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The Classical Obstacle Problem for nonlinear variational energies and related problems

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Dottorando:
Francesco Geraci

Tutore
Prof. Matteo Focardi

Coordinatore
Prof. Graziano Gentili

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Introduction

The *obstacle problem* consists in finding the minimizer of a suitable energy among all functions, with fixed boundary data, constrained to lie above a given *obstacle*. The obstacle can live in the whole domain, case denoted by *classical obstacle*, or lives on a surface of codimension one, case denoted by *thin obstacle*. In Chapters 2 and 3 we study the classical obstacle problem, in the case of quadratic energies with coefficients in Sobolev fractional spaces and linear term with a Dini-type continuity and nonlinear variational energies rispectively, while in Chapter 4 we analyse a particular case of thin obstacle i.e. the *fractional obstacle problem*, where the obstacle is laid in a hyperplane of the domain and the energies are the weighted versions of Dirichlet energy. In this context the more renowned research fields are the properties of regularity of the minimizer (see Chapter 3 which proveides details on the condition for uniqueness of minimizer) and the regularity of the *free boundary*, i.e. the boundary of the coincidence set between the minimizer and the obstacle.

The motivation for studying obstacle problems has roots in many applications for example in physics and in mechanics; some prime examples can be found in [24, 40, 63, 65, 87].

In order to introduce the subject of thesis we briefly analyse a relevant example: the behaviour of elastic membrane. Let us suppose that the membrane is stretched such that it takes the fixed position on the boundary and at the same time does not penetrate the solid obstacle. In the absence of external forces, the energy of the membrane is in its deformation energy, i.e. the energy of the membrane is proportional to its surface area; thus for the "Principle of least action" the problem consists in minimizing the area functional

$$\int_\Omega \sqrt{1+|\nabla v|^2}\,dx.$$

Taking small deformations into account, we can approximate the area functional with its linearization, so it is possible to analyse the Dirichlet energy in order to simplify the problem (in Chapter 3 we analyse the case of area functional without this simplification). Therefore, we can reduce it to the following formulation:

$$\min_{v\in\mathbb{K}_{\psi,g}} \int_\Omega |\nabla v|^2\,dx, \tag{0.1}$$

in a domain $\Omega \subset \mathbb{R}^n$, where $\mathbb{K}_{\psi,g} := \{v \in g + W^{1,2}(\Omega) : v \geq \psi\}$, ψ is the *obstacle* and the function g satisfies the condition $g \geq \psi$.

In the '60s the obstacle problem was introduced within the study of variational inequalities (see [65]); in fact it is possible to give a further interpretation of the problem. It is easy to prove that the minimizer u of problem (0.1) satisfies the following condition:

$$\int_\Omega \nabla u \nabla (v-u)\, dx \geq 0 \qquad\qquad \forall v \in \mathbb{K}_{\psi,g}. \tag{0.2}$$

Such inequalities are called *variational inequalities* and they are the inequalities that involve the first variation of some convex functionals (see (1.35) in Chapter 1). Variational inequalities are a classical topic in partial differential equations starting with the seminal works of Fichera and Stampacchia in the early '60s. This were motivated by a wide variety of applications in mechanics and other applied sciences. This subject has been developed over the last 50 years by the works of many authors; it is not realistic to give here a complete account: rather we refer the readers to the relevant books and surveys [5, 20, 27, 40, 65, 85, 87, 91, 92] for a fairly vast bibliography and its historical developments.

Standard variational methods prove that a unique minimizer exists which that we denote as u. The minimizer u, that represents the profile of the membrane in an equilibrium condition, is a solution of the partial differential equation (see Proposition 2.1.5 in Chapter 2 and Corollary 3.1.5 in Chapter 3):

$$-\Delta u = (-\Delta \psi)^+ \chi_{\{u=\psi\}}. \tag{0.3}$$

Taking the regularity of the minimizer profile into account, the equation (0.3) provides the hint to deduce that the function u is at most $C^{1,1}$; in fact if we consider the obstacle $\psi = 1-|x|^2$ in B_1 and $g \equiv 1/2$, then the $-\Delta u$ jumps from the value 0, when the membrane does not touch the obstacle, to the value -2 when the membrane coincides with the obstacle. So it is natural to suppose $\psi \in C^{1,1}$. Furthermore Brezis and Kinderlehrer [12], Caffarelli and Kinderlehrer [18] and Frehse [38], proved (also in more general framwork) that the solution u is really $C^{1,1}$.

The issue about the regularity of the free boundary $\Gamma(u) = \partial\{u=\psi\} \cap \Omega$ is usually considered very hard. Note that $\Gamma(u)$ is, a priori, an unknown datum and is a part of the problem. In the last fifty years much effort has been put into the understanding of the problem. A wide variety of issues have been analysed and new mathematical ideas have been introduced. Caffarelli in [15] introduced the so-called method of *blow up*, inspired by De Giorgi's work, in the geometric measure theory for the study of minimal surfaces, in order to prove some local properties of solutions. This method consists in the introduction of a sequence of rescaled functions of the solution

$$(u_{x_0,r})_r = \left(\frac{u(x_0+rx)}{r^2}\right)_r,$$

and in the study of the limit as $r \to 0^+$, where $x_0 \in \Gamma(u)$. This heuristically corresponds to "zooming" the profile of the function with the factor $1/r$ near the point x_0. The idea is to analyze the limit when $r \to 0^+$, which would correspond to the idea of "infinite zoom". The factor of scaling r^2, is not randomly choosen, instead with this scaling factor the function $u_{x_0,r} = \frac{u(x_0+rx)}{r^2}$ satisfies the same condition of u, in B_r instead of B_1. The exponent 2 is the Almegren frequency of solution of classical obstacle problem (cf. (4.8) in Chapter 4). The same rescaling factor is used in Chapter 2 in order to study the classical obstacle problem with more general energies. In Chapter 4 we also introduce a sequence of rescaled functions of the solution to study the properties of a suitable subset of the free boundary for the fractional obstacle problem. However this rescaling factor has a different exponent which is still the Almegren frequency of the points of the subset (cf. section 4.1 in Chapter 4).

In order to give some results, related to the regularity of the free boundary, some assumptions on the obstacle are needed: based on the work of Caffarelli and Riviére [19] we suppose that the obstacle is concave and considering Blank [6] we assume that ψ is sufficiently smooth so as to avoid that $\Gamma(u)$ is Reifenberg flat and not smooth. Thanks to the blow up method Caffarelli distinguished the points of $\Gamma(u)$ in regular and singular points, respectively denoted by $Reg(u)$ and $Sing(u)$ (cf. Definition 2.4.6 in Chapter 2). Caffarelli proved that the set of the regular points is locally the graph of a regular function, and the set of the singular points is locally contained in a C^1 manifold of lesser dimension. In Chapters 2 and 3, Theorems 2.7.1, 2.7.3 and 3.1.8 respectively, we prove the results of regularity for quadratic energies (with the coefficients in Sobolev fractional spaces and linear term with a Dini-type continuity) and nonlinear variational energies. In the Theorem 4.5.1 in Chapter 4 we show a result of regularity of suitable subset of free boundary for the fractional obstacle problem (see [21, Theorem 7.1] for the regularity result in the case of nonzero obstacles). Moreover in the recent years Alt, Caffarelli and Friedman [1], Weiss [95] and Monneau [77] introduced monotonicity formulas that turn out to be a really good tool to show the blow up property and to obtain the free boundary regularity in various problems (see [14, 20, 40, 65, 85, 87] for more detailed references and historical developments).

Recently many authors have improved classical results by replacing the Dirichlet energy with a more general variational functional and weakening the regularity of the obstacle (see [23, 30, 32–34, 75, 78, 93, 94]). Similar to the aforementioned work in our thesis we study the classical obstacle problem with quadratic energies and linear terms (that play the role of "$-\Delta\psi$") with a Dini-type continuity. We prove the regularity of minimizers and, following the approach of Weiss and Monneau, we establish the quas-imonotonicity formulas for the adjusted boundary energies. By following Caffarelli's method of blow up, and using the quasi-monotonicity formulas as well as the epiperimetric formula of Weiss, we establish the regularity of free boundary.

This regularity results is the starting point for a further generalizations. In the case of nonlinear variational energies we extend the previous regularity results thanks to a suitable linearization argument along with the regularity of free boundary (for quadratic energies with Lipschitz coefficients). To this aim we establish the optimal regularity of solutions to nonlinear, nondegenerate variational inequalities.

Finally, we study the regularity of the free boundary for the fractional obstacle problem. We prove an epiperimetric inequality and deduce the regularity of a suitable subset of the free boundary as a consequence of a decay estimate of a boundary adjusted energy "à la Weiss", the non degeneracy of solution and the uniqueness of blow up.

What has been touched upon above will be now discussed in more detail in the following compendia. Each compendium sums up the contents of a corresponding paper annexed to this thesis. The contents of Chapter 2,3 and 4 respectively are the argument of papers [48], [35] and [49], and are the result of a research activity over three years of a PhD, supported by Università di Firenze, Università di Perugia, INdAM consortium in the CIAFM, in collaboration with Matteo Focardi and Emanuele Spadaro.

Compendium of Chapter 2

In this chapter we study the problem of regularity of minimizers and of the related free boundary of the following energy

$$\mathcal{E}(v) := \int_\Omega \big(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2f(x)v(x)\big)\,dx, \tag{0.4}$$

for all positive functions with fixed boundary data, where $\Omega \subset \mathbb{R}^n$ is a smooth, bounded and open set, $n \geq 2$, $\mathbb{A} : \Omega \to \mathbb{R}^{n\times n}$ is a matrix-valued field and $f : \Omega \to \mathbb{R}$ is a function satisfying:

(I1) $\mathbb{A} \in W^{1+s,p}(\Omega; \mathbb{R}^{n\times n})$ with $s > \frac{1}{p}$ and $p > \frac{n^2}{n(1+s)-1} \wedge n$ or $s = 0$ and $p = +\infty$, where the symbol $\wedge$ indicate the minimum of the surrounded quantities;

(I2) $\mathbb{A}(x) = (a_{ij}(x))_{i,j=1,\dots,n}$ is symmetric, continuous and coercive, that is $a_{ij} = a_{ji}$ $\mathcal{L}^n$ a.e. Ω and for some $\Lambda \geq 1$ i.e.

$$\Lambda^{-1}|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \qquad \mathcal{L}^n \text{ a.e. } \Omega,\ \forall \xi \in \mathbb{R}^n; \tag{0.5}$$

(I3) f is Dini-continuous, namely if $\omega(t) = \sup_{|x-y|\leq t} |f(x) - f(y)|$ is the modulus of continuity of f, ω satisfies the following integrability condition

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty; \tag{0.6}$$

(I4) there exists $c_0 > 0$ such that $f \geq c_0$.

In Remark 2.3.8 we will justify the choice of p in hypothesis $(I1)$. We note that we are reduced to the 0 obstacle case, so $f = \mathrm{div}(\mathbb{A}\nabla\psi)$.

Later in this chapter we prove that the unique minimizer, indicated below by u, is the solution of an elliptic differential equation in divergence form, and with the classical PDE's regularity method, we deduce that u is Hölder continuous and D^2u has opportune sommability. To prove the regularity of the free boundary $\Gamma_u = \partial\{u = 0\} \cap \Omega$, we apply the method of blow up introduced by Caffarelli [15]. For all x_0 points of the free boundary $\Gamma_u = \partial\{u = 0\} \cap \Omega$ we introduce a sequence of rescaled functions and, through a $C^{1,\gamma}$ estimate of rescaled function (for a suitable $\gamma \in (0, 1)$), we prove the existence of sequence limits; these limits are called blow ups. To classify the blow ups and to prove the uniqueness of the sequence limit for all points of Γ_u we introduce a technical tool: the quasi-monotonicity formulas. To simplify the notation we introduce, for all $x_0 \in \Gamma_u$ a suitable change of variable for which, without loss of generality, we can assume:

$$x_0 = \underline{0} \in \Gamma_u, \qquad \mathbb{A}(\underline{0}) = I_n, \qquad f(\underline{0}) = 1. \tag{0.7}$$

As in [34] we introduce the auxiliary energy “à la Weiss”

$$\Phi(r) := \int_{B_1} \big(\langle \mathbb{A}(rx)\nabla u_r, \nabla u_r\rangle + 2f(rx)u_r\big)\, dx + 2\int_{\partial B_1} \Big\langle \mathbb{A}(rx)\frac{x}{|x|}, \frac{x}{|x|}\Big\rangle\, u_r^2\, d\mathcal{H}^{n-1}$$

and prove the main results of this chapter:

Theorem 0.0.1 (Weiss' quasi-monotonicity formula)**.** *Assuming that* $(I1)$*-*$(I4)$ *and* (0.7) *hold. There exist nonnegative constants* $\bar{C}_3$ *and* C_4*, independent from r, such that the function*

$$r \mapsto \Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt$$

with the constant Θ *given in equation* (2.64)*, is nondecreasing on the interval* $(0, \frac{1}{2}\mathrm{dist}(\underline{0}, \partial\Omega) \wedge 1)$*.*

More precisely, the following estimate holds true for $\mathcal{L}^1$*-a.e.* r *in such an interval:*

$$\begin{aligned}\frac{d}{dr}\left(\Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt\right)& \\ \geq \frac{2e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}}}{r^{n+2}} \int_{\partial B_r} \mu\Big(\langle \mu^{-1}\mathbb{A}\nu, \nabla u\rangle - 2\frac{u}{r}\Big)^2\, d\mathcal{H}^{n-1}&.\end{aligned} \tag{0.8}$$

In particular, the limit $\Phi(0^+) := \lim_{r\to 0^+} \Phi(r)$ *exists and it is finite and there exists a constant* $c > 0$ *such that*

$$\begin{aligned}&\Phi(r) - \Phi(0^+) \\ &\quad \geq \Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt - \Phi(0^+) - c\left(r^{1-\frac{n}{\Theta}} + \omega(r)\right).\end{aligned} \tag{0.9}$$

Theorem 0.0.2 (Monneau's quasi-monotonicity formula)**.** *Assume* $(I1)$*-*$(I3)$ *and* (0.7)*. Let* u *be the minimizer of* $\mathcal{E}$ *on* K*, with* $\underline{0} \in Sing(u)$ *(i.e.* (2.93) *holds), and* v *be a 2-homogeneous, positive, polynomial function, solution of* $\Delta v = 1$ *on* $\mathbb{R}^n$*. Then, there exists a positive constant* $C_5 = C_5(\lambda, \|\mathbb{A}\|_{W^{s,p}})$ *such that*

$$r \longmapsto \int_{\partial B_1} (u_r - v)^2\, d\mathcal{H}^{n-1} + C_5\left(r^{(1-\frac{n}{\Theta})} + \omega(r)\right) \tag{0.10}$$

is nondecreasing on $(0, \frac{1}{2}\mathrm{dist}(\underline{0}, \partial\Omega) \wedge 1)$. *More precisely,* $\mathcal{L}^1$*-a.e. on such an interval*

$$\frac{d}{dr}\left(\int_{\partial B_1}(u_r - v)^2\, d\mathcal{H}^{n-1} + C_5\left(r^{1-\frac{n}{\Theta}} + \int_0^r \frac{\omega(t)}{t}\, dt\right)\right)$$
$$\geq \frac{2}{r}\left(e^{C_3\, r^{1-\frac{n}{\Theta}}}\,\Phi(r) + C_4\int_0^r e^{c_3 t^{1-\frac{n}{\Theta}}}\left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\right) dt - \Psi_v(1)\right), \tag{0.11}$$

where $\Psi_v(1) := \int_{B_1}\left(|\nabla v|^2 + 2v\right) dx - 2\int_{\partial B_1} v^2\, d\mathcal{H}^{n-1}$.

Weiss' quasi-monotonicity formula allows us firstly, to deduce the 2-homogeneity of blow ups and together with the nondegeneracy of the solution, proven by Blank and Hao [8] in a more general setting and secondly, to determine that the blow ups are not null. Thanks to a Γ-convergence argument and according to Caffarelli's classification of blow up, in the classical case (see [14, 15, 17]), we can classify the blow up types and so distinguish the points in $\Gamma(u)$ as regular and singular (respectively $Reg(u)$ and $Sing(u)$, see Definition 2.4.6).

Following the energetic approach by Focardi, Gelli and Spadaro [34] we prove the uniqueness of blow ups for both the regular and the singular cases. In the classical framework, the uniqueness of the blow-ups can be derived, a posteriori, from the regularity properties of the free boundary (see Caffarelli [15]). In our setting we distinguish two cases: $x_0 \in Sing(u)$ and $x_0 \in Reg(u)$. In the first case, through the two quasi-monotonicity formulas and an "absurdum" argument, we prove the uniqueness of blow-ups providing a uniform decay estimate for all points in a compact subset of $Sing(u)$. In the second case, we need to introduce an assumption, probably of a technical nature, on the modulus of continuity of f (more restrictive than double Dini continuity, see [78, Definition 1.1]):

$(I3)'$ Let $\omega(t) = \sup_{|x-y|\leq t}|f(x) - f(y)|$ be the modulus of continuity of f and set $a > 2$ the following condition of integrability is valid

$$\int_0^1 \frac{\omega(r)}{r}\,|\log r|^a\, dr < \infty. \tag{0.12}$$

So, thanks to the epiperimetric inequality of Weiss [95] we obtain a uniform decay estimate for the convergence of the rescaled functions with respect to their blow up limits. We recall that Weiss [95] proved the uniqueness for regular points in $\mathbb{A} \equiv I_n$ and $f \equiv 1$. Focardi, Gelli and Spadaro [34] also had proved our same result for $\mathbb{A}$ Lipschitz continuous and f Hölder continuous. Monneau [78] proved the uniqueness of blow-ups both for regular points and for singular points with $\mathbb{A} \equiv I_n$ and f with Dini continuous modulus of mean oscillation in L^p. Therefore, without further hypotheses, in the regular case and adding double Dini continuity condition on the modulus of the mean oscillation, Monneau gave a very accurate pointwise decay estimate, providing an explicit modulus of continuity for the solution.

These results allow us to prove the regularity of the free boundary:

Theorem 0.0.3. *We assume the hypothesis* $(I1)$-$(I3)$. *The free boundary decomposes as* $\Gamma_u = Reg(u) \cup Sing(u)$ *with* $Reg(u) \cap Sing(u) = \emptyset$.

(i) Assume $(I3)'$. $Reg(u)$ *is relatively open in* $\partial\{u = 0\}$ *and for every point* $x_0 \in Reg(u)$. *there exists* $r = r(x_0) > 0$ *such that* $\Gamma_u \cap B_r(x_0)$ *is a* C^1 *hypersurface with normal versor* ς *is absolutely continuous with a modulus of continuity depending on* ρ *defined in* (2.118).

In particular if f *is Hölder continuous there exists* $r = r(x_0) > 0$ *such that* $\Gamma_u \cap B_r(x)$ *is* $C^{1,\beta}$ *hypersurface for some universal exponent* $\beta \in (0, 1)$.

(ii) $Sing(u) = \cup_{k=0}^{n-1} S_k$ *(see Definition 2.7.2) and for all* $x \in S_k$ *there exists* r *such that* $S_k \cap B_r(x)$ *is contained in a regular* k*-dimensional submanifold of* $\mathbb{R}^n$.

The natural sequel of these results is the study of the obstacle problem for nonlinear energies. The aims for future developments are presented in Chapter 3 and contained in [35] where the author, Focardi and Spadaro prove an exhaustive analysis of the free boundary for nonlinear variational energies as the outcome of analogous results for the classical obstacle problem for quadratic energies with Lipschitz coefficients.

This chapter is organized as follows: in Section 2.1 we prove the existence, the uniqueness and regularity of minimizer u. In Section 2.2 we introduce the sequence of rescaled functions, prove the existence of blow-ups and state a property of non degeneracy of solution of obstacle problem. In Sections 2.3 and 2.5 respectively we prove the quasi-monotonicity formulas of Weiss and Monneau. In Section 2.4 we prove the 2-homogeneity and the non zero value property of blow ups, we classify the blow ups and distinguish the point of the free-boundary in regular and singular. In Section 2.6we deduce the uniqueness of blow ups in case of regular and singular points. In Section 2.7 we state the the properties of regularity of free boundary.

Compendium of Chapter 3

Chapter 3 is devoted to the analysis of nonlinear energies. In order to introduce the problem, let ψ and g be given functions in $W^{1,p}(\Omega)$, $p \in (1, \infty)$, with $g \geq \psi$ $\mathcal{L}^n$ a.e. on Ω and set

$$\mathbb{K}_{\psi,g} := \{v \in g + W_0^{1,p}(\Omega) : v \geq \psi \quad \mathcal{L}^n \text{ a.e. on } \Omega\}. \tag{0.13}$$

Consider a smooth *coercive vector field* $(a_0, \mathbf{a}) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ according to [65, Definition 3.1 of Chapter IV] and [91, Chapter 4] (cf. Section 3.1 for the precise definitions and the necessary assumptions). The existence of a solution $u \in \mathbb{K}_{\psi,g}$ of the problem

$$\int_\Omega \mathbf{a}(x, u, \nabla u) \cdot \nabla(v - u)\, dx + \int_\Omega a_0(x, u, \nabla u)(v - u) dx \geq 0 \qquad \text{for all } v \in \mathbb{K}_{\psi,g}, \tag{0.14}$$

is well-known (cf. [65, Section 4 of Chapter III] if $p = 2$ and [91, Chapter 4] otherwise) and shortly recalled in Section 3.1 below. Under suitable hypotheses on the fields, classical results ensure optimal regularity for u, i.e. $u \in C^{1,1}_{loc}(\Omega)$, as long as $\psi \in C^{1,1}_{loc}(\Omega)$ (cf. for instance [91, Sections 4.5-4.6] in the quadratic case, and [92] in general).

The prototype example we have in mind is that of nonlinear variational problems

$$\min_{v \in \mathbb{K}_{\psi,g}} \int_\Omega F(x, v, \nabla v)\, dx \tag{0.15}$$

which leads to a variational inequality of the form (0.14) with $\mathbf{a} = \nabla_\xi F$ and $a_0 = \partial_z F$, under suitable assumptions on $F = F(x, z, \xi)$ such as global smoothness, convexity and p-growth in the last variable (cf. Theorem 3.1.8 below for the precise assumptions on F).

The aim of this chapter is to perform an exhaustive analysis of the *free boundary*, i.e. the set $\partial\{u = \psi\}$, for the broad class of obstacle problems introduced in (0.15), and to establish a parallel with the known results in the quadratic case as developed by Caffarelli [17], Weiss [95] and Monneau [77] (cf. Theorem 3.1.8 for the statement).

The sharp analysis and stratification of the free boundary we provide is an outcome of a suitable linearization argument (cf. Lemma 3.1.12 below) and of the analogous results, for the classical obstacle problem, for quadratic energies with Lipschitz coefficients. This was recently proven in [34] and improved in the case of coefficients, in fractional Sobolev spaces which we prove in Chapter 2 and state in Theorem 0.0.3 (cf. Theorems 2.7.1 and 2.7.3). It corresponds to the case $F(x, \xi) = \mathbb{A}(x)\xi \cdot \xi$ in (0.15), with $\mathbb{A} \in \mathrm{Lip}(\Omega, \mathbb{R}^{n \times n})$ which defines a continuous and coercive quadratic form.

As a direct outcome of Theorem 0.0.3 we shall deduce the analogous result for solutions of (0.15) (cf. Theorem 3.1.8). Furthermore, adding suitable assumptions on the data of the problem, we can provide similar conclusions in case of the vector field $\nabla_\xi F$ which is more generally *locally coercive*, thus including in our analysis, the important case of the area functional.

Non-optimal regularity for solutions is a classical topic well-known in literature, at least in the quadratic case $p = 2$, which has been established in several ways such as: by penalization methods (cf. [70], [13], [11]), by Lewy-Stampacchia inequalities (cf. [81], [80], [60], [39], [91]), by local comparison methods (cf. [52]), by introducing a substitute variational inequality (cf. [57]), and by the linearization method (see [41, 42]). By following the streamline of ideas of the latter technique introduced in [41], we provide an elementary variational proof valid in the general framework of nonlinear variational inequalities under investigation. In particular, we show that solutions of (0.14) satisfy a nonlinear elliptic PDE in divergence form and in turn, from this, suboptimal regularity can be established (for further comments cf. Section 3.2 in Chapter 3). Finally, we are able to establish optimal regularity following Gerhardt [50] (see [12, 18, 38] for the classical results). In addition, we remark that solutions to (0.14) are actually *Q-minima* of a related functional according to Giaquinta and Giusti [53, 54].

Furthermore, in the case of the area functional, we can prove that solutions to the obstacle problem are actually almost minimizers of the perimeter, thus leading by a well-known theory of minimal surfaces (cf. [90]) to estimates on the gradient of the solutions which bypass the global approach by Hartman and Stampacchia [61] exploiting the bounded slope condition and the construction of barriers.

A short summary of the contents of the chapter is resumed in what follows: in Section 3.1 we introduce the necessary definitions to state the main result of the paper, Theorem 3.1.8, and show how the latter follows directly from Theorem 0.0.3. In doing this, we shall first review almost optimal and then optimal regularity in the broader setting of solutions to variational inequalities driven by coercive vector fields as in (0.14) (cf. Theorems 3.1.4 and 3.1.6), and then develop in details the analysis of the free boundary in the variational case in (0.15). Finally, in Section 3.2 we highlight the required changes to deduce similar conclusions for the case of locally coercive vector fields, and also analyse the case of the area functional in a Riemannian manifold.

Compendium of Chapter 4

In this Chapter we study the fractional obstacle problem. It consists in the minimizing of energy

$$\mathcal{E}(v) := \int_{B_1^+} |\nabla v|^2 \, x_n^a \, dx, \tag{0.16}$$

among all functions in the class of admissible functions

$$\mathfrak{A}_g := \{v \in H^1(B_1^+, a) \,:\, v \geq 0 \textit{ on } B_1', v = g \textit{ on } (\partial B_1)^+\}, \tag{0.17}$$

where $H^1(A, \mu_a)$ is the weighted Sobolev Space and μ_a is the measure $\mu_a := |x_n|^a \, \mathcal{L}^n \llcorner B_1$ with $a \in (-1, 1)$.

Let $u \in \operatorname{argmin}_{\mathfrak{A}_g} \mathcal{E}$; we denote by $\Gamma(u) := \partial\{(\widehat{x}, 0) \in B_1' \,:\, u(\widehat{x}, 0) = 0\} \cap B_1'$ its free boundary. Caffarelli and Silvestre in [22] showed that the minimum u is the solution of

$$\begin{cases} u(\widehat{x}, 0) \geq 0 & \widehat{x} \in B_1 \\ u(\widehat{x}, x_n) = u(\widehat{x}, -x_n) & \\ \operatorname{div}(|x_n|^a \, \nabla u(\widehat{x}, x_n)) = 0 & x \in B_1 \setminus \{(\widehat{x}, 0) \,:\, u(\widehat{x}, 0) = 0\} \\ \operatorname{div}(|x_n|^a \, \nabla u(\widehat{x}, x_n)) \leq 0 & x \in B_1 \textit{ in distributional sense.} \end{cases} \tag{0.18}$$

and this problem is equivalent to the study of the classical obstacle problem in $\mathbb{R}^{n-1}$ for fractional Laplacian $(\Delta)^s$ with $s \in (0, 1)$, $a = 1 - 2s$. Silvestre in [88] proved the existence and uniqueness of the solution. Caffarelli, Silvestre and Salsa in [21] proved the regularity of a suitable subset of the free boundary. In this Chapter we give an alternative proof of their result.

In order to establish the regularity of free boundary we recall a Almgren *frequency* type function for all points (see [4] for $s = 1/2$). $x_0 \in \Gamma(u)$

$$N_a^{x_0}(r, u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2 \, |x_n|^a \, dx}{\int_{\partial B_r(x_0)} u^2 \, |x_n|^a \, d\mathcal{H}^{n-1}}. \tag{0.19}$$

Caffarelli and Silvestre [22] proved the monotonicity of function $r \mapsto N_a^{x_0}(r, u)$ and some of its properties such as, the property of being constant oevr all homogeneous functions; the two authors and Salsa [21] established the property of the frequency function of being bigger than $1+s$. Thus it is possible to define the frequency of u in x_0 as $N_a^{x_0}(0^+,u) := \lim_{r\to 0^+} N_a^{x_0}(r,u)$ and denote by $\Gamma_{1+s}(u)$ the points of the free boundary with less frequency i.e. $1+s$. In order to prove the regularity of the set $\Gamma_{1+s}(u)$, proceeding as in classical obstacle problem we introduce a sequence of rescaled functions $u_{r,x_0} = \frac{u(x_0+rx)}{r^{1+s}}$ and an auxiliary energy "à la Weiss"

$$W_{1+s}^{x_0}(r,u) := \frac{1}{r^{n+1}}\int_{B_r(x_0)} |\nabla u_r|^2\,|x_n|^a\,dx - \frac{1+s}{r^{n+1}}\int_{\partial B_r(x_0)} |u_r|^2\,|x_n|^a\,d\mathcal{H}^{n-1}, \tag{0.20}$$

that is the sum of a volume energy and a boundary energy. We note that, as in Chapter 2, the frequency of points of the free boundary examined ($1+s$ in this case and 2 in the classical obstacle problem) is the exponent of the rescaled factor of sequence $u_{x_0,r}$ and the coefficient of boundary energy. The existence of blow ups is a consequence of a gradient estimate of rescaled function in $L^2(B_1,\mu_a)$; reasoning by contradiction, thanks to properties of the frequency and the optimal regularity of the solution we prove the $(1+s)$-homogeneity of blow ups. So, according to a result of classification of Caffarelli, Salsa and Silvestre [21] we state the result of the clasification of $(1+s)$-homogeneous global solutions of the fractional obstacle, which constitute the following closed cone

$$\mathfrak{H}_{1+s} := \{\lambda\, h_e \;:\; e\in\mathbb{S}^{n-2}, \lambda\in[0,+\infty)\} \subset H^1_{loc}(\mathbb{R}^n,\mu_a),$$

with

$$h_e(x) := \left(s^{-1}\widehat{x}\cdot e - \sqrt{(\widehat{x}\cdot e)^2 + x_n^2}\right)\left(\sqrt{(\widehat{x}\cdot e)^2 + x_n^2} + \widehat{x}\cdot e\right)^s.$$

The key result presented in this chapter is a Weiss' epiperimetric inequality for fractional obstacle problem (cf. [95, Theorem 1] and Theorem 2.6.2 in Chapter 2).

Theorem 0.0.4 (Epiperimetric inequalities)**.** *Let $\underline{0}\in\Gamma_{1+s}(u)$. There exists a dimensional constant $\kappa\in(0,1)$ such that if $c\in H^1(B_1,\mu_a)$ is a function $(1+s)$-homogeneous for which $c\geq 0$ on B_1' then*

$$\inf_{v\in\mathfrak{A}_c} W_{1+s}^{\underline{0}}(1,v) \leq (1-\kappa) W_{1+s}^{\underline{0}}(1,c).$$

We follow the variational approach of Focardi and Spadaro [36], for the case $a=0$. The two authors outline the presence in their proof of two competing variational principles that contribute to the achievement of proof.

Thanks to an homogeneous argument we can reduce Theorem 0.0.4 to prove the result with an extra condition of nearness between the function c and the cone of global solution $\mathfrak{H}_{1+s}$. By contradicting the nearness assumption we obtain a quasi minimality condition for a sequence of auxiliary functionals. With an argument of Γ-convergence we inspect the Γ-limits of the sequnce of auxiliary energies and analyse their minimizer that represents the directions along which the epiperimetric inequality may fail. With variational method we obtain that such minimizers show in the same time contradictory relationship with the cone $\mathfrak{H}_{1+s}$.

The epiperimetric inequality is a key ingredient to deduce the following decay estimate of energy:

$$W^{x_0}_{1+s}(r, u) \le C\, r^{\gamma}, \tag{0.21}$$

where C and γ are positive constants. Thanks to the decay estimate (0.21) we prove a property of nondegeneration of solutions, from which we deduce that the blow ups are not null. Proceeding as in Chapter 2 we prove the uniqueness of blow ups and the regularity of $\Gamma_{1+s}(u)$.

What follows is a summary of the structure of this chapter: in section 4.1 introducing the frequency and its properties we define $\Gamma_{1+s}(u)$ the subset of free boundary with low frequency. In section 4.2 prove the existence and $(1+s)$-homogeneity of blow ups in the points in $\Gamma_{1+s}(u)$ and in section 4.3, thanks a result by [21], we characterize the $(1+s)$-homogeneous global solution of fractional obstacle problem. Scetion 4.4 is devoted to establish the epiperimetric inequality and its consequences in the framework of regularity of free boundary, a decay estimate of an auxiliary energy, the nondegeneracy of the solution and the uniqueness of the blow ups. In section4.5 we prove the regularity of $\Gamma_{1+s}(u)$.

Comparisons with existing literature

Theorems 0.0.1 and 0.0.2 generalize the results of Weiss [95] and Monneau [77]. Weiss' monotonicity formula which was proven by Weiss [95] for $\mathbb{A} \equiv I_n$ and $f \equiv 1$; in the same paper he proved the celebrated epiperimetric inequality (see Theorem 2.6.2) and gave a new way of approaching the problem of the regularity for the free boundary. In [84] Petrosyan and Shahgholian proved the monotonicity formula for $\mathbb{A} \equiv I_n$ and f double Dini modulus of continuity (but for obstacle problems with no sign condition on the solution). Lederman and Wolanski [68] provided a local monotonicity formula for the perturbated problem to achieve the regularity of Bernoulli and Stefan free boundary problem, while Ma, Song and Zhao [72] showed the formula for elliptic and parabolic systems in the case in which $\mathbb{A} \equiv I_n$ and the equations present a first order nonlinear term. Garofalo and Petrosyan in [45] proved the formula for the thin obstacle problem with a smooth obstacle. The two authors together with Smith Vega Garcia in [46] proved the result for Signorini's problem under the hypotheses $\mathbb{A} \in W^{1,\infty}$ and $f \in L^{\infty}$. Focardi, Gelli and Spadaro in [34] proved the formula for the classical obstacle problem for $\mathbb{A} \in W^{1,\infty}$ and $f \in C^{0,\alpha}$ for $\alpha \in (0, 1)$. In the same paper (under the same hypotheses of coefficients) the three authors proved a generalization of the monotonicity formula introduced by Monneau [77] to analyse the regularity singular point (see Definition 2.4.6). Monneau in [78] improved his result; he showed that his monotonicity formula holds under the hypotheses that $\mathbb{A} \equiv I_n$ and f with Dini continuous modulus of continuity (in average L^p). In [45] Garofalo and Petrosyan showed the formula of Monneau for the thin obstacle with a regular obstacle. In our work (inspired by [34]) we prove the quasi-monotonicity formulas under the hypotheses, $(I1)$-$(I4)$ improving the results with respect to current literature. As we will see in Corollary 1.1.7 if $ps > n$ the immersion $W^{1+s,p} \hookrightarrow W^{1,\infty}$

holds true. Consequently, we assume $sp \leq n$ and we obtain an original result not covered by [34] if $p > \frac{n^2}{n(1+s)-1} \wedge n$ (we can observe that $(\frac{n^2}{n(1+s)-1} \wedge n) < \frac{n}{s}$ for all $s \in \mathbb{R}$).

In order to justify the choice of regularity of the coefficients of $\mathbb{A}$ and f we discuss the hypotheses $(I1)$ and $(I3)$.

The hypothesis $(I3)$ turns out to be the best condition to obtain the uniqueness of blow up. In fact when condition (2.2) is not satisfied, Blank gave in [6] an example of non uniqueness of the blow up limit in a regular point. Monneau observed in [77] that using the symmetry $x \mapsto -x$, it is easy to transform the result of Blank in an example of non uniqueness of the blow up limit in a singular point when condition (2.2) is not satisfied. Therefore, in the same paper Monneau asked if (2.2) is a sufficient condition to ensure the uniqueness of the blow up limit in singular points (in case in which $\mathbb{A} \equiv I_n$): with Proposition 2.6.1 we answer positively to his question, not only in the Laplacian case, but also when the matrix of coefficients $\mathbb{A}$ satisfies the hypotheses $(I1)$ and $(I2)$.

Before taking into account hypothesis $(I1)$ we need to clarify the relationship between the regularity of coefficients $\mathbb{A}$, f and the regularity of the free boundary. Caffarelli [14] and Kinderlehrer and Nirenberg [64] proved that for smooth coefficients of $\mathbb{A}$ and for $f \in C^1$ the regular points are a $C^{1,\alpha}$-manifold for all $\alpha \in (0,1)$, for $f \in C^{m,\alpha}$ $Reg(u)$ is a $C^{m+1,\alpha}$-manifold with $\alpha \in (0,1)$ and if f is analytic so is $Reg(u)$. In [6] Blank proved that, in Laplacian case with f Dini continuous, the set of regular points is a C^1-manifold, but if f is C^0, but is not Dini continuous, then $Reg(u)$ is Reifenberg vanishing but not smooth. In [34] Focardi, Gelli and Spadaro proved that if $\mathbb{A} \in W^{1,\infty}$ and $f \in C^{0,\alpha}$ with $\alpha \in (0,1)$ $Reg(u)$ is a $C^{1,\beta}$-manifold with $\beta \in (0,\alpha)$. A careful inspection of the proof of [34, Theorem 4.12] shows that in the case of $\mathbb{A} \in W^{1,\infty}$ and $f \equiv 1$ the regular set turns out to be a $C^{1,\beta'}$-manifold with $\beta' \in (0, \frac{1}{2})$, so, despite the linear term being constant, the regularity improves slightly but remains in the same class. Blank and Hao in [7] proved that if $a_{i,j}, f \in VMO$, any compact set $K \subset\subset Reg(u) \cap B_{\frac{1}{2}}$ is relatively Reifenberg vanishing with respect to $Reg(u) \cap B_{\frac{1}{2}}$. So the regularity of the regular part of the free boundary turns out to be strictly related to regularity of coefficients of matrix $\mathbb{A}$ and the linear term f. In Chapter 2 we suppose the matrix $\mathbb{A} \in W^{1+s,p}$; if f is Hölder continuous we obtain that the regular part of the free boundary is a $C^{1,\beta}$-manifold for some β, while if f satisfies hypothesis $(I3)'$ we prove that $Reg(u)$ is a C^1-manifold.

So the process of weakening the regularity of coefficients goes along two directions: to obtain a strong or a weak regularity of the regular part of the free boundary. Our work forms part of the first way and with the technical hypothesis $(I3)'$ for f, which is better than Hölder continuity, and by hypothesis $(I1)$ of matrix $\mathbb{A}$, we improve the current literature. The best regularity for $\mathbb{A}$ that allows us to have a strong regularity of $Reg(u)$ still remains, to our knowledge, an open problem. Regarding the best regularity for f, from [6] we know that it is the Dini continuity; we do not reach it but we improve the already investigated condition of Hölder continuity.

Taking the epiperimetric inequality into account, Weiss proved this result in [95] in the classical obstacle case. With a similar proof Garofalo, Petrosyan and Smith Vega Garcia [46] proved the epiperimetric inequality for the Signorini's problem with variable coefficients. Independently Focardi and Spadaro [36] proved the same result for the

Signorini's problem with a variational approach; by following the approach of [36] we prove the inequality in the case of the fractional obstacle problem. Recently Garofalo, Petrosyan, Pop and Smith Vega Garcia [44] proved an epiperimetric inequality for the fractional obstacle problem with drift in the case of $s \in (1/2, 1)$. In the case whitout drift we prove the inequality for all $s \in (0, 1)$ and moreover our result is stronger as we do not need any closeness assumption to the cone of blow ups. Instead, Garofalo *et al.* prove that there exists $\kappa \in (0, 1)$ sucht that, if v is a blow up and w is a $(1 + s)$-homogeneous function near v in $H^1(B_1, \mu_a)$-norm for which $w \geq 0$ on B_1', there exists a function $\tilde{w}$, $\tilde{w} = w$ on ∂B_1, for which (cf. 2.102 in Chapter 2)

$$W^0_{1+s}(1, \tilde{w}) \leq (1 - \kappa)\, W^0_{1+s}(1, w). \tag{0.22}$$

Chapter 1

Preliminaries

1.1 Fractional Sobolev spaces

In order to fix the notation we recall the definition of fractional Sobolev spaces. See [26,71] for more detailed references.

Definition 1.1.1. For any real $\lambda \in (0,1)$ and for all $p \in [0,\infty)$ we define the space

$$W^{\lambda,p}(\Omega) := \left\{ v \in L^p(\Omega) : \frac{|v(x)-v(y)|}{|x-y|^{\frac{n}{p}+\lambda}} \in L^p(\Omega\times\Omega) \right\}, \tag{1.1}$$

i.e, an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the norm

$$\|v\|_{W^{\lambda,p}(\Omega)} = \left(\int_\Omega |v|^p \, dx + \iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n+\lambda p}} \, dx \, dy \right)^{\frac{1}{p}}.$$

If $\lambda > 1$ and not integer we indicate with $\lfloor\lambda\rfloor$ its integer part and with $\sigma = \lambda - \lfloor\lambda\rfloor$ its fractional part. In this case the space $W^{\lambda,p}$ consists of functions $u \in W^{\lfloor\lambda\rfloor,p}$ such that the distributional derivatives $D^\alpha v \in W^{\sigma,p}$ with $|\alpha| = \lfloor\lambda\rfloor$

$$W^{\lambda,p}(\Omega) := \left\{ v \in W^{\lfloor\lambda\rfloor,p}(\Omega) : \frac{|D^\alpha v(x) - D^\alpha v(y)|}{|x-y|^{\frac{n}{p}+\sigma}} \in L^p(\Omega\times\Omega),\ \forall\alpha \text{ such that } |\alpha| = \lfloor\lambda\rfloor \right\}.$$

$W^{\lambda,p}(\Omega)$ is a Banach space with the norm

$$\|v\|_{W^{\lambda,p}(\Omega)} = \left(\|v\|^p_{W^{\lfloor\lambda\rfloor,p}(\Omega)} + \sum_{|\alpha|=\lfloor\lambda\rfloor} \|D^\alpha v\|^p_{W^{\sigma,p}(\Omega)} \right)^{\frac{1}{p}}.$$

As in the classical case with λ being an integer, the space $W^{\lambda,p}$ is continuously embedded in $W^{\lambda',p}$ when $\lambda' < \lambda$; the next proposition sums up [26, Propositions 2.1, 2.2 and Corollary 2.3]

Proposition 1.1.2. *Let $p \in [1, \infty)$ and $0 < \lambda' \leq \lambda$. Let Ω an open set in $\mathbb{R}^n$ and v a measurable function. Then*

$$\|v\|_{W^{\lambda',p}(\Omega)} \leq C \, \|v\|_{W^{\lambda,p}(\Omega)}, \tag{1.2}$$

for some suitable positive constant $C = C(n, p, \lambda) \geq 1$. In particular

$$W^{\lambda,p}(\Omega) \subset W^{\lambda',p}(\Omega). \tag{1.3}$$

Di Nezza, Palatucci and Valdinoci in [26] gave a proof of a fractional version of classical extension and immersion theorem.

Theorem 1.1.3 ([26, Theorem 5.4]). *Let $p \geq 1$, $\lambda \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ an open set of class $C^{0,1}$ with bounded boundary. Then $W^{\lambda,p}(\Omega)$ is continuously embedded in $W^{\lambda,p}(\mathbb{R}^n)$, namely for all $v \in W^{\lambda,p}(\Omega)$ there exists a $\tilde{v} \in W^{\lambda,p}(\mathbb{R}^n)$ such that $v = \tilde{v}_{|\Omega}$ and*

$$\|\tilde{v}\|_{W^{\lambda,p}(\mathbb{R}^n)} \leq C \|v\|_{W^{\lambda,p}(\Omega)}, \tag{1.4}$$

where $C = C(n, p, \lambda, \Omega)$.

Theorem 1.1.4 ([26, Theorems 6.7, 6.10 and 8.2]). *Let $\lambda \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ an open set of class $C^{0,1}$ with bounded boundary. Then we can distinguish three cases:*

(i) if $\lambda p < n$ there exists a positive constant $C = C(n, p, \lambda, \Omega)$ such that for all $v \in W^{\lambda,p}(\Omega)$ we have

$$\|v\|_{L^q(\Omega)} \leq C \, \|v\|_{W^{\lambda,p}(\Omega)}, \tag{1.5}$$

for any $q \in [p, p^*]$, with $p^* := p^*(n, p, \lambda) = \frac{np}{n-\lambda p}$; i.e. $W^{\lambda,q}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, p^*]$.

(ii) if $\lambda p = n$ there exists a positive constant $C = C(n, p, \lambda, \Omega)$ such that for all $v \in W^{\lambda,p}(\Omega)$ we have

$$\|v\|_{L^q(\Omega)} \leq C \, \|v\|_{W^{\lambda,p}(\Omega)}, \tag{1.6}$$

for any $q \in [p, \infty)$; i.e. $W^{\lambda,q}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, \infty)$.

(iii) if $\lambda p > n$ there exists a positive constant $C = C(n, p, \Omega)$ such that for all $v \in W^{\lambda,p}(\Omega)$ we have

$$\|v\|_{C^{0,\alpha}(\Omega)} \leq C \, \|v\|_{W^{\lambda,p}(\Omega)}, \tag{1.7}$$

with $\alpha := \frac{\lambda p - n}{p}$

The following proposition is proved by Leoni [69] in the framework of Besov space; according to Theorem 1.1.3 we can extend [69, Theorems 14.22 and 14.32] to the case in which Ω is enough regular.

Proposition 1.1.5 ([69, Theorems 14.22 and 14.32]). *Let $v \in W^{\lambda,p}(\Omega)$ with $\lambda \in (0,1)$, $1 < p < \infty$, $p\lambda < n$ and $\Omega \subset \mathbb{R}^n$ be an open set of class $C^{0,1}$ with bounded boundary. Then for all $0 < t < \lambda$ there exists a constant $C = C(n, \lambda, p, t, \Omega)$ for which*

$$\|u\|_{W^{t,\frac{np}{n-(\lambda-t)p}}(\Omega)} \leq C \|u\|_{W^{\lambda,p}(\Omega)}, \tag{1.8}$$

$$\|u\|_{W^{t,\frac{np}{n-(1-t)p}}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \tag{1.9}$$

We state three results on the Sobolev fractional spaces useful for what follows. Theorems 1.1.6 and 1.1.8 are proved, respectively in [82] and [89], for Besov spaces; thanks to [89, Remark 3.6] and [69, Theorem 14.40] we can reformulate these results in our notations. Theorem 1.1.7 is obtained combining classical Morrey theorem, [26, Theorem 8.3] and Theorem 1.1.6.

Theorem 1.1.6 ([82, Theorem 9] and [26, Theorem 6.5]). *Let $v \in W^{\lambda,p}(\Omega)$ with $\lambda > 0$, $1 < p < \infty$, $p\lambda < n$ and $\Omega \subset \mathbb{R}^n$ be an open set of class $C^{0,1}$ with bounded boundary. Then for all $0 < t < \lambda$ there exists a constant $C = C(n, \lambda, p, t, \Omega)$ for which*

$$\|v\|_{W^{t,\frac{np}{n-(\lambda-t)p}}(\Omega)} \leq C \|v\|_{W^{\lambda,p}(\Omega)}.$$

If $t = 0$ there exists a constant $C = C(n, \lambda, p, \Omega)$ for which

$$\|v\|_{L^{\frac{np}{n-\lambda p}}(\Omega)} \leq C \|v\|_{W^{\lambda,p}(\Omega)}.$$

Proof. Let now $\lambda = \lfloor \lambda \rfloor + \sigma$ and $t = \lfloor t \rfloor + \tau$ be. We analyze the various cases:

- if $\lfloor \lambda \rfloor = \lfloor t \rfloor$ and $\tau > 0$ we have the thesis for (1.8);
- if $\lfloor \lambda \rfloor = \lfloor t \rfloor$ and $\tau = 0$ just apply (1.9) to $D^\alpha u$ with $|\alpha| = \lfloor \lambda \rfloor$;
- if $\lfloor \lambda \rfloor > \lfloor t \rfloor$ since $p(\lambda - t) < n$ also $p\sigma < n$, then applying the first item of Theorem 1.1.4 to $D^\alpha u$ with $|\alpha| = \lfloor \lambda \rfloor$ we have

  $$W^{\lambda,p}(\Omega) \hookrightarrow W^{\lfloor \lambda \rfloor, \frac{np}{n-\sigma p}}(\Omega).$$

  Now we have two cases:

  - if $\lfloor t \rfloor = \lfloor \lambda \rfloor - 1$ by (1.9)

    $$W^{\lambda,p}(\Omega) \hookrightarrow W^{\lfloor \lambda \rfloor, \frac{np}{n-\sigma p}}(\Omega) \hookrightarrow W^{t, \frac{np}{n-(\lambda-t)p}}(\Omega),$$

    because $\frac{np}{n(\lambda-t)p} = \frac{n\frac{np}{n-\sigma p}}{n-(\lfloor\lambda\rfloor-t)\frac{np}{n-\sigma p}}$;
  - if $\lfloor t \rfloor < \lfloor \lambda \rfloor - 1$ applying the classical immersion theorem to $D^a u$ wiht $|\alpha| = \lfloor t \rfloor$ we have

    $$W^{\lambda,p}(\Omega) \hookrightarrow W^{\lfloor \lambda \rfloor, \frac{np}{n-\sigma p}}(\Omega) \hookrightarrow W^{\lfloor t \rfloor+1, \frac{np}{n-(\lambda-\lfloor t \rfloor-1)p}},$$

    because $\frac{np}{n-(\lambda-\lfloor t\rfloor-1)p} = \frac{n\frac{np}{n-\sigma p}}{n-(\lfloor\lambda\rfloor-\lfloor t\rfloor-1)\frac{np}{n-\sigma p}}$. To conclude just observe that we are in the previous case with $\lfloor \lambda \rfloor = \lfloor t \rfloor + 1$.

$\square$

Theorem 1.1.7. *Let $p \in [1, \infty)$ such that $p\lambda > n$ and $\Omega \subset \mathbb{R}^n$ be an extension domain for $W^{\lambda,p}$. Then there exists a positive constant $C = C(n, p, \lambda, \Omega)$, for which*

$$\|v\|_{C^{h,\alpha}} \leq C\|v\|_{W^{\lambda,p}(\Omega)}, \tag{1.10}$$

for all $v \in L^p(\Omega)$ with $\alpha \in (0, 1)$ and some h integer with $h \leq \lfloor \lambda \rfloor$.

Proof. (i) If $\sigma p > n$, from the third item in Theorem 1.1.4$\nabla^{\lfloor \lambda \rfloor} f \in C^{0,\alpha}$ wiht $\alpha = \frac{\sigma p - n}{n}$, and holds

$$\|\nabla^{\lfloor \lambda \rfloor} f\|_{C^{0,\alpha}} \leq C\|\nabla^{\lfloor \lambda \rfloor} f\|_{W^{\sigma,p}} \leq C\|f\|_{W^{\lambda,p}}.$$

In the same way for all $k < \lfloor \lambda \rfloor$ it holds $f \in W^{k+\sigma,p}$ so

$$\|\nabla^k f\|_{C^{0,\alpha}} \leq C\|f\|_{W^{k+\sigma,p}} \overset{(1.1.2)}{\leq} C\|f\|_{W^{k+\sigma,p}}.$$

Whence, up to a product by constant

$$\|f\|_{C^{\lfloor \lambda \rfloor,\alpha}} \leq C\|f\|_{W^{\lambda,p}}.$$

(ii) If $\sigma p < n$ then $\lfloor \lambda \rfloor \geq 1$. For Corollary 1.1.6

$$\|f\|_{W^{\lfloor \lambda \rfloor, \frac{np}{n-\sigma p}}} \leq C\|f\|_{W^{\lambda,p}}. \tag{1.11}$$

We observe that $p\lambda > n$ then also $\frac{\lfloor \lambda \rfloor np}{n-\sigma p} > n$, so for Morrey theorem

$$\|f\|_{C^{\lfloor \lambda \rfloor - \lfloor \frac{n-\sigma p}{p} \rfloor - 1,\alpha}} \leq C\|f\|_{W^{\lfloor \lambda \rfloor, \frac{np}{n-\sigma p}}}.$$

Then by (1.11) we have

$$\|f\|_{C^{\lfloor \lambda \rfloor - \lfloor \frac{n}{p} - \sigma \rfloor - 1,\alpha}} \leq C\|f\|_{W^{\lambda,p}}.$$

(iii) If $\sigma p = n$ then $\lfloor \lambda \rfloor \geq 1$. For all $\sigma' < \sigma$, due to definition of fractional space and thanks to Corollary 1.1.6

$$\|f\|_{W^{\lfloor \lambda \rfloor, \frac{np}{n-\sigma' p}}} \leq \|f\|_{W^{\lambda-\sigma', \frac{np}{n-\sigma' p}}} \leq C\|f\|_{W^{\lambda,p}}. \tag{1.12}$$

Then since $\sigma' p < n$, for (ii) we have

$$\|f\|_{C^{\lfloor \lambda \rfloor - \lfloor \frac{n}{p} - \sigma' \rfloor - 1,\alpha}} \leq C\|f\|_{W^{\lambda,p}}.$$

$\square$

We shall state a trace result for fractional Sobolev spaces. (see [71]).

Theorem 1.1.8 ([89, Theorem 3.16]). *Let $n \geq 2$, $0 < p < \infty$, $\lambda > \frac{1}{p}$ and U a bounded C^k domain, $k > \lambda$ in $\mathbb{R}^n$. Then there exists a bounded operator*

$$\gamma_0 : W^{\lambda,p}(U) \longrightarrow W^{\lambda-\frac{1}{p},p}(\partial U; H^{n-1}), \tag{1.13}$$

such that $\gamma_0(v) = v_{|\partial U}$ for all functions $v \in W^{\lambda,p}(U) \cap C(\overline{U})$.

Remark 1.1.9. Let p, λ be exponents as in theorem 1.1.8, $p\lambda < n$ and $\sigma := \lambda - \lfloor \lambda \rfloor$. If $U = B_r$, we see how the constant of the trace operator changes when the radius r changes.

By taking into account Theorems 1.1.8 and 1.1.6 we have the following embeddings

$$W^{\lambda,p}(B_r) \hookrightarrow W^{\lambda-\frac{1}{p},p}(\partial B_r; H^{n-1}) \hookrightarrow L^1(\partial B_r; H^{n-1}).$$

Then, setting $v_r(y) = v(ry)$

$$\begin{aligned}
&\int_{\partial B_r} |\gamma_0(v)(x)|\, dH^{n-1} \overset{y=rx}{=} r^{n-1} \int_{\partial B_1} |\gamma_0(v_r)(y)|\, dH^{n-1} \\
&\quad\leq c(1) r^{n-1} \Bigg(\sum_{|\alpha| \leq \lfloor \lambda \rfloor} \int_{B_1} |D^\alpha v_r(y)|^p\, dx + \iint_{B_1 \times B_1} \frac{|v_r(y) - v_r(z)|^p}{|y-z|^{n+\sigma p}}\, dy\, dz \Bigg)^{\frac{1}{p}} \\
&\quad\overset{x=\frac{y}{r}}{=} c(1) r^{n-1}\, r^{-\frac{n}{p}} \Bigg(\sum_{|\alpha| \leq \lfloor \lambda \rfloor} \int_{B_r} |D^\alpha v(x)|^p\, dx + r^{\sigma p} \iint_{B_r \times B_r} \frac{|v(x) - v(w)|^p}{|x-w|^{n+\sigma p}}\, dx\, dw \Bigg)^{\frac{1}{p}} \\
&\quad\leq C r^{n-1}\, r^{-\frac{n}{p}} \|v\|_{W^{\lambda,p}(B_r)}.
\end{aligned}$$

Hence

$$\|\gamma_0(v)\|_{L^1(\partial B_r; H^{n-1})} \leq C\, r^{n-1}\, r^{-\frac{n}{p}} \|v\|_{W^{\lambda,p}(B_r)}. \tag{1.14}$$

If $p \leq n$, let $\frac{n-(\lambda-t)p}{np} < t < \lambda$, i.e. $\frac{n-\lambda p}{(n-1)p} < t < \lambda$, of note that $\lambda > \frac{n-\lambda p}{(n-1)p}$ if and only if $\lambda > \frac{1}{p}$. We infer by Theorems 1.1.6 and 1.1.8 the following

$$W^{\lambda,p}(B_r) \hookrightarrow W^{t,\frac{np}{n-(\lambda-t)p}}(B_r) \hookrightarrow W^{t-\frac{n-(\lambda-t)p}{np},\frac{np}{n-(\lambda-t)p}}(\partial B_r; H^{n-1}) \hookrightarrow L^1(\partial B_r; H^{n-1}).$$

Applying the same reasoning to deduce (1.14), in particular we achieve

$$\|\gamma_0(v)\|_{L^1(\partial B_r; H^{n-1})} \leq C\, r^{n-1}\, r^{-\frac{n-(\lambda-t)p}{p}} \|v\|_{W^{t,\frac{np}{n-(\lambda-t)p}}(B_r)}. \tag{1.15}$$

1.2 Theory of elliptic PDEs

We consider the operators L of the forms

$$L(v) = \operatorname{div}(A(x)\nabla v) + b(x) \cdot \nabla v + c(x) v, \tag{1.16}$$

where the coefficients of matrix $A = (a^{ij})$, the coefficients of the vector $b = (b^i)$ and c are assumed measurable functions on a domain $\Omega \subset \mathbb{R}^n$.

The function v satisfies in *weak* or in *generalized* sense the equation $L(v) = 0$ $(\leq 0, \geq 0)$ respectively in Ω if

$$\mathcal{L}(v,\varphi) = \int_\Omega \Big((A(x)\nabla v)\cdot\nabla\varphi - (b(x)\cdot\nabla v + c(x)v)\varphi \Big)\, dx = 0 \ \ (\leq 0, \geq 0) \tag{1.17}$$

for all non negative functions $\varphi \in C_0^1(\Omega)$. Let $h : \Omega \to \mathbb{R}^n$, $l : \Omega \to \mathbb{R}$ be locally integrable function on Ω. The function v is called a *weak* or *generalized* solution of inhomogeneous equation $Lv = D_i h^i + l$ if

$$\mathcal{L}(v,\varphi) = \int_\Omega \Big(h(x)\cdot\nabla\varphi - l(x)\varphi \Big)\, dx, \qquad\qquad \forall\varphi \in C_0^1(\Omega). \tag{1.18}$$

We state a result on the generalized Dirichlet problem. We shall assume that L is strictly elliptic, or rather there exists $\lambda > 0$ such that

$$\langle A(x)\xi,\xi\rangle \geq \lambda|\xi|^2, \quad \forall x\in\Omega,\ \xi\in\mathbb{R}^n. \tag{1.19}$$

We also assume that the coefficients are bounded: there exist $\Lambda, \nu > 0$ such that

$$\sum |a^{ij}(x)|^2 \leq \Lambda^2,\ \ \lambda^{-2}\sum(|b(x)|^2 + |c(x)|^2) + \lambda^{-1}|d(x)| < \nu^2, \quad \forall x\in\Omega. \tag{1.20}$$

A function $v \in W^{1,2}(\Omega)$ will be called solution of generalized Dirichlet problem

$$\begin{cases} Lv = \operatorname{div}(h) + l & \textit{in } \Omega \\ v = \phi & \textit{on } \partial\Omega \end{cases} \tag{1.21}$$

if v is solution of (1.18) and $v - \phi \in W_0^{1,2}(\Omega)$.

In what follows we will request a weak condition of non-positivity of c; we assume

$$\int_\Omega c\varphi\, dx \leq 0 \qquad \forall\varphi\geq 0, \varphi\in W_0^{1,1}(\Omega). \tag{1.22}$$

We state a classical uniqueness result of the weak solution.

Theorem 1.2.1 ([55, Theorem 8.3]). *Let the operator L satisfy the condition* (1.19), (1.20) *and* (1.22). *Then for $\phi \in W^{1,2}(\Omega)$ and $\operatorname{div}(h), l \in L^2(\Omega)$ the generalized Dirichlet problem, $Lv = l + \operatorname{div}(h)$ in Ω, $v = \phi$ on $\partial\Omega$ is uniquely solvable.*

The following two Theorems are a local a *priori boundness* and a *weak Harnack inequality* for supersolution.

Theorem 1.2.2 ([55, Theorems 8.17]). *Let the operator L satisfy the condition* (1.19), (1.20) *and suppose that $h \in L^q(\Omega;\mathbb{R}^n)$ and $l \in L^{\frac{q}{2}(\Omega)}$ for some $q > n$. Then if v is a $W^{1,2}$ subsolution (supersolution) of equation* (1.18) *in Ω, we have, for any ball $B_{2R}(y) \subset \Omega$ and $p > 1$*

$$\sup_{B_R(y)} v(-v) \leq C\left(R^{-\frac{n}{p}}\|v^+(v^-)\|_{L^p(B_{2R}(y))} + \lambda^{-1}\big(R^{1-\frac{n}{q}}\|h\|_{L^q} + R^{2(1-\frac{n}{q})}\|l\|_{L^{\frac{q}{2}}}\big)\right), \tag{1.23}$$

where $C = C(n, \frac{\Lambda}{\lambda}, \nu R, q, p)$.

Theorem 1.2.3 ([55, Theorems 8.18]). *Let the operator L satisfy the condition* (1.19), (1.20) *and suppose that $h \in L^q(\Omega;\mathbb{R}^n)$ and $l \in L^{\frac{q}{2}(\Omega)}$ for some $q > n$. Then if v is a $W^{1,2}$ supersolution of equation* (1.18) *in Ω, non-negative in a ball $B_{4R}(y) \subset \Omega$ and $1 < p < \frac{n}{n-2}$, we have*

$$R^{-\frac{n}{p}}\|v\|_{L^p(B_{2R}(y))} \leq C, \Big(\inf_{B_R(y)} v + \lambda^{-1}\big(R^{1-\frac{n}{q}}\|h\|_{L^q} + R^{2(1-\frac{n}{q})}\|l\|_{L^{\frac{q}{2}}}\big)\Big), \tag{1.24}$$

where $C = C(n, \frac{\Lambda}{\lambda}, \nu R, p, q)$.

The next two Theorems are the Hölder regularity results of the gradient of the solution.

Theorem 1.2.4 ([59, Theorem 3.13]). *Let $v \in W^{1,2}(\Omega)$ solve* (1.18) *with $b = c = h \equiv \underline{0}$. Assume there exist λ and Λ for which*

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \overline{\Lambda}|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{1.25}$$

and $A \in C^{0,\alpha}(\overline{B_1};\mathbb{R}^{n\times n})$, $d, l \in L^q(B_1)$ for some $q > n$ and $\alpha = 1 - \frac{n}{q} \in (0,1)$. Then $\nabla v \in C^{1,\alpha}(B_1, \mathbb{R}^n)$

Theorem 1.2.5 ([55, Theorem 8.32]). *Let $v \in C^{1,\alpha}(\Omega)$ a solution of* (1.18) *in a bounded domain $\Omega \subset \mathbb{R}^n$. Then for any subdomain $\Omega' \subset\subset \Omega$ we have*

$$\|v\|_{C^{1,\alpha}(\Omega')} \leq C\Big(\|v\|_{C^{0,\alpha}(\Omega)} + \|h\|_{C^{0,\alpha}(\Omega)} + \|l\|_{C^{0,\alpha}(\Omega)}\Big), \tag{1.26}$$

for $C = C(n, \lambda, K, \mathrm{dist}(\Omega', \partial\Omega))$, where $K \geq \max_{i,j=1,\dots,n}\{\|a^{ij}\|_{C^{0,\alpha}(\Omega)}, \|b^i, c\|_{C^0(\Omega)}\}$.

We state a result of comparable of solution:

Proposition 1.2.6 ([8, Lemma 3.4]). *Let L and $\widetilde{L}$ be divergence form elliptic operators as* (1.16) *with $b = d = 0$, $c = \underline{0}$ and $A \in L^\infty(B_1;\mathbb{R}^{n\times n})$ that satisfy* (1.25) *with their constants of ellipticity all contained in the interval of positive numbers $[\overline{\lambda}, \overline{\Lambda}]$. If*

$$\begin{aligned} Lw = \widetilde{L}\widetilde{w} &= 1 && \text{in } B_1 \\ w = \widetilde{w} &= 0 && \text{on } \partial B_1 \end{aligned} \tag{1.27}$$

Then there exists a positive constant $C_0 = C_0(n, \overline{\lambda}, \overline{\Lambda})$, such that for all $x \in B_{1/4}$ we obtain

$$C_0^{-1}w(x) \leq \widetilde{w} \leq C_0 w(x). \tag{1.28}$$

We now state a result related to operator in general form (not in divergence form). Let L' be the operator

$$L'v = \mathrm{Tr}(A(x)\nabla^2 v) + b(x)\cdot\nabla v + c(x)v, \tag{1.29}$$

where A, c and d are defined as above. If l as before is a *strong solution* of

$$L'v = l \tag{1.30}$$

is a twice differentiable function on Ω satisfying the equation (1.30) almost everywhere in Ω.

Theorem 1.2.7 ([55, Theorem 9.11]). *Let Ω be an open set in $\mathbb{R}^n$ and $v \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$, $1 < p < \infty$, a strong solution of $L'v = l$ where the coefficients of L' satisfy*

$$\begin{aligned} A \in C^0(\Omega; \mathbb{R}^{n\times n}), \quad & c \in L^\infty(\Omega;\mathbb{R}^n), \quad d \in L^\infty(\Omega), \quad l \in L^p(\Omega), \\ & \langle A(x)\xi, \xi\rangle \geq \lambda|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n, \\ & |A|, |b|, |c| \leq \Lambda'. \end{aligned} \tag{1.31}$$

Then for any domain $\Omega' \subset\subset \Omega$

$$\|v\|_{W^{2,p}(\Omega')} \leq C\Big(\|v\|_{L^p(\Omega)} + \|l\|_{L^p(\Omega)}\Big) \tag{1.32}$$

where C depends on $n, p, \lambda, \Lambda', \Omega', \Omega$ and the moduli of continuity of the coefficients of matrix A.

Corollary 1.2.8 ([55, Corollary 9.18]). *Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$, and let the operator L' be strictly elliptic in Ω with coefficients $A \in C^0(\overline{\Omega}; \mathbb{R}^{n\times n})$, $b \in L^\infty(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega)$ and $c \leq 0$. Then if $l \in L^p(\Omega)$ with $p > \frac{n}{2}$, $\phi \in C^0(\partial\Omega)$, the Dirichlet problem $L'v = l$ in Ω, $v = \phi$ on $\partial\Omega$ has a unique solution $v \in W^{2,p}_{loc}(\Omega) \cap C^0(\overline{\Omega})$.*

1.3 Coercive vector fields

Let V be a closed subspace of $W^{1,p}(\Omega)$ with $p > 1$. We introduce a non linear operator $\mathtt{A} : W^{1,p}(\Omega) \to V'$ by setting

$$\langle \mathtt{A}v, w\rangle = \int_\Omega \big(A(x, v(x), \nabla v(x)) \cdot \nabla w + A^0(x, v(x), \nabla v(x))w\big)\, dx, \tag{1.33}$$

for $v \in W^{1,p}(\Omega)$ and $w \in V$, where $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $A^0 : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are supposed to be the Carathéodory function of $x \in \Omega$ and $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$ with

$$|A(x, \eta, \xi))| \vee |A^0(x, \eta, \xi))| \leq C(|\eta|^{p-1} + |\xi|^{p-1}) + h(x), \tag{1.34}$$

for a.e. $x \in \Omega$, any $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and with $h \in L^{p'}(\Omega)$.

We analyse a *variational inequality* that is an inequality that involves a functional that has to satisfy for all functions in a suitable set. We look for $v \in K$, with K a non empty, convex subset of V such that

$$\langle \mathtt{A}(v) - F, w - v\rangle \geq 0 \qquad \textit{for } w \in K, \tag{1.35}$$

where $\mathtt{A}(v)$, hence $\mathtt{A}(v) - F$, is the Gateaux derivative at v of some convex functional in V.

In order to study this context very generally we introduce the following definitions.

Definition 1.3.1. We say that a nonlinear operator $\mathtt{A} : V \to V'$ is

- *hemicontinuous* if each real function

$$\lambda \mapsto \langle \mathtt{A}((1-\lambda)v + \lambda w), w - v\rangle \tag{1.36}$$

with $v, w \in V$, is continuous in $\mathbb{R}$;

- *monotone* if

$$\lambda \mapsto \langle \mathtt{A}(v) - \mathtt{A}(w)), v - w\rangle \geq 0 \qquad \forall v, w \in V; \tag{1.37}$$

- *strictly monotone* if the requirement

$$\lambda \mapsto \langle \mathtt{A}(v) - \mathtt{A}(w)), v - w\rangle = 0 \quad \Longrightarrow \quad v = w \tag{1.38}$$

is added to monotonicity.

- *pseudomonotone* if it is bounded and satisfies

$$\liminf_{n\to\infty} \langle \mathtt{A}(v_n), v_n - w\rangle \geq \langle \mathtt{A}(v), v - w\rangle \qquad \textit{for } w \in V \tag{1.39}$$

whenever the sequence $v_n \rightharpoonup v$ in V with

$$\limsup_{n\to\infty} \langle \mathtt{A}(v_n), v_n - w\rangle \leq 0. \tag{1.40}$$

Definition 1.3.2. A nonlinear operator $\mathtt{A} : V \to V'$ is a *Leray-Lions operator* if it is bounded and satisfies

$$\mathtt{A}(v) = \mathcal{A}(u, u) \qquad \textit{for } v \in V, \tag{1.41}$$

where $\mathcal{A} : V \times V \to V'$ has the following properties:

(i) whenever $v \in V$, the mapping $w \mapsto \mathcal{A}(v, w)$ is bounded and hemicontinuous from V to V', with

$$\langle \mathcal{A}(v, v) - \mathcal{A}(v, w), v - w\rangle \geq 0 \qquad \textit{for } w \in V; \tag{1.42}$$

(ii) whenever $w \in V$, the mapping $v \mapsto \mathcal{A}(v, w)$ is bounded and hemicontinuous from V to V';

(iii) whenever $w \in V$ $\mathcal{A}(v_n, w)$ converges weakly to $\mathcal{A}(v, w)$ in V' if $(v_n) \subset V$ is such that $v_n \rightharpoonup v$ in V and

$$\langle \mathcal{A}(v_n, v_n) - \mathcal{A}(v_n, v), v_n - v\rangle \to 0; \tag{1.43}$$

(iv) whenever $w \in V$ $\langle \mathcal{A}(v_n, w), v_n\rangle$ converges to $\langle F, v\rangle$ if $(v_n) \subset V$ is such that $v_n \rightharpoonup v$ in V, $\mathcal{A}(v_n, w) \rightharpoonup F$ in V'.

Lemma 1.3.3 ([91, Lemmas 4.12 and 4.13]). *Let $\mathtt{A} : V \to V'$ be a nonlinear operator:*

(i) If $\mathtt{A}$ is bounded, hemicontinuous and monotone then $\mathtt{A}$ is pseudomonotone.

(ii) If $\mathtt{A}$ *is Leray-Lions operator then* $\mathtt{A}$ *is pseudomonotone.*

Theorem 1.3.4 ([91, Theorem 4.17])**.** *Let* $\mathtt{A} : V \to V'$ *be a pseudomonotone satisfying the following growth condition*

$$\exists R \in (0, \infty), v_0 \in K, \|v_0\|_V < R \ : \ \langle \mathtt{A}(v) - F, v_0 - v \rangle < 0 \qquad \textit{for } \|v\|_V = R, \tag{1.44}$$

where $K \neq \emptyset$ *is a closed and convex subset of* V*. Then for any choice of* $F \in V'$ (1.35) *admits at least one solution.*

In the sequel we shall call $\mathtt{A} : W^{1,p}(\Omega) \to V'$ bounded, or hemicontinuous, or monotone if the restriction of $\mathtt{A}$ to V is such.

Definition 1.3.5. A nonlinear operator $\mathtt{A} : W^{1,p}(\Omega) \to V'$ is

- *T-monotone* if

$$\lambda \mapsto \langle \mathtt{A}(v) - \mathtt{A}(w)), (v - w)^+ \rangle \geq 0 \qquad \textit{for } v, w \in W^{1,p} \textit{ with } (v - w)^+ \in V; \tag{1.45}$$

- *strictly T-monotone* if the equality sign in the above inequality can only hold when $v \leq w$ in Ω.

Proposition 1.3.6 ([91] p. 231)**.** *T-monotonicity implies monotonicity and the uniqueness of the solution of* (1.35)*.x*

Proposition 1.3.7. *If the requirement*

$$\big(A^0(\cdot, \eta, \xi) - A^0(\cdot, \eta', \xi')\big)(\eta - \eta') + \big(A(\cdot, \eta, \xi) - A(\cdot, \eta', \xi')\big) \cdot (\xi - \xi') \geq 0 \tag{1.46}$$

a.e. in Ω*, for* $\eta, \eta' \in \mathbb{R}$ *and* $\xi, \xi' \in \mathbb{R}^n$*, is added to* (1.34) *then* $\mathtt{A}$ *is T-monotone.*

If (1.46) is weakened into

$$\big(A(\cdot, \eta, \xi) - A(\cdot, \eta, \xi') \cdot (\xi - \xi') \geq 0 \tag{1.47}$$

a.e. in Ω, for $\eta \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^n$, monotonicity can no longer be claimed. However we have the following theorem.

Theorem 1.3.8 ([91, Theorem 4.21])**.** *Let* V *be compactly embedded into* $L^p(\Omega)$ *and* $\mathtt{A}$ *be defined by* (1.33) *under assumption* (1.34)*. Suppose that* (1.47) *holds, with the strict inequality sign for* $\xi \neq \xi'$*. Then* $\mathtt{A}$ *is a Leray-Lions operator, hence a pseudomonotone operator, when restricted to* V*.*

1.3.1 Regularity theory for nonlinear operator

In this paragraph we state two results of regularity for nonlinear equation in divergence form.

Theorem 1.3.9 ([74, Theorem 6]). *Let $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ a solution of*

$$\operatorname{div} A(x, v, \nabla v) = B(x, v, \nabla v), \tag{1.48}$$

in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following condition:

$$A \text{ is differentiable with respect to } h \in \mathbb{R}^n \text{ and for all } h, \xi \in \mathbb{R}^n, z \in \mathbb{R} \text{ and a.e. } x \in \Omega$$
$$\gamma_0(\epsilon + |h|^2)^{\frac{p}{2}-1}|\xi|^2 \leq \nabla_h A(x, z, h)\xi \cdot \xi. \tag{1.49}$$

For all $h \in \mathbb{R}^n$, all $z \in \mathbb{R}$ and a.e. $x \in \Omega$

$$|\nabla_h A(x, z, h)| \leq \gamma_1(\epsilon + |h|^2)^{\frac{p}{2}-1}, \tag{1.50}$$
$$\nabla_h A(x, z, h) \cdot h \geq \gamma_2(\epsilon + |h|^2)^{\frac{p}{2}} - \mu, \tag{1.51}$$
$$|A(x, z, h)| \leq \gamma_3(1 + |h|^2)^{\frac{p-1}{2}}, \tag{1.52}$$
$$|B(x, z, h)| \leq \gamma_4(1 + |h|^2)^{\frac{p}{2}}, \tag{1.53}$$

where $p > 1$, $0 \leq \epsilon \leq 1$ and $\mu, \gamma_0, \gamma_1, \gamma_2, \gamma_3$ and γ_4 are positive constants.

There exists a bounded continuous increasing function $\sigma(t)$ with $\sigma(0) = 0$ such that for all $x, x' \in \Omega$, $z, z' \in \mathbb{R}$ and $h \in \mathbb{R}^n$ we have

$$|A(x, z, h) - A(x', z', h)| \leq \sigma(|x - x'| + |z - z'|)(1 + |h|^2)^{\frac{p-1}{2}}. \tag{1.54}$$

If there exists a $\delta > 0$ for which $\sigma(t) \leq C\, t^\delta$, then

(i) there exists a number λ_0, $0 < \lambda_0 \leq p$ such that

$$|\nabla u| \in \mathcal{L}^{p, n+\lambda_0}(\Omega'), \tag{1.55}$$

(ii) $v \in C^{1,\frac{\lambda_0}{p}}(\Omega')$.

Here λ_0 depends only on the data. The norm $\|\nabla u\|_{\mathcal{L}^{p,n+\lambda_0}(\Omega')}$ depends also on $\operatorname{dist}(\Omega', \partial\Omega)$ *and $\|v\|_{W^{1,p}(\Omega)}$.*

Theorem 1.3.10 ([67, Theorem 5.2, Chapter 4]). *Let v a bounded generalized solution of* (1.48) *with $\sup_\Omega |v| = M$, and suppose that $A(x, z, h) = (A^i(x, z, h))$, $B(x, z, h)$ with*

$i = 1, \ldots, n$ are differentiable and that they satisfy the following conditions

$$\nu(v)(1+|h|)^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{n} \frac{\partial A^i(x,v,h)}{\partial h_j}\xi_i\,\xi_j \leq \mu(v)(1+|h|)^{p-2}|\xi|^2 \tag{1.56}$$

$$\sum_{i=1}^{n}\Big(\Big|\frac{\partial A^i(x,v,h)}{\partial z}\Big| + |A^i|\Big)(1+|h|) + \sum_{i,j=1}^{n}\Big|\frac{\partial A^i(x,v,h)}{\partial x_j}\Big| + |B(x,v,h)| \leq \mu(v)(1+|h|)^p \tag{1.57}$$

$$\sum_{i,j=1}^{n}\Big|\frac{\partial A^i(x,v,h)}{\partial x_j}\Big|(1+|h|) + \sum_{j=1}^{n}\Big|\frac{\partial B(x,v,h)}{\partial h_j}\Big|(1+|h|) \tag{1.58}$$

$$+\Big|\frac{\partial B(x,v,h)}{\partial z}\Big| + \sum_{j=1}^{n}\Big|\frac{\partial B(x,v,h)}{\partial x_j}\Big| \leq \mu_1(|v|)(1+|h|)^p, \tag{1.59}$$

with $p > 1$, for $x \in \Omega$, $|v| \leq M$ and for arbitrary h. Then $v \in W^{2,2}_{loc}(\Omega)$.

1.4 Quasi-minima

We consider the functional $\mathcal{F} : W^{1,p}(\Omega) \to \mathbb{R}$

$$\mathcal{F}(v;\Omega) := \int_\Omega f(x, v(x), \nabla v(x))\, dx. \tag{1.60}$$

Let us also suppose

$$|\xi|^p - b|z|^\gamma - \vartheta(x) \leq f(x,z,\xi) \leq \mu|\xi|^p + b|z|^\gamma + \vartheta(x), \tag{1.61}$$

with ϑ a given nonnegative function and b, μ and γ nonnegative constant staifying

$$p > 1; \qquad\qquad 1 \leq \gamma \leq \begin{cases} \frac{pn}{n-p} & p < n \\ +\infty & p \geq n. \end{cases} \tag{1.62}$$

Definition 1.4.1. Let Q be a constant $Q \geq 1$. A function $v \in W^{1,p}(\Omega)$ is a Q-minimum for $\mathcal{F}$ if and only if for every function $\phi \in W^{1,p}(\Omega)$ with $\mathrm{supp}(\phi) = K \subset \Omega$ we have

$$\mathcal{F}(v, K) \leq Q\,\mathcal{F}(v + \phi, K). \tag{1.63}$$

Theorem 1.4.2 ([54, Theorem 4.1])**.** *Let $v \in W^{1,p}(\Omega)$ be a quasi-minimum for the functional $\mathcal{F}$ with condition* (1.61) *and* (1.62)*. Let us suppose $\vartheta \in L^\sigma_{loc}(\Omega)$ for some $\sigma > \frac{n}{p}$. Then v is locally bounded in Ω.*

Theorem 1.4.3 ([54, Theorem 4.2])**.** *Let the function $f(x,z,\xi)$ satisfy the growth condition*

$$|\xi|^p - \vartheta(x, M) \leq f(x,z,\xi) \leq \mu(M)|\xi|^p + \vartheta(x, M), \tag{1.64}$$

for every $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $z \in \mathbb{R}$ with $|z| \leq M$. Let $v \in W^{1,p}_{loc}(\Omega)$ be a bounded quasi-minimum for $\mathcal{F}$, and suppose that for every M, $g(\cdot, M) \in L^\sigma_{loc}$ for some $\sigma > \frac{n}{p}$. Then v is Hölder continuous in Ω.

1.5 Γ-convergence

We recall the definition of the Γ-convergence introduced by De Giorgi in a generic metric space (X, d) endowed with the topology induced by the distance d (see the books [9, 25]).

Definition 1.5.1. We say that a sequence of functionals $F_j : X \to \overline{\mathbb{R}}$ Γ-converges in X to a functional $F : X \to \overline{\mathbb{R}}$ in $x \in X$, and we write $F(x) = \Gamma\text{-}\lim_j F_j(x)$ if the following two condition hold:

(a) Γ-$\liminf$ *inequality*: for every sequence (x_j) converging to x we have

$$F(x) \le \liminf_j F(x_j); \tag{1.65}$$

(b) Γ-$\limsup$ *inequality*: there exists a sequence $(\overline{x}_j)$ converging to x we have

$$F(x) \ge \limsup_j F(\overline{x}_j); \tag{1.66}$$

We say that F_j Γ-converges to F, and write $F = \Gamma\text{-}\lim_j F_j$, if $F(x) = \Gamma\text{-}\lim_j F_j(x)$ for all $x \in X$. The functional F is called the Γ-*limit* of (F_j).

We can also define a notion of *lower* and *upper* Γ-*limit*:

Definition 1.5.2. The Γ-*lower limit* and the Γ-*upper limit* of a sequence of functionals $F_j : X \to \overline{\mathbb{R}}$ are the functionals from X into $\overline{\mathbb{R}}$ defined by

$$\begin{aligned} \Gamma\text{-}\liminf_j F_j(x) &:= \inf\{\liminf_j F_j(x_j) \ : \ x_j \to x\}, \\ \Gamma\text{-}\limsup_j F_j(x) &:= \inf\{\limsup_j F_j(x_j) \ : \ x_j \to x\} \end{aligned} \tag{1.67}$$

respectively. There exists a functional $F : X \to \overline{\mathbb{R}}$ for which $\Gamma\text{-}\liminf_j F_j = F = \Gamma\text{-}\limsup_j F_j$ if and only if F satisfies the above condition (i) and (ii) and $F = \Gamma\text{-}lim(F_j)$.

One of the main reasons for the introduction of this notion is the following fundamental theorem:

Theorem 1.5.3 ([25, Theorem 7.8])**.** *Let $F = \Gamma\text{-}\lim_j F_j$, and assume there exists a compact set $K \subset X$ such that $\inf_X F_j = \inf_K F_j$ for all j. Then there exists the minimum of F*

$$\min_X F = \lim_j \inf_X F_j. \tag{1.68}$$

Moreover, given $(x_j)_{j\in\mathbb{N}}$ a converging sequence $x_j \to x$ in X. If $\lim_j F_j(x_j) = \lim_j \inf_X F_j$ then x is a minimum point for F

Chapter 2

The classical obstacle problem for quadratic energies

2.1 The classical obstacle problem

In this section we prove the existence, the uniqueness and regularity of minimizer u.

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded and open set, $n \geq 2$, let $\mathbb{A} : \Omega \to \mathbb{R}^{n\times n}$ be a symmetric matrix-valued field and $f : \Omega \to \mathbb{R}$ be a function satisfying the following hypotheses:

(I1) $\mathbb{A} \in W^{1+s,p}(\Omega; \mathbb{R}^{n\times n})$ with $s > \frac{1}{p}$ and $p > \frac{n^2}{n(1+s)-1} \wedge n$ or $s = 0$ and $p = +\infty$, where the symbol $\wedge$ indicates the minimum of the surrounded quantities;

(I2) $\mathbb{A}(x) = (a_{ij}(x))_{i,j=1,\dots,n}$ symmetric, continuous and coercive, that is $a_{ij} = a_{ji}$ $\mathcal{L}^n$ a.e. Ω and for some $\Lambda \geq 1$ i.e.

$$\Lambda^{-1}|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \qquad \mathcal{L}^n \text{ a.e. } \Omega,\ \forall \xi \in \mathbb{R}^n; \tag{2.1}$$

(I3) f Dini-continuous, that is $\omega(t) = \sup_{|x-y|\leq t} |f(x) - f(y)|$ modulus of continuity f satisfying the following integrability condition:

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty; \tag{2.2}$$

(I4) there exists a positive constant $c_0 > 0$ such that $f \geq c_0$.

Remark 2.1.1. As we will see in Corollary 1.1.7, if $ps > n$ it holds that $W^{1+s,p} \hookrightarrow W^{1,\infty}$. Consequently, in view of [34] we assume $sp \leq n$ and we obtain an original result if $p > \frac{n^2}{n(1+s)-1} \wedge n$ (we can observe that $(\frac{n^2}{n(1+s)-1} \wedge n) < \frac{n}{s}$ for all $s \in \mathbb{R}$).

Remark 2.1.2. By (2.1), we immediately deduce that $\mathbb{A}$ is bounded. In particular, $\|\mathbb{A}\|_{L^\infty(\Omega;\mathbb{R}^n)} \leq \Lambda$.

We define, for every open $A \subseteq \Omega$ and for each function $v \in H^1(\Omega)$, the following energy:

$$\mathcal{E}[v, A] := \int_A \left(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2f(x)v(x) \right) dx, \tag{2.3}$$

with $\mathcal{E}[v, \Omega] := \mathcal{E}[v]$.

Proposition 2.1.3. *We consider the followinging minimum problem with obstacle:*

$$\inf_K \mathcal{E}[\cdot], \tag{2.4}$$

where $K \subset H^1(\Omega)$ is the weakly closed convex set given by

$$K := \{v \in H^1(\Omega) \,|\, v \geq 0 \, \mathcal{L}^n\text{-a.e. on } \Omega,\ \gamma_0(v) = g \text{ on } \partial\Omega\}, \tag{2.5}$$

with $g \in H^{\frac{1}{2}}(\partial\Omega)$ being a nonnegative function.

Then there exists a unique solution for the minimum problem (2.4).

Proof. The hypotheses $(I1)$-$(I3)$ imply that the energy $\mathcal{E}$ is coercive and strictly convex in K, therefore $\mathcal{E}$ is lower semicontinuous for the weak topology in $H^1(\Omega)$, then there exists a unique minimizer that, as we stated in the introduction, will be indicated by u. □

Now, we can fix the notation for the *coincidence set*, *non-coincidence set* and the *free boundary* by defining the following:

$$\Lambda_u := \{u = 0\}, \qquad N_u := \{u > 0\}, \qquad \Gamma_u = \partial\Lambda_u \cap \Omega. \tag{2.6}$$

We consider the functional

$$\mathcal{G}[v] := \int_\Omega \big(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2\, f(x)v^+(x) \big)\, dx, \tag{2.7}$$

defined on $H^1(\Omega)$ and prove the following:

Proposition 2.1.4. *The problem*

$$\min_{\tilde{g}+H^1_0(\Omega)} \mathcal{G}[\cdot], \tag{2.8}$$

where $\gamma_0(\tilde{g}) = g \in H^{\frac{1}{2}}(\partial\Omega)$, has a unique solution. Therefore,

$$\min_K \mathcal{E}[\cdot] = \min_{\tilde{g}+H^1_0(\Omega)} \mathcal{G}[\cdot]. \tag{2.9}$$

Proof. In order to prove the first part of the statement, it is enough to prove that $\int_\Omega 2fv^+\, dx$ is H^1 weakly continuous.

Therefore, let $v_k \rightharpoonup v$ in $H^1(\Omega)$, up to subsequences we can assume $v_k \to v$ in $L^1(\Omega)$ and a.e. in Ω. So the inequality

$$|x^+ - y^+| \leq |x - y|$$

allows us to conclude. To prove the second part of the statement we observe that $K \subset \tilde{g} + H^1(\Omega)$ and we note that $\mathcal{E}[v] = \mathcal{G}[v]$ for $v \in K$. So

$$\min_{\tilde{g}+H^1_0(\Omega)} \mathcal{G} \leq \min_K \mathcal{E}.$$

Instead, if $v \in \left(\tilde{g} + H^1_0(\Omega)\right) \setminus K$ then $v^+ \in K$ and $\mathcal{E}[v^+] = \mathcal{G}[v^+] \leq \mathcal{G}[v]$. Thus

$$\min_K \mathcal{E} \leq \min_{\tilde{g}+H^1_0(\Omega)} \mathcal{G}.$$

□

Since the function u is the minimum of $\mathcal{E}$ it is also the solution of a variational inequality (see also (1.35)). If we consider the functional

$$\langle \mathtt{E}(u), \varphi \rangle = \int_\Omega \left(\langle \mathbb{A}(x)\nabla v(x), \nabla\varphi(x) \rangle + 2\, f(x)\varphi(x) \right) dx, \tag{2.10}$$

that is the Gateaux derivative of $\mathcal{E}$ in u we have

$$\langle \mathtt{E}(u), \varphi - u \rangle \geq 0 \qquad\qquad \forall \varphi \in K. \tag{2.11}$$

Actually the minimum u satisfies the partial differential equation both in the distributional sense and a.e. on Ω. Therefore it shows good properties of regularity:

Proposition 2.1.5. *Let u be the minimum of $\mathcal{E}$ in K. Then*

$$\operatorname{div}(\mathbb{A}(x)\nabla u(x)) = f(x)\chi_{\{u>0\}}(x) \qquad \text{a.e. on } \Omega \text{ and in } \mathcal{D}'(\Omega). \tag{2.12}$$

Therefore,

(i) if $ps < n$, called $p^*(s,p) := p^* = \frac{np}{n-sp}$, we have $u \in W^{2,p^*} \cap C^{1,1-\frac{n}{p^*}}(\Omega)$;

(ii) if $ps = n$ we have $u \in W^{2,q} \cap C^{1,1-\frac{n}{q}}(\Omega)$ for all $1 < q < \infty$.

Proof. We can split the proof of the proposition into three steps:
[1] **Step 1: Preliminary equation.** Let $\varphi \in H^1_0 \cap C^0(\Omega)$ and $\varepsilon > 0$, and we consider $u + \varepsilon\varphi$, a competitor for $\mathcal{G}$. Since $u \geq 0$

$$\begin{aligned}
&0 \leq \varepsilon^{-1} \left(\mathcal{G}[u+\varepsilon\varphi] - \mathcal{G}[u] \right) \\
&\quad = \varepsilon^{-1} \left(\int_\Omega \!\left(\langle \mathbb{A}\nabla(u+\varepsilon\varphi), \nabla(u+\varepsilon\varphi) \rangle + 2\, f(u+\varepsilon\varphi)^+ \right) dx - \int_\Omega \left(\langle \mathbb{A}\nabla u, \nabla u \rangle + 2\, fu \right) dx \right) \\
&\quad = \int_\Omega \left(\varepsilon \langle \mathbb{A}\nabla\varphi, \nabla\varphi \rangle + 2\langle \mathbb{A}\nabla u, \nabla\varphi \rangle \right) dx + 2\varepsilon^{-1} \int_\Omega f \left((u+\varepsilon\varphi)^+ - u \right) dx.
\end{aligned} \tag{2.13}$$

[1] For the first part of the proof we refer to [34, Proposition 2.2].

We observe that

$$\int_\Omega f\left((u+\varepsilon\varphi)^+ - u\right)\,dx = \varepsilon\int_{\{u+\varepsilon\varphi\geq 0\}} f\varphi\,dx - \int_{\{u+\varepsilon\varphi<0\}} fu\,dx \tag{2.14}$$

and

$$0 \leq \int_{\{u+\varepsilon\varphi<0\}} fu\,dx \leq -\varepsilon\int_{\{u+\varepsilon\varphi<0\}} f\varphi\,dx = o(\varepsilon). \tag{2.15}$$

Fixing the set $A_\varphi := \{u=0\}\cap\{\varphi\geq 0\}$,

$$\chi_{\{u+\varepsilon\varphi\geq 0\}} \xrightarrow{L^1} \chi_{A_\varphi\cup\{u>0\}} \qquad for\ \varepsilon\to 0, \tag{2.16}$$

in fact, since φ is bounded,

$$\begin{aligned}\int_\Omega |\chi_{\{u+\varepsilon\varphi\geq 0\}} - \chi_{A_\varphi\cap\{u>0\}}|\,dx &= \int_\Omega |\chi_{\{u+\varepsilon\varphi\geq 0\}\setminus A_\varphi} - \chi_{\{u>0\}}|\,dx\\ &= \int_\Omega |\chi_{\{u\geq -\varepsilon\varphi\}\cap\{u>0\}} - \chi_{\{u>0\}}|\,dx\\ &= \int_\Omega \chi_{\{u>0\}\setminus\{u\geq-\varepsilon\varphi\}}\,dx = \int_\Omega \chi_{\{0<u<-\varepsilon\varphi\}}\,dx \xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

Passing to the limit as $\varepsilon\to 0$ on (2.13), thanks to (2.14), (2.15), (2.16) and applying the Lebesgue's dominated convergence Theorem we obtain

$$\int_\Omega \langle\mathbb{A}\nabla u,\nabla\varphi\rangle\,dx + \int_\Omega \varphi f\chi_{\{u>0\}\cup A_\varphi}\,dx \geq 0,$$

thus

$$\int_\Omega \langle\mathbb{A}\nabla u,\nabla\varphi\rangle\,dx + \int_\Omega \varphi f\,dx \geq \int_\Omega \varphi f\chi_{\{u=0\}\cap\{\varphi<0\}}\,dx. \tag{2.17}$$

Therefore, the distributional divergence $\mathrm{div}(\mathbb{A}(\cdot)\nabla u)$ of $\mathbb{A}(\cdot)\nabla u$ satisfies

$$\langle -\mathrm{div}(\mathbb{A}(\cdot)\nabla u) + f\mathcal{L}^n \llcorner\Omega, \varphi\rangle \geq 0 \qquad \text{for all } \varphi\in C_c^\infty(\Omega),\ \varphi\geq 0, \tag{2.18}$$

in turn implying that $\mu := -\mathrm{div}(\mathbb{A}(\cdot)\nabla u) + f\mathcal{L}^n\llcorner\Omega$ is a non-negative Radon measure.

Employing the condition (2.17) with $\pm\varphi$ we obtain

$$\int_{\{u=0\}\cap\{\varphi<0\}} \varphi f\,dx \leq \int_\Omega \varphi\,d\mu \leq \int_{\{u=0\}\cap\{\varphi>0\}} \varphi f\,dx. \tag{2.19}$$

In turn, the latter inequalities imply that $\mu \ll \mathcal{L}^n\llcorner\Omega$. Thus, if $\mu = \zeta\mathcal{L}^n\llcorner\Omega$, with $\zeta\in L^1(\Omega)$, we infer that $0\leq\zeta\leq f\chi_{\{u=0\}}$ $\mathcal{L}^n$ a.e. Ω, such that $\zeta\in L^\infty_{loc}(\Omega)$ by (I3). So, by definition $\mu = -\mathrm{div}(\mathbb{A}(\cdot)\nabla u) + f\mathcal{L}^n\llcorner\Omega$, the following equation holds

$$\mathrm{div}(\mathbb{A}(x)\nabla u(x)) = f(x) - \zeta(x) \qquad a.e.\ on\ \Omega\ and\ in\ \mathcal{D}'(\Omega). \tag{2.20}$$

Step 2: Regularity. Now, based on the previous step, we can prove (i), the regularity of u if $ps<n$. From Theorem 1.1.6 $W^{1+s,p}(\Omega)\hookrightarrow W^{1,p^*}(\Omega)$ with $p^* = \frac{np}{n-sp}$. We also

note that by the hypothesis $(I1)$ $p^* > n$, so by Morrey theorem $\mathbb{A} \in C^{0,1-\frac{n}{p^*}}(\Omega)$. Since u is the solution of (2.12), and thanks to Theorem 1.2.4, $u \in C^{1,1-\frac{n}{p^*}}_{loc}(\Omega)$. We consider the equation

$$\mathrm{Tr}(\mathbb{A}\nabla^2 v) = f - \zeta - \sum_j \mathrm{div}(a^j)\frac{\partial u}{\partial x_j} =: \phi, \tag{2.21}$$

where the symbol Tr is the *trace* of the matrix $\mathbb{A}\nabla^2 v$ and a^j denotes the j-column of $\mathbb{A}$. Since $\nabla u \in L^\infty_{loc}(\Omega)$ and $\mathrm{div}(a^j) \in L^{p^*}(\Omega)$ for all $j \in \{1, \dots, n\}$ then $\phi \in L^{p^*}_{loc}(\Omega)$. So, from Corollary 1.2.8 there exists a unique $v \in W^{2,p^*}_{loc}(\Omega)$ solution of (2.21). We observe that the identity $\mathrm{Tr}(\mathbb{A}\nabla^2 v) = \mathrm{div}(\mathbb{A}\nabla v) - \sum_j \mathrm{div}(a^j)\frac{\partial u}{\partial x_j}$ is verified. So, if we rewrite (2.21) as follows

$$\mathrm{div}(\mathbb{A}\nabla v) - \sum_j \mathrm{div}(a^j)\frac{\partial u}{\partial x_j} = \phi, \tag{2.22}$$

we have that u and v are two solutions, then by Theorem 1.2.1 we obtain $u = v$ and the thesis follows. Instead, if $ps = n$ from item (ii) of Theorem 1.1.4$\mathbb{A} \in W^{1,q}$ and so $u \in W^{2,q} \cap C^{1,1-\frac{n}{q}}(\Omega)$ for all $1 < q < \infty$. Applying the same reasoning to deduce the item (i) we obtain the item (ii) of the thesis.
Step 3: Conclusion. By the regularity $W^{2,q}$ of u we can compute the divergence in the definition of the measure μ and thanks to the unilateral obstacle condition using the locality of weak derivatives we have $\zeta_{|\{u=0\}} = f$; so $\zeta = f\chi_{\{u=0\}}$ and we conclude (2.12). $\square$

We note that thanks to the continuity of u the sets defined in (2.6) are pointwise defined and we can also write $\Gamma_u = \partial N_u \cap \Omega$.

Remark 2.1.6. The assumption $(I4)$ is not necessary in order to prove the regularity of u and that the minimum u satisfies the equation (2.12) (cf. Proposition 3.1.2, Theorem 3.1.4 and Corollary 3.1.5 in Chapter 3).

2.2 The blow up method: Existence of blow ups and nondegeneration of the solutions

In this section we shall investigate the existence of blow ups. In this connection, we need to introduce for any point $x_0 \in \Gamma_u$ a sequence of rescaled functions:

$$(u_{x_0,r})_r := \left(\frac{u(x_0 + rx)}{r^2}\right)_r. \tag{2.23}$$

We want to prove the existence of limits (in a strong sense) of this sequence as $r \to 0^+$ and define these blow ups.

We start observing that the rescaled function satisfies an appropriate PDE and satisfies a uniform W^{2,p^*} estimate. We can prove this thanks to the regularity theory for elliptic equations.

Proposition 2.2.1. *Let u be the solution to the obstacle problem* (2.4) *and $x_0 \in \Gamma_u$. Then, for every $R > 0$ there exists a constant $C > 0$ such that, for every $r \in (0, \frac{\mathrm{dist}(x_0,\partial\Omega)}{4R})$*

$$\|u_{x_0,r}\|_{W^{2,p^*}(B_R(x_0))} \leq C. \tag{2.24}$$

In particular, the functions $u_{x_0,r}$ are equibounded in $C^{1,\gamma'}$ for $\gamma' \leq \gamma := 1 - \frac{n}{p^*}$.

Proof. From (2.23) and Proposition 2.1.5 it holds

$$\mathrm{div}(\mathbb{A}(x_0 + rx)\nabla u_{x_0,r}(x)) = f(x_0 + rx)\chi_{\{u_{x_0,r}>0\}}(x) \tag{2.25}$$

a.e. on $B_{4R}(x_0)$ and on $\mathcal{D}'(B_{4R}(x_0))$, and $u_{x_0,r} \in W^{2,p^*} \cap C^{1,\gamma}(B_{4R}(x_0))$. We have $x_0 \in \Gamma_u$, then $u_{x_0,r}(\underline{0}) = 0$. Since $u_{x_0,r} \geq 0$, by Theorems 1.2.2 and 1.2.3 we have

$$\|u_{x_0,r}\|_{L^\infty(B_{4R}(x_0))} \leq C(R, x_0)\|f\|_{L^\infty(B_{4R}(x_0))}. \tag{2.26}$$

Thanks to Theorem 1.2.5 and (2.26) we obtain

$$\|u_{x_0,r}\|_{C^{1,\gamma}(B_{2R}(x_0))} \leq C\left(\|u_{x_0,r}\|_{L^\infty(B_{4R}(x_0))} + \|f\|_{L^\infty(B_{4R}(x_0))}\right) \leq C'\|f\|_{L^\infty(B_{4R}(x_0))}. \tag{2.27}$$

We observe that, as in Proposition 2.1.5, $u_{x_0,r}$ is the solution to

$$\mathrm{Tr}(\mathbb{A}(x_0+rx)\nabla^2 u_r(x)) = f(x_0+rx)\chi_{\{u_{x_0,r}>0\}} - r\sum_j \mathrm{div}(a^j(x_0+rx))\frac{\partial u_{x_0,r}}{\partial x_j}(x) =: \phi_r(x), \tag{2.28}$$

with $\phi_r \in L^{p^*}(B_{2R}(x_0))$, then according to Theorem 1.2.7

$$\|u_{x_0,r}\|_{W^{2,p^*}}(B_R(x_0)) \leq C\left(\|u_{x_0,r}\|_{L^{p^*}(B_{2R}(x_0))} + \|\phi_r\|_{L^{p^*}(B_{2R}(x_0))}\right). \tag{2.29}$$

We define $\mathrm{div}(\mathbb{A}) := (\mathrm{div}(a^j))_j$, namely the vector of divergence of the vector column of $\mathbb{A}$. Then by (2.27)

$$\begin{aligned}
\|\phi_r\|^{p^*}_{L^{p^*}(B_{2R}(x_0))} &= \int_{B_{2R}(x_0)} |f(rx)\chi_{\{u_r>0\}} - r\langle \mathrm{div}\mathbb{A}(rx), \nabla u_r(x)\rangle|^{p^*}\, dx \\
&\leq C\,\|f\|^{p^*}_{L^\infty(B_{4R}(x_0))}\left(1 + r^{p^*-n}\int_{B_{2rR}(x_0)} |\langle \mathrm{div}\mathbb{A}(y)|^{p^*}\, dy\right) \\
&\leq C\,\|f\|^{p^*}_{L^\infty(B_{4R}(x_0))}\left(1 + \Big(\frac{\mathrm{dist}\,(x_0, \partial\Omega)}{4R}\Big)^{p^*-n}\|\mathrm{div}\mathbb{A}(y)\|^{p^*}_{W^{1,p^*}(\Omega)}\right).
\end{aligned}$$

So $\|u_{x_0,r}\|_{W^{2,p^*}}(B_R(x_0)) \leq C$, where C does not depend on r. □

Corollary 2.2.2 (Existence of blow ups)**.** *Let $x_0 \in \Gamma_u$ with u the solution of* (2.4)*. Then for every sequence $r_k \downarrow 0$ there exists a subsequence $(r_{k_j})_j \subset (r_k)_k$ such that the rescaled functions $(u_{x_0,r_{k_j}})_j$ converge in $C^{1,\gamma}$. We define these limits as* blow ups.

Proof. The proof is an easy consequence of Proposition 2.2.1 and the Ascoli-Arzelà Theorem. $\square$

Remark 2.2.3. Recalling $x_0 \in \Gamma_u$ we have $u(x_0) = 0$ and $\nabla u(x_0) = 0$ so

$$\|u\|_{L^\infty(B_r)(x_0)} \leq C\, r^2 \qquad \text{and} \qquad \|\nabla u\|_{L^\infty(B_r(x_0))} \leq C\, r. \tag{2.30}$$

We note that the constant in (2.30) only depends on the constant C in (2.24) and is therefore uniformly bounded for points $x_0 \in \Gamma_u \cap K$ for each compact set $K \subset \Omega$.

As in the classical case, the solution u has a quadratic growth. The lack of regularity of the problem does not allow us to use the classic approach by Caffarelli [17] also used by Focardi, Gelli and Spadaro in [34, Lemma 4.3]. The main problem is that $\operatorname{div}(a^j)$, that is a W^{1,p^*} function, is not a priori pointwise defined, so the classical argument fails. We use a more general result of Blank and Hao in [8, Chapter 3] which we will prove explicitly for the convenience of readers.

Proposition 2.2.4 ([8, Theorem 3.9]). *Let $x_0 \in \Gamma_u$, and u be the minimum of* (2.4). *Then, there exists a constant $\theta > 0$ such that*

$$\sup_{\partial B_r(x_0)} u \geq \theta\, r^2. \tag{2.31}$$

Proof. We divide this proof into five steps.

Step 1 Let us suppose that W satisfies the condition

$$\overline{\lambda} \leq L(W) \leq \overline{\Lambda} \qquad\qquad \text{in } B_1, \tag{2.32}$$

where $W \geq 0$. Then there exists a positive constant C such that

$$\sup_{\partial B_r} W \geq W(0) + C\, r^2. \tag{2.33}$$

Let v_1 be the solution of

$$\begin{cases} L(v_1) = 0 & \text{in } B_r \\ v_1 = W & \text{on } \partial B_r, \end{cases}$$

then according to the Weak Maximum Principle we obtain

$$\sup_{\partial B_r} v_1 \geq \sup_{B_r} v_1 \geq v_1(\underline{0}). \tag{2.34}$$

Let v_2 be the solution of

$$\begin{cases} L(v_2) = L(W) & \text{in } B_r \\ v_2 = 0 & \text{on } \partial B_r. \end{cases}$$

Moreover, let $v_0 = \frac{|x|^2 - r^2}{2n}$ be the solution of

$$\begin{cases} \Delta(v_0) = 1 & \text{su } B_r \\ v_0 = 0 & \text{su } \partial B_r. \end{cases}$$

Due to Proposition 1.2.6 there exist two constants c_1, c_2 for which

$$c_1 v_0(x) \leq v_2(x) \leq c_2 v_0(x);$$

in particular

$$-v_2(\underline{0}) \geq c_2 \frac{r^2}{2n}. \tag{2.35}$$

On the other hand from definition of v_1 and v_2 we know that $W = v_1 + v_2$, so due to conditions (2.34) and (2.35) we deduce

$$\sup_{\partial B_r} W = \sup_{\partial B_r} v_1 \geq v_1(\underline{0}) = W(\underline{0}) - v_2(\underline{0}) \geq c_2 \frac{r^2}{2n}.$$

Step 2 *Let w be the solution of equation* (2.12) *in B_1, and assume that $w(0) = \varepsilon > 0$. Then $w > 0$ in a ball B_{δ_0} with $\delta_0 = C_0\sqrt{\varepsilon}$.*

According to Remark 2.2.3, if $w(y_0) = 0$, we have

$$\varepsilon = |w(y_0) - w(0)| \leq C|y_0|^2, \tag{2.36}$$

for which $|y_0| \geq C\sqrt{\varepsilon}$.

Step 3 *Let w be the solution of the equation* (2.12) *in B_2, and assume that $w(0) = \varepsilon > 0$ with $\varepsilon << 1$. Then there exists a constant $C > 0$ such that*

$$\sup_{B_1} w(x) \geq C + \varepsilon. \tag{2.37}$$

Without loss of generality, we can suppose that there exists a point $y \in B_{1/3}$ such that $w(y) = 0$. If this does not hold true from the maximum principle and *Step 1* we have

$$\sup_{B_1} w \geq \sup_{B_{1/3}} w = \sup_{\partial B_{1/3}} w \geq \varepsilon + C, \tag{2.38}$$

which is (2.37).

According to *Step 1* and *Step 2* there exists a point $y_1 \in \partial B_{\delta_0}$ such that

$$w(y_1) \geq w(0) + C\frac{\delta_0^2}{2n} = (1 + C_1)\varepsilon. \tag{2.39}$$

In the same way we can apply the result of *Step 2* to y_1 and B_{δ_1}, so we obtain a point $y_2 \in \partial B_{\delta_1}(y_1)$ for which

$$w(y_2) \geq (1 + C_1)w(x_1) \geq (1 + C_1)^2\varepsilon. \tag{2.40}$$

Repeating this argument a finite number of times we can get finite sequences $\{y_i\}$ and $\{\delta_i\}$ with $y_0 = 0$ such that

$$w(y_i) \geq (1+C_1)^i \varepsilon \qquad \textit{and} \quad \delta_i = |x_{i+1} - x_i| = C_0\sqrt{w(x_i)}. \tag{2.41}$$

We observe that as long as $y_i \in B_{1/3}$ the radii $\delta_i \leq 2/3$, due to the starting assumption of the existence of $y \in B_{1/3}$; thus x_{i+1} still belongs to B_1. Choose N as the smallest integer that satisfies the inequality

$$\sum_{i=0}^{N} \delta_i = \sum_{i=0}^{N} C_0\sqrt{\varepsilon}(1+C_1)^{\frac{i}{2}} \geq \frac{1}{3}. \tag{2.42}$$

Then

$$N \geq \frac{2\ln\left[\frac{(1+C_1)^{\frac{1}{2}}-1}{3\,C_0\sqrt{\varepsilon}} + 1\right]}{\ln(1+C_1)} - 1. \tag{2.43}$$

By putting together the inequalities (2.41) and (2.43), we deduce

$$\begin{aligned} w(y_N) \geq &\varepsilon(1+C_1)^{\frac{2\ln\left[\frac{(1+C_1)^{\frac{1}{2}}-1}{3\,C_0\sqrt{\varepsilon}}+1\right]}{\ln(1+C_1)}-1} = \frac{\varepsilon}{1+C_1}\Big(\frac{(1+C_1)^{\frac{1}{2}}-1}{3\,C_0\sqrt{\varepsilon}} + 1\Big)^2 \\ =&(\widetilde{C}_0 + \widetilde{C}_1\sqrt{\varepsilon}) \geq C_2(1+\varepsilon), \end{aligned} \tag{2.44}$$

where the last inequality is guaranteed by the hypothesis on ε for which $\varepsilon << 1$.

Step 4 *Let w be a function as in* Step 3 *and $\underline{0} \in \overline{\{w > 0\}}$. Then*

$$\sup_{\partial B_1} w \geq \theta. \tag{2.45}$$

Let $(x_i)_i \in \mathbb{N}$ be a sequence in $\{w > 0\}$ such that $x_i \to x_0$ for $i \to \infty$, and let $\varepsilon_i = w(x_i)$. From result of *Step 3* for all $i \in \mathbb{N}$ it holds that

$$\sup_{B_1(x_i)} w \geq C + \varepsilon_i, \tag{2.46}$$

where C is a positive constant that depends on the constant of Remark 2.2.3. Passing to the limit as $i \to \infty$ in the inequality (2.46) and from the maximum principle we verify (2.46).

Step 5 *Conclusion.*

Let us suppose by contradiction that there exists some $r_0 \leq 1$, such that

$$\sup_{B_{r_0}(x_0)} u(x) = \theta_1 r_0^2 < \theta r_0^2. \tag{2.47}$$

We note that for $r \leq 1$ $u_{x_0,r}(x) = \frac{u(x_0+rx)}{r^2}$ is the solution of equation (2.25) and we observe that the ellipticity of the differential operator in (2.25) is the same as in (2.12). So in particular, for $u_{x_0,r_0}(x) = \frac{u(x_0+r_0x)}{r_0^2}$ we have that for all $x \in B_1$

$$u_{x_0,r_0}(x) = \frac{u(x_0 + r_0 x)}{r_0^2} \leq \frac{1}{r_0^2} \sup_{B_{r_0}(x_0)} w = \theta_1 < \theta, \tag{2.48}$$

and this contradicts the result in *Step 4*.

□

To proceed in the analysis of the blow ups we shall prove a monotonicity formula. This will be a key ingredient to prove the 2-homogeneity of blow ups and that blow ups are non zero. Therefore it allows us to classify blow ups. This result will be the focus of Section 2.4, while the quasi-monotonicity formula will be the topic of Section 2.3.

2.3 Weiss' quasi-monotonicity formula

In this section we show that the monotonicity formula established by Weiss [95] in the Laplacian case ($\mathbb{A} \equiv I_n$) and by Focardi, Gelli and Spadaro [34] in the $\mathbb{A}$ Lipschitz continuous and f Hölder continuous case, holds in our case as well.

As in [34] we proceed by fixing the coordinates system: let $x_0 \in \Gamma_u$ be any point of free boundary, then the affine change of variables

$$x \longmapsto x_0 + f(x_0)^{-\frac{1}{2}} \mathbb{A}^{\frac{1}{2}}(x_0)x = x_0 + \mathbb{L}(x_0)x \tag{2.49}$$

leads to

$$\mathcal{E}[u,\Omega] = f^{1-\frac{n}{2}}(x_0) \det(\mathbb{A}^{\frac{1}{2}}(x_0))\ \mathcal{E}_{\mathbb{L}(x_0)}[u_{\mathbb{L}(x_0)}, \Omega_{\mathbb{L}(x_0)}],$$

with the following notations:

$$\begin{aligned}
\mathcal{E}_{\mathbb{L}(x_0)}[v, A] &:= \int_A \left(\langle \mathbb{C}_{x_0} \nabla v, \nabla v \rangle + 2 \frac{f_{\mathbb{L}(x_0)}}{f(x_0)} v \right) dx \qquad \forall\, A \subset \Omega_{\mathbb{L}(x_0)}, \\
\Omega_{\mathbb{L}(x_0)} &:= \mathbb{L}(x_0)^{-1}(\Omega - x_0), \\
u_{\mathbb{L}(x_0)}(x) &:= u(x_0 + \mathbb{L}(x_0)x), \\
f_{\mathbb{L}(x_0)} &:= f(x_0 + \mathbb{L}(x_0)x), \\
\mathbb{C}_{x_0} &:= \mathbb{A}^{-\frac{1}{2}}(x_0)\mathbb{A}(x_0 + \mathbb{L}(x_0)x)\mathbb{A}^{-\frac{1}{2}}(x_0), \\
u_{\mathbb{L}(x_0),r}(y) &:= \frac{u(x_0 + r\mathbb{L}(x_0)y)}{r^2}.
\end{aligned} \tag{2.50}$$

We observe that the image of the free boundary in the new coordinates is:

$$\Gamma_{u_{\mathbb{L}(x_0)}} = \mathbb{L}(x_0)^{-1}(\Gamma_u - x_0) \tag{2.51}$$

and we see how energy $\mathcal{E}$ is minimized by u, if and only if, the energy $\mathcal{E}_{\mathbb{L}(x_0)}$ is minimized by $u_{\mathbb{L}(x_0)}$.

Therefore, for a fixed base point $x_0 \in \Gamma_u$, we change the coordinates system and as we stated before

$$\underline{0} \in \Gamma_{u_{\mathbb{L}(x_0)}} \qquad \mathbb{C}_{x_0}(\underline{0}) = I_n \qquad f_{\mathbb{L}(x_0)}(\underline{0}) = f(x_0).$$

The point of the choice of this change of variable is that, in a neighborhood of $\underline{0}$, the functional $\mathcal{E}_{\mathbb{L}(x_0)}[v, \Omega]$ is a perturbation of $\int_\Omega (|\nabla v|^2 + 2v)\, dx$, which is the functional associated with the classical Laplacian case. We identify the two spaces in this section to simplify the ensuing calculations, then with a slight abuse of notation we reduce to (0.7):

$$x_0 = \underline{0} \in \Gamma_u, \qquad \mathbb{A}(\underline{0}) = I_n, \qquad f(\underline{0}) = 1.$$

We note that with this convention $\underline{0} \in \Omega$. In the new coordinates system we define

$$\nu(x) := \frac{x}{|x|} \quad \textit{per } x \neq \underline{0}, \qquad \mu(x) := \begin{cases} \langle \mathbb{A}(x)\nu(x), \nu(x)\rangle & \text{if } x \neq \underline{0} \\ 1 & \text{otherwise.} \end{cases} \tag{2.52}$$

We note that $\mu \in C^0(\Omega)$ by (I1) and (0.7). We prove the following result:

Lemma 2.3.1. *Let $\mathbb{A}$ be a matrix-valued field. Assume that (I1), (I2) and (0.7) hold, then*

$$\mu \in W^{1,q} \cap C^{0,1-\frac{n}{p^*}}(\Omega) \qquad \forall q < p^*, \tag{2.53}$$

and

$$\Lambda^{-1} \leq \mu(x) \leq \Lambda \qquad \forall x \in \Omega. \tag{2.54}$$

We prove a preliminary Lemma.

Lemma 2.3.2. *Let $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n \setminus \{-x\}$. Then it holds*

$$\left| (x+h)\frac{|x|}{|x+h|} - x \right| < 2|h|.$$

Proof. We observe that $(x+h)\frac{|x|}{|x+h|} \in \mathbb{S}_{|x|}$. If $|h| \geq |x|$ the thesis easily follows because for all $y \in \mathbb{S}_{|x|}$

$$|y - x| \leq 2|x| \leq 2|h|.$$

Instead, if $|h| < |x|$, we note that $(x+h)\frac{|x|}{|x+h|}$ is the projection of $x+h$ on $\mathbb{S}_{|x|}$. So, $\left|(x+h)\frac{|x|}{|x+h|} - x\right|$ takes on its maximum value if $x+h$ lies on $\widetilde{\mathbb{S}}$, where by $\widetilde{\mathbb{S}}$ we mean the $(n-1)$-sphere consisting of points for which there exists a line passing through $\underline{0}$ which is a tangent to $\mathbb{S}_{|h|(x)}$. If $(x+h)\frac{|x|}{|x+h|} \in \widetilde{\mathbb{S}^{n-1}}$ for an easy calculus of geometrical nature we obtain

$$\left| (x+h)\frac{|x|}{|x+h|} - x \right|^2 = 2|x|^2 - 2|x|\sqrt{|x|^2 - |h|^2}.$$

We want to prove that if $|h| < |x|$ then

$$2|x|^2 - 2|x|\sqrt{|x|^2 - |h|^2} \leq 4|h|^2.$$

Or rather, set $f(t) := 2|x|^2 - 2|x|\sqrt{|x|^2 - t} - 4t$ we want to prove that $f(t) \leq 0$ if $t < |x|^2$. Since $f(0) = 0$, $f(|x|^2) = -2|x|^2 < 0$ and, on $(0, |x|^2)$,

$$f'(t) \geq 0 \quad \Longleftrightarrow \quad \frac{|x|}{\sqrt{|x^2| - t}} - 4 \geq 0 \quad \Longleftrightarrow \quad t \geq \frac{15}{16}|x|^2.$$

$\square$

Proof of Lemma 2.3.1. We prove that $\mu \in W^{1,q}$ for any $q < p^*$.

We use a characterization of the Sobolev spaces (see [10, Proposition IX.3]): $\mu \in W^{1,q}(\Omega)$ if and only if there exists a constant $C > 0$ such that for every open $\omega \subset\subset \Omega$ and for any $h \in \mathbb{R}^n$ with $|h| < \operatorname{dist}(\omega, \partial\Omega)$ it holds

$$\|\tau_h \mu - \mu\|_{L^q(\omega)} \leq C\,|h|.$$

For the convexity of the function $|\cdot|^q$, remembering that $\mathbb{A}$ is $(1 - \frac{n}{p^*})$-Hölder continuous and by Lemma 2.3.2, we have

$$\begin{aligned}
\|\tau_h\mu - \mu\|^q_{L^q(\omega)} &= \int_\omega \left| \langle \mathbb{A}(x+h)\frac{x+h}{|x+h|}, \frac{x+h}{|x+h|}\rangle - \langle \mathbb{A}(x)\frac{x}{|x|}, \frac{x}{|x|}\rangle \right|^q dx \\
&= \int_\omega \Bigg| \langle (\mathbb{A}(x+h) - \mathbb{A}(x))\frac{x+h}{|x+h|}, \frac{x+h}{|x+h|}\rangle \\
&\quad + \langle (\mathbb{A}(x) - \mathbb{A}(\underline{0}))\left(\frac{x+h}{|x+h|} + \frac{x}{|x|}\right), \left(\frac{x+h}{|x+h|} - \frac{x}{|x|}\right)\rangle \Bigg|^q dx \\
&\leq 2^{q-1}\left(\int_\omega |\mathbb{A}(x+h) - \mathbb{A}(x)|^q\, dx + \int_\omega 2^q \left|\frac{\mathbb{A}(x) - \mathbb{A}(\underline{0})}{|x|}\right|^q \left|(x+h)\frac{|x|}{|x+h|} - x\right|^q \right) \\
&\leq 2^{q-1}\left(c|h|^q + 4^q|h|^q \int_\omega \frac{1}{|x|^{n\frac{q}{p^*}}}\, dx \right) = C\,|h|^q,
\end{aligned}$$

where in the last equality, we rely on $|x|^{-\frac{nq}{p^*}}$ being integrable if and only if $q < p^*$. By the Sobolev embedding Theorem, we have $\mu \in C^{0,1-\frac{n}{q}}$ for any $q < p^*$.

Thanks to the structure of μ we can earn more regularity. In particular $\mu \in C^{0,\gamma}$ with $\gamma = 1 - \frac{n}{p^*}$. We start off proving the inequality when one of the two points is $\underline{0}$:

$$\begin{aligned}
|\mu(x) - \mu(\underline{0})| &= \left| \langle \mathbb{A}(x)\frac{x}{|x|}, \frac{x}{|x|}\rangle - 1 \right| = \left| \langle \mathbb{A}(x)\frac{x}{|x|}, \frac{x}{|x|}\rangle - \langle \frac{x}{|x|}, \frac{x}{|x|}\rangle \right| \\
&= \left| \langle (\mathbb{A}(x) - \mathbb{A}(\underline{0}))\frac{x}{|x|}, \frac{x}{|x|}\rangle \right| = [A]_{C^{0,\gamma}}\,|x|^\gamma.
\end{aligned}$$

Let us assume now that $x, y \neq \underline{0}$ and prove the inequality in the remaining case. Let $z = |y|\frac{x}{|x|}$ then

$$|\mu(x) - \mu(y)| \leq |\mu(x) - \mu(z)| + |\mu(z) - \mu(y)|.$$

As $\frac{z}{|z|} = \frac{x}{|x|}$

$$|\mu(x) - \mu(z)| = \left|\langle(\mathbb{A}(x) - \mathbb{A}(z))\frac{x}{|x|}, \frac{x}{|x|}\rangle\right| \le [A]_{C^{0,\gamma}}\,|x-z|^\gamma,$$

while by $|z| = |y| = r$

$$\begin{aligned}
|\mu(z) - \mu(y)| &= \left|\langle\mathbb{A}(z)\frac{z}{r}, \frac{z}{r}\rangle - \langle\mathbb{A}(y)\frac{y}{r}, \frac{y}{r}\rangle\right| \\
&\le \left|\langle(\mathbb{A}(z) - \mathbb{A}(y))\frac{z}{r}, \frac{z}{r}\rangle\right| + \left|\langle\mathbb{A}(y))\frac{z}{r}, \frac{z}{r}\rangle - \langle\mathbb{A}(y)\frac{y}{r}, \frac{y}{r}\rangle\right| \\
&\le [\mathbb{A}]_{C^{0,\gamma}}|z-y|^\gamma + \left|\langle(\mathbb{A}(y) - \mathbb{A}(0))\frac{z+y}{r}, \frac{z-y}{r}\rangle\right| \\
&\le [\mathbb{A}]_{C^{0,\gamma}}\left(|z-y|^\gamma + 2\frac{|z-y|^{1-\gamma}}{r^{1-\gamma}}|z-y|^\gamma\right) \\
&\le [\mathbb{A}]_{C^{0,\gamma}}\left(|z-y|^\gamma + 2^{1-\gamma}|z-y|^\gamma\right) \le C[\mathbb{A}]_{C^{0,\gamma}}|z-y|^\gamma.
\end{aligned}$$

Therefore, since $|x-z| = ||x| - |y|| \le |x-y|$ and $|z-y| \le |z-x| + |x-y| \le 2|x-y|$ we have

$$|\mu(x) - \mu(y)| \le C[\mathbb{A}]_{C^{0,\gamma}}|x-y|^\gamma.$$

□

We introduce rescaled volume and boundary energies

$$\begin{aligned}
\mathcal{E}(r) := \mathcal{E}[u, B_r] &= \int_{B_r} \left(\langle\mathbb{A}(x)\nabla u(x), \nabla u(x)\rangle + 2f(x)u(x)\right)\,dx \\
&= r^{n+2}\int_{B_1} \left(\langle\mathbb{A}(rx)\nabla u_r(x), \nabla u_r(x)\rangle + 2f(rx)u_r(x)\right)\,dx
\end{aligned} \tag{2.55}$$

$$\mathscr{H}(r) := \int_{\partial B_r} \mu(x)u^2(x)\,d\mathcal{H}^{n-1} = r^{n+3}\int_{\partial B_1} \mu(rx)u_r^2(x)\,d\mathcal{H}^{n-1}. \tag{2.56}$$

We now introduce an energy "à la Weiss" combining and rescaling the terms above:

$$\Phi(r) := r^{-n-2}\mathcal{E}(r) - 2\,r^{-n-3}\mathscr{H}(r). \tag{2.57}$$

Remark 2.3.3. By (0.7), (2.55), (2.56) and Proposition 2.2.1 we have

$$\begin{aligned}
\mathcal{E}(r) &= \int_{B_r} (|\nabla u_r|^2 + 2u)\,dx + O(r^{n+2+\min(\gamma,\alpha)}) \overset{(2.30)}{=} O(r^{n+2}), \\
\mathscr{H}(r) &= \int_{\partial B_r} u^2\,d\mathcal{H}^{n-1} + O(r^{n+3+\gamma}) \overset{(2.30)}{=} O(r^{n+3}).
\end{aligned} \tag{2.58}$$

Hence, the choice of the renormalizing factors in (2.57).

To complete the notation in (2.50) we show the trasformed version of (2.52) and (2.57):

$$\begin{aligned}
\mu_{\mathbb{L}(x_0)}(y) &:= \langle \mathbb{C}_{x_0}(y)\nu(y), \nu(y)\rangle \qquad y \neq \underline{0}, \qquad\qquad \mu_{\mathbb{L}(x_0)}(\underline{0}) := 1,\\
\Phi_{\mathbb{L}(x_0)}(r) &:= \int_{B_1} \big(\langle \mathbb{C}_{x_0}(ry)\nabla u_{\mathbb{L}(x_0),r}(y), \nabla u_{\mathbb{L}(x_0),r}(y)\rangle + 2\,\frac{f_{\mathbb{L}(x_0)}(ry)}{f(x_0)} u_{\mathbb{L}(x_0),r}\big)\,dy \\
&\quad - 2\int_{\partial B_1} \mu_{\mathbb{L}(x_0)}(ry)\, u^2_{\mathbb{L}(x_0),r}(y)\,d\mathcal{H}^{n-1}
\end{aligned} \tag{2.59}$$

Remark 2.3.4. We can note by the definition above and in view of Lemma 2.3.1 $\Lambda^{-2} \leq \mu_{\mathbb{L}(x_0)}(y) \leq \Lambda^2$ and $\mu_{\mathbb{L}(x_0)} \in C^{0,\gamma}(\Omega)$.

2.3.1 Estimate of derivatives of $\mathcal{E}$ and $\mathscr{H}$

To estimate the derivative of ausiliary energy Φ we estimate the derivative of addends $\mathcal{E}$ and $\mathscr{H}$. Starting with $\mathcal{E}$, for this purpose, following Focardi, Gelli and Spadaro [34], we use a generalization of Rellich–Necas' identity due to Payne–Weinberger [34, Lemma 3.4] in order to calculate the derivative.

Lemma 2.3.5 ([34, Lemma 3.4])**.** *Let $F \in W^{1,q} \cap C^{0,1-\frac{n}{p^*}}(B_r, \mathbb{R}^n)$, $\mathbb{A} \in W^{1,p^*}(\Omega)$ with $1 \leq q < p^*$ e $w \in W^{2,p^*}$. Then it holds*

$$\begin{aligned}
&\int_{\partial B_r} \Big(\langle \mathbb{A}\nabla w, \nabla w\rangle \langle F, \nu\rangle - 2\langle \mathbb{A}\nu, \nabla w\rangle\langle F, \nabla w\rangle\Big)\,d\mathcal{H}^{n-1}\\
&\qquad = \int_{B_r} \Big(\langle \mathbb{A}\nabla w, \nabla w\rangle \mathrm{div} F - 2\langle F, \nabla w\rangle \mathrm{div}(\mathbb{A}\nabla w)\Big)\,dx \\
&\qquad\qquad + \int_{B_r} \Big(\nabla\mathbb{A} : F \otimes \nabla w \otimes \nabla w - 2\langle \mathbb{A}\nabla w, \nabla^T F \nabla w\rangle\Big)\,dx.
\end{aligned} \tag{2.60}$$

Proof. We note that the terms $\nabla\mathbb{A} : F \otimes \nabla w \otimes \nabla w$ and $\mathrm{div}(\mathbb{A}\nabla w) = (\mathrm{div}\mathbb{A})\nabla w + \mathbb{A}\nabla^2 w$ are functions in $L^{p^*}(B_r)$ and the terms $\langle \mathbb{A}\nabla w, \nabla w\rangle \mathrm{div} F$ e $\langle \mathbb{A}\nabla w, \nabla^T F\nabla w\rangle$ are in $L^q(B_r)$; so the equation (2.60) is well defined. In order to conclude, it is enough to apply the Divergence Theorem to the expression

$$\mathrm{div}(\langle \mathbb{A}\nabla w, \nabla w\rangle F - 2\langle F, \nabla w\rangle \mathbb{A}\nabla w).$$

□

Proposition 2.3.6 ([34, Proposition 3.5])**.** *There exists a constant $C_1 > 0$, $C_1 = C_1(\lambda, C, \|\mathbb{A}\|_{W^{1+s,p}(\Omega)})$, such that for $\mathcal{L}^1$-a.e. $r \in (0, \mathrm{dist}(\underline{0}, \partial\Omega))$,*

$$\begin{aligned}
\mathcal{E}'(r) &= 2\int_{\partial B_r} \mu^{-1}\langle \mathbb{A}\nu, \nabla u\rangle^2\,d\mathcal{H}^{n-1} + \frac{1}{r}\int_{B_r} \langle \mathbb{A}\nabla u, \nabla u\rangle\,\mathrm{div}(\mu^{-1}\mathbb{A}x)\,dx\\
&\quad - \frac{2}{r}\int_{B_r} f\langle \mu^{-1}\mathbb{A}x, \nabla u\rangle\,dx - \frac{2}{r}\int_{B_r} \langle \mathbb{A}\nabla u, \nabla^T(\mu^{-1}\mathbb{A}x)\nabla u\rangle\,dx\\
&\quad + 2\int_{\partial B_r} f u\,d\mathcal{H}^{n-1} + \varepsilon(r),
\end{aligned}$$

with $\varepsilon(r) \leq C_1 \mathcal{E}(r)\, r^{-\frac{n}{p^*}}$.

Proof. We consider the vector field

$$F(x) = \frac{\mathbb{A}(x)x}{r\mu(x)}.$$

Thanks to (I1) and Lemma 2.3.1, $F \in W^{1,q} \cap C^{1,1-\frac{n}{p^*}(\Omega)}$ for all $q < p^*$. We observe that

$$\langle F, \nu\rangle = 1 \qquad \text{and} \qquad \langle F, \nabla u\rangle = \mu^{-1}\langle \mathbb{A}\nu, \nabla u\rangle, \qquad\qquad \textit{on } \partial B_r.$$

For the Coarea Formula

$$\mathcal{E}'(r) = \int_{\partial B_r} \big(\langle \mathbb{A}\nabla u, \nabla u\rangle + 2fu\big)\, d\mathcal{H}^{n-1}, \qquad\qquad \mathcal{L}^1\text{-}a.e. \ \ on\ r \in (0, \operatorname{dist}(\underline{0}, \partial\Omega)).$$

According to the choice of F, Lemma 2.3.5 and equation (2.12) gives us

$$\begin{aligned}
\mathcal{E}'(r) = &\int_{\partial B_r} \big(\langle \mathbb{A}\nabla u, \nabla u\rangle + 2fu\big)\, d\mathcal{H}^{n-1} = 2\int_{\partial B_r} \langle \mathbb{A}\nu, \nabla u\rangle^2 \mu^{-1}\, d\mathcal{H}^{n-1} \\
&+ \frac{1}{r}\int_{B_r} \Big(\langle \mathbb{A}\nabla u, \nabla u\rangle \operatorname{div}(\mu^{-1}\mathbb{A}x) - 2\mu^{-1}\langle \mathbb{A}\nu, \nabla u\rangle \operatorname{div}(\mathbb{A}\nabla u)\Big)\, dx \\
&+ \frac{1}{r}\int_{B_r} \Big(\mu^{-1}\, \nabla\mathbb{A} : \mathbb{A}x \otimes \nabla u \otimes \nabla u - 2\langle \mathbb{A}\nabla u, \nabla^T(\mu^{-1}\mathbb{A}x)\nabla u\rangle\Big)\, dx + \int_{\partial B_r} 2fu\, d\mathcal{H}^{n-1} \\
&= 2\int_{\partial B_r} \langle \mathbb{A}\nu, \nabla u\rangle^2 \mu^{-1}\, d\mathcal{H}^{n-1} + \frac{1}{r}\int_{B_r} \langle \mathbb{A}\nabla u, \nabla u\rangle \operatorname{div}(\mu^{-1}\mathbb{A}x) \\
&- \frac{2}{r}\int_{B_r} \mu^{-1}\langle \mathbb{A}\nu, \nabla u\rangle f\, dx + \frac{1}{r}\int_{B_r} \mu^{-1}\, (\nabla\mathbb{A} : \mathbb{A}x \otimes \nabla u \otimes \nabla u)\, dx \\
&- \frac{2}{r}\int_{B_r} \langle \mathbb{A}\nabla u, \nabla^T(\mu^{-1}\mathbb{A}x)\nabla u\rangle\Big)\, dx + \int_{\partial B_r} 2fu\, d\mathcal{H}^{n-1}.
\end{aligned}$$

Then it is enough to prove that

$$\varepsilon(r) := \frac{1}{r}\int_{B_r} \mu^{-1}\, (\nabla\mathbb{A} : \mathbb{A}x \otimes \nabla u \otimes \nabla u)\, dx \leq C_1\, \mathcal{E}(r) r^{-\frac{n}{p^*}}.$$

In effect

$$\begin{aligned}
\varepsilon(r) = \frac{1}{r}\int_{B_r} \mu^{-1}\, (\nabla\mathbb{A} : \mathbb{A}x \otimes \nabla u \otimes \nabla u)\, dx &\leq \frac{\lambda}{r}\int_{B_r} |\nabla\mathbb{A}||\mathbb{A}(x)||x||\nabla u|^2\, dx \\
\leq \frac{C \sup \mathbb{A}}{r}\int_{B_r} |\nabla\mathbb{A}| r^3\, dx &\leq C' r^2 \Big(\int_{B_r} |\nabla\mathbb{A}|^{p^*}\, dx\Big)^{\frac{1}{p^*}} (\omega_n r^n)^{1-\frac{1}{p^*}} \\
\leq C'\, r^{n+2-\frac{n}{p^*}} \|\mathbb{A}\|_{W^{1,p}(B_r)} &\leq C_1 \mathcal{E}(r) r^{-\frac{n}{p^*}}.
\end{aligned}$$

$\square$

The next step is to estimate the derivative of $\mathscr{H}(r)$. By definition $\mathscr{H}(r)$ is a boundary integral; we follow the strategy of [34, Proposition 3.6] that consists in bringing us back to a volume integral using the divergence theorem and deriving through Coarea formula. The difficulty is that we have to integrate the function $\mathrm{div}\mathbb{A}$ on ∂B_r, but by $(I1)$ $\mathrm{div}\mathbb{A}$ is a function in $W^{s,p}(\Omega)$ with $s > \frac{1}{p}$, and it is not, a priori, well defined on ∂B_r. Then, taking into account the concept of *trace* we prove a corollary of the Coarea formula.

Proposition 2.3.7. *Let $\phi \in W^{\lambda,p}(B_1)$ with $\lambda > \frac{1}{p}$. Then for $\mathcal{L}^1$-a.e. $r \in (0,1)$ it holds that*

$$\frac{d}{dr}\left(\int_{B_r} \phi\, dx\right) = \int_{\partial B_r} \gamma_0(\phi)\, d\mathcal{H}^{n-1}, \tag{2.61}$$

where γ_0 is the trace operator given in Theorem 1.1.8 of Chapter 1.

Proof. Let $(\phi_j)_j \subset C^\infty(\overline{B_1})$ such that $\phi_j \to \phi$ in $W^{\lambda,p}(B_1)$. For each function g_j, by the Coarea formula for $\mathcal{L}^1$-a.e. $r \in (0,1)$ it holds that

$$\frac{d}{dr}\left(\int_{B_r} \phi_j\, dx\right) = \int_{\partial B_r} \phi_j\, d\mathcal{H}^{n-1}. \tag{2.62}$$

By the continuity of trace and Lebesgue's dominated convergence Theorem we have

$$\lim_j \int_{\partial B_r} \phi_j\, d\mathcal{H}^{n-1} = \int_{\partial B_r} \gamma_0(\phi)\, d\mathcal{H}^{n-1}. \tag{2.63}$$

Let us now prove that $\lim_j \frac{d}{dr}\left(\int_{B_r} \phi_j\, dx\right) = \frac{d}{dr}\left(\int_{B_r} \phi\, dx\right)$.

In this connection we define the function $G(r) := \int_{B_r} \phi\, dx$ and the sequence $G_j(r) := \int_{B_r} \phi_j\, dx$; we prove that $G_j \to G$ in $W^{1,1}((0,1))$.

We recall that by a well-know characterization, the functions in $W^{1,1}$ on an interval are absolutely continuous functions. In order to deduce that $G, G_j \in W^{1,1}$ we have to prove that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite sequence of disjoint intervals $(a_k, b_k) \subset (0,1)$ the condition $\sum_k |G_j(b_k) - G_j(a_k)| < \varepsilon$ holds if $\sum_k |b_k - a_k| < \delta$.

Therefore, we estimate as follows

$$\sum_k |G_j(b_k) - G_j(a_k)| = \sum_k \left| \int_{B_{b_k} \setminus B_{a_k}} \phi_j\, dx \right| \le \int_{\bigcup_k (B_{b_k} \setminus B_{a_k})} |\phi_j|\, dx < \varepsilon,$$

where in the last inequality, we use the absolute continuity of the integral and

$$\mathcal{L}^n(\cup_k (B_{b_k} \setminus B_{a_k}) = n\omega_n \int_{\cup_k (b_k, a_k)} r^{n-1}\, dr \le n\omega_n \int_{1-\delta}^{1} r^{n-1}\, dr \le n\omega_n \delta.$$

The previous argument holds for G as well. Thus G and G_j are differentiable $\mathcal{L}^1$-a.e. on $(0,1)$. On the other hand by the Coarea formula, we can represent the weak derivative of G_j in the following way:

$$G_j'(r) = \int_{\partial B_r} \phi_j\, d\mathcal{H}^{n-1}, \quad G'(r) = \int_{\partial B_r} \phi\, d\mathcal{H}^{n-1} \qquad \mathcal{L}^1\text{-}a.e.\ r \in (0,1).$$

Thus

$$\|G_j - G\|_{L^1((0,1))} = \int_0^1 \Big| \int_{B_r} (\phi_j - \phi)\, dx \Big|\, dr \leq \|\phi_j - \phi\|_{L^1(B_1)} \xrightarrow{j\to\infty} 0,$$
$$\|G'_j - G'\|_{L^1((0,1))} = \int_0^1 \Big| \int_{\partial B_r} (\phi_j - \phi)\, d\mathcal{H}^{n-1} \Big|\, dr \leq \|\phi_j - \phi\|_{L^1(B_1)} \xrightarrow{j\to\infty} 0,$$

therefore up to subsequence $G'_j \to G'$ $\mathcal{L}^1$-a.e. $(0,1)$; then by combining together this, (2.62) and (2.63) we have the thesis. □

We estimate the derivative of $\mathscr{H}(r)$.
We define an exponent $\Theta = \Theta(s,p,n,t_0)$, with $t_0 \in \left(\frac{n-sp}{p(n-1)}, s\right)$ as in Remark 1.1.9, for which the term $r^{-\frac{n}{\Theta}}$ is integrable. For this purpose we define:

$$\Theta = \Theta(s,p,n,t_0) = \begin{cases} p & \text{if } p > n \\ \frac{np}{n-(s-t_0)p} & \text{if } p \leq n. \end{cases} \tag{2.64}$$

Remark 2.3.8. If $p > n$ the condition is trivial. If instead $p \leq n$, the condition $\frac{np}{n-(s-t_0)p} > n$ is equivalent to $t_0 < s + 1 - \frac{n}{p}$. Now such that t_0 exists if and only if $\frac{n-sp}{p(n-1)} < s + 1 - \frac{n}{p}$ that is equivalent to demand that $p > \frac{n^2}{n(1+s)-1}$. This explains the choice of condition $(I1)$.

Proposition 2.3.9. *There exists a positive constant $C_2 = C_2(\|\mathbb{A}\|_{W^{1+s,p}})$ such that for $\mathcal{L}^1$-a.e. $r \in (0, \mathrm{dist}(\underline{0}, \partial\Omega))$ it holds that*

$$\mathscr{H}'(r) = \frac{n-1}{r}\mathscr{H}(r) + 2\int_{\partial B_r} u\langle \mathbb{A}\nu, \nabla u\rangle\, d\mathcal{H}^{n-1} + h(r), \tag{2.65}$$

with $|h(r)| \leq C_2 \mathscr{H}(r) r^{-\frac{n}{\Theta}}$, Θ defined in (2.64).

Proof. From the Divergence Theorem we write $\mathscr{H}(r)$ as volume integral

$$\begin{aligned}\mathscr{H}(r) &= \frac{1}{r}\int_{\partial B_r} u^2(x)\langle \mathbb{A}(x)x, \nu\rangle\, d\mathcal{H}^{n-1} = \frac{1}{r}\int_{B_r} \mathrm{div}\big(u^2(x)\mathbb{A}(x)x\big)\, dx \\ &= \frac{2}{r}\int_{B_r} u\nabla u \cdot \mathbb{A}(x)x\, dx + \frac{1}{r}\int_{B_r} u^2(x)\mathrm{Tr}\mathbb{A}\, dx + \frac{1}{r}\int_{B_r} u^2(x)\mathrm{div}\mathbb{A}(x)\cdot x\, dx.\end{aligned}$$

By taking Coarea formula and Proposition 2.3.7 into account, we have

$$\begin{aligned}\mathscr{H}'(r) &= -\frac{1}{r}\mathscr{H}(r) + 2\int_{\partial B_r} u\langle \mathbb{A}\nu, \nabla u\rangle\, d\mathcal{H}^{n-1} \\ &\quad + \frac{1}{r}\int_{\partial B_r} u^2\, \mathrm{Tr}\mathbb{A}\, d\mathcal{H}^{n-1} + \frac{1}{r}\int_{\partial B_r} u^2\gamma_0\big(\mathrm{div}\mathbb{A}(x)\big)\cdot x\, d\mathcal{H}^{n-1} \\ &= \frac{n-1}{r}\mathscr{H}(r) + 2\int_{\partial B_r} \mu\langle \mathbb{A}\nu, \nabla u\rangle\, d\mathcal{H}^{n-1} + h(r),\end{aligned}$$

with

$$h(r) = \frac{1}{r}\int_{\partial B_r} u^2\left(\mathrm{Tr}\mathbb{A} - n\mu\right) d\mathcal{H}^{n-1} + \frac{1}{r}\int_{\partial B_r} u^2\gamma_0\big(\mathrm{div}\mathbb{A}(x)\big)\cdot x\, d\mathcal{H}^{n-1} =: I + II.$$

We estimate separately the two terms.

For the first term let us recall that the Hölder continuity of $\mathbb{A}$ and μ, the condition (2.30) and the fact that $\mathbb{A}(\underline{0}) = I_n$ and $\mu(\underline{0}) = 1$ hold, we have:

$$\begin{aligned} |I| &= \frac{1}{r}\Big|\int_{\partial B_r} u^2 \sum_i \big(a_{ii}(x) - \mu(x)\big)\, d\mathcal{H}^{n-1}\Big| \\ &\le \frac{1}{r}\int_{\partial B_r} u^2 \sum_i \big(|a_{ii}(x) - a_{ii}(\underline{0})| + |\mu(\underline{0}) - \mu(x)|\big)\, d\mathcal{H}^{n-1} \\ &\le C' r^{n+3-\frac{n}{p^*}} \le C'\mathscr{H}(r)\, r^{-\frac{n}{p^*}}, \end{aligned} \tag{2.66}$$

where in the last inequality we use (2.58).

For the second term from Hölder inequality, by (2.30) and recalling Remark 1.1.9 according to which $\gamma_0(\mathrm{div}\mathbb{A}) \in L^1(\partial B_r)$ we have:

$$|II| \le \frac{1}{r}\int_{\partial B_r} u^2 \big|\gamma_0\big(\mathrm{div}\mathbb{A}\big)(x)\big|\, |x|\, d\mathcal{H}^{n-1} \le C' r^4 \|\gamma_0(\mathrm{div}\mathbb{A})\|_{L^1(\partial B_r)}. \tag{2.67}$$

Now we analyse separately the two cases $p > n$ and $p \le n$.

We start off with the case $p > n$. We use (1.14), (2.58) in (2.67) to obtain

$$|II| \le C\,\|\mathrm{div}\mathbb{A}\|_{W^{s,p}(B_r)}\, r^{n+3}\, r^{-\frac{n}{p}} \le C\,\|\mathrm{div}\mathbb{A}\|_{W^{p,s}(\Omega)}\, \mathscr{H}(r)\, r^{-\frac{n}{p}} \le C\,\mathscr{H}(r)\, r^{-\frac{n}{p}}. \tag{2.68}$$

If $p \ge n$ by (1.15) we have

$$\|\gamma_0(\mathrm{div}\mathbb{A})\|_{L^1(\partial B_r;\mathcal{H}^{n-1})} \le C\, r^{n-1}\, r^{-\frac{n-(s-t_0)p}{p}} \|\mathrm{div}\mathbb{A}\|_{W^{t_0,\frac{np}{n-(s-t_0)p}}(B_r)}.$$

Hence, recalling (2.67) and (2.58)

$$\begin{aligned} |II| &\le C\,\|\mathrm{div}\mathbb{A}\|_{W^{t_0,\frac{np}{n-(s-t_0)p}}(B_r)}\, r^{n+3}\, r^{-\frac{n-(s-t_0)p}{p}} \\ &\le C\,\|\mathrm{div}\mathbb{A}\|_{W^{t_0,\frac{np}{n-(s-t_0)p}}(\Omega)}\, \mathscr{H}(r)\, r^{-\frac{n-(s-t_0)p}{p}} \\ &\le C\,\|\mathrm{div}\mathbb{A}\|_{W^{s,p}(\Omega)}\, \mathscr{H}(r)\, r^{-\frac{n-(s-t_0)p}{p}} \le C\,\mathscr{H}(r)\, r^{-\frac{n-(s-t_0)p}{p}} \end{aligned} \tag{2.69}$$

So, assuming the notation introduced in (2.64), by combining together (2.66), (2.69) and (2.68), and recalling that $\Theta < p^*$, we have

$$|h(r)| \le C'\mathscr{H}(r)\, r^{-\frac{n}{p^*}} + \bar{C}\mathscr{H}(r)\, r^{-\frac{n}{\Theta}} \le C_2\mathscr{H}(r)\, r^{-\frac{n}{\Theta}}.$$

$\square$

2.3.2 Proof of Weiss's quasi-monotonicity formula

In this section we prove Weiss' quasi-monotonicity formula that is one of the main results of the chapter. The plan of proof is the same as [34, Theorem 3.7]. The difference, due to regularity of coefficients, consists in the presence of additional unbounded factors and terms produced in Proposition 2.3.6, Proposition 2.3.9 and from freezing argument: $r^{-\frac{n}{p^*}}$, $r^{-\frac{n}{\Theta}}$ and $\frac{\omega(r)}{r}$. The key observation is that, for our hypotheses, these terms are integrable, so we are able to obtain the formula alike. For completeness we report the proof with all the details.

Theorem 2.3.10 (Weiss' quasi-monotonicity formula)**.** *Assume that* $(I1)$*-*$(I4)$ *and* (0.7) *hold. There exist nonnegative constants* $\bar{C}_3$ *and* C_4 *independent from* r *such that the function*

$$r \mapsto \Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt$$

with the constant Θ *given in equation* (2.64)*, is nondecreasing on* $(0, \frac{1}{2}\mathrm{dist}(\underline{0}, \partial\Omega) \wedge 1)$.

More precisely, the following estimate holds true for $\mathcal{L}^1$*-a.e.* r *in such an interval:*

$$\begin{aligned}\frac{d}{dr}&\left(\Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt\right) \\ &\geq \frac{2e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}}}{r^{n+2}} \int_{\partial B_r} \mu\Big(\langle \mu^{-1}\mathbb{A}\nu, \nabla u\rangle - 2\frac{u}{r}\Big)^2\, d\mathcal{H}^{n-1}.\end{aligned} \tag{2.70}$$

In particular, the limit $\Phi(0^+) := \lim_{r\to 0^+}\Phi(r)$ *exists and it is finite and there exists a constant* $c > 0$ *such that*

$$\begin{aligned}&\Phi(r) - \Phi(0^+) \\ &\geq \Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt - \Phi(0^+) - c\left(r^{1-\frac{n}{\Theta}} + \omega(r)\right).\end{aligned} \tag{2.71}$$

Proof. Assume the definition of $\Phi(r)$ by (2.57):

$$\Phi(r) := r^{-n-2}\mathcal{E}(r) - 2\, r^{-n-3}\mathscr{H}(r).$$

Then for $\mathcal{L}^1$-a.e. $r \in \mathrm{dist}(\underline{0}, \partial\Omega)$ we have

$$\Phi'(r) = \frac{\mathcal{E}'(r)}{r^{n+2}} - (n+2)\frac{\mathcal{E}(r)}{r^{n+3}} - 2\frac{\mathscr{H}'(r)}{r^{n+3}} + 2(n+3)\frac{\mathscr{H}(r)}{r^{n+4}}. \tag{2.72}$$

By Proposition 2.3.6 we have

$$\begin{aligned}&\frac{\mathcal{E}'(r)}{r^{n+2}} - (n+2)\frac{\mathcal{E}(r)}{r^{n+3}} \geq \frac{2}{r^{n+2}} \int_{\partial B_r} \mu^{-1}\langle \mathbb{A}\nu, \nabla u\rangle^2\, d\mathcal{H}^{n-1} + \frac{1}{r^{n+3}} \int_{B_r} \langle \mathbb{A}\nabla u, \nabla u\rangle\, \mathrm{div}(\mu^{-1}\mathbb{A}x)\, dx \\ &- \frac{2}{r^{n+3}} \int_{B_r} f\langle \mu^{-1}\mathbb{A}x, \nabla u\rangle\, dx - \frac{2}{r^{n+3}} \int_{B_r} \langle \mathbb{A}\nabla u, \nabla^T(\mu^{-1}\mathbb{A}x)\nabla u\rangle\, dx \\ &+ \frac{2}{r^{n+2}} \int_{\partial B_r} f u\, d\mathcal{H}^{n-1} - \frac{C_1}{r^{n+2}}\frac{\mathcal{E}(r)}{r^{\frac{n}{p^*}}} - \frac{n+2}{r^{n+3}} \int_{B_r} \langle \mathbb{A}\nabla u, \nabla u\rangle\, dx - \frac{2(n+2)}{r^{n+3}} \int_{B_r} f u\, dx.\end{aligned}$$

Then, integrating by parts and given (2.12):

$$\begin{aligned}\int_{B_r}\langle \mathbb{A}\nabla u,\nabla u\rangle\,dx+\int_{B_r} fu\,dx &= \int_{B_r}\langle \mathbb{A}\nabla u,\nabla u\rangle\,dx+\int_{B_r} u\mathrm{div}(\mathbb{A}\nabla u)\,dx\\ &= \int_{\partial B_r} u\langle \mathbb{A}\nu,\nabla u\rangle\,d\mathcal{H}^{n-1}.\end{aligned}\tag{2.73}$$

Thus, applying (2.73) in four occurrences, we deduce

$$\begin{aligned}&\frac{\mathcal{E}'(r)}{r^{n+2}}-(n+2)\frac{\mathcal{E}(r)}{r^{n+3}}\geq -\frac{C_1}{r^{n+2}}\mathcal{E}(r)\,r^{-\frac{n}{p^*}}+\frac{2}{r^{n+2}}\int_{\partial B_r}\mu^{-1}\langle \mathbb{A}\nu,\nabla u\rangle^2\,d\mathcal{H}^{n-1}\\ &+\frac{1}{r^{n+3}}\int_{B_r}\Big(\langle \mathbb{A}\nabla u,\nabla u\rangle\,\mathrm{div}(\mu^{-1}\mathbb{A}x)-2\langle \mathbb{A}\nabla u,\nabla^T(\mu^{-1}\mathbb{A}x)\nabla u\rangle-(n-2)\langle \mathbb{A}\nabla u,\nabla u\rangle\Big)\,dx\\ &\qquad-\frac{2}{r^{n+3}}\int_{B_r} f\langle \mu^{-1}\mathbb{A}x,\nabla u\rangle\,dx+\frac{2}{r^{n+2}}\int_{\partial B_r} fu\,d\mathcal{H}^{n-1}\\ &\qquad\quad-\frac{4}{r^{n+3}}\int_{\partial B_r} u\langle \mathbb{A}\nu,\nabla u\rangle\,d\mathcal{H}^{n-1}-\frac{2n}{r^{n+3}}\int_{B_r} fu\,dx.\end{aligned}\tag{2.74}$$

Instead the Proposition 2.3.9 leads to

$$\begin{aligned}-2\frac{\mathscr{H}'(r)}{r^{n+3}}&+2(n+3)\frac{\mathscr{H}(r)}{r^{n+4}}\geq\\ &-\frac{2C_2}{r^{n+3}}\mathscr{H}(r)r^{-\frac{n}{\Theta}}+\frac{8}{r^{n+4}}\mathscr{H}(r)-\frac{4}{r^{n+3}}\int_{\partial B_r} u\langle \mathbb{A}\nu,\nabla u\rangle\,d\mathcal{H}^{n-1}.\end{aligned}\tag{2.75}$$

By combining together (2.74) and (2.75) and since $p^*\geq\Theta$ we finally infer that

$$\begin{aligned}&\Phi'(r)+(C_1'\vee C_2)\Phi(r)r^{-\frac{n}{\Theta}}\geq\frac{2}{r^{n+2}}\int_{\partial B_r}\mu^{-1}\langle \mathbb{A}\nu,\nabla u\rangle^2\,d\mathcal{H}^{n-1}\\ &+\frac{1}{r^{n+3}}\int_{B_r}\Big(\langle \mathbb{A}\nabla u,\nabla u\rangle\,\mathrm{div}(\mu^{-1}\mathbb{A}x)-2\langle \mathbb{A}\nabla u,\nabla^T(\mu^{-1}\mathbb{A}x)\nabla u\rangle-(n-2)\langle \mathbb{A}\nabla u,\nabla u\rangle\Big)\,dx\\ &-\frac{2}{r^{n+3}}\int_{B_r} f\langle \mu^{-1}\mathbb{A}x,\nabla u\rangle\,dx+\frac{2}{r^{n+2}}\int_{\partial B_r} fu\,d\mathcal{H}^{n-1}-\frac{4}{r^{n+3}}\int_{\partial B_r} u\langle \mathbb{A}\nu,\nabla u\rangle\,d\mathcal{H}^{n-1}\\ &-\frac{2n}{r^{n+3}}\int_{B_r} fu\,dx+\frac{8}{r^{n+4}}\mathscr{H}(r)-\frac{4}{r^{n+3}}\int_{\partial B_r} u\langle \mathbb{A}\nu,\nabla u\rangle\,d\mathcal{H}^{n-1}\\ &=\frac{2}{r^{n+2}}\int_{\partial B_r}\Big(\mu^{-1}\langle \mathbb{A}\nu,\nabla u\rangle^2+4\frac{u^2}{r^2}\mu-4\frac{u}{r}\langle \mathbb{A}\nu,\nabla u\rangle\Big)\,d\mathcal{H}^{n-1}\\ &+\frac{1}{r^{n+3}}\int_{B_r}\Big(\langle \mathbb{A}\nabla u,\nabla u\rangle\,\mathrm{div}(\mu^{-1}\mathbb{A}x)-2\langle \mathbb{A}\nabla u,\nabla^T(\mu^{-1}\mathbb{A}x)\nabla u\rangle-(n-2)\langle \mathbb{A}\nabla u,\nabla u\rangle\Big)\,dx\\ &-\frac{2}{r^{n+3}}\left(\int_{B_r} f\big(\langle \mu^{-1}\mathbb{A}x,\nabla u\rangle-nu\big)\,dx-r\int_{\partial B_r} fu\,dx\right)=:R_1+R_2+R_3.\end{aligned}\tag{2.76}$$

We estimate separately three addenda.

$$\begin{aligned} R_1 &= \frac{2}{r^{n+2}} \int_{\partial B_r} \mu\Big(\mu^{-2}\langle \mathbb{A}\nu, \nabla u\rangle^2 + 4\frac{u^2}{r^2} - 4\mu^{-1}\frac{u}{r}\langle \mathbb{A}\nu, \nabla u\rangle\Big)\, d\mathcal{H}^{n-1} \\ &= \frac{2}{r^{n+2}} \int_{\partial B_r} \mu\Big(\langle \mu^{-1}\mathbb{A}\nu, \nabla u\rangle - 2\frac{u}{r}\Big)^2\, d\mathcal{H}^{n-1}. \end{aligned} \tag{2.77}$$

Since $n = \mathrm{div}(x)$ by (2.30) we have

$$\begin{aligned} |R_2| &= \left| \frac{1}{r^{n+3}} \int_{B_r}\Big(\langle \mathbb{A}\nabla u, \nabla u\rangle \,\mathrm{div}(\mu^{-1}\mathbb{A}x - x) - 2\langle \mathbb{A}\nabla u, \nabla^T(\mu^{-1}\mathbb{A}x - x)\nabla u\rangle\Big)\, dx \right| \\ &\le \frac{C^2\,\Lambda}{r^{n+1}} \int_{B_r}\Big(|\mathrm{div}(\mu^{-1}\mathbb{A}x - x) + 2|\nabla(\mu^{-1}\mathbb{A}x - x)|\Big)\, dx \le \frac{C'\,\Lambda}{r^{n+1}} \int_{B_r}\Big(|\nabla(\mu^{-1}\mathbb{A}x - x)|\Big)\, dx, \end{aligned}$$

We estimate $|\nabla(\mu^{-1}\mathbb{A}x - x)|$:

$$\begin{aligned} |\nabla(\mu^{-1}\mathbb{A}x - x)| &= \left|\nabla\left((\mu^{-1}\mathbb{A} - I_n)x\right)\right| = |\nabla(\mu^{-1}\mathbb{A} - I_n)x + (\mu^{-1}\mathbb{A} - I_n)| \\ &= |\nabla(\mu^{-1}) \otimes \mathbb{A}x + \mu^{-1}\nabla\mathbb{A}\, x + (\mu^{-1}\mathbb{A} - I_n)| \\ &\le \Lambda r(|\nabla\mathbb{A}| + |\nabla\mu|) + C\, r^{1-\frac{n}{p^*}}, \end{aligned}$$

where in the last inequality, we use the γ-Hölder continuity of $\mathbb{A} - \mu I_n$. Thus, from Lemma 2.3.1

$$\begin{aligned} |R_2| &\le \frac{C\Lambda}{r^{n+1}} \int_{B_r} \Big(|\nabla(\mu^{-1}\mathbb{A}x - x)|\Big)\, dx \le \frac{C}{r^{n+1}} \int_{B_r} \Big(r^{1-\frac{n}{p^*}} + r(|\nabla\mathbb{A}| + |\nabla\mu|)\Big)\, dx \\ &\le Cr^{-\frac{n}{p^*}} + \frac{C}{r^n}\bigg(\Big(\int_{B_r}(|\nabla\mathbb{A}|^{p^*})\, dx\Big)^{\frac{1}{p^*}}(\omega_n r^n)^{1-\frac{1}{p^*}} + \Big(\int_{B_r}(|\nabla\mu|^q)\, dx\Big)^{\frac{1}{q}}(\omega_n r^n)^{1-\frac{1}{q}}\bigg) \le Cr^{-\frac{n}{q}}, \end{aligned}$$

for each $n < \Theta < q < p^*$, whence

$$|R_2| \le c\,\frac{\mathcal{E}(r)}{r^{n+2}} r^{-\frac{n}{q}}.$$

Moreover, from (2.56) and (2.54)

$$0 \le \frac{\mathscr{H}(r)}{r^{n+3}} \le c\|u_r\|^2_{L^\infty} \le c,$$

with a certain constant c independent from r, then

$$|R_2| \le c\left(\frac{\mathcal{E}(r)}{r^{n+2}} - 2\frac{\mathscr{H}(r)}{r^{n+3}}\right) r^{-\frac{n}{q}} + 2c\,\frac{\mathscr{H}(r)}{r^{n+3}}\, r^{-\frac{n}{q}} \le c\,\Phi(r) r^{-\frac{n}{\Theta}} + c\, r^{-\frac{n}{\Theta}}. \tag{2.78}$$

Finally, assuming that $n = \mathrm{div}x$ and using the following identity, consequence of the divergence theorem

$$\int_{B_r} \left(\langle x, \nabla u\rangle + u\mathrm{div}x\right)\, dx = r\int_{\partial B_r} u\, d\mathcal{H}^{n-1}, \tag{2.79}$$

we have

$$
\begin{aligned}
R_3 =& -\frac{2}{r^{n+3}}\left(\int_{B_r} f\big(\langle \mu^{-1}\mathbb{A}x, \nabla u\rangle - nu\big)\,dx - r\int_{\partial B_r} fu\,d\mathcal{H}^{n-1}\right)\\
=& -\frac{2}{r^{n+3}}\left(\int_{B_r}(f(x)-f(\underline{0})\big(\langle \mu^{-1}\mathbb{A}x, \nabla u\rangle + f(\underline{0})\int_{B_r}\big(\langle \mu^{-1}\mathbb{A}x, \nabla u\rangle - u\mathrm{div}x\big)\,dx\right.\\
&\left. - r\int_{\partial B_r}(f(x)-f(\underline{0}))u\,dx - r\,f(\underline{0})\int_{\partial B_r} u\,dx\right)\\
=& -\frac{2}{r^{n+3}}\left(\int_{B_r}(f(x)-f(\underline{0})\big(\langle \mu^{-1}\mathbb{A}x, \nabla u\rangle + f(\underline{0})\int_{B_r}\big(\langle \mu^{-1}\mathbb{A}x - x, \nabla u\rangle\big)\,dx\right.\\
&\left. - r\int_{\partial B_r}(f(x)-f(\underline{0}))u\,d\mathcal{H}^{n-1}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
|R_3| =& \frac{2}{r^{n+3}}\left|f(\underline{0})\int_{B_r}\big(\langle \mu^{-1}\mathbb{A}x - x, \nabla u\rangle\big)\,dx\right.\\
&\left. + \int_{B_r}(f(x)-f(\underline{0})\big(\langle \mu^{-1}\mathbb{A}x, \nabla u\rangle - r\int_{\partial B_r}(f(x)-f(\underline{0}))u\,d\mathcal{H}^{n-1}\right|\\
\leq& \frac{c}{r^{n+1}}\left(\int_{B_r}|\mathbb{A}-\mu I_n|\,dx + \int_{B_r}|f(x)-f(\underline{0})|\,dx + r\int_{\partial B_r}|f(x)-f(\underline{0})|\,d\mathcal{H}^{n-1}\right)\\
\leq& \frac{c}{r^{n+1}}(r^{n+1-\frac{n}{p^*}} + r^n\,\omega(r)) \leq c\left(r^{-\frac{n}{p^*}} + \frac{\omega(r)}{r}\right).
\end{aligned}
\tag{2.80}
$$

Now by combining together (2.76), (2.77), (2.78) and (2.80)we have

$$
\Phi'(r) + C_3\Phi(r)\,r^{-\frac{n}{\Theta}} + C_4\left(r^{-\frac{n}{\Theta}} + \frac{\omega(r)}{r}\right) \geq \frac{2}{r^{n+2}}\int_{\partial B_r}\mu\Big(\langle \mu^{-1}\mathbb{A}\nu, \nabla u\rangle - 2\frac{u}{r}\Big)^2\,d\mathcal{H}^{n-1}. \tag{2.81}
$$

Multiplying the inequality by the integral factor $e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}}$ with $\bar{C}_3 = \frac{C_3}{1-\frac{n}{\Theta}}$ we get

$$
\begin{aligned}
\Big(\Phi(r)\,e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}}\Big)' &+ C_4\left(r^{-\frac{n}{\Theta}} + \frac{\omega(r)}{r}\right)e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}}\\
&\geq \frac{2e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}}}{r^{n+2}}\int_{\partial B_r}\mu\Big(\langle \mu^{-1}\mathbb{A}\nu, \nabla u\rangle - 2\frac{u}{r}\Big)^2\,d\mathcal{H}^{n-1}
\end{aligned}
$$

whence

$$
\begin{aligned}
\frac{d}{dr}&\left(\Phi(r)\,e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4\int_0^r\left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\right)e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\,dt\right)\\
&\geq \frac{2e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}}}{r^{n+2}}\int_{\partial B_r}\mu\Big(\langle \mu^{-1}\mathbb{A}\nu, \nabla u\rangle - 2\frac{u}{r}\Big)^2\,d\mathcal{H}^{n-1}.
\end{aligned}
\tag{2.82}
$$

In particular, the quantity under the sign of the derivative, bounded by construction, is also monotonic, therefore its limit exists as $r \to 0^+$. It follows that $\Phi(0^+) := \lim_{r\to 0^+} \Phi(r)$ exists and is bounded.

Finally,

$$
\begin{aligned}
\Phi(r) - \Phi(0^+) \geq & -|\Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} - \Phi(r)| + \Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} \\
& + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t} \right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt - \Phi(0^+) - C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t} \right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt \\
\geq & -|\Phi(r)|\, c'\, r^{1-\frac{n}{\Theta}} + \Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t} \right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt \\
& - \Phi(0^+) - c' \left(r^{1-\frac{n}{\Theta}} + \omega(r) \right) \\
\geq & \Phi(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t} \right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt - \Phi(0^+) - c \left(r^{1-\frac{n}{\Theta}} + \omega(r) \right),
\end{aligned}
\tag{2.83}
$$

where in the last inequality, we use the boundedness of $\Phi(r)$. □

Remark 2.3.11. We note that from Proposition 2.2.1 the uniform boundedness of the sequence $(u_{x_0,r})_r$ in $C^{0,\gamma}(\mathbb{R}^n)$ follows. Moreover, for the base points x_0 in a compact set of Ω, the $C^{0,\gamma}$ norms, and thus the constants in the monotonicity formulae, are uniformly bounded. Indeed, as pointed out in the corresponding statements they depend on $\|\mathbb{A}\|_{W^{s,p}}(\Omega)$ and $\mathrm{dist}(x_0, \partial\Omega)$.

2.4 The blow up method: Classification of blow ups

In this section we proceed with the analysis of the blow ups showing the consequence of Theorem 2.3.10.

The first consequence is that the blow ups are 2-homogeneous, i.e. $v(tx) = t^2 v(x)$ for all $t > 0$ and for all $x \in \mathbb{R}^n$, as it is possible to deduce from the second member of 2.70 where, according to Euler's homogeneous function Theorem[2], the integral represents a distance to a 2-homogeneous function set.

Proposition 2.4.1 (2-homogeneity of blow ups [34, Proposition 4.2])**.** *Let $x_0 \in \Gamma_u$ and $(u_{x_0,r})_r$ be as in (2.23). Then, for every sequence $(r_j)_j \downarrow 0$ there exists a subsequence $(r_{j_k})_k \subset (r_j)_j$ such that the sequence $(u_{x_0,r_{j_k}})_k$ converges in $C^{1,\gamma}(\mathbb{R}^n)$ to a function $v(y) = w(\mathbb{L}^{-1}(x_0)y)$, where w is 2-homogeneous.*

Proof. We apply the Weiss quasi-monotonicity formula (2.70) to $\Phi_{\mathbb{L}(x_0)}$ on the interval

[2]Let $v : \mathbb{R}^n \to \mathbb{R}$ a differentiable function, then v is k-homogeneous with $k > 0$ if and only if $k\, v(x) = \langle \nabla v(x), x \rangle$

$(r_j r, r_j R)$ with $r \in (0,R)$ and we obtain

$$
\begin{aligned}
&\Phi_{\mathbb{L}(x_0)}(r_j R)\, e^{\bar{C}_3 (r_j R)^{1-\frac{n}{\Theta}}} - \Phi_{\mathbb{L}(x_0)}(r_j r)_{\mathbb{L}(x_0)}\, e^{\bar{C}_3 (r_j r)^{1-\frac{n}{\Theta}}} + C_4 \int_{r_j r}^{r_j R} \Big(t^{-\frac{n}{p^*}} + \frac{\omega(t)}{t} \Big) e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt \\
&\geq \int_{r_j r}^{r_j R} \frac{2 e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}}{t^{n+2}} \int_{\Theta_t} \mu_{\mathbb{L}(x_0)} \Big(\langle \mu^{-1}_{\mathbb{L}(x_0)} \mathbb{C}_{x_0} \nu, \nabla u_{\mathbb{L}(x_0)} \rangle^2 - 2 \frac{u_{\mathbb{L}(x_0)}}{t} \Big)^2 d\mathcal{H}^{n-1}\, dt \\
&\overset{t=r_j\varrho}{=} \int_r^R \frac{2 e^{\bar{C}_3 (r_j \varrho)^{1-\frac{n}{\Theta}}}}{(r_j \varrho)^{n+2}} r_j^{n-1} \int_{B_\varrho} \mu_{\mathbb{L}(x_0)}(r_j y) \cdot \\
&\qquad \cdot \Big(\langle \frac{\mathbb{C}_{x_0}(r_j y) \nu}{\mu_{\mathbb{L}(x_0)}(r_j y)}, \nabla u_{\mathbb{L}(x_0)}(r_j y) \rangle^2 - 2 \frac{u_{\mathbb{L}(x_0)}(r_j y)}{r_j \varrho} \Big)^2 d\mathcal{H}^{n-1}(y)\, r_j\, d\varrho \\
&= \int_r^R \frac{2 e^{\bar{C}_3 (r_j \varrho)^{1-\frac{n}{\Theta}}}}{\varrho^{n+2}} \int_{B_\varrho} \mu_{\mathbb{L}(x_0)}(r_j y) \cdot \\
&\qquad \cdot \Big(\langle \frac{\mathbb{C}_{x_0}(r_j y) \nu}{\mu_{\mathbb{L}(x_0)}(r_j y)}, \nabla u_{\mathbb{L}(x_0), r_j}(y) \rangle^2 - 2 \frac{u_{\mathbb{L}(x_0), r_j}(y)}{\varrho} \Big)^2 d\mathcal{H}^{n-1} d\varrho \\
&= \int_r^R \frac{2 e^{\bar{C}_3 (r_j \varrho)^{1-\frac{n}{\Theta}}}}{\varrho^{n+4}} \int_{B_\varrho} \mu_{\mathbb{L}(x_0)}(r_j y) \cdot \\
&\qquad \cdot \Big(\langle \frac{\mathbb{C}_{x_0}(r_j y) y}{\mu_{\mathbb{L}(x_0)}(r_j y)}, \nabla u_{\mathbb{L}(x_0), r_j}(y) \rangle^2 - 2 u_{\mathbb{L}(x_0), r_j}(y) \Big)^2 d\mathcal{H}^{n-1} d\varrho
\end{aligned}
\tag{2.84}
$$

Since the functions $u_{\mathbb{L}(x_0),r}$ satisfy a uniform estimate on $C^{1,\gamma}_{loc}(\mathbb{R}^n)$, for all sequences $(u_{x_0,r_j})_{r_j}$ we can extract a subsequence $(u_{x_0,r_{j_k}})_{r_{j_k}}$ that converges in $C^{1,\gamma}_{loc}$ to some function w. Then, remembering that $\mathbb{C}(\underline{0}) = I_n$ and $\mu_{\mathbb{L}(x_0)}(\underline{0}) = 1$, thanks to Lebesgue's dominate convergence Theorem we obtain

$$
0 \geq \int_r^R \frac{2}{\varrho^{n+4}} \int_{B_\varrho} \Big(\langle y, \nabla w(y) \rangle - 2w(y) \Big)^2 d\mathcal{H}^{n-1} d\varrho.
$$

Thus, it holds

$$
\int_{B_\varrho} \Big(\langle y, \nabla w(y) \rangle - 2w(y) \Big)^2 d\mathcal{H}^{n-1} = 0 \qquad \mathcal{L}^1\text{-}a.e.\ \varrho \in (r,R),
$$

and

$$
\langle \nabla w(y), y \rangle - 2w(y) = 0 \qquad \mathcal{H}^{n-1}\text{-}a.e.\ y \in \partial B_\varrho. \tag{2.85}
$$

Due to the continuity of w the condition (2.85) holds for all $y \in B_R \setminus B_r$, so for Euler's Homogeneous Function Theorem we deduce the 2-homogeneity property. Going back with respect to the change of coordinates we find the thesis. $\square$

As a second consequence, remembering Proposition 2.2.4 we can obtain that the blow ups are non zero.

Corollary 2.4.2. *Let $v(y) = w(\mathbb{L}^{-1}(x_0)y)$ be a limit of $C^{1,\gamma}$ a converging sequence of rescalings $(u_{x_0,r_j})_j$ in a free boundary point $x_0 \in \Gamma_u$, then $\underline{0} \in \Gamma_w$, i.e. $w \not\equiv 0$ in any neighborhood $\underline{0}$.*

Proof. Due to Proposition 2.2.4 for any $j \in \mathbb{N}$ there exists a $\nu_j \in \mathbb{S}^{n-1}$ such that $u_{x_0,r_j}(\nu_j) \geq \theta$. By the compactness of $\mathbb{S}^{n-1}$ we can extract a subsequence $(\nu_{j_k})_k$ such that $\nu_{j_k} \to \nu \in \mathbb{S}^{n-1}$. By the convergence in $C^{1,\gamma}$ we have that $v(\nu) \geq \theta$, if we define $\xi := \mathbb{L}^{-1}(x_0)\nu$, we get $w(\xi) \geq \theta$. As noticed in Proposition 2.4.1 w is 2-homogeneous, then in any neighborhood $\underline{0}$ there exists a point on the direction ξ on which w is strictly positive, so for any $\delta > 0$ we have $w(\delta\xi) = \delta^2 w(\xi) \geq \delta^2\theta$, and thus this Corollary is verified. □

Finally, it is possible to give a classification of blow ups. We begin by recalling the result in the classical case established by Caffarelli [14, 15, 17].

Definition 2.4.3. A *global solution* to the obstacle problem is a positive function $w \in C^{1,1}_{loc}(\mathbb{R}^n)$ solving (2.12) in the case $\mathbb{A} \equiv I_n$ and $f \equiv 1$.

The following result occurs:

Theorem 2.4.4. *Every global solution w is convex. Moreover, if $w \not\equiv 0$ and 2-homogeneous, then one of the following two cases occurs:*

(A) $w(y) = \frac{1}{2}\big(\langle y, \nu\rangle \vee 0\big)^2$ for some $\nu \in \mathbb{S}^{n-1}$, where the symbol $\vee$ denote the maximum of the surrounded quantities;

(B) $w(y) = \langle \mathbb{B}y, y\rangle$ with $\mathbb{B}$ being a symmetric, positive semidefinite matrix such that $\mathrm{Tr}\mathbb{B} = \frac{1}{2}$.

Having this result at hand, a complete classification of the blow up limits, for the obstacle problem (2.4), follows as in the classical context. The ingredient of the proof is the quasi-monotonicity formula by Weiss and a Γ-convergence argument:

Proposition 2.4.5 (Classification of blow ups [34, Proposition 4.2 and 4.5])**.** *Every blow up v_{x_0} at a free boundary point $x_0 \in \Gamma_u$ is of the form $v_{x_0} = w(\mathbb{L}^{-1}(x_0)y)$, with w a non-trivial, 2-homogeneous global solution.*

Proof. We indicate by w the limit of $(u_{\mathbb{L}(x_0),r_j})_j$ for some $r_j \searrow 0$ in $C^{1,\gamma}_{loc}$ and we consider the following energy defined on $H^1(B_1)$ by

$$\mathcal{F}_j(v) := \begin{cases} \int_{B_1} \Big(\langle \mathbb{C}_{\mathbb{L}(x_0)}(r_j y)\nabla v(y), \nabla v(y)\rangle + 2\frac{f_{\mathbb{L}(x_0)}(r_j y)}{f(x_0)} v\Big)\, dy & \text{if } v \in \mathcal{V}_j \\ \infty & \text{otherwise.} \end{cases} \tag{2.86}$$

with $\mathcal{V}_j = \{v \in H^1(B_1) : v \geq 0\ \mathcal{L}^n\text{-a.e. on } B_1,\ v_{|\partial B_1} = u_{\mathbb{L}(x_0),r_j\,|\partial B_1}\}$. By definition the function $u_{\mathbb{L}(x_0),r_j}$ is the minimum of $\mathcal{F}_j$. Remembering from (2.50) that $\mathbb{C}_{\mathbb{L}(x_0)}(\underline{0}) = I_n$ and $f_{\mathbb{L}(x_0)}(\underline{0}) = f(x_0)$, we prove that $\mathcal{F} = \Gamma(H^1)\text{-}\lim_j \mathcal{F}_j$, with

$$\mathcal{F}(v) := \begin{cases} \int_{B_1} \big(|\nabla v(y)|^2 + 2v\big)\, dy & \text{if } v \in \mathcal{V} \\ \infty & \text{otherwise,} \end{cases} \tag{2.87}$$

where $\mathcal{V} := \{v \in H^1(B_1) : v \geq 0\ \mathcal{L}^n\text{-q.o. in } B_1,\ v_{|\partial B_1} = w_{|\partial B_1}\}$.

(i) Γ-$\liminf$ *inequality.*

Let $v \in H^1(B_1)$, we prove that for all $v_j \to v$ in $H^1(B_1)$ the inequality $\mathcal{F}(v) \leq \liminf_j \mathcal{F}_j(v_j)$ holds.

Without loss of generality we can suppose $\liminf_j \mathcal{F}_j(v_j) < \infty$, so from the definition of $\mathcal{F}_j(v_j)$ we have $v \geq 0$ $\mathcal{L}^n$-a.e. in B_1 and $v_{|\partial B_1} = u_{\mathbb{L}(x_0),r_j}{}_{|\partial B_1}$ and from convergence of v_j in H^1 we have $v \geq 0$ $\mathcal{L}^n$-a.e. and $v_{|\partial B_1} = w_{|\partial B_1}$. Then remembering the continuity of $\mathbb{C}_{\mathbb{L}(x_0)}$ and $f_{\mathbb{L}(x_0)}$ and its modulus of continuity we obtain

$$
\begin{aligned}
&\mathcal{F}_j(v) = \int_{B_1} \big(\langle \mathbb{C}_{\mathbb{L}(x_0)}(r_j y)\nabla v_j(y), \nabla v_j(y)\rangle + 2\frac{f_{\mathbb{L}(x_0)}(r_j y)}{f(x_0)} v_j(y)\big)\, dy \\
&= \int_{B_1}\big(\langle(\mathbb{C}_{\mathbb{L}(x_0)}(r_j y) - \mathbb{C}_{\mathbb{L}(x_0)}(\underline{0}))\nabla v_j(y), \nabla v_j(y)\rangle \\
&\quad + 2\frac{f_{\mathbb{L}(x_0)}(r_j y) - f_{\mathbb{L}(x_0)}(\underline{0})}{f(x_0)} v_j(y)\big)\, dy + \int_{B_1} \big(|\nabla v_j(y)|^2 + 2v_j(y)\big)\, dy \\
&\geq -C\Big(r_j^\gamma \int_{B_1} |\nabla v_j(y)|^2\, dy + \omega(r_j)\int_{B_1} 2v_j(y)\, dy\Big) + \int_{B_1}\big(|\nabla v(y)|^2 + 2v(y)\big)\, dy \\
&\quad - \int_{B_1} \big(|\nabla v_j - \nabla v(y)|^2 + 2|v_j - v|\big)\, dy \\
&\geq -C\Big((r_j^\gamma + \omega(r_j))(\|v\|^2_{H^1(B_1)} + \|v\|_{L^1(B_1)}) - (\|v_j - v\|^2_{H^1(B_1)} + \|v_j - v\|_{L^1(B_1)})\Big) \\
&\quad + \mathcal{F}(v).
\end{aligned}
$$

By passing to the limit as $j \to \infty$ and to the lower limit on every sequence (v_j) we find the inequality.

(ii) Γ-$\limsup$ *inequality.*

Step 1: $v - w$ has compact support in B_1.

We want to build a recovery sequence. Let $\varepsilon_h \searrow 0$ and $(\varphi_h)_h$ be a sequence of the function for which

$$
\varphi_{h|B_{1-\varepsilon_h}} = 1, \qquad \varphi_{h|B_1^c} = 0 \qquad \text{e} \qquad |\nabla \varphi_h| \leq \frac{c}{\varepsilon_h}.
$$

We consider the sequence $(v_h^k)_{h,k}$ defined as

$$
v_h^k = \varphi_h v + (1 - \varphi_h) u_{\mathbb{L}(x_0),r_k}.
$$

So we have

$$
\begin{aligned}
&\mathcal{F}_k(v_h^k) = \int_{B_1} \big(\langle \mathbb{C}_{\mathbb{L}(x_0)}(r_k y)\nabla v_h^k(y), \nabla v_h^k(y)\rangle + 2\frac{f_{\mathbb{L}(x_0)}(r_k y)}{f(x_0)} v_h^k\big)\, dy \\
&\le \int_{B_1} \big(|\nabla v_h^k(y)|^2 + 2v_h^k(y)\big)\, dy + C\,(r_k^\gamma + \omega(r_k))(\|v\|^2_{H^1(B_1)} + \|v\|_{L^1(B_1)}) \\
&\le \int_{B_1} \Big(\varphi_h^2|\nabla v|^2 + (1-\varphi_h)^2|\nabla u_{\mathbb{L}(x_0),r_k}|^2 + |\nabla\varphi_h|^2(v - u_{\mathbb{L}(x_0),r_k})\Big)\, dy \\
&\quad + 2\int_{B_1} (\varphi_h v + (1-\varphi_h)u_{\mathbb{L}(x_0),r_k})\, dy + C\,(r_k^\gamma + \omega(r_k))(\|v\|^2_{H^1(B_1)} + \|v\|_{L^1(B_1)}).
\end{aligned}
$$

By passing to the upper limit as k we obtain

$$
\begin{aligned}
\limsup_k \mathcal{F}_k(v_h^k) \le &\int_{B_1} \big(|\nabla v|^2 + 2v\big)\, dy \\
&+ \int_{B_1\setminus B_{1-\varepsilon_h}} \big(|\nabla w|^2 + 2w\big)\, dy + \frac{1}{\varepsilon_h}\int_{B_1\setminus B_{1-\varepsilon_h}} |v-w|^2\, dy.
\end{aligned}
$$

For $h \to 0$ according to the absolute continuity of the integral and $v - w \in H^1_c(B_1)$ we deduce

$$
\lim_h \limsup_k \mathcal{F}_k(v_h^k) \le \mathcal{F}(v),
$$

With a diagonal argument we extract the recovery sequence and we conclude Step 1.

Step 2: General case.

Let $v \in w + H^1_0(B_1)$ and we extend it to w on B_1^c. We define $v^\rho(x) = \rho^2 v(\frac{x}{\rho})$ for $\rho \nearrow 1$, $\rho < 1$ and we prove that $v^\rho \to v$ in $H^1(B_1)$ and $v^\rho - w$ have compact support in B_1.

Since $v \in L^2$ there exists a sequence $v^j \in C_c(B_1)$ such that $v^j \xrightarrow{L^2} v$, then

$$
\|v - v^\rho\|_{L^2(B_1)} \le \|v - v^j\|_{L^2(B_1)} + \|v^j - (v^j)^\rho\|_{L^2(B_1)} + \|(v^j)^\rho - v^\rho\|_{L^2(B_1)}.
$$

Due to continuity and uniform boundedness of $\|v_j\|_{L^2(B_1)}$

$$
\begin{aligned}
\|v^j - (v^j)^\rho\|_{L^2(B_1)} &\le \|v^j(x) - \rho^2 v^j(x)\|_{L^2(B_1)} + \|\rho^2 v^j(x) - \rho^2 v^j\left(\frac{x}{\rho}\right)\|_{L^2(B_1)} \\
&\le (1-\rho^2)\|v^j\|_{L^2(B_1)} + \rho^2\omega_j\left(\frac{1}{\rho} - 1\right) \le C\,\widetilde{\omega}^j(1-\rho),
\end{aligned}
$$

with ω_j modulus of continuity of v_j and $\widetilde{\omega}^j(t) = t + \omega^j(t)$, while with a change of variable

$$
\begin{aligned}
\|(v^j)^\rho - v^\rho\|_{L^2(B_1)} &= \rho^2\|v^j(\rho x) - v^j(\rho x)|_{L^2(B_1)} \\
&= \rho^{2+n}\|v^j(x) - v(x)|_{L^2(B_{\frac{1}{\rho}})} \xrightarrow{\rho\to 1} \|v - v^j\|_{L^2(B_1)},
\end{aligned}
$$

where from the absolute convergence of integral we obtain the convergence. So

$$\limsup_{\rho\to 1} \|v - v^\rho\|_{L^2(B_1)} \le 2\|v - v^j\|_{L^2(B_1)}.$$

By passing to the lower limit as j we have the convergence in L^2. In the same way we deduce the convergences for the gradient; so we obtain the convergence in H^1.

In order to prove that $v^\rho - w$ have compact support in B_1, we use the 2-homogeneity of w and the fact that $v = w$ on B_1^c; for all $x \in B_\rho^c$ we have

$$(v^\rho - w)(x) = \rho^2 \left(v\left(\frac{x}{\rho}\right) - w\left(\frac{x}{\rho}\right)\right) = 0,$$

in fact $\left|\frac{x}{\rho}\right| > 1$ and $v\left(\frac{x}{\rho}\right) = w\left(\frac{x}{\rho}\right)$.

Therefore from *Step 1* for all $\rho \nearrow 1$ we have

$$\Gamma\text{-}\limsup_j \mathcal{F}_j(v^\rho) \le \mathcal{F}(v). \tag{2.88}$$

We observe that

$$\begin{aligned}\mathcal{F}(v^\rho) &= \int_{B_1} \left(|\nabla v^\rho|^2 + 2v^\rho\right) dy = \rho \int_{B_{\frac{1}{\rho}}} |\nabla v^\rho|^2 \, dx + \int_{B_{\frac{1}{\rho}}} 2v^\rho \, dx \\ &\le \int_{B_1} \left(|\nabla v|^2 + 2v\right) dx + \int_{B_{\frac{1}{\rho}} \setminus B_1} \left(|\nabla w|^2 + 2w\right) dy,\end{aligned}$$

so from absolute continuity of the integral

$$\liminf_{\rho\to 1} \mathcal{F}(v^\rho) \le \mathcal{F}(v). \tag{2.89}$$

According to semicontinuity of Γ-$\limsup$, (2.88) and (2.89) we conclude

$$\Gamma\text{-}\limsup_j \mathcal{F}_j(v) \le \liminf_{\rho\to 1} \left(\Gamma\text{-}\limsup \mathcal{F}(v^\rho)\right) \le \liminf_{\rho\to 1} \mathcal{F}(v^\rho) \le \mathcal{F}(v).$$

From Theorem 1.5.3 we have the convergence of minima, so if $\bar{v} = \min_{\mathcal{V}} \mathcal{F}$ then $u_{\mathbb{L}(x_0),r_j} \to \bar{v}$ in $H^1(B_1)$. Due to Proposition 2.4.1, up to subsequence, $(u_{\mathbb{L}(x_0),r_j})$ converge in $C^{1,\gamma}$ to some 2-homogeneous function v', and from the uniqueness of minimum we obtain $\bar{v} = v'$. We extend w and $\bar{v}$ for 2-homogeneity on B_1. Since $\bar{v}_{|\partial B_1} = w_{|\partial B_1}$ we have $\bar{v} = w$. Then w is a global, 2-homogeneous solution, and from Corollary 2.4.2, $w \not\equiv 0$. Finally we have $u_{x_0,r}(x) = u_{\mathbb{L}(x_0),r}(\mathbb{L}^{-1}(x_0)x) \to w(\mathbb{L}^{-1}(x_0)x)$, for which $u_{x_0,r}(x) \to v_{x_0}(x)$ in $C^{1,\gamma}$ with $v_{x_0}(x) = w(\mathbb{L}^{-1}(x_0)x)$.

$\square$

According to Theorem 2.4.4 we shall call a global solution of type (A) or of type (B).

The above proposition allows us to formulate a simple criterion to distinguish between regular and singular free boundary points.

Definition 2.4.6. A point $x_0 \in \Gamma_u$ is a *regular* free boundary point, and we write $x_0 \in Reg(u)$ if there exists a blow up of u at x_0 of type (A). Otherwise, we say that x_0 is *singular* and write $x_0 \in Sing(u)$.

Remark 2.4.7. Simple calculations show that $\Psi_w(1) = \theta$ for every global solution of type (A) and $\Psi_w(1) = 2\theta$ for every global solution of type (B), where Ψ_w is the energy defined in (2.91) and θ is a dimensional constant.

Remark 2.4.8. We observe that for every sequence $r_j \searrow 0$ for which $u_{\mathbb{L}(x_0),r_j} \to w$ in $C^{1,\gamma}(B_1)$ with w being a 2-homogeneous global solution then

$$\lim_{r_j \to 0} \Phi_{\mathbb{L}(x_0)}(r_j) = \Psi_w(1).$$

From Weiss' quasi-monotonicity the uniqueness of the limit follows, so $\Phi_{\mathbb{L}(x_0)}(0) = \Psi_w(1)$ for every w that is the limit of the sequence $(u_{\mathbb{L}(x_0),r})_r$. It follows that if $x_0 \in \Gamma_u$ is a regular point then $\Phi_{\mathbb{L}(x_0)}(0) = \theta$ or, equivalently every blow up at x_0 is of type (A).

2.5 Monneau's quasi-monotonicity formula

In this section we prove a Monneau's type quasi-monotonicity formula (see [77]) for singular free boundary points. The plan of proof follows [34, Theorem 3.8]. The additional difficulty is the same as Theorem 2.3.10 so for completeness we report the whole proof.

Let v be a 2-homogeneous positive polynomial, solving

$$\Delta v = 1 \qquad \text{on } \mathbb{R}^n. \tag{2.90}$$

Let

$$\Psi_v(r) := \frac{1}{r^{n+2}} \int_{B_r} \left(|\nabla v|^2 + 2v\right) dx - \frac{2}{r^{n+3}} \int_{\partial B_r} v^2 \, d\mathcal{H}^{n-1}. \tag{2.91}$$

We note that the expression of $\Psi_v(r)$ is analogous to those of Φ with coefficients frozen in $\underline{0}$ (recalling (2.57)). An integration by parts, (2.91) and the 2-homogeneity of v yields

$$\begin{aligned}
\frac{1}{r^{n+2}} \int_{B_r} |\nabla v|^2 \, dx =& \frac{1}{r^{n+2}} \int_{B_r} \left(\operatorname{div}(v\nabla v) - v\,\Delta v\right) dx \\
=& \frac{1}{r^{n+3}} \int_{\partial B_r} \langle \nabla v, x\rangle \, d\mathcal{H}^{n-1} - \frac{1}{r^{n+2}} \int_{B_r} v \, dx \\
=& \frac{1}{r^{n+3}} \int_{\partial B_r} v^2 \, d\mathcal{H}^{n-1} - \frac{1}{r^{n+2}} \int_{B_r} v \, dx = \int_{\partial B_1} v^2 \, d\mathcal{H}^{n-1} - \int_{B_1} v \, dx
\end{aligned}$$

and therefore

$$\Psi_v(r) = \Psi_v(1) = \int_{B_1} v \, dx. \tag{2.92}$$

In the next theorem we give a monotonicity formula for solutions of the obstacle problem such that $\underline{0}$ is a point of the free boundary and

$$\Phi(0^+) = \Psi_v(1) \quad \text{for some}\, v \, \text{2-homogeneous solution of (2.90).} \tag{2.93}$$

As explained in Definition 2.4.6, formula (2.93) characterizes the singular part of the free boundary.

Theorem 2.5.1 (Monneau's quasi-monotonicity formula)**.** *Assume* $(I1)$-$(I4)$ *and* (0.7)*. Let* u *be the minimizer of* $\mathcal{E}$ *on* K*, with* $\underline{0} \in Sing(u)$ *(i.e.* (2.93) *holds), and* v *be as above. Then, there exists a positive constant* $C_5 = C_5(\lambda, \|\mathbb{A}\|_{W^{s,p}})$ *such that*

$$r \longmapsto \int_{\partial B_1} (u_r - v)^2 \, d\mathcal{H}^{n-1} + C_5 \left(r^{(1-\frac{n}{\Theta})} + \omega(r) \right) \tag{2.94}$$

is nondecreasing on $(0, \frac{1}{2}\mathrm{dist}(\underline{0}, \partial\Omega) \wedge 1)$*. More precisely,* $\mathcal{L}^1$*-a.e. on such an interval*

$$\begin{aligned} \frac{d}{dr}\bigg(\int_{\partial B_1} (u_r - v)^2 \, d\mathcal{H}^{n-1} + C_5 \left(r^{1-\frac{n}{\Theta}} + \int_0^r \frac{\omega(t)}{t} \, dt \right) \bigg) \\ \geq \frac{2}{r} \bigg(e^{C_3 r^{1-\frac{n}{\Theta}}} \Phi(r) + C_4 \int_0^r e^{c_3 t^{1-\frac{n}{\Theta}}} \left(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t} \right) dt - \Psi_v(1) \bigg). \end{aligned} \tag{2.95}$$

Proof. Set $w_r = u_r - v$. As v is 2-homogeneous we have that $w_r(x) = \frac{w(rx)}{r^2}$. Assuming that from (0.7) $\mathbb{A}(\underline{0}) = I_n$, due to the Divergence Theorem and Euler's homogeneous function Theorem we find

$$\begin{aligned} &\frac{d}{dr} \int_{\partial B_1} w_r^2 \, d\mathcal{H}^{n-1} = \int_{\partial B_1} w_r \, \frac{d}{dr}(\frac{w(rx)}{r^2}) \, d\mathcal{H}^{n-1} \\ =&\frac{2}{r} \int_{\partial B_1} w_r(\langle \nabla w_r, x\rangle - 2w_r) \, d\mathcal{H}^{n-1} = \frac{2}{r} \int_{\partial B_1} w_r(\langle \nabla u_r, x\rangle - 2u_r) \, d\mathcal{H}^{n-1} \\ =&\frac{2}{r} \int_{\partial B_1} w_r(\langle \mathbb{A}(rx) \nabla u_r, x\rangle - 2u_r) \, d\mathcal{H}^{n-1} + \frac{2}{r} \int_{\partial B_1} w_r \langle (\mathbb{A}(\underline{0} - \mathbb{A}(rx))) \nabla u_r, x\rangle \, d\mathcal{H}^{n-1} \\ \geq&\frac{2}{r} \int_{\partial B_1} w_r(\langle \mathbb{A}(rx) \nabla u_r, x\rangle - 2u_r) \, d\mathcal{H}^{n-1} - C \, \|\nabla u_r\|_{L^2(\partial B_1)} \, \|w_r\|_{L^2(\partial B_1)} \, [\mathbb{A}]_{0,\gamma} \, r^{-\frac{n}{p^*}}, \end{aligned}$$

thus by (2.30)

$$\frac{d}{dr} \int_{\partial B_1} w_r^2 \, d\mathcal{H}^{n-1} \geq \frac{2}{r} \int_{\partial B_1} w_r(\langle \mathbb{A}(rx) \nabla u_r, x\rangle - 2u_r) \, d\mathcal{H}^{n-1} - C \, r^{-\frac{n}{p^*}}. \tag{2.96}$$

Using an integration by parts, and (2.90) we can rewrite the first term on the right as

$$
\begin{aligned}
&\int_{\partial B_1} w_r(\langle \mathbb{A}(rx)\nabla u_r, x\rangle - 2u_r)\, d\mathcal{H}^{n-1} \\
&\overset{(2.12)}{=} \int_{B_1} \big(\langle \mathbb{A}(rx)\nabla u_r, \nabla w_r\rangle + w_r\, f(rx)\chi_{\{u_r>0\}}(x)\big)\, dx - \int_{\partial B_1} 2\, w_r\, u_r\, d\mathcal{H}^{n-1} \\
&= \int_{B_1} \big(\langle \mathbb{A}(rx)\nabla u_r, \nabla u_r\rangle + u_r\, f(rx)\chi_{\{u_r>0\}}(x)\big)\, dx - \int_{\partial B_1} 2\, u_r^2\, d\mathcal{H}^{n-1} \\
&\quad - \int_{B_1} \big(\langle \mathbb{A}(rx)\nabla u_r, \nabla v\rangle + v\, f(rx)\chi_{\{u_r>0\}}(x)\big)\, dx + \int_{\partial B_1} 2\, v\, u_r\, d\mathcal{H}^{n-1} \\
&= \Phi(r) - \int_{B_1} f(rx)\big(u_r + v\, \chi_{\{u_r>0\}}(x)\big)\, dx + 2\int_{\partial B_1} \big(\mu(rx) - \mu(\underline{0})\big) u_r^2\, d\mathcal{H}^{n-1} \\
&\quad - \int_{B_1} \langle \mathbb{A}(rx)\nabla u_r, \nabla v\rangle\, dx + 2\int_{\partial B_1} v\, u_r\, d\mathcal{H}^{n-1} \\
&\geq \Phi(r) + \int_{B_1} (u_r + v)\, dx - \int_{B_1} \langle \nabla u_r, \nabla v\rangle\, dx - \int_{B_1} \big(f(rx) - f(\underline{0})\big)(u_r + v)\, dx \\
&\quad - \int_{B_1} \langle \big(\mathbb{A}(rx) - \mathbb{A}(\underline{0})\big)\nabla u_r, \nabla v\rangle\, dx + 2\int_{\partial B_1} \big(\mu(rx) - \mu(\underline{0})\big) u_r^2\, d\mathcal{H}^{n-1} + 2\int_{\partial B_1} v\, u_r\, d\mathcal{H}^{n-1}.
\end{aligned} \tag{2.97}
$$

Recalling the γ-Hölder continuity of $\mathbb{A}$ and μ, from the Divergence Theorem, we obtain

$$
\begin{aligned}
&\int_{\partial B_1} w_r(\langle \mathbb{A}(rx)\nabla u_r, x\rangle - 2u_r)\, d\mathcal{H}^{n-1} \\
&\geq \Phi(r) + \int_{B_1} (u_r + v)\, dx - \int_{B_1} \langle \nabla u_r, \nabla v\rangle\, dx + 2\int_{\partial B_1} v\, u_r\, d\mathcal{H}^{n-1} - c\, (r^\gamma + \omega(r)) \\
&\overset{(2.92)}{=} \Phi(r) - \Psi_v(1) + \int_{B_1} (u_r\, \Delta v)\, dx - \int_{B_1} \langle \nabla u_r, \nabla v\rangle\, dx + 2\int_{\partial B_1} v u_r\, d\mathcal{H}^{n-1} - c'\, (r^\gamma + \omega(r)) \\
&= \Phi(r) - \Psi_v(1) + \int_{B_1} (\mathrm{div}(u_r\, \nabla v)\, dx + 2\int_{\partial B_1} v\, u_r\, d\mathcal{H}^{n-1} - c'\, (r^\gamma + \omega(r)) \\
&= \Phi(r) - \Psi_v(1) + \int_{\partial B_1} u_r(\langle \nabla v, x\rangle\, dx + 2v\big)\, d\mathcal{H}^{n-1} - c'\, (r^\gamma + \omega(r)) \\
&= \Phi(r) - \Psi_v(1) - c'\, (r^\gamma + \omega(r))\, .
\end{aligned} \tag{2.98}
$$

So, by combining together (2.96) and (2.98), and assuming that $\gamma := 1 - \frac{n}{p^*}$ we deduce

$$
\frac{d}{dr}\int_{\partial B_1} w_r^2\, d\mathcal{H}^{n-1} \geq \frac{2}{r}\big(\Phi(r) - \Psi_v(1)\big) - c'\left(r^{-\frac{n}{p^*}} + \frac{\omega(r)}{r}\right).
$$

from inequality (2.71) we deduce

$$\begin{aligned}\frac{d}{dr}\int_{\partial B_1} w_r^2\, d\mathcal{H}^{n-1} \geq &\frac{2}{r}\Big(\Phi(r)e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4\int_0^r\Big(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\Big)\, e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt \\ &\qquad - c\left(r^{1-\frac{n}{\Theta}} + \omega(r)\right) - \Psi_v(1)\Big) - c'\left(r^{-\frac{n}{p^*}} + \frac{\omega(r)}{r}\right) \\ \geq &\frac{2}{r}\Big(\Phi(r)e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4\int_0^r\Big(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\Big)\, e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt - \Psi_v(1)\Big) \\ &\qquad - c''\left(r^{-\frac{n}{\Theta}} + \frac{\omega(r)}{r}\right)\end{aligned}$$

and then set $C_5 = \frac{c''}{1-\frac{n}{\Theta}}$

$$\begin{aligned}&\frac{d}{dr}\left(\int_{\partial B_1} w_r^2\, d\mathcal{H}^{n-1} + C_5\left(r^{1-\frac{n}{\Theta}} + \int_0^r \frac{\omega(t)}{t}\, dt\right)\right) \\ &\qquad\geq \frac{2}{r}\Big(\Phi(r)e^{\bar{C}_3 r^{1-\frac{n}{\Theta}}} + C_4\int_0^r\Big(t^{-\frac{n}{\Theta}} + \frac{\omega(t)}{t}\Big)\, e^{\bar{C}_3 t^{1-\frac{n}{\Theta}}}\, dt - \Psi_v(1)\Big).\end{aligned}$$

□

2.6 Blow up method: Uniqueness of the blow ups

The last remarks show that the blow up limits at the free boundary points must be of a unique type: nevertheless, this does not imply the uniqueness of the limit itself. In this section we prove the property of uniqueness of blow ups.

In view of Proposition 2.4.5, if $x \in \Gamma_u$ the blow up in x is unique with form

$$v_x(y) = \begin{cases} \frac{1}{2}\big(\langle \mathbb{L}^{-1}(x)\varsigma(x), y\rangle \vee 0\big)^2 & x \in Reg(u) \\ \langle \mathbb{L}^{-1}(x)\mathbb{B}_x\mathbb{L}^{-1}(x)y, y\rangle & x \in Sing(u). \end{cases}$$

where $\varsigma(x) \in \mathbb{S}^{n-1}$ is the blow up direction at $x \in Reg(u)$ and $\mathbb{B}_x$ is a symmetric matrix such that $\operatorname{Tr}\mathbb{B}_x = \frac{1}{2}$.

We start with the case of the singular points. Therefore, from Weiss' and Monneau's quasi-monotonicity formulae it follows that:

Proposition 2.6.1 ([34, Proposition 4.11]). *For every point $x \in Sing(u)$ there exists a unique blow up limit $v_x(y) = w(\mathbb{L}^{-1}(x)y)$. Moreover, if $K \subset Sing(u)$ is a compact subset, then, for every point $x \in K$*

$$\left\|u_{\mathbb{L}(x),r} - w\right\|_{C^1(B_1)} \leq \sigma_K(r) \qquad \forall r \in (0, r_K), \tag{2.99}$$

for some modulus of continuity $\sigma_K : \mathbb{R}^+ \to \mathbb{R}^+$ and a radius $r_K > 0$.

Proof. Without loss of generality we show the uniqueness in the case in which the base point $x \in Sing(u)$ is $\underline{0}$ and the condition (0.7) holds. We use Monneau's quasi monotonicity formula.

Suppose $u_{r_j} \to v$ in $C^{1,\gamma}(B_1 x)$ with v being a 2-homogeneous, polynomial and quadratic function such that $\mathrm{Tr}(D^2 v) = 1$. From uniform convergence

$$\lim \int_{\partial B_1} |u_{r_j} - v|^2 \, d\mathcal{H}^{n-1} = 0.$$

According to (2.95)

$$r \longmapsto \int_{\partial B_1} (u_r - v)^2 \, d\mathcal{H}^{n-1} + C_5 \left(r^{1-\frac{n}{\Theta(s)}} + \omega(r) \right)$$

is monotonic and infinitesimal if $r \searrow 0$. In particular $(u_{h_j})_j \to v$ in $C^{1,\gamma}$ for all sequences $h_j \searrow 0$, so from Uryshon's property the whole sequence converges to v. This implies the uniqueness of blow ups.

We fix a compact set K and we prove the uniform convergence in K. Let's suppose by contradiction that there exist $x_j \in K$ and $r_j \to 0$ such that the rescaled function $u_{\mathbb{L}(x_j),r_j}$ and $w_j(\cdot) = v_{x_j}(\mathbb{L}(x_j)\cdot)$, where v_{x_j} is the blow up of u in the point x_j, satisfy

$$\left\| u_{\mathbb{L}(x_j),r_j} - w_j \right\|_{C^1(B_1)} \geq \varepsilon \qquad \forall \varepsilon > 0. \tag{2.100}$$

Due to Proposition 2.24 $\|u_{\mathbb{L}(x_j),r_j}\|_{C^{1,\gamma}(B_1)} \leq C$, for all $j \in \mathbb{N}$. From Ascoli-Arzelà's Theorem, up to extract a subsequence (that we do not relabel), $(u_{\mathbb{L}(x_j),r_j})_j$ converges to some function w in $C^{1,\gamma}$. Since $x \in K$, according to Remark 2.3.11, the constants in Weiss' quasi monotonicity formula (2.70) are bounded, so reasoning as in Proposition 2.4.1 we achieve the 2-homogeneity property for w. Proceeding as in Proposition 2.4.5: we define the functional

$$\widetilde{\mathcal{F}_j}(v) := \begin{cases} \int_{B_1} \left(\langle \mathbb{C}_{\mathbb{L}(x_j)}(r_j y) \nabla v(y), \nabla v(y) \rangle + 2 \frac{f_{\mathbb{L}(x_j)}(r_j y)}{f(x_j)} v \right) dy & \textit{if } v \in \mathcal{V}_j \\ \infty & \text{otherwise}, \end{cases}$$

with $\mathcal{V}_j$ as in (2.86), we prove that $\widetilde{\mathcal{F}_j}$ Γ-converges to $\mathcal{F}$ defined in (2.87) and so we obtain that $(u_{\mathbb{L}(x_j),r_j})_j \to w$ and w is a 2-homogeneous, global solution.

Then according to (2.59), (2.91) and (2.93) we have

$$\Phi_{u_{\mathbb{L}(x_j),r_j}}(R) \xrightarrow{j \to \infty} \Psi_w(R) = \Psi_v(1) \qquad \forall\, R > 0. \tag{2.101}$$

From Weiss' formula and remembering that $x_j \in Sing(u)$ for all $j \in \mathbb{N}$ it holds

$$\Phi_{u_{\mathbb{L}(x_j),r}}(1) \xrightarrow{r \to 0} \Phi_{\mathbb{L}(x_j)}(0^+) = 2\,\theta$$

and the function

$$r \mapsto \Phi_{u_{\mathbb{L}(x_j)}}(r)\, e^{\bar{C}_3 r^{1-\frac{n}{\Theta(s)}}} + C_4 \int_0^r \left(t^{-\frac{n}{\Theta(s)}} + \frac{\omega(t)}{t} \right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta(s)}}} \, dt,$$

is not decreasing. Therefore

$$\Phi_{u_{\mathbb{L}(x_j)}}(r) \geq \frac{2\,\theta - C_4 \int_0^r \left(t^{-\frac{n}{\Theta(s)}} + \frac{\omega(t)}{t}\right) e^{\bar{C}_3 t^{1-\frac{n}{\Theta(s)}}}\, dt}{e^{\bar{C}_3 r^{1-\frac{n}{\Theta(s)}}}},$$

and if $r << 1$ we deduce

$$\Phi_{u_{\mathbb{L}(x_j)}}(r) \geq \frac{3}{2}\,\theta.$$

Then for $j >> 1$

$$\Phi_{u_{\mathbb{L}(x_j),r_j}}(1) \geq \frac{3}{2}\,\theta,$$

and from (2.101)

$$\Phi_w(1) \geq \frac{3}{2}\,\theta.$$

Remembering that by Corollary 2.4.2 $w \not\equiv 0$, we deduce that w is a non trivial, 2-homogeneous, global solution with $\Psi_w(1) \geq \frac{3}{2}\theta$, so according to Theorem 2.4.4 and Rermark 2.4.7 $w(y) = \langle \mathbb{B}y, y\rangle$ with $\mathbb{B}$ symmetric matrix and $\mathrm{Tr}(\mathbb{B}) = \frac{1}{2}$.

In order to conclude, since all norms evaluated on polynomials are equivalent, from (2.100) and Monneau's quasi monotonicity formula (that holds with the same constants because the points x_j are contained in a compact set) we deduce that

$$\begin{aligned} 0 < \varepsilon &\leq \limsup_j \left\| u_{\mathbb{L}(x_j),r_j} - w_j \right\|_{C^1(B_1)} \leq \limsup_j \|w - w_j\|_{C^1(B_1)} \\ &\leq C\, \limsup_j \|w - w_j\|_{L^2(\partial B_1)} \overset{\text{Theorem 2.5.1}}{\leq} C\, \limsup_j \left\| w - u_{\mathbb{L}(x_j),r_j} \right\|_{L^2(\partial B_1)} = 0. \end{aligned}$$

□

Next, we proceed with the case of the regular points.
We extend the energy defined in (2.91) from 2-homogeneous functions to each function $\xi \in W^{1,2}(B_1)$ by

$$\Psi_\xi(1) = \int_{B_1} \left(|\nabla \xi|^2 + 2\xi\right) dx - \int_{\partial B_1} \xi^2 \, d\mathcal{H}^{n-1}.$$

We state Weiss' celebrated epiperimetric inequality [95, Theorem 1] (recently a variational proof for the thin obstacle problem has been given in [36] and with the same approach as [49] and Chapter 3 for the fractional Laplacian):

Theorem 2.6.2 (Weiss' epiperimetric inequality)**.** *There exist $\delta > 0$ and $k \in (0,1)$ such that, for every $\varphi \in H^1(B_1)$, 2-homogeneous function, with*

$$\|\varphi - w\|_{H^1(B_1)} \leq \delta \tag{2.102}$$

for some global solution w of type (A), there exists a function $\xi \in H^1(B_1)$ such that $\xi_{|\partial B_1} = \varphi_{|\partial B_1}$, $\xi \geq 0$ and

$$\Psi_\xi(1) - \theta \leq (1-k)\left(\Psi_\varphi(1) - \theta\right), \tag{2.103}$$

where $\theta = \Psi_w(1)$ is the energy of any global solution of type (A).

For the reader's convenience we recall the definition (I3') seen in the introduction

$(I3)'$ Let $\omega(t) = \sup_{|x-y|\le t} |f(x) - f(y)|$ be the modulus of continuity of f and set $a > 2$ the following condition of integrability holds

$$\int_0^1 \frac{\omega(r)}{r} \, |\log r|^a \, dr < \infty. \tag{2.104}$$

As in [34] we prove a technical lemma that will be the key ingredient in the proof of uniqueness. With respect to [34, Lemma 4.8] the lack of regularity of $\mathbb{A}$ and f in $(I1)$-$(I3)$ does not allow to use the final diadic argument; for this reason we introduce a technical hypothesis $(I3)'$. For a clearer comprehension on behalf of the reader, we report the whole proof:

Lemma 2.6.3. *Let u be solution of* (2.4) *and we assume (I3') and* (0.7)*. If there exist radii $0 \le \varrho_0 < r_0 < 1$ such that*

$$\inf_{w} \|u_{r|\partial B_1} - w\|_{H^1(\partial B_1)} \le \delta \qquad\qquad \forall\ \varrho_0 \le r \le r_0, \tag{2.105}$$

where the infimum is taken on all global solutions w of type (A) and $\delta > 0$ is the constant of Theorem 2.6.2, then for each pair of rays ϱ, t such that $\varrho_0 \le \varrho < t \le r_0$ we have

$$\int_{\partial B_1} |u_t - u_\varrho| \, d\mathcal{H}^{n-1} \le C_7 \, \rho(t), \tag{2.106}$$

with C_7 positive constants independent of r and ϱ, while $\rho(t)$ a growing function vanishing in 0*.*

Proof. From the Divergence Theorem, (2.58) and (2.75) we can compute the derivative of $\Phi'(r)$ in the following way:

$$\begin{aligned}
\Phi'(r) =& \frac{\mathcal{E}'(r)}{r^{n+2}} - (n+2)\frac{\mathcal{E}(r)}{r^{n+3}} - 2\frac{\mathscr{H}'(r)}{r^{n+3}} + 2(n+3)\frac{\mathscr{H}(r)}{r^{n+4}} \\
\ge& \frac{1}{r^{n+2}} \int_{\partial B_r} \left(\langle \mathbb{A}\nabla u, \nabla u\rangle + 2\, f u\right) d\mathcal{H}^{n-1} - (n+2)\frac{\mathcal{E}(r)}{r^{n+3}} + \frac{8}{r^{n+4}}\mathscr{H}(r) \\
&- \frac{4}{r^{n+3}} \int_{\partial B_r} u \langle \mathbb{A}\nu, \nabla u\rangle \, d\mathcal{H}^{n-1} - C\, r^{-\frac{n}{\Theta}} \\
\ge& \frac{1}{r^{n+2}} \int_{\partial B_r} \left(|\nabla u|^2 + 2\, u\right) d\mathcal{H}^{n-1} - \frac{(n+2)}{r}\Phi(r) - \frac{2(n-2)}{r^{n+4}} \int_{\partial B_r} u^2 \, d\mathcal{H}^{n-1} \\
&- \frac{4}{r^{n+3}} \int_{\partial B_r} u \langle \nu, \nabla u\rangle \, d\mathcal{H}^{n-1} - C\left(r^{-\frac{n}{\Theta}} + \frac{\omega(r)}{r}\right) \\
=& -\frac{(n+2)}{r}\Phi(r) + \frac{1}{r}\int_{\partial B_1} \left(\left(\langle \nu, \nabla u_r\rangle - 2u_r\right)^2 + |\partial_\tau u_r|^2 + 2u_r - 2n\, u_r^2\right) d\mathcal{H}^{n-1} \\
&- C\left(r^{-\frac{n}{\Theta}} + \frac{\omega(r)}{r}\right),
\end{aligned}$$

where we denote by $\partial_\tau u_r$, the tangential derivative of u_r along ∂B_1. Let w_r be the 2-homogeneous extension of $u_{r|\partial B_1}$. We note that if ϕ is a 2-homogeneous function it holds

$$\begin{aligned}\int_{B_1}\phi(x)\,dx=&\int_0^1\int_{\partial B_t}\phi(y)\,d\mathcal{H}^{n-1}(y)\,dt=\int_0^1 t^{n+1}\int_{\partial B_1}\phi(y)\,d\mathcal{H}^{n-1}(y)\\=&\frac{1}{n+2}\int_{\partial B_1}\phi(y)\,d\mathcal{H}^{n-1}(y).\end{aligned}\tag{2.107}$$

Then a simple integration in polar coordinates, thanks to Euler's homogeneous function Theorem functions and (2.107) which give

$$\begin{aligned}\int_{\partial B_1}&\left(|\partial_\tau u_r|^2+2u_r-2n\,u_r^2\right)d\mathcal{H}^{n-1}=\int_{\partial B_1}\left(|\partial_\tau w_r|^2+2w_r+4w_r^2-2(n+2)\,w_r^2\right)d\mathcal{H}^{n-1}\\&=\int_{\partial B_1}\left(|\nabla w_r|^2+2w_r\right)-2(n+2)\int_{\partial B_1}w_r^2\,d\mathcal{H}^{n-1}\\&=(n+2)\int_{B_1}\left(|\nabla w_r|^2+2w_r\right)d\mathcal{H}^{n-1}-2(n+2)\int_{\partial B_1}w_r^2\,d\mathcal{H}^{n-1}=(n+2)\Psi_{w_r}(1).\end{aligned}$$

Therefore, we conclude that

$$\Phi'(r)\geq\frac{(n+2)}{r}\big(\Psi_{w_r}(1)-\Phi(r)\big)+\frac{1}{r}\int_{\partial B_1}\big(\langle\nu,\nabla u_r\rangle-2u_r\big)^2\,d\mathcal{H}^{n-1}-C\left(r^{-\frac{n}{\Theta}}+\frac{\omega(r)}{r}\right).\tag{2.108}$$

We can also note that, being w_r the 2-homogeneous extension of $u_{r|\partial B1}$, thanks to (2.107) and (2.105), there exists a global solution w of type (A) such that

$$\|w_r-w\|_{H^1(B_1)}\leq\frac{1}{\sqrt{n+2}}\|w_{r\partial B_1}-w\|_{H^1(\partial B_1)}\leq\delta.$$

Hence, we can apply the epiperimetric inequality (2.103) to w_r and find a function $\xi\in w_r+H_0^1(B_1)$ such that

$$\Psi_\xi(1)-\theta\leq(1-k)\big(\Psi_{w_r}(1)-\theta\big).\tag{2.109}$$

Moreover, we can assume without loss of generality (otherwise we substitute ξ with u_r) that $\Psi_\xi(1)\leq\Psi_{u_r}(1)$. Then, by the minimality of u_r in $\mathcal{E}$ with respect to its boundary conditions (0.7), (I1)-(I4) and lemma 2.3.1 we have

$$\begin{aligned}\Psi_\xi(1)=&\int_{B_1}\left(|\nabla\xi|^2+2\xi\right)dx-\int_{\partial B_1}\xi^2\,d\mathcal{H}^{n-1}\\\geq&\int_{B_1}\left(\langle\mathbb{A}(rx)\nabla\xi,\nabla\xi\rangle+2\,f(rx)\xi\right)dx-\int_{\partial B_1}\mu(rx)\,\xi^2\,d\mathcal{H}^{n-1}\\&-C\left(r^{1-\frac{n}{p^*}}+\omega(r)\right)\int_{B_1}\left(|\nabla\xi|^2+2\xi\right)dx-C\,r^\gamma\int_{\partial B_1}\xi^2\,d\mathcal{H}^{n-1}\\\geq&\Phi(r)-C\left(r^{1-\frac{n}{p^*}}+\omega(r)\right)\int_{B_1}\left(|\nabla\xi|^2+2\xi\right)dx-C\,r^\gamma\int_{\partial B_1}\xi^2\,d\mathcal{H}^{n-1}\\\geq&\Phi(r)-C\left(r^{1-\frac{n}{p^*}}+\omega(r)\right).\end{aligned}\tag{2.110}$$

From (2.109) and (2.110) we get

$$\Psi_{w_r}(1)-\Phi(r) \geq \frac{1}{1-k}(\Phi(r)-\theta-C\,r^{\beta})+\theta-\Phi(r) = \frac{k}{1-k}(\Phi(r)-\theta)-C\,\left(r^{1-\frac{n}{p^*}}+\omega(r)\right). \tag{2.111}$$

Then from (2.108) and (2.110)

$$\Phi'(r) \geq \frac{n+2}{r}\frac{k}{1-k}(\Phi(r)-\theta)-C\,\left(r^{-\frac{n}{\Theta}}+\frac{\omega(r)}{r}\right). \tag{2.112}$$

Let now $\widetilde{C}_6 \in (0,\,(1-\frac{n}{\Theta})\wedge(n+2)\frac{k}{1-k})$, then

$$\left((\Phi(r)-\theta)\,r^{-\widetilde{C}_6}\right)' \geq -C\,\left(r^{-\frac{n}{\Theta}-\widetilde{C}_6}+\frac{\omega(r)}{r^{1+\widetilde{C}_6}}\right). \tag{2.113}$$

Indeed, by taking into account (2.112)

$$\begin{aligned}
\left((\Phi(r)-\theta)\,r^{-\widetilde{C}_6}\right)' &= \Phi'(r)r^{-\widetilde{C}_6}-\widetilde{C}_6\,(\Phi(r)-\theta)\,r^{-\widetilde{C}_6-1}\\
&\geq \left(\frac{n+2}{r}\frac{k}{1-k}(\Phi(r)-\theta)-C\,\left(r^{-\frac{n}{\Theta}}+\frac{\omega(r)}{r}\right)\right)r^{-\widetilde{C}_6}-\widetilde{C}_6\,(\Phi(r)-\theta)\,r^{-\widetilde{C}_6-1}\\
&\geq (\Phi(r)-\theta)r^{-\widetilde{C}_6-1}\left((n+2)\frac{k}{1-k}-\widetilde{C}_6\right)-C\,\left(r^{-\frac{n}{\Theta}}+\frac{\omega(r)}{r}\right)r^{-\widetilde{C}_6}\\
&\geq -C\,\left(r^{-\frac{n}{\Theta}-\widetilde{C}_6}+\frac{\omega(r)}{r^{1+\widetilde{C}_6}}\right).
\end{aligned}$$

By integrating (2.113) in (t,r_0) with $t\in(s_0,r_0)$ and multiplying by $t^{\widetilde{C}_6}$ we finally get

$$t^{\widetilde{C}_6}\left[(\Phi(r)-\theta)\,r^{-\widetilde{C}_6}\right]_t^{r_0} \geq -C\,t^{\widetilde{C}_6}\int_t^{r_0}\left(r^{-\frac{n}{\Theta}-\widetilde{C}_6}+\frac{\omega(r)}{r^{1+\widetilde{C}_6}}\right)dr$$

whence

$$\begin{aligned}
\Phi(t)-\theta &\leq C\Bigg(\int_t^{r_0}\left(r^{-\frac{n}{\Theta}-\widetilde{C}_6}+\frac{\omega(r)}{r^{1+\widetilde{C}_6}}\right)dr+1\Bigg)t^{\widetilde{C}_6}\\
&\leq C\Bigg(r^{1-\frac{n}{\Theta}}+t^{\widetilde{C}_6}+t^{\widetilde{C}_6}\int_t^{r_0}\frac{\omega(r)}{r^{1+\widetilde{C}_6}}\,dr\Bigg) \leq C\,t^{\widetilde{C}_6}\left(\int_t^{r_0}\frac{\omega(r)}{r^{1+\widetilde{C}_6}}\,dr+1\right).
\end{aligned} \tag{2.114}$$

Consider now $\varrho_0<\varrho<r_0$ and estimate as follows

$$\begin{aligned}
\int_{\partial B_1}|u_t-u_\varrho|\,d\mathcal{H}^{n-1} &= \int_{\partial B_1}\left|\int_\varrho^t\frac{d}{dr}\left(\frac{u(rx)}{r^2}\right)dr\right|\,d\mathcal{H}^{n-1}\\
&\leq \int_\varrho^t r^{-2}\int_{\partial B_1}\left|\langle\nabla u(rx),x\rangle-2\frac{u(rx)}{r}\right|\,d\mathcal{H}^{n-1}\,dr\\
&= \int_\varrho^t r^{-1}\int_{\partial B_1}|\langle\nabla u_r(x),x\rangle-2\,u_r(x)|\,d\mathcal{H}^{n-1}\,dr\\
&\leq \sqrt{n\omega_n}\int_\varrho^t r^{-\frac{1}{2}}\left(r^{-1}\int_{\partial B_1}|\langle\nabla u_r(x),x\rangle-2\,u_r(x)|^2\,d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}dr.
\end{aligned} \tag{2.115}$$

Combining (2.71), (2.108), (2.111), (2.114) and Hölder inequality we have

$$\begin{aligned}
\int_{\partial B_1} |u_t - u_\varrho| \, d\mathcal{H}^{n-1} &\le C \int_\varrho^t r^{-\frac{1}{2}} \left(\Phi'(r) + C \left(r^{-\frac{n}{\Theta}} + \frac{\omega(r)}{r} \right) \right)^{\frac{1}{2}} dr \\
&\le C \left(\log \frac{t}{\varrho} \right)^{\frac{1}{2}} \left(\Phi(t) - \Phi(\varrho) + C \left(t^{1-\frac{n}{\Theta}} - \varrho^{1-\frac{n}{\Theta}} + \int_\varrho^t \frac{\omega(r)}{r} \, dr \right) \right)^{\frac{1}{2}} \\
&\le C \left(\log \frac{t}{\varrho} \right)^{\frac{1}{2}} \left((\Phi(t) - \theta) + (\theta - \Phi(\varrho)) + C \left(t^{1-\frac{n}{\Theta}} + \int_\varrho^t \frac{\omega(r)}{r} \, dr \right) \right)^{\frac{1}{2}} \\
&\le C \left(\log \frac{t}{\varrho} \right)^{\frac{1}{2}} \left(t^{\widetilde{C}_6} + t^{\widetilde{C}_6} \int_t^{r_0} \frac{\omega(r)}{r^{1+\widetilde{C}_6}} \, dr + \omega(t) + \int_\varrho^t \frac{\omega(r)}{r} \, dr \right)^{\frac{1}{2}} .
\end{aligned} \tag{2.116}$$

Now thanks to the hypothesis $(I3)'$, if $r_0 << 1$ for every $0 \le t \le r_0$ we can apply the inequality $\omega(t) \le |\log t|^{-a}$, the infinitesimal function $t^{\widetilde{C}_6} |\log t|^a$ which is growing, the inequality (2.116) and decreasing of $|\log t|^a$ we have

$$\begin{aligned}
\int_{\partial B_1} |u_t - u_\varrho| \, d\mathcal{H}^{n-1} &\le C \left(\log \frac{t}{\varrho} \right)^{\frac{1}{2}} |\log t|^{-\frac{a}{2}} \left(1 + \int_t^{r_0} \frac{\omega(r) \, |\log r|^a}{r} \, dr \right)^{\frac{1}{2}} \\
&\le C \left(\log \frac{t}{\varrho} \right)^{\frac{1}{2}} |\log t|^{-\frac{a}{2}} .
\end{aligned} \tag{2.117}$$

A simple dyadic decomposition argument then leads to the conclusion. If $\varrho \in [2^{-k}, 2^{-k+1})$ and $t \in [2^{-h}, 2^{-h+1})$ with $h < k$, applying (2.117)

$$\int_{\partial B_1} |u_t - u_\varrho| \, d\mathcal{H}^{n-1} \le C \sum_{j=h}^{k} \log(2^j)^{-\frac{a}{2}} \le C_7 \sum_{j=h}^{\infty} \frac{1}{j^{\frac{a}{2}}} =: C_7 \, \rho(t),$$

with

$$\rho(t) := \sum_{j=h}^{\infty} \frac{1}{j^{\frac{a}{2}}} \quad \text{if} \quad t \in [2^{-h}, 2^{-h+1}). \tag{2.118}$$

By taking (0.12) into account we have $a > 2$, therefore, the function $\rho(t)$ is growing and infinitesimal in 0, from which the conclusion of the lemma follows. $\square$

Checking the hypothesis of Lemma 2.6.3 it is possible to prove the uniqueness of the blow ups at regular points of the free boundary:

Proposition 2.6.4 ([34, Proposition 4.10]). *Let u be a solution to the obstacle problem* (2.12) *with f that satisfies $(I3)'$ and $x_0 \in Reg(u)$. Then, there exist constants $r_0 = r_0(x_0), \eta_0 = \eta_0(x_0)$ such that every $x \in \Gamma_u \cap B_{\eta_0}(x_0)$ is a regular point and, denoting by $v_x = w(\mathbb{L}^{-1}(x)y)$ any blow up of u in x we have*

$$\int_{\partial B_1} |u_{\mathbb{L}(x),r} - w| \, d\mathcal{H}^{n-1}(y) \le C_7 \, \rho(r) \qquad \forall \, r \in (0, r_0), \tag{2.119}$$

where C_7 is an independent constant from r and $\rho(r)$ a growing, infinitesimal function in 0. In particular, the blow up limit v_x is unique.

Proof. We indicate by $\Phi(x, r)$ the energy in (2.57) where we modify the base point from $\underline{0}$ to x, or rather we assume $B_r(x)$ as the integration domain. Due to continuity of the translation in L^p, the function $\Gamma_u \ni x \mapsto \Phi(x, r)$ is continuous, and since, from Theorem 2.3.10, we have $\Phi(x, 0^+) = \inf_r \Phi(x, r)$, we find that the function $\Gamma_u \ni x \mapsto \Phi(x, 0^+)$ is upper semicontinuous. So $Reg(u) \subset \Gamma_u$ is relatively open in Γ_u[3].

By Proposition 2.2.1, given $\bar{\eta} > 0$ such that $B_{\bar{\eta}}(x_0) \subset\subset \Omega$ and $\Gamma_u \cap B_{\bar{\eta}}(x_0) = Reg(u) \cap B_{\bar{\eta}}(x_0)$, then

$$C_8 := \sup_{x \in \Gamma_u \cap B_{\bar{\eta}}(x_0),\, r < \bar{\eta}} \|u_{\mathbb{L}(x),r}\|_{C^{1,\gamma}(\partial B_1)} < \infty.$$

Let $\delta > 0$ be the constant in Theorem 2.6.2. Due to a compactness argument if φ is a function $C^{1,\gamma}(\partial B_1)$ that satisfies $\|\varphi\|_{C^{1,\gamma}(\partial B_1)} < C_8$, then there exists an $\varepsilon > 0$ for which

$$\|\varphi\|_{L^1(\partial B_1)} \leq \varepsilon \qquad \Longrightarrow \qquad \|\varphi\|_{H^1(\partial B_1)} \leq \frac{\delta}{4}. \tag{2.120}$$

On the other hand, if the condition (2.120) does not hold we have that for all $\varepsilon > 0$ there exists a $\varphi_\varepsilon \in C^{1,\gamma}(\partial B_1)$ such that $\|\varphi_\varepsilon\|_{C^{1,\gamma}(\partial B_1)} \leq C_8$, for which $\|\varphi_\varepsilon\|_{L^1(\partial B_1)} \leq \varepsilon$ but $\|\varphi_\varepsilon\|_{H^1(\partial B_1)} > \frac{\delta}{4}$. According to Ascoli-Arzelà's Theorem it is possible to extract a subsequence $\varphi_{\varepsilon_j} \to \varphi$ in $C^{1,\gamma}(\partial B_1)$, with $\varphi \in C^{1,\gamma}(\partial B_1)$, for which $\|\varphi\|_{L^1(\partial B_1)} = 0$, so $\varphi_{|\partial B_1} = 0$ $\mathcal{H}^{n-1}$-a.e.; but $\|\varphi_\varepsilon\|_{H^1(\partial B_1)} = \lim_j \|\varphi_{\varepsilon_j}\|_{H^1(\partial B_1)} > \frac{\delta}{4}$, and this is a contradiction.

We now fix $\bar{r}_0 > 0$ such that $C_7\, \rho(\bar{r}_0) \leq \varepsilon$ and

$$\inf_w \left\| u_{\mathbb{L}(x_0),\bar{r}_0|\partial B_1} - w \right\|_{H^1(\partial B_1)} \leq \frac{\delta}{4}, \tag{2.121}$$

where the infimum is taken on all (A)-type solutions w. In order to prove the existence of threshold $\bar{r}_0$ we resort to a reductio ad absurdum: if a similar threshold does not exists we could find a sequence $r_j \to 0$ such that $\|u_{\mathbb{L}(x_0),r_j|\partial B_1} - w\|_{H^1(\partial B_1)} \geq \frac{\delta}{4}$ for every (A)-type solution w; but on the other hand $x_0 \in Reg(u)$, so (up to subsequence that we do not relabel) $(u_{\mathbb{L}(x_0),r_j})_j$ converges to type (A) blow up v of u in x_0 in $C^{1,\gamma}_{loc}$ and this is the absurdum.

From continuity of $\mathbb{A}$ and f, and thus of $\mathbb{L}$, there exists $0 < \eta_0 < \bar{\eta}$ such that for all $x \in Reg(u) \cap B_{\eta_0}(x_0)$

$$\inf_w \left\| u_{\mathbb{L}(x),\bar{r}_0|\partial B_1} - w \right\|_{H^1(\partial B_1)} \leq \frac{\delta}{2}, \tag{2.122}$$

where the infimum is taken on the same class as above. We prove that this implies that for all $x \in Reg(u) \cap B_{\eta_0}(x_0)$ and $0 < r < \bar{r}_0$

$$\inf_w \left\| u_{\mathbb{L}(x),r|\partial B_1} - w \right\|_{H^1(\partial B_1)} \leq \delta. \tag{2.123}$$

[3] in fact if $y \to x$ we have $\Phi(y, 0^+) \leq \Phi(x, 0^+) = \theta$, so for $\eta_0 << 1$ we achieve the thesis.

For this purpose we fix $x \in Reg(u) \cap B_{\eta_0}(x_0)$ and let $\varrho_0 < \bar{r}_0$ be the minimum radius such that the condition (2.123) holds for all radii $\varrho_0 \le r \le r_0$. Let us assume $\varrho_0 > 0$ and we note that according to continuity of $u, \mathbb{A}$ and f we deduce

$$\inf_w \left\| u_{\mathbb{L}(x),\varrho_0 |\partial B_1} - w \right\|_{H^1(\partial B_1)} = \delta. \tag{2.124}$$

Then, remembering that from Remark 2.3.11, since $B_{\eta_0} \subset\subset \Omega$, the constants are uniform in $\Gamma_u \cap B_{\eta_0}(x_0)$, due to Lemma 2.6.3 we obtain

$$\inf_w \left\| u_{\mathbb{L}(x),\varrho|\partial B_1} - u_{\mathbb{L}(x),t|\partial B_1} \right\|_{H^1(\partial B_1)} \le C_7\, \rho(\bar{r}_0) \qquad \forall\, \varrho, t \in [\varrho_0, \bar{r}_0].$$

Since the functions $u_{\mathbb{L}(x),\varrho}$ are equibounded in $C^{1,\gamma}(\partial B_1)$ by C_8, the condition (2.120) gives us

$$\inf_w \left\| u_{\mathbb{L}(x),\varrho|\partial B_1} - u_{\mathbb{L}(x),t|\partial B_1} \right\|_{H^1(\partial B_1)} \le \frac{\delta}{4} \qquad \forall\, \varrho, t \in [\varrho_0, \bar{r}_0].$$

In particular from (2.122) and triangle inequality we contradict the condition (2.124).

In order to conclude we observe that thanks to (2.123) we obtain (2.105) and deduce (2.106) for every $\varrho, t \in (0, \bar{r}_0)$. Moreover by passing to the limit as $\varrho \searrow 0$ in (2.106) we find

$$\int_{\partial B_1} |u_{\mathbb{L}(x),t} - w|\, d\mathcal{H}^{n-1} \le C_7\, \rho(t),$$

and we achieve the uniqueness of the blow ups. □

Remark 2.6.5. If f is α-Hölder we can prove Lemma 2.6.3 and Proposition 2.6.4 with $\rho(t) = t^{C_6}$ where $C_6 := \frac{\bar{C}_6 \wedge \alpha}{2}$.

2.7 Regularity of the free boundary

In this last section we state some regularity results of the free boundary of u, the solution of (2.4). If the matrix $\mathbb{A}$ satisfies the hypotheses $(I1)$-$(I2)$ and the linear term f satisfies the hypothesis $(I3)'$ we obtain differentiability of the free boundary in a neighborhood of any point $x \in Reg(u)$. In particular if f is Hölder we establish the $C^{1,\beta}$ regularity as in [34] where $\mathbb{A}$ is Lipschitz continuous.

Theorem 2.7.1 ([34, Theorem 4.12]). *Assume hypotheses $(I1)$, $(I2)$, $(I3)'$ and $(I4)$ hold. Let $x \in Reg(u)$. Then, there exists $r > 0$ such that $\Gamma_u \cap B_r(x)$ is hypersurface C^1 and n its normal versor is absolutely continuous with modulus of continuity depending on ρ defined in (2.118). In particular if f is Hölder continuous there exists $r > 0$ such that $\Gamma_u \cap B_r(x)$ is hypersurface $C^{1,\beta}$ for some universal exponent $\beta \in (0,1)$.*

Proof. Let $\eta_0 = \eta_0(\underline{0})$ and $r_0 = r_0(\underline{0})$ be the radii provided by Proposition 2.6.4. We can prove that there exist two constants $C > 0$ and $\beta \in (0,1)$[4] such that

$$\left| \mathbb{L}^{-1}(x) n(x) - \mathbb{L}^{-1}(z) n(z) \right| \le C\, |x - z|^{\beta}, \qquad \forall\, x, z \in Reg(u) \cap B_{\frac{\eta_0}{2}}. \tag{2.125}$$

[4] β is computable.

For this aim let $\varrho \in (0, r_0)$, then changing Coordinate system in (2.119) we have

$$\begin{aligned}\|v_x - v_z\|_{L^1(\partial B_1)} &\le \|v_x - u_{x,\varrho}\|_{L^1(\partial B_1)} + \|u_{x,\varrho} - u_{z,\varrho}\|_{L^1(\partial B_1)} + \|u_{z,\varrho} - v_z\|_{L^1(\partial B_1)} \\ &\le C\,\rho(\varrho) + \|u_{x,\varrho} - u_{z,\varrho}\|_{L^1(\partial B_1)}.\end{aligned} \tag{2.126}$$

The map $y \mapsto \mathbb{L}(y)$ is absolutely continuous with $\rho(r) + r^\gamma$ its modulus of continuity :

$$\begin{aligned}\left|\frac{\mathbb{A}^{\frac12}(y)}{f^{\frac12}(y)} - \frac{\mathbb{A}^{\frac12}(z)}{f^{\frac12}(z)}\right| &\le \left|\frac{\mathbb{A}^{\frac12}(y) f^{\frac12}(z) - \mathbb{A}^{\frac12}(z) f^{\frac12}(y)}{f^{\frac12}(y) f^{\frac12}(z)}\right| \\ &\le C' \left| |\mathbb{A}^{\frac12}(y)|\, |f^{\frac12}(y) - f^{\frac12}(z)| + f^{\frac12}(y)\, |\mathbb{A}^{\frac12}(y) - \mathbb{A}^{\frac12}(z)| \right| \\ &\le C\,(\rho(|y-z|) + |y-z|^\gamma).\end{aligned}$$

Then thanks to (2.30) we estimate the following term

$$\begin{aligned}\|u_{x,\varrho} - u_{z,\varrho}\|_{L^1(\partial B_1)} &\le \int_{\partial B_1}\int_0^1 \left|\frac{\nabla u\big(t(z+\varrho y) + (1-t)(x+\varrho y)\big)}{\varrho^2}\right| |z-x|\, dt\, d\mathcal{H}^{n-1}(y) \\ &C\,\varrho^{-2}(|z-x| + \varrho)|z-x| \le C\,|z-x|^\gamma,\end{aligned} \tag{2.127}$$

if $\varrho = |z-x|^{1-\gamma}$ and $C = C(n)$. Moreover, observing that $\|v_x\|^{\frac12}_{L^1(\partial B_1)}$ is a norm for the vector $\mathbb{L}^{-1}(x)n(x)$, and remembering that all norms in a finite vectorial space are equivalent, we obtain

$$\left|\mathbb{L}^{-1}(x)n(x) - \mathbb{L}^{-1}(z)n(z)\right| \le C\,\|v_x - v_z\|^{\frac12}_{L^1(\partial B_1)}. \tag{2.128}$$

We achieve the condition (2.125) by putting together (2.126), (2.127)) and (2.128):

$$\left|\mathbb{L}^{-1}(x)n(x) - \mathbb{L}^{-1}(z)n(z)\right| \le C\,\|v_x - v_z\|^{\frac12}_{L^1(\partial B_1)} \le C\,\sqrt{\rho(|x-z|^{1-\gamma})}. \tag{2.129}$$

We now consider the cones $C^{\mp}(x,\varepsilon)$, with $x \in Reg(u)$, given by

$$C^{\pm}(x,\varepsilon) := \left\{ y \in \mathbb{R}^n \;:\; \pm\left\langle y - x, \frac{\mathbb{A}^{-\frac12}(x)n(x)}{|\mathbb{A}^{-\frac12}(x)n(x)|}\right\rangle \ge \varepsilon\,|y-x| \right\}.$$

We prove that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in Reg(u) \cap B_{\frac{r_0}{2}}$,

$$C^+(x,\varepsilon) \cap B_\delta(x) \subset N_u \qquad \textit{and} \qquad C^-(x,\varepsilon) \cap B_\delta(x) \subset \Lambda_u. \tag{2.130}$$

Let us suppose by contradiction that there exists a sequence $(x_j)_j \subset Reg(u) \cap B_{\frac{r_0}{2}}$ such that $x_j \to x \in Reg(u) \cap \overline{B}_{\frac{r_0}{2}}$ and a sequence $(y_j)_j$ for which $y_j \in C^+(x_j,\varepsilon)$, $x_j - y_j \to 0$ and $u(y_j) = 0$.

The rescaled function u_{x_j,r_j} with $r_j = |(y_j - x_j)|$, uniformly converges to v_x. Changing the coordinate system in (2.119) and from (2.129)

$$\begin{aligned}\|u_{x_j,r_j} - v_x\|_{L^1(\partial B_1)} &\le \|u_{x_j,r_j} - v_{x_j}\|_{L^1(\partial B_1)} + \|v_{x_j} - v_x\|_{L^1(\partial B_1)}\\ &\le C\,(\rho(r_j) + \sqrt{\rho(|x - x_j|^{1-\gamma})}),\end{aligned}$$

thus $u_{x_j,r_j} \to v_x$ in $L^1(\partial B_1)$. Changing the coordinate system in Proposition 2.2.1 and since, from Remark 2.3.11, the constant in (2.24) is uniformly bounded, because $(x_j)_j \subset \overline{B}_{\frac{\eta_0}{2}}$, we have that the sequence $(u_{x_j,r_j})_j$ is bounded in $C^{1,\gamma}$. Then for all sequences we can extract a convergent subsequence in $C^{1,\gamma}$ that, for uniqueness of the limit, converges to v_x. So for Uryshon's property the whole sequence uniformly converges to v_x.

We define the sequence $z_j = r_j^{-1}(y_j - x_j)$ and we observe that $z_j \in \big(C^+(x_j,\varepsilon) - x_j\big) \cap \mathbb{S}^{n-1}$. Up to subsequence (that we do not relabel) we can suppose that $z_j \to z \in \big(C^+(x,\varepsilon) - x\big) \cap \mathbb{S}^{n-1}$. Thus

$$v_x(z) = \lim_j u_{x_j,r_j}(z_j) = \lim_j \frac{u(y_j)}{r_j^2} = 0, \tag{2.131}$$

but on the other hand there exists a $y \in C^+(x,\varepsilon)$ for which $z = y - x$, so from definition of v_x and $C^+(x,\varepsilon)$, according to (I2) and (I4)

$$\begin{aligned}v_x(z) = v_x(y-x) &= \frac{1}{2}\big(\langle y - x, \mathbb{L}^{-1}(x)n(n)\rangle \vee 0\big)^2\\ &\ge \big(\varepsilon|y-x||\mathbb{L}^{-1}(x)n(n)|\big)^2 \ge \frac{\lambda\, c_0}{2}\,\varepsilon\,|y-x| > 0,\end{aligned} \tag{2.132}$$

that gives a contradiction. Reasoning in the same way it is possible to prove that $C^-(x,\varepsilon) \cap B_\delta(x) \subset \Lambda_u$.

We show now that $\Lambda_u \cap B_{\rho_1}$ is the subgraph of a function g for a suitable constant $\rho_1 > 0$. We fix $x_0 \in Reg(u)$ and indicate by $\nu(x_0) = \frac{\mathbb{L}^{-1}(x_0)n(x_0)}{|\mathbb{L}^{-1}(x_0)n(x_0)|} = \frac{\mathbb{A}^{-\frac{1}{2}}(x)n(x)}{|\mathbb{A}^{-\frac{1}{2}}(x)n(x)|}$ the generating line of cones $C^{\pm}(x,\varepsilon)$. Let $\varphi : \mathbb{R}^{n-1} = \{x_0 + \nu(x_0)^{\perp}\} \to \mathbb{R}$ be a function defined by

$$\varphi(x') := \max\big\{t \in \mathbb{R} \,:\, (x',t) \in \Lambda_u\big\}, \qquad \forall\, x' \in \{x_0 + \nu(x_0)^{\perp}\} \,:\, |x' - x_0| \le \delta\sqrt{1-\varepsilon^2}.$$

We note that according to (4.106) the maximum exists in $[-\varepsilon\delta, \varepsilon\delta]$, and

$$\begin{aligned}(x',t) \in \Lambda_u \quad &\Longrightarrow \quad -\varepsilon\delta \le t \le \varphi(x'),\\ (x',t) \in N_u \quad &\Longrightarrow \quad \varphi(x') < t \le \varepsilon\delta.\end{aligned}$$

Therefore φ is differentiable and its normal vector $\nu(x) = \frac{\mathbb{L}^{-1}(x)n(x)}{|\mathbb{L}^{-1}(x)n(x)|}$ is absolutely continuous, in fact

$$
\begin{aligned}
|\nu(x)-\nu(y)| &= \left|\frac{\mathbb{L}^{-1}(x)n(x)}{|\mathbb{L}^{-1}(x)n(x)|}-\frac{\mathbb{L}^{-1}(y)n(y)}{|\mathbb{L}^{-1}(y)n(y)|}\right| \\
&\le \left|\frac{\mathbb{L}^{-1}(x)n(x)-\mathbb{L}^{-1}(y)n(y)}{|\mathbb{L}^{-1}(x)n(x)|}\right| + |\mathbb{L}^{-1}(y)n(y)| \left|\frac{1}{|\mathbb{L}^{-1}(x)n(x)|}-\frac{1}{|\mathbb{L}^{-1}(y)n(y)|}\right| \\
&\le \frac{C}{\sqrt{c_0\,\lambda}}\sqrt{\rho(|x-y|^{1-\gamma})} + \frac{\big||\mathbb{L}^{-1}(x)n(x)|-|\mathbb{L}^{-1}(y)n(y)|\big|}{|\mathbb{L}^{-1}(x)n(x)|} \le \frac{2\,C}{\sqrt{c_0\,\lambda}}\sqrt{\rho(|x-y|^{1-\gamma})},
\end{aligned}
$$

so $\varphi \in C^1$ and this prove the theorem. □

We are able to say less on the set of singular points. We know that below the hypotheses $(I1)$-$(I4)$, the set $Sing(u)$ is contained in the countable union of C^1 submanifold.

Definition 2.7.2. The singular stratum S_k of dimension k for $k = 0, 1, \dots, n-1$ is the subset of points $x \in Sing(u)$ for which $\mathrm{Ker}(\mathbb{B}_x) = k$.

In the following theorem we show that the set $Sing(u)$ has a stronger regularity property than rectifiabilty: we show that the singular stratum S_k is locally contained in a single submanifold. Moreover that $\cup_{k=l}^{n-1} S_k$ is a closed set for every $l = 0, 1, \dots, n-1$.

Theorem 2.7.3 ([34, Theorem 4.14]). *Assume hypotheses $(I1)$-$(I4)$. Let $x \in S_k$. Then there exists r such that $S_k \cap B_r(x)$ is contained in regular k-dimensional submanifold of $\mathbb{R}^n$.*

Proof. We divide the proof into two steps.

Step 1: The map $Sing(u) \ni x \mapsto \mathbb{L}^{-1}(x)\mathbb{B}_x\mathbb{L}^{-1}(x)$ is continuous.
We proceed as in Theorem 2.7.1 observing that $\|M\| = \int_{\partial B_{\frac{1}{2}}} |\langle My, y\rangle|\, dy$ is a norm on $\mathbb{R}^{n\times n}_{sym}$; thus we obtain

$$\left|\mathbb{L}^{-1}(x)\mathbb{B}_x\mathbb{L}^{-1}(x) - \mathbb{L}^{-1}(z)\mathbb{B}_z\mathbb{L}^{-1}(z)\right| \le C\,\|v_x - v_z\|_{L^1(\partial B_{\frac{1}{2}})}. \tag{2.133}$$

Let us fix a compact set $K \subset Sing(u)$, and let σ_K be the modulus of continuity found in Proposition 2.6.1. Then for all $x, z \in K$ let $s = |x-z|^{1-\gamma} \in (0, r_k)$ and $C > 0$ be a suitable dimensional constant. According to (2.99) and (2.127) we have

$$
\begin{aligned}
\|v_x - v_z\|_{L^1(\partial B_1)} &\le \|v_x - u_{x,s}\|_{L^1(\partial B_1)} + \|u_{x,s} - u_{z,s}\|_{L^1(\partial B_1)} + \|u_{z,s} - v_z\|_{L^1(\partial B_1)} \\
&\le C\left(\sigma_K(|x-z|^{1-\gamma}) + |x-z|^{\gamma}\right.
\end{aligned} \tag{2.134}
$$

According to (2.133) and (2.134) we deduce the continuity.

Step 2: There exists a function $\varphi \in C^2(\mathbb{R}^n)$, extension of the null function on K, such that for all $x \in K$

$$\varphi(y) - v_x(y-x) = o(|y-x|^2) \qquad \text{for } y \to x. \tag{2.135}$$

We prove that the family of translations of the blow-ups $\{v_x(\cdot - x)\}_{x\in K}$ satisfies the hypotheses of Whitney's extension Theorem [98, Theorem 3.5.7]. Precisely we show that the family of polynomials $p_x(y) := v_x(x-y)$ on varying of $x \in K$ satisfies the following conditions:

(i) $p_x(x) = 0$, for all $x \in K \cap S_k$,

(ii) $D^l(p_x - p_z)(x) = o(|x-z|^{2-l})$ for all $x, z \in K \cap S_k$, e $l = 0, 1, 2$.

The condition (i) is trivial. Instead, in order to prove the condition (ii), due to (2.99) and the uniform ellipticity of $\mathbb{L}$, we obtain

$$\|u - p_z\|_{C^0(B_r(x))} \le r^2\,\sigma_K(r), \qquad \|\nabla u - \nabla p_z\|_{C^0(B_r(x))} \le r\,\sigma_K(r) \qquad \forall r \in (0, \widetilde{r}_K), \tag{2.136}$$

with $\widetilde{r}_K$ that depends on r_K and λ. In fact by (2.99), for all $r \in (0, r_K)$

$$\begin{aligned}\sigma_K(r) &\ge \sup_{y\in B_1} \left|u_{\mathbb{L}(z),r}(y) - w_z(y)\right| = \sup_{y\in B_1} |u_{z,r}(\mathbb{L}(z)y) - w_z(y)| \\ &= \sup_{y'\in\mathbb{L}(z)(B_1)} \left|u_{z,r}(y') - v_z(y')\right| \ge \sup_{y'\in B_{\lambda^{-1/2}}} \left|\frac{u_z(ry')}{r^2} - \frac{v_z(ry')}{r^2}\right| \\ &= \frac{\lambda}{r'^2}\sup_{y\in B'_r} |u(z+y) - v_z(y)| = \frac{\lambda}{r'^2}\sup_{y'\in B'_r(z)} \left|u(y') - p_z(y')\right|,\end{aligned}$$

with $r' \in (0, \sqrt{\lambda}\,r_K)$; proceeding in the same way for the gradient, we deduce (2.136). Then, since $u(\underline{0}) = 0$ and $\nabla u(\underline{0}) = \underline{0}$ it holds that:

$$|p_x(x) - p_z(x)| = |u(x) - p_z(x)| \qquad \textit{and} \qquad |\nabla p_x(x) - \nabla p_z(x)| = |\nabla u(x) - \nabla p_z(x)|,$$

that implies the condition (ii) in the case in which $l = 0, 1$. The condition (ii) in the case $l = 2$ is limited to continuity of $\mathbb{L}^{-1}(x)\mathbb{B}_x\mathbb{L}^{-1}(x)$ proved in *Step* 1.

The condition (2.135) proves that $K \subset \{\nabla\varphi = \underline{0}\}$, in fact for all $x \in K$ and $y \to x$

$$\lim_{y\to x}\left|\frac{\varphi(x) - \varphi(y)}{|x-y|} - \underline{0}\right| = \lim_{y\to x}\left|\frac{-v_x(y-x)}{|x-y|} + \frac{o(|y-x|^2)}{|x-y|}\right| = \lim_{y\to x}\left|\frac{-v_x(y-x)}{|x-y|}\right| = 0.$$

Given $x \in K$, since $\mathrm{Rk}(\mathbb{B}_x) = k$, and for Whitney's Theorem $\nabla^2\varphi(x) = \mathbb{L}^{-1}(x)\mathbb{B}_x\mathbb{L}^{-1}(x)$, with a change of variable in $\mathbb{R}^n$ we can reduce to the case in which the first k vector of canonical basis, e_i with $i = 1, \ldots, k$, are the eigenvalue of $\nabla^2\varphi(\underline{0})$. Thus the minor $(n-k)\times(n-k)$ of $\nabla^2\varphi(\underline{0})$ in new basis, composed by the first $(n-k)$ rows and the first $(n-k)$ columns is null. Then according to implicit function Theorem we obtain that $\cap_{i=1}^{n-k}\{\partial_i\varphi = 0\}$ is a C^1 submanifold in a neighborhood of x. We conclude observing that $K \cap S_k \subset \{\nabla^2\varphi = \underline{0}\} \subset \cap_{i=1}^{n-k}\{\partial_i\varphi = 0\}$. $\square$

Chapter 3

The classical obstacle problem for non linear variational energies

3.1 Coercive vector fields

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded and open set. Consider $(a_0, \mathbf{a}) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ a smooth vector field satisfying (cf. [91, Section 4.3.2])

(H1) a_0 is Carathéodory, $\mathbf{a} \in C^{1,1}_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and there is $p \in (1, \infty)$, for which

- (i) $\big(\mathbf{a}(x,z,\xi) \cdot \xi\big) \wedge \big(a_0(x,z,\xi)z\big) \geq \lambda|\xi|^p + \lambda_1|z|^p - \phi_1(x)$ for $\mathcal{L}^n$ a.e. $x \in \Omega$, and for all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, with $\phi_1 \in L^1(\Omega)$, $\lambda > 0$ and $\lambda_1 \geq 0$;
- (ii) $|a_0(x,z,\xi)| \vee |\mathbf{a}(x,z,\xi)| \leq \Lambda(|z|^{p-1} + |\xi|^{p-1}) + \phi_2(x)$ for $\mathcal{L}^n$ a.e. $x \in \Omega$ and for all $(z,\xi) \in \mathbb{R} \times \mathbb{R}^n$, with $\Lambda > 0$ and $\phi_2 \in L^\infty(\Omega)$;
- (iii) there is a constant $\Theta > 0$ such that for all $x \in \Omega$, z, $\zeta \in \mathbb{R}$, and $\xi \in \mathbb{R}^n$
$$|\mathbf{a}(x,z,\xi) - \mathbf{a}(x,\zeta,\xi)| \leq \Theta|z - \zeta|(1 + |\xi|^{p-1});$$

(H2) for $\mathcal{L}^n$ a.e. $x \in \Omega$, and for all $z \in \mathbb{R}$, ξ, $\eta \in \mathbb{R}^n$
$$0 \leq \big(\mathbf{a}(x,z,\xi) - \mathbf{a}(x,z,\eta)\big) \cdot (\xi - \eta), \tag{3.1}$$
with strict inequality sign for $\xi \neq \eta$.

Note that strongly coercive vector fields as defined in [65, Definition 3.1 of Chapter IV] satisfy the assumptions above.

Let ψ and g be given functions in $W^{1,p}(\Omega)$, $p \in (1, \infty)$, with $g \geq \psi$ $\mathcal{L}^n$ a.e. on Ω and set
$$\mathbb{K}_{\psi,g} := \{v \in g + W^{1,p}_0(\Omega) : \ v \geq \psi \quad \mathcal{L}^n \text{ a.e. on } \Omega\}. \tag{3.2}$$

We consider the following variational inequality
$$\int_\Omega \mathbf{a}(x,u,\nabla u) \cdot \nabla(v - u)\, dx + \int_\Omega a_0(x,u,\nabla u)(v-u)dx \geq 0 \qquad \text{for all } v \in \mathbb{K}_{\psi,g}. \tag{3.3}$$

Under conditions (H1)-(H2) and supposing the obstacle ψ and the boundary datum g in $W^{1,p}(\Omega)$ and satisfying the compatibility condition $g \geq \psi$ $\mathcal{L}^n$ a.e. on Ω, the existence of solutions to (3.3) is a consequence of classical results. Indeed, consider the nonlinear operator $\mathscr{A} : W^{1,p}(\Omega) \mapsto W^{1,-p'}(\Omega)$ defined by

$$\langle \mathscr{A}(w), v\rangle := \int_\Omega \big(\widetilde{\mathbf{a}}(x, w, \nabla w) \cdot \nabla v + \widetilde{a}_0(x, w, \nabla w)\, v\big)\, dx \tag{3.4}$$

for $w \in W^{1,p}(\Omega)$ and $v \in W^{1,p}_0(\Omega)$, where for all $(x, z, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$

$$\widetilde{\mathbf{a}}(x, z, \xi) := \mathbf{a}(x, z + g(x), \xi + \nabla g(x)), \qquad \widetilde{a}_0(x, z, \xi) := a_0(x, z + g(x), \xi + \nabla g(x)).$$

Note that $\widetilde{\mathbf{a}}$ and $\widetilde{a}_0$ are Carathéodory functions on account of the regularity of $\mathbf{a}$ and a_0. Then, items (i) and (ii) in (H1) yield that $\mathscr{A}$ is coercive relative to the closed (in the norm topology of $W^{1,p}$) convex subset $\mathbb{K}_{\psi-g,0}$ of $W^{1,p}_0(\Omega)$ given by

$$\mathbb{K}_{\psi-g,0} := \{v \in W^{1,p}_0(\Omega) : \ v \geq \psi - g \quad \mathcal{L}^n \text{ a.e. on } \Omega\}.$$

More precisely, for some $w_0 \in \mathbb{K}_{\psi-g,0}$ (actually for any w_0 in this case)

$$\lim_{w \in W^{1,p}_0(\Omega),\, \|w\|_{W^{1,p}(\Omega)} \to \infty} \|w\|^{-1}_{W^{1,p}(\Omega)} \langle \mathscr{A}(w), w - w_0\rangle = +\infty.$$

Remark 3.1.1. Coercivity is clearly ensured under weaker conditions than those in item (i) of (H1) in view of Sobolev embedding theorems (cf. [56, Theorems 3.7 and 3.8])

In particular, [91, condition (4.26)] is fulfilled for any $w_0 \in \mathbb{K}_{\psi-g,0}$ and for any $R > 0$. Since the injection $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, assumption (H2) gives that $\mathscr{A}$ is a Leray-Lions operator (cf. [91, Theorem 4.21]). Existence of a solution $\widetilde{u} \in \mathbb{K}_{\psi-g,0}$ for

$$\int_\Omega \widetilde{\mathbf{a}}(x, \widetilde{u}, \nabla\widetilde{u}) \cdot \nabla(v - \widetilde{u})\, dx + \int_\Omega \widetilde{a}_0(x, \widetilde{u}, \nabla\widetilde{u})(v - \widetilde{u}) dx \geq 0 \qquad \text{for all } v \in \mathbb{K}_{\psi-g,0}$$

follows at once from [91, Lemma 4.13 and Theorem 4.17]. Therefore, $u := \widetilde{u} + g$ is a solution to (3.3).

Finally, uniqueness of solutions to (3.3) is guaranteed in case the ensuing more stringent monotonicity condition is satisfied

$$0 \leq \big(\mathbf{a}(x, z, \xi) - \mathbf{a}(x, \zeta, \eta)\big) \cdot (\xi - \eta) + \big(a_0(x, z, \xi) - a_0(x, \zeta, \eta)\big)(z - \zeta), \tag{3.5}$$

for $\mathcal{L}^n$ a.e. $x \in \Omega$, for all z, $\zeta \in \mathbb{R}$ and ξ, $\eta \in \mathbb{R}^n$, with strict inequality sign in (3.5) if $\xi \neq \eta$. Disregarding the characterization of the equality case in (3.5), the latter condition yields that the nonlinear operator $\mathscr{A}$ defined in (3.4) is monotone, actually T-monotone (cf. Theorem 1.3.6 in Chapter 1).

In the variational case in which $\mathbf{a} = \nabla_\xi F$ and $a_0 = \partial_z F$, (H2) follows from the convexity of the Lagrangian F in the gradient variable ξ, while (3.5) from the joint convexity of F in (z, ξ).

3.1.1 Regularity of solutions

In what follows we consider variational inequalities as in (3.3) for vector fields $(a_0, \mathbf{a})$ satisfying (H1)-(H2) and further assuming the following conditions on the obstacle function:

(H3) $\psi \in C^{1,1}_{loc}(\Omega)$.

Note then that

$$h := -\mathrm{div}\big(\mathbf{a}(x, \psi, \nabla\psi)\big) + a_0(x, \psi, \nabla\psi) \in L^\infty_{loc}(\Omega). \tag{3.6}$$

The key to establish optimal regularity is contained in Proposition 3.1.2 in which we switch from a variational inequality to a nonlinear elliptic PDE in divergence form. Indeed, on account of Proposition 3.1.2, in Theorem 3.1.4 we establish almost optimal regularity of solutions through classical elliptic regularity results and finally optimal regularity is achieved in Theorem 3.1.6 by means of Gerhardt's approach (cf. [50]).

Despite almost optimal regularity of solutions is a well-studied subject, we provide in Proposition 3.1.2 and Theorem 3.1.4 below a different proof that departs from the classical ones known in literature ([11, 13, 39, 52, 57, 60, 70, 81, 91]) by extending the linearization method to the general setting studied here (cf. [41, 42]). The idea is to reduce regularity for variational inequalities of the sort in (3.3) to the more standard setting of nonlinear elliptic PDEs. In the case of quadratic forms a similar argument has been established in [34], inspired by the case discussed in [95] for the Laplacian (see Theorems 2.7.1 and 2.7.3 in Chapter 2).

Proposition 3.1.2. *Let (H1)-(H3) hold true. Then, a solution $u \in \mathbb{K}_{\psi,g}$ to problem* (3.3) *solves*

$$-\mathrm{div}(\mathbf{a}(x, u, \nabla u)) + a_0(x, u, \nabla u) = \zeta(x) \tag{3.7}$$

$\mathcal{L}^n$ a.e. in Ω and in $\mathcal{D}'(\Omega)$, for some function $\zeta \in L^\infty_{loc}(\Omega)$ such that, for h defined in (3.6)*,*

$$0 \le \zeta \le h^+ \chi_{\{u=\psi\}} \quad \mathcal{L}^n \text{ a.e. in } \Omega.$$

Proof. Let $\varphi \in C^\infty_c(\Omega)$ and for all $\varepsilon > 0$ take $v_\varepsilon := (u + \varepsilon\varphi) \vee \psi \in \mathbb{K}_{\psi,g}$ as test function in (3.3). Note that in case φ is a non-negative function we obtain

$$\int_\Omega \mathbf{a}(x, u, \nabla u) \cdot \nabla\varphi \, dx + \int_\Omega a_0(x, u, \nabla u)\varphi \, dx \ge 0. \tag{3.8}$$

Therefore, the distributional divergence $\mathrm{div}(\mathbf{a}(\cdot, u, \nabla u))$ of $\mathbf{a}(\cdot, u, \nabla u)$ satisfies

$$\langle -\mathrm{div}(\mathbf{a}(\cdot, u, \nabla u)) + a_0(\cdot, u, \nabla u)\mathcal{L}^n \llcorner \Omega, \varphi\rangle \ge 0 \qquad \text{for all } \varphi \in C^\infty_c(\Omega),\ \varphi \ge 0,$$

in turn implying that $\mu := -\mathrm{div}(\mathbf{a}(\cdot, u, \nabla u)) + a_0(\cdot, u, \nabla u)\mathcal{L}^n \llcorner \Omega$ is a non-negative Radon measure.

Next, consider v_ε as above with no sign assumptions on φ, set $\Omega_\varepsilon := \{u + \varepsilon\varphi < \psi\}$, and rewrite the two addends in (3.3) respectively as follows

$$\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla(v_\varepsilon - u)dx = \varepsilon\int_\Omega \mathbf{a}(x,u,\nabla u)\cdot\nabla\varphi\,dx + \int_{\Omega_\varepsilon} \mathbf{a}(x,u,\nabla u)\cdot\nabla\big(\psi-(u+\varepsilon\varphi)\big)dx,$$

and

$$\int_\Omega a_0(x,u,\nabla u)(v_\varepsilon - u)dx = \varepsilon\int_\Omega a_0(x,u,\nabla u)\varphi\,dx + \int_{\Omega_\varepsilon} a_0(x,u,\nabla u)\big(\psi-(u+\varepsilon\varphi)\big)dx.$$

Thus, on account of the definition of the measure μ we conclude that

$$\varepsilon\int_\Omega \varphi\,d\mu \geq -\int_{\Omega_\varepsilon} \mathbf{a}(x,u,\nabla u)\cdot\nabla\big(\psi-(u+\varepsilon\varphi)\big)dx - \int_{\Omega_\varepsilon} a_0(x,u,\nabla u)\,\big(\psi-(u+\varepsilon\varphi)\big)dx.$$

By the monotonicity hypothesis on the field $\mathbf{a}$ in (H2) we have that

$$\begin{aligned}\varepsilon\int_\Omega \varphi\,d\mu \geq &-\int_{\Omega_\varepsilon} \mathbf{a}(x,u,\nabla\psi)\cdot\nabla\big(\psi-u\big)dx\\ &\qquad + \varepsilon\int_{\Omega_\varepsilon} \mathbf{a}(x,u,\nabla u)\cdot\nabla\varphi\,dx - \int_{\Omega_\varepsilon} a_0(x,u,\nabla u)\big(\psi-(u+\varepsilon\varphi)\big)dx\end{aligned}$$

and therefore we infer that

$$\begin{aligned}\varepsilon\int_\Omega \varphi\,d\mu \geq &\underbrace{-\int_{\Omega_\varepsilon}\Big(\mathbf{a}(x,\psi,\nabla\psi)\cdot\nabla\big(\psi-(u+\varepsilon\varphi)\big) + a_0(x,\psi,\nabla\psi)\big(\psi-(u+\varepsilon\varphi)\big)\Big)dx}_{=:I_\varepsilon^{(1)}}\\ &\underbrace{+\,\varepsilon\int_{\Omega_\varepsilon}\big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,\psi,\nabla\psi)\big)\cdot\nabla\varphi\,dx + \varepsilon\int_{\Omega_\varepsilon}\big(a_0(x,u,\nabla u)-a_0(x,\psi,\nabla\psi)\big)\varphi\,dx}_{=:I_\varepsilon^{(2)}}\\ &\underbrace{+\int_{\Omega_\varepsilon}\big(\mathbf{a}(x,\psi,\nabla\psi)-\mathbf{a}(x,u,\nabla\psi)\big)\cdot\nabla\big(\psi-u\big)dx + \int_{\Omega_\varepsilon}\big(a_0(x,\psi,\nabla\psi)-a_0(x,u,\nabla u)\big)\big(\psi-u\big)\big)dx.}_{=:I_\varepsilon^{(3)}}\end{aligned} \tag{3.9}$$

We deal with the three terms above separately. We start off with the first term that we rewrite as

$$I_\varepsilon^{(1)} = -\int_\Omega \Big(\mathbf{a}(x,\psi,\nabla\psi)\cdot\nabla\big((\psi-(u+\varepsilon\varphi))\vee 0\big) + a_0(x,\psi,\nabla\psi)\big((\psi-(u+\varepsilon\varphi))\vee 0\big)\Big)dx.$$

Being $u \geq \psi$ $\mathcal{L}^n$ a.e. in Ω and $\varphi \in C_c^\infty(\Omega)$, we have $\Omega_\varepsilon \subset\subset \Omega$, so that $(\psi-(u+\varepsilon\varphi))\vee 0 \in W_0^{1,p}(\Omega)$. By taking this into account, together with the condition $\psi \in C^{1,1}_{loc}(\Omega)$ (cf. (H3)), item (ii) in (H1) and an integration by parts yield, recalling that $h = -\mathrm{div}\big(\mathbf{a}(x,\psi,\nabla\psi)\big) + a_0(x,\psi,\nabla\psi)$,

$$
\begin{aligned}
I_\varepsilon^{(1)} &= \int_\Omega \big(\mathrm{div}(\mathbf{a}(x,\psi,\nabla\psi)) - a_0(x,\psi,\nabla\psi)\big)\big((\psi-(u+\varepsilon\varphi))\vee 0\big)dx \\
&= -\int_{\Omega_\varepsilon} h\,\big((\psi-(u+\varepsilon\varphi)\big)\,dx \geq -\int_{\Omega_\varepsilon} h^+\,\big(\psi-(u+\varepsilon\varphi)\big)\,dx \geq \varepsilon\int_{\Omega_\varepsilon} h^+\,\varphi\,dx \qquad (3.10)
\end{aligned}
$$

where in the last but one equality we have used that $\psi-(u+\varepsilon\varphi)\geq 0$ $\mathcal{L}^n$ a.e. on Ω_ε and in the last one that $u \geq \psi$ $\mathcal{L}^n$ a.e. on Ω. In turn, the latter condition implies that

$$
\mathcal{L}^n\big((\{u=\psi\}\cap\{\varphi<0\})\setminus\Omega_\varepsilon\big) = \mathcal{L}^n\big(\Omega_\varepsilon\setminus\big(\{0\leq u-\psi\leq\varepsilon\|\varphi\|_{L^\infty(\Omega)}\}\cap\{\varphi<0\}\big)\big) = 0,
$$

so that $\chi_{\Omega_\varepsilon}\to\chi_{\{u=\psi\}\cap\{\varphi<0\}}$ in $L^1(\Omega)$, for every $\varphi\in C_c^\infty(\Omega)$. Therefore, from (3.10) we infer

$$
\liminf_{\varepsilon\to0^+}\varepsilon^{-1}I_\varepsilon^{(1)} \geq \int_{\{u=\psi\}\cap\{\varphi<0\}} h^+\,\varphi\,dx. \qquad (3.11)
$$

In addition, by the Dominated convergence theorem and by the locality of the weak gradient, we conclude that for every $\varphi\in C_c^\infty(\Omega)$

$$
\begin{aligned}
\lim_{\varepsilon\to0^+}\varepsilon^{-1}I_\varepsilon^{(2)} = &\int_{\{u=\psi\}\cap\{\varphi<0\}} \big(\mathbf{a}(x,u,\nabla u)-\mathbf{a}(x,\psi,\nabla\psi)\big)\cdot\nabla\varphi\,dx \\
&+\int_{\{u=\psi\}\cap\{\varphi<0\}} \big(a_0(x,u,\nabla u)-a_0(x,\psi,\nabla\psi)\big)\varphi\,dx = 0. \qquad (3.12)
\end{aligned}
$$

Finally, to deal with $I_\varepsilon^{(3)}$ we use item (iii) in (H1) to deduce that

$$
\begin{aligned}
I_\varepsilon^{(3)} \geq &-\varepsilon\Theta\|\varphi\|_{L^\infty(\Omega)}\int_{\Omega_\varepsilon}(1+|\nabla\psi|^{p-1})|\nabla(\psi-u)|\,dx \\
&-\varepsilon\|\varphi\|_{L^\infty(\Omega)}\int_{\Omega_\varepsilon}|a_0(x,u,\nabla u)-a_0(x,\psi,\nabla\psi)|\,dx.
\end{aligned}
$$

Therefore, by the quoted convergence of $\chi_{\Omega_\varepsilon}$ and by the locality of the weak gradient, as in (3.11) and (3.12), we conclude that

$$
\liminf_{\varepsilon\to0^+}\varepsilon^{-1}I_\varepsilon^{(3)} \geq 0. \qquad (3.13)
$$

Resuming, by (3.11), (3.12) and (3.13), passing to the limit as $\varepsilon\downarrow0^+$ in (3.9) divided by $\varepsilon>0$, we infer that

$$
\int_\Omega\varphi\,d\mu \geq \int_{\{u=\psi\}\cap\{\varphi<0\}} h^+\,\varphi\,dx.
$$

By approximation (and by applying the argument above to $-\varphi$) we infer that for every $\varphi\in C_c^0(\Omega)$

$$
\int_{\{u=\psi\}\cap\{\varphi<0\}} h^+\,\varphi\,dx \leq \int_\Omega\varphi\,d\mu \leq \int_{\{u=\psi\}\cap\{\varphi>0\}} h^+\,\varphi\,dx.
$$

In turn, the latter inequalities imply that $\mu \ll \mathcal{L}^n \llcorner \Omega$. Thus, if $\mu = \zeta \mathcal{L}^n \llcorner \Omega$, with $\zeta \in L^1(\Omega)$, we infer that $0 \le \zeta \le h^+ \chi_{\{u=\psi\}}\, \mathcal{L}^n$ a.e. Ω, so that $\zeta \in L^\infty_{loc}(\Omega)$ by (3.6).

In conclusion, as by definition $\mu = -\mathrm{div}(\mathbf{a}(\cdot, u, \nabla u)) + a_0(\cdot, u, \nabla u)\mathcal{L}^n \llcorner \Omega$, equation (3.7) follows at once. □

Remark 3.1.3. One can prove that a solution u of (3.3) is a Q-minimum of a lower order perturbation of the p-Dirichlet energy from the conclusions of Proposition 3.1.2 as argued in [54] (cf. also [56, Chapter 6]). More precisely, let $\mathscr{G} : \mathcal{B}(\Omega) \times W^{1,p}(\Omega) \to [0, \infty)$ be

$$\mathscr{G}(w, A) := \int_A G\big(x, w(x), \nabla w(x)\big)\, dx,$$

where $A \in \mathcal{B}(\Omega)$, the class of Borel subsets of Ω, and $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ is the Carathéodory integrand defined by

$$G(x, z, \xi) := |\xi|^p + |z|^p + |\nabla\psi(x)|^p + |\phi_2(x)|^{\frac{p}{p-1}} + |\phi_1(x)| + |a_0(x, u(x), \nabla u(x))|^{\frac{p}{p-1}}.$$

Then, there is a constant $Q = Q(p, \lambda, \Lambda) > 1$ such that

$$\mathscr{G}(u, K) \le Q\, \mathscr{G}(w, K) \tag{3.14}$$

for all $w \in g + W^{1,p}_0(\Omega)$ such that $K := \mathrm{spt}(w - u) \subset\subset \Omega$. Note that $|a_0(\cdot, u(\cdot), \nabla u(\cdot))|^{\frac{p}{p-1}} \in L^1(\Omega)$ by item (ii) in (H1). The direct methods for regularity introduced by Giaquinta and Giusti [53, 54] imply that $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1]$ under suitable assumptions on ϕ_1, ϕ_2, a_0 and p (cf. [42] and Section 1.4 in Chapter 1).

Actually, we can establish (3.14) a priori, directly from (3.3) by taking the family of test functions $v = w \vee \psi$ with w as above by means of items (i) and (ii) in (H1).

Finally, we recall that under the standing assumptions on $(\mathbf{a}, a_0)$ upper semicontinuity and approximate continuity of ψ suffice to establish continuity of solutions (cf. [76]). In particular, this shows that the sets $\{u > \psi\}$ and Ω_ε, $\varepsilon > 0$ suitable, in the proof of Proposition 3.1.2 are actually open.

We are now ready to deduce almost optimal regularity for solutions to (3.3) from standard elliptic regularity provided item (iii) in (H1) and (H2) are substituted by the more restrictive

(iii)′ there is a constant $\Theta > 0$ such that for all x, $y \in \Omega$, z, $\zeta \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$|\mathbf{a}(x, z, \xi) - \mathbf{a}(y, \zeta, \xi)| \le \Theta(|x - y| + |z - \zeta|)(1 + |\xi|^{p-1})$$

(H2)′ there is $\nu > 0$ such that for $\mathcal{L}^n$ a.e. $x \in \Omega$, and for all $z \in \mathbb{R}$, ξ, $\eta \in \mathbb{R}^n$

$$\nu^{-1}(1+|\xi|+|\eta|)^{p-2}\, |\xi-\eta|^2 \le \big(\mathbf{a}(x, z, \xi)-\mathbf{a}(x, z, \eta)\big)\cdot(\xi-\eta) \le \nu(1+|\xi|+|\eta|)^{p-2}\, |\xi-\eta|^2; \tag{3.15}$$

On account of (3.7) in Proposition 3.1.2 suboptimal regularity follows.

Theorem 3.1.4 (Almost optimal regularity). *Let (H1) (with (iii)$'$ in place of (iii)), (H2)$'$ and (H3) hold true. Let $u \in \mathbb{K}_{\psi,g}$ be a solution to problem* (3.3)*, then $u \in W^{2,q}_{loc} \cap C^{1,\alpha}_{loc}(\Omega)$ for all $q \in [1, \infty)$ and all $\alpha \in (0, 1)$.*

Proof. By taking into account that u solves (3.7) (cf. Proposition 3.1.2), classical elliptic regularity for nonlinear elliptic equations in divergence form yield that $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$ (cf. [73, Section 3], [74, Chapter 5]).

It is also classical to prove that $u \in W^{2,2}_{loc}(\Omega)$ (cf. [67, Chapter 4, Theorem 5.2]) and by differentiation, on account of the $C^{1,\alpha}_{loc}$ regularity already established and (H1)-(H2)$'$, that the weak derivatives of u satisfy a linear uniformly elliptic PDE with Hölder coefficients and right hand side being the divergence of a field in $L^{\infty}_{loc}(\Omega, \mathbb{R}^n)$. Therefore, we may apply standard L^q-regularity estimates (cf. [56, Theorem 10.15]) to conclude that $u \in W^{2,q}_{loc} \cap C^{1,\alpha}_{loc}(\Omega)$ for all $q \in [1, \infty)$ and all $\alpha \in (0, 1)$. □

Corollary 3.1.5. *Under the assumptions of Theorem 3.1.4 the function ζ in* (3.7) *of Proposition 3.1.2 actually equals $h^+\chi_{\{u=\psi\}}$ $\mathcal{L}^n$ a.e. on Ω.*

Proof. By the $W^{2,q}$ regularity of u and the $C^{1,1}_{loc}$ regularity of $\mathbf{a}$, one can compute the divergence in the definition of the measure μ and use the locality of weak derivatives to conclude. □

Optimal $C^{1,1}_{loc}$ regularity of solutions follows at once from Gerhardt's result [50] provided a_0 is locally Lipschitz continuous.

Theorem 3.1.6 (Optimal regularity). *Let (H1) (with (iii)$'$ in place of (iii)), (H2)$'$ and (H3) hold true, and assume $g \in C^2(\overline{\Omega})$ with $\psi < g$ on $\partial\Omega$, and $a_0 \in C^{0,1}_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$.*

If $u \in \mathbb{K}_{\psi,g}$ is a solution to problem (3.3)*, then $u \in C^{1,1}_{loc}(\Omega)$.*

Proof. The proof is essentially that of [50] despite the forcing term, i.e. $a_0(\cdot, u, \nabla u)$ in our case, is not in $C^{0,1}$ as required in the statement there. Nevertheless, a careful inspection of that proof shows that the slightly weaker assumption $a_0(\cdot, u, \nabla u) \in W^{1,q}_{loc}(\Omega)$ for all $q \in [1, \infty)$ actually suffices (cf. formula (16) there). In our setting this property is an immediate outcome of the regularity hypothesis on a_0 and Theorem 3.1.4 above. □

Remark 3.1.7. We point out that for $p \neq 2$ the study of degenerate fields $\mathbf{a}$ deserves additional efforts. Optimal regularity of solutions to (3.3) with $\mathbf{a}(\xi) = |\xi|^{p-2}\xi$ and $a_0(x, z) = f(x)z$, $f \in L^{\infty}(\Omega)$, has been established only recently in [2] (cf. the bibliography there for more detailed references, and also the results in [41]). That paper also deals with the case $\psi \in C^{1,\beta}(\Omega)$, $\beta \in (0, 1)$, that is not covered by our methods. More precisely, it is established there that solutions are $C^{1,\beta\wedge 1/(p-1)}_{loc}(\Omega)$, $\beta \in (0, 1]$, and actually $C^{1,\beta}_{loc}$ in the homogeneous setting $f \equiv 0$.

Building upon Proposition 3.1.2 and a careful analysis of the estimates in [74, Chapter 5] one can actually show that $u \in C^{1,\alpha}_{loc}(\Omega)$, for all $\alpha \in (0, \frac{1}{p-1}] \cap (0, 1)$ for fields satisfying (H1) and the degenerate analogue of (H2)$'$.

We end this subsection pointing out that the conclusions of Proposition 3.1.2 and Theorems 3.1.4 and 3.1.6 extend to the more general setting of fields a_0 satisfying the so called *unnatural* growth conditions following the terminology of Giusti [56] (cf. formula (6.15) there), of which item (ii) in (H1) is a simple instance.

This claim is also true in case a_0 satisfies the *natural* growth conditions (cf. [56, formula (6.18)]) provided bounded solutions are taken into account. Existence of such solutions is guaranteed for bounded obstacles and bounded boundary data, for instance.

3.1.2 Free boundary regularity in the variational case

We are now ready to state and prove the main result of the paper. From now on we restrict to the variational case, in which $\mathbf{a} = \nabla_\xi F$ and $a_0 = \partial_z F$ for suitable integrands F. In this framework the problem (3.3) is equivalent to

$$\min_{v \in \mathbb{K}_{\psi,g}} \int_\Omega F(x, v, \nabla v)\, dx. \tag{3.16}$$

We need to rephrase assumptions (H1), and (H2)$'$ terms of the energy density F itself. In passing we note that item (i) in (H1) is not needed provided F satisfies suitable convexity and growth conditions in view of the Direct Method of the Calculus of Variations. Indeed, item (i) in (H1) has been used only in the proof of existence of solutions to (3.3).

Theorem 3.1.8. *Let $\Omega \subset \mathbb{R}^n$ be smooth, bounded and open, and $p \in (1, \infty)$. Assume (H3) for ψ, and $g \in C^2(\overline{\Omega})$ with $\psi < g$ on $\partial\Omega$.*

Let $F \in C^{2,1}_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be satisfying

$$c_1|\xi|^p - \phi(x) \le F(x, z, \xi) \le c_2|\xi|^p + c_3|z|^{p^*} + \phi(x) \tag{3.17}$$

for all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, for $\mathcal{L}^n$ a.e. $x \in \Omega$, where $\phi \in L^1(\Omega)$, c_1, $c_2 > 0$ and $c_3 \ge 0$, and p^* is the Sobolev exponent of p (thus p^* is any exponent if $p \ge n$).

Suppose that items (ii), (iii)$'$ in (H1) are satisfied by $\mathbf{a} = \nabla_\xi F$ and $a_0 = \partial_z F$, and in addition assume $F(x, z, \cdot)$ to be uniformly convex in (x, z) w.r.to ξ, i.e. there exists $\nu > 1$ such that for all $x \in \Omega$, $z \in \mathbb{R}$ and ξ, $\eta \in \mathbb{R}^n$

$$\nu^{-1}(1 + |\eta|)^{p-2}|\xi|^2 \le \nabla^2_\xi F(x, z, \eta)\xi \cdot \xi \le \nu(1 + |\eta|)^{p-2}|\xi|^2. \tag{3.18}$$

Then, the minimum problem in (3.16) *has (at least) a solution $u \in \mathbb{K}_{\psi,g}$, and, moreover, every solution belongs to $C^{1,1}_{loc}(\Omega)$.*

Let $u \in \mathbb{K}_{\psi,g}$ be a solution. If, moreover, ψ satisfies

(H4) for some constant $c_0 > 0$ we have for $\mathcal{L}^n$ a.e. on Ω

$$h = -\operatorname{div}\big(\nabla_\xi F(x, \psi, \nabla\psi)\big) + \partial_z F(x, \psi, \nabla\psi) \ge c_0 > 0;$$

(H5) for some $\alpha \in (0, 1)$

$$\operatorname{div}\big(\nabla_\xi F(\cdot, u, \nabla\psi)\big) \in C^{0,\alpha}_{loc}(\Omega),$$

then the free boundary decomposes as $\partial\{u = \psi\} \cap \Omega = \mathrm{Reg}(u) \cup \mathrm{Sing}(u)$*, where* $\mathrm{Reg}(u)$ *and* $\mathrm{Sing}(u)$ *are called its regular and singular part, respectively. Moreover,* $\mathrm{Reg}(u) \cap \mathrm{Sing}(u) = \emptyset$ *and*

(i) $\mathrm{Reg}(u)$ *is relatively open in* $\partial\{u = \psi\}$ *and, for every point* $x_0 \in \mathrm{Reg}(u)$*, there exist* $r = r(x_0) > 0$ *and* $\beta = \beta(x_0) \in (0,1)$ *such that* $\mathrm{Reg}(u) \cap B_r(x_0)$ *is a* $C^{1,\beta}$ *submanifold of dimension* $n-1$*;*

(ii) $\mathrm{Sing}(u) = \cup_{k=0}^{n-1} S_k$*, with* S_k *contained in the union of at most countably many submanifolds of dimension* k *and class* C^1*.*

Remark 3.1.9. In case $F = F(x, \xi)$ the structural conditions imposed on F, i.e. convexity and (3.17), imply item (ii) in (H1) (cf. [56, Lemma 5.2]). Therefore, besides uniform convexity, the only nontrivial assumption on F is (iii)$'$ in (H1). In turn, the latter is clearly satisfied in the autonomous case $F = F(\xi)$.

Remark 3.1.10. Assumption (H4) corresponds to the well-known concavity assumption on the obstacle function ψ in the case of the Laplacian, or better to the localized form of such a condition introduced in [19]. Simple examples show that (H4) is a necessary request to expect regular free boundaries.

Remark 3.1.11. In view of the regularity assumptions on F and the optimal regularity of u, assumption (H5) is basically a hypothesis on the obstacle ψ that can be enforced by assuming more regularity on ψ itself. For instance, it is implied by taking $\psi \in C^{2,\alpha}_{loc}(\Omega)$.

Finally, non trivial examples show that a qualified continuity hypothesis on the relevant operator calculated on the obstacle function, weaker than Hölder continuity imposed in (H5), is actually necessary to conclude free boundary regularity already in the classical case of the Laplacian (cf. [6, 77]).

To establish Theorem 3.1.8 we introduce the ensuing linearization; in this way we rewrite the PDE in (3.7) as a locally uniform elliptic equation with suitable locally Lipschitz continuous matrix coefficients in case the gradient of the solution itself shares such a regularity.

Lemma 3.1.12. *Let (H1)-(H4) hold true, and let* $u \in C^{1,1}_{loc}(\Omega)$ *be a solution of* (3.16)*. Then, there exists a symmetric matrix field* $\mathbb{A} : \Omega \to \mathbb{R}^{n\times n}$ *such that*

$$\mathrm{div}\big(\mathbb{A}(x)\nabla(u-\psi)\big) = \big(-\mathrm{div}(\nabla_\xi F(x,u,\nabla\psi)) + \partial_z F(x,u,\nabla u)\big)\,\chi_{\{u>\psi\}} \tag{3.19}$$

$\mathcal{L}^n$ *a.e. in* Ω *and in* $\mathcal{D}'(\Omega)$*; with* $\mathbb{A}$ *satisfying*

(i) $\mathbb{A} \in C^{0,1}_{loc}(\Omega, \mathbb{R}^{n\times n})$,

(ii) for all $K \subset\subset \Omega$ *there is* $\lambda_K \geq 1$ *for which*

$$\lambda_K^{-1}|\xi|^2 \leq \mathbb{A}(x)\xi\cdot\xi \leq \lambda_K|\xi|^2 \quad \text{for all } x \in K \text{ and for all } \xi \in \mathbb{R}^n. \tag{3.20}$$

Proof. We start off rewriting the Euler-Lagrange equation (3.7) as follows

$$\operatorname{div}\big(\nabla_\xi F(x,u,\nabla u)-\nabla_\xi F(x,u,\nabla\psi)\big)=\big(-\operatorname{div}(\nabla_\xi F(x,u,\nabla\psi))+\partial_z F(x,u,\nabla u)\big)\,\chi_{\{u>\psi\}}. \tag{3.21}$$

In claiming the last equality we have used Corollary 3.1.5, assumption (H4) and the inclusion

$$\{u=\psi\}\subseteq\{\nabla u=\nabla\psi\},$$

consequence of the unilateral obstacle condition $u\geq\psi$ on Ω and the regularity of both u and ψ. Then set $w:=u-\psi$, and note that for all x in Ω

$$\begin{aligned}\nabla_\xi F(x,u(x),&\nabla u(x))-\nabla_\xi F(x,u(x),\nabla\psi(x))\\ &=\nabla_\xi F(x,u(x),\nabla w(x)+\nabla\psi(x))-\nabla_\xi F(x,u(x),\nabla\psi(x))\\ &=\Big(\int_0^1\nabla_\xi^2 F\big(x,u(x),\nabla\psi(x)+t\nabla w(x)\big)dt\Big)\nabla w(x)=:\mathbb{A}(x)\nabla w(x).\end{aligned} \tag{3.22}$$

From (3.21) and (3.22), we conclude that w satisfies (3.19). Moreover, being $u\,,\psi\in C^{1,1}_{loc}(\Omega)$ and $F\in C^{2,1}_{loc}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$, we deduce that item (i) in the statement is satisfied, as well. Moreover, for all $x\in\Omega$ and for all $\xi\in K$, $K\subset\mathbb{R}^n$ a compact set, we have

$$\begin{aligned}\nu^{-1}(2^{p-2}\wedge 1)|\xi|^2&\int_0^1\big(1+|\nabla\psi(x)+t\nabla w(x)|\big)^{p-2}\,dt\leq\mathbb{A}(x)\xi\cdot\xi\\ &=\int_0^1\nabla_\xi^2 F\big(x,u(x),\nabla\psi(x)+t\nabla w(x)\big)\xi\cdot\xi\,dt\leq\|\nabla_\xi^2 F\|_{L^\infty(K\times B_{r_K}\times B_{r_K},\mathbb{R}^{n\times n})}|\xi|^2,\end{aligned}$$

with $r_K:=\sup_K(|u|+|\nabla\psi|+|\nabla w|)$. The inequality on the left hand side above is an easy consequence of the coercivity condition in (3.15). Ellipticity then easily follows if $p\geq 2$, for $p\in(1,2)$ instead we use that $u\,,\psi\in C^{1,1}_{loc}(\Omega)$. Finally, the upper bound in (3.20) follows easily in both cases. The conclusion then follows. □

We are ready to prove Theorem 3.1.8 as a direct consequence of Lemma 3.1.12, Theorems 2.7.1 and 2.7.3 in Chapter 2.

Proof of Theorem 3.1.8. Existence of solutions to (3.16) follows from [56, Theorem 4.5] thanks to the convexity of $\xi\mapsto F(x,z,\xi)$ and the growth conditions (3.17). The former guarantees lower semicontinuity of the associated functional in the weak $W^{1,p}$ topology, the latter ensures its coercivity over $\mathbb{K}_{\psi,g}$. Therefore, the Direct Method of the Calculus of Variations applies.

Moreover, any minimizer u is $C^{1,1}_{loc}(\Omega)$. To this aim, it suffices to note that u satisfies the PDE in (3.7), since the derivation of the latter is independent from item (i) in (H1). Note that assumption (H2)$'$ corresponds to (3.18).

Hence, in view of Lemma 3.1.12, to conclude the free boundary analysis we only need to check that, locally in Ω, we may apply Theorems 2.7.1 and 2.7.3with matrix field $\mathbb{A}$ as above, with

$$f:=-\operatorname{div}(\nabla_\xi F(x,u,\nabla\psi))+\partial_z F(x,u,\nabla u),$$

with 0 obstacle and with boundary datum $g-\psi$. Indeed, thanks to (3.19), $w=u-\psi$ is the minimizer of the quadratic energy

$$\mathscr{E}[v]=\int_\Omega \big(\mathbb{A}(x)\nabla v(x)\cdot\nabla v(x)+2f(x)\,v(x)\big)\,dx$$

over $\mathbb{K}_{g-\psi,0}$. In addition, note that $\partial\{w=0\}\cap\Omega=\partial\{u=\psi\}\cap\Omega$.

With the aim of applying Theorems 2.7.1 and 2.7.3 we first recall that $\{u=\psi\}\subseteq\{\nabla u=\nabla\psi\}$, being $u\geq\psi$ on Ω. Thus, given $\Omega'\Subset\Omega$ and any $\varepsilon>0$, the set $\Omega'_\varepsilon:=\{0\leq u-\psi<\varepsilon\}\cap\{|\nabla(u-\psi)|<\varepsilon\}\cap\Omega'$ is open and such that $\{u=\psi\}\cap\Omega'\subset\Omega'_\varepsilon$ in view of the remark above. Moreover, as $h=-\mathrm{div}(\nabla_\xi F(x,\psi,\nabla\psi))+\partial_z F(x,\psi,\nabla\psi)\geq c_0>0$ (cf. (H4)), we have on Ω'_ε

$$\begin{aligned}
&f\geq h-\|h-f\|_{L^\infty(\Omega'_\varepsilon)}\\
&\geq c_0-\|\partial_z F(\cdot,\psi,\nabla\psi)-\partial_z F(\cdot,u,\nabla u)\|_{L^\infty(\Omega'_\varepsilon)}-\|\mathrm{div}(\nabla_\xi F(\cdot,\psi,\nabla\psi))-\mathrm{div}(\nabla_\xi F(\cdot,u,\nabla\psi))\|_{L^\infty(\Omega'_\varepsilon)}\\
&\qquad\geq c_0-\omega_{\partial_z F}(2\varepsilon)-\omega_{\nabla^2_{x,\xi}F}(\varepsilon)-\|\nabla u\|_{L^\infty(\Omega'_\varepsilon,\mathbb{R}^n)}\omega_{\nabla^2_{z,\xi}F}(\varepsilon)\\
&\qquad\qquad-\varepsilon\|\nabla^2_{z,\xi}F(\cdot,\psi,\nabla\psi)\|_{L^\infty(\Omega'_\varepsilon)}-\|\nabla^2\psi\|_{L^\infty(\Omega'_\varepsilon,\mathbb{R}^{n\times n})}\omega_{\nabla^2_\xi F}(\varepsilon),
\end{aligned}$$

denoting with ω_ϑ a modulus of continuity of the relevant function ϑ on Ω' (recall that $F\in C^{2,1}_{loc}$). Therefore, we can choose $\varepsilon>0$ sufficiently small in order to accomplish the condition $f\geq {c_0}/2>0$ on Ω'_ε. In addition, $f\in C^{0,\alpha}_{loc}(\Omega)$ by hypotheses (H3), (H5) and by Theorem 3.1.4. Hence, all the conditions in the statement of Theorems 2.7.1 and 2.7.3 are satisfied on the open set Ω'_ε, thus the conclusions follow straightforwardly. $\square$

3.2 Locally coercive vector fields

The analysis in Section 3.1 does not cover many cases of interest, most relevantly that of the area functional where

$$F(\xi)=\sqrt{1+|\xi|^2},\qquad \mathbf{a}(\xi)=\nabla F(\xi)=\frac{\xi}{\sqrt{1+|\xi|^2}}.$$

The latter vector field clearly does not fulfill (3.15) in (H2)$'$ being F strictly but not uniformly convex. Moreover, for such a vector field also the existence of solutions to the corresponding variational inequality is not guaranteed in general and requires additional conditions on the set Ω, on the obstacle ψ and on the boundary datum g (cf. [65, Section 4 of Chapter IV]), [56, Chapter 1] and the references therein). The same considerations hold more generally for *locally coercive* vector fields $\mathbf{a}$ (cf. [65, Section 4 of Chapter IV] in the autonomous case and Theorem 3.2.1 below).

Assuming a priori the existence of a solution and its global Lipschitz continuity, the next result due to Gerhardt implies its global $C^{1,1}$ regularity.

Theorem 3.2.1 (Theorem 0.1 [51])**.** *Let Ω be of class $C^{3,\alpha}$, for some $\alpha\in(0,1)$, $g\in C^{2,1}(\overline{\Omega})$ and $\psi\in C^{1,1}(\overline{\Omega})$. Let $a_0\in C^{1,1}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$, and assume that $\mathbf{a}(\cdot,\cdot,\xi)$ is*

$C^{1,1}(\Omega \times \mathbb{R}, \mathbb{R}^n)$ *for all* $\xi \in \mathbb{R}^n$, *that* $\mathbf{a}(x, z, \cdot)$ *is* $C^{2,1}(\mathbb{R}^n, \mathbb{R}^n)$ *for all* $(x, z) \in \Omega \times \mathbb{R}$, *and that for all* $(x, z, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$

$$\partial_\xi \mathbf{a}(x, z, \eta)\xi \cdot \xi > 0 \qquad \text{for all } \xi \neq 0.$$

If $u \in C^{0,1}(\Omega)$ *is a solution of the variational inequality in* (3.3) *over the set*

$$\{v \in C^{0,1}(\Omega) : \, v \geq \psi \ \text{ on } \Omega, \quad v = g \ \text{ on } \partial\Omega\}$$

then $u \in C^{1,1}(\Omega)$.

Therefore, with Theorem 3.2.1 at hand, if a locally coercive vector field corresponds to an integrand F satisfying hypothesis (H5) of Theorem 3.1.8 we can argue as in Lemma 3.1.12 and in the second part of the proof of Theorem 3.1.8 itself to conclude the same stratification result for the free boundary of a solution u. Note that, in particular, this claim holds for the area functional in the Euclidean space (cf. [65, Section 5 of Chapter V] for the two dimensional case, and [14]).

3.2.1 The area functional in a Riemannian manifold

Similarly, we would like to discuss here the case of the obstacle problem for the area functional in a Riemannian manifold, that naturally enters several geometric applications (cf., *e.g.*, [79]). Indeed, to the best of our knowledge a comprehensive stratification result of the free boundary points in this case has not appeared elsewhere. Since we aim here at a local regularity result, we assume that

(M1) our manifold is parametrized by a single chart $\Sigma := B^n_{r_0} \times (-r_0, r_0) \subset \mathbb{R}^n \times \mathbb{R}$, for some $r_0 > 0$;

(M2) the metric tensor g satisfies $g(0) = I$ and $\nabla g(0) = 0$ (where ∇ denotes the Levi-Civita connection);

(M3) the obstacle $\psi \in C^{1,1}(B^n_{r_0}, (-r_0, r_0))$ with $\psi(0) = |\nabla\psi(0)| = 0$;

We consider the following obstacle problem:

$$\min_{v \in \mathbb{K}_{\psi,g}} \ \mathrm{vol}_g\big(\mathrm{graph}(v)\big), \tag{3.23}$$

where $\mathbb{K}_{\psi,g} := \big\{v \in C^{0,1}(B^n_{r_0}, (-r_0, r_0)) \, : \, v \geq \psi, \, v|_{\partial B^n_{r_0}} = g\big\}$ for some $g \in C^{0,1}(\partial B^n_{r_0})$ with $g \geq \psi|_{\partial B^n_{r_0}}$, $\mathrm{graph}(v) := \big\{(x, v(x)) \, : \, x \in B^n_{r_0}\big\} \subset \mathbb{R}^n \times \mathbb{R}$ and $\mathrm{vol}_g\big(\mathrm{graph}(v)\big)$ is the area (n-dimensional measure) of the Lipschitz submanifold associated to the graph of v. In local coordinates, one can express the area of $\mathrm{graph}(u)$ in the following way: let $G : B^n_{r_0} \to \Sigma$ be given by $G(x) = \big(x, u(x)\big)$ and

$$JG(x) := \sqrt{\det\big(DG(x)^T g(G(x)) DG(x)\big)};$$

then

$$\mathrm{vol}_g\big(\mathrm{graph}(u)\big) = \int_{B^n_{r_0}} JG(x)\,dx.$$

More explicitly, the matrix $M(x) := DG(x)^T g(G(x))DG(x)$ has entries for $i, j = 1, \ldots, n$

$$\begin{aligned} M_{ij}(x) :=& g_{ij}\big(x, u(x)\big) + g_{j(n+1)}\big(x, u(x)\big)\,\partial_i u(x) \\ &+ g_{i(n+1)}\big(x, u(x)\big)\,\partial_j u(x) + g_{(n+1)(n+1)}\partial_i u(x)\,\partial_j u(x). \end{aligned}$$

As for the case of a flat metric, the existence of solutions to (3.23) is not always guaranteed and several conditions for it should be verified. However we do not investigate this problem in the present note, but we assume that we are given a solution $u \in C^{0,1}(B^n_{r_0}, (-r_0, r_0))$ and moreover we assume that

(M4) $u \in C^{1,\alpha}(B^n_{r_0}, (-r_0, r_0))$ for some $\alpha > 0$, and $u(0) = |\nabla u(0)| = 0$.

Remark 3.2.2. A comment regarding the assumption (M4) is necessary. The natural setting for the study of obstacle problems in Riemannian manifolds is that of the so called "parametric minimal surfaces" theory, *i.e.* the theory of Caccioppoli sets minimizing the perimeter among all sets which contain (or are contained in) a given obstacle. In this setting the existence issue for the obstacle problem is a simple consequence of the compactness property of Caccioppoli sets, although in general the graphical property would not be ensured.

On the other hand, around points of the free boundary of the solutions it is simple to check that one can choose normal coordinates in such a way that hypotheses (M1)–(M4) are matched. In particular, the hypothesis (M4) is a consequence of the *almost minimizing* property of the solutions to the parametric obstacle problem and of a Bernstein theorem (cf. [79, Section 6.1.2] and [90]), and therefore it is not restrictive to assume it.

In order to better understand the structure of the area functional, we can follow the strategy in [79] and look at the first variations of the functional

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0^+} \mathrm{vol}_g\big(\mathrm{graph}(u + \varepsilon\phi)\big) \geq 0, \tag{3.24}$$

for every $\phi \in C^\infty_c(B^n_{r_0})$ such that $\phi|_{\Lambda_u} \geq 0$ where $\Lambda_u := \{u = \psi\}$. By following the computations in [79] we infer that the inequality (3.24) reads as

$$\int_{B^n_{r_0}} \phi\, Lu\, dx \leq 0 \quad \forall\, \phi \in C^\infty_c(B^n_{r_0}),\ \phi|_{\Lambda_u} \geq 0, \tag{3.25}$$

where

$$Lu(x) := \mathrm{div}\Big(A\big(x, u(x), \nabla u(x)\big)\nabla u(x) + b\big(x, u(x), \nabla u(x)\big)\Big) - f\big(x, u(x), \nabla u(x)\big),$$

and A, b and f are given by the following formulas (the Einstein convention of repeated indices is consistently employed in the sequel):

(1) $A = (a^{ij})_{i,j=1,\dots,n} : B^n_{r_0} \times (-r_0, r_0) \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is given by

$$a^{ij}(x,z,\xi) := g_{(n+1)(n+1)}(x,z) h^{ij}(x,z,\xi),$$

and $(h^{ij})_{i,j=1,\dots,n}$ is the inverse of the matrix $(h_{ij})_{i,j=1,\dots,n}$ with

$$\begin{aligned} h_{ij}(x,z,\xi) := g_{ij}(x,z) + \xi_i\, g_{j(n+1)}(x,z) + \xi_j\, g_{(n+1)i}(x,z) \\ + \xi_i\, \xi_j\, g_{(n+1)(n+1)}(x,z) \qquad \forall\, i,j = 1,\dots,n, \end{aligned}$$

(note that $(h_{ij})_{i,j=1,\dots,n}$ is non-singular for small enough $|x|, |z|, |\xi|$);

(2) $b = (b^i)_{i=1,\dots,n} : B^n_{r_0} \times (-r_0, r_0) \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$b^i(x,z,\xi) := g_{j(n+1)}(x,z) h^{ji}(x,z,\xi);$$

(3) $f : B^n_{r_0} \times (-r_0, r_0) \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\begin{aligned} f(x,z,\xi) := h^{ij}\, \xi_i\, \Gamma^k_{(n+1)(n+1)}\, g_{jk} + h^{ij}\, \xi_j\, \xi_i\, \Gamma^k_{(n+1)(n+1)}\, g_{k(n+1)} \\ + h^{ij}\, \Gamma^k_{i(n+1)}\, g_{jk} + h^{ij}\, \xi_j\, \Gamma^k_{i(n+1)}\, g_{k(n+1)}, \end{aligned}$$

where to simplify the notation we have written $h^{ij} = h^{ij}\big(x,z,\xi\big)$, $g_{ij} = g_{ij}\big(x,z\big)$ and $\Gamma^k_{ij} = \Gamma^k_{ij}\big(x,z\big)$ denote the Christoffel symbols.

Note that (3.25) reads as a differential inequality of the form (3.3) where

$$\mathbf{a}(x,z,\xi) = A(x,z,\xi)\xi + b(x,z,\xi) \quad \text{and} \quad a_0(x,z,\xi) = f(x,z,\xi).$$

We now verify that there exists $s_0 < r_0$ such that $\mathbf{a}$ and a_0 above satisfy the conditions of Theorem 3.1.6 as long as $|x| + |z| + |\xi| < s_0$, *i.e.* (H1) with (iii)$'$ replacing (iii) and $p = 2$, (H2)$'$ for $p = 2$.

For what concerns (H1), we note that $\mathbf{a}$ and a_0 are smooth functions in their domains and therefore (i), (ii) and (iii)$'$ clearly follows for $|x| + |z| + |\xi| < s_0$ after choosing ϕ_1 and ϕ_2 suitable constants.

Similarly, the upper bound of (H2)$'$ follows from the regularity of $\mathbf{a}$. For what concerns the coercivity condition we start estimating as follows (we write h^{-1} for the inverse of the matrix $h = (h_{ij})$):

$$\begin{aligned} \big(\mathbf{a}(z,x,\xi) - \mathbf{a}(z,x,\eta)\big) \cdot (\xi - \eta) &= \big(A(x,z,\xi)\xi - A(x,z,\eta)\eta\big) \cdot (\xi - \eta) \\ &\quad + \big(b(x,z,\xi) - b(x,z,\eta)\big) \cdot (\xi - \eta) \\ &= g_{(n+1)(n+1)}(x,z)\big(h^{-1}(x,z,\xi)\xi - h^{-1}(x,z,\eta)\eta\big) \cdot (\xi - \eta) \\ &\quad + g_{j(n+1)}(x,z)\big(h^{ji}(x,z,\xi) - h^{ji}(x,z,\eta)\big) \cdot (\xi_i - \eta_i). \end{aligned} \tag{3.26}$$

Next note that, since $g(0) = \mathrm{I}$, then for every $\kappa > 0$ one can find s_0 sufficiently small such that

$$\big|g_{j(n+1)}(x,z)\big(h^{ji}(x,z,\xi) - h^{ji}(x,z,\eta)\big) \cdot (\xi_i - \eta_i)\big| \le \kappa\, |\xi - \eta|^2. \tag{3.27}$$

On the other hand, we can estimate the first addendum in (3.26) in the following way:

$$\begin{aligned}\big(h^{-1}(x,z,\xi)\xi - h^{-1}(x,z,\eta)\eta\big)\cdot(\xi-\eta) &= h^{-1}(x,z,\xi)(\xi-\eta)\cdot(\xi-\eta)\\ &\quad + \big(h^{-1}(x,z,\xi) - h^{-1}(x,z,\eta)\big)\,\eta\cdot(\xi-\eta).\end{aligned} \tag{3.28}$$

We can use the fact that $h^{-1}(0,0,0) = \mathrm{I}$ and the regularity of h^{-1} to get that, if $|x|+|z|+|\xi| < s_0$ for some suitably small s_0, then

$$\begin{aligned}&\big(h^{-1}(x,z,\xi)\xi - h^{-1}(x,z,\eta)\eta\big)\cdot(\xi-\eta)\\ &\qquad\geq \frac{1}{2}\,|\xi-\eta|^2 - \big|h^{-1}(x,z,\xi) - h^{-1}(x,z,\eta)\big|\,|\eta|\,|\xi-\eta|\\ &\qquad\geq \Big(\frac{1}{2} - \mathrm{Lip}(h^{-1})\,s_0\Big)\,|\xi-\eta|^2.\end{aligned} \tag{3.29}$$

Using the fact that $g_{(n+1)(n+1)}(0,0) = 1$, we then conclude the lower bound in (H2)$'$ by choosing a suitable s_0 fulfilling all the requests above. Note also that (3.5) is also satisfied because a_0 does not depend on z.

Therefore, if we assume that (H3) is satisfied, in view of (M4) we can apply Theorem 3.1.6 to $u|_{B^n_{s_0}}$, and deduce that our solution $u|_{B^n_{s_0}}$ has the optimal regularity $C^{1,1}(B^n_{s_0})$.

Finally, we can consider the regularity of the free boundary of u in $B^n_{s_0}$, which can be now obtained by the use of classical arguments. Indeed, since now u has second derivatives almost everywhere, we can also rewrite the operator in the following form (the convention of summation over repeated indices is used):

$$Lu = c^{ij}\big(x,u(x),\nabla u(x)\big)\,\partial_{ij}u + d\big(x,u(x),\nabla u(x)\big), \tag{3.30}$$

where

$$c^{ij}(x,z,\xi) = \partial_{\xi_i}\mathbf{a}_j(x,z,\xi)$$

and

$$d(x,z,\xi) = \mathrm{div}_x\mathbf{a}(x,z,\xi) + \partial_z\mathbf{a}(x,z,\xi)\cdot\xi - a_0(x,z,\xi).$$

By a simple manipulation of the equation (3.7) it follows then that

$$-c^{ij}\big(x,\psi(x),\nabla\psi(x)\big)\partial_{ij}\big(u(x)-\psi(x)\big) \tag{3.31}$$

$$\begin{aligned}&= \Big(L\psi(x) + d\big(x,u(x),\nabla u(x)\big) - d\big(x,\psi(x),\nabla\psi(x)\big)\Big)\chi_{\{u>\psi\}}\\ &\quad + \Big(c^{ij}\big(x,u(x),\nabla u(x)\big) - c^{ij}\big(x,\psi(x),\nabla\psi(x)\big)\Big)\partial_{ij}u(x)\,\chi_{\{u>\psi\}}.\end{aligned} \tag{3.32}$$

Moreover, we also deduce from the regularity of $\mathbf{a}$ and a_0 that, up to reducing eventually s_0, the function $w := u-\psi$ satisfies the following obstacle problem

$$\mathbb{A}^{ij}(x)\partial_{ij}w(x) = q(x)\chi_{\{w>0\}}, \tag{3.33}$$

where the matrix field $\mathbb{A}^{ij}(x) = c^{ij}\big(x, \psi(x), \nabla\psi(x)\big)$ is uniformly elliptic, and

$$\begin{aligned} q(x) = &-L\psi(x) - \Big(d\big(x, u(x), \nabla u(x)\big) - d\big(x, \psi(x), \nabla\psi(x)\big)\Big)\chi_{\{u>\psi\}} \\ &- \Big(c^{ij}\big(x, u(x), \nabla u(x)\big) - c^{ij}\big(x, \psi(x), \nabla\psi(x)\big)\Big)\partial_{ij}u(x)\,\chi_{\{u>\psi\}}. \end{aligned}$$

By additionally assuming (H4), we have that $-L\psi(x) \geq c_0 > 0$ and $q > c_0/2 > 0$. Furthermore, if the obstacle $\psi \in C^{2,\alpha}$ for some $\alpha > 0$ then $q \in C^{0,\alpha}$ (where, for the last claim, the Schauder estimates for the second derivatives of w in $\{w > 0\}$ are used (cf. [55, Theorem 6.2]), and the regularity of u which implies that $|\nabla u(x) - \nabla\psi(x)| \leq C\,\mathrm{dist}(x, \{u = \psi\})$).

Now, by using the regularity results for such obstacle problem in [14, 77] we can easily conclude the following final result.

Theorem 3.2.3. *Let (Σ, g) be a Riemannian manifold satisfying conditions (M1) and (M2), and let u be satisfying (M4) and be a solution to the obstacle problem for the area functional with respect to an obstacle $\psi \in C^{2,\alpha}(B^n_{r_0}, (-r_0, r_0))$ satisfying (M3) and such that $-L\psi(x) \geq c_0 > 0$.*

Then, there exists $s_0 > 0$ such that $u \in C^{1,1}(B^n_{s_0}, (-r_0, r_0))$ and the free boundary decomposes as $\partial\{u = \psi\} \cap B^n_{s_0} = \mathrm{Reg}(u) \cup \mathrm{Sing}(u)$, where $\mathrm{Reg}(u)$ and $\mathrm{Sing}(u)$ are called its regular and singular part, respectively. Moreover, $\mathrm{Reg}(u) \cap \mathrm{Sing}(u) = \emptyset$ and

(i) $\mathrm{Reg}(u)$ *is relatively open in $\partial\{u = \psi\}$ and, for every point $x_0 \in \mathrm{Reg}(u)$, there exist $r = r(x_0) > 0$ and $\beta = \beta(x_0) \in (0, 1)$ such that $\mathrm{Reg}(u) \cap B_r(x_0)$ is a $C^{1,\beta}$ submanifold of dimension $n - 1$;*

(ii) $\mathrm{Sing}(u) = \cup_{k=0}^{n-1} S_k$*, with S_k contained in the union of at most countably many submanifolds of dimension k and class C^1.*

Remark 3.2.4. Recalling that the operator L is the first variation of the area functional, the condition (H4) can be read as the geometric property of the obstacle ψ of having the mean curvature vector "pointing downward", i.e. on the opposite side with respect to the graph of u.

Chapter 4

An epiperimetric inequality for the fractional obstacle problem

We consider the minimum of

$$\mathcal{E}(v) := \int_{B_1^+} |\nabla v|^2 \, x_n^a \, dx, \tag{4.1}$$

among all functions in the class of admissible functions

$$\mathfrak{A}_g := \{v \in H^1(B_1^+, \mu_a) \, : \, v \geq 0 \; on \; B_1', v = g \; on \; (\partial B_1)^+\}, \tag{4.2}$$

where $H^1(A, \mu_a) := \overline{C^\infty(A)}^{\|\cdot\|_{H^1(A,\mu_a)}}$ with $\|v\|_{H^1(A,\mu_a)} = \left(\int_A v^2 \, d\mu_a + \int_A |\nabla v|^2 \, d\mu_a\right)^{\frac{1}{2}}$, $\mu_a := |x_n|^a \, \mathcal{L}^n \llcorner B_1$ and $a \in (-1, 1)$.

Let $u \in \operatorname{argmin}_{\mathfrak{A}_g} \mathcal{E}$; we denote by $\Lambda(u)$ its coincidence set, $\Lambda(u) := \{\widehat{x} \in B_1' \, : \, u(\widehat{x}, 0) = 0\}$, and by $\Gamma(u)$ its free boundary $\Gamma(u) := \partial\Lambda(u)$ in B_1' topology.

Caffarelli and Silvestre in [22] showed that the minimum u is the solution of

$$\begin{cases} u(\widehat{x}, 0) \geq 0 & \widehat{x} \in (B_1')^+ \\ \operatorname{div}(x_n^a \, \nabla u(\widehat{x}, x_n)) = 0 & x_n > 0 \\ \lim_{x_n \to 0^+} x_n^a \partial_n u(\widehat{x}, x_n) = 0 & u(\widehat{x}, 0) > 0 \\ \lim_{x_n \to 0^+} x_n^a \partial_n u(\widehat{x}, x_n) \leq 0 & \widehat{x} \in (B_1)^+, \end{cases} \tag{4.3}$$

and this problem is related to the study of the classical obstacle problem in $\mathbb{R}^{n-1}$ for fractional Laplacian $(\Delta)^s$ with $s \in (0, 1)$, $a = 1 - 2s$. In particular, for all v solution of $\operatorname{div}(x_n^a \, \nabla v(\widehat{x}, x_n)) = 0$ on B_1^+, with an appropriate extension to whole $\mathbb{R}^n$, there exists the limit $\lim_{x_n \to 0^+} x_n^a \partial_n v(\widehat{x}, x_n)$ and $\lim_{x_n \to 0^+} x_n^a \partial_n v(\widehat{x}, x_n) = C(-\Delta)^s f(\widehat{x})$ with f the trace of v on B_1' and C a constant depending on n and s (cf. [22]).

For $x_n > 0$, $u(\widehat{x}, x_n)$ is smooth so the second condition in (4.3) holds in the classical sense, while the third and fourth condition in (4.3) hold in the weak sense. By Silvestre [88] $u(\widehat{x}, 0) \in C^{0,\alpha}$ with $\alpha < s$, in particular if $a < \alpha < s$ the limit $\lim_{x_n \to 0^+} x_n^a \partial_n u(\widehat{x}, x_n)$ can be considered in the classical sense. By [88] we also know that $\partial_{ee} u \geq 0$ for all

$e \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1} \times \{0\}$, or rather u is semiconvex in the variable $\widehat{x}$; in the case in which the obstacle $\varphi \neq 0$, Silvestre shows that $\partial_{ee} u \geq -\sup |D^2 \varphi|$.

The function u, the solution of (4.3), can be extended by simmetrization $u(\widehat{x}, x_n) = u(\widehat{x}, -x_n)$. So, as shown in [22] we can rewrite the problem (4.3) as

$$\begin{cases} u(\widehat{x}, 0) \geq 0 & \widehat{x} \in B_1 \\ u(\widehat{x}, x_n) = u(\widehat{x}, -x_n) & \\ \operatorname{div}(|x_n|^a \nabla u(\widehat{x}, x_n)) = 0 & x \in B_1 \setminus \{(\widehat{x}, 0) \, : \, u(\widehat{x}, 0) = 0\} \\ \operatorname{div}(|x_n|^a \nabla u(\widehat{x}, x_n)) \leq 0 & x \in B_1 \text{ in distributional sense.} \end{cases} \tag{4.4}$$

In order to simplify the notation, we introduce the following symbol:

$$R_a(\psi) := \lim_{\varepsilon \to 0^+} \varepsilon^a \partial_n \psi(\widehat{x}, \varepsilon) \tag{4.5}$$

for all functions ψ which are solutions for

$$\begin{cases} \psi(\widehat{x}, x_n) = \psi(\widehat{x}, -x_n) & \\ L_a(\psi) := \operatorname{div}(|x_n|^a \nabla \psi(\widehat{x}, x_n)) = 0 & \{x_n \neq 0\}. \end{cases} \tag{4.6}$$

Silvestre in [88] proved the existence and the uniqueness of the solution. Caffarelli, Silvestre and Salsa in [21] proved the regularity of a part of the free boundary. In what follows, we shall state a uniform estimate on the solution u, so we report a quantitative result stated in [37, Theorem 2.1]

Theorem 4.0.5. *For every boundary datum $g \in H^1(B_1, \mu_a)$ that respects the condition of compatibility with the problem, i.e. $g(\widehat{x}, x_n) = g(\widehat{x}, -x_n)$ and $g(\widehat{x}, 0) \geq 0$, there exists a unique solution u to the fractional obstacle problem* (4.4). *Moreover, $\partial_{x_i} u \in C^s(B_1)$ for $i = 1, \dots, n-1$ and $|x_n|^a \partial_{x_n} u \in C^s(\overline{B_1^+})$, and*

$$\|u\|_{C^{1+s}(\overline{B^+_{1/2}})} := \|u\|_{C^0(\overline{B^+_{1/2}})} + \|\nabla_{\widehat{x}} u\|_{C^s(\overline{B^+_{1/2}})} + \| |x_n|^a \partial_{x_n} u\|_{C^s(\overline{B^+_{1/2}})} \leq C \|u\|_{L^2(B_1^+, \mu_a)}. \tag{4.7}$$

In this Chapter we prove Weiss' epiperimetric inequality for the fractional obstacle problem (cf. [95, Theorem 1] and Theorem 2.6.2 in Chapter 2) and its main consequence in the framework of the regularity of the free boundary. A similar result was recently proved by Garofalo, Petrosyan, Pop and Smith Vega Garcia [44] in the case of the fractional obstacle problem with drift for $s \in (1/2, 1)$. Their statement requires an extra hypotesis of closeness (cf. conditon 2.102 in Chapter 2). We bypass this hypotesis with an argument of homogeneity (cf. Section 4.4.1).

In particular, by introducing a frequency formula it is possible to identify the set of points with low frequency that we denote by $\Gamma_{1+s}(u)$. According to a classification result of Caffarelli, Salsa and Silvestre [21] we classify the global solutions. Following the approach of [34] we set a sequence of rescaled functions in a point with lower frequency (with rescaled factor r^{1+s} where $1+s$ is the Almegren frequency) and an auxiliary energy (cf. (2.57) in Chapter 2), and we prove a decay estimate of the energy through the epiperimetric inequalitiy. The epiperimetric inequality, together with the deacy estimate, provide a non degeneracy of the solution and a rate of convergence of the rescaled functions. We prove the uniqueness of the blow up for all points in $\Gamma_{1+s}(u)$ and finally we obtain the regularity of $\Gamma_{1+s}(u)$.

4.1 Frequency formula

Let $x_0 \in \Gamma(u)$ and $r \in (0, 1 - |x_0|)$; let $N^{x_0}(r, u)$ be the frequency function defined by

$$N_a^{x_0}(r,u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a}{\int_{\partial B_r(x_0)} u^2 \, |x_n|^a \, d\mathcal{H}^{n-1}} \tag{4.8}$$

if $u|_{\partial B_r(x_0)} \not\equiv 0$. We recall the monotonicity result due to Caffarelli and Silvestre [22].

Theorem 4.1.1. *(i) The frequency function $N_a^{x_0}(r,u)$ is monotone non decreasing in the variable r for all $r \in (0, 1 - |x_0|)$.*

(ii) For all points $x_0 \in \Gamma(u)$ the function $N^{x_0}(r,u) = \lambda$ for all $r \in (0, 1 - |x_0|)$ if and only if $u(x_0 + \cdot)$ is λ-homogeneous.

(iii) If $u(x_0 + \cdot)$ is λ-homogeneous then $\lambda \geq 1 + s$.

(iv) $N_a^{x_0}(r,u) \geq 1 + s$ for all $x_0 \in \Gamma_u$ and $r \in (0, 1 - |x_0|)$.

Proof. As far as the proof of (i) and (ii) is concerned, we refer to [22, Theorem 6.1].
(iii) Suppose by contradiction that v is λ-homogeneous with $1 < \lambda < 1 + s$ (if $\lambda \leq 1$ then $u \notin C^1(B_1)$). By Theorem 4.0.5 $\partial_{x_i} u$ is s-Hölder continuous for every $i = 1, \ldots, n-1$, and by Euler's Theorem, $\partial_{x_i} u(x_0 + \cdot)$ is $(\lambda - 1)$-homogeneous. Let $x \in \partial B_1$ and let i be an index such that $\partial_{x_i} u(x_0 + x) \neq 0$. Then x and i exist, otherwise the function $u(x_0 + \cdot)$ would be constantly 0 on B_1'. So x_0 is an internal point of the conicidence set $\Lambda(u)$ and this is a contradiction. Then thanks to the $(\lambda - 1)$-homogeneity

$$\begin{aligned}
\sup_{y,z \in B_1} \frac{|\partial_{x_i} u(x_0 + y) - \partial_{x_i} u(x_0 + z)|}{|y - z|^s} &\geq \lim_{\varepsilon \to 0} \frac{|\partial_{x_i} u(x_0 + 2\,\varepsilon x) - \partial_{x_i} u(x_0 + \varepsilon x)|}{\varepsilon^s} \\
&= |2^{\lambda - 1} - 1| \, |\partial_{x_i} u(x_0 + x)| \lim_{\varepsilon \to 0} \varepsilon^{\lambda - 1 - s} = +\infty
\end{aligned}$$

but this is in contradiction with s-Hölder continuity of $\partial_{x_i} u$.
(iv) As regards the proof of (iv), see Remark 4.2.6. $\square$

Thanks to Theorem item (i) in 4.1.1 it is possible to define the limit $N_a^{x_0}(0^+, u) := \lim_{r \to 0^+} N_a^{x_0}(r, u)$. We denote by $\Gamma_{1+s}(u)$ the subset of points of free boundary with frequency $1 + s$:

$$\Gamma_{1+s}(u) := \{x_0 \in \Gamma_u \ : \ N_a^{x_0}(0^+, u) = 1 + s\}. \tag{4.9}$$

Note that from the monotonicity of the frequency and by the upper semicontinuity of the function $x \mapsto N_a^x(0^+, u)$ (in fact it is the infimum of continuous functions $N_a^x(r, u)$) the set $\Gamma_{1+s} \subset \Gamma_u$ is open in the relative topology.

Next, we prove a fractional version of the Divergence Theorem that will be used much time in the chapter.

Theorem 4.1.2 (Divergence Theorem). *Let $\varphi \in H^1(B_1, \mu_a)$ and ψ be a solution of* (4.6), *then*

$$\int_{B_1} \nabla\psi \cdot \nabla\varphi d\mu_a = \int_{\partial B_1} \varphi\, \nabla\psi \cdot x\, |x_n|^a\, d\mathcal{H}^{n-1} - 2\int_{B_1'} \varphi R_a(\psi)\, d\mathcal{H}^{n-1} \tag{4.10}$$

Proof. Let $\varphi \in H^1(B_1, \mu_a)$ and ψ be a solution of (4.6), integrating by parts, we obtain

$$\begin{aligned}
\int_{B_1} \nabla\psi \cdot \nabla\varphi d\mu_a &= \int_{B_1\cap\{|x_n|\le\varepsilon\}} \nabla\psi \cdot \nabla\varphi d\mu_a + \int_{B_1\setminus\{|x_n|\le\varepsilon\}} \nabla\psi \cdot \nabla\varphi d\mu_a \\
&= \int_{B_1\cap\{|x_n|\le\varepsilon\}} \nabla\psi \cdot \nabla\varphi\, d\mu_a + \int_{B_1\setminus\{|x_n|\le\varepsilon\}} \mathrm{div}(\varphi\nabla\psi|x_n|^a)\, dx - \int_{B_1\setminus\{|x_n|\le\varepsilon\}} \varphi\mathrm{div}(\nabla\psi|x_n|^a)\, dx \\
&= \int_{B_1\cap\{|x_n|\le\varepsilon\}} \nabla\psi \cdot \nabla\varphi\, d\mu_a + \int_{\partial(B_1\setminus\{|x_n|\le\varepsilon\})} \varphi|x_n|^a\, \nabla\psi \cdot \nu\, dx,
\end{aligned}$$

where in the last equation we used the classical Divergence Theorem and the third condition in (4.6) that hold in the classical sense far off the hyper-plane $\{x_n = 0\}$. Then, by computing further, we obtain

$$\begin{aligned}
&\int_{B_1\cap\{|x_n|\le\varepsilon\}} \nabla\psi \cdot \nabla\varphi\, d\mu_a + \int_{\partial(B_1\setminus\{|x_n|\le\varepsilon\})} \varphi|x_n|^a\, \nabla\psi \cdot \nu\, dx \\
&\quad = \int_{B_1\cap\{|x_n|\le\varepsilon\}} \nabla\psi \cdot \nabla\varphi\, d\mu_a + \int_{\partial B_1\setminus\{|x_n|\le\varepsilon\}} \varphi|x_n|^a\, \nabla\psi \cdot x\, d\mathcal{H}^{n-1} \\
&\qquad - \int_{B_1\setminus\{|x_n|=\varepsilon\}} \varphi\varepsilon^a\, \partial_n\psi(\widehat{x}, \varepsilon)\, d\mathcal{H}^{n-1}
\end{aligned}$$

Passing to limit as $\varepsilon \to 0$, since $\nabla\psi \cdot \nabla\varphi$ is locally integrable with respect to the measure μ_a, the first integral goes to 0 for absolute continuity of measure, instead $\varphi|x_n|^a\, \nabla\psi \cdot x$ and $|\varepsilon^a \frac{\partial}{\partial x_n}\psi(\widehat{x}, \varepsilon)| \le C(1 + |\widehat{x}|)$ are integrable respect to the Lebesgue measure. So by Lebesgue's dominated convergence Theorem we obtain what follows:

$$\int_{B_1} \nabla\psi \cdot \nabla\varphi d\mu_a = \int_{\partial B_1} \varphi\, \nabla\psi \cdot x\, |x_n|^a\, d\mathcal{H}^{n-1} - 2\int_{B_1'} \varphi R_a(\psi)\, d\mathcal{H}^{n-1}$$

□

We introduce the notation:

$$D_a^{x_0}(r) = \int_{B_r(x_0)} |\nabla u|^2\, d\mu_a \qquad\qquad H_a^{x_0}(r) = \int_{\partial B_r(x_0)} u^2\, |x_n|^a\, d\mathcal{H}^{n-1}$$

and we omit to write the point x_0 if $x_0 = \underline{0}$.

All functions $H_a^{x_0}(\cdot), D_a^{x_0}(\cdot)$ and $N_a^{x_0}(\cdot)$ are absolutely continuous functions of the radius, so they are differentiable a.e..

We prove two properties of $H_a^{x_0}(r)$ (see [1, Lemma 2], [36, A.2.Lemma] for the case $a = 0$).

Lemma 4.1.3. *(i) The function*

$$(0, 1 - |x_0|) \ni r \mapsto \frac{H_a^{x_0}(r)}{r^{n+2}} \tag{4.11}$$

is nondecreasing and in particular

$$H_a^{x_0}(r) \leq \frac{H_a^{x_0}(1 - |x_0|)}{(1 - |x_0|)^{n+2}} r^{n+2} \qquad \forall\ 0 < r < 1 - |x_0|. \tag{4.12}$$

(ii) Let $x_0 \in \Gamma_{1+s}$. For all $\varepsilon > 0$ there exists an $r_0(\varepsilon)$ such that

$$H_a^{x_0}(r) \geq \frac{H_a^{x_0}(r_0)}{r_0^{n+2+\varepsilon}}\, r^{n+2+\varepsilon} \qquad \forall\ 0 < r < r_0. \tag{4.13}$$

Proof. (i) We proceed along a two-step argument. Let $x_0 \in \Gamma_{1+s}(u)$, we remember that $x_0 = (\widehat{x_0}, 0)$. We can compute the derivative of $\frac{H_a^{x_0}(r)}{r^{n-2s}}$.

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{1}{r^{n-2s}}\, H_a^{x_0}(r)\right) = \frac{d}{dr}\left(\frac{1}{r^{n-2s}} \int_{\partial B_r(x_0)} u^2\, |x_n|^a \, d\mathcal{H}^{n-1}\right) \\
&\overset{x=x_0+ry}{=} \frac{d}{dr}\left(\frac{r^{n-1}}{r^{n-2s}} \int_{\partial B_1} u^2(x_0 + ry)\, |ry_n|^a \, d\mathcal{H}^{n-1}\right) = \frac{d}{dr}\left(\int_{\partial B_1} u^2(x_0 + ry)\, |y_n|^a \, d\mathcal{H}^{n-1}\right) \\
&= \int_{\partial B_1} 2\, u(x_0 + ry) \langle \nabla u(x_0 + ry), y\rangle\, |y_n|^a \, d\mathcal{H}^{n-1} \overset{(4.10)}{=} 2r \int_{B_1} |\nabla u(x_0 + ry)|^2\, |y_n|^a \, dy \\
&\overset{ry=x_0+x}{=} 2\, r^{1-n} \int_{B_r(x_0)} |\nabla u(x)|^2 \left|\frac{x_n}{r}\right|^a dy = \frac{2}{r^{n-2s}} \int_{B_r(x_0)} |\nabla u(x)|^2 \, d\mu_a,
\end{aligned} \tag{4.14}$$

where in the third line we have used the Divergence Theorem and the third conditon of (4.3) for which $uR_a(u) = 0$ in B_1'. Next we compute the derivative of $\frac{H_a^{x_0}(r)}{r^{n+2}}$

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{H_a^{x_0}(r)}{r^{n+2}}\right) = \frac{d}{dr}\left(\frac{H_a^{x_0}(r)}{r^{n-2s}} \frac{1}{r^{2(1+s)}}\right) \\
&= \frac{2}{r^{n-2s+2(1+s)}} \int_{B_r(x_0)} |\nabla u(x)|^2 \, d\mu_a + \frac{(-2)(1+s)}{r^{n-2s+2(1+s)+1}} \int_{\partial B_r(x_0)} u^2\, |x_n|^a \, d\mathcal{H}^{n-1} \\
&= 2\, r^{-n-3}\left(r \int_{B_r(x_0)} |\nabla u(x)|^2 \, d\mu_a - (1+s) \int_{\partial B_r(x_0)} u^2\, |x_n|^a \, d\mathcal{H}^{n-1}\right),
\end{aligned} \tag{4.15}$$

then, according to item (i) in Theorem 4.1.1 and recalling that $x_0 \in \Gamma_{1+s}(u)$ we can deduce that $r^{-(n+2)}\, H_a^{x_0}(r)$ is nondecreasing.

In particular

$$\frac{H_a^{x_0}(r)}{r^{n+2}} \le \frac{H_a^{x_0}(1-|x_0|)}{(1-|x_0|)^{n+2}} \qquad\qquad \forall\, 0<r<1-|x_0|.$$

(ii) Let $r_0 = r_0(\varepsilon)$ be a radius such that for all $r < r_0$ it holds that $N_a^{x_0}(u) \le (1+s)+\varepsilon/2$. Then, thanks to (4.14), we obtain

$$N_a^{x_0}(r,u) = \frac{r}{2}\frac{d}{dr}\log\left(\frac{H_a^{x_0}(r)}{r^{n-2s}}\right) \le (1+s)+\varepsilon/2.$$

So, integrating on (r, r_0) we have

$$\log\left(\frac{H_a^{x_0}(r_0)}{H_a^{x_0}(r)}\left(\frac{r}{r_0}\right)^{n-2s}\right) \le (2(1+s)+\varepsilon)\log\frac{r_0}{r}$$

and

$$H_a^{x_0}(r) \ge H_a^{x_0}(r_0)\left(\frac{r}{r_0}\right)^{n+2+\varepsilon}. \qquad \square$$

We now prove a generalization (in the fractional setting) of the Rellich formula (cf. Lemma 2.3.5):

Proposition 4.1.4 (Rellich formula)**.** *Let v be a solution of* (4.4)*. Then it holds that:*

$$\int_{\partial B_r} |\nabla v|^2\,|x_n|^a\,d\mathcal{H}^{n-1} = \frac{n-2+a}{r}\int_{B_r}|\nabla v|^2\,d\mu_a + 2\int_{\partial B_r}\left(\langle \nabla v, \frac{x}{r}\rangle\right)^2 |x_n|^a\,d\mathcal{H}^{n-1}.$$

Proof. We apply the Divergence Theorem and the third conditon of (4.3) for which $uR_a(u) = 0$ in B_1' and develop

$$\operatorname{div}\left(|\nabla v|^2\frac{x}{r}\,|x_n|^a - 2\,\langle\nabla v, \frac{x}{r}\rangle \nabla v\,|x_n|^a\right). \qquad \square$$

Similarly as Propositions 2.3.6 and 2.3.9 we compute the derivative of the volume and boundary energies.

Lemma 4.1.5. *The following formulas hold:*

(i) $(H_a^{x_0})'(r) = \frac{n-2s}{r}H(r) + 2\int_{\partial B_r} u\nabla u\cdot\nu\,|x_n|^a\,d\mathcal{H}^{n-1}$*;*

(ii) $(D_a^{x_0})'(r) = \frac{n-2+a}{r}D(r) + 2\int_{\partial B_r}(\nabla u\cdot\nu)^2\,|x_n|^a\,d\mathcal{H}^{n-1}$*;*

(iii) $D_a^{x_0}(r) = \int_{\partial B_r} u\nabla u\cdot\nu\,|x_n|^a\,d\mathcal{H}^{n-1}$*;*

Proof. (i) By changing variables and recalling that $a = 1 - 2s$, we obtain

$$(H_a^{x_0})'(r) = \frac{d}{dr}\left(\int_{\partial B_r(x_0)} u^2\,|x_n|^a\,d\mathcal{H}^{n-1}\right) \overset{x=x_0+ry}{=} \frac{d}{dr}\left(r^{n-1}\int_{\partial B_1} u^2(x_0+ry)\,|ry_n|^a\,d\mathcal{H}^{n-1}(y)\right)$$

$$= \frac{n-2s}{r}r^{n-2s}\int_{\partial B_1} u^2(x_0+ry)\,|y_n|^a\,d\mathcal{H}^{n-1} + 2\,r^{n-1}\int_{\partial B_1} 2\,u(x_0+ry)\nabla u(x_0+ry)\cdot y\,|ry_n|^a\,d\mathcal{H}^{n-1}$$

$$\overset{y=\frac{x-x_0}{ry}}{=} \frac{n-2s}{r}H_a^{x_0}(r) + 2\,r^{n-1}\int_{\partial B_r(x_0)} 2\,u\nabla u(x_0+ry)\cdot y\,|ry_n|^a\,d\mathcal{H}^{n-1}$$

$$= \frac{n-2s}{r}H_a^{x_0} + 2\int_{\partial B_r(x_0)} u\nabla u\cdot\nu\,|x_n|^a\,d\mathcal{H}^{n-1}$$

(ii) From Coarea and Rellich Formulas we obtain

$$(D_a^{x_0})'(r) = \frac{d}{dr}\left(\int_{B_r(x_0)} |\nabla u|^2\,d\mu_a\right) \overset{Coarea Formula}{=} \int_{\partial B_r(x_0)} |\nabla u|^2\,d\mu_a$$

$$\overset{Prop.4.1.4}{=} \frac{n-2+a}{r}D_a^{x_0}(r) + 2\int_{\partial B_r(x_0)} (\nabla u\cdot\nu)^2\,|x_n|^a\,d\mathcal{H}^{n-1}.$$

(iii) In order to prove the formula, it is enough to apply the the Divergence Theorem and the third conditon of (4.3) for which $uR_a(u) = 0$ in B_1'. $\square$

4.2 The blow up method: existence and $(1+s)$-homogeneity of blow ups

In order to study the properties of the free boundary, we investigate the properties of the blow up limits. Proceeding as in Chapter 2 we shall consider a suitable sequence of rescaled functions of the solution u. Let $x_0 \in \Gamma_{1+s}(u)$, we set

$$u_{x_0,r}(x) := \frac{u(rx)}{r^{1+s}}, \tag{4.16}$$

if $x_0 = \underline{0}$ we denote $u_r(x)$ in the place of $u_{\underline{0},r}(x)$. Note that in the choice of the rescaling factor in (4.16) we follow the same approach as in [36] and [44], which is different with respect to the previous approach used in [4] to analyse the fractional Laplacian obstacle problem.

The first step in the analysis of blow ups is to prove the existence of the limits of the sequence $(u_{x_0,r})_r$ for all $x_0 \in \Gamma_{1+s}(u)$. In order to prove their existence, we state the equiboundedness of $(u_{x_0,r})_r$ with respect to the weighted Dirichlet energy.

Proposition 4.2.1 (Existence of blow ups). *Let $u \in H^1(B_1,\mu_a)$ be the solution of (4.4) and let $x_0 \in \Gamma_{1+s}(u)$. Then for every sequence $r_k \downarrow 0$ there exists a subsequence $(r_{k_j})_j \subset (r_k)_k$ such that the rescaled functions $(u_{x_0,r_{k_j}})_j$ converge in $L^2(B_{1-|x_0|},\mu_a)$.*

Proof. According to Theorem 4.1.1 and Lemma 4.1.3, we obtain the following:

$$\begin{aligned}\int_{B_1} |\nabla u_{x_0,r}|^2\, d\mu_a &\overset{x_0+rx=y}{=} \frac{\int_{B_r(x_0)} |\nabla u(y)|^2 r^{-2s}\, |y_n|^a r^{-a}\, dx}{r^n} = \frac{\int_{B_r(x_0)} |\nabla u(y)|^2\, |y_n|^a\, dx}{r^{n+1}} \\ &= \frac{r\, \int_{B_r(x_0)} |\nabla u(y)|^2\, |y_n|^a\, dx}{\int_{\partial B_r(x_0)} u^2 |y_n|^a\, d\mathcal{H}^{n-1}}\; \frac{\int_{\partial B_r(x_0)} u^2 |y_n|^a\, d\mathcal{H}^{n-1}}{r^{n+2}} \\ &\overset{(4.12)}{\leq} N_a^{x_0}(r,u) H_a^{x_0}(1-x_0) \leq N_a^{x_0}(1,u) H_a^{x_0}(1-|x_0|),\end{aligned} \tag{4.17}$$

where in the last inequalitiy we have used the first item of Theorem 4.1.1. Therefore, thanks to [62, Theorem 1.31] for every subsequence of radii $r_k \searrow 0$, there exists an extracted subsequence $r_{k_j} \searrow 0$ such that $u_{x_0,r_{k_j}} \to u_0$ in $L^2(B_{1-|x_0|}, \mu_a)$ as $j \to +\infty$. □

Remark 4.2.2. In view of (4.17), the Theorem of convergence of traces and the estimate (4.7), similarly to Remark 2.2.3 of Chapter 2, we deduce

$$\|u\|_{L^\infty(B_1)} \leq C r^{1+s} \qquad\qquad \|\nabla u\|_{L^\infty(B_1;\mathbb{R}^n)} \leq C r^s \tag{4.18}$$

Similarly to (2.57) in Chapter 2 we consider an energy “à la Weiss” introduced in [95], used in [45] and [36] for fractional Laplacian (see [44] for a version in the fractional Laplacian problem with drift and [34, 48] for a version in the classical obstacle problem with quadratic energies with variable coefficients):

$$W_{1+s}^{x_0}(r,u) = \frac{1}{r^n+1} \int_{B_r(x_0)} |\nabla u|^2\, |x_n|^a\, dx - \frac{1+s}{r^{n+2}} \int_{\partial B_r(x_0)} u^2\, |x_n|^a\, d\mathcal{H}^{n-1}. \tag{4.19}$$

We note that

$$W_{1+s}^{x_0}(r,u) = \frac{H_a^{x_0}(r)}{r^{n+2}} (N_a^{x_0}(r,u) - (1+s)),$$

thus if $x_0 \in \Gamma_{1+s}(u)$ by (4.9) and Lemma 4.1.3 (which guarantees the boundedness of $\frac{H_a^{x_0}(r)}{r^{n+2}}$) we have

$$\lim_{r \searrow 0} W_{1+s}^{x_0}(r,u) = 0$$

and due to Theorem 4.1.1, we obtain

$$W_{1+s}^{x_0}(r,u) \geq 0.$$

Moreover, the function $W_{1+s}^{x_0}(\cdot, u)$ satisfies a monotonicity formula in the same essence as Weiss' monotonicity formula proved in [95] (cf. Theorem 2.3.10 in Chapter 2). For a similar proof see [44, Theorem 3.5].

Proposition 4.2.3 (Weiss' monotonicity formula)**.** *Let $x_0 \in \Gamma_{1+s}(x_0)$ and u be a solution of Problem* (4.3)*; then the function $r \mapsto W_{1+s}^{x_0}(r,u)$ is nondecreasing. In particular, the following formula holds:*

$$\frac{d}{dr} W_{1+s}^{x_0}(r,u) = \frac{2}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s) u_r)^2\, |x_n|^a\, d\mathcal{H}^{n-1}$$

Proof. Similarly to Teorem 2.3.10 in Chapter 2, thanks to Lemma 4.1.5 we compute the derivative of $W^{x_0}_{1+s}(r,u)$

$$\begin{aligned}
\frac{d}{dr}W^{x_0}_{1+s}(r,u) &= \frac{d}{dr}\left(\frac{1}{r^n+1}D^{x_0}_a(r) - \frac{1+s}{r^{n+2}}H^{x_0}_a(r)\right)\\
&= \frac{(n+1)}{r^{n+2}}D^{x_0}_a(r) + \frac{1}{r^{n+1}}(D^{x_0}_a)'(r) + \frac{(1+s)(n+2)}{r^{n+2}}H^{x_0}_a(r) - \frac{(1+s)}{r^{n+2}}(H^{x_0}_a)'(r)\\
&= -\frac{4(1+s)}{r^{n+2}}\int_{\partial B_r(x_0)} u\nabla u\cdot\nu\,|x_n|^a\,d\mathcal{H}^{n-1} + \frac{2}{r^{n+1}}\int_{\partial B_r(x_0)}(\nabla u\cdot\nu)^2\,|x_n|^a\,d\mathcal{H}^{n-1}\\
&+ \frac{2(1+s)^2}{r^{n+1}}\int_{\partial B_r(x_0)} u^2\,|x_n|^a\,d\mathcal{H}^{n-1} = \frac{2}{r^{n+1}}\int_{\partial B_r(x_0)}\left(\nabla u\cdot\nu - \frac{(1+s)}{r}u\right)^2|x_n|^a\,d\mathcal{H}^{n-1}.
\end{aligned}$$

We conclude the thesis with a change of variable. □

Next, we prove the homogeneity property of blow ups. Unlike Proposition 2.4.1 in Chapter 2 where we prove the homogeneity of blow ups thanks to Weiss' monotonicity formula, here we prove the result through properties of the frequency function and the optimal regularity of the solution.

Proposition 4.2.4 ($(1+s)$-homogeneity of blow ups). *Let $u\in H^1(B_1,\mu_a)$ be a solution of Problem (4.4). Let $x_0\in\Gamma_{1+s}(u)$ and $(u_{x_0,r})_r$ be a sequence of rescaled functions. Then, for every sequence $(r_j)_j\downarrow 0$ there exists a subsequence $(r_{j_k})_k\subset(r_j)_j$ such that the sequence $(u_{x_0,r_{j_k}})_k$ converges in $C^{1+\alpha}(\mathbb{R}^n)$ (see (4.7)) for all $\alpha<s$ to u_{x_0} a $(1+s)$-homogeneous function.*

Proof. According to the quantitative estimate (4.7) and Poincaré inequality we have

$$\begin{aligned}
\sup_k \|u_{x_0,r_k}\|_{C^{1+s}(B_{1/2})} &\overset{(4.7)}{\leq} \sup_k \|u_{x_0,r_k}\|_{L^2(B_1,\mu_a)}\\
&\leq C\left(\sup_k \|u_{x_0,r_k}\|_{L^2(\partial B_1,|x_n|^a)} + \sup_k \|\nabla u_{x_0,r_k}\|_{L^2(B_1,|x_n|^a;\mathbb{R}^n)}\right)
\end{aligned}$$

Due to Lemma 4.1.3(i), we have

$$\|u_{x_0,r_k}\|^2_{L^2(\partial B_1,|x_n|^a)} = \frac{H^{x_0}_a(r_k)}{r_k^{n+2}} \overset{(4.12)}{\leq} \frac{H^{x_0}_a(1-|x_0|)}{(1-|x_0|)^{n+2}},$$

while, since $x_0\in\Gamma_{1+s}(u)$,

$$\|\nabla u_{x_0,r_k}\|^2_{L^2(B_1,|x_n|^a;\mathbb{R}^n)} = \frac{D^{x_0}_a(r_k)}{r_k^{n+1}} \overset{(4.8)}{=} \frac{H^{x_0}_a(r_k)}{N^{x_0}_a(r_k)\,r_k^{n+2}} \leq \frac{H^{x_0}_a(1-|x_0|)}{(1+s)\,(1-|x_0|)^{n+2}}$$

where in the last inequality we used the inequality (4.12) and the Theorem 4.1.1(i). Thus

$$\sup_k \|\nabla u_{x_0,r_k}\|^2_{L^2(B_1,|x_n|^a;\mathbb{R}^n)} + \sup_k \|u_{x_0,r_k}\|^2_{L^2(\partial B_1,|x_n|^a)} \leq \frac{H_a(1)}{(1-|x_0|)^{n+2}}\left(\frac{1}{1+s}+1\right) < \infty.$$

Then thanks to the Ascoli-Arzelà Theorem there exists a subsequence (that we do not relabel) u_{x_0,r_k} and $u_{x_0} \in C^{1+s}(B_{1/2})$ such that $\|u_{x_0,r_k} - u_{x_0}\|_{C^{1+\alpha}(B_{1/2})}$ converge to 0 for all $\alpha < s$.

Using a Γ-convergence argument we deduce that u_{x_0} is a solution of Problem (4.4). For all $k \in \mathbb{N}$, u_{x_0,r_k} is the minimizer (with respect to its boundary data) of

$$F_k(v) = \int_{B_{1-|x_0|}(x_0)} |\nabla v|^2 \, |y_n|^a \, dy.$$

Let

$$F_0(v) = \int_{B_{1-|x_0|}(x_0)} |\nabla v|^2 \, |y_n|^a \, dy$$

and let w be its minimum with respect to the boundary data of u_{x_0}. We observe that F_k $\Gamma(H^1(B_1, \mu_a))$- converge to F_0, then by Theorem 1.5.3 $u_{x_0,r_k} \to w$, but $u_{x_0,r_k} \to u_{x_0}$ in C^{1+s} so $w = u_{x_0}$.

In order to conclude the proof, we show that w_0 is $(1+s)$-homogeneous.
We note that for every $\delta > 0$ we can fix $\rho > 0$ such that $N_a^{x_0}(\rho, u) \le (1+s) + \delta$. So for $k >> 1$, for every $t \in (0,1)$ (such that $t\,r_k < \rho$)

$$\begin{aligned} N_a(t, u_{x_0,r_k}) =& N_a^{x_0}(t, u_{r_k}) = N_a^{x_0}(t\,r_k, u) - N_a^{x_0}(\rho, u) + N_a^{x_0}(\rho, u) \le (1+s) + \delta, \\ & N_a^{x_0}(t, u_{r_k}) = N_a^{x_0}(t\,r_k, u) \ge 1+s \end{aligned} \tag{4.20}$$

where we resort to Theorem 4.1.1. Now, from the convergence of u_{x_0,r_k} to u_{x_0} and thanks to the arbitrariness of δ, we obtain $N_a(t, u_{x_0}) \equiv 1+s$; then, by Theorem 4.1.1(ii), u_{x_0} is $(1+s)$-homogeneous. □

Remark 4.2.5. Proceeding as in Proposition 2.4.1 in Chapter 2 and thanks to Proposition 4.2.3 we can obtain the same result.

Remark 4.2.6. By proceeding in the same way, we can prove Theorem 4.1.1(iv) as well:

Proof of Theorem 4.1.1(iv). Let $x_0 \in \Gamma(u)$ and $\lambda = N_a^{x_0}(0^+, u)$. Then, if $r_k \searrow 0$ is a suitable sequence of radii, for all $\delta > 0$ we can fix $\rho > 0$ such that $N_a^{x_0}(\rho, u) \le \lambda + \delta$. So, proceeding in much the same way as in (4.20), we deduce

$$\lambda \le N_a(t, u_{x_0}, r_k) \le \lambda + \delta,$$

thus, by the strong convergence of u_{x_0,r_k}, to its blow up w_0 and by the arbitrariness of δ, we have $N_a(t, w_0) \equiv \lambda$. So, by the second item of Theorem 4.1.1 w_0 is λ-homogeneous and by Theorem 4.1.1(iii) $\lambda \ge 1+s$. □

4.3 Classification of the $(1+s)$-homogeneous global solutions

Let h_e be the function defined by

$$h_e(x) := \left(s^{-1}\widehat{x} \cdot e - \sqrt{(\widehat{x} \cdot e)^2 + x_n^2}\right) \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e\right)^s. \tag{4.21}$$

Then the following properties hold:

(i) $h_e(\widehat{x}, x_n) = h_e(\widehat{x}, -x_n)$;

(ii) $h_e(x) \geq 0$ on $\{x_n = 0\}$ and $h_e = 0$ on $\{x_n = 0,\, \widehat{x} \cdot e \leq 0\}$;

(iii) $\partial_e h_e(x) = \frac{1-s^2}{s} \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e \right)^s$;

(iv) $\partial_n h_e(x) = -(1+s)x_n \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e \right)^{s-1}$;

(v) h_e is solution of (4.6);

(vi)
$$R_a h_e(\widehat{x}) = \begin{cases} 0 & \widehat{x} \cdot e \geq 0 \\ -(1+s)\,(2\,|\widehat{x} \cdot e|)^{1-s} & \widehat{x} \cdot e < 0. \end{cases} \tag{4.22}$$

In particular, we obtain a complementarity property
$$h_e(\widehat{x}, x_n)\, R_a h_e(\widehat{x}) = 0 \qquad \text{on } \{x_n = 0\} \tag{4.23}$$

Proof. Properties (i) and (ii) are straightforward.
(iii) Let $e = (\lambda_1, \ldots, \lambda_{n-1}, 0)$, $e \in \mathbb{S}^{n-2} := \mathbb{S}^{n-1} \cap \{x_n = 0\} \subset \mathbb{R}^n$. In order to semplify the notation we introuce the function $\rho_e : \mathbb{R}^n \to \mathbb{R}$ defined by $\rho_e(x) := \sqrt{(\widehat{x} \cdot e)^2 + x_n^2}$. We observe that $\partial_e \rho_e(x) = \langle \nabla \rho_e(x), e \rangle = \frac{\widehat{x} \cdot e}{\rho_e(x)}$, so

$$\begin{aligned} \partial_e h_e(x) = &\left(s^{-1} - \frac{\widehat{x} \cdot e}{\rho_e(x)} \right) (\rho_e(x) + \widehat{x} \cdot e)^s \\ &+ s\left(s^{-1}\widehat{x} \cdot e - \rho_e(x)\right) (\rho_e(x) + \widehat{x} \cdot e)^{s-1} \left(1 + \frac{\widehat{x} \cdot e}{\rho_e(x)} \right) \\ = &(\rho_e(x) + \widehat{x} \cdot e)^{s-1} \left(s^{-1} - s\right) (\rho_e \widehat{x} \cdot e) = \left(s^{-1} - s\right) (\rho_e \widehat{x} \cdot e)^s . \end{aligned}$$

(iv) With the same notation as above, we calculate $\partial_n h_e$:

$$\begin{aligned} \partial_n h_e(x) &= -\frac{x}{\rho_e(x)} (\rho_e(x) + \widehat{x} \cdot e)^s + s\left(s^{-1}\widehat{x} \cdot e - \rho_e\right) \frac{x_n}{\rho_e} (\rho_e(x) + \widehat{x} \cdot e)^{s-1} \\ &= -(1+s)x_n\, (\rho_e(x) + \widehat{x} \cdot e)^{s-1} . \end{aligned}$$

(v) In order to calculate $\operatorname{div}(|x_n|^a \nabla u(\widehat{x}, x_n))$, we use the same notation as in item (iii) and, by resorting to the same calculus, we obtain

$$\partial_i h_e(x) = \lambda_i (s^{-1} - s)\, (\rho_e(x) + \widehat{x} \cdot e)^s ,$$

and, in turn,
$$\nabla h_e(x) = \begin{pmatrix} \lambda_1\, \partial_e h_e(x) \\ \vdots \\ \lambda_{n-1}\, \partial_e h_e(x) \\ \partial_n h_e(x) \end{pmatrix} .$$

Next, we calculate the second derivative of $h_e(x)$ in the variable x_i with $i = 1, \dots, n-1$:

$$\partial_{ii} h_e(x) = \lambda_i (s^{-1} - s) \partial_i \left((\rho_e(x) + \widehat{x} \cdot e)^s \right) = \lambda_i^2 \, s \, (s^{-1} - s) \, (\rho_e(x) + \widehat{x} \cdot e)^{s-1} \left(1 + \frac{\widehat{x} \cdot e}{\rho_e(x)} \right)$$

and subsequently, in order to calculate the derivative in the variable x_n, we write

$$\partial_n (x_n^a \, \partial_n h_e(x)) = -(1 - s^2) x^{1-2s} \, (\rho_e(x) + \widehat{x} \cdot e)^{s-2} \left(2\rho_e(x) + 2\widehat{x} \cdot e - \frac{x_n^2}{\rho_e(x)} \right).$$

so, on $\{x_n \neq 0\}$ we obtain

$$\begin{aligned} \operatorname{div} \left(|x_n|^a \, \nabla h_e(x) \right) = & \sum_{i=1}^{n-1} \lambda_i^2 (1 - s^2) \, (\rho_e(x) + \widehat{x} \cdot e)^{s-1} \left(1 + \frac{\widehat{x} \cdot e}{\rho_e(x)} \right) |x_n|^a \\ & - (1 - s^2) |x_n|^a \, (\rho_e(x) + \widehat{x} \cdot e)^{s-2} \left(2\rho_e(x) + 2\widehat{x} \cdot e - \frac{x_n^2}{\rho_e(x)} \right) = 0. \end{aligned}$$

(vi) By property (iv) and recalling that $a = 1 - 2s$ we have

$$\begin{aligned} \lim_{x_n \to 0^+} x_n^a \partial_n h_e(\widehat{x}, x_n) &= \lim_{x_n \to 0^+} -(1+s) x_n \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e \right)^{s-1} x_n^{1-2s} \\ &= \lim_{x_n \to 0^+} -(1+s) \left(\frac{x_n^2}{\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e} \right)^{1-s}. \end{aligned}$$

Using the Taylor expansion of the second order of $\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e$ we have

$$R_a h_e(\widehat{x}) = \begin{cases} 0 & \widehat{x} \cdot e \geq 0 \\ -(1+s) \, (2 \, |\widehat{x} \cdot e|)^{1-s} & \widehat{x} \cdot e < 0. \end{cases} \qquad \square$$

In view of properties above, h_e is a solution of problem (4.3), so by [88] $\partial_{\tau\tau} h_e \geq 0$ for any vector $\tau \in \mathbb{S}^n \subset \mathbb{R}^{n-1} \times \{0\}$. So, thanks to its $(1+s)$-homogeneity, h_e is a solution of

$$\begin{cases} v(\widehat{x}, 0) \geq 0 & \widehat{x} \in \mathbb{R}^{n-1} \\ v(\widehat{x}, x_n) = v(\widehat{x}, -x_n) & \\ \operatorname{div}(|x_n|^a \, \nabla v(\widehat{x}, x_n)) = 0 & x \in \mathbb{R}^n \setminus \{(\widehat{x}, 0) \, : \, u(\widehat{x}, 0) = 0\} \\ \operatorname{div}(|x_n|^a \, \nabla v(\widehat{x}, x_n)) \leq 0 & x \in \mathbb{R}^n \text{ in distributional sense} \\ \partial_{\tau\tau} v \geq 0 & \text{for any vector } \tau \in \partial B_1'. \end{cases} \tag{4.24}$$

According to [21, Proposition 5.5], the function h_e is, up to a rotation and the product by scalar, the unique $(1+s)$-homogeneous, global solution of (4.24).

We consider the closed convex cone of $(1+s)$-homogeneous global solutions :

$$\mathfrak{H}_{1+s} := \{ \lambda \, h_e \ : \ e \in \mathbb{S}^{n-2}, \lambda \in [0, +\infty) \} \subset H^1_{loc}(\mathbb{R}^n, \mu_a). \tag{4.25}$$

Caffarelli, Salsa and Sivestre [21] proved that $\mathfrak{H}_{1+s} \setminus \underline{0}$ is the set of blow-ups in the regular points of the free-boundary with lower frequency.

We note that $\mathfrak{H}_{1+s}$ is a closed cone in $H^1_{loc}(\mathbb{R}^n, \mu_a)$. The restriction

$$\mathfrak{H}_{1+s}|_{B_1} := \{ v|_{B_1} \ : \ v \in \mathfrak{H}_{1+s} \} \subset H^1(B_1, \mu_a)$$

is a closed set, and $\mathfrak{H}_{1+s} \setminus \{0\}$ is parameterized by a $(n-1)$-manifold by the map

$$\begin{array}{ccc} \mathbb{S}^{n-2} \times (0,\infty) & \xrightarrow{\Phi} & \mathfrak{H}_{1+s} \setminus \{0\} \\ (e,\lambda) & \longmapsto & \lambda\, h_e. \end{array}$$

Next we can introduce the tangent plane to space $\mathfrak{H}_{1+s}$ in every point $\lambda\, h_e$ as

$$T_{\lambda\, h_e}\mathfrak{H}_{1+s} := \{d_{(e,\lambda)}\Phi(\xi,\alpha) \ : \ \xi \cdot e_n = \xi \cdot e = 0, \alpha \in \mathbb{R}\} \tag{4.26}$$

We compute the derivative of the map Φ in a point of $\mathbb{S}^{n-2} \times (0,\infty)$:

$$d_{(e,\lambda)}\Phi(\xi,\alpha) = \frac{d}{dt} h_{\sigma(t)}|_{t=0} \tag{4.27}$$

with $\sigma(t) = \frac{e+t\xi}{\|e+t\xi\|}$, a curve on $\mathbb{S}^{n-2}$ such that $\sigma(0) = e$ and $\sigma'(0) = \xi$. By (4.21) we have

$$h_{\sigma(t)}(x) := \left(s^{-1}\widehat{x} \cdot \sigma(t) - \sqrt{(\widehat{x} \cdot \sigma(t))^2 + x_n^2}\right)\left(\sqrt{(\widehat{x} \cdot \sigma(t))^2 + x_n^2} + \widehat{x} \cdot \sigma(t)\right)^s.$$

In order to compute (4.27), we start by noting that

$$\frac{d}{dt}\widehat{x} \cdot \sigma(t) = \sum_{i=1}^{n-1} x_i \frac{d}{dt}\sigma_i(t) = \widehat{x} \cdot \xi.$$

To simplify the notation we denote $\rho_\sigma(t) := \sqrt{(\widehat{x} \cdot \sigma(t))^2 + x_n^2}$, so we obtain

$$\begin{aligned} \frac{d}{dt} h_{\sigma(t)}{}_{|t=0} =& \frac{d}{dt}\left(\left(s^{-1}\widehat{x} \cdot \sigma(t) - \rho_\sigma(t)\right)\left(\rho_\sigma(t) + \widehat{x} \cdot \sigma(t)\right)^s\right)_{|t=0} \\ =& (s^{-1} - s)\, \widehat{x} \cdot \xi \left(\rho_\sigma(t) + \widehat{x} \cdot \sigma(t)\right)^s_{|t=0} \\ =& (s^{-1} - s)\, \widehat{x} \cdot \xi \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e\right)^s. \end{aligned}$$

Then, we can rewrite (4.26) as

$$T_{\lambda\, h_e}\mathfrak{H}_{1+s} := \{\alpha h_e + v_{e,\xi} \ : \ \xi \cdot e_n = \xi \cdot e = 0, \alpha \in \mathbb{R}\}$$

where the function $v_{e,\xi}$ is defined as follows:

$$v_{e,\xi} = \widehat{x} \cdot \xi \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e\right)^s.$$

We highlight some properties of function $\psi \in \mathfrak{H}_{1+s}$. For all $\varphi \in H^1(B_1, \mu_a)$, integrating by parts, according to Theorem 4.1.2 and Euler's homogeneous function Theorem we obtain

$$\begin{aligned} \int_{B_1} \nabla\psi \cdot \nabla\varphi d\mu_a &= \int_{\partial B_1} \varphi\, \nabla\psi \cdot x\, |x_n|^a\, d\mathcal{H}^{n-1} - 2\int_{B_1'} \varphi R_a(\psi)\, d\mathcal{H}^{n-1} \\ &= (1+s)\int_{\partial B_1} \varphi\psi\, |x_n|^a\, d\mathcal{H}^{n-1} - 2\int_{B_1'} \varphi R_a(\psi)\, d\mathcal{H}^{n-1}. \end{aligned} \tag{4.28}$$

Remark 4.3.1. The first variation of functional $W^0_{1+s}(1,\cdot)$ in a point $\psi \in \mathfrak{H}_{1+s}$ along a direction $\varphi \in H^1(B_1,\mu_a)$ is

$$\begin{aligned}\delta W^0_{1+s}(1,\psi)[\varphi] :=& \lim_{t\to 0}\left(\frac{W^0_{1+s}(1,+t\varphi)-W^0_{1+s}(1,\psi)}{t}\right)\\ =&2\int_{B_1}\nabla\psi\cdot\nabla\varphi\,d\mu_a-2(1+s)\int_{\partial B_1}\psi\,\varphi\,|x_n|^a\,d\mathcal{H}^{n-1}.\end{aligned}$$

Then, by (4.28)

$$\delta W^0_{1+s}(1,\psi)[\varphi]=-4\int_{B_1'}\varphi R_a(\psi)(\widehat{x})\,d\mathcal{H}^{n-1}, \tag{4.29}$$

by (4.23)

$$\delta W^0_{1+s}(1,\psi)[\psi]=0, \tag{4.30}$$

so we can infer that

$$W^0_{1+s}(1,\psi)=\frac{1}{2}\delta W^0_{1+s}(1,\psi)[\psi]=0 \qquad\qquad \forall\psi\in\mathfrak{H}_{1+s}. \tag{4.31}$$

4.4 The epiperimetric inequality and its consequences

In this section we prove an epiperimetric inequalitiy for the points in $\Gamma_{1+s}(u)$, and its main consequences in the framework of the regularity of the free boundary. In Paragraph 4.4.1 we prove the epiperimetric inequality. In Paragraph 4.4.2 we establish a decay estimate for adjusted boundary energy. In Paragraphs 4.4.3 and 4.4.4 we prove the nondegeneracy of the solution and the uniqueness of the blow ups in $\Gamma_{1+s}(u)$ respectively.

4.4.1 Epiperimetric inequality

We now state the main result of this chapter: the *epiperimetric inequality* “à la Weiss” for the thin obstacle in the case of the fractional Laplacian. This result is a key ingredient in our approach to the decay of the boundary adjusted energy and to the uniqueness of blow ups (see [36] for the classical case of Laplacian $s=1/2$).

Garofalo, Petrosyan, Pop and Smith Vega Garcia [44] proved a similar epiperimetric inequality for the fractional obstacle problem with drift in the case of $s\in(1/2,1)$. Their statement requires an extra hypotesis of closeness between the function c and a blow up limit (cf. the conditon 2.102 in Chapter 2). We bypass this hypotesis with an argument of homogeneity (cf. Theorem **??** and the comment write before).

In this paragraph we state and prove the epiperimetric inequality. For the convenience of readers, the proof will be split into several steps.

Theorem 4.4.1 (Epiperimetric inequality)**.** *There exists a dimensional constant $\kappa\in(0,1)$ such that if $c\in H^1(B_1,\mu_a)$ is a $(1+s)$-homogeneous function with $c\geq 0$ on B_1' and $c(\widehat{x},x_n)=c(\widehat{x},-x_n)$ then*

$$\inf_{v\in\mathfrak{A}_c} W^0_{1+s}(v)\leq(1-\kappa)W^0_{1+s}(c). \tag{4.32}$$

Proof. Without loss of generality it is possible to suppose that the function c satisfies the follows condition

$$\mathrm{dist}_{H^1(B_1,\mu_a)}(c,\mathfrak{H}_{1+s})<\delta. \tag{4.33}$$

In fact, according to the $(1+s)$-homogeneity of c and recalling that $\mathfrak{H}_{1+s}$ is a cone, for all $\delta>0$ there exists a constant $\gamma>0$ such that

$$\mathrm{dist}_{H^1(B_1,\mu_a)}(\gamma c,\mathfrak{H}_{1+s})<\delta.$$

We can observe that if $v\in\mathfrak{A}_{\gamma c}$ then $\gamma^{-1}v\in\mathfrak{A}_c$. So, if we prove inequality (4.32) for the function γc, or rather

$$\inf_{v\in\mathfrak{A}_{\gamma c}} W^0_{1+s}(1,v)\le(1-\kappa)W^0_{1+s}(1,\gamma c),$$

then, thanks to $W^0_{1+s}(1,\gamma c)=\gamma^2 W^0_{1+s}(1,c)$ we infer

$$\inf_{w\in\mathfrak{A}_c} W^0_{1+s}(1,w)\le(1-\kappa)W^0_{1+s}(1,c).$$

To simplify the notation we denote the functional $W^0_{1+s}(1,\cdot)$ by $\mathcal{G}(\cdot)$.
We argue by contradiction. Let us suppose the existence of sequences of positive numbers $\kappa_j,\delta_j\downarrow 0$ and a sequence of $(1+s)$-homogeneous functions $c_j\in H^1(B_1,\mu_a)$ with $c_j\ge 0$ on B_1' such that

$$\mathrm{dist}_{H^1(B_1,\mu_a)}(c_j,\mathfrak{H}_{1+s})=\delta_j, \tag{4.34}$$

$$(1-\kappa)\mathcal{G}(c)\le\inf_{v\in\mathfrak{A}_c}\mathcal{G}(v). \tag{4.35}$$

In particular, fixing $h:=h_{e_n}$, up to change of coordinate depending on j, we assume that there exists $\lambda_j\ge 0$ for which $\psi_j:=\lambda_j h$ is the point satisfiying the minimum distance between c_j and $\mathfrak{H}_{1+s}$, or rather

$$\|\psi_j-c_j\|_{H^1(B_1,\mu_a)}=\mathrm{dist}_{H^1(B_1,\mu_a)}(c_j,\mathfrak{H}_{1+s})=\delta_j,\qquad\forall j\in\mathbb{N}. \tag{4.36}$$

We split the proof into some intermediate steps.

Step 1: Auxiliary functionals. We can rewrite (4.35) and interpret this inequality as a condition of quasi-minimality for a sequence of new functionals. Setting $j\in\mathbb{N}$, let $v\in\mathfrak{A}_{c_j}$, we use (4.29) (applied twice to ψ_j with test functions $c_j-\psi_j$ and $v-\psi_j$) and (4.31); we can rewrite (4.35):

$$\begin{aligned}(1-\kappa_j)&\left(\mathcal{G}(c_j)-\mathcal{G}(\psi_j)-\delta\mathcal{G}(\psi_j)[c_j-\psi_j]-4\int_{B_1'}(c_j-\psi_j)R_a(\psi_j)\,d\mathcal{H}^{n-1}\right)\\&\le\mathcal{G}(v)-\mathcal{G}(\psi_j)-\delta\mathcal{G}(\psi_j)[v-\psi_j]-4\int_{B_1'}(v-\psi_j)R_a(\psi_j)\,d\mathcal{H}^{n-1}.\end{aligned} \tag{4.37}$$

We can observe that $\mathcal{G}(v_1) - \mathcal{G}(v_2) - \delta\mathcal{G}(v_2)[v_1 - v_2] = \mathcal{G}(v_1 - v_2)$, then for all $v \in \mathfrak{A}_{c_j}$ (4.37) can be rewritten as

$$
\begin{aligned}
(1-\kappa_j)\left(\mathcal{G}(c_j - \psi_j) - 4\int_{B_1'} (c_j - \psi_j) R_a(\psi_j)\, d\mathcal{H}^{n-1}\right) \\
\leq \mathcal{G}(v - \psi_j) - 4\int_{B_1'} (v - \psi_j) R_a(\psi_j)\, d\mathcal{H}^{n-1}.
\end{aligned}
\tag{4.38}
$$

Next we define new sequences of functions

$$
z_j := \frac{c_j - \psi_j}{\delta_j} \tag{4.39}
$$

(recalling that $\psi_j = \lambda_j h$), positive numbers $\theta_j := \frac{\lambda_j}{\delta_j}$ and sets $\mathcal{B}_j := \{z \in z_j + H^1_0(B_1, \mu_a) : (z + \theta_j h)|_{B_1'} \geq 0\}$. Now we introduce a sequence of auxiliary functionals $\mathcal{G}_j : L^2(B_1, \mu_a) \to (-\infty, +\infty]$

$$
\mathcal{G}_j(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 \, |x_n|^a \, dx - (1+s)\int_{\partial B_1} z_j^2 \, |x_n|^a \, d\mathcal{H}^{n-1} - 4\theta_j \int_{B_1'} z R_a(h)\, d\mathcal{H}^{n-1} & \textit{if } z \in \mathcal{B}_j \\ +\infty & \textit{otherwise.} \end{cases}
\tag{4.40}
$$

We can observe that the second term in the formula above does not depend on z but only on its boundary datum $z|_{\partial B_1} = z_j|_{\partial B_1}$.

We can rewrite (4.38) with the new notation and obtain

$$
\begin{aligned}
(1-\kappa_j)\left(\mathcal{G}(\delta_j z_j) - 4\delta_j \int_{B_1'} z_j R_a(\lambda_j h)\, d\mathcal{H}^{n-1}\right) \\
\leq \mathcal{G}(\lambda_j z) - 4\delta_j \int_{B_1'} z R_a(\lambda_j h)\, d\mathcal{H}^{n-1}
\end{aligned}
$$

and dividing by δ_j^2 we obtain the condition of quasi-minimality for z_j with respect to $\mathcal{G}_j$:

$$
(1-\kappa_j)\mathcal{G}_j(z_j) \leq \mathcal{G}_j(z) \qquad \forall z \in L^2(B_1, \mu_a). \tag{4.41}
$$

Therefore we note that by the very definitions of z_j and δ_j we have

$$
\|z_j\|_{H^1(B_1,a} = 1. \tag{4.42}
$$

So, by the compactness of Sobolev embedding from $H^1(B_1, \mu_a)$ into the space $L^2(B_1, \mu_a)$ [62, Theorem 1.31], the trace operator from $H^1(B_1, \mu_a)$ into the space $L^2(B_1')$ [31, Theorem 3.4], and the trace operator from $H^1(B_1, \mu_a)$ into $L^2(\partial B_1, |x_n|\mathcal{H}^{n-1})$ [86, Lemma 2], we may extract a subsequence (that we do not rename) such that

(a) $(z_j)_{j\in\mathbb{N}}$ converges weakly in $H^1(B_1, \mu_a)$ to some z_∞;

(b) the sequences of traces $z_j|_{B'_1}$ and $z_j|_{\partial B_1}$ converge respectively in $L^2(B'_1)$ and $L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$;

(c) θ_j has a limit $\theta \in [0, \infty]$.

Step 2: First property of $(\mathcal{G}_j)_{j\in\mathbb{N}}$. In this step we establish the equi-coercivity and some other properties of the family $(\mathcal{G}_j)_{j\in\mathbb{N}}$.

We observe that for all $w \in \mathcal{B}_j$, since $w|_{\partial B_1} = z_j|_{\partial B_1}$ and $hR_a(h)(\widehat{x}) = 0$, it holds that

$$-\int_{B'_1} wR_a(h)(\widehat{x})\, d\mathcal{H}^{n-1} = -\int_{B'_1} (w + \theta_j h) R_a(h)(\widehat{x})\, d\mathcal{H}^{n-1} + \theta_j \int_{B'_1} hR_a(h)(\widehat{x})\, d\mathcal{H}^{n-1} \geq 0 \tag{4.43}$$

where we used (4.22) for which $R_a(h)(\widehat{x}) \leq 0$ and the condition $w \in \mathcal{B}_j$ for which $(w + \theta_j h)_{|B'_1} \geq 0$. Then from the definition of (4.40) we have

$$\int_{B_1} |\nabla w|^2 \, d\mu_a - (1+s) \int_{\partial B_1} z_j^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \leq \mathcal{G}_j(w). \tag{4.44}$$

This establishes the equi-coercivity of the sequence $\mathcal{G}_j$, in fact from (4.42), thanks to strong convergence of traces, we obtain

$$\liminf_{j\in\mathbb{N}} \mathcal{G}_j(z_j) \geq -(1+s) \int_{\partial B_1} z_\infty^2 \, |x_n|^a \, dx - 4\theta \int_{B'_1} z_\infty R_a(h)\, d\mathcal{H}^{n-1};$$

while if $\theta = +\infty$ from (4.42) and (4.44) we conclude that

$$\liminf_{j\in\mathbb{N}} \mathcal{G}_j(z_j) \geq -(1+s) \int_{\partial B_1} z_\infty^2 \, |x_n|^a \, dx.$$

Note that it is not restrictive (up to subsequence) to assume that $\mathcal{G}_j(z_j)$ has a limit in $(-\infty, +\infty]$. Finally we can observe that

$$\lim_{j\to\infty} \mathcal{G}_j(z_j) = +\infty \qquad \Longleftrightarrow \qquad \lim_{j\to\infty} \theta_j \int_{B'_1} z_j R_a(h)\, d\mathcal{H}^{n-1} = +\infty. \tag{4.45}$$

Step 3: Asymptotic analysis of $(\mathcal{G}_j)_{j\in\mathbb{N}}$. In this step we prove a result of Γ-convergence for the family of functionals $(\mathcal{G}_j)_{j\in\mathbb{N}}$.

We can distinguish three cases:

(1) If $\theta \in [0, +\infty)$, then $(z_\infty + \theta h)|_{B'_1} \geq 0$ and $\Gamma(L^2(B_1, \mu_a))$-$\lim \mathcal{G}_j = \mathcal{G}_\infty^{(1)}$ with

$$\mathcal{G}_\infty^{(1)}(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 \, |x_n|^a \, dx - (1+s) \int_{\partial B_1} z_\infty^2 \, |x_n|^a \, d\mathcal{H}^{n-1} - 4\theta \int_{B'_1} z R_a(h)\, d\mathcal{H}^{n-1} & \text{if } z \in \mathcal{B}_\infty^{(1)} \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{B}^{(1)}_\infty := \{z \in z_\infty + H^1_0(B_1,\mu_a) \ : \ (z+\theta h)|_{B_1'} = 0\}$.
(2) If $\theta = +\infty$ and $\lim_j \mathcal{G}_j(z_j) < \infty$, then $z_\infty|_{B_1^{',-}} = 0$ (where $B_1^{',-} = B' \cap \{x_{n-1} \leq 0\}$) and $\Gamma(L^2(B_1,\mu_a))$-$\lim \mathcal{G}_j = \mathcal{G}^{(2)}_\infty$ with

$$\mathcal{G}^{(2)}_\infty(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 \, |x_n|^a \, dx - (1+s) \int_{\partial B_1} z_\infty^2 \, |x_n|^a \, d\mathcal{H}^{n-1} & \text{if } z \in \mathcal{B}^{(2)}_\infty \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{B}^{(2)}_\infty := \{z \in z_\infty + H^1_0(B_1,\mu_a) \ : \ z|_{B_1^{',-}} = 0\}$. We note that the third addendum of $\mathcal{G}_j$ is identically zero in $\mathcal{B}^{(2)}_\infty$, while if $z \in \mathcal{B}_j \setminus \mathcal{B}^{(2)}_\infty$ the sequence $\mathcal{G}_j(z)$ diverges; this heuristically justifies the choice of $\mathcal{G}^{(2)}_\infty(z)$ and $\mathcal{B}^{(2)}_\infty$.
(3) If $\theta = +\infty$ and $\lim_j \mathcal{G}_j(z_j) = +\infty$, then $\Gamma(L^2(B_1,\mu_a))$-$\lim \mathcal{G}_j = \mathcal{G}^{(3)}_\infty$ with

$$\mathcal{G}^{(3)}_\infty(z) = +\infty \qquad \text{on } L^2(B_1,\mu_a).$$

For the reader's convenience we recall the Definition of Γ-limit (see Definition 1.5.1); the equality $\Gamma(L^2(B_1,\mu_a))$-$\lim \mathcal{G}_j = \mathcal{G}^{(i)}_\infty$ with $i = 1,2,3$ is satisfied if the two following conditions hold:

(a) for all sequences $(w_j)_j \subset L^2(B_1,\mu_a)$ and $w \in L^2(B_1,\mu_a)$ such that $w_j \to w$ in $L^2(B_1,\mu_a)$ it holds
$$\liminf_j \mathcal{G}_j(w_j) \geq \mathcal{G}^{(i)}_j(w) \tag{4.46}$$

(b) for all $w \in L^2(B_1,\mu_a)$ there exists a sequence $(w_j)_j \subset L^2(B_1,\mu_a)$ such that $w_j \to w$ in $L^2(B_1,\mu_a)$ and
$$\limsup_j \mathcal{G}_j(w_j) \leq \mathcal{G}^{(i)}_j(w). \tag{4.47}$$

Proof of the Γ*-convergence: case (1).*
(a) Without loss of generality we may suppose $\liminf_j \mathcal{G}_j(w_j) = \lim_j \mathcal{G}_j(w_j) < +\infty$. Taking (4.44) into account, we deduce

$$\int_{B_1} |\nabla w_j|^2 \, d\mu_a \leq \mathcal{G}_j(w_j) + (1+s) \int_{\partial B_1} w_j^2 \, |x_n|^a \, d\mathcal{H}^{n-1},$$

then, since $w_j \to w$ in $L^2(B_1,\mu_a)$ we have $\sup_j \|w_j\|^1_H(B_1,\mu_a) < +\infty$, so from [62, Theorem 1.31] $\nabla w_j \rightharpoonup \nabla w$ in $L^2(B_1,\mu_a)$. Then the respective traces converge in $L^2(\partial B_1,\mu_a)$ [86, Lemma 2] and $L^2(B_1')$ [31, Theorem 3.4]. Hence, we obtain $(w+\theta h)|_{B_1'} \geq 0$ and, in particular, since $w_j|_{B_1'} = z_j|_{B_1'}$ then $w|_{B_1'} = z_\infty|_{B_1'}$ and so $z_\infty \in \mathcal{B}^{(1)}_\infty$. At this point thanks to the convergence of traces of w_j and weak semicontinuity of the norm of the gradient in $L^2(B_1,\mu_a)$ we have (4.46).

(b) We observe that it is sufficient to prove the inequality for $w \in \mathcal{B}_\infty^{(1)}$ with

$$\mathrm{supp}(w - z_\infty) \subset B_\rho \qquad \text{for some } \rho \in (0,1). \tag{4.48}$$

If we want to deal with the general case, we consider the function

$$w_t(x) = t^{1+s}\left(w\left(\frac{x}{t}\right)\chi_{B_1}\left(\frac{x}{t}\right) + z_\infty\left(\frac{x}{t}\right)\chi_{B_{1/t}\setminus\overline{B_1}}\left(\frac{x}{t}\right)\right) \quad \text{with } t < 1.$$

It is easy to prove that $w_t \in H^1(B_1, \mu_a)$ and $\mathrm{supp}(w_t - z_\infty) \subset B_t$; moreover, $w_t \to w$ in $H^1(B_1, \mu_a)$ (for a similar procedure see Proposition 2.4.1 in Chapter 2). If (4.47) holds for all w_t, resorting to a diagonalization argument we obtain (4.47) for w. Therefore for a Uryshon's type property it is sufficient to prove the following property: fixing w as in (4.48), for all sub sequences $j_k \uparrow +\infty$ there exists an extract subsequence $j_{k_l} \uparrow +\infty$ and there exists $w_l \to w$ in $L^2(B_1, \mu_a)$ such that [1]

$$\limsup_l \mathcal{G}_{j_{k_l}}(w_l) \le \mathcal{G}_\infty^{(1)}(w).$$

Setting $r \in (\rho, 1)$ let $R := \frac{1+r}{2}$ and let $\varphi \in C_c^1(B_1)$ be a cut-off function such that

$$\varphi|_{B_r} \equiv 1, \qquad \varphi|_{B_1\setminus\overline{B_R}} \equiv 0, \qquad \|\nabla\varphi\|_{L^\infty} \le \frac{4}{1-r}.$$

We define

$$w_k^r := \varphi\left(w + (\theta - \theta_{j_k})h\right) + (1-\varphi)z_{j_k} \tag{4.49}$$

and we verify that $w_k^r \in \mathcal{B}_{j_k}$. In fact $w \in \mathcal{B}_\infty^{(1)}$, $z_{j_k} \in \mathcal{B}_{j_k}$ and

$$w_k^r + \theta_{j_k}h = \varphi(w + \theta h) + (1-\varphi)(z_{j_k} + \theta_{j_k}h) \ge 0.$$

Therefore, since $\theta_{j_k} \to \theta \in [0, +\infty)$ we have $w_k^r \to \varphi w + (1-\varphi)z_\infty$ in $L^2(B_1, \mu_a)$. Thanks to the convergence of traces of z_{j_k} in $L^2(B_1')$ it is enough to prove the upper bound inequality for the first addendum of $\mathcal{G}_j$ and $\mathcal{G}_\infty^{(1)}$ respectively. From (4.49), we can infer

$$\begin{aligned}\int_{B_1} |\nabla w_k^r|^2\, d\mu_a \le &\int_{B_r} |\nabla w + (\theta - \theta_{j_k})\nabla h|^2\, d\mu_a \\ &+ \underbrace{\int_{B_R\setminus\overline{B_r}} |\nabla w_k^r|^2\, d\mu_a}_{:=I_k} + \int_{B_1\setminus\overline{B_R}} |\nabla z_{j_k}|^2\, d\mu_a.\end{aligned} \tag{4.50}$$

[1] Let us suppose by contradiction that there exists w such that

$$\Gamma\text{-}\limsup_j \mathcal{G}_j(w) > \mathcal{G}_\infty^{(1)}(w),$$

if $(w_j)_{j\in\mathbb{N}}$ is a sequence that achieves the Γ-lim sup, i.e. $\limsup_j \mathcal{G}_j(w_j) = \Gamma\text{-}\limsup_j \mathcal{G}_j(w_j)$, and j_k is a subsequence for which $\limsup_j \mathcal{G}_j(w_j) = \limsup_k \mathcal{G}_{j_k}(w_{j_k})$, by assumption then there exists j_{k_l} such that

$$\lim_l \mathcal{G}_{j_{k_l}}(w_{j_{k_l}}) \le \mathcal{G}_\infty^{(1)}(w),$$

leading to a contradiction.

Since $r > \rho$, from assumption (4.48), we estimate the term I_k as follows[2]

$$
\begin{aligned}
I_k \leq & 3\int_{B_R\setminus\overline{B_r}} \varphi^2|\nabla w + (\theta-\theta_{j_k})\nabla h|^2\, d\mu_a \\
& + 3\int_{B_R\setminus\overline{B_r}}(1-\varphi)^2|\nabla z_{j_k}|^2\, d\mu_a + 3\int_{B_R\setminus\overline{B_r}} |\nabla\varphi|^2|z_\infty - z_{j_k} + (\theta-\theta_{j_k})\nabla h|^2\, d\mu_a
\end{aligned}
$$

So

$$
\begin{aligned}
& \limsup_k \int_{B_1} |\nabla w_k^r|^2\, d\mu_a \\
& \leq \int_{B_r} |\nabla w|^2\, d\mu_a + 3\int_{B_R\setminus\overline{B_r}} |\nabla w|^2\, d\mu_a + 4\limsup_k \int_{B_1\setminus\overline{B_r}} |\nabla z_{j_k}|^2\, d\mu_a
\end{aligned}
\tag{4.51}
$$

By the $(1+s)$-homogeneity of z_{j_k}, we deduce

$$
\begin{aligned}
\int_{B_1\setminus\overline{B_r}}|\nabla z_{j_k}|^2\, d\mu_a &= \int_r^1 \int_{\partial B_t} |\nabla z_{j_k}|^2\, |x_n|^a\, d\mathcal{H}^{n-1}\, dt \\
&= \int_r^1 t^n \int_{\partial B_1} |\nabla z_{j_k}|^2\, |x_n|^a\, d\mathcal{H}^{n-1}\, dt = \frac{1-r^{n+1}}{n+1}\int_{\partial B_1} |\nabla z_{j_k}|^2\, |x_n|^a\, d\mathcal{H}^{n-1}
\end{aligned}
$$

which leads us to

$$
\int_{\partial B_1} |\nabla z_{j_k}|^2\, |x_n|^a\, d\mathcal{H}^{n-1} = \frac{n+1}{1-(1/2)^{n+1}}\int_{B_1\setminus\overline{B_r}} |\nabla z_{j_k}|^2\, d\mu_a \overset{(4.42)}{\leq} 2(n+1)
$$

in turn implying

$$
\int_{B_1\setminus\overline{B_r}}|\nabla z_{j_k}|^2\, d\mu_a \leq 2\,(1-r)\,(n+1). \tag{4.52}
$$

We apply this construction to a subsequence $r_l \uparrow 1$ and $R_l := \frac{1+r_l}{2}$ and with a diagonal argument we obtain a subsequence $w_l \to w$ in $L^2(B_1, \mu_a)$. Thanks to (4.51) and (4.52)

$$
\begin{aligned}
& \limsup_l \int_{B_1} |\nabla w_l|^2\, d\mu_a \\
& \qquad \leq \int_{B_1} |\nabla w|^2\, d\mu_a + 3\limsup_l \int_{B_{R_l}\setminus\overline{B_{r_l}}} |\nabla w|^2\, d\mu_a + 4\limsup_l \int_{B_1\setminus\overline{B_{r_l}}} |\nabla z_{j_l}|^2\, d\mu_a \\
& \qquad \leq \int_{B_1} |\nabla w|^2\, d\mu_a + \lim_l 8\,(1-r_l)(n+1) = \int_{B_1} |\nabla w|^2\, d\mu_a,
\end{aligned}
$$

and this provides the conclusion.

Proof of the Γ*-convergence: case (2).*

(a) Without loss of generality we assume that

$$
\liminf_j \mathcal{G}_j(w_j) = \lim_j \mathcal{G}_j(w_j) < +\infty. \tag{4.53}
$$

[2] $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$

Let $w_j \to w$ in $L^2(B_1, \mu_a)$, since $w_j \in \mathcal{B}_j$ and (4.53), then $w \geq 0$ on $B_1'^{,-}$. From (4.43), we obtain

$$0 \leq -\theta_j \int_{B_1'} w_j R_a(h)\, d\mathcal{H}^{n-1} \leq \mathcal{G}_j(w_j) + (1+s) \int_{\partial B_1} z_j^2\, |x_n|^a\, d\mathcal{H}^{n-1}$$
$$\leq \sup_j \left(\mathcal{G}_j(w_j) + (1+s) \int_{\partial B_1} z_j^2\, |x_n|^a\, d\mathcal{H}^{n-1} \right) < +\infty.$$

Then dividing by θ_j, the convergence of traces leads us to

$$\int_{B_1'} w R_a(h)\, d\mathcal{H}^{n-1} = \lim_j \int_{B_1'} w_j R_a(h)\, d\mathcal{H}^{n-1} = 0$$

From (4.22) we deduce that $w|_{B_1'^{,-}} = 0$, or rather $w \in \mathcal{B}_\infty^{(2)}$. In particular also $z_\infty \in \mathcal{B}_\infty^{(2)}$ because $\sup_j \mathcal{G}_j(z_j) < +\infty$. Then, according to the semicontinuity of the norm $H^1(B_1, \mu_a)$ with respect to weak convergence of gradient, the convergence of w_j in $L^2(B_1, \mu_a)$ and the convergence of traces in $L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$ we obtain the Γ-$\liminf$ inequality (4.46).

(b) Now we prove the inequality (4.47). With the same argument used in case (1) we can consider the case of $w \in \mathcal{B}_\infty^{(2)}$ for which (4.48) holds and for which for all $j_k \uparrow +\infty$ we find a subsequence $j_{k_l} \uparrow +\infty$ and a sequence $w_l \to w$ in $L^2(B_1, \mu_a)$ such that

$$\limsup_l \mathcal{G}_{j_{k_l}}(w_l) \leq \mathcal{G}_\infty^{(2)}. \tag{4.54}$$

We introduce the positive Radon measures

$$\nu_k := |\nabla z_{j_k}|^2\, |x_n|^a\, \mathcal{L}^n \llcorner B_1 - 4\theta_{j_k}(z_{j_k} + \theta_{j_k} h)\, R_a(h) \mathcal{H}^{n-1} \llcorner B_1'^{,-}.$$

Assuming that $k >> 1$, we obtain

$$\nu_k(B_1) = \mathcal{G}_{j_k}(z_{j_k}) + (1+s) \int_{\partial B_1} z_{j_k}^2\, |x_n|^a\, d\mathcal{H}^{n-1} \leq \sup_j \mathcal{G}_j(z_j) + C \sup_j \|z_j\|_{H^1(B_1, \mu_a)} < \infty,$$

which leads us to

$$\sup_k \nu_k(B_1) = \Lambda_0 < +\infty.$$

In order to prove $\nu_k(B_\rho) = \rho^{n+1} \nu(B_1)$ we observe that setting $\rho \in (0,1)$ by $(1+s)$-

homogeneity of z_{j_k} we obtain

$$\begin{aligned}
\int_{B_\rho} |\nabla z_{j_k}|^2 \, d\mu_a &= \int_0^\rho dt \int_{\partial B_t} |\nabla z_{j_k}|^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \\
&\overset{x=ty}{=} \int_0^\rho t^{n-1} \int_{\partial B_1} |\nabla z_{j_k}(ty)|^2 \, |ty_n|^a \, d\mathcal{H}^{n-1}(y) \, dt \\
&= \int_0^\rho t^n \int_{\partial B_1} |\nabla z_{j_k}(y)|^2 \, |y_n|^a \, d\mathcal{H}^{n-1}(y) \, dt = \frac{\rho^{n+1}}{n+1} \int_{\partial B_1} |\nabla z_{j_k}(y)|^2 \, |y_n|^a \, d\mathcal{H}^{n-1}(y) \, dt \\
&= \rho^{n+1} \int_0^1 t^n \, dt \int_{\partial B_1} |\nabla z_{j_k}(y)|^2 \, |y_n|^a \, d\mathcal{H}^{n-1}(y) \, dt \\
&= \rho^{n+1} \int_0^1 t^{n-1} \, dt \int_{\partial B_1} |\nabla z_{j_k}(ty)|^2 \, |ty_n|^a \, d\mathcal{H}^{n-1}(y) \, dt \\
&\overset{ty=x}{=} \rho^{n+1} \int_0^1 \int_{\partial B_t} |\nabla z_{j_k}(x)|^2 \, |x_n|^a \, d\mathcal{H}^{n-1}(y) \, dt = \rho^{n+1} \int_{B_1} |\nabla z_{j_k}|^2 \, d\mu_a,
\end{aligned}$$

and

$$\begin{aligned}
\int_{B'_\rho} z_{j_k} \, R_a(h)(\widehat{x}) d\mathcal{H}^{n-1} &= \int_0^\rho \int_{\partial B'_t} z_{j_k} \, R_a(h)(\widehat{x}) d\mathcal{H}^{n-2} \\
&\overset{\widehat{x}=t\widehat{y}}{=} \int_0^\rho t^{n-2} \, dt \int_{\partial B'_1} z_{j_k}(t\widehat{y}, 0) \, R_a(h)(t\widehat{y}) d\mathcal{H}^{n-2}(\widehat{y}) \\
&= \int_0^\rho t^{n-2} \, dt \int_{\partial B'_1} z_{j_k}(t\widehat{y}, 0) \, \lim_{\varepsilon \to 0} (t\varepsilon)^a \frac{\partial h}{\partial x_n}(t\widehat{y}, t\varepsilon) d\mathcal{H}^{n-2}(\widehat{y}) \\
&= \int_0^\rho t^{n-2+1+s+1-2s+s} \, dt \int_{\partial B'_1} z_{j_k}(\widehat{y}, 0) \, R_a(h)(\widehat{y}) d\mathcal{H}^{n-2}(\widehat{y}) \\
&= \frac{\rho^{n+1}}{n+1} \int_{\partial B'_1} z_{j_k}(\widehat{y}, 0) \, R_a(h)(\widehat{y}) d\mathcal{H}^{n-2}(\widehat{y}) \\
&= \rho^{n+1} \int_{B'_1} z_{j_k} \, R_a(h)(\widehat{x}) d\mathcal{H}^{n-1}
\end{aligned}$$

where in the last equality we did the previous calculus again in reverse order. Since $\nu_k(B_1) < \infty$ then $\nu_k(\partial B_\rho) = 0$ with $\rho \in (0,1) \setminus I$ where I is a set at the most countable. Thus

$$\nu_k(B_{\rho_1} \setminus B_{\rho_2}) \leq \Lambda_0(\rho_1^{n+1} - \rho_2^{n+1}) \leq c(n, \Lambda_0)(\rho_1 - \rho_2), \tag{4.55}$$

for all $0 < \rho_1 \leq \rho_2 < 1$ such that $\rho_1, \rho_2 \in (0,1) \setminus I$. Repeating the argument in (4.48) we prove the Γ-$\limsup$ inequality for function $w \in \mathcal{B}_\infty^{(2)}$ for which there exists some $\rho \in (0,1)$ such that $\{w \not\equiv z_\infty\} \subset\subset B_\rho$. We extend w on $\mathbb{R}^n$ as z_∞ in B_ρ^c and we indicate the extension by w again. We fix $\varepsilon > 0$ and introduce the following auxiliary tools.

Due to the definition of $H^1(B_1, \mu_a)$ as $\overline{C^\infty(B_1)}^{\|\cdot\|_{H^1(B_1,\mu_a)}}$ (cf. [62, Section 1.9 and Lemma 1.15]) there exists a function $v_\delta \in C^\infty(B_1)$ such that

$$\|v^\delta - w\|_{H^1(B_1,\mu_a)} < \delta(\varepsilon) \qquad \textit{with } \delta(\varepsilon) = o(\varepsilon). \tag{4.56}$$

Let $w^\varepsilon(x) := w(x - 3\varepsilon e_{n-1})$ be the translated function along the direction e_{n-1}. Since $w \in \mathcal{B}^{(2)}_\infty$, we observe that

$$w^\varepsilon(x) = 0 \quad \Longleftrightarrow \quad x - 3\varepsilon e_{n-1} \in \{(\widehat{x}, 0) \ : \ x_{n-1} \leq 0\} \quad \Longleftrightarrow \quad x \in \{(\widehat{x}, 0) \ : \ x_{n-1} \leq 3\varepsilon\}.$$

Let I_σ be the set defined as

$$I_\sigma = \{x \in B_1 \ : \ \mathrm{dist}(x, B_1^{',-}) < \sigma\} \tag{4.57}$$

Let ϕ_ε and χ_ε be two cut-off functions such that

$$\begin{aligned} \phi_\varepsilon \in C^\infty_c(I_{3\varepsilon}), \quad \phi_{\varepsilon\ |I_{2\varepsilon}} \equiv 1, \quad \|\nabla \phi_\varepsilon\|_{L^\infty(B_1)} \leq \frac{C}{\varepsilon} \\ \chi_\varepsilon \in C^\infty_c(B_{1-\varepsilon}), \quad \chi_{\varepsilon\ |B_{1-2\varepsilon}} \equiv 1, \quad \|\nabla \chi_\varepsilon\|_{L^\infty(B_1)} \leq \frac{C}{\varepsilon}. \end{aligned} \tag{4.58}$$

For all $0 < \varepsilon << 1$ we build the sequence of functions

$$w_k^{(\varepsilon)} := \chi_\varepsilon(\phi_\varepsilon w^\varepsilon + (1 - \phi_\varepsilon) v^\delta) + (1 - \chi_\varepsilon) z_{j_k}.$$

Then we can at once infer

$$w_k^{(\varepsilon)} \in z_{j_k} + W^{1,2}_0(B_1)$$

and since we can write

$$w_k^{(\varepsilon)} + \theta_{j_k} h := \chi_\varepsilon(\phi_\varepsilon(v_\delta + \theta_{j_k} h) + (1 - \phi_\varepsilon)(w^\tau + \theta_{j_k} h)) + (1 - \chi_\varepsilon)(z_{j_k} + \theta_{j_k} h),$$

we prove that $w_k^{(\varepsilon)} \in \mathcal{B}_{j_k}$: $w_k^{(\varepsilon)}$ is a convex combination of functions v_δ, w^τ and z_{j_k} with boundary data as z_{j_k} and every addendum is bigger than $-\theta_{j_k} h$ restricted to B_1'. In fact

(i) by definition $z_{j_k} + \theta_{j_k} h \geq 0$ in B_1';

(ii) if $x \in \mathrm{supp}(\phi_\varepsilon) \cap B_1'$ then $x_{n-1} < 3\varepsilon$. Thus $w^\varepsilon(x) = 0$ then $\phi_\varepsilon(x)(w^\tau(x) + \theta_{j_k} h(x)) = \phi_\varepsilon(x)\theta_{j_k} h(x) \geq 0$;

(iii) if $x \in \mathrm{supp}(1 - \phi_\varepsilon) \cap B_1'$ then $x_{n-1} \geq 2\varepsilon$, so $h(\widehat{x}, 0) > 0$ and as $\theta_{j_k} \to +\infty$ $v^\delta(x) + \theta_{j_k} h(x) \geq -\|v_\delta\|_{L^\infty(B_1)} + \theta_{j_k} h(x) \geq 0$ for $k > k_\delta$.

So $w_k^{(\varepsilon)} \in \mathcal{B}_{j_k}$ for $k > k_\delta$.
Next, consider,

$$\begin{aligned} J_k^\varepsilon &:= -4\theta_{j_k} \int_{B_1'} w_k^{(\varepsilon)} R_a(h)\, d\mathcal{H}^{n-1} \\ I_k^\varepsilon &:= \int_{B_1} |\nabla w_k^{(\varepsilon)}|^2\, d\mu_a, \end{aligned}$$

respectively the trace term and the volume term of the energy of $w_k^{(\varepsilon)}$. By definition we have

$$J_k^\varepsilon \le -4\theta_{j_k}\int_{B'_{1-\varepsilon}}(\phi_\varepsilon w^\varepsilon + (1-\phi_\varepsilon)v^\delta)R_a(h)\,d\mathcal{H}^{n-1} - 4\theta_{j_k}\int_{B'_1\setminus B'_{1-2\varepsilon}} z_{j_k}R_a(h)d\mathcal{H}^{n-1} = J_k^{(1)} + J_k^{(2)}.$$

According to (i), (4.23) and (4.55) we deduce

$$0 \le \sup_k J_k^{(2)} \le \sup_k \nu_k(B_1 \setminus B_{1-2\varepsilon}) \le C\,2\varepsilon. \tag{4.59}$$

Instead, due to (ii), the function $w^\varepsilon_{|B'_1\cap G_{3\varepsilon}} = 0$ and from definitions of $I_{2\varepsilon}$ and h we have $R_a(h)_{|B'_{1-\varepsilon}\setminus I_{2\varepsilon}} = 0$. From this we infer

$$0 \le J_k^{(1)} \le -4\theta_{j_k}\left(\int_{B'_{1-\varepsilon}\cap I_{3\varepsilon}} w^\varepsilon R_a(h)d\mathcal{H}^{n-1} + \int_{B'_{1-\varepsilon}\setminus I_{3\varepsilon}} v^\delta R_a(h)d\mathcal{H}^{n-1}\right) = 0. \tag{4.60}$$

Putting (4.59) and (4.60) together yields

$$\limsup_{k\to\infty} J_k^\varepsilon \le C\varepsilon. \tag{4.61}$$

In order to estimate the functional I_k^ε we observe that

$$\begin{aligned}
I_k^\varepsilon &\le \int_{B_{1-2\varepsilon}} |\nabla(\phi_\varepsilon w^\varepsilon + (1-\phi_\varepsilon)\,v^\delta)|^2\,d\mu_a + c\int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} |\nabla(\phi_\varepsilon w^\varepsilon + (1-\phi_\varepsilon)v^\delta)|^2\,d\mu_a \\
&\quad + c\int_{B_1\setminus B_{1-2\varepsilon}} |\nabla z_{j_k}|^2\,d\mu_a + \frac{c}{\varepsilon^2}\int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} |(\phi_\varepsilon w^\varepsilon + (1-\phi_\varepsilon)v^\delta - z_{j_k}|^2\,d\mu_a \\
&= I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)}.
\end{aligned}$$

We estimate the four addenda separately. From condition (4.55), we can infer

$$\sup_k I_k^{(3)} \le \sup_k \nu_k(B_1 \setminus B_{1-2\varepsilon})C\,\varepsilon. \tag{4.62}$$

We now estimate the first term; recalling that $\phi_{\varepsilon|I^c_{3\varepsilon}} = 0$

$$\begin{aligned}
I_k^{(1)} &= \int_{B_{1-2\varepsilon}\setminus I_{3\varepsilon}} |\nabla v^\delta|^2\,d\mu_a + \int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} \left|\nabla\left(\phi_\varepsilon(w^\varepsilon - v^\delta)\right)\nabla v^\delta\right|^2\,d\mu_a \\
&\le \int_{B_{1-2\varepsilon}\setminus I_{3\varepsilon}} |\nabla v^\delta|^2\,d\mu_a + c\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |\nabla v^\delta|^2\,d\mu_a \\
&\quad + \int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |\nabla(w^\varepsilon - v^\delta)|^2\,d\mu_a + \frac{c}{\varepsilon^2}\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |v^\delta - w^\varepsilon|^2\,d\mu_a \\
&\le \int_{B_{1-2\varepsilon}\setminus I_{3\varepsilon}} |\nabla v^\delta|^2\,d\mu_a + c\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |\nabla(v^\delta - \nabla w)|^2\,d\mu_a \\
&\quad + c\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |\nabla w|^2\,d\mu_a + \frac{c}{\varepsilon^2}\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} (|v^\delta - w|^2 + |w - w^\varepsilon|^2)\,d\mu_a.
\end{aligned} \tag{4.63}$$

Taking the last addendum above into account, we notice that for all φ smooth functions and $\tau > 0$

$$|\varphi(x - \tau e_{n-1}) - \varphi(x)| \leq \tau \int_0^1 |\nabla \varphi|(x - \tau t e_{n-1})\, dt.$$

Then, by a simple application of Fubini's theorem we deduce

$$\frac{c}{\varepsilon^2} \int_{B_{1-2\varepsilon} \cap I_{3\varepsilon}} |\varphi(x - \tau e_{n-1}) - \varphi(x)|^2\, d\mu_a \leq c \frac{\tau^2}{\varepsilon^2} \int_{(B_{1-2\varepsilon} \cap I_{3\varepsilon}) + [0,\tau] e_{n-1}} |\nabla \varphi|^2\, d\mu_a$$

where $(B_{1-2\varepsilon} \cap I_{3\varepsilon}) + [0, \tau] e_{n-1}$ denotes the Minkowski sum between sets. So, thanks to a density argument and for $\tau = 3\varepsilon$ we infer

$$\frac{c}{\varepsilon^2} \int_{G_{2\varepsilon} \setminus G_{3\varepsilon}} |w - w^\varepsilon|^2\, d\mu_a \leq c \int_{(B_{1-2\varepsilon} \cap I_{3\varepsilon}) + [0,\tau] e_{n-1}} |\nabla w|^2\, d\mu_a.$$

So, from (4.63), according to (4.56), the continuity of translation in L^2 and the absolute continuity of the integral, and observing that $\mathcal{L}^n((B_{1-2\varepsilon} \cap I_{3\varepsilon}) + [0, \tau] e_{n-1}) = O(\varepsilon)$ we obtain

$$I_k^{(1)} \leq \int_{B_{1-2\varepsilon} \setminus I_{3\varepsilon}} |\nabla v_\varepsilon|^2\, d\mu_a + O(\varepsilon). \tag{4.64}$$

Reasoning in the same way as in the estimate of $I_k^{(1)}$ we obtain

$$I_k^{(2)} \leq O(\varepsilon) \tag{4.65}$$

Since $\mathrm{supp}\phi_\varepsilon \subset G_{2\varepsilon}$ and recalling that by condition (4.48), if we choose ε sufficiently small such that $\rho < 1 - 5\varepsilon$, $\mathrm{supp}(w^{3\varepsilon} - z_\infty^{3\varepsilon}) \subset B_{1-2\varepsilon}$, we obtain

$$\begin{aligned} I_k^{(4)} &\leq \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} |\phi_\varepsilon(w^\varepsilon - v^\delta)|^2\, d\mu_a + \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} |v^\delta - z_{j_k}|^2\, d\mu_a \\ &\leq \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} (|w^\varepsilon - w|^2 + |w - v^\delta|^2 + |w - z_{j_k}|^2)\, d\mu_a \end{aligned}$$

So, proceeding as in estimate of $I_k^{(1)}$ and recalling that $\mathrm{supp}(w - z_\infty) \subset B_\rho$ for ε sufficiently small we deduce

$$\limsup_{k \to \infty} I_k^{(4)} \leq \limsup_{k \to \infty} \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon} \setminus B_{1-2\varepsilon}} |z_\infty - z_{j_k}|^2\, d\mu_a + O(\varepsilon) \leq O(\varepsilon). \tag{4.66}$$

Then putting together the estimates in (4.62), (4.64), (4.65) and (4.66) leads to

$$\limsup_{k \to \infty} I_k^\varepsilon \leq \int_{B_{1-2\varepsilon} \setminus I_{3\varepsilon}} |\nabla v_\varepsilon|^2\, d\mu_a + O(\varepsilon).$$

So, since

$$w_k^{(\varepsilon)} \xrightarrow{k \to \infty} \chi_\varepsilon(\phi_\varepsilon v_{\delta_\varepsilon} + (\phi_\varepsilon) w^{3\varepsilon}) + (1 - \chi_\varepsilon) z_\infty =: w^{(\varepsilon)} \qquad \textit{in } L^2(B_1, \mu_a)$$

and

$$w^{(\varepsilon)} \xrightarrow{\varepsilon \to 0} w \qquad in \ L^2(B_1, \mu_a),$$

we conclude by the lower semicontinuity of the Γ-lim sup

$$\begin{aligned}\Gamma\!-\limsup_{k\to\infty} \mathcal{G}_{j_k}(w) &\le \liminf_{\varepsilon\to 0}\left(\Gamma\!-\limsup_{k\to\infty} \mathcal{G}_{j_k}(w^{(\varepsilon)})\right)\\ &\le \limsup_{\varepsilon\to 0}\left(\limsup_{k\to\infty}(I_k^\varepsilon + J_k^\varepsilon)\right) \le \int_{B_1} |\nabla w|^2 \, d\mu_a,\end{aligned}$$

that provides the thesis.

Proof of the Γ*-convergence: case (3).*
(a) From (4.41), we immediately have

$$\liminf_j \mathcal{G}_j(w_j) \ge \liminf_j (1-\kappa_j)\mathcal{G}_j(z_j) = +\infty = \mathcal{G}_\infty^{(3)}.$$

(b) This is trivial, in fact $\liminf_j \mathcal{G}_j(w_j) \le +\infty = \mathcal{G}_\infty^{(3)}$.

Step 4: Improving the convergence of $(z_j)_j \in \mathbb{N})$ **if** $\lim_j \mathcal{G}_j(z_j) < +\infty$**.** Using a standard result of Γ-convergence we show that $z_j \to z_\infty$ in $H^1(B_1, \mu_a)$.
For equi-coercivity of $\mathcal{G}_j$ seen in (4.44), [21, Lemma 2.10] (a version of Poincaré inequality for weighted Sobolev spaces) and $\|z_j\|_{H^1(B_1,\mu_a)} = 1$ we have

$$\|w\|_{H^1(B_1,\mu_a)} \le C\sqrt{\mathcal{G}_j(w)+1},$$

so every minimizing sequence converges weakly in $H^1(B_1, \mu_a)$ and thanks to [58, Theorem 8.1] converges strongly in $L^2(B_1, \mu_a)$. Since $\mathcal{G}_j$ is semicontinuous with respect to weak topology of $H^1(B_1, \mu_a)$ there exists ζ_j minimizer of $\mathcal{G}_j$. Taking into account Theorem 1.5.3, with $i = 1, 2$ there exists $\zeta_\infty \in H^1(B_1, \mu_a)$ such that

$$\zeta_j \to \zeta_\infty, \qquad in \ L^2(B_1, \mu_a) \tag{4.67}$$

$$\mathcal{G}_j(\zeta_j) \to \mathcal{G}_\infty^{(i)}(\zeta_\infty), \tag{4.68}$$

$$\zeta_\infty \text{ is the unique minimizer of } \mathcal{G}_\infty^{(i)}, \tag{4.69}$$

where due to (4.69) we have used the strict convexity of $\mathcal{G}_\infty^{(i)}$. Therefore using the strong convergence of traces in $L^2(\partial B_1, |x_n|^a\mathcal{H}^{n-1})$ and $L^2(B_1')$, then from the estimates

$$\mathcal{G}_j(\zeta_j) \le \mathcal{G}_j(z_j) \le \sup_j \mathcal{G}_j(z_j) < \infty, \tag{4.70}$$

and (4.69) we obtain

$$\int_{B_1} |\nabla \zeta_j|^2 \, d\mu_a \to \int_{B_1} |\nabla \zeta_\infty|^2 \, d\mu_a,$$

which implies $\zeta_j \to \zeta_\infty$ in $H^1(B_1, \mu_a)$. According to (4.41) and (4.70), z_j is an almost minimizer of $\mathcal{G}_j$ in the following sense

$$0 \le \mathcal{G}_j(z_j) - \mathcal{G}_j(\zeta_j) \le \kappa_j \mathcal{G}_j(z_j) \le \kappa_j \sup_j \mathcal{G}_j(z_j).$$

Since $\kappa_j \downarrow 0$ and $z_j \rightharpoonup z_\infty$ in $H^1(B_1, \mu_a)$, (4.68) and Step 3 yield that

$$\mathcal{G}_\infty^{(i)}(z_\infty) \le \liminf_j \mathcal{G}_j(z_j) = \lim_j \mathcal{G}_j(\zeta_j) = \mathcal{G}_\infty^{(i)}(\zeta_\infty), \tag{4.71}$$

with $i = 1, 2$. From (4.41), we infer

$$\mathcal{G}_j(z_j) \le \frac{1}{1 - k_j} \mathcal{G}_j(\zeta_j);$$

from this, by (4.71) and by strong convergence of traces we obtain

$$\liminf_j \int_{B_1} |\nabla z_j|^2 \, d\mu_a = \int_{B_1} |\nabla z_\infty|^2 \, d\mu_a,$$

that with the weak convergence of in $H^1(B_1, \mu_a)$ proves the convergence

$$z_j \to z_\infty \qquad\qquad in \;\; H^1(B_1, \mu_a).$$

In particular

$$\|z_\infty\|_{H^1(B_1, \mu_a)} = 1. \tag{4.72}$$

Step 5: Case (1) cannot occur. We recall the properties of z_∞:

(i) $\|z_\infty\|_{H^1(B_1,\mu_a)} = 1$;

(ii) z_∞ is $(1+s)$-homogeneous and even with respect to $\{x_n = 0\}$;

(iii) z_∞ is the unique minimizer of $\mathcal{G}_\infty^{(1)}$ with respect to its boundary data;

(iv) $z_\infty \in \mathcal{B}_\infty^{(1)} = \{z \in z_\infty + H_0^1(B_1, \mu_a) \, : \, (z + \theta h)|_{B_1'} \ge 0\}$.

These properties imply that

$$w_\infty := z_\infty + \theta h$$

is the minimizer of $\int_{B_1} |\nabla \cdot|^2 \, d\mu_a$ among all functions $w \in w_\infty + H_0^1(B_1, \mu_a)$ and $w|_{B_1'} \ge 0$ in the sense of the trace. So, w_∞ is the solution of the fractional obstacle problem. To prove this claim, for all $z \in \mathcal{B}_\infty^{(1)}$ we consider $w := z + \theta h$ and, recalling (4.29), we have

$$\begin{aligned}
\mathcal{G}_\infty^{(1)}(z) &= \int_{B_1} |\nabla w|^2 \, d\mu_a - \theta^2 \int_{B_1} |\nabla h|^2 \, d\mu_a - (1+s) \int_{B_1} z_\infty^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \\
&\quad - 2\theta \int_{B_1} \nabla w \cdot \nabla h \, d\mu_a - 4\theta \int_{B_1'} z \lim_{\varepsilon \to 0} \left(\varepsilon^a \, \frac{\partial h}{\partial x_n}(\widehat{x}, \varepsilon) \right) d\mathcal{H}^{n-1} \\
&\overset{(4.29)}{=} \int_{B_1} |\nabla w|^2 \, d\mu_a - \theta^2 \int_{B_1} |\nabla h|^2 \, d\mu_a - (1+s) \int_{B_1} z_\infty^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \\
&\quad - 2(1+s) \int_{\partial B_1} z_\infty \, h \, |x_n|^a \, d\mathcal{H}^{n-1}.
\end{aligned}$$

Since $\mathcal{G}^{(1)}_\infty(z_\infty) \le \mathcal{G}^{(1)}_\infty(z)$ for all $z \in \mathcal{B}^{(1)}_\infty$ then

$$\int_{B_1} |\nabla w_\infty|^2 \, d\mu_a \le \int_{B_1} |\nabla w|^2 \, d\mu_a \qquad\qquad \forall w \in w_\infty + H^1_0(B_1, \mu_a).$$

Using the $(1+s)$-homogeneity and [21, Proposition 5.5], the result of classification of the global solutions, we deduce that $w_\infty = \lambda_\infty h_{\nu_\infty} \in \mathfrak{H}_{1+s}$ for some $\lambda_\infty \ge 0$ and $\nu_\infty \in \mathbb{S}^{n-2}$.

Thanks to (4.36) we have the contradiction: from $z_j \rightharpoonup z_\infty$ in $H^1(B_1, \mu_a)$ and (4.39) we have

$$\frac{c_j}{\delta_j} = \theta_j h + z_j \to \theta h + z_\infty \in \mathfrak{H}_{1+s} \qquad\qquad in \;\; H^1(B_1, \mu_a), \tag{4.73}$$

so for $j >> 1$

$$\mathrm{dist}_{H^1(B_1,\mu_a)}(c_j, \mathfrak{H}_{1+s}) \le \|c_j - \delta_j \lambda_\infty h_{\nu_\infty}\|_{H^1(B_1,\mu_a)} \overset{(4.73)}{=} o(\delta_j) < \delta_j = \mathrm{dist}_{H^1(B_1,\mu_a)}(c_j, \mathfrak{H}_{1+s})$$

where we have used that $\delta_j \lambda_\infty h_{\nu_\infty} \in \mathfrak{H}_{1+s}$.

Step 6: Case (3) cannot occur. To prove that case (3) cannot occur, we conveniently scale the energies so as to get a non trivial Γ-limit for the rescaled functionals ultimately leading to a contradiction.

By means (4.45), since $\lim_j \mathcal{G}_j(z_j) = +\infty$, we have

$$\gamma_j := -4\theta_j \int_{B_1'} z_j \, R_a(h)(\widehat{x}) \, d\mathcal{H}^{n-1} \uparrow +\infty. \tag{4.74}$$

Moreover $z_j \to z_\infty$ in $L^2(B_1')$ and (4.43) give us

$$\lim_j \frac{\gamma_j}{\theta_j} = -4 \lim_j \int_{B_1'} z_j \, R_a(h) \, d\mathcal{H}^{n-1} = -4 \int_{B_1'} z_\infty \, R_a(h) \, d\mathcal{H}^{n-1} \in [0, \infty)$$

so

$$\theta_j \gamma_j^{-1/2} \uparrow +\infty. \tag{4.75}$$

Then we rescale the functional $\mathcal{G}_j$ dividing by γ_j. For all $z \in \mathcal{B}_j$ we consider $\gamma_j^{-1}\mathcal{G}_j(z)$ and we note that

$$\gamma_j^{-1}\mathcal{G}_j(z) = \widetilde{\mathcal{G}}_j(\gamma_j^{-1/2} z) \tag{4.76}$$

with

$$\widetilde{\mathcal{G}}_j(w) = \begin{cases} \displaystyle\int_{B_1} |\nabla w|^2 \, d\mu_a - (1+s) \int_{\partial B_1} w^2 \, |x_n|^a \, d\mathcal{H}^{n-1} - 4 \frac{\theta_j}{\gamma_j^{1/2}} \int_{B_1'} w \, R_a(h) \, d\mathcal{H}^{n-1} & w \in \widetilde{\mathcal{B}}_j \\ +\infty & \textit{otherwise}, \end{cases}$$

where

$$\widetilde{\mathcal{B}}_j := \{ w \in \gamma_j^{-1/2} z_j + H^1_0(B_1, \mu_a) \; : \; (w + \theta_j \gamma_j^{-1/2} h)|_{B_1'} \ge 0 \}.$$

Setting $\widetilde{z}_j := \gamma_j^{-1/2} z_j$, due to (4.42) and $\gamma_j \uparrow +\infty$, we have $\widetilde{z}_j \to 0$ in $H^1(B_1,\mu_a)$. Moreover the condition (4.76) and the definition of γ_j (4.74) yield

$$\widetilde{\mathcal{G}}_j(\widetilde{z}_j) = \frac{\int_{B_1} |\nabla w|^2 \, d\mu_a - (1+s)\int_{\partial B_1} w^2 \, |x_n|^a \, d\mathcal{H}^{n-1}}{\gamma_j} + 1 = 1 + O(\gamma_j^{-1}). \tag{4.77}$$

Thanks to (4.76) we can rewrite the inequalities (4.41) as

$$(1-\kappa_j)\widetilde{\mathcal{G}}_j(\widetilde{z}_j) \leq \widetilde{\mathcal{G}}_j(\widetilde{z}) \qquad \forall \widetilde{z} \in \widetilde{\mathcal{B}}_j.$$

In particular, by taking into consideration (4.75), $\widetilde{z}_j \to 0$ in $H^1(B_1,\mu_a)$, and (4.77) (in other words $\lim_j \widetilde{\mathcal{G}}_j(\widetilde{z}_j) < \infty$) we proceed as in case (2) of Step 3 establishing that

$$\Gamma(L^2(B_1,\mu_a))\text{-}\lim_j \widetilde{\mathcal{G}}_j = \widetilde{\mathcal{G}_\infty},$$

with

$$\widetilde{\mathcal{G}_\infty}(\widetilde{z}) = \begin{cases} \displaystyle\int_{B_1} |\nabla \widetilde{z}|^2 \, d\mu_a & w \in \widetilde{\mathcal{B}}_\infty \\ +\infty & \text{otherwise,} \end{cases}$$

where $\widetilde{\mathcal{B}}_\infty = \{\widetilde{z} \in H^1_0(B_1,\mu_a) \ : \ \widetilde{z}|_{B_1'} = 0\}$. From Step 4 and the convergence $\widetilde{z}_j \to 0$ in $H^1(B_1,\mu_a)$, the null function turns out to be the unique minimizer of $\widetilde{\mathcal{G}_\infty}$ and $\lim_j \widetilde{\mathcal{G}}_j(\widetilde{z}_j) \to \widetilde{\mathcal{G}_\infty}(0) = 0$; this is in contradiction with (4.77).

To prove the theorem we have only to exclude case (2) of Step 3. In what follows, we suppose the hypothesis of case (2) of Step 3: $\theta = +\infty$ and $\lim_j \mathcal{G}_j(z_j) < +\infty$. In the following steps we exhibit further properties of the limit z_∞.

Step 7: An orthogonality condition. By evaluating that ψ_j is a point of minimal distance between c_j and $\mathfrak{H}_{1+s}$, we prove that z_∞ is orthogonal to the tangent space $T_h\mathfrak{H}_{1+s}$.

From the hypothesis $\theta = +\infty$ we deduce that $\lambda_j > 0$ for $j >> 1$. Therefore, by the condition of minimal distance (4.36), we deduce that for all $\nu \in \mathbb{S}^{n-2}$ and $\lambda \geq 0$,

$$\|c_j - \psi_j\|_{H^1(B_1,\mu_a)} \leq \|c_j - \lambda\, h_\nu\|_{H^1(B_1,\mu_a)},$$

and thanks to definition of z_j in (4.39) it holds

$$\delta_j \|z_j\|_{H^1(B_1,\mu_a)} \leq \|\psi_j - \lambda\, h_\nu + \delta z_j\|_{H^1(B_1,\mu_a)}$$

or in the same way

$$-\|\psi_j - \lambda\, h_\nu\|^2_{H^1(B_1,\mu_a)} \leq 2\delta_j \langle z_j, \psi_j - \lambda h_\nu\rangle_{H^1(B_1,\mu_a)}. \tag{4.78}$$

Now we suppose $(\lambda,\nu) \neq (\lambda_j, e_{n-1})$ and renormalizing (4.78) we obtain

$$-\|\psi_j - \lambda\, h_\nu\|_{H^1(B_1,\mu_a)} \leq 2\delta_j \langle z_j, \frac{\psi_j - \lambda h_\nu}{\|\psi_j - \lambda\, h_\nu\|_{H^1(B_1,\mu_a)}}\rangle_{H^1(B_1,\mu_a)}$$

and by passing to the limit $(\lambda,\nu)\to(\lambda_j,e_{n-1})$, reminding the definition of tangent space $T\mathfrak{H}_{1+s}$ in (4.26), we deduce

$$\langle z_j,\zeta\rangle\geq 0 \qquad \zeta\in T_{\psi_j}\mathfrak{H}_{1+s}=T_h\mathfrak{H}_{1+s}, \tag{4.79}$$

where we used $\lambda_j>0$ in the computation of the tangent vector. By choosing the sequence $(\lambda,\nu)\to(\lambda_j,e_{n-1})$ such that $\lim\frac{\psi_j-\lambda h_\nu}{\|\psi_j-\lambda h_\nu\|_{H^1(B_1,\mu_a)}}=-\zeta$ we obtain $\langle z_j,\zeta\rangle\leq 0$ thus

$$\langle z_j,\zeta\rangle=0 \qquad \zeta\in T_h\mathfrak{H}_{1+s}. \tag{4.80}$$

So, taking the limit $j\to+\infty$ we conclude

$$\langle z_\infty,\zeta\rangle=0 \qquad \zeta\in T_h\mathfrak{H}_{1+s}. \tag{4.81}$$

Step 8: Identification of z_∞ in case (2). There exist real constants $a_0,\dots,a_{n-2}$ such that

$$z_\infty=a_0h+\left(\sum_{i=1}^{n-2}a_ix_i\right)\left(\sqrt{x_{n-1}^2+x_n^2}+x_{n-1}\right)^s, \tag{4.82}$$

or rather $z_\infty\in T_h\mathfrak{H}_{1+s}$. For its proof we follow an argument given in [44, Lemma A.3] that we recall below for the reader's convenience.

The minimum z_∞ is the solution of

$$\begin{cases} L_az_\infty=0 & B_1\setminus B'^{,-}_1\\ z_\infty=0 & B'^{,-}_1, \end{cases}$$

We note that for all multi-indices $\alpha\in\mathbb{N}^{n-2}$ the derivative $\partial_\alpha z_\infty$ is the solution of

$$\begin{cases} L_a\partial_\alpha z_\infty=0 & B_1\setminus B'^{,-}_1\\ \partial_\alpha z_\infty=0 & B'^{,-}_1, \end{cases} \tag{4.83}$$

According to [29, Lemma 2.4.1] and [21, Proposition 2.3] the derivative $\partial_\alpha z_\infty$ are bounded in $B_{1/2}$, thanks to [29, Theorems 2.3.12 and 2.4.6] they are also continuous in $B_{1/2}\setminus\{x_{n-1}=x_n=0\}$. We consider the second derivative $\partial_{ij}z_\infty$ with $i,j=1,\dots,n-2$: since z_∞ is $(1+s)$-homogeneous, the function $\partial_{ij}z_\infty$ is $(s-1)$-homogeneous; as $0<s<1$ from the boundedness of the derivative we deduce

$$\partial_{ij}z_\infty=0 \qquad in\ B_1 \qquad \forall i,j=1,\dots,n-2. \tag{4.84}$$

The solution z_∞ is a smooth function in $B^+_{1/2}$ and $B^-_{1/2}$ because the coefficients of the strictly elliptic operator L_a are smooth in these domains. Thus, fixed x_{n-1} and x_n, we can write the first order Taylor polynomial of $z_\infty(\cdot,x_{n-1},x_n)$ in $(\underline{0}',x_{n-1},x_n)$

$$z_\infty(x',x_{n-1},x_n)=c_0(x_{n-1},x_n)+\sum_{i=1}^{n-2}c_i(x_{n-1},x_n)x_i,$$

with $c_0(x_{n-1}, x_n) = z_\infty(\underline{0}', x_{n-1}, x_n)$ and $c_i(x_{n-1}, x_n) = \partial_i z_\infty(\underline{0}', x_{n-1}, x_n)$. By definition the function $c_0(x_{n-1}, x_n)$ is $(1+s)$-homogeneous and the functions $c_i(x_{n-1}, x_n)$ are s-homogeneous. Since z_∞ and $\partial_i z_\infty$ are continuous in $B_{1/2} \setminus \{x_{n-1} = x_n = 0\}$ the function $c_0(x_{n-1}, x_n)$ and $c_i(x_{n-1}, x_n)$ are continuous in $\mathcal{B}_{1/2} \setminus \{x_{n-1} = 0\}$ with $\mathcal{B}_{1/2} := \{(x_{n-1}, x_n) \in \mathbb{R}^2 \ : \ x_{n-1}^2 + x_n^2 < 1/4\}$. Thanks to homogeneity with positive degree $c_0(x_{n-1}, x_n)$ and $c_i(x_{n-1}, x_n)$ are continuous in $\mathcal{B}_{1/2}$.

Taking into account (4.84), for all $i = 1, \ldots, n-2$ we obtain

$$\begin{aligned} c_i(x_{n-1}, x_n) &= \partial_i z_\infty(x', x_{n-1}, x_n) \\ c_0(x_{n-1}, x_n) &= z_\infty(x', x_{n-1}, x_n), \end{aligned}$$

thus $c_i, c_0 \in H^1(\mathcal{B}^\pm_{1/2}, |x_n|^a \mathcal{L}^2)$ and are solutions of (4.83) on $\mathcal{B}^\pm_{1/2}$. Since $c_i(x_{n-1}, x_n)$ is s-homogeneous there exist some constants $(\tilde{a}_i)_{i=1,\ldots,n-2}$ such that $c_i(x_{n-1}, 0) = \tilde{a}_i x_{n-1}^s$ when $x_{n-1} > 0$ and similarly since $c_0(x_{n-1}, x_n)$ is $(1+s)$-homogeneous, there exists a constant $\tilde{a}_0$ such that $c_0(x_{n-1}, 0) = \tilde{a}_0 x_{n-1}^s$ when $x_{n-1} > 0$.

We show that

$$c_i(x_{n-1}, x_n) = \frac{\tilde{a}_i}{2^s} \left(x_{n+1} + \sqrt{x_{n-1}^2 + x_n^2} \right)^s. \tag{4.85}$$

Passing to polar coordinates we can write $c_i(x_{n-1}, x_n) = d_i(r, \theta) = r^s \varphi_i(\theta)$. From $L_a c_i = 0$ we deduce that the function φ_i is the solution of the following second order ordinary differential equation

$$\begin{cases} \sin\theta \varphi_{\theta\theta} + a\cos\theta\varphi_\theta + (a(1+s)x + (1+s)^2)\sin\theta\varphi = 0 & \textit{in } (0, \pi) \\ \varphi(0) = \frac{\tilde{a}_i}{2^s} \\ \varphi(\pi) = 0, \end{cases}$$

and so it has a unique solution. Resorting to a direct calculation, we can verify that the function

$$\varphi_i(\theta) = \frac{\tilde{a}_i}{2^s}(\cos\theta + 1)^s$$

is solution for all $\theta \in [0, \pi]$. So the function $c_i(x_{n-1}, x_n)$ satisfies (4.85).

By proceeding in the same way we prove that the function $c_0(x_{n-1}, x_n)$ can be written as

$$c_0(x_{n-1}, x_n) = \frac{\tilde{a}_0}{2^s(s-1)} \left(x_{n+1} + \sqrt{x_{n-1}^2 + x_n^2} \right)^s \left(x_{n+1} - \sqrt{x_{n-1}^2 + x_n^2} \right),$$

and this provides the conclusion to the proof of the step.

Step 9: Case (2) cannot occur. We use the result of Step 4, 7 and 8 to deduce the contradiction.

From (4.82) we deduce that $z_\infty \in T_h\mathfrak{H}_{1+s}$, by using it as a test function in (4.81), the condition of orthogonality of Step 7 implies

$$\langle z_\infty, \zeta \rangle = 0 \qquad\qquad \zeta \in T_h\mathfrak{H}_{1+s}.$$

Then we have $z_\infty = 0$ but this is in contradiction with (4.72).

In this way we exclude the occurrence of case (2) of Step 3, thus providing the conclusion of the proof of the theorem. $\square$

In what follows we show some important consequences of epiperimetric inequality.

4.4.2 Decay of the boundary adjusted energy

The following proposition establishes a decay estimate for the boundary adjusted energy. In this connection the epiperimetric inequality allows us to estimate from below, up to a constant, the difference between the energy $W^{0}_{1+s}(1,\cdot)$ evaluated respectively in the $(1+s)$-homogeneous extension of $u_{r|\partial B_1}$ and in u_r with $W^{0}_{1+s}(1,u_r)$; in this way we obtain a differential inequality from which we deduce the decay estimate.

Proposition 4.4.2 (Decay of the boundary adjusted energy)**.** *Let $x_0 \in \Gamma_{1+s}(u)$. There exists a constant $\gamma > 0$ for which the following property holds:*
for every compact set $K \subset B'_1$ there exists a positive constant $C > 0$ such that

$$W^{x_0}_{1+s}(r,u) \le C\, r^{\gamma}, \tag{4.86}$$

for all radii $0 < r < \operatorname{dist}(K, \partial B_1)$ and for all $x_0 \in \Gamma_{1+s}(u) \cap K$.

Proof. Let us assume $x_0 = \underline{0} \in \Gamma_{1+s}(u)$. In the same way as in the proof of Theorem 2.3.10 in Chapter 2 and thanks to Lemma 4.1.5, we calculate the derivative of the boundary adjusted energy $W_{1+s}(\cdot,u)$

$$\begin{aligned}
\frac{d}{dr}W^{0}_{1+s}(r,u) &= \frac{d}{dr}\left(\frac{1}{r^{n+1}}D_a(r) - \frac{(1+s)}{r^{n+2}}H_a(r)\right)\\
&= -\frac{(n+1)}{r^{n+2}}D_a(r) + \frac{1}{r^{n+1}}D'_a(r) - \frac{(1+s)}{r^{n+2}}H'_a(r) + \frac{(1+s)(n+2)}{r^{n+3}}H_a(r)\\
&= -\frac{(n+1)}{r^{n+2}}D_a(r) + \frac{1}{r^{n+1}}D'_a(r) - \frac{(1+s)(n-2s)}{r^{n+3}}H_a(r)\\
&\quad - \frac{2(1+s)}{r^{n+2}}D_a(r) + \frac{(1+s)(n+2)}{r^{n+3}}H_a(r)\\
&= -\frac{n+1}{r}W^{0}_{1+s}(r,u) - \frac{(1+s)(n+1)}{r^{n+3}}H_a(r) + \frac{1}{r^{n+1}}D'_a(r)\\
&\quad - \frac{2(1+s)}{r^{n+2}}D_a(r) + \frac{2(1+s)^2}{r^{n+3}}H_a(r)\\
&= -\frac{n+1}{r}W^{0}_{1+s}(r,u) - \frac{(1+s)(n+1)}{r^{n+3}}H_a(r) + I.
\end{aligned} \tag{4.87}$$

According to Lemma 4.1.5 and to the definition of rescaled functions (4.16), we can write

$$
\begin{aligned}
I =& \frac{1}{r^{n+2}} \int_{\partial B_r} |\nabla u|^2 \, |x_n|^a \, d\mathcal{H}^{n-1} + \frac{2(1+s)^2}{r^{n+3}} \int_{\partial B_r} u^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \\
& - \frac{2(1+s)}{r^{n+2}} \int_{\partial B_r} u \nabla u \cdot \frac{x}{r} \, |x_n|^a \, d\mathcal{H}^{n-1} \\
\overset{x=ry}{=} & \frac{1}{r^2} \int_{\partial B_1} |\nabla u(ry)|^2 \, |ry_n|^a \, d\mathcal{H}^{n-1} + \frac{2(1+s)^2}{r^4} \int_{\partial B_1} u^2(ry) \, |ry_n|^a \, d\mathcal{H}^{n-1} \\
& - \frac{2(1+s)}{r^3} \int_{\partial B_1} u(ry) \nabla u(ry) \cdot y \, |ry_n|^a \, d\mathcal{H}^{n-1} \\
=& \frac{1}{r} \int_{\partial B_1} \left(|\nabla u_r|^2 + 2(1+s)^2 u_r^2 - 2(1+s) u_r \nabla u_r \cdot y \right) \, |y_n|^a \, d\mathcal{H}^{n-1} \\
=& \frac{1}{r} \int_{\partial B_1} \Big((\nabla u_r \cdot \nu - (1+s) u_r)^2 + |\nabla_\theta u_r|^2 + (1+s)^2 u_r^2 \Big) \, |y_n|^a \, d\mathcal{H}^{n-1}
\end{aligned}
\tag{4.88}
$$

where by $\nabla_\theta u_r$ we denote the differential of u_r in the tangent direction to ∂B_1. Let c_r be the $(1+s)$-homogeneous extension of $u_{r|\partial B_1}$

$$
c_r(x) := |x|^{1+s} u_r \left(\frac{x}{|x|} \right).
$$

Thus, according to $(1+s)$-homogeneity and by Euler's homogeneous function Theorem and recalling that $H_a(r) = r^{n+2} H_a(1)$ and $W^{0}_{1+s}(1, u_r) = W^{0}_{1+s}(r, u)$, by putting together the equations (4.87) and (4.88), we deduce

$$
\begin{aligned}
\frac{d}{dr} W^{0}_{1+s}(r,u) =& -\frac{n+1}{r} W^{0}_{1+s}(r,u) - \frac{(n+1)(1+s)}{r} \int_{\partial B_1} u^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \\
& + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s) u_r)^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \\
& + \frac{1}{r} \int_{\partial B_1} \left(|\nabla_\theta u_r|^2 + (1+s)^2 u_r^2 \right) \, |x_n|^a \, d\mathcal{H}^{n-1} \\
=& -\frac{n+1}{r} W^{0}_{1+s}(r,u) + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s) u_r)^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \\
& + \frac{1}{r} \int_{\partial B_1} \left(|\nabla_\theta c_r|^2 - (1+s)(n-s) c_r^2 \right) \, |x_n|^a \, d\mathcal{H}^{n-1} \\
=& -\frac{n+1}{r} W^{0}_{1+s}(r,u) + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s) u_r)^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \\
& + \frac{1}{r} \int_{\partial B_1} \left(|\nabla c_r|^2 - (1+s)(n+1) c_r^2 \right) \, |x_n|^a \, d\mathcal{H}^{n-1} \\
=& \frac{n+1}{r} W^{0}_{1+s}(1, c_r) - \frac{n+1}{r} W^{0}_{1+s}(1, u_r) + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s) u_r)^2 \, |x_n|^a \, d\mathcal{H}^{n-1}.
\end{aligned}
$$

So, by Proposition 4.2.3 we have

$$
\frac{d}{dr} W^{0}_{1+s}(r,u) = \frac{n+1}{2\,r} \Big(W^{0}_{1+s}(1, c_r) - W^{0}_{1+s}(1, u_r) \Big).
$$

Then, according to the epiperimetric inequality stated in Theorem 4.4.1, with the same argument used in Lemma 2.6.3 of Chapter 2, we obtain

$$\frac{d}{dr}W^{\underline{0}}_{1+s}(r,u) \geq \frac{(n+1)\kappa}{r(1-\kappa)}W^{\underline{0}}_{1+s}(1,u_r) = \frac{(n+1)\kappa}{r(1-\kappa)}W^{\underline{0}}_{1+s}(r,u),$$

and integrating this inequality in $(0, r_0)$ we have

$$W^{\underline{0}}_{1+s}(r,u) \leq W^{\underline{0}}_{1+s}(1,u)\, r^{\gamma},$$

with $\gamma := \frac{(n+1)\kappa}{r(1-\kappa)}$. □

Remark 4.4.3. In order to prove the Poposition 4.4.2 the Weiss' monotonicity formula is not necessary.

4.4.3 Nondegeneracy of the solution

In order to deduce the non degeneracy property of the solution we note that the inequality (4.13) is not enough. We show an improved version of (4.13) as a consequence of epiperimetric inequality and decay estimate of energy above.

Proposition 4.4.4 (Nondegeneracy)**.** *Let $u \in H^1(B_1, \mu_a)$ be a solution of Problem* (4.4)*. Let's assume that $\underline{0} \in \Gamma_{1+s}(u)$. Then there exists a constant $H_0 > 0$ for which*

$$H_a(r) \geq H_0\, r^{n+2} \qquad \forall 0 < r < 1. \tag{4.89}$$

Proof. Taking (4.15) into account, we obtain

$$\frac{d}{dr}\left(\log\left(\frac{H_a(r)}{r^{n+2}}\right)\right) = \frac{r^{n+2}}{H_a(r)}\,2\frac{rD_a(r)-(1+s)H(r)}{r^{n+3}} = \frac{2\,r^{n+1}}{H(r)}\,W^{\underline{0}}_{1+s}(r,u). \tag{4.90}$$

Let γ be the constant of Proposition 4.4.2 and let $\varepsilon = \frac{\gamma}{2}$ be the exponent in the second item in Lemma 4.1.3. Then by means of (4.86), (4.13) and (4.90) we infer that there exists a positive constant $C = C(\gamma) > 0$ such that for all $0 < r < r_0$, where r_0 is given in the second item in Lemma 4.1.3, it holds

$$\begin{aligned}\frac{d}{dr}\left(\log\left(\frac{H_a(r)}{r^{n+2}}\right)\right) &= \frac{2\,r^{n+1}}{H_a(r)}\,W^{\underline{0}}_{1+s}(r,u) \overset{(4.86)}{\leq} \frac{2\,r^{n+1}}{H_a(r)}\,C\,r^{\gamma} \\ &\overset{(4.13)}{\leq} \frac{2\,r^{n+1}\,C\,r^{\gamma}\,r_0^{n+2+\gamma/2}}{r^{n+2+\gamma/2}\,H_a(r_0)} \leq C\,r^{\gamma/2-1}.\end{aligned} \tag{4.91}$$

By integrating the differential inequality above, we obtain that the function

$$r \to \frac{H_a(r)}{r^{n+2}\,e^{\frac{2C}{\gamma}}\,r^{\frac{\gamma}{2}}} \tag{4.92}$$

is nonincreasing. In fact integrating (4.91) on (t_0, t_1) we obtain

$$\begin{aligned}\log\left(\frac{H_a(t_2)}{t_2^{n+2}}\right) - \log\left(\frac{H_a(t_1)}{t_1^{n+2}}\right) &\leq \frac{2\,C}{\gamma}\left(t_2^{\gamma/2} - t_1^{\gamma/2}\right)\\ \frac{H_a(t_2)}{H_a(t_1)}\left(\frac{t_1}{t_2}\right)^{n+2} &\leq e^{2\frac{C}{\gamma}\left(t_2^{\gamma/2}-t_1^{\gamma/2}\right)}\\ \frac{H_a(t_2)}{t_2^{n+2}\,e^{\frac{2C}{\gamma}\,t_2^{\frac{\gamma}{2}}}} &\leq \frac{H_a(t_1)}{t_1^{n+2}\,e^{\frac{2C}{\gamma}\,t_1^{\frac{\gamma}{2}}}}.\end{aligned} \tag{4.93}$$

In particular we conclude that

$$\lim_{r\to 0}\frac{H_a(r)}{r^{n+2}\,e^{\frac{2C}{\gamma}\,r^{\frac{\gamma}{2}}}} = \lim_{r\to 0}\frac{H_a(r)}{r^{n+2}} =: H_0,$$

and if $r << 1$, according to nonincreasing and with inequality $1 - x \leq e^{-x}$ (true if $0 < x << 1$), we can deduce

$$H_0 \geq \frac{H_a(r)}{r^{n+2}\,e^{\frac{2C}{\gamma}\,r^{\frac{\gamma}{2}}}} \geq \frac{H_a(r)}{r^{n+2}}\left(1 - \frac{2\,C}{\gamma}\,r^{\frac{\gamma}{2}}\right) > 0. \tag{4.94}$$

Then thanks to the monotonicity of function $\frac{H_a(r)}{r^{n+2}}$ proved in the first item in Lemma 4.1.3 we provide the thesis

$$\frac{H_a(r)}{r^{n+2}} \geq H_0. \qquad \square$$

By means of the nondegeneracy condition (4.89), for all $x_0 \in \Gamma_{1+s}(u)$, we deduce

$$\int_{\partial B_1} u_{x_0,r}^2\,|x_n|^a\,dx \geq H_0,$$

and if $(u_{x_0,r_k})_{k\in\mathbb{N}}$ is a sequence that converges to u_0 in $L^2(B_1, \mu_a)$, a blow up function in x_0, due to estimate (4.17) and the convergence of the traces in [86, Lemma 2], we obtain the convergence of the traces of u_{x_0,r_k} on ∂B_1; thus

$$\int_{\partial B_1} u_0\,|x_n|^a\,dx \geq H_0 > 0.$$

So we infer $u_0 \not\equiv 0$ for all u_0 blow up functions in a point of $\Gamma_{1+s}(u)$.

So, in view of Propositions 4.4.2, 4.4.4 and [21, Proposition 5.5] we can deduce the following result of the classification of blow ups.

Proposition 4.4.5 (Classification of blow ups)**.** *Let u be a solution of problem* (4.4)*. Let u_0 be a blow up of u in point $x_0 \in \Gamma_{1+s}(u)$. Then there exist a constant $\lambda > 0$ and a vector $e \in \mathbb{S}^{n-2}$ such that $u_0 = \lambda\,h_e$.*

4.4.4 The blow up method: Uniqueness of blow ups

By summarizing what we have been showing so far, due to estimate (4.17) and to [62, Theorem 1.31], for all $x_0 \in \Gamma_{1+s}(u)$ and for all sequences $r_k \to 0$ there exists at least a subsequence (that we do not relabel in what follows) such that $u_{x_0,r_k} \rightharpoonup u_{x_0}$ in $H^1(B_1, \mu_a)$ for some non trivial functions $u_{x_0} \in H^1(B_1, \mu_a)$. It is easy to prove that u_{x_0} is a solution of Problem (4.4). Furthermore u_{x_0} is $(1+s)$-homogeneous. According to Proposition 4.4.5,the result of the classification of blow ups, we obtain $u_{x_0} \in \mathfrak{H}_{1+s}$.

With the next Proposition we prove that the blow up is unique, i.e. for all $x_0 \in \Gamma_{1+s}(u)$ there exists a function u_{x_0} such that for all $r_k \to 0$ the sequence $(u_{x_0,r_k})_{k\in\mathbb{N}}$ converges to u_{x_0} in $L^2(B_1, \mu_a)$. This is again a consequence of epiperimetric inequality. In particular, the epiperimetric inequality provides an explicit rate of convergence of the rescaled function $u_{x_0,r}$.

Proposition 4.4.6. *Let u be a solution of Problem (4.4) and let $K \subset\subset B_1'$. Then there exists a positive constant $C > 0$ such that for all $x_0 \in \Gamma_{1+s}(u) \cap K$ the following inequality holds:*

$$\int_{\partial B_1} |u_{x_0,r} - u_{x_0}|\, |x_n|^a \, d\mathcal{H}^{n-1} \leq C\, r^{\frac{\gamma}{2}},$$

where $\gamma > 0$ is the constant defined in Proposition 4.4.2. In particular the blow up is unique.

Proof. Let $0 < \rho < r < r_0$ be positive radii. By proceeding as in (2.115) and (2.110) in Lemma 2.6.3 of Chapter 2 we obtain

$$\begin{aligned}
&\int_{\partial B_1} |u_{x_0,r} - u_{\rho,x_0}|\, |x_n|^a \, d\mathcal{H}^{n-1} \\
&\qquad \leq C \int_\rho^r t^{-\frac{1}{2}} \left(t^{-1} \int_{\partial B_1} |\nabla u_{t,x_0} \cdot x - (1+s) u_{t,x_0}|^2 \, |x_n|^a \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} dt.
\end{aligned}$$

By means of formula (4.1.4), the Cauchy-Schwartz inequality and decay estimate (4.86), we infer

$$\begin{aligned}
\int_{\partial B_1} |u_{x_0,r} - u_{x_0,\rho}|\, |x_n|^a \, d\mathcal{H}^{n-1} &\overset{(4.1.4)}{\leq} C \int_\rho^r t^{-\frac{1}{2}} \left(\frac{d}{dr} W^{0}_{1+s}(t,u) \right)^{\frac{1}{2}} \\
&\leq C \left(\log\left(\frac{r}{\rho}\right) \right)^{-\frac{1}{2}} \left(W^{0}_{1+s}(r,u) - W^{0}_{1+s}(\rho,u) \right)^{\frac{1}{2}} \\
&\overset{(4.86)}{\leq} C \left(\log\left(\frac{r}{\rho}\right) \right)^{-\frac{1}{2}} r^{\frac{\gamma}{2}}.
\end{aligned} \tag{4.95}$$

Let $h, k \in \mathbb{N}$ be such that $2^{-k} < \rho < r < 2^{-h+1}$, with the same dyadic argument as in

Lemma 2.6.3, applying (4.95) to $\rho = \frac{r}{2} = 2^{-i}$, we have

$$\int_{\partial B_1} |u_{x_0,r} - u_{x_0,\rho}|\, |x_n|^a \, d\mathcal{H}^{n-1} = \sum_{i=h}^{k} \int_{\partial B_1} |u_{x_0,2^{-i+1}} - u_{x_0,2^{-i}}|\, |x_n|^a \, d\mathcal{H}^{n-1}$$
$$\leq C \sum_{i=h}^{k} (\sqrt{\log 2})\, 2^{(-i+1)\frac{\gamma}{2}} \leq \sum_{i=h+1}^{\infty} 2^{-i\frac{\gamma}{2}} \leq C\, 2^{-h\frac{\gamma}{2}} \leq C\, r^{\frac{\gamma}{2}}.$$

Passing to limit as $\rho \to 0$ and eventually changing the value of constant C, we provide the conclusion of the thesis. □

4.5 The regularity of the free boundary

Thanks to the uniqueness of blow ups we can give a proof of the $C^{1,\alpha}$ regularity of $\Gamma_{1+s}(u)$ the subset of the free boundary with lower frequency (cf. Theorem 2.7.1 in Chapter 2).

Theorem 4.5.1. *Let $u \in H^1(B_1, \mu_a)$ be a solution of Problem 4.4. Then, there exists a constant $\alpha > 0$ such that for all $x_0 \in \Gamma_{1+s}(u)$ there exists a radius $r = r(x_0)$ for which $\Gamma_{1+s}(u) \cap B'_r(x_0)$ is a $C^{1,\alpha}$ regular $(n-2)$-submanifold in B'_1.*

Proof. Without loss of generality we can suppose that $\underline{0} \in \Gamma_{1+s}(u)$ and prove the regularity of $\Gamma_{1+s}(u)$ in a neighborhood of $\underline{0}$. According to the openness of $\Gamma_{1+s}(u)$ there exists a radius $\rho > 0$ such that $\Gamma(u) \cap B'_\rho = \Gamma_{1+s}(u) \cap B'_\rho$. By means of the uniqueness of blow ups proved in Proposition 4.4.6 and the blow up classification result stated in Proposition 4.4.5, we infer that for every $x_0 \in \Gamma_{1+s}(u) \cap B'_\rho$ the blow up in x_0 has the following form

$$u_{x_0} = \lambda_{x_0}\, h_{e(x_0)} \in \mathfrak{H}_{1+s} \tag{4.96}$$

for some $\lambda_{x_0} > 0$ and $e(x_0) \in \mathbb{S}^{n-2}$, where $h_{e(x_0)}$ and $\mathfrak{H}_{1+s}$ are defined respectively in (4.21) and (4.25).

The first step consists in the proof of the Hölder continuity of the function $x_0 \mapsto \lambda_{x_0}$. By improving the inequalities (4.91), taking Proposition 4.4.2 and Proposition 4.4.4 into account, we obtain

$$\frac{d}{dr}\left(\log\left(\frac{H_a(r)}{r^{n+2}}\right)\right) = \frac{2\, r^{n+1}}{H_a(r)}\, W^{\underline{0}}_{1+s}(r,u) \overset{(4.86)}{\leq} \frac{2\, r^{n+1}}{H_a(r)}\, C\, r^{\gamma} \overset{(4.89)}{\leq} \frac{2\, r^{n+1}}{H_0\, r^{n+2}}\, C\, r^{\gamma} \leq C\, r^{\gamma-1}, \tag{4.97}$$

for all $r \in (0,1)$. Due to strong convergence of rescaled functions in $L^2(\partial B_1, |x_n|^a\, \mathcal{H}^{n-1})$, we have

$$\lambda_{x_0} = c_0 \lim_{r \searrow 0} \frac{H_a^{x_0}(r)}{r^{n+2}},$$

with $c_0 > 0$ being a dimensional constant. In fact

$$\begin{aligned}
\lim_{r\searrow 0}\frac{H_a^{x_0}(r)}{r^{n+2}} &= \lim_{r\searrow 0}\frac{\int_{\partial B_r(x_0)} u^2\,|x_n|^a\,d\mathcal{H}^{n-1}(x)}{r^{n+2}} \overset{x=x_0+ry}{=} \lim_{r\searrow 0}\int_{\partial B_1} u^2_{x_0,r}\,|y_n|^a\,d\mathcal{H}^{n-1}(y)\\
&=\int_{\partial B_1} u^2_{x_0}\,|y_n|^a\,d\mathcal{H}^{n-1}(y) = \lambda_{x_0}\int_{\partial B_1} h^2_{e(x_0)}\,|y_n|^a\,d\mathcal{H}^{n-1}(y)\\
&=\lambda_{x_0}\int_{\partial B_1} h^2\,|y_n|^a\,d\mathcal{H}^{n-1}(y) = \lambda_{x_0}c_0^{-1}.
\end{aligned}$$

By the integration of differential inequality (4.97) and proceeding as in (4.93) and (4.94) in Proposition 4.4.4 we obtain

$$c_0\,\frac{H_a^{x_0}(r)}{r^{n+2}} - \lambda_{x_0} \le C\,r^\gamma, \qquad \forall r \in (0,1).$$

Moreover, for $x_0, y_0 \in \Gamma_{1+s}(u)\cap B'_\rho$ and $r = |x_0-y_0|^{1-\theta}$ with $\theta := \frac{\gamma}{1+\gamma}$ we have

$$\begin{aligned}
&\int_{\partial B_1} |u_{x_0,r} - u_{y_0,r}|\,|x_n|^a\,d\mathcal{H}^{n-1}\\
&\quad\le r^{-(1+s)}\int_{\partial B_1}\int_0^1 |\nabla u(t(x_0+rx)+(1-t)(y_0+rx))|\,|x_0-y_0|\,|x_n|^a\,dt\,d\mathcal{H}^{n-1}\\
&\quad\le C\,r^{-1}\,|x_0-y_0| \le C\,|x_0-y_0|^\theta,
\end{aligned} \tag{4.98}$$

where in the first inequality of the last line we used the condition of growth (4.18). So we can conclude that for $r = |x_0-y_0|^{1-\theta}$ with $\theta := \frac{\gamma}{1+\gamma}$ it yields as follows:

$$\begin{aligned}
|\lambda_{x_0} - \lambda_{y_0}| &\le \left|\lambda_{x_0} - c_0\,\frac{H_a^{x_0}(r)}{r^{n+2}}\right| + c_0\left|c_0\,\frac{H_a^{x_0}(r)}{r^{n+2}} - c_0\,\frac{H_a^{y_0}(r)}{r^{n+2}}\right| + \left|c_0\,\frac{H_a^{y_0}(r)}{r^{n+2}} - \lambda_{y_0}\right|\\
&\le Cr^\gamma + C\int_{B_1} |u^2_{x_0,r} - u^2_{y_0,r}|\,|x_n|^a\,d\mathcal{H}^{n-1}\\
&\le Cr^\gamma + C\int_{B_1} |u_{x_0,r} - u_{y_0,r}|\,|u_{x_0,r} + u_{y_0,r}|\,|x_n|^a\,d\mathcal{H}^{n-1}\\
&\le C\,r^\gamma + C\int_{B_1} |u_{x_0,r} - u_{y_0,r}|\,|x_n|^a\,d\mathcal{H}^{n-1} \overset{(4.98)}{\le} C\,|x_0-y_0|^\theta
\end{aligned} \tag{4.99}$$

where, thanks to (4.18), we could use the uniform equiboundedness of $u_{\cdot,r}$.

The second step consists in the proof of Hölder continuity of the function $x_0 \mapsto e(x_0)$. We can observe that by definition (4.21), if x_0, y_0 are as above, we infer

$$\begin{aligned}
|e(x_0) - e(y_0)| &\le C\int_{\partial B'_1} |y\cdot e(x_0)\chi_{\{y\cdot e(x_0)>0\}} - y\cdot e(y_0)\chi_{\{y\cdot e(y_0)>0\}}|\,d\mathcal{H}^{n-2}\\
&= C\int_{\partial B'_1} |h_{e(x_0)} - h_{e(y_0)}|\,d\mathcal{H}^{n-2}.
\end{aligned}$$

In fact, let β be the angle between $e(x_0)$ and $e(y_0)$; by Chord Theorem $|e(x_0) - e(y_0)| = 2\sin\frac{\beta}{2}$, thus if $0 \le \beta \le \frac{\pi}{2}$ by resorting to geometric reasoning, we have

$$
\begin{aligned}
&\int_{\partial B_1'} |y \cdot e(x_0)\chi_{\{y\cdot e(x_0)>0\}} - y \cdot e(y_0)\chi_{\{y\cdot e(y_0)>0\}}|\, d\mathcal{H}^{n-2} \\
&= \int_{\partial B_1' \cap \{y\cdot e(x_0)>0, y\cdot e(y_0)>0\}} |y \cdot (e(x_0) - e(y_0))|\, d\mathcal{H}^{n-2} \\
&+ \mathcal{H}^{n-2}\left((\partial B_1' \cap \{y \cdot e(x_0) > 0, y \cdot e(y_0) \le 0\}) \cup (\partial B_1' \cap \{y \cdot e(x_0) \le 0, y \cdot e(y_0) > 0\}) \right) \\
&\ge \int_{\partial B_1' \cap \{y\cdot e(x_0)>0, y\cdot e(y_0)>0\}} \cos\frac{\beta}{2} |(e(x_0) - e(y_0))|\, d\mathcal{H}^{n-2} + C \sin\frac{\beta}{2} \\
&\ge C|(e(x_0) - e(y_0))|,
\end{aligned}
$$

and for $\frac{\pi}{2} \le \beta \le \pi$ the argument is similar. In order to prove a Hölder estimate for the map $x \mapsto e(x)$ we study the quantity $\int_{\partial B_1'} |h_{e(x_0)} - h_{e(y_0)}|\, d\mathcal{H}^{n-2}$. By Trace Theorem [31, Theorem 3.4] we have

$$
\int_{B_1'} |u_{x_0} - u_{y_0}|^2\, d\mathcal{H}^{n-1} \le C \left(\int_{B_1} |u_{x_0} - u_{y_0}|^2\, d\mu_a + \int_{B_1} |\nabla u_{x_0} - \nabla u_{y_0}|^2\, d\mu_a \right). \tag{4.100}
$$

Since u_{x_0} and u_{y_0} are solutions of Problem (4.4), we have

$$
\begin{aligned}
&\int_{B_1} L_a(u_x)\, u_{x_0}\, dx = 0 \qquad\qquad \textit{with } x = x_0 \textit{ or } x = y_0, \\
&\int_{B_1} L_a(u_{x_0})\, u_{y_0}\, dx \le 0, \\
&\int_{uB_1} L_a(u_{y_0})\, u_{x_0}\, dx \le 0,
\end{aligned}
$$

so we obtain

$$
\int_{B_1} L_a(u_{x_0} - u_{y_0})(u_{x_0} - u_{y_0})\, dx \ge 0.
$$

Integrating by parts, we infer

$$
\int_{B_1} |\nabla u_{x_0} - \nabla u_{y_0}|^2\, d\mu_a \le \int_{\partial B_1} (\nabla(u_{x_0} - u_{y_0}) \cdot \nu)\, (u_{x_0} - u_{y_0})\, d\mu_a
$$

and recalling that the blow ups are $(1+s)$-homogeneous, due to Euler's homogeneous Theorem, iy yields:

$$
\int_{B_1} |\nabla(u_{x_0} - u_{y_0})|^2\, d\mu_a \le \int_{\partial B_1} (u_{x_0} - u_{y_0})^2\, d\mu_a \le C \int_{B_1} (u_{x_0} - u_{y_0})^2\, d\mu_a. \tag{4.101}
$$

Next, by conditions (4.100) and (4.101) we obtain

$$
\begin{aligned}
\int_{B_1'} |u_{x_0} - u_{y_0}|^2\, d\mathcal{H}^{n-1} &\le C \int_{B_1} (u_{x_0} - u_{y_0})^2\, d\mu_a \\
&\le C \int_{\partial B_1} |u_{x_0} - u_{y_0}|\, d\mu_a,
\end{aligned} \tag{4.102}
$$

where in the last inequality we have used the $(1+s)$-homogeneity and the uniform boundedness of $u_{0,x}$. Thus, thanks to Proposition 4.4.6 and (4.98) we have

$$\begin{aligned}
&\int_{\partial B_1} |u_{x_0} - u_{y_0}|\, d\mu_a \\
&\leq \int_{\partial B_1} |u_{x_0} - u_{x_0,r}|\, d\mu_a + \int_{\partial B_1} |u_{x_0,r} - u_{y_0,r}|\, d\mu_a + \int_{\partial B_1} |u_{y_0,r} - u_{y_0}|\, d\mu_a \\
&\leq Cr^{\frac{\gamma}{2}} + C|x_0 - y_0|^{\theta} \leq |x_0 - y_0|^{\frac{\theta}{2}}.
\end{aligned} \tag{4.103}$$

Finally, since

$$\begin{aligned}
|h_{e(x_0)} - h_{e(y_0)}| = \left|\frac{u_{x_0}}{\lambda_{x_0}} - \frac{u_{y_0}}{\lambda_{y_0}}\right| \leq \left|\frac{u_{x_0}}{\lambda_{x_0}} - \frac{u_{x_0}}{\lambda_{y_0}}\right| + \left|\frac{u_{x_0}}{\lambda_{y_0}} - \frac{u_{y_0}}{\lambda_{y_0}}\right| \\
\leq C\left(|\lambda_{e(x_0)} - \lambda_{e(y_0)}| + |u_{x_0} - u_{y_0}|\right),
\end{aligned}$$

according to (4.99), (4.102) and (4.103) we infer

$$|e(x_0) - e(y_0)| \leq C \int_{\partial B_1'} |h_{e(x_0)} - h_{e(y_0)}|\, d\mathcal{H}^{n-2} \leq C\,|x_0 - y_0|^{\theta}. \tag{4.104}$$

In what follows we show that, as in Theorem 2.7.1 in Chapter 2, the vector $e(x)$ plays the role of "normal vector to surface". In this connection we introduce the cones centred in $x_0 \in \Gamma_{1+s}(u)$ given by

$$C^{\pm}(x_0, \varepsilon) := \left\{x \in \mathbb{R}^{n-1} \times \{0\} \,:\, \pm\langle x - x_0, e(x_0)\rangle \geq \varepsilon|x - x_0|\right\}. \tag{4.105}$$

We prove that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_0 \in \Gamma_{1+s}(u) \cap B_{\frac{\varrho}{2}}$,

$$C^{+}(x_0, \varepsilon) \cap B_\delta(x) \subset N_u \qquad \textit{and} \qquad C^{-}(x_0, \varepsilon) \cap B_\delta(x) \subset \Lambda_u. \tag{4.106}$$

Let us suppose by contradiction that there exists a sequence $(x_j)_{j\in\mathbb{N}} \subset \Gamma_{1+s}(u) \cap B_{\frac{\varrho}{2}}$ such that $x_j \to x \in \Gamma_{1+s}(u) \cap \overline{B}_{\frac{\varrho}{2}}$ and a sequence $(y_j)_{j\in\mathbb{N}} \subset C^{+}(x_j, \varepsilon)$, for which $x_j - y_j \to 0$ and $u(y_j) = 0$.

According to optimal regularity of solution (4.7) and (4.103) the rescaled function u_{r_j,x_j} with $r_j = |y_j - x_j|$, converges uniformly to u_{x_0}.

We define the sequence $z_j = r_j^{-1}(y_j - x_j)$ and we observe that $z_j \in \big(C^{+}(x_j, \varepsilon) - x_j\big) \cap \mathbb{S}^{n-1}$. Up to subsequence (that we do not relabel) we can suppose that $z_j \to z \in \big(C^{+}(x, \varepsilon) - x_0\big) \cap \mathbb{S}^{n-1}$. Thus

$$u_{x_0}(z) = \lim_j u_{r_j,x_j}(z_j) = \lim_j \frac{u(y_j)}{r_j^2} = 0,$$

but on the other hand there exists a $y \in C^{+}(x, \varepsilon)$ for which $z = y - x_0$, so by definition of u_{x_0}, $h_{e(x_0)}$ and $C^{+}(x, \varepsilon)$ to be found in (4.96), (4.21) and (4.105) respectively we deduce

$$u_{x_0}(z) = \lambda_{x_0} h_{e(x_0)}(y - x_0) = \lambda_{x_0}\, 2^s(s^{-1} - 1)\langle x - x_0, e(x_0)\rangle^{1+s} \geq C\varepsilon^{1+s}|x - x_0|^{1+s} > 0,$$

which gives a contradiction. Reasoning in the same way, it is possible to prove that $C^-(x,\varepsilon) \cap B_\delta(x) \subset \Lambda_u$. We now conclude showing that $\Gamma_{1+s}(u) \cap B_{\rho_1}$ is the subgraph of a function φ for a suitable constant $\rho_1 > 0$. Fixing $x_0 \in \Gamma_{1+s}(u)$, we recall that $e(x_0)$ is the generating line of cones $C^\pm(x,\varepsilon)$. Let $\varphi : \mathbb{R}^{n-2} = \{x_0 + e(x_0)^\perp\} \to \mathbb{R}$ be a function defined by

$$\varphi(x') := \max \big\{ t \in \mathbb{R} \,:\, (x', t, 0) \in \Lambda_u \big\}, \qquad \forall\, x' \in \{x_0 + e(x_0)^\perp\} \,:\, |x' - x_0| \leq \delta\sqrt{1-\varepsilon^2}.$$

We note that according to (4.105) the maximum exists in $[-\varepsilon\delta, \varepsilon\delta]$, and

$$\begin{aligned} (x', t, 0) \in \Lambda_u \qquad &\Longrightarrow \qquad -\varepsilon\delta \leq t \leq \varphi(x'), \\ (x', t, 0) \in N_u \qquad &\Longrightarrow \qquad \varphi(x') < t \leq \varepsilon\delta. \end{aligned}$$

Therefore φ is differentiable and due to (4.104) its normal vector $e(x_0)$ is Hölder continuous; so $\varphi \in C^{1,\alpha}$ and in this way we have provided the conclusion to the proof of the theorem. $\square$

Bibliography

[1] Alt, H. W.; Caffarelli, L. A.; Friedman, A., *Variational problems with two phases and their free boundaries.* Trans. Amer. Math. Soc. 282 (1984), no. 2, 431–461.

[2] Andersson, J.; Lindgren, E.; Shahgholian, H., *Optimal regularity for the obstacle problem for the p-Laplacian.* J. Diff. Equations 259 (2015), Issue 6, 2167–2179.

[3] Anzellotti, G., *On the $C^{1,\alpha}$-regularity of ω-minima of quadratic functionals.* Boll. Un. Mat. Ital. C (6) 2 (1983), no. 1, 195–212.

[4] Athanasopoulos, I.; Caffarelli, L. A.; Salsa, S., *The structure of the free boundary for lower dimensional obstacle problems.* Amer. J. Math. 130 (2008), no. 2, 485–498.

[5] Baiocchi, C.; Capelo, A., *Variational and quasivariational inequalities. Applications to free boundary problems.* John Wiley & Sons, Inc., New York, 1984. ix+452 pp.

[6] Blank, I., *Sharp results for the regularity and stability of the free boundary in the obstacle problem.* Indiana Univ. Math. J. 50 (2001), no. 3, 1077–1112.

[7] Blank, I.; Hao, Z., *Reifenberg flatness of free boundaries in obstacle problems with VMO ingredients.* Calc. Var. Partial Differential Equations 53 (2015), no. 3-4, 943–959.

[8] Blank, I.; Hao, Z., *The mean value theorem and basic properties of the obstacle problem for divergence form elliptic operators.* Comm. Anal. Geom. 23 (2015), no. 1, 129–158.

[9] Braides, A., *Γ-convergence for beginners.* Oxford Lecture Series in Mathematics and its Applications, 22. Oxford University Press, Oxford, 2002. xii+218 pp.

[10] Brezis, H. R., *Analisi funzionale.* Ed. Liguori, Napoli, (1986). xv+419pp.

[11] Brezis, H. R., *Problèmes unilatéraux.* J. Math. Pures Appl. (9) 51 (1972), 1–168.

[12] Brezis, H. R.; Kinderlehrer, D., *The smoothness of solutions to nonlinear variational inequalities.* Indiana Univ. Math. J. 23 (1973/74), 831–844.

[13] Brezis, H. R.; Stampacchia, G., *Sur la régularité de la solution d'inéquations elliptiques.* (French) Bull. Soc. Math. France 96 (1968), 153–180.

[14] Caffarelli, L. A., *The regularity of free boundaries in higher dimensions.* Acta Math. 139 (1977), no. 3-4, 155–184.

[15] Caffarelli, L. A., *Compactness methods in free boundary problems.* Comm. Partial Differential Equations 5 (1980), no. 4, 427–448.

[16] Caffarelli, L. A., The obstacle problem revisited. Lezioni Fermiane. [Fermi Lectures] Accademia Nazionale dei Lincei, Rome; Scuola Normale Superiore, Pisa, 1998. ii+54 pp.

[17] Caffarelli, L. A., *The obstacle problem revisited.* J. Fourier Anal. Appl. 4 (1998), no. 4-5, 383–402.

[18] Caffarelli, L. A.; Kinderlehrer, D., *Potential methods in variational inequalities.* J. Analyse Math. 37 (1980), 285–295.

[19] Caffarelli, L. A.; Riviére, N. M., Smoothness and analyticity of free boundaries in variational inequalities. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 3 (1976), no. 2, 289–310.

[20] Caffarelli, L.; Salsa, S., *A geometric approach to free boundary problems.* Graduate Studies in Mathematics, 68. American Mathematical Society, Providence, RI, 2005. x+270 pp.

[21] Caffarelli, L. A.; Salsa, S.; Silvestre, L., *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian.* Invent. Math. 171 (2008), no. 2, 425–461.

[22] Caffarelli, L.; Silvestre, L., *An extension problem related to the fractional Laplacian.* Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245–1260.

[23] Cerutti, M. C.; Ferrari, F.; Salsa, S., *Two-phase problems for linear elliptic operators with variable coefficients: Lipschitz free boundaries are* $C^{1,\gamma}$. Arch. Ration. Mech. Anal. 171 (2004), no. 3, 329–348.

[24] Chipot, M., *Variational inequalities and flow in porous media.* Applied Mathematical Sciences, 52. Springer-Verlag, New York, 1984. vii+118 pp.

[25] Dal Maso, G., *An introduction to* Γ*-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Inc., Boston, MA, 1993. xiv+340 pp.

[26] Di Nezza, E.; Palatucci, G.; Valdinoci, E., *Hitchhiker's guide to the fractional Sobolev spaces.* Bull. Sci. Math. 136 (2012), no. 5, 521–573.

[27] Duvaut, G.; Lions, J. L., *Les inéquations en méchanique et en physique.* Dunod, Paris, (1972).

[28] Evans, L. C.; Gariepy, R. F., *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992. viii+268 pp.

[29] Fabes, E. B.; Kenig, C. E.; Serapioni, R. P., *The local regularity of solutions of degenerate elliptic equations.* Comm. Partial Differential Equations 7 (1982), no. 1, 77–116.

[30] Ferrari, F.; Salsa, S., *Regularity of the free boundary in two-phase problems for linear elliptic operators.* Adv. Math. 214 (2007), no. 1, 288–322.

[31] Focardi, M., *Homogenization of random fractional obstacle problems via* Γ*-convergence.* Comm. Partial Differential Equations 34 (2009), no. 10-12, 1607–1631.

[32] Focardi, M., *Aperiodic fractional obstacle problems.* Adv. Math. 225 (2010), no. 6, 3502–3544.

[33] Focardi, M., *Vector-valued obstacle problems for non-local energies.* Discrete Contin. Dyn. Syst. Ser. B 17 (2012), no. 2, 487–507.

[34] Focardi, M.; Gelli, M. S.; Spadaro, E., *Monotonicity formulas for obstacle problems with Lipschitz coefficients.* Calc. Var. Partial Differential Equations 54 (2015), no. 2, 1547–1573.

[35] Focardi, M.; Geraci, F.; Spadaro, E., *The classical obstacle problem for nonlinear variational energies.* Nonlinear Analysis (2016), 10.1016/j.na.2016.10.020.

[36] Focardi, M.; Spadaro, E., *An epiperimetric inequality for the thin obstacle problem.* Adv. Differential Equations 21 (2015), no 1-2, 153-200.

[37] Focardi, M.; Spadaro, E., *On the dimension of the free boundary for the fractional obstacle problem* Preprint 2016

[38] Frehse, J., *On the regularity of the solution of a second order variational inequality.* Boll. Un. Mat. Ital. (4) 6 (1972), 312–315.

[39] Frehse, J.; Mosco, U., *Irregular obstacles and quasivariational inequalities of stochastic impulse control.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 9 (1982), no. 1, 105–157.

[40] Friedman, A., *Variational principles and free-boundary problems.* Second edition. Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1988. x+710 pp.

[41] Fuchs, M., *Hölder continuity of the gradient for degenerate variational inequalities.* Nonlinear Anal. TMA 15 (1990), No. 1, 85–100.

[42] Fuchs, M.; Mingione G., *Full* $C^{1,\alpha}$*-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth.* Manuscripta Math. 102 (2000), 227–250.

[43] Garofalo, N.; Petrosyan, A.; Smit Vega Garcia, M., *An epiperimetric inequality approach to the regularity of the free boundary in the Signorini problem with variable coefficients.* J. Math. Pures Appl. (9) 105 (2016), no. 6, 745–787.

[44] Garofalo, N., Petrosyan, A., Pop, C. A., Smit Vega Garcia, M., *Regularity of the free boundary for the obstacle problem for the fractional Laplacian with drift.* Annales de l'Institut Henri Poincare. Annales: Analyse Non Lineaire/Nonlinear Analysis. DOI: 10.1016/j.anihpc.2016.03.001

[45] Garofalo, N.; Petrosyan, A., *Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem.* Invent. Math. 177 (2009), no. 2, 415–461.

[46] Garofalo, N.; Petrosyan, A.; Smit Vega Garcia, M., *An epiperimetric inequality approach to the regularity of the free boundary in the Signorini problem with variable coefficients.* J. Math. Pures Appl. (9) 105 (2016), no. 6, 745–787.

[47] Garofalo, N.; Smit Vega Garcia, M., *New monotonicity formulas and the optimal regularity in the Signorini problem with variable coefficients.* Adv. Math. 262 (2014), 682–750.

[48] Geraci, F., *The classical obstacle problem with coefficients in fractional Sobolev spaces.* Preprint 2016.

[49] Geraci, F., *An epiperimetric inequality for the fractional obstacle problem.* In progress

[50] Gerhardt, C., *Regularity of solutions of nonlinear variational inequalities.* Arch. Ration. Mech. Anal. 52 (1973), 389–393.

[51] Gerhardt, C., *Global $C^{1,1}$-regularity for solutions of quasilinear variational inequalities.* Arch. Ration. Mech. Anal. 89 (1985), 83–92.

[52] Giaquinta, M., *Remarks on the regularity of weak solutions to some variational inequalities.* Math. Z. 177 (1981), 15–31.

[53] Giaquinta, M.; Giusti, E., *On the regularity of the minima of variational integrals.* Acta Math. 148 (1982), 31–46.

[54] Giaquinta, M.; Giusti, E., *Quasiminima.* Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 2, 79–107.

[55] Gilbarg, D.; Trudinger, N. S., *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp.

[56] Giusti, E., *Direct methods in the calculus of variations.* World Scientific Publishing Co., Inc., River Edge, NJ, 2003. viii+403 pp.

[57] Gustafsson, B., *A simple proof of the regularity theorem for the variational inequality of the obstacle problem.* Nonlinear Anal. 10 (1986), no. 12, 1487–1490.

[58] Hajlasz, P.; Koskela, P., *Sobolev met Poincaré*. Mem. Amer. Math. Soc. 145 (2000), no. 688, x+101 pp.

[59] Han, Q.; Lin, F., *Elliptic partial differential equations*. Second edition. Courant Lecture Notes in Mathematics, 1. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2011. x+147 pp.

[60] Hanouzet, B.; Joly, J. L., *Méthodes d'ordre dans l'interprétation de certaines inéquations variationnelles et applications*. (French) J. Funct. Anal. 34 (1979), no. 2, 217–249.

[61] Hartman, P.; Stampacchia, G., *On some non-linear elliptic differential-functional equations*. Acta Math. 115 (1966), 271–310.

[62] Heinonen, J.; Kilpeläinen, T.; Martio, O., *Nonlinear potential theory of degenerate elliptic equations*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. vi+363 pp.

[63] Kikuchi, N.; Oden, J. T., *Contact problems in elasticity: a study of variational inequalities and finite element methods*. SIAM Studies in Applied Mathematics, 8. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988. xiv+495 pp.

[64] Kinderlehrer, D.; Nirenberg, L., *Regularity in free boundary problems*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 4 (1977), no. 2, 373–391.

[65] Kinderlehrer, D.; Stampacchia, G., *An introduction to variational inequalities and their applications*. Pure and Applied Mathematics, 88. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. xiv+313 pp.

[66] Kufner, A., *Weighted Sobolev spaces*. Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], 31. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1980. 151 pp.

[67] Ladyzhenskaya, O.; Uraltseva, N., *Linear and quasilinear elliptic equations*. Academic Press, New York-London 1968 xviii+495 pp.

[68] Lederman, C.; Wolanski, N., *A local monotonicity formula for an inhomogeneous singular perturbation problem and applications*. Ann. Mat. Pura Appl. (4) 187 (2008), no. 2, 197–220.

[69] Leoni, G., *A first course in Sobolev spaces*. Graduate Studies in Mathematics, 105. American Mathematical Society, Providence, RI, 2009. xvi+607 pp.

[70] Lewy, H.; Stampacchia, G., *On the regularity of the solution of a variational inequality*. Comm. Pure Appl. Math. 22 (1969), 153–188.

[71] Lions, J. L.; Magenes, E., *Problemi ai limiti non omogenei. III*. Ann. Scuola Norm. Sup. Pisa (3) 15 1961 41–103.

[72] Ma, L.; Song, X.; Zhao, L., *New monotonicity formulae for semi-linear elliptic and parabolic systems.* Chin. Ann. Math. Ser. B 31 (2010), no. 3, 411–432.

[73] Manfredi, J. J., *Regularity for minima of functionals with p-growth.* J. Differ. Equations 76 (1988), 203–212.

[74] Manfredi, J. J., *Regularity of the gradient for a class of nonlinear possibly degenerate elliptic equations.* PhD. Thesis. University of Washington, St. Louis.

[75] Matevosyan, N.; Petrosyan, A., *Almost monotonicity formulas for elliptic and parabolic operators with variable coefficients.* Comm. Pure Appl. Math. 64 (2011), no. 2, 271–311.

[76] Michael, J. H.; Ziemer, W. P., *Interior regularity for solutions to obstacle problems.* Nonlinear Anal. 10 (1986), no. 12, 1427–1448.

[77] Monneau, R., *On the number of singularities for the obstacle problem in two dimensions.* J. Geom. Anal. 13 (2003), no. 2, 359–389.

[78] Monneau, R., *Pointwise estimates for Laplace equation. Applications to the free boundary of the obstacle problem with Dini coefficients.* J. Fourier Anal. Appl. 15 (2009), no. 3, 279–335.

[79] Mondino, A.; Spadaro, E., *On a isoperimetric-isodiametric inequality.* Preprint (2016): arXiv:1603.05263.

[80] Mosco, U., *Implicit variational problems and quasi variational inequalities.* Nonlinear operators and the calculus of variations (Summer School, Univ. Libre Bruxelles, Brussels, 1975), pp. 83–156. Lecture Notes in Math., Vol. 543, Springer, Berlin, 1976.

[81] Mosco, U.; Troianiello, G. M., *On the smoothness of solutions of unilateral Dirichlet problems.* (Italian summary) Boll. Un. Mat. Ital. (4) 8 (1973), 57–67.

[82] Muramatu, T., *On Besov spaces and Sobolev spaces of generalized functions definded on a general region.* Publ. Res. Inst. Math. Sci. 9 (1973), 325–396.

[83] Petrosyan, A.; Pop, C. A., Optimal regularity of solutions to the obstacle problem for the fractional Laplacian with drift. (English summary) J. Funct. Anal. 268 (2015), no. 2, 417–472.

[84] Petrosyan, A.; Shahgholian, H., *Geometric and energetic criteria for the free boundary regularity in an obstacle-type problem.* Amer. J. Math. 129 (2007), no. 6, 1659–1688.

[85] Petrosyan, A.; Shahgholian, H.; Uraltseva, N., *Regularity of free boundaries in obstacle-type problems.* Graduate Studies in Mathematics, 136. American Mathematical Society, Providence, RI, 2012. x+221 pp.

[86] Pflüger, K., *Compact traces in weighted Sobolev spaces.* Analysis (Munich) 18 (1998), no. 1, 65–83.

[87] Rodrigues, J.-F., *Obstacle problems in mathematical physics.* North-Holland Mathematics Studies, 134. Notas de Matemática [Mathematical Notes], 114. North-Holland Publishing Co., Amsterdam, 1987. xvi+352 pp.

[88] Silvestre, L., *Regularity of the obstacle problem for a fractional power of the Laplace operator.* Comm. Pure Appl. Math. 60 (2007), no. 1, 67–112.

[89] Schneider, C., *Trace operators in Besov and Triebel-Lizorkin spaces.* Z. Anal. Anwend. 29 (2010), no. 3, 275–302.

[90] Tamanini, I., *Regularity results for almost minimal oriented hypersurfaces in* $\mathbb{R}^N$. Quaderni del Dipartimento di Matematica dell'Università del Salento (1984).

[91] Troianiello, G. M., *Elliptic differential equations and obstacle problems.* The University Series in Mathematics. Plenum Press, New York, 1987. xiv+353 pp.

[92] Ural'tseva, N. N., *Regularity of solutions of variational inequalities.* Russian Math. Surveys 42:6 (1987), 191–219.

[93] Wang, P.-Y., *Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. I. Lipschitz free boundaries are* $C^{1,\alpha}$. Comm. Pure Appl. Math. 53 (2000), no. 7, 799–810.

[94] Wang, P.-Y., *Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. II. Flat free boundaries are Lipschitz.* Comm. Partial Differential Equations 27 (2002), no. 7-8, 1497–1514.

[95] Weiss, G. S., *A homogeneity improvement approach to the obstacle problem.* Invent. Math. 138 (1999), no. 1, 23–50.

[96] Weiss, G. S., *An obstacle-problem-like equation with two phases: pointwise regularity of the solution and an estimate of the Hausdorff dimension of the free boundary.* Interfaces Free Bound. 3 (2001), no. 2, 121–128.

[97] Singer, I.; Thorpe J., *Lezioni di topologia elementare e di Geometria*, (1980), Boringheri. Torino.

[98] Ziemer, W. (1989). *Weakly differentiable functions, Sobolev spaces and functions of bounded variation.* Grad. Texts in Math., 120.

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