UNIVERSITÀ
DEGLI STUDI
FIRENZE

DOTTORATO DI RICERCA
IN MATEMATICA

CICLO XXVIII

Coordinatore Prof. Alberto Gandolfi

Instanton bundles and their moduli spaces

Settore Scientifico Disciplinare MAT/03

Dottorando:
Davide Vanzo

Davide Vanzo

Tutore
Prof. Giorgio M. Ottaviani

Giorgio Ottaviani

Coordinatore
Prof. Alberto Gandolfi

Anni 2012/2016

Contents

Chapter 1

Introduction

A mathematical instanton bundle on $\mathbb{P}^3$ is a particular algebraic bundle of rank 2. Its importance arises from quantum physics; in fact these particular bundles correspond (through the Penrose-Ward transform) to self dual solutions of the Yang-Mills equation over the real sphere S^4 ([Ati79], [AW77]).
Penrose-Ward transform has been generalized on higher dimensional odd projective space by Salamon ([Sal84]) giving the possibility of defining an instanton bundle on $\mathbb{P}^{2n+1}$.
There are several equivalent ways to define an instanton, the one we will use more throughout this work is the following:
given three complex vector spaces L, M and N of dimension respectively $k, 2n+2k$ and k, an instanton bundle E over $\mathbb{P}^{2n+1}$ with $c_2 = k$ is a stable bundle of rank $2n$ which appears as a cohomology bundle of a monad

$$L \otimes \mathcal{O}(-1) \xrightarrow{B^t} M \otimes \mathcal{O} \xrightarrow{A} N \otimes \mathcal{O}(1) \qquad (\star)$$

Hartshorne and Hirschowitz showed on $\mathbb{P}^3$ that the general instanton bundle has the nice property of having natural cohomology, i.e. for each $t \in \mathbb{Z}$ at most one of the cohomology groups $H^i(E(t))$ for i=0,. . . ,3 is nonzero ([HH82]).
Thanks to the properties inherited from the monad, if we fix k as above, the moduli space of instanton bundles on $\mathbb{P}^{2n+1}$ of charge k (denoted from now on by $MI_{n,k}$) is an open subset of the quasi-projective variety of the stable bundles on $\mathbb{P}^{2n+1}$ of given rank and Chern classes.
Actually this space is still quite unknown. Most of the investigations focus their attention on $n = 1$.
In this case there have been some big improvements in recent days that led to complete answers to smoothness and reducibility:
Jardim and Verbitsky proved the smoothness and the dimension $(8k-3)$ of $MI_{1,k}$ for every k, confirming a 30-year old conjecture ([JV14]). In order to prove this result the two authors used a completely new technique which deals with quaternions,

more precisely they equip $MI_{1,k}$ with a structure called Trihyperkähler reduction, which is the quotient of a trisymplectic structure by the action of a Lie group.
Before this result the only known cases were when $k \leq 5$:
the case $k = 1$ is due to Barth ([Bar77]). Case $k = 2$ was proved by Hartshorne ([Har78]). Ellinsgrud and Stromme settled the case $k = 3$ ([Em81]), while the smoothness when $k = 4$ was proved by Le Potier ([Pot83]). Finally the case $k = 5$ was proved by Katsylo and Ottaviani ([KO03]) and Coanda, Tikhomirov and Trautmann ([CTT03]).
A similar situation happened for the irreducibility of $MI_{1,k}$: until 2003 the only known cases where when $k \leq 5$ (same references as in the smoothness case except for $k = 4$, which is due to Barth ([Bar81]), but the big enhancement came when Tikhomirov proved in two different works that, for k odd first and then for k even, $MI_{1,k}$ is irreducible ([Tik12], [Tik13])
In conclusion to the case of $\mathbb{P}^3$ there are two works by Bruzzo, Markushevich and Tikhomirov ([BMT12], [BMT16]) in which, generalizing the definition of instanton bundles on $\mathbb{P}^{2n+1}$ to any rank greater or equal than $2n$, they exhibit an irreducible component for each moduli space. The proof of the second work relates this component to a particular class of instanton bundles called 't Hooft instantons; these bundles will be studied throughout this work.
If we drop the condition $n = 1$, the things get worse: in a series of works Ancona and Ottaviani proved that $MI_{n,2}$ is smooth and irreducible, while $MI_{2,k}$ is singular for $3 \leq k \leq 8$ and reducible for $4 \leq k \leq 8$ ([AO95], [AO00]).
The two authors conjectured also that for $n \geq 2$ and $k \geq 3$ the moduli space of instanton bundles is singular, conjecture proved by Mirò-Roig and Orus-Lacort ([MROL97]).
So far we have talked about instanton bundles in general, but there is an interesting subset of them which are called symplectic instanton bundles. In order to define these particular instantons it is proper to observe that if E is an instanton bundle then so it is E^*, indeed the dual of ($\star$) is still a monad, and if the maps referred to E are (A, B), then the maps referred to E^* are (B, A). An instanton bundle E is called symplectic if there exists a symplectic isomorphism between E and E^*. It is easy to show that in the case E is symplectic the monad can be rewritten in the following way:

$$N^* \otimes \mathcal{O}(-1) \xrightarrow{JA^t} M \otimes \mathcal{O} \xrightarrow{A} N \otimes \mathcal{O}(1) \tag{$\star\star$}$$

where J is a skewsymmetric isomorphism.
We can then consider the moduli space of symplectic instanton bundles with n and k fixed, and we will denote it from now on with $MIS_{n,k}$.
In the case $n = 1$ there is no difference between $MI_{1,k}$ and $MIS_{1,k}$, namely every instanton bundle on $\mathbb{P}^3$ is symplectic, but for $n \geq 2$ this is not true anymore.

A surprisingly property that holds both for $MI_{n,k}$ and $MIS_{n,k}$ is that they are affine ([CO02]). Let's sketch the proof of the latter:
the idea is to realize the moduli space as a GIT-quotient of an affine variety. In order to do that we need to focus our attention on the space $Hom(M \otimes V^*, N)$ (where we set $V = H^0(\mathcal{O}(1))$).
Inside that space we can take the subvariety $\mathcal{Q}$ given by the nondegenerate matrices for which ($\star\star$) is a complex, or, in other words, the matrices A for which $AJA^t = 0$ (where J is a fixed skewsymmetric matrix) and such that everytime $A(m \otimes v^*) = 0$ has a solution then either $m = 0$ or $v^* = 0$. The space $GL(N) \times Sp(M)$ acts on $\mathcal{Q}$ by $(g, s) \cdot A = gAs$.
The heart of the matter is that $\mathcal{Q}$ is affine and $GL(N) \times Sp(M)$ modulo $\pm(id, id)$ acts freely on it. Hence the moduli space $MIS_{n,k} = \mathcal{Q}/GL(N) \times Sp(M)$ is affine too because it is the quotient of an affine variety by a reductive group. A similar construction can be applied for $MI_{n,k}$.
A way to study the moduli space of instanton bundles is to exploit Kodaira-Spencer theory and small deformations:
let E be an instanton bundle (not necessarily symplectic) and $Ad\ E$ be the adjoint bundle: therefore the Zariski tangent space to the moduli space at $[E]$ (the isomorphism class at which E belongs to) is isomorphic to $H^1(Ad\ E)$, moreover there exists an important analytic morphism called Kuranishi map:

$$\phi_E : \ H^1(Ad\ E) \longrightarrow H^2(Ad\ E)$$

The zero locus of this map is the analytic germ of the moduli space at $[E]$. Hence the moduli space is smooth at $[E]$ if and only if the Kuranishi map is the zero map. This condition is obviously satisfied when $H^2(Ad\ E) = 0$.
When we are dealing with $MI_{n,k}$ then $Ad\ E$ is equal to $End\ E/\mathcal{O}$, hence the Kuranishi map becomes:

$$\phi_E : \ H^1(End\ E) \longrightarrow H^2(End\ E)$$

while when E is symplectic and we want to study the germ of $MIS_{n,k}$ at $[E]$, the Kuranishi map turns out to be:

$$\phi_E : \ H^1(S^2E) \longrightarrow H^2(S^2E)$$

To get an explicit description of the last map (the case non symplectic is similar) we need to use the map Φ (introduced in proposition 3.1.5):
indeed the symplectic Kuranishi map lifts through the diagram

$$\begin{array}{ccc} ker(\Phi) & \xrightarrow{\widetilde{\phi_E}} & H^0(\wedge^2 N \otimes \mathcal{O}(2)) \\ \downarrow & & \downarrow \\ H^1(S^2E) & \xrightarrow{\phi_E} & H^2(S^2E) \end{array}$$

where $\widetilde{\phi_E}(B) = BJB^t$.
Moreover it is an easy calculation (obtained by splitting the monad into two short exact sequences) to obtain the following estimates:

$$h^1(End\ E) - h^2(End\ E) = -k^2\binom{2n-1}{2} + 8kn^2 + 1 - 4n^2$$
$$h^1(S^2E) - h^2(S^2E) = -\frac{k^2}{2}\binom{2n-1}{2} + k\left(\frac{10n^2+5n+1}{2}\right) - 2n^2 - n$$

Let's now give some concrete examples of instanton bundles specifying the matrix A inside the monad:
maybe one of the first and most studied kind of instanton bundle is the so called special symplectic instanton (or Okonek-Spindler), which has the following $k \times (2n+2k)$ matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & y_0 & \cdots & y_{n-1} & y_n \\ 0 & x_0 & \cdots & \cdots & x_n & 0 & \cdots & 0 & 0 & \cdots & 0 & y_0 & \cdots & \cdots & y_n & 0 \\ \vdots & & \ddots & & & & & & & & & & & \iddots & & \vdots \\ 0 & \cdots & 0 & x_0 & x_1 & \cdots & \cdots & x_n & y_0 & \cdots & \cdots & y_{n-1} & y_n & 0 & \cdots & 0 \end{pmatrix}$$

In ([OT94]) and in ([Dio98]) the following results are computed for these particular instantons:

$$h^1(End\ E) = 4(3n-1)k + (2n-5)(2n-1)$$
$$h^1(S^2E) = (10n-2)k + (4n^2 - 10n + 3)$$

Hence, for example, for $n = 2$ and $k = 3$ $MIS_{2,3}$ is smooth of dimension 53 at the points corresponding to special bundles (because $h^2(S^2E) = 0$).
As a second example we can introduce one of the main objects of this work: the 't Hooft bundles. They were first introduced on $\mathbb{P}^3$, then in ([Ott96]) Ottaviani introduces the 't Hooft instanton bundles for $n \geq 2$ (for a precise definition go to 3.1.7):
in order to build these bundles we need to pick $k+n$ codimension 2 linear subspaces, say $\{\xi_i = \omega_i = 0\}$ for $i = 1, \ldots, k$ and $\{z_j = \eta_j = 0\}$ for $j = 1, \ldots, n$, then the following matrix describes an instanton bundle:

$$\begin{pmatrix} D(\xi_i) & aD(z_j) & D(\omega_i) & aD(\eta_j) \end{pmatrix}$$

where $D(\xi_i)$ and $D(\omega_i)$ are diagonal $k \times k$ matrices, $D(z_j)$ and $D(\eta_j)$ are diagonal $n \times n$ matrices and a is a $k \times n$ generic matrix.
Due to this construction we get the following property on $h^0(E(1))$: it is proved that for a generic 't Hooft bundle we have $h^0(E(1)) = n$.
Moreover it is conjectured in the paper that these bundles represent an irreducible component of their moduli space. In order to prove this conjecture it would be sufficient to prove that, apart from a finite number of cases,

$$h^1(S^2E) = 5kn + 4n^2.$$

Indeed, taking for granted this result, Ottaviani exhibits a basis of $H^1(S^2E)$ and shows that the Kuranishi map sends to zero all its elements, proving that 't Hooft instantons are smooth points of the relative symplectic moduli space.
The third example of a symplectic instanton bundle is given by the Rao-Skiti family. These bundles were introduced on $\mathbb{P}^3$ independently by Rao and Skiti in 1997 but their generalization is given in ([CHMRS14]). The matrix associated to this bundle can be divided into two $k \times (n+k)$ blocks: the first block is exactly the same as the Okonek-Spindler one, while the second block is a persymmetric matrix (see definition 3.1.13). This implies that the Okonek-Spindler bundle is a particular Rao-Skiti. The generic Rao-Skiti, differently from the symplectic 't Hooft, has $H^0(E(1)) = 0$.
These two last examples play an important role inside the symplectic moduli space, indeed if Ottaviani's conjecture was true, this would imply furthermore that the moduli space of symplectic instanton bundles on $\mathbb{P}^{2n+1}$, with $n \geq 2$ and k sufficiently large, is reducible: indeed in the paper [CHMRS14] the authors prove, among other things, that there are Rao-Skiti bundles which are not limit of 't Hooft bundles.

This work is divided into three chapters: the first one focuses on Steiner bundles; these objects may represent the kernel bundle defined in ($\star$). First we introduce a Steiner bundle of 't Hooft type called S^* (which is related to 't Hooft instanton), then the first result of this work is proving the behaviour of $H^1(S^2(S^*(-1)))$ and $H^1(S^2(S^*))$ for these particular bundles (see Theorem 2.1.7). The possibility of defining these bundles over $\mathbb{P}^n$ for every n allows us to proceed with an inductive proof. The idea is indeed to restrict a bundle to an hyperplane, the restricted bundle is still a Steiner 't Hooft bundle, and this permits to exploit the inductive step.
In the second chapter we introduce two families of symplectic instanton bundles: The Rao-Skiti and the 't Hooft bundles. We introduce a new family of instanton bundles which is a sort of generalization of the symplectic 't Hooft family: indeed these new bundles are not necessary symplectic, but the matrice (A, B) have the same structure than a symplectic 't Hooft (see definition 3.1.16). In proposition 3.1.18 we evaluate the dimension of this family.
We end the chapter showing the possible structures of the symplectic isomorphism J for the examples introduced above. From these results we can state that the two components (Rao-Skiti and symplectic 't Hooft) are at least path connected (see remark 3.2.8).
The last chapter is divided into two parts: the first one is devoted to a possible path to prove Ottaviani's conjecture on symplectic 't Hooft bundles. More precisely the method is the one used for Steiner bundles, but in this case we need

to restrict our bundle with a codimension two variety, and this leads us to deal with sheaves which are not bundles. Moreover we decided to apply this method to a subset of symplectic 't Hooft bundles which, for k sufficiently large, seem to behave exactly like a generic symplectic 't Hooft bundle. The matrix A associated to these particular bundles has the same form described before, but in this case the ξ_i's and the z_i's depend only on a part of variables while the ω_i's and the η_i's depend on the others.
In the second part we show some of the computational results obtained using Macaulay2 (version 1.7) [GS] and we attach the scripts created.

The aim of this thesis is therefore to study families of instanton bundles and their beaviour. The new results obtained in this work are the dimensions of $H^1(S^2(S^*))$ and $H^1(S^2(S^*(-1)))$ for S^* Steiner bundle of 't Hooft type. As said this result could be very useful in order to prove Ottaviani conjecture. Moreover we have introduced a new family of instanton bundles (which is the generalization of the symplectic 't Hooft) and studied the dimension of this family. Through an example we have seen that along the fiber of the Okonek-Spindler instanton there is also a particular 't Hooft bundle, this fact allows us to conclude that the Rao-Skiti component and the 't Hooft component are connected. Finally the computational results obtained so far show that the Rao-Skiti component is smooth both for the symplectic and the generic case, moreover the dimension of $H^1(S^2(E))$ is the same.

Chapter 2

Steiner Bundles

2.1 First properties and statement of the Theorem

Definition 2.1.1. A Steiner bundle S over $\mathbb{P}^n = \mathbb{P}(V)$ is a rank n vector bundle which appears in an exact sequence of the following type:

$$0 \longrightarrow S^* \to W \otimes \mathcal{O} \xrightarrow{A} I \otimes \mathcal{O}(1) \longrightarrow 0 \tag{2.1}$$

where W and I are complex vector spaces respectively of dimension $n+k$ and k.

Proposition 2.1.2. Let S be a Steiner bundle. Then

i) $h^1(S^2S^*) - h^2(S^2S^*) = n\left(-k^2\frac{(n-1)}{4} + k\frac{5n+3}{4} - \frac{(n+1)}{2}\right) =: p(k,n)$

ii) $h^1(S^2S^*(-1)) - h^2(S^2S^*(-1)) = -k^2\frac{(n-1)}{2} + k\frac{3n+1}{2} =: q(k,n)$

Proof. $h^0(S^2S^*) = h^0(S^2S^*(-1)) = 0$ because S is stable ([AO94] or [BS92]), $h^i(S^2S^*) = h^i(S^2S^*(-1)) = 0 \quad \forall i \geq 3$ follows from the exact sequence

$$0 \longrightarrow S^2S^* \longrightarrow S^2W \otimes \mathcal{O} \longrightarrow W \otimes I \otimes \mathcal{O}(1) \longrightarrow \wedge^2 I \otimes \mathcal{O}(2) \longrightarrow 0 \tag{2.2}$$

□

Definition 2.1.3. A Steiner bundle is called of 't Hooft type if the map $A \in Hom(W, I \otimes V)$ is given in a convenient system of coordinates by the matrix

$$[D(\xi_i)|a \cdot D(z_i)] \tag{2.3}$$

where $a = \left(a_i^j\right)$ is a $k \times n$ matrix with complex entries, $D(\xi_i)$ is a diagonal $k \times k$ matrix with diagonal entries degree 1 forms $\xi_1, \ldots, \xi_k$, $D(z_j)$ is a diagonal $n \times n$ matrix with diagonal entries degree 1 forms $z_1, \ldots, z_n$.

From now on when we write Steiner bundle we will imply of 't Hooft type.

Notation 2.1.4. When $A \in Hom(W, I \otimes V)$ we denote by $A^t \in Hom(I^*, W^* \otimes V)$ the dual map. Moreover we set $A = \sum_{i=0}^{n} A_i z_i$ where $A_i \in Hom(W, I)$.

Remark 2.1.5. Let be given a Steiner bundle, there exists a degree 1 form z_0 such that $(z_0, \ldots, z_n)$ is a system of coordinates and from now on we will use x_i instead of z_i. Furthermore we set $\xi_i = \xi_{i0}x_0 + \ldots + \xi_{in}x_n$.

Remark 2.1.6. For $k = 1$ we have $S \simeq T\mathbb{P}^n(-1)$. Hence we may suppose from now on $k \geq 2$.

Our main result of this chapter is the following

Theorem 2.1.7. *Let S be a generic Steiner bundle of 't Hooft type over $\mathbb{P}^n$ and let $k \geq 2$. Then*
i) $h^1(S^2S^*(-1)) = \max\{q(k, n), k + n\}$
ii) $h^1(S^2S^*) = \max\{p(k, n), n(k + n)\}$

Remark 2.1.8. Let $k \geq 2$. The following hold

$$q(k, n) \leq k + n \iff \frac{2n}{n-1} \leq k$$

$$p(k, n) \leq n(k + n) \iff \begin{cases} \frac{3n+1}{n-1} \leq k \\ or\ k = 2 \end{cases}$$

Remark 2.1.9. The Theorem is true for $n = 1$. In fact over $\mathbb{P}^1$ we have $S^* = \mathcal{O}(-k)$ and $S^2S^* = \mathcal{O}(-2k)$. In this case $q(k, 1) = 2k$ and $p(k, 1) = 2k - 1$.

We will prove the theorem by induction on n. The proof needs the computation of the syzygies of a certain module. It turns out that

i) it is convenient to prove by induction a stronger form of the theorem, namely we will make explicit a basis of the vector spaces $H^1(S^2S^*(-1))$. Such a basis is helpful also to prove ii) of the theorem. In fact it follows that when $p(k, n) \leq n(k + n)$ the natural map $H^1(S^2S^*(-1)) \otimes V \longrightarrow H^1(S^2S^*)$ is surjective, although in general $H^2(S^2S^* \otimes \Omega^1) \neq 0$

ii) The induction step is straightforward for $n \geq 2$ but there are technical problems in the step from $n = 1$ to $n = 2$. In order to overcome these difficulties we will have to make explicit a basis of $H^1(S^2S^*(-1))$ even for $n = 1$.

Proposition 2.1.10. Let S be a Steiner bundle corresponding to $A \in Hom(W, I \otimes V)$. Let

$$\begin{array}{cccc} \phi: & Hom(I^*, W) & \to & \wedge^2 I \otimes V \\ & B & \mapsto & AB - B^t A^t \end{array}$$

The following are true

$$H^1(S^2 S^*(-1)) = \ker\ \phi$$
$$H^2(S^2 S^*(-1)) = coker\ \phi$$

Proof. Straightforward from the exact sequence (2.2) twisted by $\mathcal{O}(-1)$. □

Proposition 2.1.11. Let S be a Steiner bundle corresponding to $A \in Hom(W, I \otimes V)$. Let

$$\begin{array}{cccc} \Phi: & Hom(I^*, W \otimes V) & \to & \wedge^2 I \otimes S^2 V \\ & B & \mapsto & AB - B^t A^t \end{array}$$

The following are true

$$H^1(S^2 S^*) \simeq \ker\ \Phi / S^2 W$$
$$H^2(S^2 S^*) \simeq coker\ \Phi$$

Moreover the embedding $S^2 W \to \ker\ \Phi$ has the matrix form $\Sigma \mapsto \Sigma A^t$ with Σ symmetric.

Proof. Straightforward from the exact sequence (2.2)

2.2 Explicit description over $\mathbb{P}^1$

Proposition 2.2.1. Let $S = \mathcal{O}(-k)$ be the Steiner bundle over $\mathbb{P}^1$ corresponding to a map $A \in Hom(W, I \otimes V)$. A basis of $H^1(\mathbb{P}^1, S^2 S^*(-1))$ is given by the following $2k$ elements of $Hom(I^*, W)$ (expressed in the dual basis of I^*):

i) E_i with only nonzero entry at place (i, i) equal to 1, for $i = 1, \ldots, k$.

ii) B_i with (j, i)-th entry given by

$$\begin{cases} \frac{-a_j \xi_{i0}}{\xi_{i0}\xi_{j1} - \xi_{i1}\xi_{j0}} & \text{for} \quad j = 1, \ldots, k \quad j \neq i \\ 0 & \text{for} \quad j = i \\ 1 & \text{for} \quad j = k+1 \end{cases}$$

with (i,j)-th entry given by

$$\begin{cases} \frac{-a_j\xi_{j0}}{\xi_{i0}\xi_{j1}-\xi_{i1}\xi_{j0}} & \text{for} \quad j=1,\dots,k \quad j\neq i \\ 0 & \text{for} \qquad\qquad j=i \end{cases}$$

and with all other entries equal to zero, for $i=1,\dots,k$.

Proof. It is straightforward to check that AE_i and AB_i are symmetric matrices. Moreover the $2k$ matrices E_i, B_j are linearly independent. Hence they constitute a basis of the $2k$ dimensional space $H^1(\mathbb{P}^1, S^2S^*(-1))$.

□

Proposition 2.2.2. Let $S=\mathcal{O}(-k)$ be the Steiner bundle of 't Hooft type over $\mathbb{P}^1=\mathbb{P}(V)$ corresponding to a map $A\in Hom(W, I\otimes V)$. Let assume moreover that ξ_{i0} and a_i are different from zero for every i.

i) The k-dimensional subspace of $I\otimes V$ generated by $I\otimes x_1$ surjects over the $(k-1)$-dimensional vector space $H^1(\mathbb{P}^1, S^*)=(I\otimes V)/W$, where in the quotient $bx_1\sim b'x_1$ if and only if $b-b'\in\langle a^1\rangle$ (a^1 is the first column of a).

ii) The $2k$-dimensional subspace of $K\subset W\otimes I\otimes V$ generated by $\langle B_i\otimes V\rangle$ for $i=1,\dots,k$ surjects over the $(2k-1)$-dimensional vector space $H^1(\mathbb{P}^1, S^2S^*)=K/S^2W$, where $\sum_{i=1}^k h_iB_i\sim\sum_{i=1}^k h'_iB_i$ if and only if $[h_1-h'_1,\dots,h_k-h'_k]\in\langle a^1x_1\rangle$.

Proof. To prove i) we consider the embedding of W in $I\otimes V$: the equation $vx_1=Aw$, where $v\in I$ and $w\in W$, gives the conditions

$$\begin{cases} w_i\xi_{i0} &= 0 \\ w_i\xi_{i1}+w_{k+1}a_i^1 &= v_i \end{cases}$$

hence i).

In order to prove ii) consider that

$$H^1(\mathbb{P}^1, S^2S^*(-1))\otimes V\longrightarrow H^1(\mathbb{P}^1, S^2S^*)$$

is surjective (H^2 vanishes over 1-dimensional projective spaces). By using the prop 2.2.1 the $4k$ dimensional subspace of K generated by $\langle B_i\otimes V\rangle$ and $\langle E_j\otimes V\rangle$ surjects over $H^1(\mathbb{P}^1, S^2S^*)$. The equation $\sum_{i=1}^k h_iB_i+\sum_{i=1}^k e_iE_i=SA^t$, with S symmetric, gives the conditions

$$h_i=s_{i,k+1}\xi_{i0}x_0+(s_{i,k+1}\xi_{i1}+s_{k+1,k+1}a_i^1)x_1$$

$$e_i = s_{i,i}\xi_{i0}x_0 + (s_{i,i}\xi_{i1} + s_{i,k+1}a_i^1)x_1$$

which are satisfied by some S if and only if

$$\begin{cases} e_{i1}\xi_{i0} - e_{i0}\xi_{i1} = h_{i0}a_i^1 \\ a_i^1\xi_{i0} = M(h_{i0}\xi_{i1} - h_{i1}\xi_{i0}) \end{cases}$$

with M independent by i.
Now ii) follows.

□

2.3 Restriction to hyperplanes

Let $\mathbb{P}^{n-1}$ be the hyperplane given by the equation $x_n = 0$. Let $A_{|\mathbb{P}^{n-1}} \in Hom(W, I \otimes V')$ (where $V' = \langle x_0, \ldots, x_{n-1}\rangle$) be given by substituting $x_n = 0$ in A. In matrix form the last column of $A_{|\mathbb{P}^{n-1}}$ is zero and we set $A_{|\mathbb{P}^{n-1}} = \left[A'|0\right]$. We set $W = W' \oplus \mathbb{C}$, so that $A' \in Hom(W', I \otimes V')$. Then $S_{|\mathbb{P}^{n-1}} \simeq S' \oplus \mathcal{O}$ where S' is again a Steiner bundle of 't Hooft type, appearing in the exact sequence:

$$0 \longrightarrow S'^* \longrightarrow W' \otimes \mathcal{O} \xrightarrow{A'} I \otimes \mathcal{O}(1) \longrightarrow 0$$

In particular $S^2S^*_{|\mathbb{P}^{n-1}} = S^2S'^* \oplus S'^* \oplus \mathcal{O}$

Consider the cohomology sequence associated to the sequence

$$0 \longrightarrow S^2S^*(-2) \longrightarrow S^2S^*(-1) \longrightarrow S^2S^*(-1)_{|\mathbb{P}^{n-1}} \longrightarrow 0 \tag{2.4}$$

It follows that $H^1(S^2S^*(-1))$ is the kernel of the boundary map δ

$$H^1(S^2S^*(-1)_{|\mathbb{P}^{n-1}}) \simeq H^1(\mathbb{P}^{n-1}, S^2S'^*(-1)) \oplus I \xrightarrow{\delta} H^2(S^2S^*(-2)) \simeq \wedge^2 I \tag{2.5}$$

Let

$$\begin{array}{cccc} \phi_{|\mathbb{P}^{n-1}}: & Hom(I^*, W) & \longrightarrow & \wedge^2 I \otimes V' \\ & B & \mapsto & \left(A_{|\mathbb{P}^{n-1}}\right) B - B^t \left(A_{|\mathbb{P}^{n-1}}\right)^t \end{array}$$

By restricting (2.2) to $\mathbb{P}^{n-1}$ it follows

$$H^1(S^2S^*_{|\mathbb{P}^{n-1}}(-1)) \simeq \ker\ \phi_{|\mathbb{P}^{n-1}}.$$

The decomposition

$$Hom(I^*, W) = Hom(I^*, W') \oplus I$$

induces in a natural way the splitting

$$H^1(S^2S^*_{|\mathbb{P}^{n-1}}(-1)) = H^1(S^2S'^*(-1)) \oplus H^1(S'^*(-1)).$$

Proposition 2.3.1. Given $B' \in H^1(\mathbb{P}^{n-1}, S^2S'^*(-1)) \subset Hom(I^*, W')$ represented by a $(k+n-1)\times k$ matrix and $b \in I$ represented by a $1\times k$ matrix, let us construct $B \in H^1(S^2S^*(-1)_{|\mathbb{P}^{n-1}}) \subset Hom(I^*, W)$ by stacking b as last row under B', that is $B := \begin{bmatrix} B' \\ b \end{bmatrix}$. Then the boundary map in (2.5) is given by

$$\delta(B', b) = A_nB - B^tA_n^t$$

Proof. It is a diagram chase into the following

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \wedge^2 I \arrow[r, "\simeq"] \arrow[d, "\cdot x_n"] & H^2(S^2S^*(-2)) \arrow[d] \\
0 \arrow[r] & H^1(S^2S^*(-1)) \arrow[r] \arrow[d] & W\otimes I \arrow[r, "\phi"] \arrow[d, "\simeq"] & \wedge^2 I\otimes V \arrow[r] \arrow[d] & H^2(S^2S^*(-1)) \arrow[d] \\
0 \arrow[r] & H^1(S^2S^*(-1)_{|\mathbb{P}^{n-1}}) \arrow[r] & W\otimes I \arrow[r, "\phi_{|\mathbb{P}^{n-1}}"] \arrow[d] & \wedge^2 I\otimes V' \arrow[r] \arrow[d] & H^2(S^2S^*(-1)_{|\mathbb{P}^{n-1}}) \arrow[d] \\
 & & 0 & 0 & 0
\end{tikzcd}$$

□

Consider the cohomology sequence associated to the sequence

$$0 \longrightarrow S^2S^*(-1) \longrightarrow S^2S^* \longrightarrow S^2S^*_{|\mathbb{P}^{n-1}} \longrightarrow 0. \tag{2.6}$$

Let

$$\begin{array}{cccc} \Phi_{|\mathbb{P}^{n-1}}: & Hom(I^*, W\otimes V') & \longrightarrow & \wedge^2 I\otimes S^2V' \\ & B & \mapsto & \left(A_{|\mathbb{P}^{n-1}}\right)B - B^t\left(A_{|\mathbb{P}^{n-1}}\right)^t \end{array}$$

and denote $K_{|\mathbb{P}^{n-1}} := \ker\ \Phi_{|\mathbb{P}^{n-1}}$. In particular the decomposition

$$Hom(I^*, W\otimes V') = Hom(I^*, W'\otimes V') \oplus (I\otimes V')$$

induces

$$K_{|\mathbb{P}^{n-1}} = K' \oplus (I\otimes V')$$

$$K'/S^2W' \simeq H^1(\mathbb{P}^{n-1}, S^2S'^*), \qquad I\otimes V'/W' \simeq H^1(\mathbb{P}^{n-1}, S'^*)$$

Proposition 2.3.2. Given $B' \in K' \subset Hom(I^*, W' \otimes V')$ represented by a $(k+n-1) \times k$ matrix with linear entries and $b \in I \otimes V'$ represented by a $1 \times k$ matrix let us construct $B \in Hom(I^*, W \otimes V)$ by stacking b as last row under B', that is $B := \begin{bmatrix} B' \\ b \end{bmatrix}$. The boundary map

$$H^1(S^2 S^*_{|\mathbb{P}^{n-1}}) \simeq H^1(\mathbb{P}^{n-1}, S^2 S'^*) \oplus H^1(\mathbb{P}^{n-1}, S'^*) \xrightarrow{\delta} H^2(S^2 S^*(-1))$$

fits into the following commutative diagram:

$$\begin{tikzcd}
K' \oplus (I \otimes V') \arrow[r, "\delta'"] \arrow[d] & \wedge^2 I \otimes V \arrow[d] \\
H^1(\mathbb{P}^{n-1}, S^2 S'^*) \oplus H^1(\mathbb{P}^{n-1}, S'^*) \arrow[r, "\delta"] & H^2(S^2 S^*(-1))
\end{tikzcd}$$

where $\delta'(B', b) = A_n B - B^t A_n^t$.

Proof. It is a diagram chase into the following

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & W \otimes I \arrow[r, "\phi"] \arrow[d, "\cdot x_n"] & \wedge^2 I \otimes V \arrow[r] \arrow[d, "\cdot x_n"] & H^2(S^2 S^*(-1)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & W \otimes I \otimes V \arrow[r, "\Phi"] \arrow[d] & \wedge^2 I \otimes S^2 V \arrow[r] \arrow[d] & H^2(S^2 S^*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_{|\mathbb{P}^{n-1}} \arrow[r] & W \otimes I \otimes V' \arrow[r, "\Phi_{|\mathbb{P}^{n-1}}"] \arrow[d] & \wedge^2 I \otimes S^2 V' \arrow[r] \arrow[d] & H^2(S^2 S^*_{|\mathbb{P}^{n-1}}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

□

2.4 Proof of the Theorem

Proposition 2.4.1. Let S be a Steiner bundle of 't Hooft type over $\mathbb{P}^n$ with $n \geq 2$. Let $q(k,n) \leq k+n$. $H^1(S^2 S^*(-1))$ has dimension $n+k$ and all its elements consist of the following elements of $Hom(I^*, W)$ (expressed in the dual basis of I^*):

$$[D(e_i) | a \cdot D(f_j)]^t \tag{2.7}$$

where a is the $k \times n$ matrix with constant entries that appears in A, $D(e_i)$ is a diagonal $k \times k$ matrix with constant diagonal entries e_i, $D(f_j)$ is a diagonal $n \times n$ matrix with constant diagonal entries f_j.

Proof. It is straightforward to check that the subspace of $Hom(I^*, W)$ spanned by the elements (2.7) has dimension $n + k$ and it is contained in $H^1(S^2S^*(-1))$.

We first prove the proposition for $n = 2$. In this case the prop. 2.2.1 is needed. We consider in the above construction $B' = \sum_{i=1}^k p_i E_i + \sum_{i=1}^k q_i B_i$ (here p_i, $q_j \in \mathbb{C}$). We denote by a^1 and a^2 the two columns of a. Hence the boundary operator assumes the nice form:

$$\delta(B', b) = (a^2 \cdot b - b^t \cdot (a^2)^t) + Q$$

where the (i,j)-th entry of Q is $(q_j a_i^1 - q_i a_j^1)\frac{(\xi_{i2}\xi_{j0} - \xi_{i0}\xi_{j2})}{(\xi_{i0}\xi_{j1} - \xi_{j0}\xi_{i1})}$

In particular δ vanishes on the k-dimensional space spanned by $(B', b) = (\sum_{i=1}^k p_i E_i, 0)$.

$\delta(B', b) = 0$ is a linear system in the $2k$ unknowns b_i, q_j. The matrix $\binom{k}{2} \times 2k$ of this system divides into two blocks $\binom{k}{2} \times k$ each one of the form (here we are considering the case $k = 7$)

$$\begin{pmatrix}
X & X & & & & & \\
X & & X & & & & \\
X & & & X & & & \\
X & & & & X & & \\
X & & & & & X & \\
X & & & & & & X \\
 & X & X & & & & \\
 & X & & X & & & \\
 & X & & & X & & \\
 & X & & & & X & \\
 & X & & & & & X \\
 & & X & X & & & \\
 & & X & & X & & \\
 & & X & & & X & \\
 & & X & & & & X \\
 & & & X & X & & \\
 & & & X & & X & \\
 & & & X & & & X \\
 & & & & X & X & \\
 & & & & X & & X \\
 & & & & & X & X
\end{pmatrix} \tag{2.8}$$

If we label a row with the couple (i,j) (with $i<j$) to which it is referred, i and j are the only non-zero entries in the row.
To be more precise: the $i-$th entry of the row (i,j) of the first block is $-a^2_j$ and the $j-$th entry is a^2_i, while the $i-$th entry of the row (i,j) of the second block is $-a^1_j\frac{\xi_{i2}\xi_{j0}-\xi_{i0}\xi_{j2}}{\xi_{i0}\xi_{j1}-\xi_{j0}\xi_{i1}}$ and the $j-$th entry is $-a^1_i\frac{\xi_{i2}\xi_{j0}-\xi_{i0}\xi_{j2}}{\xi_{i0}\xi_{j1}-\xi_{j0}\xi_{i1}}$.
Considerning these blocks separately, it is easy to see that the row (i,j) with $i>1$ is a linear combination of the rows $(1,i)$ and $(1,j)$, hence every block has rank $k-1$. Moreover, if the ξ_i are generic and $2(k-1)\leq\binom{k}{2}$, this matrix has rank $2(k-1)$ and the solutions of the system are spanned by the two obvious ones:

$$\begin{cases} b=(a^2)^t \\ q=0 \end{cases} \qquad \begin{cases} b=0 \\ q=a^1 \end{cases}$$

The above inequality is equivalent to $k\geq 4$ that is to $q(k,2)\leq k+2$. It follows that with the assumptions of the theorem $\dim H^1(\mathbb{P}^2,S^2S^*(-1))=k+2$ and this proves the case $n=2$.
The case $n\geq 3$ is easier. By induction the boundary map δ applies to the matrix B which is obtained by stacking $B'=[D(e_i)|a\cdot D(f_j)]^t$ with $b\in I$ (where $D(e_i)$ is a diagonal $k\times k$ matrix and $D(f_j)$ is a diagonal $(n-1)\times(n-1)$ matrix) and

$$\delta(B',b)=(a^n\cdot b-b^t\cdot(a^n)^t)$$

so that the kernel of δ has dimension $(k+n-1)+1=k+n$ as we wanted.

□

Let $n=2$. Let δ' as in the prop. 2.3.2. We have again $\delta'(\sum p_iE_i,0)=0$ where p_i are homogeneous polynomials of degree 1. With the notations of the prop. 2.2.2, taking $b\in I\otimes x_1$, the (p,q)-entry of $\delta'(\sum_{i=1}^k h_iB_i,b)$ is

$$(h_{0q}a^1_p-h_{0p}a^1_q)\frac{(\xi_{p2}\xi_{q0}-\xi_{p0}\xi_{q2})}{(\xi_{p0}\xi_{q1}-\xi_{q0}\xi_{p1})}x_0+\left[b_qa^2_p-b_pa^2_q+(h_{1q}a^1_p-h_{1p}a^1_q)\frac{(\xi_{p2}\xi_{q0}-\xi_{p0}\xi_{q2})}{(\xi_{p0}\xi_{q1}-\xi_{q0}\xi_{p1})}\right]x_1.$$

The (p,q) entry of $\phi(C)=AC-C^tA^t$ is

$$(c_{pq}\xi_{p0}-c_{qp}\xi_{q0})\,x_0+\left(c_{pq}\xi_{p1}-c_{qp}\xi_{q1}+c_{k+1,q}a^1_p-c_{k+1,p}a^1_q\right)x_1+$$
$$+\left(c_{pq}\xi_{p2}-c_{qp}\xi_{q2}+c_{k+2,q}a^2_p-c_{k+2,p}a^2_q\right)x_2.$$

Lemma 2.4.2. Let $n=2$ and $p(k,2)\leq 2k+4$, that is $k\geq 7$. With the notations of the prop. 2.2.2, a basis for the solutions of the system

$$\delta'(\sum_{i=1}^k h_iB_i,b)=\phi(C) \qquad (2.9)$$

in the unknowns h_i, b, C is given by the $2k+7$ solutions:

$$\left\{ \begin{array}{cc} h_{0p} = \gamma_p \xi_{p0} & h_{1p} = \gamma_p \xi_{p1} \\ c_{pq} = \frac{(\gamma_q \xi_{q0} a^1_p - \gamma_p \xi_{p0} a^1_q)\xi_{q2}}{\xi_{q0}\xi_{p1} - \xi_{p0}\xi_{q1}} & c_{k+1,p} = -\gamma_p \xi_{p2} \\ c_{k+2,p} = 0 & b = 0 \end{array} \right. \quad \forall (\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \quad (k \text{ solutions})$$

$$h_{0p} = a^1_p \quad h_{1p} = 0 \quad b = 0 \quad C = 0 \qquad (k+1-th)$$

$$h_{0p} = 0 \quad h_{1p} = 0 \quad b_p = a^2_p \quad C = 0 \qquad (k+2-th)$$

$$c_{pq} = \frac{a^2_p a^2_q}{\xi_{p0}} \quad c_{k+1,p} = 0 \quad c_{k+2,p} = \frac{a^2_p \xi_{p2}}{\xi_{p0}} \quad b_p = a^2_p \frac{\xi_{p1}}{\xi_{p0}} \qquad (k+3-th)$$

$$h_{1p} = a^1_p, \text{all other unknowns equal to zero} \qquad (k+4-th)$$

$$b_p = a^1_p \quad c_{k+1,p} = -a^2_p, \text{all other unknowns equal to zero} \qquad (k+5-th)$$

$$c_{k+1,p} = a^1_p, \text{all other unknowns equal to zero} \qquad (k+6-th)$$

$$c_{k+2,p} = a^2_p, \text{all other unknowns equal to zero} \qquad (k+7-th)$$

and the other k solutions are given by $c_{pq} = \delta_{pq}$ and all other unknowns equal to zero.

Proof. It is straightforward to check that the expressions in the statement are solutions. Let observe first that the c_{pp} unknowns are free and this fact corresponds to the last k solutions.
In order to prove that there are no other solutions, let us denote

$$r(p,q) := \frac{(h_{0q}a^1_p - h_{0p}a^1_q)\xi_{q2}}{\xi_{q0}\xi_{p1} - \xi_{p0}\xi_{q1}} \quad \text{for } 1 \leq p, \quad q \leq k$$

and define new unknowns $\lambda(p,q)$ by the equation

$$c_{pq} = r(p,q) + \lambda(p,q)\xi_{q0} \quad \text{for } 1 \leq p, \quad q \leq k$$

The equation (2.9) for the coefficients of x_0 implies

$$c_{pq}\xi_{p0} - c_{qp}\xi_{q0} = r(p,q)\xi_{p0} - r(q,p)\xi_{q0}$$

and it follows $\lambda(p,q) = \lambda(q,p)$.

The equation (2.9) for the coefficients of x_2 implies

$$\lambda(p,q)\xi_{q0}\xi_{p2} - \lambda(q,p)\xi_{p0}\xi_{q2} + c_{k+2,q}a_p^2 - c_{k+2,p}a_q^2 = 0$$

that is

$$\lambda(p,q)\left(\xi_{q0}\xi_{p2} - \xi_{p0}\xi_{q2}\right) + c_{k+2,q}a_p^2 - c_{k+2,p}a_q^2 = 0,$$

hence $\lambda(p,q)$ can be uniquely determined by the other unknowns.

In particular we get

$$c_{pq}\xi_{p1} - c_{qp}\xi_{q1} = (h_{0q}a_p^1 - h_{0p}a_q^1)\frac{(\xi_{q2}\xi_{p1} - \xi_{q1}\xi_{p2})}{\xi_{q0}\xi_{p1} - \xi_{p0}\xi_{q1}} + (a_p^2 c_{k+2,q} - a_q^2 c_{k+2,p})\frac{(\xi_{q0}\xi_{p1} - \xi_{q1}\xi_{p0})}{\xi_{q0}\xi_{p2} - \xi_{p0}\xi_{q2}} \tag{2.10}$$

The last group of equations is given by the coefficients of x_1 in (2.9) and by using (2.10) we are left with the following $\binom{k}{2}$ equations in the $5k$ unknowns h_{0p}, h_{1p}, b_q, $c_{k+1,p}$, $c_{k+2,p}$

$$\begin{aligned} -(h_{0q}a_p^1 - h_{0p}a_q^1)\frac{(\xi_{q2}\xi_{p1} - \xi_{q1}\xi_{p2})}{\xi_{q0}\xi_{p1} - \xi_{p0}\xi_{q1}} + (a_p^2 c_{k+2,q} - a_q^2 c_{k+2,p})\frac{(\xi_{q0}\xi_{p1} - \xi_{q1}\xi_{p0})}{\xi_{q0}\xi_{p2} - \xi_{p0}\xi_{q2}} + \\ -(h_{1q}a_p^1 - h_{1p}a_q^1)\frac{(\xi_{q0}\xi_{p2} - \xi_{q2}\xi_{p0})}{\xi_{q0}\xi_{p1} - \xi_{p0}\xi_{q1}} - c_{k+1,q}a_p^1 + c_{k+1,p}a_q^1 + b_q a_p^2 - b_p a_q^2 = 0 \end{aligned}$$

The matrix $\binom{k}{2} \times 5k$ of this system has rank $4k-7$. One possible way to show this fact is the following: this matrix divides into five blocks $\binom{k}{2} \times k$ each one of the same form and property described in (2.8). To fix the ideas we set again $k = 7$ and we order these five blocks in the following way: coefficients of h_{0p}, coefficients of h_{1p}, coefficients of $c_{k+1,p}$, coefficients of $c_{k+2,p}$ and coefficients of b_p.

We now perform Gaussian elimination to the first block (taking into account the entire matrix), recalling that the row (i,j) $(i > 1)$ is a linear combination of the rows $(1,i)$ and $(1,j)$. In such a way we get $k-1$ rows linearly independent: indeed after $k-1$ rows the first block will have only zero entries while the other blocks will have the following form:

$$\begin{pmatrix}
X & X & X & & & & \\
X & X & & X & & & \\
X & X & & & X & & \\
X & X & & & & X & \\
X & X & & & & & X \\
X & & X & X & & & \\
X & & X & & X & & \\
X & & X & & & X & \\
X & & X & & & & X \\
X & & & X & X & & \\
X & & & X & & X & \\
X & & & X & & & X \\
X & & & & X & X & \\
X & & & & X & & X \\
X & & & & & X & X
\end{pmatrix}$$

We now perform Gaussian elimination to the second block from the $k-th$ row. It is still true that in this block the row (i,j) (now $i>2$) is a linear combination of the rows $(2,i)$ and $(2,j)$. If we act in this way we get other $k-2$ rows that are linearly independent because the entries of the second block after these rows become all zero. Moreover also the third block after $k-2$ rows has only zero entries, while the last two blocks have this form:

$$\begin{pmatrix}
X & X & X & X & & & \\
X & X & X & & X & & \\
X & X & X & & & X & \\
X & X & X & & & & X \\
X & X & & X & X & & \\
X & X & & X & & X & \\
X & X & & X & & & X \\
X & X & & & X & X & \\
X & X & & & X & & X \\
X & X & & & & X & X
\end{pmatrix}$$

The third step is to leave the next $k-3$ rows and perform Gaussian elimination to the fourth block, the behavior is different than before: indeed after these $k-3$

rows the form of the fourth block is the following:

$$\begin{pmatrix} X & X & X & . & . & . & . \\ X & X & X & & & & \\ X & X & X & & & & \\ X & X & X & & & & \\ X & X & X & & & & \\ X & X & X & & & & \end{pmatrix}$$

while the fifth block

$$\begin{pmatrix} X & X & X & X & X & & \\ X & X & X & X & & X & \\ X & X & X & X & & & X \\ X & X & X & & X & X & \\ X & X & X & & X & & X \\ X & X & X & & & X & X \end{pmatrix}$$

The last step (similar to the others) gives another $k-4$ linearly independent rows. In such a way we end up with $4k-10$ rows which are linearly independent and other $\binom{k}{2}-4k+10$ rows, each one of them with at most seven entries different from zero. It is possible to see that the first three rows are linearly independent (there are at least three if $k \geq 7$).
Hence we can conclude that the $\binom{k}{2} \times 5k$ matrix of this system has rank $4k-7$ when $4k-7 \leq \binom{k}{2}$, which is equivalent to $p(k,2) \leq 2k+4$. This proves the lemma. $\square$

Proposition 2.4.3. Let S be a Steiner bundle of 't Hooft type over $\mathbb{P}^n$ with $n \geq 2$. Let $p(k,n) \leq n(k+n)$.

i) $\ker\ \Phi$ (see 2.1.11) is spanned by S^2W and by the following elements of $\ker\ \Phi \subset Hom(I^*, W \otimes V)$ (expressed in the dual basis of I^*):

$$[D(e_i)|a \cdot D(f_j)]^t \tag{2.11}$$

where a is the $k \times n$ matrix with constant entries appearing in (3.7), $D(e_i)$ is a diagonal $k \times k$ matrix with linear diagonal entries e_i, $D(f_j)$ is a diagonal $n \times n$ matrix with linear diagonal entries f_j.

ii)$H^1(S^2S^*) = Ker\ \Phi/S^2W$ has dimension $n(k+n)$.

iii) the natural map $H^1(S^2S^*(-1)) \otimes V \longrightarrow H^1(S^2S^*)$ is surjective.

Proof. It is straightforward to check that the elements in (2.11) belong to ker Φ. Let $Z \subset \ker \Phi$ be the linear span of the elements in (2.11). Then dim $Z = (n+1)(k+n)$. i) and ii) are equivalent because $Z \cap S^2W$ is given by the diagonal matrices Σ (as in the prop. 2.1.11) and has dimension $k+n$.
iii) follows by i) and by the prop. 2.4.1.
Consider the boundary map

$$H^1(S^2S^*_{|\mathbb{P}^{n-1}}) \xrightarrow{\delta} H^2(S^2S^*(-1))$$

We first prove i) for $n=2$. In this case the prop. 2.2.2 is needed.

By the prop. 2.3.2 and the lemma 2.4.2 we check that the kernel of δ has dimension $k+3$ and it is spanned by the first $k+3$ solutions of the lemma 2.4.2. In fact the other solutions of the lemma are zero when projected on H^1. Hence the cohomology sequence associated to (2.6)

$$\begin{array}{ccccccccc} & & \mathbb{C} & & \mathbb{C}^{k+2} & & & & \\ & & || & & || & & & & \\ 0 & \longrightarrow & H^0(S^2S^*_{|\mathbb{P}^1}) & \longrightarrow & H^1(S^2S^*(-1)) & \longrightarrow & H^1(S^2S^*) & \longrightarrow & \end{array}$$

$$\longrightarrow H^1(S^2S^*_{|\mathbb{P}^1}) \xrightarrow{\delta} H^2(S^2S^*(-1))$$

gives

$$h^1(S^2S^*) = (k+3) + (k+2) - 1 = 2k+4$$

as we wanted.

i) and ii) can now be proved by induction on n. We remark that $p(k,n) \leq n(k+n)$ implies $q(k,n) \leq k+n$ and $p(k,n-1) \leq (n-1)(k+n-1)$.

The cohomology sequence associated to (2.6) is

$$\begin{array}{ccccccccc} & & \mathbb{C} & & \mathbb{C}^{n+k} & & & & \\ & & || & & || & & & & \\ 0 & \longrightarrow & H^0(S^2S^*_{|\mathbb{P}^{n-1}}) & \longrightarrow & H^1(S^2S^*(-1)) & \longrightarrow & H^1(S^2S^*) & \longrightarrow & \end{array}$$

$$\longrightarrow H^1(S^2S^*_{|\mathbb{P}^{n-1}}) \xrightarrow{\delta} H^2(S^2S^*(-1))$$

The kernel of the boundary map δ described by the proposition 2.3.2 contains by the inductive hypothesis the subspace $H^1(S^2S'^*) \oplus 0$ and is given precisely by $H^1(S^2S'^*) \oplus (a_n^tV')$ which has dimension $(n-1)(k+n-1)+n$. It follows

$$h^1(S^2S^*) = (n-1)(k+n-1) + n + (n+k) - 1 = n(k+n)$$

as we wanted.

$\square$

proof of the Theorem 2.1.7

The part i) of the Theorem follows from the prop. 2.4.1 for $q(k,n) \leq k+n$. Only the case $n=2$, $k=3$ is left out. This case can be checked by a direct computation or by using a computer. The part ii) of the Theorem follows from the prop. 2.4.3 for $p(k,n) \leq n(k+n)$. Only the cases

$$k = 2,3 \qquad n \geq 2$$

$$k = 4 \qquad n = 2,3,4$$

$$k = 5,6 \qquad n = 2$$

are left out and in these cases we have to prove that $H^2(S^2S^*) = 0$. The case $k=2$ is contained in the following lemma 2.4.4. The case $k=3$ follows because Schwarzenberger bundles C satisfy $H^2(S^2C^*) = 0$ [Dio98] and semicontinuity applies. The remaining five cases can be checked by direct computations or also by using a computer.

Lemma 2.4.4. Let S be a Steiner bundle with $k=2$.

i) $H^i(S^*(1)) = 0$

ii) $H^2(S^2S^*) = 0$

Proof. By the assumption $c_1(S^*) = -2$, hence

$$S^*(1) \simeq \wedge^{n-1}S(-1)$$

The $(n-1)$-th wedge power of the dual of (2.1) gives

$$S^{n-1}I^*(-n) \longrightarrow \ldots \longrightarrow \wedge^{n-2}W^* \otimes I^*(-2) \longrightarrow \wedge^{n-1}W^*(-1) \longrightarrow \wedge^{n-1}S(-1) \longrightarrow 0$$

and from this sequence i) follows. By tensoring (2.1) by S^* we get

$$0 \longrightarrow S^* \otimes S^* \longrightarrow S^* \otimes W \longrightarrow S^*(1) \otimes I \longrightarrow 0$$

By using i) it follows $H^2(S^* \otimes S^*) = 0$ and hence ii) is proved.

□

Chapter 3

Instanton bundles

The Steiner bundles we have studied in the first chapter have a strong link with the instanton bundles we are going to study. Indeed to an instanton bundle can be associated a couple of matrices (A, B) modulo a group action (see remark 3.1.2 for details), where A can represent a Steiner bundle.
In this chapter we will introduce different kind of instanton bundles, trying to study the relations between them in the moduli space.

3.1 Families of Instanton bundles

Definition 3.1.1. An instanton bundle E over $\mathbb{P}^{2n+1} = \mathbb{P}(V)$ with $c_2 = k$ is a stable bundle of rank $2n$ which appears as a cohomology bundle of a monad

$$L \otimes \mathcal{O}(-1) \xrightarrow{B^t} M \otimes \mathcal{O} \xrightarrow{A} N \otimes \mathcal{O}(1) \tag{3.1}$$

where L, N and M are complex vector spaces respectively of dimension k, k and $2n + 2k$. From now on we will denote E also with the couple (A, B).

If E is symplectic, the monad (3.1) can be written in the following form:

$$N^* \otimes \mathcal{O}(-1) \xrightarrow{JA^t} M \otimes \mathcal{O} \xrightarrow{A} N \otimes \mathcal{O}(1) \tag{3.2}$$

where $J : M^* \longrightarrow M$ is the $(2n + 2k) \times (2n + 2k)$ skewsymmetric matrix of the form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

If we set $S^* = \ker A$ we get these two exact sequences

$$0 \longrightarrow N^* \otimes \mathcal{O}(-1) \xrightarrow{JA^t} S^* \longrightarrow E \longrightarrow 0 \tag{3.3}$$

$$0 \longrightarrow S^* \longrightarrow M \otimes \mathcal{O} \xrightarrow{A} N \otimes \mathcal{O}(1) \longrightarrow 0 \tag{3.4}$$

Remark 3.1.2. The Lie group $GL(k) \times GL(k) \times GL(2n+2k)$ acts on the pairs of matrices (A, B) which define an instanton bundle in the following way:

$$\Big((\alpha, \beta, \gamma), (A, B)\Big) \mapsto (\beta A \gamma^{-1}, \alpha B \gamma^t)$$

Hence two instanton bundles are isomorphic if and only if they lie in the same orbit of this action.
A similar condition holds for symplectic instanton bundle, in this case the Lie group is $GL(k) \times Symp(2n+2k)$ and the action is given by:

$$\Big((\alpha, \gamma), (A, J)\Big) \mapsto (\alpha A \gamma, J)$$

The next propositions contain some results which hold for every symplectic instanton bundle, the analogous results for generical instanton bundles can be found in [Ott96]:

Proposition 3.1.3. Let E be a symplectic instanton bundle with $c_2 = k$. Then
i) $h^1(S^2E) - h^2(S^2E) = -\frac{k^2}{2}\binom{2n-1}{2} + k(\frac{10n^2+5n+1}{2}) - 2n^2 - n =: p(k, n)$.
ii) $h^1(S^2E(-1)) - h^2(S^2E(-1)) = -k^2(n-1) + k(3n+1) =: q(k, n)$.

Proof. $h^0(S^2E) = h^0(S^2E(-1)) = 0$ because E is stable. Furthermore using (3.3) and (3.4) we get

$$0 \longrightarrow S^2(S^*) \longrightarrow S^2M \otimes \mathcal{O} \longrightarrow M \otimes N \otimes \mathcal{O}(1) \longrightarrow \Lambda^2 N \otimes \mathcal{O}(2) \longrightarrow 0 \tag{3.5}$$

$$0 \longrightarrow \Lambda^2 N^* \otimes \mathcal{O}(-2) \longrightarrow N^* \otimes S^*(-1) \longrightarrow S^2S^* \longrightarrow S^2E \longrightarrow 0 \tag{3.6}$$

The result now follows from a direct computation.

□

Proposition 3.1.4. Let E be a symplectic instanton bundle corresponding to $A \in Hom(M, N \otimes V)$, and $S^* = \ker A$. Let

$$\begin{aligned} \phi: \quad Hom(N^*, M^*) \quad &\rightarrow \quad \wedge^2 N \otimes V \\ B \quad &\mapsto \quad AJB + B^tJA^t \end{aligned}$$

The following are true

i) $H^1(S^2S^*(-1)) = \ker\ \phi$

ii) $H^2(S^2S^*(-1)) = coker\ \phi$

iii) $H^1(S^2E(-1)) = H^1(S^2S^*(-1))$

iv) $H^2(S^2E(-1)) = H^2(S^2S^*(-1))$

Proof. i) and ii) result from the exact sequence (3.5) twisted by $\mathcal{O}(-1)$, while iii) and iv) come from (3.6) twisted by $\mathcal{O}(-1)$.

□

Proposition 3.1.5. Let E be an instanton bundle corresponding to $A \in Hom(M, N \otimes V)$. Let

$$\begin{array}{rccc} \Phi: & Hom(N^*, M^* \otimes V) & \to & \wedge^2 N \otimes S^2 V \\ & B & \mapsto & AJB + B^t J A^t \end{array}$$

The following are true

i) $H^1(S^2S^*) \simeq \ker\ \Phi/\mathbf{Sp}(M)$

ii) $H^2(S^2S^*) \simeq coker\ \Phi$

iii) $H^1(S^2E) = H^1(S^2S^*)/End\ N^*$

iv) $H^2(S^2E) = H^2(S^2S^*)$

Proof. In order to prove i) first notice that $S^2M \cong \mathbf{Sp}(M)$ through the map

$$P \mapsto PJ.$$

Moreover the space $\mathbf{Sp}(M)$ can be imbedded in $\ker\ \Phi$ in this way:

$$\gamma \mapsto \gamma A^t.$$

Then i) and ii) come from the exact sequence (3.5).
The space $End\ N^*$ can be seen as a subspace of $\ker\ \Phi$ by the imbedding:

$$\alpha \mapsto A^t \alpha.$$

Then iii) and iv) follow from the exact sequence (3.6).

□

Remark 3.1.6. Summing up the proposition 3.1.5, we can see the space $Q = End\ N^* \oplus \mathbf{Sp}(M)$ as a subspace of ker Φ:

$$(\alpha, \gamma) \mapsto A^t\alpha + \gamma A^t,$$

hence we have

$$H^1(S^2E) = \text{ker } \Phi/Q.$$

We are now ready to define the two families studied mostly in this work: symplectic 't Hooft instantons and symplectic Rao-Skiti instantons.For any projective space of odd dimension these concepts have been introduced in [Ott96] and in [CHMRS14] respectevely. Furthermore we will introduce a generalization of the symplectic 't Hooft giving a bigger family of instanton (still called 't Hooft) not necessarily symplectic.
Let's introduce the symplectic 't Hooft bundles on $\mathbb{P}^3$ first:
the idea is to build a bundle E such that $E(1)$ has a section vanishing on $k+1$ disjoint lines.
This will imply that for a 't Hooft instanton $h^0(E(1)) > 0$.
Let's take $k+1$ lines $\{z = \eta = 0\}$ and $\{\xi_i = \omega_i = 0\}$ for $i = 1, \ldots, k$, where z, η, ξ_i and ω_i are generic linear forms (hence they define disjoint lines). Then we can build the following matrix:

$$A = \begin{pmatrix} \xi_1 & & & a_1 z & \omega_1 & & & a_1\eta \\ & \ddots & & \vdots & & \ddots & & \vdots \\ & & \xi_k & a_k z & & & \omega_k & a_k\eta \end{pmatrix}$$

where $a_i \in \mathbb{C}$ are generic.
This is an instanton bundle called of 't Hooft type.
If we introduce the following matrix

$$C = \begin{pmatrix} -\omega_1 & & & & \xi_1 & & & \\ & \ddots & & & & \ddots & & \\ & & -\omega_k & & & & \xi_k & \\ & & & -\eta & & & & z \end{pmatrix}$$

then $AC^t = 0$, hence it defines $k+1$ sections of the kernel bundle (twisted by $\mathcal{O}(1)$), that is one section of $E(1)$. And this section vanishes where the rank of C is not maximum.
Conversely, by the Serre correspondence, an instanton bundle E over $\mathbb{P}^3$ that has a section of $E(1)$ vanishing on $k+1$ disjoint lines comes from a matrix of the above form.

Furthermore it could happen that these $k+1$ lines are contained in a quadric: in this case $E(1)$ has one more section; indeed an independent section can be found in the following way:
say q the quadric that contains the $k+1$ lines, there exist linear form s_i, t_i with $i = 1, \ldots, k+1$ such that

$$q = s_i \xi_i + t_i \omega_i = s_{k+1} z + t_{k+1} \eta,$$

then by adding the following row to C

$$(a_1 s_1, \ldots, a_k s_k, -s_{k+1}, a_1 t_1, \ldots, a_k t_k, -t_{k+1})$$

we get the new section.
So if the $k+1$ lines are all contained in a quadric we get $h^0(E(1)) = 2$, otherwise we have $h^0(E(1)) = 1$.
Hence both the case $h^0(E(1)) = 1$ and $h^0(E(1)) = 2$ are covered by 't Hooft instanton bundles.
From the description given by the lines it is clear that 't Hooft bundles corresponding to $k+1$ lines lying on a smooth quadric are $SL(2)$ invariant. About the matter there is a classification of instanton bundles on $\mathbb{P}^3$ which are $SL(2)$ invariant in ([Fae07]).

Definition 3.1.7. A symplectic instanton bundle is called of 't Hooft type if the map $A \in Hom(M, N \otimes V)$ is given in a convenient system of coordinates by the matrix

$$[D(\xi_i)|a \cdot D(z_i)|D(\omega_i)|a \cdot D(\eta_i)] \tag{3.7}$$

where $a = \left(a_i^j\right)$ is a $k \times n$ matrix with complex entries, $D(\xi_i)$ and $D(\omega_i)$ are diagonal $k \times k$ matrices with diagonal entries degree 1 generic forms $\xi_1, \ldots, \xi_k, \omega_1, \ldots, \omega_k$, $D(z_j)$ and $D(\eta_j)$ are diagonal $n \times n$ matrices with diagonal entries degree 1 generic forms $z_1, \ldots, z_n, \eta_1, \ldots, \eta_n$.

Remark 3.1.8. Let be given a symplectic instanton bundle of 't Hooft type, there exist degree 1 forms z_0, η_0 such that $(z_0, \ldots z_n, \eta_0, \ldots, \eta_n)$ is a system of coordinates which will be denoted for the rest of the chapter by $(x_0, \ldots x_n, y_0, \ldots, y_n)$.

Notation 3.1.9. When $A \in Hom(M, N \otimes V)$ defines an instanton bundle we set $A = \sum_{i=0}^n A_i x_i + \sum_{i=0}^n \tilde{A}_i y_i$ where $A_i, \tilde{A}_i \in Hom(M, N)$.

Remark 3.1.10. An equivalent way to describe a symplectic 't Hooft instanton bundle is giving the matrix A in the following way:

$$A = [a \cdot D(\xi_i)|a \cdot D(\mu_i)]$$

where a is now a $k \times (k+n)$ matrix with complex entries and $D(\xi_i), D(\mu_i)$ are $(k+n) \times (k+n)$ diagonal matrices with linear entries.

Proposition 3.1.11. Let E be a generic symplectic 't Hooft instanton on $\mathbb{P}^{2n+1}$ with $c_2 = k \geq 3$.
Then $H^0(E(1)) = n$.

Proof. see Theorem 3.7 of [Ott96]. $\square$

Proposition 3.1.12. Symplectic 't Hooft bundles depend on $5kn+4n^2$ parameters for $k \geq 3$.

Proof. see Theorem 3.8 of [Ott96]. $\square$

Let now introduce the Rao-Skiti instanton bundles:

Definition 3.1.13. A symplectic instanton bundle is called Rao-Skiti if the map $A \in Hom(M, N \otimes V)$ is given in a convenient system of coordinates by the matrix

$$A = [F|H] \tag{3.8}$$

where F is a $k \times (n+k)$ matrix of the form

$$F = \begin{pmatrix} x_0 & x_1 & \cdots & x_n & 0 & \cdots & \cdots & 0 \\ 0 & x_0 & x_1 & \cdots & x_n & 0 & \cdots & 0 \\ 0 & 0 & x_0 & x_1 & \cdots & x_n & \ddots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_0 & x_1 & \cdots & x_n \end{pmatrix}$$

and H is a persymmetric $k \times (n+k)$ matrix of linear forms $h_{ij} \in H^0(\mathcal{O}(1))$, that is, a matrix such that $h_{ij} = h_{st}$ if $i+j = s+t$.

The analogous result of proposition 3.1.11 and 3.1.12 are the following:

Proposition 3.1.14. Let E be a generic symplectic Rao-Skiti instanton on $\mathbb{P}^{2n+1}$ with $c_2 = k \geq 3$.
Then $H^0(E(1)) = 0$.

Proof. see proposition 3.13 of [CHMRS14]. $\square$

Proposition 3.1.15. Rao-Skiti bundles depend on $(4n+2)k + 4n^2 + 2n - 4$.

Proof. see remark 3.14 of [CHMRS14].

□

As announced let give a generalization of the symplectic 't Hooft bundles:

Definition 3.1.16. An instanton bundle is called of 't Hooft type if the map $A \in Hom(M, N \otimes V)$ is given in a convenient system of coordinates by the matrix

$$A = [D(\xi_i)|a \cdot D(z_i)|D(\omega_i)|a \cdot D(\eta_i)] \tag{3.9}$$

where $a = \left(a_i^j\right)$ is a $k \times n$ matrix with complex entries, $D(\xi_i)$ and $D(\omega_i)$ are diagonal $k \times k$ matrices with diagonal entries degree 1 generic forms $\xi_1, \ldots, \xi_k, \omega_1, \ldots, \omega_k$, $D(z_j)$ and $D(\eta_j)$ are diagonal $n \times n$ matrices with diagonal entries degree 1 generic forms $z_1, \ldots, z_n, \eta_1, \ldots, \eta_n$, and $B \in Hom(M^*, L^* \otimes V)$ is given by

$$B = [-D(\omega_i)| - b \cdot D(\eta_i)|D(\xi_i)|b \cdot D(z_i)] \tag{3.10}$$

where $b = (b_i^j)$ is a $k \times n$ matrix with complex entries.

Remark 3.1.17. An equivalent way to describe a 't Hooft instanton bundle is giving the couple (A, B) in the following way

$$A = [a \cdot D(\xi_i)|a \cdot D(\mu_i)] \qquad B = [-b \cdot D(\mu_i)|b \cdot D(\xi_i)]$$

where a, b are now $k \times (k + n)$ matrices with complex entries and $D(\xi_i), D(\mu_i)$ are $(k + n) \times (k + n)$ diagonal matrices with linear entries.

Obviously the behaviour of $H^0(E(1))$ for this class of instanton bundles is the same as for the class of symplectic 't Hooft bundles. Let now prove the analogous of proposition 3.1.12:

Proposition 3.1.18. 't Hooft bundles depend on $(6n-1)k+4n^2-n+1$ parameters for $k \geq 5$.

Proof. The idea of the proof is similar to the one in Theorem 3.8 in [Ott96]. We use the description of a 't Hooft bundle shown in remark 3.1.17: The matrices a, b can be reduced to matrices with the first block $k \times k$ equal to the identity matrix, moreover we can arrange the first row of the second block to be $(1, \ldots, 1)$ hence each one depends on $(k - 1)n$ parameters. While every ξ_i, μ_j depends on $2n + 2$ parameters. So the couple (A, B) of matrices which describes a 't Hooft bundle depends on: $2(k - 1)n + (2n + 2)(2n + 2k) = k(6n + 4) + 2n(2n + 1)$ parameters. Now the group $G = GL(k) \times GL(k) \times GL(2n + 2k)$ acts over (A, B) by

$$\Big((\alpha, \beta, \gamma), (A, B)\Big) \mapsto (\beta A\gamma^{-1}, \alpha B\gamma^t)$$

In order to evaluate the isotropy subgroups of this action it is sufficient to study the actions of the three groups separately: let's focus first on α (β is the same). α sends the couple (A, B) to the couple $(A, \alpha B)$ where

$$\alpha B = (-\alpha \cdot b \cdot D(\mu_i) | \alpha \cdot b \cdot D(\xi_i))$$

In order to preserve the structure of $\alpha \cdot b$, α must be diagonal and the first element α_{11} must be equal to 1, so there are $k-1$ free parameters.
Let's study now the action of $GL(2n+2k)$ over (A, B). Let's first divide γ into four square matrices of size $n+k$ each, say

$$\gamma = \begin{pmatrix} D^1 & D^2 \\ D^3 & D^4 \end{pmatrix}$$

where $D^i = (d^i_{j,k})$.
So (A, B) is sent in the couple $(A\gamma^{-1}, B\gamma^t)$. Focusing on the second part we get

$$B\gamma^t = \big(- b(D(\mu_i)D^{1^t} - D(\xi_i)D^{2^t}) | b(-D(\mu_i)D^{3^t} + D(\xi_i)D^{4^t}) \big)$$

This leads to the condition that $D(\mu_i)D^{1^t} - D(\xi_i)D^{2^t}$ and $-D(\mu_i)D^{3^t} + D(\xi_i)D^{4^t}$ are diagonal matrices. Hence if $i \neq j$ we get the following:

$$\begin{cases} \mu_i d^1_{ji} - \xi_i d^2_{ji} & = 0 \\ -\mu_i d^3_{ji} + \xi_i d^4_{ji} & = 0 \end{cases}$$

Thanks to the generality of the linear forms, this leads to $2n+2$ conditions that are satisfied if and only if $d^i_{jk} = 0$ for every i, j, k with $j \neq k$.
This proves that the matrices D^i are all diagonal.
Moreover from $B\gamma^t$ we get the linear forms associated to the new instanton bundle, that is:

$$\begin{cases} \tilde{\mu_i} & = & \mu_i d^1_{ii} - \xi_i d^2_{ii} \\ \tilde{\xi_i} & = & -\mu_i d^3_{ii} + \xi_i d^4_{ii} \end{cases} \qquad \forall i = 1 \dots n+k.$$

Studying $A\gamma^{-1}$, knowing that

$$\gamma^{-1} = \begin{pmatrix} (D^1D^4 - D^2D^3)^{-1}D^4 & -(D^1D^4 - D^2D^3)^{-1}D^2 \\ -(D^1D^4 - D^2D^3)^{-1}D^3 & (D^1D^4 - D^2D^3)^{-1}D^1 \end{pmatrix}$$

we get

$$\begin{cases} \tilde{\mu_i} & = & \frac{1}{d^1_{ii}d^4_{ii} - d^2_{ii}d^3_{ii}}(-\xi_i d^2_{ii} + \mu_i d^1_{ii}) \\ \tilde{\xi_i} & = & \frac{1}{d^1_{ii}d^4_{ii} - d^2_{ii}d^3_{ii}}(\xi_i d^4_{ii} - \mu_i d^3_{ii}) \end{cases}$$

In order to get a 't Hooft bundle it is sufficient to satisfy the $k+n-1$ conditions

$$d^1_{ii}d^4_{ii} - d^2_{ii}d^3_{ii} = \lambda$$

where $\lambda \in \mathbb{C}$.
Summing up, the dimension of the isotropy group of the whole action is $3n+3k+1+2(k-1)$. Subtracting this dimension from the number of parameters of the couple (A, B) we get the result.

□

3.2 Different symplectic structures

So far we have investigated on the different structures a symplectic instanton can have changing the matrix associated to it and fixing the matrix associated to the symplectic isomorphism. Now let's see how many skew-symmetric matrices J satisfy the equation $AJA^t = 0$ in the unknonwn J.

Notation 3.2.1. For the rest of the section $J = (j_{lt})$ will be a $(2n+2k) \times (2n+2k)$ skew-symmetric matrix often used in the following matrix form

$$J = \begin{pmatrix} J^1 & J^2 \\ -J^{2^t} & J^3 \end{pmatrix}$$

and I_t represents the $t \times t$ identity matrix.

Proposition 3.2.2. Let A be a $k \times (2n+2k)$ matrix representing a generic symplectic 't Hooft type instanton.
The dimension of the space of solutions of the equation $AJA^t = 0$ is $k+n$ and the solutions are the following

$$J^1 = J^3 = 0, \qquad J^2 = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \alpha_{n+k} \end{pmatrix}$$

with $\alpha_i \in \mathbb{C}$.

Proof. It is easy to see that these matrices are solutions.
In order to study separately the three blocks of J we need to prove the proposition for a non generic 't Hooft bundle, i.e. let's suppose that the linear forms ξ_i's and z_i's are only in the variables x_i's while the μ_i's and the η_i's are in the y_i's (following the same notations used in 3.1.7). Then if the result holds for this particular 't Hooft by semicontinuity we can conclude.
If we call X the first block of the matrix A and Y the second block we get the following equation

$$XJ^1X^t + XJ^2Y^t - YJ^{2^t}X^t + YJ^3Y^t = 0$$

So J^1 is multiplied by quadratic forms in the x_i's, J^3 in the y_i's and J^2 in the forms x_iy_j. Let's focus first on J^1 (the procedure for J^3 is the same). We have $(n+k)^2$ variables, for simplicity we are going to call these matrices $e_{i,j}$ for $i, j = 1, \dots, k+n$, and $e_{i,j}$ is the matrix that is everywhere zero except for the position (i, j) where is equal to 1. We can divide these matrices into three groups: $e_{i,j}$ with both indexes less or equal than k, with one greater than k and with both indexes grater than k. When we perform $Xe_{i,j}X^t$ for a matrix of the first group we get a $k \times k$ matrix everywhere zero a part from place (i, j), moreover the quadratic form in this place contains the monomial x_0^2. No other matrix shares this property, hence the k^2 matrices of the first group must be multiplied by zero in order to get the result.
Let's see now how the matrices of the second group behave: taking $e_{i,j+k}$ (with $i \leq k$ and $j \leq n$) we get a $k \times k$ matrix that has only the row i different from zero and all the forms are multiple by x_j. While if we take $e_{i+k,j}$ (with $i \leq n$ and $j \leq k$) $Xe_{i+k,j}X^t$ has only the $j-$th column different from zero and the forms are multiple by x_i.
let's take $e_{i,j+k}$ with i, j fixed. $e_{j+k,t}$ is the only matrix that shares in place (i, t) the monomial x_0x_j after multiplication. So we get one condition on each $e_{j+k,t}$ for every $t = 1, \dots, k$. But if we focus on $e_{\tilde{i},j+k}$, the conditions are again over the $e_{j+k,t}$ matrices, and these conditions are independent from the previous ones if $i \neq \tilde{i}$. So one more time the $2nk$ matrices of the second group must be multiplied by zero in order to get the result. We are left with the matrices of the form $e_{i+k,j+k}$ with $i, j \leq n$. $Xe_{i+k,j+k}X^t$ is a matrix generally nowhere zero and multiple of the monomial x_ix_j. Hence only the matrices linked to $e_{i+k,j+k}$ and $e_{j+k,i+k}$ share the same monomial, but thanks to the generality of a these two matrices are not multiple, therefore we can conclude that $J^1 = 0$.
To prove that J^2 is diagonal we proceed keeping the same notation relative to $e_{i,j}$ and dividing these matrices again into the same groups we created before, only now ignoring $e_{i,i}$ for every i because we already know that these are solutions.
When we perform the calculation of $Xe_{i,j}Y^t - Ye_{j,i}X^t$ (a skew-symmetric matrix) with $i, j \leq k$ we see that this matrix has only places (i, j) and (j, i) different from zero, and in these places all the possible degree two monomials appear. Moreover between all these matrices only $e_{i,j}$ and $e_{j,i}$ give a matrix that in place (i, j) has the monomial x_0y_0. So let's keep in mind that the coefficients of $e_{i,j}$ and $e_{j,i}$ must be multiple, however we can not conclude yet that they are zero. Let's focus our attention on the matrices of the form $e_{i,j+k}$ and $e_{i+k,j}$ now: for $e_{i,j+k}$ (with $i \leq k$ and $j \leq n$) the resultant matrix has only the $i-$th row and the $i-$th column nonzero, moreover the monomials that appear in every place of this matrix are multiple of y_j, similarly $e_{j+k,i}$ creates a matrix with the same property except the fact that the monomials are multiple of x_j.
Finally $e_{i+k,j+k}$ (with $i \neq j$) creates a scalar matrix multiplied by x_iy_j.

So fixing $\tilde{i} \neq \tilde{j}$, the matrices $e_{i,j}$ such that $Xe_{i,j}Y^t - Ye_{j,i}X^t$ is different from zero in the position $(\tilde{i}, \tilde{j})$ are exactly n^2+n+2, indeed there are 2 from the first group, $2n$ from the second and $n^2 - n$ from the third one, moreover these last $n^2 - n$ matrices are the same for every position.
We have already said that there must be a relation between the 2 matrices of the first group, the same happens among the matrices of the second group: indeed these $2n$ matrices can be coupled by being the only two matrices in this group that after multiplication in place $(\tilde{i}, \tilde{j})$ they present a monomial of the form x_0y_t or x_ty_0 where $t = 1, \ldots, n$.
So these matches solve the problem for the quadratic monomials where at least one between x_0 and y_0 occurs. We have still n^2+1 free variables and n^2 equations (the other degree two monomials). In order to solve this system we are forced to use the free variables obtained by the matrices of the third group. But if we change position (if $k > 2$ there are more positions) the equations change and we have not enough free variables to use. This completes the proof.

□

Proposition 3.2.3. Let A be the $k \times (2n+2k)$ matrix of the following form

$$A = \begin{pmatrix} x_0 & x_1 & \cdots & x_n & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & y_0 & \cdots & y_{n-1} & y_n \\ 0 & x_0 & \cdots & \cdots & x_n & 0 & \cdots & 0 & 0 & \cdots & 0 & y_0 & \cdots & \cdots & y_n & 0 \\ \vdots & & \ddots & & & & & & & & & & & \iddots & & \vdots \\ 0 & \cdots & 0 & x_0 & x_1 & \cdots & \cdots & x_n & y_0 & \cdots & \cdots & y_{n-1} & y_n & 0 & \cdots & 0 \end{pmatrix}$$

(Special symplectic instanton).
The dimension of the space of solutions of the equation $AJA^t = 0$ is $2n+2k-1$ and the solutions are the following

$$J^1 = J^3 = 0, \qquad J^2 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n+k} \\ \alpha_{n+k+1} & \alpha_1 & \alpha_2 & \ddots & \alpha_{n+k-1} \\ \alpha_{n+k+2} & \alpha_{n+k+1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_1 & \alpha_2 \\ \alpha_{2n+2k-1} & \alpha_{2n+2k-2} & \cdots & \alpha_{n+k+1} & \alpha_1 \end{pmatrix}.$$

with $\alpha_i \in \mathbb{C}$.

Proof. First of all by calculation it is easy to see that for every α_i the matrices described above are solutions of $AJA^t = 0$.
In order to prove that there are no others we can treat separately J^1, J^2 and J^3:

indeed if we set $A = (X|Y)$, where X and Y are $k \times (n+k)$ matrices dealing respectively with the x_i's and y_i's, we get that

$$AJA^t = XJ^1X^t - YJ^{2^t}X^t + XJ^2Y^t + YJ^3Y^t.$$

Hence J^1 and J^3 are multiplied by quadratic forms in the x_i's and in the y_i's respectively, while J^2 is multiplied by quadratic forms in both variables.
Let's focus again first on J^1:
we need to solve the system $XJ^1X^t = 0$ in the unknown J^1. The matrix associated to this linear system is $(\binom{n+1}{2}k^2) \times (n+k)^2$ so fixing n and for k sufficiently large it is enough to show that the matrix has maximal rank, but it is immediate to see that this matrix is already reduced in echelon form and all the lines are nonzero, hence it has maximal rank.
This again implies that also $J^3 = 0$. we are left to see which are the solutions of the system $XJ^2Y^t - YJ^{2^t}X^t = 0$.
if we build the matrix that represents the linear morphism $XJ^2Y^t - YJ^{2^t}X^t$ it is immediate to see that every row of this matrix has only two entries different from zero, more precisely every row give us a condition $j^2_{(i,j)} = j^2_{(k,t)}$ if $i = k + l$ and $j = t + l$ for suitable $l \in \mathbb{Z}$.

□

Proposition 3.2.4. Let A be a $k \times (2n + 2k)$ matrix representing a generic Rao-Skiti instanton.
The only solution of the equation $AJA^t = 0$ is when

$$J = \begin{pmatrix} 0 & \alpha I_{n+k} \\ -\alpha I_{n+k} & 0 \end{pmatrix}$$

with $\alpha \in \mathbb{C}$.

Proof. This matrix for every α is a solution.
Here we apply the idea used in 3.2.2, we need to suppose that the linear forms in the persymmetric matrix P are only in the variables y_i's. In this way we can treat the blocks of J separately. $J^1 = 0$ because we can apply the same procedure used in 3.2.3; moreover also $J^3 = 0$ indeed the second block (the one named Y) of the special symplectic instanton is a specialization of a persymmetric matrix, hence if there are no solutions, except the trivial one, for this kind of matrix there is none for the general case. For the same reason J^2 must be a linear combination of the matrices in the space of solutions described in 3.2.3, i.e. band matrices $(n + k) \times (n + k)$ that have only one band different from zero and all the elements in that band are equal. Indeed J^2 must be a solution of the equation $XJ^2P^t - PJ^{2^t}X^t = 0$, which is a generalization of the equation stated for J^2 in 3.2.3. The idea of the proof is to show that the only possibility for J^2 is being a

multiple of the identity.
Let's call

$$S_1 = \begin{pmatrix} 0 & \dots & 0 & 0 & \alpha_1 \\ \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & \dots & 0 & \alpha_2 & 0 \\ \vdots & \ddots & 0 & 0 & \alpha_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\vdots$$

$$S_{2n+2k-2} = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_{2n+2k-2} & 0 & 0 & 0 & 0 \\ 0 & \alpha_{2n+2k-2} & 0 & 0 & 0 \end{pmatrix}, \; S_{2n+2k-1} = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_{2n+2k-1} & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can divide these matrices into two groups: the upper triangular and the lower triangular (we already know that S_{n+k} belongs to the space of solution hence we can get rid of it). Let's focus first on the upper triangular group. Evaluating the matrix $T_i = XS_iP^t - PS_i^tX^t$ for $i = 1, \dots, n+k-1$ we can classify it in the following way: T_1 is the only matrix among the T_i's that has in place $(1, 2)$ quadratic forms in x_0y_t for $t = 0, \dots, n$. Similarly T_2 is the only one that has quadratic forms in the same monomials in place $(2, 3)$. More generally T_i, for $i = 1, \dots, k-1$, is the only matrix that has these forms in place $(i, i+1)$.
For the other n matrices it is sufficient to focus our attention on place $(k-1, k)$; indeed T_{k+i} is the only matrix which presents in that place the monomials x_iy_t.
The lower triangular matrices give the following matrices: $U_i = XS_{i+n+k}P^t - PS_{i+n+k}^tX^t$ for $i = 1, \dots, n+k-1$. These can be classified exactly in the same way, more precisely U_i and T_i are the only matrices that share the property described above for T_i. Hence in order to get a linear combination of the T_i's and the U_j's equal to zero it should happen that, for i fixed, T_i and U_i must satisfy $n+1$ equations if we focus on the entry of the matrix where they share their property (the number of monomials of degree two that they have in common), but this, thanks to the generality of the persymmetric matrix, gives only the trivial solution if $n > 1$.

□

Remark 3.2.5. The spaces of solutions studied in the previous propositions could be used to understand the fibre of the map that associates to a couple (A, B) representing an instanton bundle the matrix A that represents a Steiner bundle.

On a generic Rao-Skiti the map is one to one, while the fibre on the special symplectic instanton is more complex, the aim for the rest of the section is to study it.
The first thing we can see is that inside this fibre there is also a 't Hooft bundle associated to a different symplectic isomorphism.

Proposition 3.2.6. Let A be the matrix representing the special symplectic instanton bundle, moreover let

$$\tilde{I} = \left(\begin{array}{c|c} I_{n+k} & 0_{n+k} \\ \hline 0_{n+k} & K \end{array}\right), \quad \tilde{V} = \left(\begin{array}{c|c} V & 0_{n+k} \\ \hline 0_{n+k} & V \end{array}\right)$$

where

$$K = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & 0 & \iddots & 0 \\ 0 & \iddots & 0 & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

is a $(n+k) \times (n+k)$ matrix and

$$V = \begin{pmatrix} {t_1}^0 & \cdots & {t_{n+k}}^0 \\ \vdots & & \vdots \\ {t_1}^{n+k-1} & \cdots & {t_{n+k}}^{n+k-1} \end{pmatrix}$$

is a $(n+k) \times (n+k)$ Vandermonde matrix with $t_i \in \mathbb{C}$.
Then if we set $\tilde{A} = A\tilde{I}\tilde{V}$ we get that $\tilde{A}J\tilde{A}^t = 0$, furthermore $\tilde{A}$ represents a symplectic 't Hooft instanton bundle.

Proof. In order to see that $\tilde{A}J\tilde{A}^t = 0$ it is sufficient to observe that the matrix $\tilde{I}\tilde{V}J\tilde{V}^t\tilde{I}^t$ is of the form shown in proposition 3.2.3. Moreover, exploiting the calculation, we can see that $\tilde{A}$ is a symplectic 't Hooft bundle with linear forms:

$$\xi_i = x_0 + {t_i}^1 x_1 + \cdots + {t_i}^n x_n, \quad \mu_i = y_n + {t_i}^1 y_{n-1} + \cdots + {t_i}^n y_0.$$

□

The following lemma will be useful to understand deeper the fibre on the special symplectic bundle:

Lemma 3.2.7. Every skew symmetric matrix $\tilde{J}$ belonging to the space of solutions of proposition 3.2.3 (keeping the same notations) can be expressed in the following form $\tilde{J} = QJQ^t$ where

$$Q = \left(\begin{array}{c|c} I_{n+k} & 0_{n+k} \\ \hline 0_{n+k} & {J_2}^t \end{array}\right)$$

and J is the classical skew symmetric matrix.

Proof. it is enough to observe that

$$QJQ^t = \left(\begin{array}{c|c} 0_{n+k} & J^2 \\ \hline -J^{2^t} & 0_{n+k} \end{array}\right)$$

$\square$

Remark 3.2.8. One important thing to notice is that moving inside the fibre of the special symplectic bundle doesn't imply that we are keeping the same isomorphism class inside the moduli space, i.e. the matrices of the form of Q described in lemma 3.2.7 are not always symplectic: indeed if we evaulate the Kuranishi map with [GS] on the two instanton bundles (which lie in the same fibre) described in proposition 3.2.6 we see that it is different, more precisely both the germs are spanned by the same number of quadrics but the generic quadrics inside these spaces have different rank. Nevertheless what we can say is that moving along that fibre allows us to move inside the symplectic moduli space in a path connected way. Hence we can join the special symplectic instanton (which of course lies in the Rao-Skiti component) to an element which lies on the symplectic 't Hooft component concluding that these two components are path connected.

Chapter 4

Wishful thinking

This final chapter is devoted to some conjectures and computational results. From 3.1 to 3.3 it is shown a possible path to prove Ottaviani's conjecture in [Ott96] (Theorem 4.4): this conjecture describes the space $H^1(S^2(E))$ when E is a generic symplectic 't Hooft bundle.
The basic idea is to apply the same method used in Chapter 1 for Steiner bundles, but in this case the main obstacle is that the inductive step is two and not one: this implies that we must deal with a sheaf which is not a bundle.
In 3.4 we study the same space whenever E is a Rao-Skiti instanton bundle.

As explained in Remark 1.5 of [Ott96] we can identify the tangent space at $MIS_{n,k}$ (the moduli space of symplectic instanton bundles of charge k in $\mathbb{P}^{2n+1}$) in E with $H^1(S^2(E))$, so solving these conjectures would give an important comprehension of the tangent space at the moduli space of symplectic instanton bundles.

4.1 Tangent space at a symplectic 't Hooft bundle

The following theorem is divided into two parts, the first one will be proved later in the work while the second part remains a conjecture:

Theorem 4.1.1. Let E be a symplectic instanton bundle of 't Hooft type over $\mathbb{P}^{2n+1}$ and let $k \geq 2$. Then, with the notations used in proposition 3.1.3,

i) $h^1(S^2E(-1)) = \max\{q(k,n), 2(k+n)\}$

ii) $h^1(S^2E) = \max\{p(k,n), 5kn + 4n^2\}$

Remark 4.1.2. Let $k \geq 2$. The following hold

$$q(k,n) \leq 2(k+n) \iff \frac{2n}{n-1} \leq k$$

$$p(k,n) \leq 5kn + 4n^2 \Longleftrightarrow \begin{cases} k \geq 9 & for \quad n = 2 \\ k \geq 6 & for \quad n = 3 \\ k \geq 5 & for \quad n = 4,5 \\ k \geq 4 & for \quad n \geq 6 \end{cases}$$

Remark 4.1.3 (Variables division)**.** In order to use the results obtained in Chapter 1 for our conjecture we need to deal with a special form of A, a matrix defining a symplectic 't Hooft bundle. More precisely:

$$A = [D(\xi_i)|a \cdot D(z_i)|D(\omega_i)|a \cdot D(\eta_i)]$$

where now the forms ξ_i's depend only on the $x_0, \ldots, x_n$ variables while the ω_i's on the $x_{n+1}, \ldots, x_{2n+1}$. this means that the two blocks of A depend on different variables.
Computational results confirm that if k is small there is a loss of generality in using this particular form of A, however, if k is enough big, the $2k$ linear forms regain generality. We will see later using [GS] that for $n = 2$ if $k \geq 11$ these bundles have the same behaviour as the generic one.
So from now on we will use this form of A when we describe a symplectic 't Hooft bundle.

Remark 4.1.4 (Proof)**.** The first part of the theorem 4.1.1 is an easy consequence of theorem 2.1.7: indeed we can split V into $V_1 = \langle x_0, \ldots, x_n \rangle$ and $V_2 = \langle x_{n+1}, \ldots, x_{2n+1} \rangle$ and see $A = \left[A_1 | A_2 \right]$ where $A_i \in Hom(W, I \otimes V_i)$ (returning to the notations used in (2.1)). Then we can consider these two exact sequences:

$$0 \longrightarrow S_1^* \to W \otimes \mathcal{O} \xrightarrow{A_1} I \otimes \mathcal{O}(1) \longrightarrow 0$$

$$0 \longrightarrow S_2^* \to W \otimes \mathcal{O} \xrightarrow{A_2} I \otimes \mathcal{O}(1) \longrightarrow 0$$

Hence S_1 and S_2 are two Steiner bundles of 't Hooft type over $\mathbb{P}^n$.
Furthermore we can split the matrix of the map in prop 3.1.4 into two parts:

$$\begin{array}{cccc} \phi : & Hom(N^*, M^*) & \to & \wedge^2 N \otimes V \\ & B = \begin{bmatrix} B_2 \\ B_1 \end{bmatrix} & \mapsto & AJB + B^t J A^t \end{array}$$

In order to find the kernel of the map above it is sufficient to solve these two independent conditions:

$$A_1 B_1 - B_1^t A_1^t = 0 \quad A_2 B_2 - B_2^t A_2^t = 0,$$

and this means that $B_i \in H^1(S^2 S_i^*(-1))$, more precisely the map

$$\begin{array}{rccc} \varphi: & H^1(S^2 S_1^*(-1)) \oplus H^1(S^2 S_2^*(-1)) & \to & H^1(S^2 S^*(-1)) \\ & B_1 \oplus B_2 & \mapsto & \begin{bmatrix} B_2 \\ B_1 \end{bmatrix} \end{array}$$

is a bijection.
Hence we can conclude.

4.2 Explicit description over $\mathbb{P}^3$

We know that the moduli space of instanton bundles with $c_2 = k$ over $\mathbb{P}^3$ is irreducible and smooth of dimension $8k-3$. Moreover every instanton is symplectic. Hence if we take E instanton bundle of 't Hooft type associated to A, with the same notations as in definition 3.1.7, we know that $h^1(S^2E) = 8k-3$. In the next proposition we make explicit a basis of $H^1(S^2E)$, but before that we need to introduce some matrices:
First set for every $i = 1 \dots k$:
C_i' with (j,i)-th entry given by

$$\begin{cases} \frac{-a_j \xi_{i0}}{\xi_{i0}\xi_{j1} - \xi_{i1}\xi_{j0}} & \text{for} \quad j = 1, \dots, k \quad j \neq i \\ 0 & \text{for} \quad j = i \\ 1 & \text{for} \quad j = k+1 \end{cases}$$

with (i,j)-th entry given by

$$\begin{cases} \frac{-a_j \xi_{j0}}{\xi_{i0}\xi_{j1} - \xi_{i1}\xi_{j0}} & \text{for} \quad j = 1, \dots, k \quad j \neq i \\ 0 & \text{for} \quad j = i \end{cases}$$

and
D_i' with (j,i)-th entry given by

$$\begin{cases} \frac{-a_j \omega_{i2}}{\omega_{i2}\omega_{j3} - \omega_{i3}\omega_{j2}} & \text{for} \quad j = 1, \dots, k \quad j \neq i \\ 0 & \text{for} \quad j = i \\ 1 & \text{for} \quad j = k+1 \end{cases}$$

with (i,j)-th entry given by

$$.\begin{cases} \frac{-a_j \omega_{j2}}{\omega_{i2}\omega_{j3} - \omega_{i3}\omega_{j2}} & \text{for} \quad j = 1, \dots, k \quad j \neq i \\ 0 & \text{for} \quad j = i \end{cases}$$

Then we can construct the following matrices (same notations used in remark 4.1.4):

$$C_i = \begin{bmatrix} 0 \\ C'_i \end{bmatrix}, \quad D_i = \begin{bmatrix} D'_i \\ 0 \end{bmatrix} \quad \forall i = 1 \dots k.$$

The next proposition is the equivalent of proposition 2.2.2:

Proposition 4.2.1. ★ NOTAZIONI Let assume that ξ_{i0}, ω_{i2} and a_i are different from zero for every i. Then

i) The $2k$-dimensional subspace of $N \otimes V$ generated by $N \otimes \langle x_1, x_3 \rangle$ surjects over the $(2k-2)$-dimensional vector space $H^1(E) = (N \otimes V)/M$, where in the quotient $(v^1 x_1 + v^2 x_3) \sim 0$ iff $v^1 \in \langle a^1 \rangle$ or $v^2 \in \langle a^1 \rangle$.

ii) The $8k$-dimensional subspace $K \subset M \otimes N^* \otimes V$ generated by $\langle C_1, \dots, C_k, D_1, \dots, D_k \rangle \otimes V$ surjects over the $(8k-3)$-dimensional vector space $H^1(S^2 E)$.

Proof. The proof of i) is the same as the first part of proposition 2.2.2: indeed the equation $v^1 x_1 + v^2 x_3 = Am$, where $v^i \in N$ and $m \in M$, gives the conditions

$$\begin{cases} m_i \xi_{i0} & = & 0 \\ m_{k+1+i} \omega_{i2} & = & 0 \\ m_i \xi_{i1} + m_{k+1} a_i^1 & = & v_i^1 \\ m_{k+1+i} \omega_{i3} + m_{2k+2} a_i^1 & = & v_i^2 \end{cases}$$

hence i).

We already know that $h^1(S^2 E) = 8k - 3$, moreover it is straightforward to see that K is contained in $\ker \Phi$. So, to prove ii), it is sufficient to see that K in $H^1(S^2 E)$ has dimension $8k - 3$.
The equation

$$\sum_{i=1}^{k} (c_i C_i + d_i D_i) = A^t \alpha + S J A^t$$

where $c_i, d_j \in V$, $\alpha \in End\ N^*$ and S is a $(2k+2) \times (2k+2)$ symmetric matrix, gives the following conditions (for every $j = 1 \dots k$):
if $i = j$ we get

$$\begin{cases} \xi_{jo}(\alpha_{jj} - s_{j,j+k+1}) & = & 0 \\ \xi_{j1}(\alpha_{jj} - s_{j,j+k+1}) - s_{j,2k+2} a_j^1 & = & 0 \\ \omega_{j2} s_{jj} & = & 0 \\ \omega_{j3} s_{jj} + s_{j,k+1} a_j^1 & = & 0 \end{cases}$$

obtaining that $s_{j,k+1} = s_{j,2k+2} = 0$.
Similarly, if $i = j + k + 1$, we get the conditions $s_{j+k+1,k+1} = s_{j+k+1,2k+2} = 0$.
If $i = k + 1$ or $i = 2k + 2$ we obtain respectively

$$\begin{cases} d_{0j} &= 0 \\ d_{1j} &= \sum_t^k a_t^1 \alpha_{tj} - s_{k+1,2k+2} a_j^1 \\ d_{2j} &= 0 \\ d_{3j} &= s_{k+1,k+1} a_j^1 \end{cases}$$

$$\begin{cases} c_{0j} &= 0 \\ c_{1j} &= -s_{2k+2,2k+2} a_j^1 \\ c_{2j} &= 0 \\ c_{3j} &= \sum_t^k a_t^1 \alpha_{tj} + s_{k+1,2k+2} a_j^1 \end{cases}$$

Hence, in order to have $\sum_{i=1}^k (c_i C_i + d_i D_i) \sim 0$, we get these $k+3$ independent conditions (the first two are exactly the same as the second part of proposition 2.2.2):

- $[c_1, \dots, c_k] \in \langle a^1 x_1 \rangle$, and $d_i = 0$;
- $[d_1, \dots, d_k] \in \langle a^1 x_3 \rangle$, and $c_i = 0$;
- $[c_{31} - d_{11}, \dots, c_{3k} - d_{1k}] \in \langle a^1 \rangle$, and all other variables equal to 0.

□

4.3 Restriction to a codimension 2 variety

Let $H \cong \mathbb{P}^{2n-1}$ be the variety given by the equations $x_n = 0$ and $x_{2n+1} = 0$. Let $A_{|\mathbb{P}^{2n-1}} \in Hom(M, N \otimes V')$ be given by substituting $x_n = 0$ and $x_{2n+1} = 0$ in A. In matrix form there are two columns of $A_{|\mathbb{P}^{2n-1}}$ which are zero (more precisely the $(n+k)-$th and the last one) and we set A' the matrix obtained by A deleting these two columns. We set $M = M' \oplus \mathbb{C}^2$, so that $A' \in Hom(M', N \otimes V')$ defines an instanton bundle E'.
If we call $\mathfrak{I}_H$ its sheaf of ideals, we get the following exact sequence

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\theta_1} \mathcal{O}(-1)^2 \xrightarrow{\theta_2} \mathfrak{I}_H \longrightarrow 0 \tag{4.1}$$

with

$$\theta_1 = \begin{pmatrix} x_{2n+1} \\ -x_n \end{pmatrix} \qquad \theta_2 = \begin{pmatrix} x_n & x_{2n+1} \end{pmatrix}.$$

Together with the defining sequence of H we get

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\theta_1} \mathcal{O}(-1)^2 \xrightarrow{\theta_2} \mathcal{O} \longrightarrow \mathcal{O}_H \longrightarrow 0$$

Tensoring the previous sequence with S^2E we thus obtain

$$0 \longrightarrow S^2E(-2) \longrightarrow S^2E(-1)^2 \longrightarrow S^2E \longrightarrow S^2E_{|H} \longrightarrow 0 \tag{4.2}$$

and this leads to the following two long exact sequences:

$$0 \longrightarrow H^1(S^2E(-1))^2 \longrightarrow H^1(\mathfrak{F}) \longrightarrow H^2(S^2E(-2)) \xrightarrow{\varphi} H^2(S^2E(-1))^2 \longrightarrow H^2(\mathfrak{F}) \longrightarrow 0 \tag{4.3}$$

$$\begin{aligned} 0 \longrightarrow H^0(S^2E_{|H}) \longrightarrow H^1(\mathfrak{F}) \longrightarrow H^1(S^2E) \longrightarrow H^1(S^2E_{|H}) \xrightarrow{\delta} \\ \longrightarrow H^2(\mathfrak{F}) \longrightarrow H^2(S^2E) \longrightarrow H^2(S^2E_{|H}) \longrightarrow 0 \end{aligned} \tag{4.4}$$

where $\mathfrak{F}$ is the sheaf which appears when we split (4.2) into two short exact sequences.
Let's focus on (4.3) first, especially on φ:

$$\begin{array}{rccc} \varphi: & \wedge^2 N & \to & H^2(S^2E(-1))^2 \\ & T & \mapsto & (Tx_{2n+1}, -Tx_n) \end{array}$$

In order to study the kernel of this map, first we study the case $n = 2$. If we set

$$\begin{array}{rccc} \varphi_1: & \wedge^2 N & \to & H^2(S^2E(-1)) \\ & T & \mapsto & -Tx_2 \end{array}$$

we have the following

Lemma 4.3.1. A basis for the solutions of the system

$$\varphi_1(T) = \phi(B)$$

(with ϕ and B as in remark 4.1.4 and $B_0 = 0$) in the unknown $B_1 = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$, where $C_0 = (c^0_{ij})$ is a $k \times k$ matrix, $C_1 = (c^1_{i,j})$ is a $2 \times k$ matrix and T is a generic skew-symmetric $k \times k$ matrix, is given by the $3k$ solutions (for every $1 \le t \le k$):

$$c^0_{t,t} = 1, \text{ all other unknowns equal to zero} \qquad \text{(k solutions)}$$

$$c^1_{2,t} = 1, \text{ all other unknowns equal to zero} \qquad \text{(k solutions)}$$

$$\left\{\begin{array}{lll} c^1_{1,t} = 1 & & \\ c^0_{t,i} = a_{i,1}\frac{\xi_{i,0}}{\xi_{t,1}\xi_{i,0}-\xi_{t,0}\xi_{i,1}} & c^0_{i,t} = c^0_{t,i}\frac{\xi_{t,0}}{\xi_{i,0}} & 0 < i < t \\ c^0_{j,t} = -a_{j,1}\frac{\xi_{t,0}}{\xi_{j,1}\xi_{t,0}-\xi_{j,0}\xi_{t,1}} & c^0_{t,j} = c^0_{j,t}\frac{\xi_{j,0}}{\xi_{t,0}} & t < j \leq k \\ \text{all other unknowns equal to zero} & & \end{array}\right. \qquad (k \text{ solutions})$$

Proof. It is straightforward to see that the expressions written above are solutions and are linearly independent. Moreover, for every (i,j) with $1 \leq i < j \leq k$, we have:

$$\left\{\begin{array}{ll} \xi_{i,1}c^0_{i,j} - \xi_{j,1}c^0_{j,i} + a_{i,1}c^1_{1,j} - a_{j,1}c^1_{1,i} & = 0 \\ \xi_{i,0}c^0_{i,j} - \xi_{j,0}c^0_{j,i} & = 0 \end{array}\right.$$

So there are $k^2 - k$ equations and $k^2 + 2k$ variables. In order to prove the lemma it is sufficient to show that the matrix associated to this system has maximal rank. Just to fix the ideas, we show the shape of the matrix when $k = 3$, in this case we have the following:

$$\begin{pmatrix} \xi_{1,1} & & -\xi_{2,1} & & & & -a_{2,1} & a_{1,1} & & \\ \xi_{1,0} & & -\xi_{2,0} & & & & & & & \\ & \xi_{1,1} & & & -\xi_{3,1} & & -a_{3,1} & & a_{1,1} & \\ & \xi_{1,0} & & & -\xi_{3,0} & & & & & \\ & & & \xi_{2,1} & & -\xi_{3,1} & & -a_{3,1} & & a_{2,2} \\ & & & \xi_{2,0} & & -\xi_{3,0} & & & & \end{pmatrix}$$

Focusing on the left part of the matrix (the one which refers to $c^0_{i,j}$) it is immediate to see that this submatrix has maximal rank.

$\square$

Lemma 4.3.2. Set Y the space of the solutions of the system in lemma 4.3.1. Then, if $k \geq 4$, the dimension of $\phi(Y)$ is $2k-2$. Moreover a basis is given by the following matrices multiplied by x_2 (for every $1 \leq t \leq k-1$):

$$\begin{pmatrix} & & & a^2_1 & & & \\ & 0 & & \vdots & & 0 & \\ & & & a^2_{t-1} & & & \\ -a^2_1 & \cdots & -a^2_{t-1} & 0 & a^2_{t+1} & \cdots & a^2_k \\ & & & -a^2_{t+1} & & & \\ & 0 & & \vdots & & 0 & \\ & & & -a^2_k & & & \end{pmatrix}, \tag{4.5}$$

$$
\begin{pmatrix}
 & & & a_1^1\xi(t,1) & & & \\
 & 0 & & \vdots & & 0 & \\
 & & & a_{t-1}^1\xi(t,t-1) & & & \\
-a_1^1\xi(t,1) & \cdots & -a_{t-1}^1\xi(t,t-1) & 0 & -a_{t+1}^1\xi(t,t+1) & \cdots & -a_k^1\xi(t,k) \\
 & & & a_{t+1}^1\xi(t,t+1) & & & \\
 & 0 & & \vdots & & 0 & \\
 & & & a_k^1\xi(t,k) & & &
\end{pmatrix}
\tag{4.6}
$$

where the only rows and columns which appear are the $t-$eth and $\xi(i,j) = \frac{\xi_{i,0}\xi_{j,2}}{\xi_{i,1}\xi_{j,0}-\xi_{i,0}\xi_{j,1}}$.

Proof. In order to get easier calculation we can observe that if we take B in Y then $AJB + B^tJA^t = (A_2JB + B^tJA_2^t)x_2$.
The first k solutions in lemma 4.3.1 are sent to 0 by ϕ.
The second k solutions are sent by ϕ to the matrices of the form (4.5) multiplied by x_2 with $1 \leq t \leq k$, but for $t = k$ the matrix we get is a linear combination of the others (just to fix the ideas, in fact everyone is a linear combination of the remaining $k-1$ matrices). We call these matrices $v_1, \dots, v_{k-1}$.
The same happens for the matrices of the form (4.6) multiplied by x_2 which are image of the last k solutions. These matrices will be denoted by $w_1, \dots, w_{k-1}$.
Now we are left to prove that these $2k-2$ matrices are linearly independent.
In order to do it we write $\alpha_1 v_1 + \ldots + \alpha_{k-1}v_{k-1} + \beta_1 w_1 + \ldots + \beta_{k-1}w_{k-1} = 0$. There are $2k-2$ variables and $\frac{k(k-1)}{2}$ conditions. If $k \geq 4$ there are at least $2k-2$ conditions, moreover, thanks to the generality of a and $\xi_{i,j}$, these are independent. So the only solution is the one with all the coefficients equal to 0, hence the matrices are independent.

□

Now we are able to study $\ker\varphi$:

Proposition 4.3.3. The dimension of $\ker\varphi$ is $k-1$. Moreover a basis is given by the matrices of the form (4.5).

Proof. We have already studied $\ker\varphi_1$. If we set

$$\begin{array}{rccc} \varphi_2: & \wedge^2 N & \to & H^2(S^2E(-1)) \\ & T & \mapsto & Tx_5 \end{array}$$

then a basis of $\ker\varphi_2$ is given by the same matrices as in lemma 4.3.2 replacing $\xi_{i,j}$ with $\omega_{i,2+j}$.
Obviously $\ker\varphi = \ker\varphi_1 \cap \ker\varphi_2$, hence the matrices of the form (4.5) belong to $\ker\varphi$. In order to prove that there are no other elements in this intersection we can apply the same argument used in the last part of the proof of lemma 4.3.2 applied to the $k-1$ matrices of the form (4.6) in $\ker\varphi_1$ and the $k-1$ matrices of the same form in $\ker\varphi_2$. Consequently we get that these matrices are linearly independent, hence we can conclude.

□

Now we return to the general case: first of all we still split φ in two:

$$\begin{array}{rccc} \varphi_1: & \wedge^2 N & \to & H^2(S^2E(-1)) \\ & T & \mapsto & -Tx_n \end{array} \qquad \begin{array}{rccc} \varphi_2: & \wedge^2 N & \to & H^2(S^2E(-1)) \\ & T & \mapsto & Tx_{2n+1} \end{array}$$

Let's give the equivalent of lemma 4.3.1 and lemma 4.3.2 when $n > 2$, because, in this case, the things change:

Lemma 4.3.4. Assume $n = 3$ and $k \geq 4$ or $n \geq 4$ and $k \geq 3$. Then a basis for the solutions of the system

$$\varphi_1(T) = \phi(B)$$

(with ϕ and B as in remark 4.1.4 and $B_0 = 0$) in the unknown $B_1 = \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$, where $C_0 = (c^0_{ij})$ is a $k \times k$ matrix, $C_1 = (c^1_{i,j})$ is a $n \times k$ matrix and T is a generic skew-symmetric $k \times k$ matrix, is given by the $2k + n - 1$ solutions (for every $1 \leq t \leq k$ and $1 \leq r \leq n-1$):

$$c^0_{t,t} = 1, \text{ all other unknowns equal to zero} \qquad \text{(k solutions)}$$

$$c^1_{n,t} = 1, \text{ all other unknowns equal to zero} \qquad \text{(k solutions)}$$

$$c^1_{r,j} = a_{j,r} \text{ all other unknowns equal to zero} \qquad (n-1 \text{ solutions})$$

Proof. The expressions written above are independent solutions. In order to see that there are no others we write the system which corresponds to $\varphi_1(T) = \phi(B)$:

$$\begin{cases} \xi_{i,r}c^0_{i,j} - \xi_{j,r}c^0_{j,i} + a_{i,r}c^1_{r,j} - a_{j,r}c^1_{r,i} & = 0 \\ \xi_{i,0}c^0_{i,j} - \xi_{j,0}c^0_{j,i} & = 0 \end{cases}$$

for every $1 \leq r \leq n-1$ and every $1 \leq i < j \leq k$.
Hence the matrix of this system is $\frac{nk(k-1)}{2} \times (k^2 + kn)$. we can split this matrix into two submatrices: the one of the c^0_{ij} which is $\frac{nk(k-1)}{2} \times k^2$, and the one of the c^1_{ij} which is $\frac{nk(k-1)}{2} \times kn$.
if we focus on the rows referred to an r fixed then the second matrix has the same properties of (2.8), then we can perform Gaussian elimination with the same method (taking into account the whole matrix). Swapping properly the rows we get the first $(n-1)(k-1)$ rows linearly independent, and the remaining rows have only zero entries under the c^1_{ij}. The remaining variables involved in the system are the c^0_{ij}, with $i \neq j$, so they are $k(k-1)$. Thanks to the generality of a and ξ_{ij} the remaining rows have rank $k(k-1)$ provided that there are enough equations left, more precisely provided that

$$\frac{nk(k-1)}{2} - (n-1)(k-1) \geq k^2 - k$$

and this inequality is true if $n = 3$ and $k \geq 4$ or $n \geq 4$ and $k \geq 3$.

□

Thanks to the next lemma we can observe that, despite the differences in $\ker \varphi_1$ between the case $n = 2$ and the others, $\ker \varphi$ shows the same structure:

Lemma 4.3.5. Set Y the space of the solutions of the system in lemma 4.3.4. Then, with the same assumptions stated in lemma 4.3.4, the dimension of $\phi(Y)$ is $k-1$. Moreover a basis is given by the following matrices multiplied by x_n (for every $1 \leq t \leq k-1$):

$$\begin{pmatrix} & & & a^n_1 & & & \\ & 0 & & \vdots & & 0 & \\ & & & a^n_{t-1} & & & \\ -a^n_1 & \cdots & -a^n_{t-1} & 0 & a^n_{t+1} & \cdots & a^n_k \\ & & & -a^n_{t+1} & & & \\ & 0 & & \vdots & & 0 & \\ & & & -a^n_k & & & \end{pmatrix}, \tag{4.7}$$

Proof. the proof is the same as in lemma 4.3.2 observing that the $n-1$ solutions found in lemma 4.3.4 are sent to 0 by ϕ.

□

Returning to (4.4), we focus our attention on

$$\delta : \quad H^1(S^2(E_{|H})) \quad \rightarrow \quad H^2(\mathfrak{F})$$

We know by Proposition 3.4 in [Ott96] that $E_{|\mathbb{P}^{2n+1-2}} \simeq E' \oplus \mathcal{O}^2$ where E' is a 't Hooft bundle over $\mathbb{P}^{2n+1-2}$. Hence we have the following natural decomposition:

$$H^1(S^2 E_{|H}) \simeq H^1(S^2 E') \oplus H^1(E')^2.$$

Moreover:
let

$$\begin{array}{cccc} \Phi_{|\mathbb{P}^{2n-1}}: & Hom(N^*, M^* \otimes V') & \longrightarrow & \wedge^2 N \otimes S^2 V' \\ & B & \mapsto & \big(A_{|\mathbb{P}^{2n-1}}\big) JB + B^t J \big(A_{|\mathbb{P}^{2n-1}}\big)^t \end{array}$$

and denote $K_{|\mathbb{P}^{2n-1}} := \ker\ \Phi_{|\mathbb{P}^{2n-1}}$. In particular the decomposition

$$Hom(N^*, M^* \otimes V') = Hom(N^*, M'^* \otimes V') \oplus (N \otimes V')^2$$

induces the following splitting

$$K_{|\mathbb{P}^{n-1}} = K' \oplus (N \otimes V')^2$$

.

In the same spirit of proposition 2.3.2 we have:

Proposition 4.3.6. Given $B' = \begin{bmatrix} B'_2 \\ B'_1 \end{bmatrix} \in Hom(N^*, M'^* \otimes V')$ represented by a $(2k + 2n - 2) \times k$ matrix with linear entries and $b_1, b_2 \in N \otimes V'$ represented by a $1 \times k$ matrix with linear entries let us construct $B \in Hom(N^*, W^* \otimes V)$ in the following way: $B := \begin{bmatrix} B'_2 \\ b_2 \\ B'_1 \\ b_1 \end{bmatrix}$.
Then the boundary map

$$H^1(S^2 E_{|\mathbb{P}^{2n-1}}) \simeq H^1(\mathbb{P}^{2n-1}, S^2 E') \oplus H^1(\mathbb{P}^{2n-1}, E')^2 \xrightarrow{\delta} H^2(\mathfrak{F})$$

fits into the following commutative diagram

$$\begin{array}{ccc} K' \oplus (N \otimes V')^2 & \xrightarrow{\delta'} & (\wedge^2 N \otimes V)^2 \\ \downarrow & & \downarrow \\ H^1(\mathbb{P}^{2n-1}, S^2 E') \oplus H^1(\mathbb{P}^{2n-1}, E')^2 & \xrightarrow{\delta} & H^2(\mathfrak{F}) \end{array}$$

$$\textit{where} \qquad \delta'(B', b_1, b_2) = \Big(A_n JB + B^t J A_n^t, A_{2n+1} JB + B^t J A_{2n+1}^t \Big)$$

Proof. The same diagram chasing applied in proposition 2.3.2

Let's split δ' of prop 4.3.6 into two (keeping the same notations):

$$\begin{array}{rccc} \delta'_1 : & K' \oplus (N \otimes V')^2 & \longrightarrow & \wedge^2 N \otimes V \\ & (B', b_1, b_2) & \mapsto & A_n JB + B^t JA^t_n \end{array}$$

$$\begin{array}{rccc} \delta'_2 : & K' \oplus (N \otimes V')^2 & \longrightarrow & \wedge^2 N \otimes V \\ & (B', b_1, b_2) & \mapsto & A_{2n+1} JB + B^t JA^t_{2n+1} \end{array}$$

From here to the end of the section the propositions staten are without proof, hence they have to be considered conjectures .

First we study $\mathbb{P}^5$:

Lemma 4.3.7. Let $n = 2$. Using the same notation as in proposition 4.2.1, a basis for the solution of the system

$$\delta'_1\left(\sum_{i=1}^{k} c_i C_i, v^1 x_1 + v^2 x_4, 0 \right) = AJS + S^t JA^t + Tx_5 \tag{4.8}$$

in the unknowns $c_i = c_{0i}x_0 + c_{1i}x_1 + c_{3i}x_3 + c_{4i}x_4$,
$v^1, v^2 \in \mathbb{C}^k$, $S \in N \otimes M^*$ and $T \in \wedge^2 N$
is given by the $8k + 7$ solutions:
(in every solution the unknowns omitted are supposed to be taken equal to zero)

$$\left\{ \begin{array}{rr} c_{0p} = \gamma_p \xi_{p0} & c_{1p} = \gamma_p \xi_{p1} \\ s_{p+2+k,q} = -\frac{(\gamma_q \xi_{q0} a^1_p - \gamma_p \xi_{p0} a^1_q)\xi_{q2}}{\xi_{p0}\xi_{q1} - \xi_{p1}\xi_{q0}} & s_{2+2k+1,p} = -\gamma_p \xi_{p2} \end{array} \right. \quad \forall (\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k$$

$(k \text{ solutions})$

$$c_{0p} = a^1_p \qquad (k + 1 - th)$$

$$v^1_p = a^2_p \qquad (k + 2 - th)$$

$$s_{2+k+p,q} = \frac{a^2_p a^2_q}{\xi_{p0}} \quad s_{2+2k+2,p} = \frac{a^2_p \xi_{p2}}{\xi_{p0}} \quad v^1_p = a^2_p \frac{\xi_{p1}}{\xi_{p0}} \qquad (k + 3 - th)$$

$$c_{1p} = a^1_p \qquad (k + 4 - th)$$

$$v^1_p = a^1_p \quad s_{2+2k+1,p} = -a^2_p \qquad (k + 5 - th)$$

$$s_{2+2k+1,p} = a_p^1 \qquad (k+6-th)$$

$$s_{2+2k+2,p} = a_p^2 \qquad (k+7-th)$$

$$s_{2+k+p,p} = \gamma_p \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \qquad (k \text{ solutions})$$

$$s_{p,p} = \gamma_p \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \qquad (k \text{ solutions})$$

$$s_{k+2,p} = \gamma_p \quad t_{pq} = a_q^2\gamma_p - a_p^2\gamma_q \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \qquad (k \text{ solutions})$$

$$\begin{cases} s_{k+1,p} = \gamma_p \\ s_{pq} = -\frac{\omega_{q3}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}}(a_q^1\gamma_p - a_p^1\gamma_q) \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \\ t_{pq} = -\frac{\omega_{p3}\omega_{q5}-\omega_{p5}\omega_{q3}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}}(a_q^1\gamma_p - a_p^1\gamma_q) \end{cases} \qquad (k \text{ solutions})$$

$$\begin{cases} v_p^2 = \gamma_p \\ s_{pq} = -\frac{\omega_{q3}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}}(a_q^2\gamma_p - a_p^2\gamma_q) \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \\ t_{pq} = -\frac{\omega_{p3}\omega_{q5}-\omega_{p5}\omega_{q3}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}}(a_q^2\gamma_p - a_p^2\gamma_q) \end{cases} \qquad (k \text{ solutions})$$

$$\begin{cases} c_{4p} = \gamma_p \\ s_{pq} = \frac{\xi_{p0}\xi_{q2}-\xi_{p2}\xi_{q0}}{\xi_{p0}\xi_{q1}-\xi_{p1}\xi_{q0}}\frac{\omega_{q3}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}}(a_q^1\gamma_p - a_p^1\gamma_q) \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \\ t_{pq} = \frac{\xi_{p0}\xi_{q2}-\xi_{p2}\xi_{q0}}{\xi_{p0}\xi_{q1}-\xi_{p1}\xi_{q0}}\frac{\omega_{p3}\omega_{q5}-\omega_{p5}\omega_{q3}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}}(a_q^1\gamma_p - a_p^1\gamma_q) \end{cases}$$

$$(k \text{ solutions})$$

$$\begin{cases} c_{3p} = \gamma_p \\ s_{pq} = -\frac{\xi_{p0}\xi_{q2}-\xi_{p2}\xi_{q0}}{\xi_{p0}\xi_{q1}-\xi_{p1}\xi_{q0}}\frac{\omega_{q4}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}}(a_q^1\gamma_p - a_p^1\gamma_q) \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \\ t_{pq} = -\frac{\xi_{p0}\xi_{q2}-\xi_{p2}\xi_{q0}}{\xi_{p0}\xi_{q1}-\xi_{p1}\xi_{q0}}\frac{\omega_{p4}\omega_{q5}-\omega_{p5}\omega_{q4}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}}(a_q^1\gamma_p - a_p^1\gamma_q) \end{cases}$$

$$(k \text{ solutions})$$

Proof. (*sketch*) Let's write more explicit the system (4.8):
for every $1 \leq i < j \leq k$ the coefficients of the x_i's of the left part of the equation are:

$$\begin{cases} -\frac{\xi_{i2}\xi_{j0}-\xi_{i0}\xi_{j2}}{\xi_{i0}\xi_{j1}-\xi_{i1}\xi_{j0}}(a_j^1 c_{0i} - a_i^1 c_{0j}) \\ -\frac{\xi_{i2}\xi_{j0}-\xi_{i0}\xi_{j2}}{\xi_{i0}\xi_{j1}-\xi_{i1}\xi_{j0}}(a_j^1 c_{1i} - a_i^1 c_{1j}) + a_i^2 v_j^1 - a_j^2 v_i^1 \\ 0 \\ -\frac{\xi_{i2}\xi_{j0}-\xi_{i0}\xi_{j2}}{\xi_{i0}\xi_{j1}-\xi_{i1}\xi_{j0}}(a_j^1 c_{3i} - a_i^1 c_{3j}) \\ -\frac{\xi_{i2}\xi_{j0}-\xi_{i0}\xi_{j2}}{\xi_{i0}\xi_{j1}-\xi_{i1}\xi_{j0}}(a_j^1 c_{4i} - a_i^1 c_{4j}) + a_i^2 v_j^2 - a_j^2 v_i^2 \\ 0 \end{cases}$$

While on the right part

$$\begin{cases} \xi_{i0}s_{2+k+i,j} - \xi_{j0}s_{2+k+j,i} \\ \xi_{i1}s_{2+k+i,j} - \xi_{j1}s_{2+k+j,i} + a_i^1 s_{2+2k+1,j} - a_j^1 s_{2+2k+1,i} \\ \xi_{i2}s_{2+k+i,j} - \xi_{j2}s_{2+k+j,i} + a_i^2 s_{2+2k+2,j} - a_j^2 s_{2+2k+2,i} \\ \omega_{j3}s_{ji} - \omega_{i3}s_{ij} \\ \omega_{j4}s_{ji} - \omega_{i4}s_{ij} + a_j^1 s_{k+1,i} - a_j^1 s_{k+1,j} \\ \omega_{j5}s_{ji} - \omega_{i5}s_{ij} + a_j^2 s_{k+2,i} - a_i^2 s_{k+2,j} + t_{ij} \end{cases}$$

We can split the system into two: the equations which come from the coefficients of x_0, x_1, x_2 and the others; indeed there is no unknown involved in both of these groups.
Hence we can notice that the first part is exactly the same as the one found in lemma 2.4.2, so the first $2k+7$ solutions written above are the contribution of this part.
We are left to prove that the matrix of the second part of the system has maximal rank.

Lemma 4.3.8. With the same assumptions of lemma 4.3.7, a basis for the solution of the system

$$\delta_2'\left(\sum_{i=1}^{k} d_i D_i, 0, r^1 x_1 + r^2 x_4\right) = AJU + U^t J A^t - T x_2$$

in the unknowns $d_i = d_{0i}x_0 + d_{1i}x_1 + d_{3i}x_3 + d_{4i}x_4$,
$r^1, r^2 \in \mathbb{C}^k$, $U \in N \otimes M^*$ and $T \in \wedge^2 N$
is given by the $8k+7$ solutions:

$$\begin{cases} d_{3p} = \gamma_p \omega_{p3} & d_{4p} = \gamma_p \omega_{p4} \\ u_{pq} = -\frac{(\gamma_q \omega_{q3} a_p^1 - \gamma_p \omega_{p3} a_q^1)\omega_{q5}}{\omega_{p3}\omega_{q4} - \omega_{p4}\omega_{q3}} & u_{k+1,p} = -\gamma_p \omega_{p5} \end{cases} \quad \forall(\gamma_1, \dots, \gamma_k) \in \mathbb{C}^k$$

$$(k \text{ solutions})$$

$$d_{3p} = a_p^1 \qquad (k+1-th)$$

$$r_p^2 = a_p^2 \qquad (k+2-th)$$

$$u_{pq} = \frac{a_p^2 a_q^2}{\omega_{p3}} \quad u_{k+2,p} = \frac{a_p^2 \omega_{p5}}{\omega_{p3}} \quad r_p^2 = -a_p^2 \frac{\omega_{p4}}{\omega_{p3}} \qquad (k+3-th)$$

$$d_{4p} = a_p^1 \qquad (k+4-th)$$

$$r_p^2 = a_p^1 \quad u_{k+1,p} = -a_p^2 \qquad (k+5-th)$$

$$u_{k+1,p} = a_p^1 \qquad (k+6-th)$$

$$u_{k+2,p} = a_p^2 \qquad (k+7-th)$$

$$u_{p,p} = \gamma_p \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \qquad (k \text{ solutions})$$

$$u_{2+k+p,p} = \gamma_p \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \qquad (k \text{ solutions})$$

$$u_{2+2k+2,p} = \gamma_p \quad t_{pq} = a_p^2 \gamma_q - a_q^2 \gamma_p \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \qquad (k \text{ solutions})$$

$$\begin{cases} u_{2+2k+1,p} = \gamma_p \\ u_{2+k+p,q} = -\frac{\xi_{q0}}{\xi_{p0}\xi_{q1} - \xi_{p1}\xi_{q0}}(a_q^1 \gamma_p - a_p^1 \gamma_q) \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \\ t_{pq} = \frac{\xi_{p0}\xi_{q2} - \xi_{p2}\xi_{q0}}{\xi_{p0}\xi_{q1} - \xi_{p1}\xi_{q0}}(a_q^1 \gamma_p - a_p^1 \gamma_q) \end{cases} \qquad (k \text{ solutions})$$

$$\begin{cases} r_p^1 = \gamma_p \\ u_{2+k+p,q} = -\frac{\xi_{q0}}{\xi_{p0}\xi_{q1} - \xi_{p1}\xi_{q0}}(a_q^2 \gamma_p - a_p^2 \gamma_q) \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \\ t_{pq} = \frac{\xi_{p0}\xi_{q2} - \xi_{p2}\xi_{q0}}{\xi_{p0}\xi_{q1} - \xi_{p1}\xi_{q0}}(a_q^2 \gamma_p - a_p^2 \gamma_q) \end{cases} \qquad (k \text{ solutions})$$

$$\begin{cases} d_{1p} = \gamma_p \\ u_{2+k+p,q} = \frac{\xi_{q0}}{\xi_{p0}\xi_{q1} - \xi_{p1}\xi_{q0}} \frac{\omega_{p3}\omega_{q5} - \omega_{p5}\omega_{q3}}{\omega_{p3}\omega_{q4} - \omega_{p4}\omega_{q3}}(a_q^1 \gamma_p - a_p^1 \gamma_q) \quad \forall(\gamma_1, \ldots, \gamma_k) \in \mathbb{C}^k \\ t_{pq} = -\frac{\xi_{p0}\xi_{q2} - \xi_{p2}\xi_{q0}}{\xi_{p0}\xi_{q1} - \xi_{p1}\xi_{q0}} \frac{\omega_{p3}\omega_{q5} - \omega_{p5}\omega_{q3}}{\omega_{p3}\omega_{q4} - \omega_{p4}\omega_{q3}}(a_q^1 \gamma_p - a_p^1 \gamma_q) \end{cases}$$

$$(k \text{ solutions})$$

$$\begin{cases} d_{0p} = \gamma_p \\ u_{2+k+p,q} = -\frac{\xi_{q1}}{\xi_{p0}\xi_{q1}-\xi_{p1}\xi_{q0}} \frac{\omega_{p3}\omega_{q5}-\omega_{p5}\omega_{q3}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}} (a^1_q \gamma_p - a^1_p \gamma_q) \quad \forall (\gamma_1, \dots, \gamma_k) \in \mathbb{C}^k \\ t_{pq} = \frac{\xi_{p1}\xi_{q2}-\xi_{p2}\xi_{q1}}{\xi_{p0}\xi_{q1}-\xi_{p1}\xi_{q0}} \frac{\omega_{p3}\omega_{q5}-\omega_{p5}\omega_{q3}}{\omega_{p3}\omega_{q4}-\omega_{p4}\omega_{q3}} (a^1_q \gamma_p - a^1_p \gamma_q) \end{cases}$$

(k solutions)

In order to study the kernel of the map δ in proposition 4.3.6 we need to combine the solutions given in 4.3.7 and 4.3.8, taking into account that the variables t_{ij} must be the same.
Once we get the dimension of the kernel of δ we have almost finished indeed we can proceed in the same way we have done in proposition 2.4.3 having all the dimensions we need.

4.4 Computational results

Throughout the whole work I used the software Macaulay2 [GS] (version 1.7). This program allowed me to picture the structure of the moduli space of instanton bundles in particular cases.
The next tables will sum up the behaviour in $\mathbb{P}^5$ of both the generic 't Hooft bundle and the generic Rao-Skiti bundle.
Symplectic t'Hooft bundle:

k	6	7	8	9	10	11	12
$h^1(S^2(E))$	89	95	98	106	116	126	136
$h^1(End(E))$	94	101	105	114	125	136	147

Rao-Skiti bundle:

k	6	7	8	9	10	11	12
$h^1(S^2(E))$	89	95	98	106	116	126	136
$h^1(End(E))$	89	95	98	106	116	126	136

Remark 4.4.1. The first thing to notice is that the values of $h^1(S^2(E))$, when E is 't Hooft, agree with Ottaviani conjecture.
In $\mathbb{P}^5$ these two families of instantons seem to share the same dimension of $H^1(S^2(E))$, $10k+16$, while the difference is when we study the moduli space of instantons (not necessarily symplectic): indeed the dimension of the tangent space on a 't Hooft bundle increases while the one on a Rao-Skiti remains unchanged: this means that

inside the Rao-Skiti component the two moduli spaces are the same, on the contrary the component which includes the 't Hooft bundles is different somehow. Moreover the Kuranishi map is always identically zero evaluated on a Rao-Skiti bundle, both for the symplectic case and the general case. This does not happen for 't Hooft bundles: in fact they are a smooth point in $MIS_{2,k}$ but for $k < 9$ are singular in $MI_{2,k}$. While for $k \geq 9$, when it is supposed that their dimension stabilizes to $10k + 16$, they become a smooth point too.

In the next section we are attaching the scripts used to achieve these results. The thesis is hence concluded.

4.4.1 Scripts

```
-- Input: n integer.
-- Output: none.
-- The program creates: kk a field (could be Q or Z_p).
--                      S polynomial ring in 2n+2
--                      indeterminates over kk.
--                      I ideal (vars S)
--                      S'=S/I^2
--                      S''=S/I^3
--                      S'''=S/I^4
-- Functions used: none.
```

```
start=(n)->
(
 --kk=QQ;
 kk=ZZ/32749;
 S=kk[x_0..x_n,y_0..y_n];
 Ide=ideal(vars S);
 S'=S/Ide^2;
 S''=S/Ide^3;
 S'''=S/Ide^4;
 use S;
)
```

```
-- Input:  n, k integers.
-- Output: A,J matrices.
-- A defines the special symplectic instanton of quantic number k
-- on P^{2n+1}, ie AJA^t=0.
-- Functions used: none
------------------------

specialinstanton=(n,k)->
(
 use S;
 var1=matrix{{y_0..y_n}};
 var2=matrix{{x_0..x_n}};
 I=map S^(n+k);
 O=I-I;
 J=matrix{{O,I},{-I,O}};
 for j to (k-1) do
    (
     zeros1=map(S^1,S^j,0);
     zeros2=map(S^1,S^(2*k-2-2*j),0);
     if j!=0 then vect=zeros1|var2 else vect=var2;
     if j!=k-1 then vect=vect|zeros2|var1 else vect=vect|var1;
     if j!=0  then vect=vect|zeros1;
     if j==0 then A=vect else A=A||vect;
    );
 A=map(S^{k:1},S^{2*n+2*k:0},A);
 return (A,J);
)

--------------------------
-- Input: n,k integers; alpha, beta rational.
-- Output: A,J matrices.
-- A(alpha,beta) defines the symplectic special instanton
-- of quantic number k on P^{2n+1} of the form described in
-- [AO00] p. 98
-- Functions used: specialinstanton.
--------------------------

specialinstantonpar=(n,k,alpha,beta)->
(
 (A,J)=specialinstanton(n,k);
```

```
 G=mutableMatrix(A);
 G1=G_(0..k-1,0..n+k-1);
 G1_(0,1)=G1_(0,1)*alpha;
 G1'=matrix(G1);
 G2=G_(0..k-1,n+k..2*n+2*k-1);
 for i to k-1 do
      if i<k-1-i then
           rowSwap(G2,i,k-i-1);
 G2_(0,1)=G2_(0,1)*beta;
 J'=mutableIdentity(S,n+k);
 rowSwap(J',0,1);
 for i to n+k-3 do
      if i<(n+k-3)/2 then
           rowSwap(J',2+i,n+k-i-1);
 G2'=matrix(G2);
 K=matrix(J');
 G=G1'|(G2'*(-K));
 A=matrix(G);
 A= map(S^{k:1},S^{2*n+2*k:0},A);
 return (A,J);
)

-----------------------------
-- Input: n,k integers.
-- Output: A,J matrices.
-- A defines a (random) 't Hooft symplectic instanton
-- of quantic number k on P^{2n+1}, ie AJA^t=0.
-- The form of A is the one described in definition 2.1.7
-- Functions used: none.
-----------------------------

tHooft=(n,k)->
(
 T=vars S;
 a=random(kk^k,kk^n);
 a=substitute(a,S);
 D=mutableIdentity(S,k);
 xi=random(kk^k,kk^(2*n+2));
 I=map S^(n+k);
 O=I-I;
```

```
J=matrix{{O,I},{-I,O}};
for j to k-1 do
    (
     element_j=0;
     for t to 2*n+1 do
          (
               element_j=element_j+xi_(j,t)*T_(0,t);
          );
     D_(j,j)=D_(j,j)*element_j;
    );
D=matrix(D);
D1=mutableIdentity(S,n);
for j to n-1 do
    (
     D1_(j,j)=D1_(j,j)*T_(0,j+1);
    );
D1=matrix(D1);
A1=(D|(a*D1));
A1=map(S^{k:1},S^{n+k:0},A1);
omega=random(kk^k,kk^(2*n+2));
D=mutableIdentity(S,k);
for j to k-1 do
    (
     element_j=0;
     for t to 2*n+1 do
          (
           element_j=element_j+omega_(j,t)*T_(0,t);
          );
     D_(j,j)=D_(j,j)*element_j;
    );
D=matrix(D);
D1=mutableIdentity(S,n);
for j to n-1 do
    (
     D1_(j,j)=D1_(j,j)*T_(0,n+2+j);
    );
D1=matrix(D1);
A2=(D|(a*D1));
A2=map(S^{k:1},S^{n+k:0},A2);
A=A1|A2;
```

```
 return (A,J)
)
```

```
-- Input: n,k integers.
-- Output: A,J matrices.
-- A defines a symplectic 't Hooft instanton
-- of quantic number k on P^{2n+1}, ie AJA^t=0
-- The form of (A,B) is the one described in remark 2.1.10
-- Functions used: none.
```

```
tHooft1=(n,k)->
(
 T=vars S;
 a=random(kk^k,kk^(n+k));
 a=substitute(a,S);
 I=map S^(n+k);
 O=I-I;
 J=matrix{{O,I},{-I,O}};
 D1=mutableIdentity(S,k+n);
 D2=mutableIdentity(S,k+n);
 xi=random(kk^(k+n),kk^(2*n+2));
 mu=random(kk^(k+n),kk^(2*n+2));
 for j to k+n-1 do
      (
       element1_j=0;
       element2_j=0;
       for t to 2*n+1 do
            (
             element1_j=element1_j+xi_(j,t)*T_(0,t);
             element2_j=element2_j+mu_(j,t)*T_(0,t);
            );
       D1_(j,j)=D1_(j,j)*element1_j;
       D2_(j,j)=D2_(j,j)*element2_j;
      );
 D1=matrix(D1);
 D2=matrix(D2);
 A=a*(D1|D2);
 A=map(S^{k:1},S^{2*n+2*k:0},A);
```

```
 return (A,J)
)
```

```
-- Input: n,k integers.
-- Output: A,B matrices.
-- A,B defines a (random) generic 't Hooft instanton
-- of quantic number k on P^{2n+1}, ie AB^t=0.
-- The form of (A,B) is the one described in definition 2.1.16
-- Functions used: none.
```

```
generictHooft=(n,k)->
(
 T=vars S;
 a=random(kk^k,kk^n);
 a=substitute(a,S);
 b=random(kk^k,kk^n);
 b=substitute(b,S);
 D=mutableIdentity(S,k);
 xi=random(kk^k,kk^(n+1));
 for j to k-1 do
      (
       element_j=0;
       for t to n do
            (
             element_j=element_j+xi_(j,t)*T_(0,t);
            );
       D_(j,j)=D_(j,j)*element_j;
      );
 D=matrix(D);
 D1=mutableIdentity(S,n);
 for j to n-1 do
      (
       D1_(j,j)=D1_(j,j)*T_(0,j+1);
      );
 D1=matrix(D1);
 A1=(D|(a*D1));
 B2=(D|(b*D1));
 omega=random(kk^k,kk^(n+1));
```

```
 D=mutableIdentity(S,k);
 for j to k-1 do
      (
       element_j=0;
       for t to n do
            (
             element_j=element_j+omega_(j,t)*T_(0,n+1+t);
            );
       D_(j,j)=D_(j,j)*element_j;
      );
 D=matrix(D);
 D1=mutableIdentity(S,n);
 for j to n-1 do
      (
       D1_(j,j)=D1_(j,j)*T_(0,n+2+j);
      );
 D1=matrix(D1);
 A2=(D|(a*D1));
 B1=-(D|(b*D1));
 A=A1|A2;
 B=B1|B2;
 A=map(S^{k:1},S^{2*n+2*k:0},A);
 B=map(S^{k:1},S^{2*n+2*k:0},B);
 return (A,B)
)

-------------------------------------------------
-- Input: n,k integers.
-- Output: A,B matrices.
-- A,B defines a (random) generic 't Hooft instanton
-- of quantic number k on P^{2n+1}, ie AB^t=0.
-- The form of (A,B) is the one described in remark 2.1.17
-- Functions used: none.
-------------------------------------------------

generictHooft1=(n,k)->
(
 T=vars S;
 a=random(kk^k,kk^(n+k));
 a=substitute(a,S);
```

```
 D1=mutableIdentity(S,k+n);
 D2=mutableIdentity(S,k+n);
 xi=random(kk^(k+n),kk^(n+1));
 mu=random(kk^(k+n),kk^(n+1));
 for j to k+n-1 do
      (
       element1_j=0;
       element2_j=0;
       for t to n do
            (
             element1_j=element1_j+xi_(j,t)*T_(0,t);
             element2_j=element2_j+mu_(j,t)*T_(0,t+n+1);
            );
       D1_(j,j)=D1_(j,j)*element1_j;
       D2_(j,j)=D2_(j,j)*element2_j;
      );
 D1=matrix(D1);
 D2=matrix(D2);
 A=a*(D1|D2);
 B=a*((-D2)|D1);
 A=map(S^{k:1},S^{2*n+2*k:0},A);
 B=map(S^{k:1},S^{2*n+2*k:0},B);
 return (A,B)
)
```

```
-- Input: n,k integers.
-- Output: A,J matrices.
-- A defines a (random) Rao-Skiti symplectic instanton
--of quantic number k on P^{2n+1}, ie AJA^t=0.
-- The form of A is the one described in definition 2.1.13
-- Functions used: none.
```

```
RSinstanton=(n,k)->
(
 T=vars S;
 vars1=matrix{{T_(0,0)..T_(0,n)}};
 I=map S^(n+k);
 O=I-I;
```

```
 J=matrix{{O,I},{-I,O}};
 firstmat=mutableMatrix(S,k,n+k);
 for i to n do
     for j to k-1 do
        firstmat_(j,j+i)=T_(0,i);
 firstmat=matrix(firstmat);
 firstmat=map(S^{k:1},S^{n+k:0},firstmat);
 xi=random(kk^(2*k+n-1),kk^(2*n+2));
 for i to 2*k+n-2 do
     (
      f_i=0;
      for j to 2*n+1 do
           f_i=f_i+xi_(i,j)*T_(0,j);
     );
 secondmat=mutableMatrix(S,k,n+k);
 for i to k-1 do
     for j to n+k-1 do
          (
           secondmat_(i,j)=f_(i+j);
          );
 secondmat=matrix(secondmat);
 secondmat=map(S^{k:1},S^{n+k:0},secondmat);
 RSmat=firstmat|secondmat;
 return (RSmat,J)
)

----------------------------
-- Input: n,k integers.
-- Output: J matrix.
-- J is a (random) matrix which has the property AJA^t=0
--if A defines a symplectic 'tHooft bundle.
-- The form of J is the one described in proposition 2.2.3
-- Functions used: none.
----------------------------

JtHooft=(n,k)->
(
 use S;
 Id=map S^(n+k);
 I=map S^(n+k);
```

```
 O=I-I;
 I=mutableMatrix(I);
 for i to n+k-1 do
      I_(i,i)= random kk;
 I=matrix(I);
 J=matrix{{O,I},{-transpose(I),O}};
 return J;
)
```

```
-- Input: n,k integers.
-- Output: J matrix.
-- J is a (random) matrix which has the property AJA^t=0,
--if A defines the special symplectic instanton bundle.
-- The form of J is the one described in proposition 2.2.2
-- Functions used: none.
```

```
Jspecial=(n,k)->
(
 use S;
 Id=map S^(n+k);
 I=map S^(n+k);
 O=I-I;
 I=mutableMatrix(I);
 for cont to 2*n+2*k-2 do
      listarandom_cont= random kk;
 for i to n+k-1 do
      for j to n+k-1 do
            I_(i,j)=listarandom_(i-j+n+k-1);
 I=matrix(I);
 J= (O|I)||((-transpose I)|O);
 return J;
)
```

```
-- Input: C matrix
-- Output: siz matrix.
-- The columns of siz are a set of generators
-- of the degree 1 syzigies of C extracting
```

```
-- those which derive from degree 0 syzigies.
-- Functions used: none
------------------------------
createsyz=(C)->
(
 n=floor((numColumns(vars S)-2)/2);
 R=syz (C,DegreeLimit =>1);
 stop=(numColumns R)-1;
 t=betti(R);
 num=t_(1,{0},0);
 for i to num-1 do
      (
       vett=R_{i};
       for j to n do
            (
             vett1=x_j**vett;
             vett2=y_j**vett;
             if (i,j)==(0,0) then
                   siz=vett1|vett2
                               else
                   siz=vett1|vett2|siz;
            );
      );
 siz=siz|R_{num..stop};
 return siz;
)

-------------------------
-- Input: deg, lenght integers; RING ring.
-- Output: bas matrix.
-- The columns of bas are a basis of the space of homogeneous
-- vectors of degree deg in RING^lenght.
-- Functions used: none.
-------------------------

creategenerators=(deg,RING,lenght)->
(
 vett=basis(deg,RING);
 nmcol=numColumns(vett);
 T=RING^lenght;
```

```
 for i to lenght-1 do
     (
      for j to nmcol-1 do
          (
           vett1=T_{i}**vett_(0,j);
           if (i,j)==(0,0) then
                bas=vett1
                           else
                bas=vett1|bas;
          );
     );
 return bas;
)

------------------------
-- Input: deg, sqrtlenght integers; RING ring.
-- Output: basisskewsymm matrix.
-- The columns of basisskewsymm are a basis
-- of the space of homogeneous skew-symmetric
-- matrices (sqrtlenght x sqrtlenght) of degree deg.
-- Functions used: none.
------------------------

createskewsymm=(deg,RING,sqrtlenght)->
(
 vett=basis(deg,RING);
 nmcol=numColumns(vett);
 for i to sqrtlenght-1 do
     (
      for j from i+1 to sqrtlenght-1 do
          (
           U=mutableMatrix(S,sqrtlenght,sqrtlenght);
           U_(i,j)=1;
           U_(j,i)=-1;
           U=matrix(U);
           for w to nmcol-1 do
               (
                U1=U*vett_(0,w);
                newvett=map(S^{sqrtlenght^2:0},S^{1:-deg},
                    reshape (S^(sqrtlenght^2),S^1,transpose U1));
```

```
                        if (i,j,w)==(0,1,0) then
                               basisskewsymm=newvett
                                                    else
                               basisskewsymm=basisskewsymm|newvett;
                      );
               );
        );
 return basisskewsymm;
)
```

```
-- Input: deg, sqrtlenght (must be even) integers; RING ring.
-- Output: basissymp matrix.
-- The columns of basissymp are a basis
-- of the space of homogeneous symplectic
-- matrices (sqrtlenght x sqrtlenght) of degree deg.
-- Functions used: none.
```

```
createsympl=(deg,RING,sqrtlenght)->
(
 vett=basis(deg,RING);
 nmcol=numColumns(vett);
 control=0;
 for i to sqrtlenght-1 do
        (
         for j to sqrtlenght-1 do
               (
                U=mutableMatrix(RING,sqrtlenght,sqrtlenght);
                if i< floor(sqrtlenght/2) then
                     (
                      if j< floor (sqrtlenght/2) then
                          (
                           U_(i,j)=1;
                           U_(j+floor(sqrtlenght/2),
                               i+floor(sqrtlenght/2))=-1;
                          )
                                                 else
                          (
                           U_(i,j)=1;
```

```
                    U_(j-floor(sqrtlenght/2),
                        i+floor(sqrtlenght/2))=1;
                   )
              )
                                         else
              (
               if j< floor(sqrtlenght/2) then
                   (
                    U_(i,j)=1;
                    U_(j+floor(sqrtlenght/2),
                        i-floor(sqrtlenght/2))=1;
                   )
                                               else
                    control=1;
              );
          U=matrix(U);
          if control==0 then
              (
               for w to nmcol-1 do
                   (
                    U1=U*vett_(0,w);
                    newvett=map(RING^{sqrtlenght^2:0},
                       RING^{1:-deg}, reshape (RING^(sqrtlenght^2),
                       RING^1,transpose U1));
                    if (i,j,w)==(0,0,0) then
                         basissymp=newvett
                                        else
                         basissymp=basissymp|newvett;
                   );
              )
                          else
               control=0;
          );
     );
 return basissymp;
)
```

```
-- Input: deg, sqrtlenght integers; RING ring.
-- Output: basissymm matrix.
```

```
-- The columns of basissymm are a basis
-- of the space of homogeneous symmetric
-- matrices (sqrtlenght x sqrtlenght) of degree deg.
-- Functions used: none.
-----------------------

createsymm=(deg,RING,sqrtlenght)->
(
 vett=basis(deg,RING);
 nmcol=numColumns(vett);
 for i to sqrtlenght-1 do
     (
      for j from i to sqrtlenght-1 do
          (
           U=mutableMatrix(S,sqrtlenght,sqrtlenght);
           U_(i,j)=1;
           U_(j,i)=1;
           U=matrix(U);
           for w to nmcol-1 do
               (
                U1=U*vett_(0,w);
                newvett=map(S^{sqrtlenght^2:0},S^{1:-deg},
                     reshape (S^(sqrtlenght^2),S^1,transpose U1));
                if (i,j,w)==(0,0,0) then
                     basissymm=newvett
                                    else
                     basissymm=basissymm|newvett;
               );
          );
     );
 return basissymm;
)

-----------------------
-- Input: A,B matrices.
-- Output: H1endE module.
-- A and B must be matrices which define E, an instanton bundle.
-- H1endE is H^1(End E) seen as an S'-module.
-- Functions used: createsyz.
-----------------------
```

```
H1=(A,B)->
(
 use S;
 k=numRows(A);
 n=(numColumns(A)-2*k)//2;
 r=2*k+2*n;
 T1=map(S^k,S^k,1);
 T2=map(S^r,S^r,1);
 N1=T1**B;
 G=entries(A);
 for i to k-1 do
      if i==0 then
           N2=T1**(matrix({G_i}))
                else
           N2=N2||(T1**(matrix({G_i})));
 C=N1|N2;
 C=map(S^{k^2:1},S^{2*k*r:0},C);
 O=map(S^(k*r),S^(k^2),0);
 M1=T1**transpose A;
 M4=transpose N1;
 for i to k-1 do
      if i==0 then
      M2=T2**(matrix({G_i}))
                else
      M2=M2||(T2**(matrix({G_i})));
 for i to k-1 do
      (
       for j to (r-1) do
            if j==0 then
                el=-B_(i,j)*T2
                     else
                el=el|(-B_(i,j)*T2);
       if i==0 then
           M3=el
                else
           M3=M3||el;
      );
 G=(M1|M2|O)||(O|M3|M4);
 G=map(S^{2*k*r:0},S^{2*k^2+r^2:-1},G);
```

```
 siz=createsyz(C);
 siz'=substitute(siz,S');
 W=substitute (image G,S');
 V=substitute (image siz,S');
 H1endE=V/W;
 dimH1endE=rank H1endE;
 return H1endE;
)
```

```
-- Input: A,B matrices.
-- Output: H2endE module.
-- A and B must be matrices which define E, an instanton bundle.
-- H2endE is H^2(End E) seen as an S''-module.
-- Functions used: creategenerators
```

```
H2=(A,B)->
(
 use S;
 k=numRows(A);
 n=(numColumns(A)-2*k)//2;
 r=2*k+2*n;
 N1=T1**B;
 G=entries(A);
 for i to k-1 do
     if i==0 then
          N2=T1**(matrix({G_i}))
               else
          N2=N2||(T1**(matrix({G_i})));
 C=N1|N2;
 C=map(S^{k^2:1},S^{2*k*r:0},C);
 bas=creategenerators(1,S,2*k*r);
 mat=map (S^{k^2:0},S^{numColumns bas:-2},C*bas);
 bas=creategenerators(2,S,k^2);
 Z=substitute(image mat,S'');
 bas'=substitute(image bas, S'');
 H2endE=bas'/Z;
 dimH2endE=rank H2endE;
 return H2endE;
```

```
)

------------------------------------------------
-- Input: A,J matrices.
-- Output: H1S2E module.
-- A must be a matrix which define E, a symplectic instanton bundle.
-- H1S2E is H^1(S^2 E) seen as an S'-module.
-- Functions used: createsyz
------------------------------------------------

SH1=(A,J)->
(
 use S;
 k=numRows(A);
 n=(numColumns(A)-2*k)//2;
 r=2*k+2*n;
 d=k+n;
 idk=id_(S^k);
 idd=id_(S^d);
 nul=map(S^d,S^(d*d),0);
 mat1=idk**transpose A;
 for i to k-1 do
      (
       mint=(idd**A^{i}_{0..d-1})||((-A^{i}_{d..r-1})**idd);
       if i==0 then
            mat2=mint
                else
            mat2=mat2||mint;
      );
 for i to k-1 do
      (
       mint=nul||(idd**A^{i}_{0..d-1});
       if i==0 then
             mat3=mint
                else
             mat3=mat3||mint;
      );
 for i to k-1 do
      (
       mint=(idd**A^{i}_{d..r-1})||nul;
       if i==0 then
```

```
            mat4=mint
                else
            mat4=mat4||mint;
      );
 endsympaction=mat1|mat2|mat3|mat4;
 -- the columns of endsympaction are a generating set
 -- of the action of  End_k \oplus Symp_r given by (a,b)--> aA+Ab
 endsympaction=map(S^{numRows(endsympaction):1},
               S^{numColumns(endsympaction):0},endsympaction);
 endsympaction=substitute( image endsympaction,S');
 AA=A*transpose J;
 for i to k-1 do
      for j to k-1 do
          (
           O=map(S^{1:1},S^{k:0},0);
           O=mutableMatrix O;
           O_(0,i)=1;
           O=matrix O;
           O=O**AA^{j};
           O1=map(S^{1:1},S^{k:0},0);
           O1=mutableMatrix O1;
           O1_(0,j)=1;
           O1=matrix O1;
           O1= -(O1**AA^{i});
           KK=O+O1;
           if (i,j)==(0,0) then
                newm=KK
                              else
                newm=newm||KK;
          );
 newm=map(S^{numRows(newm):1},S^{r*k:0},newm);
 -- newm is the matrix representing the morphism A'-->A'JA^t+AJA'^t.
 siz=createsyz(newm);
 siz=map(S^{numRows(siz):1},S^{numColumns(siz):0},siz);
 siz=substitute(image siz,S');
 H1S2E=siz/endsympaction;
 dimH1S2E=rank H1S2E;
 return H1S2E;
)
```

```
-----------------------------------------
-- Input: A,J matrices.
-- Output: H1S2E module.
-- A must be a matrix which define E, a symplectic instanton bundle.
-- H1S2E is H^1(S^2 E) seen as an S'-module.
-- This function does the same as SH1 but for every J symplectic.
-- Functions used: createsympl
-----------------------------------------

JSH1=(A,J)->
(
 use S;
 d=k+n;
 r=2*k+2*n;
 basissymp=createsympl(0,S,2*n+2*k);
 for i to numColumns(basissymp)-1 do
      (
       mm=transpose reshape(S^(2*n+2*k),
           S^(2*n+2*k),matrix(basissymp_i));
       mm=A*mm;
       mm=reshape(S^(k*(2*n+2*k)),S^1,mm);
        if i==0 then
             endsympaction=mm
                  else
             endsympaction=endsympaction|mm;
      );
 for i to k-1 do
      for j to k-1 do
           (
            alpha=mutableMatrix(S,k,k);
            alpha_(i,j)=1;
            alpha=matrix(alpha);
            mm=alpha*A;
            mm=reshape(S^(k*(2*n+2*k)),S^1,mm);
            endsympaction=endsympaction|mm;
           );
 -- the columns of endsympaction are a generating set
 -- of the action of  End_k \oplus Symp_r given by (a,b)--> aA+Ab
 endsympaction=map(S^{numRows(endsympaction):1},
                  S^{numColumns(endsympaction):0},endsympaction);
```

```
endsympaction=substitute( image endsympaction,S');
numero=floor(k*(2*n+2*k));
T=S[j_1..j_numero];
B=transpose reshape(T^k,T^(2*n+2*k),vars T);
matrice=A*J*B+transpose(B)*J*transpose(A);
numero1=floor(k*(k-1)/2);
matricemut=mutableMatrix(T,1,numero1);
for i from 1 to numero1 do
    (
     div=i;
     quot=k-1;
     i1=0;
     while ((div-1)//quot)>0 do
          (
           div=div-quot;
           quot=quot-1;
           i1=i1+1;
          );
     i2=i1+div;
     matricemut_(0,i-1)=matrice_(i1,i2);
    );
matricemut=matrix(matricemut);
stringa=coefficients matricemut;
matfin=transpose stringa_1;
variabili=stringa_0;
zeromap=map(T^numero1,T^1,0);
for i to numero-1 do
    (
     if i>numColumns(variabili)-1 then
          (
           variabili=variabili|matrix(j_(i+1));
           matfin=matfin|zeromap;
          );
     if variabili_(0,i)!=j_(i+1) then
          (
           if i==0 then
               (
                variabili=matrix(j_1)|variabili;
                matfin=zeromap|matfin;
               );
```

```
            if i!=0 and i!=numero-1 then
                (
                 variabili=submatrix(variabili,{0},
                          {0..i-1})|matrix(j_(i+1))|
                          submatrix(variabili,{0},
                          {i..(numColumns(variabili)-1)});
                 matfin=submatrix(matfin,{0..numrows(matfin)-1},
                       {0..i-1})|zeromap|submatrix(
                       matfin,{0..numrows(matfin)-1},
                       {i..(numColumns(matfin)-1)});
                );
           );
     );
 matfin=sub(matfin,S);
 matfin= map(S^{numRows(matfin):1},S^{numColumns(matfin):0},matfin);
 siz=createsyz(matfin);
 siz=map(S^{numRows(siz):1},S^{numColumns(siz):0},siz);
 siz=substitute(image siz,S');
 H1S2E=siz/endsympaction;
 dimH1S2E=rank H1S2E;
 return H1S2E;
)

-------------------------------------
-- Input: A,J matrices.
-- Output: H2S2E module.
-- A must be a matrix which define E, a symplectic instanton bundle.
-- H2S2E is H^2(S^2 E) seen as an S''-module.
-- Functions used: creategenerators, createantisymm, dimk.
-------------------------------------
SH2=(A,J)->
(
 use S;
 k=numRows(A);
 n=(numColumns(A)-2*k)//2;
 r=2*k+2*n;
 b=createskewsymm(2,S,k);
 domain=creategenerators(1,S,k*r);
 ncol=numColumns domain;
 AA=A*transpose J;
```

```
for i to k-1 do
    for j to k-1 do
        (
         O=map(S^{1:1},S^{k:0},0);
         O=mutableMatrix O;
         O_(0,i)=1;
         O=matrix O;
         O=O**AA^{j};
         O1=map(S^{1:1},S^{k:0},0);
         O1=mutableMatrix O1;
         O1_(0,j)=1;
         O1=matrix O1;
         O1= -(O1**AA^{i});
         KK=O+O1;
          if (i,j)==(0,0) then
              newm=KK
                              else
              newm=newm||KK;
        );
rows=numRows(newm);
newm=map(S^{rows:1},S^{r*k:0},newm);
for i to ncol-1 do
     (
      im=newm*domain_{i};
      im=map(S^{k^2:0},S^{1:-2}, im);
       if i==0 then
           mat=im
                else
           mat=mat|im;
     );
base'=substitute(image b,S'');
Z=substitute(image mat,S'');
H2S2E= base'/Z;
dimH2S2E=rank H2S2E;
return H2S2E;
)
```

```
-- Input: A,B matrices
-- Output: quadratic forms and their rank.
```

```
-- Given a couple of matrices (A,B) which define an
-- instanton bundle E the program computes the Kuranishi map
-- from H^1(End E) to H^2(End E). The quadratic forms
-- represent local equations of the
-- moduli space of instanton bundles near E.
-- Functions used: H1, H2.
---------------------------------------------

kura=(A,B)->
(
 use S;
 k=numRows(A);
 n=(numColumns(A)-2*k)//2;
 r=2*k+2*n;
 H1E=H1(A,B);
 H2E=H2(A,B);
 baseH1=substitute(mingens(H1E),S);
 nbase=numColumns(baseH1);
 for i to nbase-1 do
     (
      z_i=map(S^{k:0},S^{r:-1},transpose(
               reshape(S^r,S^k,baseH1_{i}^{0..k*r-1})));
      z'_i=map(S^{r:-1},S^{k:-2},
            reshape(S^r,S^k,baseH1_{i}^{k*r..2*k*r-1}));
     );
 contatore=0;
 for i to nbase-1 do
     for j from i  to nbase-1 do
          if (z_i*z'_j+z_j*z'_i)!=0 then
             (
              P_(i,j)=map(S^{k^2:0},S^{1:-2}, reshape (S^(k^2),S^1,
                      transpose (z_i*z'_j+z_j*z'_i)));
              if isSubquotient(image P_(i,j),image mat)==false then
                 (
                  if contatore==0 then
                    (
                     mapk=P_(i,j);
                     contatore=1;
                     indici=matrix{{i,j}};
                    )
```

```
                                   else
                        (
                         mapk=mapk|P_(i,j);
                         indici=indici||matrix{{i,j}};
                        )
                    );
               );
if contatore==0 then
    (
     mapk=0;
     dimkura=0;
    )
                    else
    (
     mapk'=substitute(mapk,S'');
    );
if mapk!=0 then
    (
     temp=syz ((mapk|mingens image mat),DegreeLimit=>2);
     matkura=gens kernel transpose submatrix(
               temp,0..(numcols mapk-1),0..numcols temp-1);
     h1e=floor(dimH1endE)-1;
     R=kk[w_0..w_h1e];
     matkk=sub(matkura,R);
     for i from 0 to (numcols matkura-1) do
          (
           quad_i=0;
           for j from 0 to (numrows matkura-1) do
               quad_i=quad_i+matkk_(j,i)*w_(indici_(j,0))*
               w_(indici_(j,1));
          );
     ideale=ideal(0);
     for i from 0 to (numcols matkura-1) do
          (
           ideale=ideal{ideale,quad_i};
           print(quad_i);
           print("rank ",rank diff(transpose basis(1,R),
                 diff(basis(1,R),quad_i)));
          );
    );
```

```
)

------------------------------------
-- Input: A,J matrices
-- Output: quadratic forms and their rank.
-- Given a matrix A which define a symplectic
-- instanton bundle E the program compiles the Kuranishi map
-- from H^1(S^2E) to H^2(S^2E). The quadratic forms
-- represent local equations of the
-- moduli space of symplectic instanton bundles near E.
-- Functions used: SH1, SH2.
------------------------------------

Skura=(A,B)->
(
 k=numRows(A);
 n=(numColumns(A)-2*k)//2;
 r=2*k+2*n;
 use S;
 H1E=SH1(A,B);
 H2E=SH2(A,B);
 baseH1=substitute(mingens(H1E),S);
 nbase=numColumns(baseH1);
 for i to nbase-1 do
      z_i=map(S^{k:0},S^{r:-1},transpose(
           reshape(S^r,S^k,baseH1_{i}^{0..k*r-1})));
 contatore=0;
 for i to nbase-1 do
      for j from i to nbase-1 do
           if (z_i*J*transpose z_j+z_j*J*transpose z_i)!=0 then
                (
                 P_(i,j)=map(S^{k^2:0},S^{1:-2},
                         reshape (S^(k^2),S^1,transpose(
                         z_i*J*transpose z_j+z_j*J*transpose z_i)));
                 if isSubquotient(image P_(i,j),image mat)==false then
                      if contatore==0 then
                         (
                          mapk=P_(i,j);
                          contatore=1;
                          indici=matrix{{i,j}};
```

```
                        )
                                        else
                        (
                         mapk=mapk|P_(i,j);
                         indici=indici||matrix{{i,j}};
                        )
                  );
if contatore==0 then
   (
    mapk=0;
    dimkura=0;
   )
              else
   (
    mapk'=substitute(mapk,S'');
   );
if mapk!=0 then
   (
    temp=syz ((mapk|mingens image mat),DegreeLimit=>2);
    matkura= gens kernel transpose submatrix(
            temp,0..(numcols mapk-1),0..numcols temp-1);
    h1e=floor(dimH1S2E)-1;
    R=kk[w_0..w_h1e];
    matkk=sub(matkura,R);
    for i from 0 to (numcols matkura-1) do
        (
         quad_i=0;
         for j from 0 to (numrows matkura-1) do
             quad_i=quad_i+matkk_(j,i)*w_(indici_(j,0))*
                   w_(indici_(j,1));
        );
    ideale=ideal(0);
    for i from 0 to (numcols matkura-1) do
        (
         ideale=ideal{ideale,quad_i};
         print(quad_i);
         print("rank ",rank diff(transpose basis(1,R),
              diff(basis(1,R),quad_i)));
        );
   );
```

)

Bibliography

[ADHM78] M. F. Atiyah, V. G. Drinfeld, N. J. Hitchin, and Yu. I. Manin. Construction of instantons. *Phys. Lett. A 65, 185-187*, 1978.

[AO94] V. Ancona and G. Ottaviani. Stability of special instanton bundles on P^{2n+1}. *Trans. Amer. Math. Soc. 341, 677-693*, 1994.

[AO95] V. Ancona and G. Ottaviani. On moduli of instanton bundles on P^{2n+1}. *Pacific Journal of Math. vol 171, no 2*, 1995.

[AO00] V. Ancona and G. Ottaviani. On the irreducible components of the moduli space of instanton bundles on P^5. *Geometry seminars, papers of a conference on real and complex analytic geometry, University of Bologna, November 1997*, 2000.

[Ati79] M. F. Atiyah. Geometry of Yang-Mills fields. *Scuola Normale Superiore, Pisa*, 1979.

[AW77] M. F. Atiyah and R. S. Ward. Instantons and algebraic geometry. *Comm. Math. Phys 55, 117-124*, 1977.

[Bar77] W. Barth. Some properties of stable rank-2 vector bundles on P^n. *Math. Ann. 226 125-150*, 1977.

[Bar81] W. Barth. Irreducibility of the space of mathematical instanton bundles with rank 2 and $c_2 = 4$. *Math. Ann. 258 n 1, 81106*, 1981.

[Bar82] W. Barth. Irreducibility of the space of mathematical instanton bundles with rank 2 and $c_2 = 4$. *Math. Ann. 258 81-106*, 1982.

[BMT12] U. Bruzzo, D. Markushevich, and A. S. Tikhomirov. Moduli of symplectic instanton vector bundles of higher rank on projective space P^3. *A.S. centr.eur.j.math. 10: 1232. doi:10.2478/s11533-012-0062-2*, 2012.

[BMT16] U. Bruzzo, D. Markushevich, and A. S. Tikhomirov. Symplectic instanton bundles on P^3 and 't Hooft instantons. *A.S. European Journal of Mathematics 2: 73. doi:10.1007/s40879-015-0082-0*, 2016.

[BS92] G. Bohnhorst and H. Spindler. The stability of certain vector bundles on $\mathbb{P}^n$. *Proc. Bayreuth Conference (Complex algebraic varieties), LNM 1507, 39-50*, 1992.

[CHMRS14] L. Costa, N. Hoffmann, R. M. Mirò-Roig, and A. Schmitt. Rational families of instanton bundles on P^{2n+1}. *Algebraic Geometry 2 229-260*, 2014.

[CO02] L. Costa and G. Ottaviani. Nondegenerate multidimensional matrices and instanton bundles. *Transactions of the American Mathematical Society 355 n1 49-55*, 2002.

[CTT03] I. Coanda, A. S. Tikhomirov, and G. Trautmann. Irreducibility and smoothness of the moduli space of mathematical 5-instanton over P^3. *Internat. J. Math. 14 1-45*, 2003.

[Dio98] C. Dionisi. Symplectic small deformations of special instanton bundles on P^{2n+1}. *Annali Mat. pura appl. 175 285-293*, 1998.

[Em81] G. Ellinsgrud and S. A. Strømme. Stable rank 2 vector bundles on P^3 with $c_1 = 0$ and $c_2 = 3$. *Math. Ann. 255 123-135*, 1981.

[Fae07] D. Faenzi. Homogeneous instanton bundles on P^3 for the action of SL(2). *Journal of Geometry and Physics 57 2146-2157*, 2007.

[GS] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. *Available at http://www.math.uiuc.edu/macaulay2/.*

[GS94] L. Gruson and M. Skiti. 3-instantons et reseaux de quadriques. *Mathematische Annalen 298 253-274*, 1994.

[Har78] R. Hartshorne. Stable vector bundles of rank 2 on P^3. *Math. Ann. 238 229-280*, 1978.

[HH82] R. Hartshorne and A. Hirschowitz. Cohomology of a general instanton bundle. *Annales scientifiques de l'E.N.S. 4 serie, tome 15 n2 365-390*, 1982.

[JV14] M. Jardim and M. Verbitsky. Trihyperkähler reduction and instanton bundles on $\mathbb{P}^3$. *To appear in Compositio Math. Preprint arXiv:1103.4431*, 2014.

[Kat03] P. I. Katsylo. Rationality of the moduli variety of mathematical instantons with $c_2 = 5$. Lie groups, their discrete subgroups, and invariant theory. *Transformation Groups vol 8 no 2 147-158*, 2003.

[KO03] P. I. Katsylo and G. Ottaviani. Regularity of the moduli space of instanton bundles $\mathrm{MI}_{P^3}(5)$. *Transformation Groups vol 8 no 2 147-158*, 2003.

[MROL97] R. M. Mirò-Roig and J. A. Orus-Lacort. On the smoothness of the moduli space of mathematical instanton bundles. *Compositio Matematica 105 109-119*, 1997.

[OT94] G. Ottaviani and G. Trautmann. The tangent space at a special symplectic instanton bundle. *Manuscr. Math. 85 97-107*, 1994.

[Ott96] G. Ottaviani. Real and complex 't Hooft instanton bundles over $\mathbb{P}^{2n+1}(C)$. *Rend. Semin. Mat. Fis. Milano 66, 166-199*, 1996.

[Pot83] J. Le Potier. Sur l'espace de modules des fibres de Yang et Mills. *Mathematics and physics 65-137, Progr. Math. 37*, 1983.

[Sal84] S. M. Salamon. Quaternionic structures and twistor spaces. *Global Riemannian Geometry (N.J. Willmore and T.J. Hitchin, eds), Ellis Horwood*, 1984.

[Tik12] A. S. Tikhomirov. Moduli of mathematical instanton vector bundles with odd c_2 on projective space. *Izv. Math. 76 991*, 2012.

[Tik13] A. S. Tikhomirov. Moduli of mathematical instanton vector bundles with even c_2 on projective space. *Izv. Math. 77*, 2013.