-5}t^{-1}+3q^{-3}+2q^{-1}t^1+q^1t^2)$
$11a_{208}$	$q^{-3}+q^{-5}+u^2(q^{-19}t^{-8}+2q^{-17}t^{-7}+5q^{-15}t^{-6}+$ $+6q^{-13}t^{-5}+9q^{-11}t^{-4}+8q^{-9}t^{-3}+8q^{-7}t^{-2}+7q^{-5}t^{-1}+3q^{-3}+2q^{-1}t^1+q^1t^2)$
$11a_{209}$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+2q^{-7}t^{-4}+6q^{-5}t^{-3}+$ $+8q^{-3}t^{-2}+11q^{-1}t^{-1}+12q^1+10q^3t^1+10q^5t^2+6q^7t^3+3q^9t^4+q^{11}t^5)$
$11a_{210}$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+q^{-7}t^{-4}+3q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+5q^{-1}t^{-1}+6q^1+5q^3t^1+5q^5t^2+3q^7t^3+2q^9t^4+q^{11}t^5)$
$11a_{211}$	$q^{-1}+q^{-3}+u^2(q^{-17}t^{-8}+q^{-15}t^{-7}+3q^{-13}t^{-6}+$ $+3q^{-11}t^{-5}+5q^{-9}t^{-4}+5q^{-7}t^{-3}+5q^{-5}t^{-2}+5q^{-3}t^{-1}+2q^{-1}+2q^1t^1+q^3t^2)$
$11a_{212}$	$q^5+q^3+u^2(q^3t^{-1}+2q^5+5q^7t^1+9q^9t^2+$ $+12q^{11}t^3+12q^{13}t^4+13q^{15}t^5+10q^{17}t^6+7q^{19}t^7+4q^{21}t^8+q^{23}t^9)$
$11a_{213}$	$q^5+q^3+u^2(q^3t^{-1}+2q^5+5q^7t^1+9q^9t^2+$ $+11q^{11}t^3+12q^{13}t^4+12q^{15}t^5+9q^{17}t^6+7q^{19}t^7+3q^{21}t^8+q^{23}t^9)$

Name	BN
$11a_{214}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+q^{-3}t^{-2}+4q^{-1}t^{-1}+3q^1+$ $+5q^3t^1+6q^5t^2+4q^7t^3+5q^9t^4+2q^{11}t^5+2q^{13}t^6+q^{15}t^7\big)$
$11a_{215}$	$q^5+q^3+u^2\big(q^1t^{-2}+3q^3t^{-1}+5q^5+8q^7t^1+$ $+10q^9t^2+11q^{11}t^3+10q^{13}t^4+9q^{15}t^5+5q^{17}t^6+3q^{19}t^7+q^{21}t^8\big)$
$11a_{216}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+3q^{-1}t^{-2}+6q^1t^{-1}+9q^3+$ $+11q^5t^1+12q^7t^2+12q^9t^3+9q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$
$11a_{217}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+6q^{-9}t^{-4}+$ $+8q^{-7}t^{-3}+11q^{-5}t^{-2}+12q^{-3}t^{-1}+10q^{-1}+9q^1t^1+6q^3t^2+3q^5t^3+q^7t^4\big)$
$11a_{218}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+8q^{-7}t^{-3}+10q^{-5}t^{-2}+11q^{-3}t^{-1}+10q^{-1}+8q^1t^1+6q^3t^2+3q^5t^3+q^7t^4\big)$
$11a_{219}$	$q^{-1}+q^{-3}+u^2\big(q^{-17}t^{-8}+q^{-15}t^{-7}+4q^{-13}t^{-6}+$ $+4q^{-11}t^{-5}+7q^{-9}t^{-4}+7q^{-7}t^{-3}+6q^{-5}t^{-2}+7q^{-3}t^{-1}+3q^{-1}+2q^1t^1+q^3t^2\big)$
$11a_{220}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+3q^7t^1+5q^9t^2+$ $+6q^{11}t^3+7q^{13}t^4+7q^{15}t^5+5q^{17}t^6+4q^{19}t^7+2q^{21}t^8+q^{23}t^9\big)$
$11a_{221}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+q^{-11}t^{-5}+4q^{-9}t^{-4}+$ $+4q^{-7}t^{-3}+6q^{-5}t^{-2}+7q^{-3}t^{-1}+5q^{-1}+5q^1t^1+3q^3t^2+2q^5t^3+q^7t^4\big)$
$11a_{222}$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+q^{-9}t^{-5}+4q^{-7}t^{-4}+$ $+5q^{-5}t^{-3}+7q^{-3}t^{-2}+9q^{-1}t^{-1}+7q^1+7q^3t^1+5q^5t^2+3q^7t^3+q^9t^4\big)$

Name	BN
$11a_{223}$	$q^7+q^5+u^2\big(q^5t^{-1}+q^7+3q^9t^1+4q^{11}t^2+$ $+6q^{13}t^3+5q^{15}t^4+6q^{17}t^5+5q^{19}t^6+3q^{21}t^7+2q^{23}t^8+q^{25}t^9\big)$
$11a_{224}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+3q^7t^1+5q^9t^2+$ $+7q^{11}t^3+7q^{13}t^4+7q^{15}t^5+6q^{17}t^6+4q^{19}t^7+2q^{21}t^8+q^{23}t^9\big)$
$11a_{225}$	$q^{-3}+q^{-5}+u^2\big(q^{-15}t^{-6}+q^{-13}t^{-5}+3q^{-11}t^{-4}+$ $+3q^{-9}t^{-3}+4q^{-7}t^{-2}+4q^{-5}t^{-1}+3q^{-3}+3q^{-1}t^1+2q^1t^2+q^3t^3+q^5t^4\big)$
$11a_{226}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+q^{-11}t^{-5}+3q^{-9}t^{-4}+$ $+4q^{-7}t^{-3}+5q^{-5}t^{-2}+6q^{-3}t^{-1}+5q^{-1}+4q^1t^1+3q^3t^2+2q^5t^3+q^7t^4\big)$
$11a_{227}$	$q^7+q^5+u^2\big(3q^{11}t^2+6q^{13}t^3+8q^{15}t^4+12q^{17}t^5+$ $+11q^{19}t^6+12q^{21}t^7+9q^{23}t^8+6q^{25}t^9+3q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{228}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+8q^{-3}t^{-2}+10q^{-1}t^{-1}+11q^1+10q^3t^1+9q^5t^2+6q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{229}$	$q^3+q^1+u^2\big(q^1t^{-1}+q^3+3q^5t^1+5q^7t^2+$ $+5q^9t^3+6q^{11}t^4+5q^{13}t^5+4q^{15}t^6+3q^{17}t^7+q^{19}t^8+q^{21}t^9\big)$
$11a_{230}$	$q^3+q^1+u^2\big(q^1t^{-1}+q^3+2q^5t^1+3q^7t^2+$ $+4q^9t^3+4q^{11}t^4+3q^{13}t^5+3q^{15}t^6+2q^{17}t^7+q^{19}t^8+q^{21}t^9\big)$
$11a_{231}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+7q^{-5}t^{-2}+7q^{-3}t^{-1}+9q^{-1}+6q^1t^1+6q^3t^2+4q^5t^3+q^7t^4+q^9t^5\big)$

Name	BN
$11a_{232}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+2q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+8q^{-5}t^{-2}+9q^{-3}t^{-1}+10q^{-1}+9q^1t^1+8q^3t^2+5q^5t^3+3q^7t^4+q^9t^5\big)$
$11a_{233}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+7q^{-3}t^{-2}+$ $+11q^{-1}t^{-1}+13q^1+14q^3t^1+14q^5t^2+11q^7t^3+7q^9t^4+4q^{11}t^5+q^{13}t^6\big)$
$11a_{234}$	$q^9+q^7+u^2\big(q^{13}t^2+2q^{15}t^3+q^{17}t^4+3q^{19}t^5+$ $+2q^{21}t^6+3q^{23}t^7+2q^{25}t^8+2q^{27}t^9+q^{29}t^{10}+q^{31}t^{11}\big)$
$11a_{235}$	$q^7+q^5+u^2\big(2q^{11}t^2+3q^{13}t^3+4q^{15}t^4+6q^{17}t^5+$ $+5q^{19}t^6+6q^{21}t^7+4q^{23}t^8+3q^{25}t^9+q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{236}$	$q^7+q^5+u^2\big(2q^{11}t^2+4q^{13}t^3+5q^{15}t^4+8q^{17}t^5+$ $+8q^{19}t^6+8q^{21}t^7+6q^{23}t^8+5q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{237}$	$q^5+q^3+u^2\big(3q^9t^2+4q^{11}t^3+6q^{13}t^4+8q^{15}t^5+$ $+7q^{17}t^6+7q^{19}t^7+5q^{21}t^8+4q^{23}t^9+q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{238}$	$q^5+q^3+u^2\big(2q^9t^2+3q^{11}t^3+4q^{13}t^4+5q^{15}t^5+$ $+5q^{17}t^6+5q^{19}t^7+3q^{21}t^8+3q^{23}t^9+q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{239}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+4q^{-1}t^{-2}+8q^1t^{-1}+12q^3+$ $+15q^5t^1+16q^7t^2+16q^9t^3+12q^{11}t^4+8q^{13}t^5+4q^{15}t^6+q^{17}t^7\big)$
$11a_{240}$	$q^9+q^7+u^2\big(q^{13}t^2+3q^{15}t^3+2q^{17}t^4+5q^{19}t^5+$ $+4q^{21}t^6+5q^{23}t^7+4q^{25}t^8+3q^{27}t^9+2q^{29}t^{10}+q^{31}t^{11}\big)$

Name	BN
$11a_{241}$	$q^7+q^5+u^2\big(2q^{11}t^2+4q^{13}t^3+5q^{15}t^4+8q^{17}t^5+$ $+7q^{19}t^6+8q^{21}t^7+6q^{23}t^8+4q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{242}$	$q^7+q^5+u^2\big(q^{11}t^2+2q^{13}t^3+2q^{15}t^4+4q^{17}t^5+$ $+3q^{19}t^6+4q^{21}t^7+3q^{23}t^8+2q^{25}t^9+q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{243}$	$q^5+q^3+u^2\big(2q^9t^2+3q^{11}t^3+4q^{13}t^4+6q^{15}t^5+$ $+5q^{17}t^6+5q^{19}t^7+4q^{21}t^8+3q^{23}t^9+q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{244}$	$q^7+q^5+u^2\big(3q^{11}t^2+6q^{13}t^3+8q^{15}t^4+12q^{17}t^5+$ $+12q^{19}t^6+12q^{21}t^7+9q^{23}t^8+7q^{25}t^9+3q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{245}$	$q^7+q^5+u^2\big(q^{11}t^2+3q^{13}t^3+3q^{15}t^4+6q^{17}t^5+$ $+6q^{19}t^6+6q^{21}t^7+5q^{23}t^8+4q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{246}$	$q^5+q^3+u^2\big(q^9t^2+2q^{11}t^3+2q^{13}t^4+3q^{15}t^5+$ $+3q^{17}t^6+3q^{19}t^7+2q^{21}t^8+2q^{23}t^9+q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{247}$	$q^3+q^1+u^2\big(q^7t^2+q^9t^3+q^{11}t^4+q^{13}t^5+$ $+q^{15}t^6+q^{17}t^7+q^{19}t^8+q^{21}t^9+q^{25}t^{11}\big)$
$11a_{248}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+4q^{-11}t^{-5}+7q^{-9}t^{-4}+$ $+10q^{-7}t^{-3}+13q^{-5}t^{-2}+13q^{-3}t^{-1}+12q^{-1}+9q^1t^1+6q^3t^2+3q^5t^3+q^7t^4\big)$
$11a_{249}$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+3q^{-9}t^{-5}+4q^{-7}t^{-4}+$ $+7q^{-5}t^{-3}+9q^{-3}t^{-2}+9q^{-1}t^{-1}+9q^1+7q^3t^1+5q^5t^2+3q^7t^3+q^9t^4\big)$

Name	BN
$11a_{250}$	$q^5+q^3+u^2\big(q^{-1}t^{-3}+q^1t^{-2}+4q^3t^{-1}+4q^5+$ $+6q^7t^1+7q^9t^2+6q^{11}t^3+6q^{13}t^4+4q^{15}t^5+2q^{17}t^6+q^{19}t^7\big)$
$11a_{251}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+8q^{-3}t^{-2}+10q^{-1}t^{-1}+11q^1+10q^3t^1+9q^5t^2+6q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{252}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+5q^{-1}t^{-2}+8q^1t^{-1}+$ $+9q^3+11q^5t^1+10q^7t^2+9q^9t^3+6q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{253}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+8q^{-3}t^{-2}+10q^{-1}t^{-1}+11q^1+10q^3t^1+9q^5t^2+6q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{254}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+5q^{-1}t^{-2}+8q^1t^{-1}+$ $+9q^3+11q^5t^1+10q^7t^2+9q^9t^3+6q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{255}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+3q^{-11}t^{-5}+6q^{-9}t^{-4}+$ $+9q^{-7}t^{-3}+11q^{-5}t^{-2}+12q^{-3}t^{-1}+11q^{-1}+8q^1t^1+6q^3t^2+3q^5t^3+q^7t^4\big)$
$11a_{256}$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+3q^{-9}t^{-5}+5q^{-7}t^{-4}+$ $+8q^{-5}t^{-3}+10q^{-3}t^{-2}+11q^{-1}t^{-1}+10q^1+8q^3t^1+6q^5t^2+3q^7t^3+q^9t^4\big)$
$11a_{257}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+6q^{-3}t^{-2}+7q^{-1}t^{-1}+8q^1+7q^3t^1+6q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5\big)$
$11a_{258}$	$q^3+q^1+u^2\big(q^{-7}t^{-5}+q^{-5}t^{-4}+3q^{-3}t^{-3}+4q^{-1}t^{-2}+$ $+5q^1t^{-1}+6q^3+5q^5t^1+5q^7t^2+4q^9t^3+2q^{11}t^4+q^{13}t^5\big)$

Name	BN
$11a_{259}$	$q^7+q^5+u^2(q^5t^{-1}+q^7+3q^9t^1+4q^{11}t^2+$ $+6q^{13}t^3+6q^{15}t^4+6q^{17}t^5+5q^{19}t^6+4q^{21}t^7+2q^{23}t^8+q^{25}t^9)$
$11a_{260}$	$q^5+q^3+u^2(q^3t^{-1}+q^5+3q^7t^1+4q^9t^2+$ $+5q^{11}t^3+6q^{13}t^4+5q^{15}t^5+4q^{17}t^6+3q^{19}t^7+q^{21}t^8+q^{23}t^9)$
$11a_{261}$	$q^{-3}+q^{-5}+u^2(q^{-17}t^{-7}+3q^{-15}t^{-6}+6q^{-13}t^{-5}+$ $+8q^{-11}t^{-4}+10q^{-9}t^{-3}+11q^{-7}t^{-2}+9q^{-5}t^{-1}+8q^{-3}+4q^{-1}t^1+3q^1t^2+q^3t^3)$
$11a_{262}$	$q^{-1}+q^{-3}+u^2(q^{-15}t^{-7}+2q^{-13}t^{-6}+4q^{-11}t^{-5}+$ $+6q^{-9}t^{-4}+8q^{-7}t^{-3}+9q^{-5}t^{-2}+8q^{-3}t^{-1}+7q^{-1}+4q^1t^1+3q^3t^2+q^5t^3)$
$11a_{263}$	$q^9+q^7+u^2(q^{13}t^2+4q^{15}t^3+3q^{17}t^4+7q^{19}t^5+$ $+6q^{21}t^6+6q^{23}t^7+6q^{25}t^8+4q^{27}t^9+2q^{29}t^{10}+q^{31}t^{11})$
$11a_{264}$	$q^3+q^1+u^2(q^{-3}t^{-3}+3q^{-1}t^{-2}+5q^1t^{-1}+8q^3+$ $+10q^5t^1+11q^7t^2+11q^9t^3+8q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7)$
$11a_{265}$	$q^1+q^{-1}+u^2(q^{-5}t^{-3}+3q^{-3}t^{-2}+5q^{-1}t^{-1}+$ $+7q^1+8q^3t^1+9q^5t^2+8q^7t^3+6q^9t^4+4q^{11}t^5+2q^{13}t^6+q^{15}t^7)$
$11a_{266}$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+5q^{-7}t^{-4}+9q^{-5}t^{-3}+$ $+13q^{-3}t^{-2}+17q^{-1}t^{-1}+17q^1+16q^3t^1+13q^5t^2+8q^7t^3+4q^9t^4+q^{11}t^5)$
$11a_{267}$	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+4q^{-11}t^{-5}+8q^{-9}t^{-4}+$ $+12q^{-7}t^{-3}+15q^{-5}t^{-2}+16q^{-3}t^{-1}+15q^{-1}+11q^1t^1+8q^3t^2+4q^5t^3+q^7t^4)$

Name	BN
$11a_{268}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+3q^{-3}t^{-3}+5q^{-1}t^{-2}+9q^1t^{-1}+$ $+10q^3+11q^5t^1+11q^7t^2+9q^9t^3+6q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{269}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+4q^{-3}t^{-3}+6q^{-1}t^{-2}+10q^1t^{-1}+$ $+11q^3+12q^5t^1+12q^7t^2+9q^9t^3+6q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{270}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+9q^{-1}t^{-1}+10q^1+11q^3t^1+11q^5t^2+8q^7t^3+6q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{271}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+4q^1t^{-1}+7q^3+11q^5t^1+$ $+13q^7t^2+15q^9t^3+13q^{11}t^4+10q^{13}t^5+7q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
$11a_{272}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+10q^{-1}t^{-1}+11q^1+12q^3t^1+12q^5t^2+9q^7t^3+6q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{273}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+3q^{-13}t^{-6}+7q^{-11}t^{-5}+$ $+10q^{-9}t^{-4}+12q^{-7}t^{-3}+14q^{-5}t^{-2}+12q^{-3}t^{-1}+10q^{-1}+6q^1t^1+3q^3t^2+q^5t^3\big)$
$11a_{274}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+3q^{-7}t^{-4}+7q^{-5}t^{-3}+$ $+10q^{-3}t^{-2}+13q^{-1}t^{-1}+14q^1+12q^3t^1+11q^5t^2+7q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{275}$	$q^5+q^3+u^2\big(q^3t^{-1}+2q^5+5q^7t^1+8q^9t^2+$ $+10q^{11}t^3+11q^{13}t^4+10q^{15}t^5+8q^{17}t^6+6q^{19}t^7+2q^{21}t^8+q^{23}t^9\big)$
$11a_{276}$	$q^5+q^3+u^2\big(q^3t^{-1}+3q^5+6q^7t^1+10q^9t^2+$ $+13q^{11}t^3+13q^{13}t^4+13q^{15}t^5+10q^{17}t^6+7q^{19}t^7+3q^{21}t^8+q^{23}t^9\big)$

Name	BN
$11a_{277}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+3q^{-1}t^{-2}+6q^1t^{-1}+7q^3+$ $+11q^5t^1+11q^7t^2+10q^9t^3+9q^{11}t^4+5q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$
$11a_{278}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+3q^{-3}t^{-2}+7q^{-1}t^{-1}+$ $+8q^1+11q^3t^1+12q^5t^2+10q^7t^3+9q^9t^4+5q^{11}t^5+3q^{13}t^6+q^{15}t^7\big)$
$11a_{279}$	$q^3+q^1+u^2\big(q^{-7}t^{-5}+2q^{-5}t^{-4}+4q^{-3}t^{-3}+5q^{-1}t^{-2}+$ $+7q^1t^{-1}+7q^3+6q^5t^1+6q^7t^2+4q^9t^3+2q^{11}t^4+q^{13}t^5\big)$
$11a_{280}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+5q^{-5}t^{-3}+$ $+6q^{-3}t^{-2}+8q^{-1}t^{-1}+9q^1+7q^3t^1+7q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5\big)$
$11a_{281}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+3q^{-3}t^{-3}+6q^{-1}t^{-2}+10q^1t^{-1}+$ $+11q^3+13q^5t^1+12q^7t^2+10q^9t^3+7q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{282}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+3q^{-9}t^{-4}+5q^{-7}t^{-3}+$ $+8q^{-5}t^{-2}+10q^{-3}t^{-1}+10q^{-1}+9q^1t^1+8q^3t^2+5q^5t^3+3q^7t^4+q^9t^5\big)$
$11a_{283}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+3q^{-13}t^{-6}+6q^{-11}t^{-5}+$ $+9q^{-9}t^{-4}+11q^{-7}t^{-3}+13q^{-5}t^{-2}+11q^{-3}t^{-1}+9q^{-1}+6q^1t^1+3q^3t^2+q^5t^3\big)$
$11a_{284}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+4q^{-1}t^{-2}+8q^1t^{-1}+11q^3+$ $+14q^5t^1+15q^7t^2+14q^9t^3+11q^{11}t^4+7q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$
$11a_{285}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+4q^{-7}t^{-4}+7q^{-5}t^{-3}+$ $+10q^{-3}t^{-2}+13q^{-1}t^{-1}+13q^1+12q^3t^1+10q^5t^2+6q^7t^3+3q^9t^4+q^{11}t^5\big)$

Name	BN
$11a_{286}$	$q^3+q^1+u^2(q^{-5}t^{-4}+3q^{-3}t^{-3}+6q^{-1}t^{-2}+9q^1t^{-1}+$ $+11q^3+12q^5t^1+11q^7t^2+10q^9t^3+6q^{11}t^4+3q^{13}t^5+q^{15}t^6)$
$11a_{287}$	$q^1+q^{-1}+u^2(q^{-9}t^{-5}+3q^{-7}t^{-4}+7q^{-5}t^{-3}+$ $+11q^{-3}t^{-2}+14q^{-1}t^{-1}+15q^1+14q^3t^1+12q^5t^2+8q^7t^3+4q^9t^4+q^{11}t^5)$
$11a_{288}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+4q^{-5}t^{-3}+9q^{-3}t^{-2}+$ $+13q^{-1}t^{-1}+16q^1+17q^3t^1+16q^5t^2+13q^7t^3+8q^9t^4+4q^{11}t^5+q^{13}t^6)$
$11a_{289}$	$q^1+q^{-1}+u^2(q^{-7}t^{-4}+3q^{-5}t^{-3}+6q^{-3}t^{-2}+$ $+9q^{-1}t^{-1}+11q^1+12q^3t^1+11q^5t^2+9q^7t^3+6q^9t^4+3q^{11}t^5+q^{13}t^6)$
$11a_{290}$	$q^1+q^{-1}+u^2(q^{-11}t^{-6}+3q^{-9}t^{-5}+6q^{-7}t^{-4}+$ $+8q^{-5}t^{-3}+11q^{-3}t^{-2}+12q^{-1}t^{-1}+10q^1+9q^3t^1+6q^5t^2+3q^7t^3+q^9t^4)$
$11a_{291}$	$q^7+q^5+u^2(3q^{11}t^2+4q^{13}t^3+5q^{15}t^4+9q^{17}t^5+$ $+7q^{19}t^6+8q^{21}t^7+6q^{23}t^8+4q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11})$
$11a_{292}$	$q^5+q^3+u^2(4q^9t^2+6q^{11}t^3+8q^{13}t^4+11q^{15}t^5+$ $+10q^{17}t^6+10q^{19}t^7+7q^{21}t^8+5q^{23}t^9+2q^{25}t^{10}+q^{27}t^{11})$
$11a_{293}$	$q^{-3}+q^{-5}+u^2(q^{-15}t^{-6}+2q^{-13}t^{-5}+4q^{-11}t^{-4}+$ $+4q^{-9}t^{-3}+7q^{-7}t^{-2}+6q^{-5}t^{-1}+5q^{-3}+5q^{-1}t^1+3q^1t^2+2q^3t^3+q^5t^4)$
$11a_{294}$	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+7q^{-7}t^{-3}+10q^{-5}t^{-2}+10q^{-3}t^{-1}+9q^{-1}+8q^1t^1+5q^3t^2+3q^5t^3+q^7t^4)$

Name	BN
$11a_{295}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+2q^{-17}t^{-7}+5q^{-15}t^{-6}+ +6q^{-13}t^{-5}+9q^{-11}t^{-4}+9q^{-9}t^{-3}+8q^{-7}t^{-2}+7q^{-5}t^{-1}+4q^{-3}+2q^{-1}t^1+q^1t^2\big)$
$11a_{296}$	$q^{-1}+q^{-3}+u^2\big(q^{-17}t^{-8}+2q^{-15}t^{-7}+5q^{-13}t^{-6}+ +6q^{-11}t^{-5}+9q^{-9}t^{-4}+9q^{-7}t^{-3}+8q^{-5}t^{-2}+8q^{-3}t^{-1}+4q^{-1}+2q^1t^1+q^3t^2\big)$
$11a_{297}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+4q^{-11}t^{-5}+8q^{-9}t^{-4}+ +11q^{-7}t^{-3}+14q^{-5}t^{-2}+15q^{-3}t^{-1}+13q^{-1}+10q^1t^1+7q^3t^2+3q^5t^3+q^7t^4\big)$
$11a_{298}$	$q^7+q^5+u^2\big(3q^{11}t^2+5q^{13}t^3+7q^{15}t^4+11q^{17}t^5+ +10q^{19}t^6+11q^{21}t^7+8q^{23}t^8+6q^{25}t^9+3q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{299}$	$q^5+q^3+u^2\big(3q^9t^2+4q^{11}t^3+6q^{13}t^4+8q^{15}t^5+ +7q^{17}t^6+8q^{19}t^7+5q^{21}t^8+4q^{23}t^9+2q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{300}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+3q^{-7}t^{-4}+7q^{-5}t^{-3}+ +9q^{-3}t^{-2}+12q^{-1}t^{-1}+13q^1+11q^3t^1+10q^5t^2+6q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11a_{301}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+4q^{-11}t^{-5}+9q^{-9}t^{-4}+ +12q^{-7}t^{-3}+16q^{-5}t^{-2}+17q^{-3}t^{-1}+15q^{-1}+12q^1t^1+8q^3t^2+4q^5t^3+q^7t^4\big)$
$11a_{302}$	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+3q^{-15}t^{-6}+7q^{-13}t^{-5}+ +10q^{-11}t^{-4}+12q^{-9}t^{-3}+14q^{-7}t^{-2}+12q^{-5}t^{-1}+10q^{-3}+6q^{-1}t^1+4q^1t^2+q^3t^3\big)$
$11a_{303}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+3q^{-3}t^{-2}+7q^{-1}t^{-1}+ +9q^1+11q^3t^1+13q^5t^2+11q^7t^3+9q^9t^4+6q^{11}t^5+3q^{13}t^6+q^{15}t^7\big)$

Name	BN
$11a_{304}$	$q^5+q^3+u^2\big(q^3t^{-1}+2q^5+5q^7t^1+7q^9t^2+ +9q^{11}t^3+10q^{13}t^4+9q^{15}t^5+7q^{17}t^6+5q^{19}t^7+2q^{21}t^8+q^{23}t^9\big)$
$11a_{305}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+3q^{-9}t^{-4}+5q^{-7}t^{-3}+ +9q^{-5}t^{-2}+10q^{-3}t^{-1}+11q^{-1}+10q^1t^1+8q^3t^2+6q^5t^3+3q^7t^4+q^9t^5\big)$
$11a_{306}$	$q^{-3}+q^{-5}+u^2\big(q^{-17}t^{-7}+3q^{-15}t^{-6}+5q^{-13}t^{-5}+ +7q^{-11}t^{-4}+8q^{-9}t^{-3}+9q^{-7}t^{-2}+7q^{-5}t^{-1}+6q^{-3}+3q^{-1}t^1+2q^1t^2+q^3t^3\big)$
$11a_{307}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+3q^{-11}t^{-5}+ +5q^{-9}t^{-4}+6q^{-7}t^{-3}+7q^{-5}t^{-2}+6q^{-3}t^{-1}+5q^{-1}+3q^1t^1+2q^3t^2+q^5t^3\big)$
$11a_{308}$	$q^7+q^5+u^2\big(q^5t^{-1}+q^7+2q^9t^1+4q^{11}t^2+ +5q^{13}t^3+5q^{15}t^4+6q^{17}t^5+4q^{19}t^6+4q^{21}t^7+2q^{23}t^8+q^{25}t^9\big)$
$11a_{309}$	$q^5+q^3+u^2\big(q^3t^{-1}+2q^5+3q^7t^1+6q^9t^2+ +7q^{11}t^3+7q^{13}t^4+8q^{15}t^5+5q^{17}t^6+4q^{19}t^7+2q^{21}t^8+q^{23}t^9\big)$
$11a_{310}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+2q^7t^1+4q^9t^2+ +4q^{11}t^3+5q^{13}t^4+5q^{15}t^5+3q^{17}t^6+3q^{19}t^7+q^{21}t^8+q^{23}t^9\big)$
$11a_{311}$	$q^3+q^1+u^2\big(q^1t^{-1}+2q^3+3q^5t^1+6q^7t^2+ +6q^9t^3+6q^{11}t^4+6q^{13}t^5+4q^{15}t^6+3q^{17}t^7+q^{19}t^8+q^{21}t^9\big)$
$11a_{312}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+3q^{-13}t^{-6}+5q^{-11}t^{-5}+ +8q^{-9}t^{-4}+9q^{-7}t^{-3}+10q^{-5}t^{-2}+9q^{-3}t^{-1}+7q^{-1}+4q^1t^1+2q^3t^2+q^5t^3\big)$

Name	BN
$11a_{313}$	$q^1+q^{-1}+u^2\big(q^{-13}t^{-7}+2q^{-11}t^{-6}+3q^{-9}t^{-5}+ +5q^{-7}t^{-4}+5q^{-5}t^{-3}+6q^{-3}t^{-2}+6q^{-1}t^{-1}+4q^1+3q^3t^1+2q^5t^2+q^7t^3\big)$
$11a_{314}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+4q^{-1}t^{-2}+8q^1t^{-1}+10q^3+ +14q^5t^1+14q^7t^2+13q^9t^3+11q^{11}t^4+6q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$
$11a_{315}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+7q^{-3}t^{-2}+ +10q^{-1}t^{-1}+12q^1+13q^3t^1+12q^5t^2+10q^7t^3+6q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{316}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+ +7q^{-1}t^{-1}+9q^1+10q^3t^1+9q^5t^2+8q^7t^3+5q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{317}$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+2q^{-9}t^{-5}+5q^{-7}t^{-4}+ +7q^{-5}t^{-3}+9q^{-3}t^{-2}+11q^{-1}t^{-1}+9q^1+8q^3t^1+6q^5t^2+3q^7t^3+q^9t^4\big)$
$11a_{318}$	$q^7+q^5+u^2\big(3q^{11}t^2+6q^{13}t^3+8q^{15}t^4+11q^{17}t^5+ +11q^{19}t^6+11q^{21}t^7+8q^{23}t^8+6q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{319}$	$q^7+q^5+u^2\big(3q^{11}t^2+5q^{13}t^3+7q^{15}t^4+10q^{17}t^5+ +9q^{19}t^6+10q^{21}t^7+7q^{23}t^8+5q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{320}$	$q^5+q^3+u^2\big(3q^9t^2+4q^{11}t^3+7q^{13}t^4+9q^{15}t^5+ +8q^{17}t^6+9q^{19}t^7+6q^{21}t^8+5q^{23}t^9+2q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{321}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+2q^{-17}t^{-7}+6q^{-15}t^{-6}+ +7q^{-13}t^{-5}+10q^{-11}t^{-4}+10q^{-9}t^{-3}+9q^{-7}t^{-2}+8q^{-5}t^{-1}+4q^{-3}+2q^{-1}t^1+q^1t^2\big)$

Name	BN
$11a_{322}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+3q^{-11}t^{-5}+7q^{-9}t^{-4}+$ $+9q^{-7}t^{-3}+12q^{-5}t^{-2}+13q^{-3}t^{-1}+11q^{-1}+9q^{1}t^{1}+6q^{3}t^{2}+3q^{5}t^{3}+q^{7}t^{4}\big)$
$11a_{323}$	$q^{-1}+q^{-3}+u^2\big(q^{-9}t^{-4}+2q^{-7}t^{-3}+4q^{-5}t^{-2}+$ $+5q^{-3}t^{-1}+6q^{-1}+6q^{1}t^{1}+6q^{3}t^{2}+5q^{5}t^{3}+3q^{7}t^{4}+2q^{9}t^{5}+q^{11}t^{6}\big)$
$11a_{324}$	$q^{-1}+q^{-3}+u^2\big(q^{-17}t^{-8}+2q^{-15}t^{-7}+5q^{-13}t^{-6}+$ $+5q^{-11}t^{-5}+8q^{-9}t^{-4}+8q^{-7}t^{-3}+7q^{-5}t^{-2}+7q^{-3}t^{-1}+3q^{-1}+2q^{1}t^{1}+q^{3}t^{2}\big)$
$11a_{325}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+2q^{-11}t^{-5}+4q^{-9}t^{-4}+$ $+6q^{-7}t^{-3}+7q^{-5}t^{-2}+8q^{-3}t^{-1}+7q^{-1}+5q^{1}t^{1}+4q^{3}t^{2}+2q^{5}t^{3}+q^{7}t^{4}\big)$
$11a_{326}$	$q^{1}+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+7q^{-3}t^{-2}+$ $+10q^{-1}t^{-1}+13q^{1}+14q^{3}t^{1}+13q^{5}t^{2}+11q^{7}t^{3}+7q^{9}t^{4}+4q^{11}t^{5}+q^{13}t^{6}\big)$
$11a_{327}$	$q^{3}+q^{1}+u^2\big(q^{-1}t^{-2}+4q^{1}t^{-1}+8q^{3}+11q^{5}t^{1}+$ $+15q^{7}t^{2}+16q^{9}t^{3}+14q^{11}t^{4}+12q^{13}t^{5}+7q^{15}t^{6}+4q^{17}t^{7}+q^{19}t^{8}\big)$
$11a_{328}$	$q^{5}+q^{3}+u^2\big(q^{3}t^{-1}+3q^{5}+6q^{7}t^{1}+9q^{9}t^{2}+$ $+12q^{11}t^{3}+12q^{13}t^{4}+12q^{15}t^{5}+9q^{17}t^{6}+6q^{19}t^{7}+3q^{21}t^{8}+q^{23}t^{9}\big)$
$11a_{329}$	$q^{5}+q^{3}+u^2\big(4q^{9}t^{2}+6q^{11}t^{3}+9q^{13}t^{4}+12q^{15}t^{5}+$ $+11q^{17}t^{6}+12q^{19}t^{7}+8q^{21}t^{8}+6q^{23}t^{9}+3q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{330}$	$q^{-3}+q^{-5}+u^2\big(q^{-15}t^{-6}+2q^{-13}t^{-5}+4q^{-11}t^{-4}+$ $+5q^{-9}t^{-3}+7q^{-7}t^{-2}+7q^{-5}t^{-1}+6q^{-3}+5q^{-1}t^{1}+4q^{1}t^{2}+2q^{3}t^{3}+q^{5}t^{4}\big)$

Name	BN
$11a_{331}$	$q^{-1}+q^{-3}+u^2(q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+7q^{-7}t^{-3}+9q^{-5}t^{-2}+10q^{-3}t^{-1}+8q^{-1}+7q^{1}t^{1}+5q^{3}t^{2}+2q^{5}t^{3}+q^{7}t^{4})$
$11a_{332}$	$q^{1}+q^{-1}+u^2(q^{-7}t^{-4}+4q^{-5}t^{-3}+9q^{-3}t^{-2}+$ $+12q^{-1}t^{-1}+15q^{1}+16q^{3}t^{1}+14q^{5}t^{2}+12q^{7}t^{3}+7q^{9}t^{4}+3q^{11}t^{5}+q^{13}t^{6})$
$11a_{333}$	$q^{1}+q^{-1}+u^2(q^{-13}t^{-7}+q^{-11}t^{-6}+2q^{-9}t^{-5}+$ $+4q^{-7}t^{-4}+4q^{-5}t^{-3}+5q^{-3}t^{-2}+5q^{-1}t^{-1}+4q^{1}+3q^{3}t^{1}+2q^{5}t^{2}+q^{7}t^{3})$
$11a_{334}$	$q^{9}+q^{7}+u^2(q^{13}t^{2}+2q^{15}t^{3}+2q^{17}t^{4}+4q^{19}t^{5}+$ $+3q^{21}t^{6}+4q^{23}t^{7}+3q^{25}t^{8}+3q^{27}t^{9}+q^{29}t^{10}+q^{31}t^{11})$
$11a_{335}$	$q^{7}+q^{5}+u^2(2q^{11}t^{2}+3q^{13}t^{3}+4q^{15}t^{4}+6q^{17}t^{5}+$ $+6q^{19}t^{6}+6q^{21}t^{7}+4q^{23}t^{8}+4q^{25}t^{9}+q^{27}t^{10}+q^{29}t^{11})$
$11a_{336}$	$q^{7}+q^{5}+u^2(2q^{11}t^{2}+2q^{13}t^{3}+3q^{15}t^{4}+5q^{17}t^{5}+$ $+4q^{19}t^{6}+5q^{21}t^{7}+3q^{23}t^{8}+3q^{25}t^{9}+q^{27}t^{10}+q^{29}t^{11})$
$11a_{337}$	$q^{5}+q^{3}+u^2(3q^{9}t^{2}+4q^{11}t^{3}+6q^{13}t^{4}+7q^{15}t^{5}+$ $+7q^{17}t^{6}+7q^{19}t^{7}+4q^{21}t^{8}+4q^{23}t^{9}+q^{25}t^{10}+q^{27}t^{11})$
$11a_{338}$	$q^{9}+q^{7}+u^2(q^{13}t^{2}+3q^{15}t^{3}+3q^{17}t^{4}+5q^{19}t^{5}+$ $+5q^{21}t^{6}+6q^{23}t^{7}+4q^{25}t^{8}+4q^{27}t^{9}+2q^{29}t^{10}+q^{31}t^{11})$
$11a_{339}$	$q^{7}+q^{5}+u^2(q^{11}t^{2}+2q^{13}t^{3}+3q^{15}t^{4}+4q^{17}t^{5}+$ $+4q^{19}t^{6}+5q^{21}t^{7}+3q^{23}t^{8}+3q^{25}t^{9}+q^{27}t^{10}+q^{29}t^{11})$

Name	BN
$11a_{340}$	$q^7+q^5+u^2(2q^{11}t^2+3q^{13}t^3+5q^{15}t^4+7q^{17}t^5+ + 6q^{19}t^6+8q^{21}t^7+5q^{23}t^8+4q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11})$
$11a_{341}$	$q^5+q^3+u^2(2q^9t^2+2q^{11}t^3+4q^{13}t^4+5q^{15}t^5+ + 4q^{17}t^6+5q^{19}t^7+3q^{21}t^8+3q^{23}t^9+q^{25}t^{10}+q^{27}t^{11})$
$11a_{342}$	$q^5+q^3+u^2(q^9t^2+q^{11}t^3+2q^{13}t^4+2q^{15}t^5+ + 2q^{17}t^6+2q^{19}t^7+q^{21}t^8+2q^{23}t^9+q^{27}t^{11})$
$11a_{343}$	$q^3+q^1+u^2(2q^7t^2+q^9t^3+2q^{11}t^4+2q^{13}t^5+ + 2q^{15}t^6+2q^{17}t^7+q^{19}t^8+2q^{21}t^9+q^{25}t^{11})$
$11a_{344}$	$q^5+q^3+u^2(q^3t^{-1}+2q^5+4q^7t^1+8q^9t^2+ + 10q^{11}t^3+10q^{13}t^4+11q^{15}t^5+8q^{17}t^6+6q^{19}t^7+3q^{21}t^8+q^{23}t^9)$
$11a_{345}$	$q^3+q^1+u^2(q^1t^{-1}+2q^3+3q^5t^1+6q^7t^2+ + 7q^9t^3+7q^{11}t^4+7q^{13}t^5+5q^{15}t^6+4q^{17}t^7+2q^{19}t^8+q^{21}t^9)$
$11a_{346}$	$q^5+q^3+u^2(q^{-1}t^{-3}+2q^1t^{-2}+4q^3t^{-1}+5q^5+ + 7q^7t^1+7q^9t^2+7q^{11}t^3+6q^{13}t^4+4q^{15}t^5+2q^{17}t^6+q^{19}t^7)$
$11a_{347}$	$q^3+q^1+u^2(q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+6q^3+ + 8q^5t^1+10q^7t^2+8q^9t^3+7q^{11}t^4+5q^{13}t^5+2q^{15}t^6+q^{17}t^7)$
$11a_{348}$	$q^5+q^3+u^2(q^1t^{-2}+4q^3t^{-1}+5q^5+9q^7t^1+ + 11q^9t^2+12q^{11}t^3+11q^{13}t^4+9q^{15}t^5+6q^{17}t^6+3q^{19}t^7+q^{21}t^8)$

Name	BN
$11a_{349}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+4q^1t^{-1}+6q^3+10q^5t^1+$ $+12q^7t^2+13q^9t^3+12q^{11}t^4+9q^{13}t^5+6q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
$11a_{350}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+4q^{-5}t^{-3}+7q^{-3}t^{-2}+$ $+12q^{-1}t^{-1}+14q^1+15q^3t^1+15q^5t^2+11q^7t^3+8q^9t^4+4q^{11}t^5+q^{13}t^6\big)$
$11a_{351}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+4q^{-5}t^{-3}+7q^{-3}t^{-2}+$ $+11q^{-1}t^{-1}+13q^1+13q^3t^1+13q^5t^2+10q^7t^3+6q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11a_{352}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+4q^{-3}t^{-3}+5q^{-1}t^{-2}+9q^1t^{-1}+$ $+10q^3+10q^5t^1+11q^7t^2+8q^9t^3+5q^{11}t^4+3q^{13}t^5+q^{15}t^6\big)$
$11a_{353}$	$q^7+q^5+u^2\big(3q^{11}t^2+5q^{13}t^3+7q^{15}t^4+10q^{17}t^5+$ $+10q^{19}t^6+10q^{21}t^7+7q^{23}t^8+6q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{354}$	$q^5+q^3+u^2\big(3q^9t^2+4q^{11}t^3+7q^{13}t^4+8q^{15}t^5+$ $+8q^{17}t^6+9q^{19}t^7+5q^{21}t^8+5q^{23}t^9+2q^{25}t^{10}+q^{27}t^{11}\big)$
$11a_{355}$	$q^9+q^7+u^2\big(q^{13}t^2+2q^{15}t^3+2q^{17}t^4+3q^{19}t^5+$ $+3q^{21}t^6+4q^{23}t^7+2q^{25}t^8+3q^{27}t^9+q^{29}t^{10}+q^{31}t^{11}\big)$
$11a_{356}$	$q^7+q^5+u^2\big(2q^{11}t^2+3q^{13}t^3+5q^{15}t^4+6q^{17}t^5+$ $+6q^{19}t^6+7q^{21}t^7+4q^{23}t^8+4q^{25}t^9+q^{27}t^{10}+q^{29}t^{11}\big)$
$11a_{357}$	$q^7+q^5+u^2\big(2q^{11}t^2+3q^{13}t^3+5q^{15}t^4+7q^{17}t^5+$ $+7q^{19}t^6+8q^{21}t^7+5q^{23}t^8+5q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11}\big)$

Name	BN
$11a_{358}$	$q^7+q^5+u^2(q^{11}t^2+q^{13}t^3+2q^{15}t^4+2q^{17}t^5+$ $+2q^{19}t^6+3q^{21}t^7+q^{23}t^8+2q^{25}t^9+q^{29}t^{11})$
$11a_{359}$	$q^5+q^3+u^2(2q^9t^2+2q^{11}t^3+4q^{13}t^4+4q^{15}t^5+$ $+4q^{17}t^6+4q^{19}t^7+2q^{21}t^8+3q^{23}t^9+q^{27}t^{11})$
$11a_{360}$	$q^5+q^3+u^2(2q^9t^2+2q^{11}t^3+4q^{13}t^4+4q^{15}t^5+$ $+4q^{17}t^6+5q^{19}t^7+2q^{21}t^8+3q^{23}t^9+q^{25}t^{10}+q^{27}t^{11})$
$11a_{361}$	$q^5+q^3+u^2(2q^9t^2+2q^{11}t^3+5q^{13}t^4+5q^{15}t^5+$ $+5q^{17}t^6+6q^{19}t^7+3q^{21}t^8+4q^{23}t^9+q^{25}t^{10}+q^{27}t^{11})$
$11a_{362}$	$q^3+q^1+u^2(2q^7t^2+q^9t^3+3q^{11}t^4+2q^{13}t^5+$ $+3q^{15}t^6+3q^{17}t^7+q^{19}t^8+3q^{21}t^9+q^{25}t^{11})$
$11a_{363}$	$q^3+q^1+u^2(2q^7t^2+q^9t^3+3q^{11}t^4+2q^{13}t^5+$ $+2q^{15}t^6+3q^{17}t^7+q^{19}t^8+2q^{21}t^9+q^{25}t^{11})$
$11a_{364}$	$q^9+q^7+u^2(q^{13}t^2+q^{15}t^3+q^{17}t^4+2q^{19}t^5+$ $+q^{21}t^6+2q^{23}t^7+q^{25}t^8+2q^{27}t^9+q^{31}t^{11})$
$11a_{365}$	$q^7+q^5+u^2(2q^{11}t^2+2q^{13}t^3+3q^{15}t^4+4q^{17}t^5+$ $+4q^{19}t^6+4q^{21}t^7+2q^{23}t^8+3q^{25}t^9+q^{29}t^{11})$
$11a_{366}$	$q^5+q^3+u^2(3q^9t^2+4q^{11}t^3+6q^{13}t^4+6q^{15}t^5+$ $+7q^{17}t^6+6q^{19}t^7+3q^{21}t^8+4q^{23}t^9+q^{27}t^{11})$

Name	BN
$11a_{367}$	$q^{11}+q^{9}+u^{2}\big(q^{17}t^{3}+q^{21}t^{5}+q^{25}t^{7}+q^{29}t^{9}+q^{33}t^{11}\big)$
$11n_1$	$q^{-1}+q^{-3}+u^{2}\big(q^{-17}t^{-8}+q^{-15}t^{-7}+2q^{-13}t^{-6}+$ $+2q^{-11}t^{-5}+2q^{-9}t^{-4}+2q^{-7}t^{-3}+2q^{-5}t^{-2}+q^{-3}t^{-1}\big)$
$11n_2$	$q^{5}+q^{3}+u^{2}\big(q^{3}t^{-1}+q^{5}+3q^{7}t^{1}+4q^{9}t^{2}+$ $+5q^{11}t^{3}+5q^{13}t^{4}+4q^{15}t^{5}+3q^{17}t^{6}+2q^{19}t^{7}\big)$
$11n_3$	$q^{-1}+q^{-3}+u^{2}\big(q^{-9}t^{-4}+2q^{-7}t^{-3}+3q^{-5}t^{-2}+$ $+4q^{-3}t^{-1}+3q^{-1}+3q^{1}t^{1}+3q^{3}t^{2}+q^{5}t^{3}+q^{7}t^{4}\big)$
$11n_4$	$q^{1}+q^{-1}+u^{2}\big(q^{-3}t^{-2}+2q^{-1}t^{-1}+3q^{1}+4q^{3}t^{1}+$ $+4q^{5}t^{2}+4q^{7}t^{3}+3q^{9}t^{4}+2q^{11}t^{5}+q^{13}t^{6}\big)$
$11n_5$	$q^{3}+q^{1}+u^{2}\big(2q^{1}t^{-1}+2q^{3}+5q^{5}t^{1}+6q^{7}t^{2}+$ $+6q^{9}t^{3}+6q^{11}t^{4}+4q^{13}t^{5}+3q^{15}t^{6}+q^{17}t^{7}\big)$
$11n_6$	$q^{1}+q^{-1}+u^{2}\big(q^{-11}t^{-6}+q^{-9}t^{-5}+2q^{-7}t^{-4}+$ $+2q^{-5}t^{-3}+2q^{-3}t^{-2}+q^{-3}+2q^{-1}t^{-1}+q^{1}+q^{1}+q^{3}t^{1}+q^{3}t^{2}+q^{7}t^{4}\big)$
$11n_7$	$q^{3}+q^{1}+u^{2}\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^{1}t^{-1}+5q^{3}+$ $+5q^{5}t^{1}+6q^{7}t^{2}+5q^{9}t^{3}+3q^{11}t^{4}+2q^{13}t^{5}\big)$
$11n_8$	$q^{-3}+q^{-5}+u^{2}\big(q^{-15}t^{-6}+2q^{-13}t^{-5}+4q^{-11}t^{-4}+$ $+4q^{-9}t^{-3}+5q^{-7}t^{-2}+4q^{-5}t^{-1}+3q^{-3}+2q^{-1}t^{1}+q^{1}t^{2}\big)$

Name	BN
$11n_9$	$q^7+q^5+u^2\big(q^5t^{-1}+q^9t^1+q^{11}t^2+q^{11}+$ $+q^{13}t^3+q^{13}+q^{15}t^4+2q^{15}+q^{17}t^5+2q^{17}t^6+q^{19}t^7+q^{21}t^8+q^{23}t^9\big)$
$11n_{10}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+2q^{-17}t^{-7}+4q^{-15}t^{-6}+$ $+5q^{-13}t^{-5}+6q^{-11}t^{-4}+5q^{-9}t^{-3}+5q^{-7}t^{-2}+3q^{-5}t^{-1}+q^{-3}\big)$
$11n_{11}$	$q^3+q^1+u^2\big(q^1t^{-1}+2q^3+3q^5t^1+5q^7t^2+$ $+5q^9t^3+4q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11n_{12}$	$q^3+q^1+u^2\big(q^3t^1+q^5t^2+q^7+q^9t^3+2q^9t^4+$ $+q^{11}t^5+q^{13}t^6+q^{15}t^7\big)$
$11n_{13}$	$q^7+q^5+u^2\big(q^5t^{-1}+q^9t^1+q^{11}t^2+q^{13}t^3+$ $+q^{15}t^4+q^{17}t^5+q^{21}t^7\big)$
$11n_{14}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+q^{-17}t^{-7}+3q^{-15}t^{-6}+$ $+3q^{-13}t^{-5}+4q^{-11}t^{-4}+4q^{-9}t^{-3}+3q^{-7}t^{-2}+2q^{-5}t^{-1}+q^{-3}\big)$
$11n_{15}$	$q^3+q^1+u^2\big(q^1t^{-1}+q^3+2q^5t^1+3q^7t^2+$ $+3q^9t^3+3q^{11}t^4+2q^{13}t^5+q^{15}t^6+q^{17}t^7\big)$
$11n_{16}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+2q^7t^1+3q^9t^2+$ $+3q^{11}t^3+3q^{13}t^4+3q^{15}t^5+q^{17}t^6+q^{19}t^7\big)$
$11n_{17}$	$q^{-1}+q^{-3}+u^2\big(q^{-17}t^{-8}+q^{-15}t^{-7}+3q^{-13}t^{-6}+$ $+3q^{-11}t^{-5}+4q^{-9}t^{-4}+4q^{-7}t^{-3}+3q^{-5}t^{-2}+3q^{-3}t^{-1}+q^{-1}\big)$

Name	BN
$11n_{18}$	$q^1+q^{-1}+u^2\big(q^{-1}t^{-1}+q^1+2q^3t^1+3q^5t^2+$ $+2q^7t^3+3q^9t^4+2q^{11}t^5+q^{13}t^6+q^{15}t^7\big)$
$11n_{19}$	$q^{-1}+q^{-3}+u^2\big(q^{-5}t^{-2}+q^{-5}t^{-1}+q^{-1}t^1+q^1t^2+q^5t^4\big)$
$11n_{20}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-3}+q^{-5}t^{-2}+2q^{-3}t^{-1}+2q^{-1}+$ $+2q^1t^1+2q^3t^2+q^5t^3+q^7t^4\big)$
$11n_{21}$	$q^1+q^{-1}+u^2\big(q^{-3}t^{-2}+2q^{-1}t^{-1}+3q^1+4q^3t^1+$ $+4q^5t^2+4q^7t^3+3q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
$11n_{22}$	$q^3+q^1+u^2\big(q^3+2q^5t^1+4q^7t^2+5q^9t^3+$ $+4q^{11}t^4+5q^{13}t^5+3q^{15}t^6+2q^{17}t^7+q^{19}t^8\big)$
$11n_{23}$	$q^5+q^3+u^2\big(q^1t^{-2}+q^3t^{-1}+2q^5+2q^7t^1+$ $+2q^9t^2+3q^{11}t^3+q^{13}t^4+2q^{15}t^5\big)$
$11n_{24}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-4}+q^{-3}t^{-3}+2q^{-1}t^{-2}+2q^1t^{-1}+$ $+2q^3+2q^5t^1+q^7t^2+q^9t^3\big)$
$11n_{25}$	$q^3+q^1+u^2\big(q^1t^{-1}+q^3+3q^5t^1+4q^7t^2+$ $+4q^9t^3+4q^{11}t^4+3q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11n_{26}$	$q^1+q^{-1}+u^2\big(q^{-1}t^{-1}+2q^1+2q^3t^1+4q^5t^2+$ $+3q^7t^3+3q^9t^4+3q^{11}t^5+q^{13}t^6+q^{15}t^7\big)$

Name	BN
$11n_{27}$	$q^7+q^5+u^2\big(q^5t^{-1}+2q^9t^1+q^{11}t^2+2q^{13}t^3+$ $+2q^{15}t^4+q^{15}+q^{17}t^5+q^{19}t^6\big)$
$11n_{28}$	$q^1+q^{-1}+u^2\big(q^{-1}t^{-1}+q^3t^1+2q^5t^2+q^7t^3+$ $+2q^9t^4+q^{11}t^5+q^{13}t^6+q^{15}t^7\big)$
$11n_{29}$	$q^3+q^1+u^2\big(2q^1t^{-1}+2q^3+3q^5t^1+5q^7t^2+$ $+4q^9t^3+4q^{11}t^4+3q^{13}t^5+q^{15}t^6+q^{17}t^7\big)$
$11n_{30}$	$q^5+q^3+u^2\big(q^3t^{-1}+2q^7t^1+2q^9t^2+3q^{11}t^3+$ $+3q^{13}t^4+2q^{15}t^5+2q^{17}t^6+q^{19}t^7\big)$
$11n_{31}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^7t^1+q^9t^2+q^9+q^{11}t^3+$ $+q^{11}+q^{13}t^4+q^{13}+q^{15}t^5+2q^{15}t^6+q^{17}t^7+q^{19}t^8+q^{21}t^9\big)$
$11n_{32}$	$q^1+q^{-1}+u^2\big(2q^{-5}t^{-3}+3q^{-3}t^{-2}+5q^{-1}t^{-1}+$ $+6q^1+5q^3t^1+6q^5t^2+4q^7t^3+2q^9t^4+q^{11}t^5\big)$
$11n_{33}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+4q^1t^{-1}+3q^3+$ $+4q^5t^1+5q^7t^2+3q^9t^3+3q^{11}t^4+q^{13}t^5\big)$
$11n_{34}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+q^{-5}t^{-3}+2q^{-3}t^{-2}+$ $+q^{-3}+q^{-1}t^{-1}+q^{-1}+q^1+q^1+q^3t^1+2q^3t^2+q^5t^3+q^7t^4+q^9t^5\big)$
$11n_{35}$	$q^5+q^3+u^2\big(2q^7t^1+4q^9t^2+7q^{11}t^3+7q^{13}t^4+$ $+8q^{15}t^5+7q^{17}t^6+5q^{19}t^7+3q^{21}t^8+q^{23}t^9\big)$

Name	BN
$11n_{36}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+5q^3+$ $+5q^5t^1+6q^7t^2+5q^9t^3+3q^{11}t^4+2q^{13}t^5\big)$
$11n_{37}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+2q^{-5}t^{-3}+2q^{-3}t^{-2}+$ $+2q^{-1}t^{-1}+2q^1+q^3t^1+q^5t^2\big)$
$11n_{38}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-5}+q^{-3}t^{-3}+q^{-1}t^{-2}+q^{-1}t^{-1}+q^3+q^5t^2\big)$
$11n_{39}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-4}+q^{-1}t^{-2}+q^{-1}+q^1t^{-1}+$ $+2q^1+2q^3+q^5t^1+3q^5t^2+3q^7t^3+2q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
$11n_{40}$	$q^3+q^1+u^2\big(2q^3+3q^5t^1+6q^7t^2+7q^9t^3+$ $+6q^{11}t^4+7q^{13}t^5+4q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
$11n_{41}$	$q^5+q^3+u^2\big(q^1t^{-2}+2q^3t^{-1}+3q^5+4q^7t^1+$ $+4q^9t^2+5q^{11}t^3+3q^{13}t^4+3q^{15}t^5+q^{17}t^6\big)$
$11n_{42}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+q^{-5}t^{-3}+2q^{-3}t^{-2}+$ $+q^{-3}+q^{-1}t^{-1}+q^{-1}+q^1+q^1+q^3t^1+2q^3t^2+q^5t^3+q^7t^4+q^9t^5\big)$
$11n_{43}$	$q^5+q^3+u^2\big(2q^7t^1+4q^9t^2+7q^{11}t^3+7q^{13}t^4+$ $+8q^{15}t^5+7q^{17}t^6+5q^{19}t^7+3q^{21}t^8+q^{23}t^9\big)$
$11n_{44}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+5q^3+$ $+5q^5t^1+6q^7t^2+5q^9t^3+3q^{11}t^4+2q^{13}t^5\big)$

Name	BN
$11n_{45}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-4}+q^{-1}t^{-2}+q^{-1}+q^1t^{-1}+$ $+2q^1+2q^3+q^5t^1+3q^5t^2+3q^7t^3+2q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
$11n_{46}$	$q^3+q^1+u^2\big(2q^3+3q^5t^1+6q^7t^2+7q^9t^3+$ $+6q^{11}t^4+7q^{13}t^5+4q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
$11n_{47}$	$q^5+q^3+u^2\big(q^1t^{-2}+2q^3t^{-1}+3q^5+4q^7t^1+$ $+4q^9t^2+5q^{11}t^3+3q^{13}t^4+3q^{15}t^5+q^{17}t^6\big)$
$11n_{48}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+2q^{-5}t^{-3}+2q^{-3}t^{-2}+$ $+3q^{-1}t^{-1}+2q^1+q^3t^1+2q^5t^2\big)$
$11n_{49}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-5}+q^{-3}t^{-3}+q^{-1}t^{-2}+q^{-1}t^{-1}+$ $+q^3+q^3t^1+q^5t^2+q^9t^4\big)$
$11n_{50}$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+q^{-9}t^{-5}+2q^{-7}t^{-4}+$ $+2q^{-5}t^{-3}+2q^{-3}t^{-2}+2q^{-1}t^{-1}+q^1+q^3t^1\big)$
$11n_{51}$	$q^1+q^{-1}+u^2\big(q^{-1}t^{-1}+q^1+q^3t^1+3q^5t^2+$ $+2q^7t^3+2q^9t^4+2q^{11}t^5+q^{13}t^6+q^{15}t^7\big)$
$11n_{52}$	$q^3+q^1+u^2\big(2q^1t^{-1}+2q^3+4q^5t^1+5q^7t^2+$ $+5q^9t^3+5q^{11}t^4+3q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11n_{53}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-7}t^{-4}+2q^{-5}t^{-3}+3q^{-3}t^{-2}+$ $+3q^{-1}t^{-1}+3q^1+2q^3t^1+2q^5t^2+q^7t^3\big)$

Name	BN
$11n_{54}$	$q^3+q^1+u^2\big(q^3+q^5t^1+3q^7t^2+4q^9t^3+$ $+3q^{11}t^4+4q^{13}t^5+2q^{15}t^6+2q^{17}t^7+q^{19}t^8\big)$
$11n_{55}$	$q^1+q^{-1}+u^2\big(2q^{-3}t^{-2}+3q^{-1}t^{-1}+4q^1+5q^3t^1+$ $+5q^5t^2+5q^7t^3+3q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
$11n_{56}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+q^{-3}t^{-3}+2q^{-1}t^{-2}+3q^1t^{-1}+$ $+2q^3+3q^5t^1+2q^7t^2+2q^9t^3+q^{11}t^4\big)$
$11n_{57}$	$q^7+q^5+u^2\big(q^5t^{-1}+q^9t^1+q^{11}t^2+q^{11}+$ $+q^{13}t^3+q^{15}t^4+q^{15}+q^{17}t^5+q^{17}t^6\big)$
$11n_{58}$	$q^{-1}+q^{-3}+u^2\big(q^{-7}t^{-3}+2q^{-5}t^{-2}+2q^{-3}t^{-1}+$ $+3q^{-1}+2q^1t^1+3q^3t^2+2q^5t^3+q^7t^4+q^9t^5\big)$
$11n_{59}$	$q^5+q^3+u^2\big(q^7t^1+2q^9t^2+4q^{11}t^3+4q^{13}t^4+$ $+5q^{15}t^5+4q^{17}t^6+3q^{19}t^7+2q^{21}t^8+q^{23}t^9\big)$
$11n_{60}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+2q^1t^{-1}+2q^3+$ $+2q^5t^1+3q^7t^2+2q^9t^3+q^{11}t^4+q^{13}t^5\big)$
$11n_{61}$	$q^5+q^3+u^2\big(q^1t^{-2}+q^3t^{-1}+q^5+2q^7t^1+$ $+q^7+q^9t^2+2q^{11}t^3+q^{13}t^4+q^{13}+q^{15}t^5\big)$
$11n_{62}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+q^{-3}t^{-2}+2q^{-1}t^{-1}+3q^1+$ $+2q^3t^1+3q^5t^2+2q^7t^3+q^9t^4+q^{11}t^5\big)$

Name	BN
$11n_{63}$	$q^3+q^1+u^2\big(q^5t^1+2q^7t^2+3q^9t^3+3q^{11}t^4+$ $+3q^{13}t^5+3q^{15}t^6+2q^{17}t^7+q^{19}t^8+q^{21}t^9\big)$
$11n_{64}$	$q^5+q^3+u^2\big(q^{-1}t^{-3}+2q^3t^{-1}+q^5+q^7t^1+$ $+2q^9t^2+q^{11}t^3+q^{13}t^4+q^{15}t^5\big)$
$11n_{65}$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+q^{-9}t^{-5}+3q^{-7}t^{-4}+$ $+2q^{-5}t^{-3}+3q^{-3}t^{-2}+3q^{-1}t^{-1}+q^1+2q^3t^1\big)$
$11n_{66}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+3q^1t^{-1}+4q^3+6q^5t^1+$ $+6q^7t^2+7q^9t^3+5q^{11}t^4+3q^{13}t^5+2q^{15}t^6\big)$
$11n_{67}$	$q^1+q^{-1}+u^2\big(q^{-3}t^{-3}+q^1t^{-1}+q^3+q^3t^1+$ $+q^5+q^7t^2+q^7t^3+2q^9t^4+q^{11}t^5+q^{13}t^6+q^{15}t^7\big)$
$11n_{68}$	$q^3+q^1+u^2\big(2q^1t^{-1}+2q^3+4q^5t^1+6q^7t^2+$ $+5q^9t^3+5q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11n_{69}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+3q^7t^1+3q^9t^2+$ $+4q^{11}t^3+4q^{13}t^4+3q^{15}t^5+2q^{17}t^6+q^{19}t^7\big)$
$11n_{70}$	$q^3+q^1+u^2\big(q^{-1}t^{-3}+2q^3t^{-1}+q^5+q^7t^1+$ $+2q^9t^2+q^9t^3+q^{13}t^4\big)$
$11n_{71}$	$q^3+q^1+u^2\big(2q^3+2q^5t^1+5q^7t^2+6q^9t^3+$ $+4q^{11}t^4+6q^{13}t^5+3q^{15}t^6+2q^{17}t^7+q^{19}t^8\big)$

Name	BN
$11n_{72}$	$q^5+q^3+u^2\big(2q^7t^1+3q^9t^2+7q^{11}t^3+6q^{13}t^4+$ $+7q^{15}t^5+7q^{17}t^6+4q^{19}t^7+3q^{21}t^8+q^{23}t^9\big)$
$11n_{73}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-4}+q^{-1}t^{-2}+q^1t^{-1}+q^1+$ $+q^3+q^5t^1+q^5t^2+2q^7t^3+q^9t^4+q^{11}t^5+q^{13}t^6\big)$
$11n_{74}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-4}+q^{-1}t^{-2}+q^1t^{-1}+q^1+$ $+q^3+q^5t^1+q^5t^2+2q^7t^3+q^9t^4+q^{11}t^5+q^{13}t^6\big)$
$11n_{75}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+3q^{-11}t^{-5}+$ $+6q^{-9}t^{-4}+4q^{-7}t^{-3}+6q^{-5}t^{-2}+5q^{-3}t^{-1}+2q^{-1}+2q^1t^1\big)$
$11n_{76}$	$q^{-3}+q^{-5}+u^2\big(q^{-13}t^{-5}+3q^{-11}t^{-4}+2q^{-9}t^{-3}+$ $+5q^{-7}t^{-2}+3q^{-5}t^{-1}+3q^{-3}+3q^{-1}t^1+q^1t^2+q^3t^3\big)$
$11n_{77}$	$q^9+q^7+u^2\big(q^{15}t^3+3q^{17}+q^{19}t^5+2q^{19}t^6+$ $+3q^{21}t^7+3q^{23}t^8+2q^{25}t^9+2q^{27}t^{10}+q^{29}t^{11}\big)$
$11n_{78}$	$q^5+q^3+u^2\big(q^1t^{-2}+q^3t^{-1}+3q^5+3q^7t^1+$ $+3q^9t^2+5q^{11}t^3+2q^{13}t^4+3q^{15}t^5+q^{17}t^6\big)$
$11n_{79}$	$q^1+q^{-1}+u^2\big(q^{-3}t^{-3}+2q^1t^{-1}+q^3+q^5t^1+2q^7t^2+q^{11}t^4\big)$
$11n_{80}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+q^{-11}t^{-5}+2q^{-9}t^{-4}+$ $+2q^{-7}t^{-3}+2q^{-5}t^{-2}+q^{-5}+2q^{-3}t^{-1}+q^{-1}+q^{-1}t^1+q^1t^2+q^5t^4\big)$

Name	BN
$11n_{81}$	$q^7+q^5+u^2\big(q^5t^{-1}+3q^9t^1+q^{11}t^2+3q^{13}t^3+$ $+3q^{15}t^4+q^{15}+q^{17}t^5+2q^{19}t^6\big)$
$11n_{82}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+q^{-3}t^{-3}+q^{-1}t^{-2}+2q^1t^{-1}+$ $+q^3+q^5t^1+q^7t^2+q^9t^3\big)$
$11n_{83}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+3q^{-3}t^{-2}+$ $+4q^{-1}t^{-1}+4q^1+4q^3t^1+3q^5t^2+2q^7t^3+q^9t^4\big)$
$11n_{84}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+2q^{-11}t^{-5}+$ $+3q^{-9}t^{-4}+3q^{-7}t^{-3}+3q^{-5}t^{-2}+2q^{-3}t^{-1}+q^{-1}\big)$
$11n_{85}$	$q^1+q^{-1}+u^2\big(q^{-3}t^{-2}+2q^{-1}t^{-1}+3q^1+3q^3t^1+$ $+4q^5t^2+4q^7t^3+2q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
$11n_{86}$	$q^1+q^{-1}+u^2\big(q^{-1}t^{-1}+2q^1+2q^3t^1+3q^5t^2+$ $+3q^7t^3+2q^9t^4+2q^{11}t^5+q^{13}t^6\big)$
$11n_{87}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+3q^{-11}t^{-5}+$ $+4q^{-9}t^{-4}+4q^{-7}t^{-3}+5q^{-5}t^{-2}+3q^{-3}t^{-1}+2q^{-1}+q^1t^1\big)$
$11n_{88}$	$q^7+q^5+u^2\big(q^5t^{-1}+q^9t^1+q^{11}t^2+q^{13}t^3+$ $+q^{15}t^4+q^{15}+q^{17}t^5\big)$
$11n_{89}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+2q^{-17}t^{-7}+4q^{-15}t^{-6}+$ $+4q^{-13}t^{-5}+6q^{-11}t^{-4}+5q^{-9}t^{-3}+4q^{-7}t^{-2}+3q^{-5}t^{-1}+q^{-3}\big)$

Name	BN
$11n_{90}$	$q^5+q^3+u^2\big(q^3t^{-1}+q^5+2q^7t^1+3q^9t^2+$ $+4q^{11}t^3+3q^{13}t^4+3q^{15}t^5+2q^{17}t^6+q^{19}t^7\big)$
$11n_{91}$	$q^{-1}+q^{-3}+u^2\big(q^{-17}t^{-8}+q^{-15}t^{-7}+2q^{-13}t^{-6}+$ $+2q^{-11}t^{-5}+3q^{-9}t^{-4}+2q^{-7}t^{-3}+2q^{-5}t^{-2}+2q^{-3}t^{-1}\big)$
$11n_{92}$	$q^1+q^{-1}+u^2\big(q^{-3}t^{-1}+q^{-1}+q^1t^1+2q^3t^2+$ $+q^5t^3+q^7t^4+q^9t^5\big)$
$11n_{93}$	$q^7+q^5+u^2\big(2q^{11}t^2+3q^{13}t^3+3q^{15}t^4+5q^{17}t^5+$ $+3q^{19}t^6+4q^{21}t^7+2q^{23}t^8+q^{25}t^9\big)$
$11n_{94}$	$q^1+q^{-1}+u^2\big(q^{-3}t^{-2}+2q^{-1}t^{-1}+4q^1+4q^3t^1+$ $+5q^5t^2+5q^7t^3+3q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11n_{95}$	$q^5+q^3+u^2\big(q^7t^1+2q^9t^2+3q^{11}t^3+3q^{13}t^4+$ $+3q^{15}t^5+2q^{17}t^6+2q^{19}t^7\big)$
$11n_{96}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-4}+q^{-3}t^{-3}+q^{-1}t^{-2}+2q^1t^{-1}+$ $+q^1+q^3+q^5t^1+q^5+q^7t^2+q^7t^3+q^{11}t^5\big)$
$11n_{97}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+q^{-5}t^{-3}+q^{-3}t^{-2}+2q^{-1}t^{-1}+$ $+q^1+q^1+q^3t^1+q^5t^2+q^5t^3+q^7t^4+q^{11}t^6\big)$
$11n_{98}$	$q^1+q^{-1}+u^2\big(2q^{-7}t^{-4}+2q^{-5}t^{-3}+5q^{-3}t^{-2}+$ $+6q^{-1}t^{-1}+5q^1+6q^3t^1+4q^5t^2+3q^7t^3+q^9t^4\big)$

Name	BN
$11n_{99}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+q^{-13}t^{-6}+3q^{-11}t^{-5}+$ $+3q^{-9}t^{-4}+3q^{-7}t^{-3}+4q^{-5}t^{-2}+2q^{-3}t^{-1}+2q^{-1}\big)$
$11n_{100}$	$q^{1}+q^{-1}+u^2\big(q^{-3}t^{-2}+3q^{-1}t^{-1}+2q^{1}+4q^{3}t^{1}+$ $+4q^{5}t^{2}+3q^{7}t^{3}+3q^{9}t^{4}+q^{11}t^{5}+q^{13}t^{6}\big)$
$11n_{101}$	$q^{-1}+q^{-3}+u^2\big(q^{-9}t^{-4}+q^{-7}t^{-3}+3q^{-5}t^{-2}+$ $+3q^{-3}t^{-1}+3q^{-1}+3q^{1}t^{1}+2q^{3}t^{2}+2q^{5}t^{3}+q^{7}t^{4}\big)$
$11n_{102}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-7}+q^{-9}t^{-5}+q^{-7}t^{-4}+$ $+q^{-7}t^{-3}+q^{-5}+q^{-3}t^{-2}+q^{-3}t^{-1}+q^{-1}+q^{3}t^{2}\big)$
$11n_{103}$	$q^{-3}+q^{-5}+u^2\big(2q^{-15}t^{-6}+3q^{-13}t^{-5}+5q^{-11}t^{-4}+$ $+5q^{-9}t^{-3}+6q^{-7}t^{-2}+5q^{-5}t^{-1}+3q^{-3}+2q^{-1}t^{1}+q^{1}t^{2}\big)$
$11n_{104}$	$q^{7}+q^{5}+u^2\big(q^{5}t^{-1}+q^{9}t^{1}+q^{11}t^{2}+q^{11}+$ $+q^{13}t^{3}+q^{15}t^{4}+2q^{15}+q^{17}t^{5}+q^{17}t^{6}+q^{21}t^{8}\big)$
$11n_{105}$	$q^{5}+q^{3}+u^2\big(q^{7}t^{1}+3q^{9}t^{2}+5q^{11}t^{3}+5q^{13}t^{4}+$ $+7q^{15}t^{5}+5q^{17}t^{6}+4q^{19}t^{7}+3q^{21}t^{8}+q^{23}t^{9}\big)$
$11n_{106}$	$q^{-1}+q^{-3}+u^2\big(2q^{-5}t^{-2}+q^{-3}t^{-1}+2q^{-1}+2q^{1}t^{1}+$ $+2q^{3}t^{2}+2q^{5}t^{3}+q^{7}t^{4}+q^{9}t^{5}\big)$
$11n_{107}$	$q^{5}+q^{3}+u^2\big(q^{1}t^{-2}+q^{3}t^{-1}+q^{5}+2q^{7}t^{1}+$ $+q^{9}t^{2}+2q^{11}t^{3}+q^{13}t^{4}+q^{15}t^{5}\big)$

Name	BN
$11n_{108}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+2q^{-17}t^{-7}+5q^{-15}t^{-6}+$ $+5q^{-13}t^{-5}+7q^{-11}t^{-4}+6q^{-9}t^{-3}+5q^{-7}t^{-2}+4q^{-5}t^{-1}+q^{-3}\big)$
$11n_{109}$	$q^{-3}+q^{-5}+u^2\big(q^{-19}t^{-8}+q^{-17}t^{-7}+4q^{-15}t^{-6}+$ $+4q^{-13}t^{-5}+5q^{-11}t^{-4}+5q^{-9}t^{-3}+4q^{-7}t^{-2}+3q^{-5}t^{-1}+q^{-3}\big)$
$11n_{110}$	$q^{1}+q^{-1}+u^2\big(2q^{-1}t^{-1}+2q^{1}+3q^{3}t^{1}+4q^{5}t^{2}+$ $+3q^{7}t^{3}+3q^{9}t^{4}+2q^{11}t^{5}+q^{13}t^{6}\big)$
$11n_{111}$	$q^{3}+q^{1}+u^2\big(q^{-1}t^{-3}+q^{3}t^{-1}+q^{5}+q^{5}t^{1}+$ $+q^{7}+q^{9}t^{2}+2q^{9}t^{3}+q^{11}t^{4}+q^{13}t^{5}+q^{15}t^{6}\big)$
$11n_{112}$	$q^{3}+q^{1}+u^2\big(q^{1}t^{-1}+2q^{3}+3q^{5}t^{1}+5q^{7}t^{2}+$ $+5q^{9}t^{3}+4q^{11}t^{4}+4q^{13}t^{5}+2q^{15}t^{6}+q^{17}t^{7}\big)$
$11n_{113}$	$q^{-1}+q^{-3}+u^2\big(q^{-17}t^{-8}+q^{-15}t^{-7}+3q^{-13}t^{-6}+$ $+2q^{-11}t^{-5}+3q^{-9}t^{-4}+3q^{-7}t^{-3}+2q^{-5}t^{-2}+2q^{-3}t^{-1}\big)$
$11n_{114}$	$q^{1}+q^{-1}+u^2\big(q^{-5}t^{-3}+2q^{-3}t^{-2}+4q^{-1}t^{-1}+$ $+4q^{1}+4q^{3}t^{1}+5q^{5}t^{2}+3q^{7}t^{3}+2q^{9}t^{4}+q^{11}t^{5}\big)$
$11n_{115}$	$q^{1}+q^{-1}+u^2\big(q^{-7}t^{-4}+3q^{-5}t^{-3}+4q^{-3}t^{-2}+$ $+7q^{-1}t^{-1}+6q^{1}+6q^{3}t^{1}+6q^{5}t^{2}+3q^{7}t^{3}+2q^{9}t^{4}\big)$
$11n_{116}$	$q^{1}+q^{-1}+u^2\big(q^{-9}t^{-5}+q^{-5}t^{-3}+q^{-3}t^{-2}+q^{-3}t^{-1}+$ $+q^{1}+q^{1}t^{1}+q^{3}t^{2}+q^{7}t^{4}\big)$

Name	BN
$11n_{117}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+q^{-1}t^{-2}+3q^1t^{-1}+2q^3+$ $+3q^5t^1+3q^7t^2+2q^9t^3+2q^{11}t^4\big)$
$11n_{118}$	$q^5+q^3+u^2\big(q^7t^1+q^9t^2+2q^{11}t^3+2q^{13}t^4+$ $+2q^{15}t^5+q^{17}t^6+q^{19}t^7\big)$
$11n_{119}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+5q^{-3}t^{-2}+6q^{-1}t^{-1}+6q^1+4q^3t^1+4q^5t^2+2q^7t^3\big)$
$11n_{120}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+3q^1t^{-1}+3q^3+$ $+4q^5t^1+4q^7t^2+3q^9t^3+2q^{11}t^4+q^{13}t^5\big)$
$11n_{121}$	$q^{-3}+q^{-5}+u^2\big(q^{-15}t^{-6}+q^{-13}t^{-5}+4q^{-11}t^{-4}+$ $+3q^{-9}t^{-3}+4q^{-7}t^{-2}+4q^{-5}t^{-1}+2q^{-3}+2q^{-1}t^1+q^1t^2\big)$
$11n_{122}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+q^{-13}t^{-6}+2q^{-11}t^{-5}+$ $+2q^{-9}t^{-4}+2q^{-7}t^{-3}+3q^{-5}t^{-2}+q^{-3}t^{-1}+q^{-1}\big)$
$11n_{123}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+4q^{-3}t^{-2}+5q^{-1}t^{-1}+5q^1+3q^3t^1+3q^5t^2+q^7t^3\big)$
$11n_{124}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+4q^3+$ $+5q^5t^1+5q^7t^2+4q^9t^3+3q^{11}t^4+q^{13}t^5\big)$
$11n_{125}$	$q^3+q^1+u^2\big(q^1t^{-1}+2q^3+4q^5t^1+5q^7t^2+$ $+6q^9t^3+5q^{11}t^4+4q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$

Name	BN
$11n_{126}$	$q^7+q^5+u^2\big(2q^{11}t^2+2q^{13}t^3+2q^{15}t^4+q^{15}+$ $+4q^{17}t^5+q^{19}t^6+2q^{21}t^7+q^{23}t^8\big)$
$11n_{127}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+3q^{-11}t^{-5}+$ $+5q^{-9}t^{-4}+4q^{-7}t^{-3}+5q^{-5}t^{-2}+4q^{-3}t^{-1}+2q^{-1}+q^1t^1\big)$
$11n_{128}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+$ $+3q^3+3q^5t^1+3q^7t^2+2q^9t^3+q^{11}t^4\big)$
$11n_{129}$	$q^{-1}+q^{-3}+u^2\big(q^{-13}t^{-6}+q^{-11}t^{-5}+3q^{-9}t^{-4}+$ $+3q^{-7}t^{-3}+4q^{-5}t^{-2}+4q^{-3}t^{-1}+2q^{-1}+2q^1t^1+q^3t^2\big)$
$11n_{130}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+2q^{-7}t^{-4}+3q^{-5}t^{-3}+$ $+4q^{-3}t^{-2}+5q^{-1}t^{-1}+4q^1+3q^3t^1+3q^5t^2+q^7t^3\big)$
$11n_{131}$	$q^{-1}+q^{-3}+u^2\big(q^{-11}t^{-5}+3q^{-9}t^{-4}+4q^{-7}t^{-3}+$ $+6q^{-5}t^{-2}+6q^{-3}t^{-1}+5q^{-1}+4q^1t^1+3q^3t^2+q^5t^3\big)$
$11n_{132}$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+q^{-9}t^{-5}+2q^{-7}t^{-4}+$ $+2q^{-5}t^{-3}+2q^{-3}t^{-2}+2q^{-1}t^{-1}+q^1+q^3t^1\big)$
$11n_{133}$	$q^5+q^3+u^2\big(q^1t^{-2}+2q^3t^{-1}+q^5+3q^7t^1+$ $+q^7+2q^9t^2+2q^{11}t^3+2q^{13}t^4+q^{13}+q^{15}t^5\big)$
$11n_{134}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+3q^{-11}t^{-5}+$ $+4q^{-9}t^{-4}+4q^{-7}t^{-3}+4q^{-5}t^{-2}+3q^{-3}t^{-1}+2q^{-1}\big)$

Name	BN
$11n_{135}$	$q^5 + q^3 + u^2\big(q^3t^{-1} + q^7t^1 + q^9t^2 + q^9 + q^{11}t^3 + $ $+ q^{13}t^4 + q^{13} + q^{15}t^5 + q^{15}t^6 + q^{19}t^8\big)$
$11n_{136}$	$q^7 + q^5 + u^2\big(2q^{11}t^2 + 4q^{13}t^3 + 4q^{15}t^4 + 6q^{17}t^5 + $ $+ 5q^{19}t^6 + 5q^{21}t^7 + 3q^{23}t^8 + 2q^{25}t^9\big)$
$11n_{137}$	$q^{-3} + q^{-5} + u^2\big(2q^{-15}t^{-6} + 2q^{-13}t^{-5} + 5q^{-11}t^{-4} + $ $+ 4q^{-9}t^{-3} + 5q^{-7}t^{-2} + 5q^{-5}t^{-1} + 2q^{-3} + 2q^{-1}t^1 + q^1t^2\big)$
$11n_{138}$	$q^1 + q^{-1} + u^2\big(q^{-7}t^{-5} + 2q^{-3}t^{-3} + q^{-1}t^{-2} + q^1t^{-1} + 2q^3 + q^7t^2\big)$
$11n_{139}$	$q^1 + q^{-1} + u^2\big(q^7t^3 + q^{11}t^5 + q^{13}t^6 + q^{17}t^8\big)$
$11n_{140}$	$q^{-1} + q^{-3} + u^2\big(2q^{-13}t^{-6} + q^{-11}t^{-5} + 4q^{-9}t^{-4} + $ $+ 4q^{-7}t^{-3} + 4q^{-5}t^{-2} + 5q^{-3}t^{-1} + 2q^{-1} + 2q^1t^1 + q^3t^2\big)$
$11n_{141}$	$q^1 + q^{-1} + u^2\big(q^{-9}t^{-5} + 2q^{-5}t^{-3} + q^{-3}t^{-2} + 2q^{-1}t^{-1} + $ $+ 2q^1 + 2q^5t^2\big)$
$11n_{142}$	$q^1 + q^{-1} + u^2\big(q^{-9}t^{-5} + q^{-7}t^{-4} + 3q^{-5}t^{-3} + 2q^{-3}t^{-2} + $ $+ 3q^{-1}t^{-1} + 3q^1 + q^3t^1 + 2q^5t^2\big)$
$11n_{143}$	$q^1 + q^{-1} + u^2\big(q^{-3}t^{-3} + q^1t^{-1} + q^1 + q^3 + q^3t^1 + $ $+ q^5 + q^7t^2 + 2q^7t^3 + q^9t^4 + q^{11}t^5 + q^{13}t^6\big)$

Name	BN
$11n_{144}$	$q^5+q^3+u^2\big(q^7t^1+3q^9t^2+5q^{11}t^3+5q^{13}t^4+$ $+6q^{15}t^5+5q^{17}t^6+4q^{19}t^7+2q^{21}t^8+q^{23}t^9\big)$
$11n_{145}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-4}+q^{-1}t^{-2}+q^{-1}+q^1t^{-1}+$ $+q^1+q^3+q^5t^1+2q^5t^2+q^7t^3+q^9t^4+q^{11}t^5\big)$
$11n_{146}$	$q^3+q^1+u^2\big(q^3+2q^5t^1+5q^7t^2+5q^9t^3+$ $+5q^{11}t^4+6q^{13}t^5+3q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
$11n_{147}$	$q^5+q^3+u^2\big(q^1t^{-2}+2q^3t^{-1}+2q^5+3q^7t^1+$ $+3q^9t^2+3q^{11}t^3+2q^{13}t^4+2q^{15}t^5\big)$
$11n_{148}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+3q^{-1}t^{-2}+4q^1t^{-1}+6q^3+$ $+6q^5t^1+6q^7t^2+6q^9t^3+3q^{11}t^4+2q^{13}t^5\big)$
$11n_{149}$	$q^5+q^3+u^2\big(q^1t^{-2}+2q^3t^{-1}+q^5+3q^7t^1+$ $+2q^9t^2+3q^{11}t^3+2q^{13}t^4+q^{15}t^5+q^{17}t^6\big)$
$11n_{150}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+3q^1t^{-1}+4q^3+6q^5t^1+$ $+6q^7t^2+7q^9t^3+5q^{11}t^4+3q^{13}t^5+2q^{15}t^6\big)$
$11n_{151}$	$q^3+q^1+u^2\big(q^{-1}t^{-3}+q^3t^{-1}+q^5+2q^5t^1+$ $+2q^7+q^9t^2+3q^9t^3+3q^{11}t^4+2q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11n_{152}$	$q^3+q^1+u^2\big(q^{-1}t^{-3}+q^3t^{-1}+q^5+2q^5t^1+$ $+2q^7+q^9t^2+3q^9t^3+3q^{11}t^4+2q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$

Name	BN
$11n_{153}$	$q^1+q^{-1}+u^2\big(q^{-7}t^{-4}+2q^{-5}t^{-3}+3q^{-3}t^{-2}+$ $+5q^{-1}t^{-1}+4q^1+5q^3t^1+4q^5t^2+2q^7t^3+2q^9t^4\big)$
$11n_{154}$	$q^3+q^1+u^2\big(2q^1t^{-1}+3q^3+5q^5t^1+7q^7t^2+$ $+7q^9t^3+6q^{11}t^4+5q^{13}t^5+3q^{15}t^6+q^{17}t^7\big)$
$11n_{155}$	$q^3+q^1+u^2\big(q^{-5}t^{-4}+2q^{-3}t^{-3}+2q^{-1}t^{-2}+5q^1t^{-1}+$ $+3q^3+4q^5t^1+4q^7t^2+2q^9t^3+2q^{11}t^4\big)$
$11n_{156}$	$q^1+q^{-1}+u^2\big(2q^{-3}t^{-2}+4q^{-1}t^{-1}+5q^1+6q^3t^1+$ $+7q^5t^2+6q^7t^3+4q^9t^4+3q^{11}t^5+q^{13}t^6\big)$
$11n_{157}$	$q^1+q^{-1}+u^2\big(q^{-9}t^{-5}+3q^{-7}t^{-4}+4q^{-5}t^{-3}+$ $+5q^{-3}t^{-2}+6q^{-1}t^{-1}+5q^1+4q^3t^1+3q^5t^2+q^7t^3\big)$
$11n_{158}$	$q^5+q^3+u^2\big(q^1t^{-2}+2q^3t^{-1}+2q^5+4q^7t^1+$ $+3q^9t^2+4q^{11}t^3+3q^{13}t^4+2q^{15}t^5+q^{17}t^6\big)$
$11n_{159}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+3q^{-13}t^{-6}+4q^{-11}t^{-5}+$ $+6q^{-9}t^{-4}+6q^{-7}t^{-3}+6q^{-5}t^{-2}+5q^{-3}t^{-1}+3q^{-1}+q^1t^1\big)$
$11n_{160}$	$q^3+q^1+u^2\big(2q^1t^{-1}+3q^3+4q^5t^1+6q^7t^2+$ $+6q^9t^3+5q^{11}t^4+4q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11n_{161}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+2q^1t^{-1}+3q^3+5q^5t^1+$ $+5q^7t^2+6q^9t^3+4q^{11}t^4+3q^{13}t^5+2q^{15}t^6\big)$

Name	BN
$11n_{162}$	$q^3+q^1+u^2\big(q^5t^1+3q^7t^2+4q^9t^3+4q^{11}t^4+$ $+5q^{13}t^5+4q^{15}t^6+3q^{17}t^7+2q^{19}t^8+q^{21}t^9\big)$
$11n_{163}$	$q^3+q^1+u^2\big(q^{-1}t^{-2}+4q^1t^{-1}+5q^3+7q^5t^1+$ $+8q^7t^2+8q^9t^3+6q^{11}t^4+4q^{13}t^5+2q^{15}t^6\big)$
$11n_{164}$	$q^5+q^3+u^2\big(q^3t^{-1}+2q^5+3q^7t^1+3q^9t^2+$ $+5q^{11}t^3+3q^{13}t^4+3q^{15}t^5+2q^{17}t^6\big)$
$11n_{165}$	$q^1+q^{-1}+u^2\big(2q^{-5}t^{-3}+4q^{-3}t^{-2}+6q^{-1}t^{-1}+$ $+7q^1+7q^3t^1+7q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11n_{166}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+2q^{-1}t^{-2}+4q^1t^{-1}+4q^3+$ $+5q^5t^1+5q^7t^2+4q^9t^3+3q^{11}t^4+q^{13}t^5\big)$
$11n_{167}$	$q^3+q^1+u^2\big(q^1t^{-1}+3q^3+3q^5t^1+6q^7t^2+$ $+6q^9t^3+4q^{11}t^4+5q^{13}t^5+2q^{15}t^6+q^{17}t^7\big)$
$11n_{168}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+3q^{-1}t^{-2}+4q^1t^{-1}+6q^3+$ $+6q^5t^1+6q^7t^2+6q^9t^3+3q^{11}t^4+2q^{13}t^5\big)$
$11n_{169}$	$q^7+q^5+u^2\big(2q^{11}t^2+2q^{13}t^3+2q^{15}t^4+4q^{17}t^5+$ $+2q^{19}t^6+3q^{21}t^7+q^{23}t^8+q^{25}t^9\big)$
$11n_{170}$	$q^{-1}+q^{-3}+u^2\big(2q^{-13}t^{-6}+2q^{-11}t^{-5}+5q^{-9}t^{-4}+$ $+5q^{-7}t^{-3}+5q^{-5}t^{-2}+6q^{-3}t^{-1}+3q^{-1}+2q^1t^1+q^3t^2\big)$

Name	BN
$11n_{171}$	$q^5+q^3+u^2\big(3q^9t^2+4q^{11}t^3+5q^{13}t^4+6q^{15}t^5+$ $+5q^{17}t^6+5q^{19}t^7+2q^{21}t^8+2q^{23}t^9\big)$
$11n_{172}$	$q^1+q^{-1}+u^2\big(q^{-11}t^{-6}+2q^{-9}t^{-5}+3q^{-7}t^{-4}+$ $+4q^{-5}t^{-3}+4q^{-3}t^{-2}+4q^{-1}t^{-1}+3q^1+2q^3t^1+q^5t^2\big)$
$11n_{173}$	$q^5+q^3+u^2\big(q^1t^{-2}+2q^3t^{-1}+3q^5+3q^7t^1+$ $+4q^9t^2+4q^{11}t^3+2q^{13}t^4+3q^{15}t^5\big)$
$11n_{174}$	$q^5+q^3+u^2\big(2q^7t^1+5q^9t^2+7q^{11}t^3+8q^{13}t^4+$ $+9q^{15}t^5+7q^{17}t^6+6q^{19}t^7+3q^{21}t^8+q^{23}t^9\big)$
$11n_{175}$	$q^5+q^3+u^2\big(q^3t^{-1}+2q^5+3q^7t^1+5q^9t^2+$ $+6q^{11}t^3+5q^{13}t^4+5q^{15}t^5+3q^{17}t^6+2q^{19}t^7\big)$
$11n_{176}$	$q^{-1}+q^{-3}+u^2\big(q^{-15}t^{-7}+2q^{-13}t^{-6}+4q^{-11}t^{-5}+$ $+5q^{-9}t^{-4}+5q^{-7}t^{-3}+6q^{-5}t^{-2}+4q^{-3}t^{-1}+3q^{-1}+q^1t^1\big)$
$11n_{177}$	$q^3+q^1+u^2\big(q^{-3}t^{-3}+3q^{-1}t^{-2}+5q^1t^{-1}+6q^3+$ $+7q^5t^1+7q^7t^2+6q^9t^3+4q^{11}t^4+2q^{13}t^5\big)$
$11n_{178}$	$q^3+q^1+u^2\big(2q^3+4q^5t^1+7q^7t^2+8q^9t^3+$ $+8q^{11}t^4+8q^{13}t^5+5q^{15}t^6+4q^{17}t^7+q^{19}t^8\big)$
$11n_{179}$	$q^1+q^{-1}+u^2\big(q^{-5}t^{-3}+3q^{-3}t^{-2}+5q^{-1}t^{-1}+$ $+6q^1+6q^3t^1+7q^5t^2+5q^7t^3+3q^9t^4+2q^{11}t^5\big)$

Name	BN
$11n_{180}$	$q^7+q^5+u^2\big(2q^{11}t^2+3q^{13}t^3+4q^{15}t^4+5q^{17}t^5+$ $+4q^{19}t^6+5q^{21}t^7+2q^{23}t^8+2q^{25}t^9\big)$
$11n_{181}$	$q^5+q^3+u^2\big(2q^9t^2+2q^{11}t^3+4q^{13}t^4+4q^{15}t^5+$ $+3q^{17}t^6+4q^{19}t^7+q^{21}t^8+2q^{23}t^9\big)$
$11n_{182}$	$q^1+q^{-1}+u^2\big(3q^{-5}t^{-3}+4q^{-3}t^{-2}+7q^{-1}t^{-1}+$ $+8q^1+7q^3t^1+8q^5t^2+5q^7t^3+3q^9t^4+q^{11}t^5\big)$
$11n_{183}$	$q^7+q^5+u^2\big(q^{13}t^3+2q^{13}t^4+2q^{15}+q^{17}t^5+$ $+2q^{17}t^6+3q^{19}t^7+q^{21}t^8+2q^{23}t^9+q^{25}t^{10}\big)$
$11n_{184}$	$q^3+q^1+u^2\big(2q^3+4q^5t^1+6q^7t^2+8q^9t^3+$ $+7q^{11}t^4+7q^{13}t^5+5q^{15}t^6+3q^{17}t^7+q^{19}t^8\big)$
$11n_{185}$	$q^5+q^3+u^2\big(2q^7t^1+5q^9t^2+8q^{11}t^3+8q^{13}t^4+$ $+10q^{15}t^5+8q^{17}t^6+6q^{19}t^7+4q^{21}t^8+q^{23}t^9\big)$

5. The computation of the s-invariant for some families of links

case	< 0	> 0	(Lb12) upper	(s-ineq) upper	(Lb12) lower	(s-ineq) lower
a.	h,k,p,r,t	—	$2(h+k-1)$	$2(h+k)$	$2(h+k-3)$	$2(h+k-1)$
b.	h,k,p,r	t	$2(h+k)$	$2(h+k+1)$	$2(h+k-2)$	$2(h+k)$
c.	h,k,r,t	p	$2(h+k)$	$2(h+k+1)$	$2(h+k-2)$	$2(h+k)$
d.	h,p,r,t	k	$2(h+k-1)$	$2(h+k)$	$2(h+k-4)$	$2(h+k-2)$
e.	h,k,p	t,r	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-1)$	$2(h+k+1)$
f.	h,p,r	t,k	$2(h+k)$	$2(h+k+1)$	$2(h+k-3)$	$2(h+k-1)$
g.	h,k,r	t,p	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-2)$	$2(h+k+1)$
h.	k,p,r	t,h	$2(h+k)$	$2(h+k+1)$	$2(h+k-3)$	$2(h+k-1)$
i.	p,r,t	h,k	$2(h+k-2)$	$2(h+k-1)$	$2(h+k-4)$	$2(h+k-2)$
j.	h,r,t	p,k	$2(h+k-1)$	$2(h+k)$	$2(h+k-3)$	$2(h+k-1)$
k.	h,k	p,r,t	$2(h+k+2)$	$2(h+k+3)$	$2(h+k)$	$2(h+k+2)$
l.	h,r	k,p,t	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-2)$	$2(h+k)$
m.	k,r	h,p,t	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-2)$	$2(h+k)$
n.	r,t	h,k,p	$2(h+k-1)$	$2(h+k)$	$2(h+k-3)$	$2(h+k-1)$
o.	p,h	k,t,r	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-2)$	$2(h+k)$
p.	p,t	h,k,r	$2(h+k-1)$	$2(h+k)$	$2(h+k-3)$	$2(h+k-1)$
q.	r	h,k,p,t	$2(h+k)$	$2(h+k+1)$	$2(h+k-2)$	$2(h+k)$
r.	h	k,p,t,r	$2(h+k+2)$	$2(h+k+3)$	$2(h+k-1)$	$2(h+k+1)$
s.	p	h,k,r,t	$2(h+k)$	$2(h+k+1)$	$2(h+k-2)$	$2(h+k)$
t.	—	h,k,p,r,t	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-1)$	$2(h+k+1)$

TABLE 1. Lower and upper bounds on the s invariant of the link $\lambda(h,k,p,r,t)$.

Consider the oriented link diagram $L(h,k,p,r,t)$, drawn in Figure 2. Our aim is to compute the s-invariant, as far as possible, for this family of links. Of course the value of s will depend on the sign of the parameters h,k,p,r,t. The writhe and the number of circles in the oriented resolution of $L(h,k,p,r,t)$ are easily computed and are

$$w(L(h,k,r,t,p)) = 2(h+k+p+t+r), \quad o(L(h,k,r,t,p)) = 2(|r|+|p|+|t|) - 1.$$

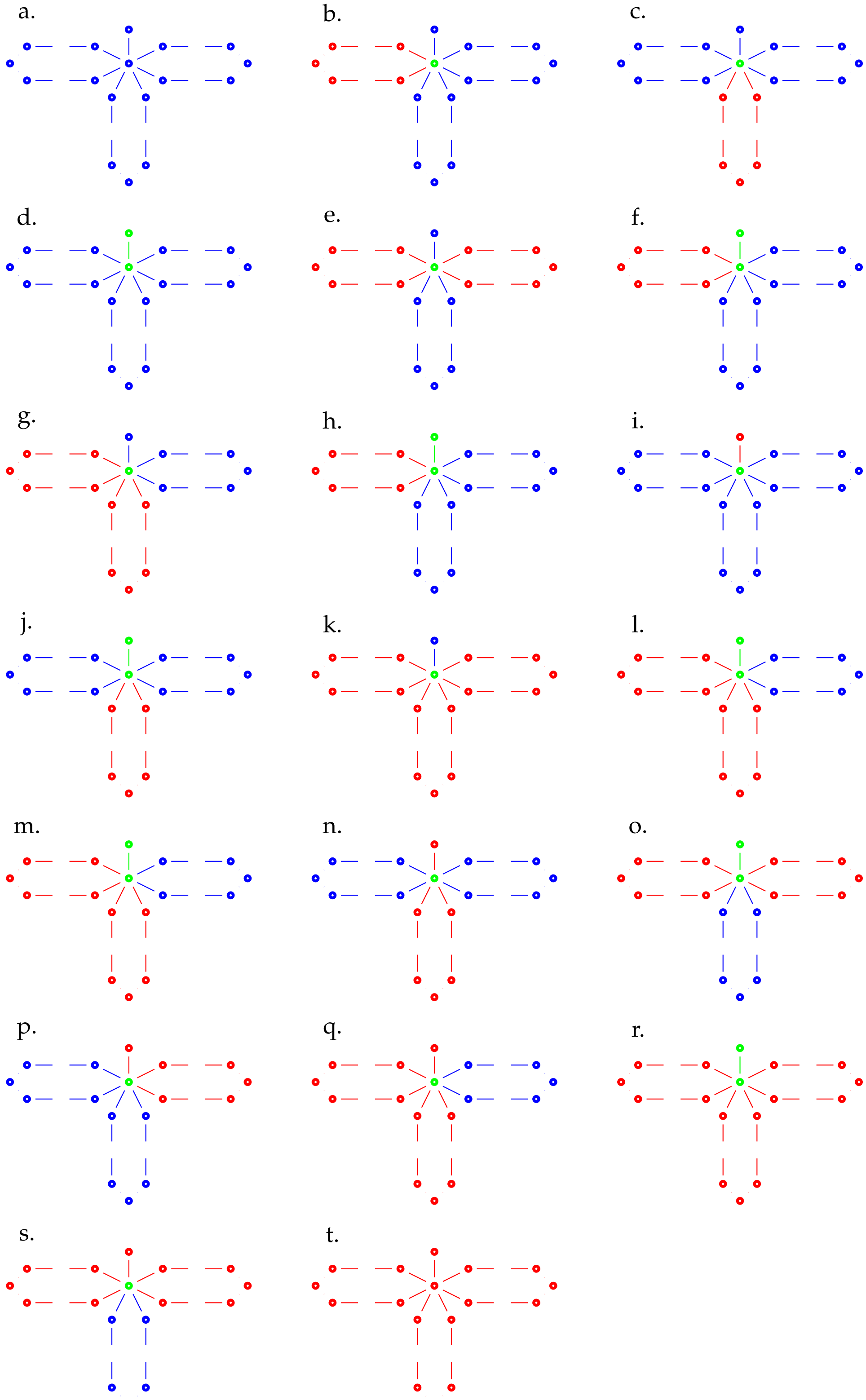
The other quantities appearing in the Bennequin s-inequalities are listed in Table 1, Table 2 and Table 3. Finally, we list all the simplified Tait graphs coloured in such a way that negative edges and vertices are blue, positive edges and vertices are red and neutral edges and vertices are green

case	<0	>0	o_+	o_-	ℓ_-	ℓ_+	s_-	s_+	δ^-	δ^+	
a.	h,k,p,r,t	—-	0	$2(\|p\|+\|r\|+\|t\|)-1$	1	0	1	$2(\|p\|+\|r\|+\|t\|)-1$	0	0	$\surd$
b.	h,k,p,r	t	$2\|t\|-1$	$2(\|r\|+\|p\|)-1$	0	0	$2\|t\|$	$2(\|r\|+\|p\|)$	0	0	$\surd$
c.	h,k,r,t	p	$2\|p\|-1$	$2(\|r\|+\|t\|)-1$	0	0	$2\|p\|$	$2(\|r\|+\|t\|)$	0	0	$\surd$
d.	h,p,r,t	k	0	$2(\|p\|+\|r\|+\|t\|)-3$	0	0	1	$2(\|p\|+\|r\|+\|t\|)-2$	0	0	$\surd$
e.	h,k,p	t,r	$2(\|r\|+\|t\|)-2$	$2\|p\|$	0	0	$2(\|r\|+\|t\|)-1$	$2\|p\|+1$	0	0	$\surd$
f.	h,p,r	t,k	$2\|t\|-1$	$2(\|r\|+\|p\|)-2$	0	0	$2\|t\|$	$2(\|r\|+\|p\|)-1$	0	0	$\surd$
g.	h,k,r	t,p	$2(\|p\|+\|t\|)-2$	$2\|r\|$	0	0	$2(\|p\|+\|t\|)-1$	$2\|r\|+1$	0	0	$\surd$
h.	k,p,r	t,h	$2\|t\|-1$	$2(\|r\|+\|p\|)-2$	0	0	$2\|t\|$	$2(\|r\|+\|p\|)-1$	0	0	$\surd$
i.	p,r,t	h,k	1	$2(\|p\|+\|r\|+\|t\|)-3$	0	0	2	$2(\|p\|+\|r\|+\|t\|)-2$	0	0	$\surd$
j.	h,r,t	p,k	$2\|p\|-1$	$2(\|r\|+\|t\|)-2$	0	0	$2\|p\|$	$2(\|r\|+\|t\|)-1$	0	0	$\surd$
k.	h,k	p,r,t	$2(\|p\|+\|r\|+\|t\|)-3$	1	0	0	$2(\|p\|+\|r\|+\|t\|)-2$	2	0	0	$\surd$
l.	h,r	k,p,t	$2(\|t\|+\|p\|)-2$	$2\|r\|-1$	0	0	$2(\|t\|+\|p\|)-1$	$2\|r\|$	0	0	$\surd$
m.	k,r	h,p,t	$2(\|t\|+\|p\|)-2$	$2\|r\|-1$	0	0	$2(\|t\|+\|p\|)-1$	$2\|r\|$	0	0	$\surd$
n.	r,t	h,k,p	$2\|p\|$	$2(\|r\|+\|t\|)-2$	0	0	$2\|p\|+1$	$2(\|r\|+\|t\|)-1$	0	0	$\surd$
o.	p,h	k,t,r	$2(\|t\|+\|r\|)-2$	$2\|p\|-1$	0	0	$2(\|t\|+\|r\|)-1$	$2\|p\|$	0	0	$\surd$
p.	p,t	h,k,r	$2\|r\|$	$2(\|t\|+\|p\|)-2$	0	0	$2\|r\|+1$	$2(\|t\|+\|p\|)-1$	0	0	$\surd$
q.	r	h,k,p,t	$2(\|t\|+\|p\|)-1$	$2\|r\|-1$	0	0	$2(\|t\|+\|p\|)$	$2\|r\|$	0	0	$\surd$
r.	h	k,p,t,r	$2(\|p\|+\|r\|+\|t\|)-3$	0	0	0	$2(\|p\|+\|r\|+\|t\|)-2$	1	0	0	$\surd$
s.	p	h,k,r,t	$2(\|r\|+\|t\|)-1$	$2\|p\|-1$	0	0	$2(\|r\|+\|t\|)$	$2\|p\|$	0	0	$\surd$
t.	—-	h,k,p,r,t	$2(\|p\|+\|r\|+\|t\|)-1$	0	0	1	$2(\|p\|+\|r\|+\|t\|)-1$	1	0	0	$\surd$

TABLE 2.

case	<0	>0	$w+o$	$w-o$	(Lb12) upper	(s-ineq) upper	(Lb12) lower	(s-ineq) lower
a.	h,k,p,r,t	—	$2(h+k)-1$	$2(h+k+2p+2r+2t)+1$	$2(h+k-1)$	$2(h+k)$	$2(h+k-3)$	$2(h+k-1)$
b.	h,k,p,r	t	$2(h+k+2t)-1$	$2(h+k+2p+2r)+1$	$2(h+k)$	$2(h+k+1)$	$2(h+k-2)$	$2(h+k)$
c.	h,k,r,t	p	$2(h+k+2p)-1$	$2(h+k+2t+2r)+1$	$2(h+k)$	$2(h+k+1)$	$2(h+k-2)$	$2(h+k)$
d.	h,p,r,t	k	$2(h+k)-1$	$2(h+k+2p+2r+2t)+1$	$2(h+k-1)$	$2(h+k)$	$2(h+k-4)$	$2(h+k-2)$
e.	h,k,p	t,r	$2(h+k+2r+2t)-1$	$2(h+k+2p)+1$	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-1)$	$2(h+k+1)$
f.	h,p,r	t,k	$2(h+k+2t)-1$	$2(h+k+2p+2r)+1$	$2(h+k)$	$2(h+k+1)$	$2(h+k-3)$	$2(h+k-1)$
g.	h,k,r	t,p	$2(h+k+2t+2p)-1$	$2(h+k+2r)+1$	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-2)$	$2(h+k+1)$
h.	k,p,r	t,h	$2(h+k+2t)-1$	$2(h+k+2p+2r)+1$	$2(h+k)$	$2(h+k+1)$	$2(h+k-3)$	$2(h+k-1)$
i.	p,r,t	h,k	$2(h+k)-1$	$2(h+k+2t+2r+2p)+1$	$2(h+k-2)$	$2(h+k-1)$	$2(h+k-4)$	$2(h+k-2)$
j.	h,r,t	p,k	$2(h+k+2p)-1$	$2(h+k+2t+2r)+1$	$2(h+k-1)$	$2(h+k)$	$2(h+k-3)$	$2(h+k-1)$
k.	h,k	p,r,t	$2(h+k+2t+2p+2r)-1$	$2(h+k)+1$	$2(h+k+2)$	$2(h+k+3)$	$2(h+k)$	$2(h+k+2)$
l.	h,r	k,p,t	$2(h+k+2t+2p)-1$	$2(h+k+2r)+1$	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-2)$	$2(h+k)$
m.	k,r	h,p,t	$2(h+k+2t+2p)-1$	$2(h+k+2r)+1$	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-2)$	$2(h+k)$
n.	r,t	h,k,p	$2(h+k+2p)-1$	$2(h+k+2t+2r)+1$	$2(h+k-1)$	$2(h+k)$	$2(h+k-3)$	$2(h+k-1)$
o.	p,h	k,t,r	$2(h+k+2r+2t)-1$	$2(h+k+2p)+1$	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-2)$	$2(h+k)$
p.	p,t	h,k,r	$2(h+k+2r)-1$	$2(h+k+2p+2t)+1$	$2(h+k-1)$	$2(h+k)$	$2(h+k-3)$	$2(h+k-1)$
q.	r	h,k,p,t	$2(h+k+2t+2p)-1$	$2(h+k+2r)+1$	$2(h+k)$	$2(h+k+1)$	$2(h+k-2)$	$2(h+k)$
r.	h	k,p,t,r	$2(h+k+2p+2r+2t)-1$	$2(h+k)+1$	$2(h+k+2)$	$2(h+k+3)$	$2(h+k-1)$	$2(h+k+1)$
s.	p	h,k,r,t	$2(h+k+2t+2r)-1$	$2(h+k+2p)+1$	$2(h+k)$	$2(h+k+1)$	$2(h+k-2)$	$2(h+k)$
t.	—	h,k,p,r,t	$2(h+k+2p+2r+2t)-1$	$2(h+k)+1$	$2(h+k+1)$	$2(h+k+2)$	$2(h+k-1)$	$2(h+k+1)$

TABLE 3.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APPENDIX C

Self-duality of the Bar-Nathan Frobenius Algebra

Let R be a field, and recall that BN is defined as

$$A = R[U]\langle x_+, x_- \rangle,\ \varepsilon : A \to R : P(U)x_+ + Q(U)x_- \mapsto Q(U),$$

and the multiplication on A is given by

$$m(x_+, y) = m(y, x_+) = y, \quad m(x_-, x_-) = Ux_-,$$

for each $y \in A$. It turns out that the co-multiplication is given by

$$\Delta(x_+) = x_+ \otimes x_- + x_- \otimes x_+ - Ux_+ \otimes x_+,$$

and

$$\Delta(x_-) = x_- \otimes x_-.$$

Finally, the unit map is defined by

$$\iota(1_{R[U]}) = x_+.$$

As mentioned before, we have a natural, non degenerate, pairing (cf. Proposition 1.1) given by the Frobenius algebra structure, and this pairing defines an isomorphism of $R_{\mathcal{F}}$-modules as follows

$$\Phi : A_{\mathcal{F}} \longrightarrow A^*_{\mathcal{F}} :\ \alpha \mapsto (\alpha, \cdot) =: \alpha^*.$$

Let us compute x_+^* and x_-^*. Since $A_{\mathcal{F}}$ is free[1] it suffices to find their values on x_+ and x_-. Simple computations show that

$$x_-^*[x_+] = \varepsilon(m(x_-, x_+)) = 1, \qquad x_-^*[x_-] = \varepsilon(m(x_-, x_-)) = U,$$

and

$$x_+^*[x_+] = \varepsilon(m(x_+, x_+)) = 0, \qquad x_+^*[x_-] = \varepsilon(m(x_+, x_-)) = 1.$$

Remark 67. Notice that the canonical duals $\varphi_\pm$, defined by

$$\varphi_\pm(x_\pm) = 1,\ \varphi_\pm(x_\mp) = 0,$$

are such that:

$$\varphi_- = x_+^*, \quad \varphi_+ = x_\bullet^*,$$

where $x_\bullet = x_- - Ux_+$. In particular, $deg(x_-^*) = -1$, and $deg(x_+^*) = 1$. This imples that BN and BN^* are isomorphic as graded Frobenius algebras.

It is not difficult to check that the multiplication table in A_{BN^*} is the following.

[1] A projective module of finite type over a PID is always free, see for example [**31**, Chapter III Section 7].

m_{BN^*}	x_+^*	x_-^*
x_+^*	x_+^*	x_-^*
x_-^*	x_-^*	Ux_-^*

Remark 68. The fact that the multiplication table is as above implies that

$$\iota_{BN^*}(1_{R_{BN}}) = x_+^*.$$

As an example, we will verify that $m_{BN^*}(x_+^*, x_+^*) = x_+^*$, leaving the other cases to the reader.

$$m_{BN^*}(x_+^*, x_+^*)[x_+] = (x_+^* \otimes x_+^*)[\Delta_{BN}(x_+)] =$$
$$= x_+^*[x_+]x_+^*[x_-] + x_+^*[x_-]x_+^*[x_+] - Ux_+^*[x_+]x_+^*[x_+] =$$
$$= 0 \cdot 1 + 1 \cdot 0 - U0 \cdot 0 = 0.$$
$$m_{BN^*}(x_+^*, x_+^*)[x_-] = (x_+^* \otimes x_+^*)[\Delta_{BN}(x_-)] = x_+^*[x_-]x_+^*[x_-] = 1.$$

Finally, one has to show that $\varepsilon_{BN^*}(\alpha^*) = \varepsilon_{BN}(\alpha)$, and this completes the proof of the self-duality property.

$$\varepsilon_{BN^*}(x_+^*) = x_+^*[\iota_{BN}(1)] = 0, \quad \varepsilon_{BN^*}(x_-^*) = x_-^*[\iota_{BN}(1)] = 1.$$