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Seismic vulnerability of pointed arches under rigid body assumption Numerical and experimental evaluations

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To my family and to my closest friends

for their advice and their patience, because they always understood.

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Abstract

Conservation of cultural heritage against seismic risk constitutes one of the major challenges of the scientific community, which is actually engaged in refining operative solutions for practitioners as well as theoretical mechanical models. The response of special architectural elements, like arches, domes and vaults, has attracted the interest of historic scientists, but still today comprehensive and general formulations lack of a full dynamic perspective. Among architectural elements, the arch is certainly an iconography of mechanics applied to architecture. Indeed, extensive investigations are available in the literature on the response of circular arches to vertical loads, and a few complete dynamic models for horizontal acceleration load can be found as well. Pointed arches, even though spread in seismic prone areas, received much less of interest.

An amazing case study on which working on, consisting in a giant pointed arched system located in Afghanistan, motivated a seminal interest on the issue, toughened by the lacking literature. For these reasons, this dissertation reports on a parametric analysis of the vulnerability of pointed arches made of two circular arcs, which is actually the simplest thinkable pointed arch. The analysis considers variations of arch slenderness and sharpness that result from different positions of centres of circular arcs.

Firstly, the arch is addressed as a rigid macro-block system, and limit analysis with the kinematic approach is exploited to determine the collapse acceleration through Non-Linear Programming optimisation. The pattern of hinges at collapse differs considerably from the one that occurs for circular arches. Moreover, the effect of arch slenderness on collapse accelerations turns out to be significantly conditioned by sharpness. Acceleration necessary to initiate motion grows with the rise, as opposed to what occurs for circular shapes. Dynamic behaviour of pointed arches for rectangular shaped and harmonic inputs are investigated as well transforming arch mechanisms into four-bar linkages. Systematic integration of the non-linear form of the distinctive ODE of the problem revealed that failure during second half cycle of motion occurs form most of the profiles. Such a trend would have never been tackled in the framework of linearised motion, which however provides more conservative estimations. Moreover, failure during second half cycle of motion for harmonic inputs, especially for low and medium frequencies, is found to be deeply influenced by the adopted impact model and, more importantly, by position of hinges. A dedicated sensitivity analysis validates the procedure predicting failure of circular arches as a particular case of pointed.

Considering also a micro-block approach, a wide experimental campaign addressed the equivalent static and full dynamic response of a set of 11 reduced scale model of pointed arches made of prismatic Autoclaved Aerated Concrete blocks. Global geometric characteristics of models are the same considered in the macro-block approach. Tilt tests and shake table tests uncovered the inherent sliding vulnerability of these profiles.

Thus, a kinematic model capable to consider sliding, independently from the adopted friction coefficient can be represented through a two-bar model hinged at ground and connected by a slider; outcomes tackle global sliding mechanism of thick and sharp profiles. A similar aim justified the use of the Distinct Element Method through the commercial code 3DEC. As for equivalent static tests, range of friction coefficient necessary to initiate a hinging mechanism vary with variation in geometry of the profile, and most important, for thick and sharp profiles, the rocking mechanism can hardly be activated unless a perfect hinging interface is not assumed.

Regarding dynamic tests, harmonic pulses with frequency ranging between 2Hz and 10Hz have been considered and results of analytical, numerical and experimental models have been compared. Given the stated vulnerability of pointed arches to crown sliding, the four bar linkage model will be always lacking a fundamental aspect, especially for sharp and stocky profiles subjected to high frequency inputs.

Future investigations should address vulnerability to complete time histories and in a probabilistic framework, sliding phenomena when overloading is considered and 3D structures.

Table of contents

List of figures			xiii
List of tables			xix
1	Intr	duction	1
	1.1	Context	1
	1.2	Bounds to rigid body assumption for masonry	2
	1.3	Motivations and objectives	4
	1.4	Outline of the thesis	6
2	Stat	of the art	7
	2.1	Overview	7
	2.2	Systems with unilateral constraints	8
		2.2.1 The rocking block	8
		2.2.2 Rigid body assemblies	15
		2.2.3 Inverted pendulum versus oscillator	17
	2.3	Arches under horizontal actions	19
		2.3.1 Overview	19
		2.3.2 Equivalent static input	19
		2.3.3 Dynamic input	23
	2.4	The statics of pointed arches	25
	2.5	Summary	29

3	Ana	alytical modelling of pointed arches forced by horizontal actions	31
	3.1	Overview	31
	3.2	Geometrical model	31
	3.3	The pointed rocking arch	34
		3.3.1 Onset of motion for equivalent static actions	34
		3.3.2 Dynamic response for acceleration inputs	38
	3.4	Summary	62
4	Exp	perimental evaluations on reduced-scale block models	65
	4.1	Specimens	65
	4.2	Tilt tests	65
		4.2.1 Test apparatus	65
		4.2.2 Results	66
	4.3	Shake table tests	72
		4.3.1 Test apparatus	72
		4.3.2 Results	74
	4.4	Summary	84
5	Ana	alytical and numerical interpretation of tests results	87
	5.1	A simplified rocking-sliding model for tilt test outcomes	87
	5.2	The DE method for the interpretation of static and dynamics tests $\ . \ .$	92
	5.3	Summary	103
6	A c	ase study: The Noh-Gunbad Mosque in Balkh, Afghanistan	109
	6.1	Overiview	109
	6.2	Static consistency of the arched system of the Mosque	114
	6.3	Physical-mineralogical and mechanical characterization of materials $\ . \ .$	118
	6.4	Vulnerability assessment	121
	6.5	Seismic Hazard and safety assessment	122

		6.5.1	Afghan seismicity	. 122
		6.5.2	Arches safety assessment	. 126
	6.6	Conclu	sions	. 128
7	Con	cluding	g remarks and future research	129
	7.1	Main f	indings	. 129
	7.2	Contri	butions	. 131
	7.3	Future	research	. 131
Re	eferei	nces		133
AĮ	open	dix A	Minimization code	147
AĮ	open	dix B	Flow chart of the computing loop to solve motion problem	n151
AĮ	open	dix C	Rectangular shaped acceleration input	153

List of figures

2.1	Rigid block subjected to unilateral contact, (Sinopoli et al., 1998)	9
2.2	The rocking block and its moment rotation diagram, (Zhang and Makris, 2001)	10
2.3	Possible models for joint definition in Discrete Element Methods, (Lemos, 2007), u_s , shear displacement, u_n , exaggerated overlapping, F_s , shear force, elastic and sliding component for the soft contact model, F_n , normal contact force	18
2.4	Behaviour of the rigid no-tension material, (Como, 2010) $\ldots \ldots \ldots$	21
2.5	Collapse mechanism for a circular arch, (Clemente, 1998) \ldots	22
2.6	Four-bar linkage applied to a circular arch: l_{-} , length of links connecting hinges; θ_{-} slope of the links; r_{-} , massless bars connecting hinges to centres of mass of each macro-block; ψ_{-} slopes of r_{-} with respect to l_{-} ; F_{i} and F_{iv} , reactions exerted by impulsive forces assumed on the corner of new hinges forming after an impact has occurred. Arabic numbering of lower cases refers to configuration before impact, Roman numbering to after impact	24
2.7	Early examples of pointed arches	26
2.8	Baghdad Gate, Raqqa, Syria, image courtesy of Groundhopping Merse- burg, (Merseburg, 2016)	27
2.9	Constructive solutions for keystone in pointed arches, (Ungewitter, 1890)	28

3.1	Geometrical model of the pointed arch. Shape parameters: R_p , mean radius of the pointed arch; e , eccentricity; t , thickness; R_c , radius of the corresponding circular arch. Parameters of the analytical model: R_e and R_i , extrados and intrados radii; $E(\beta)$, extrados profile; $I(\beta)$, intrados profile; O , principal reference system origin; O_L and O_R local reference systems origins. β_n , δ_n and γ_n angles identifying hinge positions and plane that separates two blocks; ρ_j polar distance identifying the centre of mass, G_j ; K surface of half keystone defined by ring thickness varying from t_v to t_r	32
3.2	Multiplier of loads for varying thickness for circular and pointed arches	37
3.3	Hinge positions, β_n , versus Sd ; trend of load multiplier, λ , in dotted lines, and position of hinges for circular arches, (Clemente, 1998), in grey continuous lines. Circles, triangles and squares represent the position of the first, second and third hinge respectively $\ldots \ldots \ldots \ldots \ldots$	39
3.4	Four-bar linkage applied to pointed arch: l_{-} , length of links connecting hinges; θ_{-} slope of the links; r_{-} , massless bars connecting hinges to centres of mass of each macro-block; ψ_{-} slopes of r_{-} with respect to l_{-} ; F_{i} and F_{iv} , reactions exerted by impulsive forces assumed on the corner of new hinges forming after an impact has occurred. Arabic numbering of lower cases refers to configuration before impact, Roman numbering to after impact	40
3.5	Failure domains <i>input duration</i> - <i>acceleration</i> for Pulse Type A for three Sh values and increasing Sd	45
3.6	Failure domains <i>input duration</i> - <i>acceleration</i> for Pulse Type B, duration of first interval of the input is considered, for three Sh values and increasing Sd ; and for fixed Sh and Sd values and increasing R_c . Black filled pointers indicate failure without impact.	51
3.7	Comparison among failure domains for Pulse Type A and B for slender of three Sh values. Black filled pointers indicate failure without impact.	52
3.8	Comparison among failure domains for Pulse Type A and B for stocky profiles of three Sh values $\ldots \ldots \ldots$	53
3.9	Failure domains for sinusoidal input and minimised layout	55

3.10	Load multipliers for Oppenheim's circular arch, (Oppenheim, 1992), i.e. $Sh = 0$, and for pointed arches with same mechanism layout $\ldots \ldots$	56
3.11	Fixed mechanism layout. Failure domains <i>input duration</i> - <i>acceleration</i> for Pulse Type A for three Sh values and increasing Sd	57
3.12	Fixed mechanism layout. Failure domains <i>input duration - acceleration</i> for Pulse Type B, duration of first interval of the input is considered, for four Sh ratios and increasing Sd . Black filled pointers indicate failure without impact	58
3.13	Fixed mechanism layout. Failure domains for sinusoidal input	59
3.14	Sensitivity analysis results for different inputs	61
3.15	Failure domains for Pulse Type A for a circular and a pointed arch with different mechanism orientation	62
4.1	Setting up tilt test	66
4.2	Tested specimens	67
4.3	Preparation of models	68
4.4	Results of tilt tests and comparison between multiplier of loads as deduced from tilt tests, λ_{exp} , and evaluated through procedure of Section 3.3.1, λ_{min} , for different Sh and Sd values. Error bars indicate ranging outcomes of repeated tests	69
4.5	Results of tilt test for $Sh = 0.2$	70
4.6	Results of tilt test for $Sh = 0.6$	70
4.7	Results of tilt test for $Sh = 1$	71
4.8	Position of hinges for tilt tests. Circles, triangles and squares represent the position of the first, second and third hinge respectively. Empty triangular pointers represent sliding joints and dashed lines indicate the angular width occupied by the crown block	73
4.9	Setting up shake table test	74
4.10	Results of shake table tests for $Sh = 0.2$	75
4.11	Results of shake table tests for $Sh = 0.6$	76
4.12	Results of shake table tests for $Sh = 1$	77

4.13	Dynamic tests for $Sh = 0.2$	78
4.14	Dynamic tests for $Sh = 0.2$	78
4.15	Sh06Sd010 - 2Hz, 2 nd half cycle \ldots \ldots \ldots \ldots \ldots \ldots \ldots	79
4.16	Dynamic tests for $Sh = 0.6$ and low thickness $\ldots \ldots \ldots \ldots \ldots$	80
4.17	Dynamic tests for $Sh = 0.6$ and $Sd = 0.20$	81
4.18	Dynamic tests for $Sh = 0.6$ and $Sd = 0.25$	81
4.19	Dynamic tests for $Sh = 1$ and low thickness $\ldots \ldots \ldots \ldots \ldots$	82
4.20	Sh1Sd015 failure for low frequency range	82
4.21	Dynamic tests for $Sh = 1$ and medium to high thickness $\ldots \ldots \ldots$	83
4.22	Dynamic tests for $Sh = 1$ and high thickness	83
5.1	Load multipliers evaluated through minimization procedure, Section 3.3.1, for whole ring thickness, λ_{1t} , and 80% ring thickness, $\lambda_{0.8t}$, superposed to experimental values, λ_{exp}	90
5.2	Two bar model for the sliding pointed arch	91
5.3	Load multipliers evaluated through minimization procedure, Section 3.3.1, for whole ring thickness, λ_{1t} , and 80% ring thickness, $\lambda_{0.8t}$, superposed to experimental values, λ_{exp} and evaluations of the two bar model	93
5.4	Determination of acceleration necessary for collapse in 3Dec	97
5.5	Load multipliers evaluated through 3Dec superposed to experimental values, Section 4.2.1, and values deduced from minimization procedure, Section 3.3.1	98
5.6	Rocking-sliding threshold for three Sharpness for increasing ring thickness	99
5.7	Determination of natural frequencies of tested specimens in 3Dec	99
5.8	Failure for experimental sine pulse input. Comparison among shake table tests, Section 4.3, values deduced from minimization procedure, Section 3.3.1, and evaluated through dynamic analyses in 3Dec for increasing Sh and $Sd = 0.15$	101

5.9	Failure for experimental sine pulse input. Comparison among shake table tests, Section 4.3, values deduced from minimization procedure, Section 3.3.1, and evaluated through dynamic analyses in 3Dec for increasing Sh and $Sd = 0.20 \ldots \ldots$
5.10	Sh02Sd015-8Hz, 3Dec modelling of the response, hinging during 1 st half cycle of motion and successive hinge moving
5.11	Sh06Sd015-8Hz, 3Dec modelling of the response, hinging during 1 st half cycle of motion and successive block twisting during 2 nd half cycle of motion
5.12	Sh1Sd015-8Hz, 3Dec modelling of the response, partial sliding and chaotic block twisting during 2^{nd} half cycle of motion $\ldots \ldots \ldots$
6.1	General view of the mosque by Pugachenkova (1968)
6.2	Plan of the Mosque
6.3	View of the still standing arches
6.4	Views of the still standing arches of the Mosque
6.5	Historical images of the mosque by Pugachenkova (1968) \ldots 113
6.6	Metal roof to protect the site work
6.7	Detail of the gypsum decoration
6.8	View of the perimeter wall
6.9	Detail of the deep crack in the west arch ring
6.10	Detail of the arrangement of bricks at crown
6.11	Detail of the arrangement of bricks in the columns
6.12	Specimens cut from sample 3
6.13	Mechanisms involving arch ring or half and full height of abutments of the arches of Noh-Gunbad mosque
6.14	Failure domains for rectangular shaped pulse Type A and B for the Noh-Gunbad Mosque
6.15	Afghan seismicity, Ambraseys and Bilham (2003)

6.16	Location of main activities in Afghanistan recently recorded by USGS, Ambraseys and Bilham (2003)	4
6.17	Preliminary hazard curves for the Mazar-e Sharif area, Boyd et al. (2007). The solid black line is the seismic hazard curve resulting from a combina- tion of all sources. The dashed-dot curves reflect contributions to seismic hazard using the ground-motion relation of Ambraseys et al. (1996). Solid curves represent Western United States ground-motion relations. The red curves are the contribution to seismic hazard from fault sources that are characteristic, and the green curves are for Gutenberg-Richter. The blue curves represent contributions from background seismicity less than 50-km depth, while the cyan curves represent contributions from seismicity between 50- and 100-km depth (solid) and 100- and 250-km	6
6.18	Strengthening interventions on the arches of Noh-Gunbad mosque, Abassi et al. (2016)	7

List of tables

3.1	Values of the Coefficient of Restitution, COR_{im} , for three values of Sh and Sd and collapse hinge layout as illustrated in Figure 3.2 49
4.1	Geometrical characteristics of reduced scale models and results of tilt tests 67
5.1	Employed stiffness and stiffness-proportional constant for Rayleigh damp- ing in 3Dec analyses
6.1	Results of mechanical tests on the Noh-Gunbad specimens
6.2	Mineralogical composition of the Noh-Gunbad specimens. *Data ob- tained through calcimetry test
6.3	Clay composition of Adobe bricks of the Noh-Gunbad mosque $\ . \ . \ . \ . \ 120$
6.4	Multiplier of loads and mechanism layout for the arches of Noh-Gunbad mosque
6.5	Safety assessment for different mechanism layouts in reinforced and unreinforced configurations of the arches of Noh-Gunbad mosque 128

Chapter 1

Introduction

1.1 Context

Conservation of Cultural Heritage constitutes one of the greatest challenges of our days, which requires a sound knowledge path of the building and the contest that produced it. Intrinsic earthquake vulnerability of historical monuments, which include "great works of art but also to more modest works of the past which have acquired cultural significance with the passing of time" (Venice Charter Art.1), is intrinsically connected with a structural or proportion design for vertical loads only. Among others, this aspect has motivated the spread of important research efforts worldwide in recent years and clearly, Italy, due to the presence of a vast cultural heritage asset and a medium seismic risk overall the territory, has naturally represented a productive environment.

As the latest seismic events in Italy have confirmed (Umbria Marche,1997; L'Aquila, 2009; Emilia Romagna, 2012; Amatrice, Ussi and Norcia, 2016), the response of ordinary and monumental historical buildings to earthquake actions is deeply influenced by masonry quality, location, scale and morphological arrangement of the building. Historic buildings that belong to the dwelling context, indeed, present compact dimensions and standardisation in shape, construction techniques and materials. Monumental structures, especially those connected to religious activities, instead, show a morphological complexity and related by-part response, elevated geometric slenderness (due to the absence of horizontal slabs) and, most important, distinctive architectural elements such as arches, vaults, domes.

Handling the mechanical response of curved structural elements attracted the interest of famous scientists from da Vinci and Galilei to Hooke and de La Hire, just

to cite a few, and is certainly still intriguing in view of contemporary computational potential.

In modern times, investigations on masonry behaviour received new attention after the famous works by Jaques Heyman in the late 60s. At present, analytical and numerical tools aimed at seismic assessment of masonry structures divide into two groups. In the first, seismic action is represented by a set of equivalent static forces exerted by inertial loads. Thus, it comprises linear and non-linear static analyses of continuously homogenised media and Limit Analysis procedures assuming rigid plastic material. The second group of tools considers a time-varying acceleration applicable as the input function for linear and nonlinear dynamic analyses of continuously homogenised media or single- or multi- DOF systems of rigid bodies.

The duality deformable continuous media or rigid plastic (no-tension) materials represents a clear indication of assumptions made on masonry quality but also of objectives of assessment, namely stability and ultimate limit state or damage and serviceability limit state.

Modern computational potentiality and surveying activities on post-seismic scenarios enable new strategies to be detected and tested. Results already available in the literature are constantly updated with new contributions revealing how many aspects in this field, such as non-linear mechanical behaviour, strengthening techniques and experimental testing, still need to be deepened.

1.2 Bounds to rigid body assumption for masonry

The intrinsic composite nature of masonry, the extreme variety of materials used, quality and texture significantly affects the mechanical performance of the overall structure. However, masonry types are clearly distinguishable, and this idea was clear to workers and architects through instructions of historical treatises and traditional know-how. Masonry quality level depends on the fulfilment grade of the *rule of art*:

- regular blocks
- horizontal bed joints
- staggering of head-joints
- presence of transversal blocks, i.e. diatono

High-quality masonry behaves as a nearly non-deformable monolith. For these masonries, units are highly stronger than joints, and the non-linear behaviour of blocks is so negligible in the overall response that rocking phenomena become relevant, deep fractures for block separation can occur, and small displacements assumption could become unsuitable. For these reasons, high-quality realisations, i.e. stone ashlar masonries, can be regarded as a system of rigid bodies connected by interfaces where deformations occur. The no-tension rigid body assumption enables to exploit limit analysis and dynamics of rigid bodies to model the seismic performance.

On the other hand, for masonries with relatively little difference between strengths of the unit and joint, e.g. brick masonries, and for masonries with low strength of units, e.g. Adobe, the assumption of a deformable medium can be useful and has motivated the spreading of a fruitful research field grounded on a micromechanics approach. Such a method considers masonry as a composite medium constituted by mortar matrix and block inclusions. For realisations with regular microstructure, a reliable assessment of linear and non-linear mechanical properties of the homogenised medium can be carried out. Nonetheless, for most of the historical masonries, the assumption of a periodic microstructure could be inappropriate and local configurations become central in the response. Thus, although recent results in this field exhibited encouraging perspectives, challenges of the micromechanical approach, e.g. definition of phases and choice of homogenization model, make it nearly prohibitive for practitioners.

The aim underpinning the assumption of a deformable continuum is the definition of adequate constitutive laws of material to adequately model damage states and related evolution, which is the fundamental objective in current structural investigations, in the so-called displacement capacity assessment. At large scale, for regular masonries homogeneously diffused, non-linear analyses on Finite Element Models constitute a reliable tool to simulate first cracks, the evolution of damage and failure, given a great modeller's expertise.

As a matter of fact, the most spread analysis tool for seismic assessment of whichever building, Finite Element Modelling, has been conceived for structures that remain connected during response and fail due to the strength of material rather than loss of equilibrium and dislocation of parts.

For masonry buildings, crushing and cracking failure is a resource exploitable prevented that rocking phenomena did not occur. In other words, failure for loss of equilibrium, mostly governed by mass velocity, is the first failure cause, which drives the stability of structures much before the in-plane damage. Thus, safety assessment through the definition of a security level can be considered the first step to being achieved.

This general trend has to be checked every time and cannot be applied straightforwardly and thoroughly, but permit a reliable specification on assessment strategies. Indeed, it is worth noting that the exposition carried out so far is just convenient for framing current approaches and is clearly not aimed at defining a good or a bad framework, but at underlining how the proper strategy has to be chosen case by case.

As previously mentioned, arched structures require an accurate and expensive building process and, most of the times, represent a recognition of the *rule of art*, thus a masonry very close to the Vitruvian *Opus Isodomum* with regularised or squared units and thin mortar joints.

In this framework, the assumption of a rigid no-tension medium has seemed the most appropriate and has been applied throughout this work. Thus, the stability of pointed arches has been investigated concerning load levels necessary for collapse, preferred failure modes and safety margins.

1.3 Motivations and objectives

A long-time literature regards safety assessment of arches subjected to point or distributed loads, although seismic actions have received much less attention. The focus of these studies is mainly the response of circular arch possibly for its apparent geometrical simplicity and due to a homogeneous diffusion in Western culture.

Nonetheless, arched structures with shapes different from circular are widespread as well, both in Europe, with Gothic style, and less significantly in Western cultures, with the Gothic revival style. As concerns non-Western constructive traditions, circular arches are instead nearly missing whereas kaleidoscopic variations of pointed arches coexist. To the best of knowledge, systematic investigations, which define preferred collapse modes, parameters influencing the response and safety assessment criteria for pointed arches subjected to horizontal actions, have not been carried out so far.

Personal concern for seismic risk mitigation of cultural heritage, lack of research on the issue and the opportunity to analyse an enthralling case study motivated the work carried out in this thesis. The research group at the Department of Architecture was involved in the investigations necessary for the conservation of an early Muslim architecture located in Balkh, Afghanistan. The unusual traits of the ruins of the Noh Gumbad Mosque can hardly be compared to any other monumental architecture for historical, aesthetical, cultural and, clearly, mechanical reasons. The still standing arched system, formed by three massive columns and two orthogonal pointed arches, used to stand out on site for the appalling conservation state due to past earthquake events. Conservation works have commenced in 2011 and ended in 2014 as regards the arched system, but the restoration site is still going on for the various operations necessary to hand this monument over to Afghan People.

Starting from investigations on this specific monument, the lack of studies on the subject has forthwith appeared clear and motivated the in-depth research reported in this thesis, which answers the question: What are the peculiarities of the seismic response of pointed arches? In detail, the objectives can be listed as follows:

- to define the influence of the constructive technique and material choice
- to define role of geometrical variations
- to test suitability of assessment methods validated for circular arches
- to define assessment criteria

Thus, a set of 48 pointed arches made of two circular arcs has been defined considering variations of geometrical profile and main constructive features. Both numerical and experimental testing have been carried out and compared. Limit analysis, dynamics of SDOF mechanisms, Distinct element models and experimental investigations were the exploited tools through permitted the definition of general and somehow unexpected considerations about main vulnerabilities. To this end, various in-house routines have been set and are intended to be further developed to become user-friendly, with the aim of releasing useful analysis tools in the practice of conservation of cultural heritage.

An experimental investigation has been wholly carried out at the official testing laboratory provided by the Department of Architecture, University of Florence. As regards the exploitation of the Distinct Element Method, the use of the commercial code 3DEC was possible at the Department of Engineering, Cambridge University during a fruitful three-months cooperation.

1.4 Outline of the thesis

The rest of the thesis manuscript is organised into five chapters. Chapter 2 is dedicated to the review of state of the art on the dynamics of systems made of one or more rigid bodies, the differences between rocking and oscillating structures, main results on the response of masonry arches subjected to horizontal actions and the statics of pointed arches.

The operational part of the thesis is divided into three chapters. In particular, Chapter 3 reports on results obtained modelling the rocking arch as a four-hinge chain with equivalent static and dynamic inputs. Evaluation of load multipliers with kinematic approach has been carried out through minimization of virtual work equation. The value of the multiplier so computed is not affected by any apriori assumption about block discretisation since the arch profile is treated as a piecewise continuous curve and hinges can occur anywhere along the profile. Evaluation of dynamic response of rocking pointed arches has been achieved through the integration of motion equations of the four-bar linkage scheme. Link dimensions were derived from mechanism layouts specified by the minimization procedure. Rocking spectra are reported for the set of arches with collapse layout arising from the minimization procedure and for the same group of arches but with fixed hinge positions. Finally, a sensitivity analysis for two kinds of inputs is proposed and commented.

Chapter 4 reports results of static and dynamic tests on a set of arches made of Autoclaved Aerated Concrete blocks. Chapter 5 deals with analytical and numerical modelling of tests through (i) investigation of load multipliers through an original kinematic model made of two hinged bars capable of sliding and (ii) using the commercial code 3Dec.

Finally, Chapter 6 and 7 are dedicated to the case study of the Noh Gumbad Mosque in Balkh, Afghanistan and to concluding remarks on main contributions of the research and future developments.

Chapter 2

State of the art

2.1 Overview

In this chapter, results of scientific research relevant to this work are organised into three sections. Section 1 reviews investigations on the rocking block with the aim of highlighting main difficulties in handling motion equations of one of the seemingly simplest rigid dynamic model that can represent the unitary masonry cell. Second section of this Chapter deals with main procedures acknowledged for the safety assessment of masonry round arches under horizontal loads in the framework of rigid body assumption and the last section lights up limited results available on the response of pointed arches under vertical loads.

A parallel and complementary research field employs a micro-mechanics approach to the masonry like materials. Homogenization techniques are continuously developing and thus it seems necessary to mention the most important trends of research, despite not being further considered in the next sections.

Assuming of masonry as a homogenised deformable medium is the assumption adopted in numerous investigations, mainly after (Pande et al., 1989) and (Suquet, 1987). In a micro-mechanics framework, masonry is a composite medium constituted by a mortar matrix with block inclusions, and homogenization techniques permit to define an averaged continuous material. For masonries with periodic micro-structure, homogenization techniques enable a reliable assessment of mechanical properties through two classes of parameters, i.e. geometry of the micro-structure and mechanical characteristics of constituents. In particular, the model proposed by Pande et al. (1989) to evaluate constitutive parameters of periodic masonries is a two-step procedure that

assumes first a transition medium homogenized through a Mori-Tanaka approach, neglecting the presence of horizontal bed joints, then, the lamination theory is applied to complete homogenization. Such a procedure is adopted also by Maier et al. (1991) and Pietruszczak and Niu (1992) to evaluate damage process of masonry. In (Anthone, 1995), assuming micro periodicity and perfect interfaces and making use of homogenization techniques in association with the finite element method, the overall elastic properties of an in-plane loaded masonry are derived from brick and mortar characteristics. Successive studies, (Cecchi and Sab, 2002; De Buhan and de Felice, 1997; Luciano and Sacco, 1997), starting from a micro mechanical analysis, focus on damage models and ultimate strength, and has constituted the basis of a recent and vast literature reviewed and critically compared in (Baraldi et al., 2014; Lourenco et al., 2007). Nonetheless, for most of masonry types, the assumption of regular micro-structure could be inappropriate, as highlighted in (Feo et al., 2016). Thus, an approach that considers statistical distribution of phases and finiteness of the investigated body becomes indispensable, as proposed in (Luciano and Willis, 2005, 2006).

2.2 Systems with unilateral constraints

2.2.1 The rocking block

A variety of structures undergoes uplift, rocking and separation under the action of strong earthquakes, as reported by early investigations, (Kirkpatrick, 1927; Milne, 1885; Perry, 1881), and this circumstance has motivated an important number of studies on the subject that have been applied also to masonry structures only relatively recently, (Ferreira et al., 2015). Rocking phenomena are inherent to masonry structures considering their discrete nature, from a micro-block perspective, and for their response characterised by local mechanisms considering a macro-block approach.

The rocking block, RB, is a rigid-body dynamic model subjected to unilateral constraints. The position of the representative block showed in Figure 2.1 can be considered through a fixed reference system Oxy to which associate unit vectors **N** and **V**. The unilateral constraint condition does not change the degrees of freedom of the system, defines a non-linear dependence among coordinates or velocities and imposes bounded variations to the instantaneous position of the body when it reaches the boundary of the allowed configurations, i.e. impenetrability of ground, (Sinopoli, 1997).

In particular, it must be assured that any point $P \in [A, B]$ of the constrained boundary has a positive component of virtual displacement along **N**, given the generalised coordinate of the system $\delta \mathbf{q}$:

$$\delta \mathbf{r}_P^N = \mathbf{N}_P \wedge \delta \mathbf{q} \ge 0 \tag{2.1}$$



Fig. 2.1 Rigid block subjected to unilateral contact, (Sinopoli et al., 1998)

A rigid body resting on ground subjected to its weight and a horizontal acceleration can initiate rocking, sliding or mixed sliding-rocking, provided sufficient acceleration and friction levels, (Shenton III, 1996).

The main feature of RB is the piecewise defined differential form of equation of motion, depending on the alternation of rotation sign, and the stiffness trend in the moment-rotation diagram, Figure2.2, which sets this dynamic model apart from a traditional moment resisting structure. In particular, RB has infinite stiffness until rocking onset, coinciding with an unbalancing moment equal to mg R sin α . From that point on, stiffness becomes negative and restoring moment decreases monotonically until reaching zero for a critical value of the rotation (coincident with the slenderness angle of the block, $\theta_{\rm cr} = \alpha = \frac{b}{h}$. The time spent to gain the limit-displaced configuration is effectively a quarter of the rocking period, considering a complete cycle of motion that experienced after an impact has occurred and the block has returned in the initial configuration.

Pure rocking motion is examined in the pioneering work by Housner (1963) assuming impossibility of sliding or bouncing and linearising the problem to small angle oscillations induced by rectangular shaped or sinusoidal pulses. The results of Housner's work uncovered (i) a scale effect so that the smaller of two geometrically similar blocks topples under an excitation for which the bigger survives and (ii) given two acceleration pulses of a certain intensity the longer one is more capable of inducing overturning. In Housner's model, at each inelastic impact (no bouncing) the block starts rocking in the opposite direction after a sudden loss in angular velocity, expressed by the coefficient of restitution and evaluated comparing the moment of momentum before and after impact. In so doing, the parameters obtained by Housner (or theoretical parameters) depend only on the block geometry and mass, and not specifically on the material of the block or the base.

The broad applicability of the rocking block model and the challenges uncovered by Housner's work boosted the proliferation of several contributions. A first group focuses on the stability of the rocking block for harmonic forcing functions dealing with both non-linear form and linearised form of the equation of motion, (EOM). In this respect, it is worth underlining that linearising EOM enables more rapid computations but can lead to errors, even for slender blocks, when slow and small amplitude forcing harmonic input are considered, (Allen and Duan, 1995).



Fig. 2.2 The rocking block and its moment rotation diagram, (Zhang and Makris, 2001)

In the seminal work by Spanos and Koh (1984), authors develop an analytical method to determine harmonic and subharmonic steady state response of slender and stocky rocking blocks and carry out stability analysis to define discretely safe regions for variation of the restitution coefficient. Tso and Wong (1989) define analytically phase angles necessary to get two main classes of harmonic and sub-harmonic steady state

modes (in-phase and out-of-phase modes) and four regions in the forcing amplitudefrequency space. Then, the stability of the computed steady-state modes is investigated through direct integration of several time histories and experimental testing,(Wong and Tso, 1989). Significant results confirm the geometrical scale effect and the impossibility of the in-phase steady state mode to be stable. Successively, Hogan (1989, 1990, 1994) determines, in the frequency amplitude space, regions leading to instability for a wide set of orbits of period nT (with T forcing period) and also the symmetry breaking bifurcations leading to period doubling and chaotic response and a relatively good agreement with previous results (Tso and Wong, 1989; Wong and Tso, 1989) is found.

The chaotic response to harmonic excitations of the rocking block becomes the focus of a series of successive investigations, given the existence of responses that could not be accounted for by the classical analytical methods. In particular, in (Yim and Lin, 1991a,b), by means of an approximated method based on Melnikov function, two added response modes, i.e. quasi-periodic and chaotic, are presented and examined besides the "classic" harmonics, as arisen from the extreme sensitivity of the linearised system to slight changes in the initial conditions and to transition of governing equations at impact. In addition, the inclusion of vertical harmonic forcing makes the quasi-periodic and chaotic responses dominant, especially for un-damped or weakly damped systems, as experimental investigations of those years, (Aslam et al., 1980; Wong and Tso, 1989), revealed.

In the framework of dynamical systems theory, Bruhn and Koch (1991) demonstrate the existence of transverse heteroclinic points in the Poincaré map and give an analytical form to determine the stable and unstable manifolds of the periodic solutions for a slender block for all system parameters. In (Iyengar and Roy, 1996), the the addition of vertical harmonic acceleration permits to define also a set of homoclinic bifurcations of the one period orbit for slender blocks in the non-linear formulation of the EOM.

In (Lin and Yim, 1996), previous deterministic investigations are extended with a approach to get the correspondent of the Melnikov distance for the slender rocking block forced with a periodic excitation and a parameter-dependant noise perturbation, which is found to increase the boundary of possible chaotic domains. To get a lower threshold in the parameter space that separated stability from the bounded chaotic response. More recently, Lenci and Rega (2006) report linearized and nonlinear versions of failure charts for the direct overturning of the block without any transient oscillation for unknown or free-from-phase-angle excitations, as occurs for earthquake excitations.

It is thus clear that an inherent characteristic of the dynamical system RB is the chaotic response, which occurs for specific combinations of *forcing characteristics* - *system parameters* and independently from geometric features of the block. Given the extreme sensitivity of stability to system parameters and forcing input for both the non-linear and the linearised version of the EOM, in a parallel investigation field, researchers often have adopted a probabilistic framework to achieve a more operative perspective in earthquake engineering.

In (Yim et al., 1980), authors propose a numerical approach to integrating the nonlinear form of the EOM through a fourth-order Runge-Kutta scheme. Input acceleration consists of a set of artificially generated ground motions built to recreate the properties of four recorded time histories scaling a white noise signal in terms of frequency content and intensity. Results of systematic integration for varying parameters enable the definition of cumulative probability functions of overturning, which are much less sensitive to the variation of system parameters.

An effective methodology to simulate earthquake input is presented in (Makris and Roussos, 2000), where authors simplify near fault ground motions to single or multiple pulse signals so that the comparison of analytical and numerical solutions of the time histories can be carried out. Zhang and Makris (2001) consider a single pulse excitation to represent near-fault ground motions, define two main response modes, namely with or without impact before the collapse, and identify the so-called temporary recovery interval in the stable zone of the rocking spectrum.

In (Sorrentino et al., 2006), a set of 20 accelerograms is employed to determine the most influent factors on overturning probability choosing among kinematic and energy parameters. Results show that the so-called scale effect is more noticeable for high amplitude signals and that the parameter best correlating overturning probability is the PGV, since it can summarize both frequency and amplitude contents of signals.

In (DeJong, 2012), with the aim of detecting systematic trends for parameter changing in a probabilistic perspective input energy necessary to maximise rocking motion is defined and maps of overturning probability for a set of earthquakes with given intensity are provided; an alternative procedure to evaluate the maximizing input energy is reported in (Casapulla, 2016).

It is worth underling that rocking motion could be difficult to activate in reality, given the possible non-perfect rigidity or slenderness of blocks, the presence of mixedmode motions and the actual face-to-face (instead of point-to-point) interface. Indeed, experimental evidence, (Aslam et al., 1980), reports poor correlation between tested and expected coefficient of restitution highlighting how other phenomena like sliding, crushing, bouncing can intervene contemporarily with rocking.

With this aim, Ishiyama (1982) comprises bouncing and sliding at impact instant assuming, in addition to standard parameters, also a tangent coefficient of restitution and static and dynamic friction coefficients for edge-to-ground and face-to-ground interfaces.

Identification of five possible response modes i.e. rest, slide, slide-rock, rock and free flight, and related analytical formulations are provided in (Shenton III and Jones, 1991a); closed form solutions for the linearised EOM for the slide-rock steady state response to harmonic forcing are reported in (Shenton III and Jones, 1991b) making use of Coulomb friction coefficient. Dynamic and static friction coefficients are included to model the slide-rock response of the single and the three-rocking block in (Augusti and Sinopoli, 1992; Sinopoli and Sepe, 1993).

In (Lipscombe and Pellegrino, 1993), a comparison on discrepancies between experimental results from other authors, (Aslam et al., 1980; Muto et al., 1960; Priestley et al., 1978), and their free-rocking tests is carried out to uncover boundaries of applicability of linearisation of motion associated with momentum-conservation model for impact, suggesting the inclusion of bouncing.

Later, Scalia and Sumbatyan (1996) assume the possibility of a bidirectional sliding and investigate on the evolution of slide-rock mode relating critical rotation value with minimum coefficient of friction. Shenton III (1996) focuses on the necessary condition to activate one of the five response modes from at rest condition. In (Pompei et al., 1998), the threshold separating the stick rocking from sliding mode are investigated in the first period response for harmonic forcing. Results highlight that friction coefficient preventing from sliding is sufficiently low for slender blocks to induce a dominant rocking response and that the value of the critical rotation angle surpassing the slenderness coefficient α is higher for the slip mode than for the rocking mode. Taniguchi (2002) extends criteria for initiation of rocking or slipping to the non-linear form of EOM and the response of a set of blocks is evaluated scaling two real earthquakes motions, for different system parameters, among which static and dynamic friction coefficients are included.

Jeong et al. (2003) include a vertical component to the forcing acceleration of the EOM of rocking blocks that can also slide. Results of a wide parametric analysis reveal that a decrease in the friction coefficient widens the chaotic response region

in Poincaré sections and that sliding, in addition to increasing the region of chaotic response, changes the shape and the dimension of the attractor.

Despite the significant research effort produced on the issue, the rocking block problem remains a challenging task and criteria for a seismic safety assessment pivoting on a robust experimental data set are still lacking. Many authors concentrated on the refinement of analytical and numerical models, while very few regarded the experimental nature of the rocking block. First attempts to test such a sensitive dynamical system, (Aslam et al., 1980; Fielder et al., 1997), showed difficulties in providing quantitative estimations of results due to a poor repeatability of the tests. In particular, in (Aslam et al., 1980), experimental investigations on a set of rocking blocks subjected to artificially generated inputs and comparison with numerically estimated results reveal how the coefficient of restitution evaluated through conservation of momentum can sensibly underestimate dissipation capacity and how hardly predictable can be the global response.

In (Peña et al., 2007), a wide experimental campaign on single, stacked and trilithonlayout blocks subjected to free rocking, harmonic and chaotic motions is reported. Results of experimental tests highlight that system parameters evaluated through the Housner's model can sensibly differ from those deduced from tests; in particular, slenderness angle is measured to be lower than theoretical value and dynamic critical rotation angle is always greater than the statically evaluated one, which turns out to offer a conservative estimation in the design process. In addition, in Peña et al. (2008) authors compare outcomes of the rocking response to DEM simulations and complex coupled rocking rotations (CCRR) method, previously proposed by same authors, (Prieto and Lourenço, 2005), which relies on the complex number definition of the independent variable to overcome drawbacks connected to the piecewise nature of the classical differential form.

In (Zhang et al., 2014), a multiple impact model, named LBZ, that incorporates flexibility effects through a distribution law and frictional response is proposed and demonstrated to be effective for rocking objects of different scales; prediction of the LZB impact model are compared with results of experimental tests from Peña et al. (2008, 2007) finding good accordance.

Recent results provided complete solutions of the 3D rocking problem (Chatzis and Smyth, 2012; Konstantinidis and Makris, 2007; Zulli et al., 2012) and influence of crushing on the response through experimental testing (Costa et al., 2013).

Finally, the seismic vulnerability of unanchored objects for pure sliding, likely to occur for stocky blocks, is investigated by Choi and Tung (2002) to determine values of sliding displacement. Results of the experimental investigation on maximum sliding displacement experienced by stocky blocks are reported by Chaudhuri and Hutchinson (2005). On this experimental basis, the same authors in (Hutchinson and Chaudhuri, 2006), develop seismic fragility curves for sliding displacement overcoming, by means of a complete representation of the considered time histories, the sensitivity recorded by Lopez Garcia and Soong (2003) when vertical forcing is couple with to horizontal.

In (Konstantinidis and Makris, 2009, 2010), static and dynamic tests on full and reduced scale laboratory equipment undergoing, planar rotation, uplift and sliding are presented. Results highlight the predominant sliding behaviour and provide refined fragility curves for sliding, in a probabilistic approach in order to define an engineering demand parameter depending on the seismic hazard level.

2.2.2 Rigid body assemblies

Compared to results available for the single rocking block, very few works analyse the rocking behaviour of systems with multiple degrees of freedom, (Ferreira et al., 2015), indeed even for the simple case of two blocks the rocking problem becomes very complex and extending the derivation to several block systems would be intensive, (D'Ayala and Shi, 2011).

The "simple" case of a two rocking blocks systems, where sliding not is not allowed is examined in (Allen et al., 1986; D'Ayala and Shi, 2011; Gabellieri et al., 2013; Kounadis et al., 2012; Psycharis, 1990; Spanos et al., 2001); equations of motion for each vibration mode are derived, criteria for initiation of rocking motion and transition between modes are given. Three stacked blocks that can only rock are addressed in (Kounadis and Papadopoulos, 2016; Sinopoli and Sepe, 1993) with an analytical approach. Instead, a multi-drums column able to rock and slide is considered in (Konstantinidis and Makris, 2005) through a commercial Discrete Element code; evaluations on the response for pulse type acceleration and full time-history are compared over the response of a dynamically similar single rocking block, highlighting the beneficial role of sliding among drums to dissipate energy.

To analyse the response of rigid body assemblies, such as low bond masonry structures efforts can be more efficiently expedited incorporating concepts of discrete element techniques, (Spanos et al., 2001; Winkler et al., 1995), and contact dynamics, (Portioli and Cascini, 2016). Indeed, the Discrete Element Method, (Cundall and Strack, 1979), constitutes a very efficient tool to implement an idealization of the discontinuous nature of masonry, which can drive the mechanical response. The method is particularly efficient to define the response of little and detailed models, such as those assembled during laboratory tests, (Lemos, 2007).

It is worth noting that the discontinuous nature of masonry can be represented also through FEM models, in a micro-modelling approach, where masonry is a continuum cut by joints. Anyway, FEM and DEM tools start from different perspectives, according to which the focus is on the extension of the element or on the boundary, respectively. Increased computation capabilities permit to include in DEM codes also deformable meshed elements, while FEM codes are constantly refining integrations schemes dedicated to interface elements. Similar names are often associated with the Discrete Element Method, such as discontinuous deformation analysis, rigid block analysis, discrete-finite elements method. However, the common underpinning idea is the possibility to model strong geometrical and physical non linearities, such as sliding or separation, (Giamundo et al., 2014; Lemos, 2007).

Seminal formulation of the Discrete Element Method has been formalised by Cundall in the Seventies, (Cundall, 1971; Cundall and Strack, 1979), with the aim of evaluating the stability of predetermined slopes in hard rock. The method proposed has been implemented in successive versions of the most spread commercial tool Universal Distinct Element (UDEC) and 3-dimensional Distinct Element Code (3DEC) distributed by ITASCA (www.itascacg.com). The first UDEC release, (Cundall, 1980; Lemos et al., 1985), was conceived for two-dimensional problems of jointed mass, and has been extended to applications in particle flow research, (Walton et al., 1988). Extensions to three-dimensional problems are considered after in (Cundall, 1988; Hart et al., 1988). The discrete element method can be defined through requirements and characteristics that set it apart form FEM, (Cundall and Hart, 1992; Lemos, 2007):

- Blocks are generally rigid, and system deformation capacity is lumped at joints
- Point contacts or edge-to-edge contacts represent interaction among blocks
- Finite displacements and rotations of discrete bodies, including complete detachment, are allowed
- New contacts can be automatically detected as the calculation progresses
• Time-stepping algorithms are employed for solving both quasi static and dynamic problems

In particular, contacts in DE models generally behave according to the so-called Soft Contact model so that normal and shear stiffness linking joint stress to block displacement are defined. However, differently from the Hard Contact model, i.e. limit analysis models, in DEM a small overlap occurs when the joints are compressed, (Lemos, 2007), Figure 2.3. Conversely, in the Non-Smooth Contact Dynamics (NSCD) model reported in (Acary and Jean, 1998) and the Discontinuous Deformation Analysis (DDA) model proposed in (Goodman and Shi, 1988) no overlapping is allowed.

For discrete modelling of masonry, normal stiffness of contacts can be addressed as an allowable penalty coefficient to be accurately fine-tuned, (DeJong, 2009), and represents deformability of mortar joints and blocks or deformability of blocks only, depending on the kind of masonry. Joint definition is complete defining a Coulomb friction criterion for shear stiffness and Rayleigh damping properties. Potentialities of the method embody its major sensitivity, indeed, numerous investigations have been carried out exploiting Distinct Element Analyses, starting from simple rocking block assemblies to simulate furniture overturning, (Peña et al., 2007; Winkler et al., 1995), to more complex masonry structures, (Azevedo et al., 2000; Drei and Fontana, 2003; Psycharis et al., 2003, 2000). However, extreme attention is necessary to set system parameters, e.g. joint stiffness, friction coefficient and damping ratio as highlighted by DeJong (2009); Dimitri et al. (2011); Lemos (1998, 2007); Sarhosis et al., 2002; Sarhosis and Sheng, 2014; Tóth et al., 2009).

2.2.3 Inverted pendulum versus oscillator

In (Priestley et al., 1978), with the aim of developing an operational procedure to define the response of rocking structures exploiting standard displacement and acceleration spectra, it is assumed that a rocking block can be represented through an equivalent single-degree-of-freedom (SDOF) oscillator with constant damping and period related to the first cycle of free rocking motion. However, closed form solutions of the rocking problem are governed by hyperbolic functions, while the oscillator obeys to trigonometric functions. In (Makris and Konstantinidis, 2003), authors disclaim the eventuality of replacing rocking structures with "equivalent" SDOF oscillators given the deep dynamic



Fig. 2.3 Possible models for joint definition in Discrete Element Methods, (Lemos, 2007), u_s , shear displacement, u_n , exaggerated overlapping, F_s , shear force, elastic and sliding component for the soft contact model, F_n , normal contact force

difference and propose the use of rocking spectra as an additional measure to the response spectra for the safety estimations of slender and rigid structures.

Also very recent studies, (Dar et al., 2013, 2016), have confirmed the shortcomings connected to approximate methods. In particular, Dar et al. (2016) highlight that even the simpli-fied method proposed by ASCE for the seismic design criteria in nuclear facilities, (American Society of Civil Engineers, 2005), to estimate the possibility of rocking of un-anchored objects leads to estimations less conservative than those obtainable through to commercial numerical tools.

Nonetheless, the simplified approach of (Priestley et al., 1978) received acknowledgements in the scientific community, justifying a series of further publications, (Doherty et al., 2002, 2000), also very recent, (Calderini and Lagomarsino, 2014; Calderini et al., 2015; Lagomarsino, 2015; Lagomarsino and Resemini, 2009), where, in some cases, authors explicitly state the limited reliability of the comparison. As for National standard codes, FEMA 356, (American Society of Civil Engineers, 2007), Italian design code, (Ministero Infrastrutture e Trasporti, 2008a,b), and Eurocode 8, (EN, 2005), foresee strength-based or displacement-based procedures, neglecting the dynamic component of the rocking mechanism and adopting, for example, a safety coefficient of 2, (Ministero Infrastrutture e Trasporti, 2008a,b), on the values of the acceleration level to withstand. Given the sensitivity of rocking structures to velocity properties of the input, approaching the safety assessment in a force-based or displacement-based framework will be always defective in tacking effectively the RB response and possibly other approaches, namely energetic, should be followed, (Sorrentino et al., 2017).

2.3 Arches under horizontal actions

2.3.1 Overview

Long-time known instructions of historical treatises link masonry quality level to the fulfilment grade of the *rule of art*, (Giuffrè, 1990; Giuffrè and Carocci, 1997, 1999; Giuffrè et al., 2000). High quality masonries, typical of monumental or representative buildings, (Caniggia and Maffei, 2001), behave as a monolith with negligible deformation states and an inherent low bond level between units and joints, (Giamundo et al., 2014). Therefore, simple structures in stone masonry, like arches, can be effectively modelled as rigid-labile system with unilateral constraints, for which a stable equilibrium state can be defined and safety margins can be evaluated as a stability parameter, (Sinopoli, 1987, 1997).

Investigation on the statics of circular masonry arches is hundreds of years old, and after the seminal works by Heyman, (Heyman, 1966, 1969), a new interest motivated the spreading of a broad literature, (Albuerne and Huerta, 2010; Foce, 2005, 2007; Gilbert, 2007; Gilbert et al., 2006; Gilbert and Melbourne, 1994; Heyman, 1969; Portioli et al., 2014; Roca et al., 2010; Sinopoli et al., 1997, 1998) and references reported therein, just to cite main review works about classical approaches and recent developments. As for the effect of horizontal actions, much less effort has been made with recent increased interest.

Given a structural system characterised by multiple rigid bodies, earthquake loading can be assumed through an equivalent static approach, i.e. applying a set of equivalent static forces, or through a dynamic approach, imposing a time dependant acceleration input. Therefore, the literature review reported hereafter is so organised.

2.3.2 Equivalent static input

The determination of the equivalent static acceleration necessary for an arch to form a mechanism can be treated in the framework of limit analysis as proposed by Heyman, (Baker and Heyman, 1969; Heyman, 1966; Horne, 1979), which considers:

• Infinite compressive strength (for masonry structures compression level ranges around a tenth of compressive strength)

- Nil tensile strength (inherently true for dry blocks masonry and a safe assumption for all other kinds of masonry)
- Impossibility of sliding among blocks (infinite friction coefficient)
- Load set linearly increased by a load factor, λ , (no random change allowed)
- Displacements experienced by the structure are small enough not to change the geometry of the structure
- Enforcement of virtual power equation permit the determination of the load factor λ

Furthermore, according to the *Safe Theorem* of limit analysis (static or lower bound approach), among the set of admissible virtual displacements, if the yield condition is satisfied on each point of the structure, the limit condition of equilibrium is associated with the maximum value of the load factor, $\lambda \leq \lambda_c$, and zero virtual work.

The Unsafe Theorem (kinematic or upper bound approach) states that among the set of admissible mechanisms of the structure, the collapse condition is associated with the mechanism with lowest load factor, $\lambda \geq \lambda_c$, and zero virtual work.

The Uniqueness Theorem associates any value of the load factor λ , for which equilibrium, mechanism and yield conditions are satisfied, the unique collapse load factor, λ_c . Uniqueness theorem enables the evaluation of the collapse factor either through one of the approaches.

Main potentiality of the static approach is the direct definition of the thrust in structure, but it requires an iterative procedure. The major drawback of the kinematic approach is that hinge localization depends on discretization in blocks, raising a tradeoff between precision and computational effort (Sinopoli et al., 1998). However, in the analysis of damaged structures, kinematic mechanisms directly correlate observed fracture patterns.

Heyman's constitutive model of material can be represented through a uni-axial law as shown in Figure 2.4a. The elementary masonry cell is constituted by two blocks of height h and related admissible forces are defined by a vector $\boldsymbol{\sigma}^T = [V, N, M]$; $\boldsymbol{\sigma} \in \mathbb{R}^3$. Stress vector, $\boldsymbol{\sigma}$ should lie within the space region defined by two planes $\boldsymbol{\Pi}$, orthogonal to the plane $\boldsymbol{V} = 0$, whose traces are $M = \pm h/2N$, since any shearing force can act on blocks without induce sliding, (Como, 2010), Figure 2.4b.



(a) Uni-axial constitutive law. Infinite com-(b) Elementary resisting cell. Admissible pressive strength, nil tensile strength and displacements among blocks: complete seponly positive strain allowed, i.e. interpene- aration and relative rotation, and related tration among blocks cannot occur admissible forces

Fig. 2.4 Behaviour of the rigid no-tension material, (Como, 2010)

The virtual contact law, which governs both the static and the dynamic problem, from which one can determine the stability of equilibrium, (Sinopoli et al., 1998), requires a positive component to the interface of the virtual displacement of the generic point P, as sated by Equation 2.1. Thus, admissible virtual displacements between two blocks are only those in the direction of the outward normal to the interface.

When the assumption of infinite friction is removed, the use of linear programming algorithms and the uniqueness of the solution is still possible assuming a linearisation of the yielding criterion in an associated flow rule framework, (Drucker, 1953a,b; Kooharian, 1952; Livesley, 1978), which implies convexity of the yielding surface and normality between generalised strains and yielding surface. However, experimental and numerical tests, (Lourenço and Ramos, 2004), have showed that the associated flow rule assumption can lead to overestimations, especially for large structures. Therefore, with the aim of framing a general approach for masonry walls, in (Baggio and Trovalusci, 1998; Fishwick, 1996) the assumption of an associated flow rule is removed and the problem is directly solved as a non linear constrained optimization, which turns out to be particularly stiff to be handled when the number of blocks heightens. Instead, the novel approach proposed in (Ferris and Tin-Loi, 2001), considers complementarity constraints, which substantially solve the limit analysis problem of a set of frictional rigid blocks considering contemporaneously positivity of the work done, compatibility, equilibrium conditions and the constitutive law.

Standard limit analysis problems applied to masonry arches, considering the limited number of blocks representing the problem, can be addressed as a linear programming problem, (Charnes et al., 1959; Dorn and Greenberg, 1957), thus exploiting robust algorithms, e.g. Simplex, as applied in (Caporale et al., 2006; Gilbert and Melbourne, 1994) for the case of vertical loads.

The hinging mechanism state for an arbitrarily loaded arch is reached when a set of hinges, capable to transform the originally redundant structure (single span arch has three redundancies) in a one degree of freedom mechanism, forms. The condition for the activation of a mechanism induced by self-weight and a set of horizontal forces, proportional to self-weight and affected by a multiplier, occurs when the thrust line becomes tangent to the arch profile on four points, resulting in a mechanism of three blocks and four alternate (intrados – extrados) hinges, (Foraboschi, 2001), Figure 2.5. For symmetric structures symmetrically loaded, five hinges are necessary to activate a mechanism. Load configuration causing the occurrence of the mechanism is indeed the ultimate load and the related thrust line and mechanism configuration are the only possible, (Clemente, 1998; Sinopoli et al., 1998).



Fig. 2.5 Collapse mechanism for a circular arch, (Clemente, 1998)

In (Clemente, 1998; Raithel, 1998), authors conduct a parametric study on circular arches subjected to constant acceleration input through successive iterations of the solution of the mechanism problem and evaluating the related thrust forces. In so doing, it has to be checked whether the trust line is effectively passing through the supposed hinges, if not, sections which maximise the distance between the thrust line and the arch profile become the new position of hinges. Main results on the issue can be summarised as follows:

• Load factor λ increases directly with the increase in ring thickness

- Load factor λ increases directly with the decrease of the so-called embrace angle,
 i.e. angle subtended by the arch profile
- Last conclusion enables to assume that any sub-mechanism cannot form within each rotating macro-block
- One of the extremal hinges always forms at one of the abutment

In (Foraboschi, 2001), a model for the description of the response of a masonry arch subjected to a seismic action characterised by both horizontal and vertical components, proposing a mechanism approach for the solution of the macro-block problem. However, any direct application is available to compare results of this method with (Clemente, 1998).

In (De Luca et al., 2004), the minimization problem is by-passed exploiting outputs of linear FEM analyses on a set of circular arches and systems arch - abutment. Tension distribution suggests the area where possible positions of hinges can be found. Then, through an iterative procedure carried out on an CAD basis, it is possible to reduce hinge positions to a reduced group, constituting the basis of a novel simplified method for the estimation of the kinematic multiplier. Such a procedure cannot bring to the estimation of the minimum of the multipliers, as stated by authors, but gives a value reasonably near to the lowest, actually turning out to be the potentiality of the method.

More recently, in (Alexakis and Makris, 2014), authors apply a variational formulation of the principle of stationary potential energy to define minimum ring thickness of circular to sustain horizontal actions, finding out that the value is influenced by the direction of the ruptures in the ring, namely radial or vertical.

2.3.3 Dynamic input

Extension to acceleration inputs forcing an arch essentially is owed to the seminal paper by Oppenheim, (Oppenheim, 1992), where a four-bar linkage mechanism represents the circular arch mechanism, similarly to (Allen et al., 1986) for rectangular portals. By means of Oppenheim's model, the whole oscillation cycle of a rocking arch is analysed and refined in (De Lorenzis et al., 2007; DeJong, 2009; DeJong et al., 2008) taking into account the coefficient of restitution affecting rotational velocity after impact. Recently, in (De Santis and de Felice, 2014), the four-bar scheme is exploited to validate a fibre-beam approach for the investigation of a seven-span bridge. The four-bar linkage Erdman and Sandor (1997) is a closed chain linkage that can be usefully exploited to investigate the response of a four-hinge mechanism of an arch. The method, firstly proposed in Oppenheim (1992), is then applied in De Lorenzis et al. (2007); De Santis and de Felice (2014); DeJong et al. (2008); DeJong and Dimitrakopoulos (2014) on circular arches. The scheme of Figure 2.6 represents the three rotating blocks as rigid links.



Fig. 2.6 Four-bar linkage applied to a circular arch: l_{-} , length of links connecting hinges; θ_{-} slope of the links; r_{-} , massless bars connecting hinges to centres of mass of each macro-block; ψ_{-} slopes of r_{-} with respect to l_{-} ; F_{i} and F_{iv} , reactions exerted by impulsive forces assumed on the corner of new hinges forming after an impact has occurred. Arabic numbering of lower cases refers to configuration before impact, Roman numbering to after impact

In (Oppenheim, 1992), no evluation of arch motion during the second half cycle of motion is carried out, thus avoiding to model impact at interfaces. On the other hand, in (De Lorenzis et al., 2007; DeJong et al., 2008) an impact model for the rocking arch is proposed, applying the principle of impulse and momentum (Kane and Levinson, 1985; Shenton III and Jones, 1991a) so deriving a conservative evaluation of the coefficient of restitution, e.

Recently, the model proposed by Oppenheim, (Oppenheim, 1992), and exploited in (De Lorenzis et al., 2007; DeJong, 2009; DeJong et al., 2008) is considered also in (DeJong and Ochsendorf, 2010), where authors confirm that when the content of a time history has a clear distinct pulse, prediction of the model proposed by Oppenheim are accurate. However, when the content of the time history develops through successive pulses, these can have an amplifying effect on the rocking motion. Considered the sensitivity of the response to small changes of the series of pulses, a statistical approach becomes necessary to evaluate the probability of overturning for a set of ground motions of an expected intensity.

Differently from the Oppenheim-based model, in (Sinopoli, 2010), a refined semianalytical model for the assessment of the onset of motion of a rocking macro-block arch with unilateral constraints is presented. However, main difficulties encountered for a broad application of the method are: sensitivity in the choice of the Lagrangian parameter governing motion, necessity of solving the sub-problem for friction forces at contacts in a static framework to re-evaluate the successive step of the dynamic solution.

2.4 The statics of pointed arches

Exposition on the evolution of pointed arches as architectonic element, diffusion among cultures with related mutations in shape and constructive methods, (Shelby, 1969), is beyond the scope of this work, nonetheless, the few considerations reported hereafter underline the complexity of a research field that still needs further investigation.

In western countries, pointed arches are generally connected with Gothic Style, although its exact origin is an issue of ongoing historical research (Mark, 1982). However, more recent investigations, (Creswell, 1989), relate the origin of this architectural to the pre Islamic Sassanid Great Persia (Second Persian Empire, 224-651 A.D.) and it was only at the beginning of the VIII Century that this shape was transferred to the Islamic Umayyad world.

Two early Umayyad examples are dated back between 706-715 A.D., few years after Mahomet death: the Great Mosque of Damascus in Syria, Figures 2.7c, 2.7d, and the Qusayr 'Amra, (namely the little palace of Amra) in the North East part of Jordan, Figures 2.7a, 2.7b. These early examples does not differ a lot from a circular shape. The distance between the centres of two circular arcs is evaluated between 1/10 and 1/20 of the span and the centres of the arcs are raised from the springing plane (Creswell, 1989; Warren, 1991).

In this first dissemination period in addition to two-centred arches, four-centred arches appeared, like the Baghdad Gate in Raqqa (Syria, 772 A.D.), 2.8. The Gothic



(a) Outside view of Qusayr 'Amra, (b) Mosaic detail of Qusayr 'Amra,
 Jordan, image courtesy of Kenneth Jordan, image courtesy of Kenneth
 Zuckerman, (Zuckerman, 2016)
 Zuckerman, (Zuckerman, 2016)



(c) View of the Great Mosquée of (d) Great Mosquée of Damascus, Damascus, Syria, image courtesy of Syria, image courtesy of Nasser Rab-Bernard Gagnon, (Bernard, 2016) bat, (Rabbat, 2016)

Fig. 2.7 Early examples of pointed arches

style inherited the potentiality of this shape, which enables minor thrusts, nearly four centuries after.



Fig. 2.8 Baghdad Gate, Raqqa, Syria, image courtesy of Groundhopping Merseburg, (Merseburg, 2016)

Comprehensive historical treatises on Gothic style, (Ungewitter, 1890), report on constructive techniques for pointed arches. Geometric discontinuity at the keystone, Figure 2.9, used to be obtained through different constructive techniques in different stylistic periods. The simplest and roughest solution, implemented in brickwork, fills in the keystone space with smaller bricks and thicker joints, 2.9a. Sophisticated solutions, implemented in stonework, exploited stereotomy. The keystone used to be modelled such that left and right faces were sloped radially and intrados and extrados faces were effectively carved, 2.9b. Alternatively, two mirrored voussoirs were carved to fill in the space comprised by the last radial slope and the vertical joint, actually attaining two keystone voussoirs 2.9c.

The general idea that pointed arches could be sustained by thinner buttresses and induce lower thrusts was clear since the 16th Century, (Romano and Ochsendorf, 2010) and references reported therein. In modern times, very few works address the analysis of pointed arches resulting from the composition of two circular arcs. In (Romano and Ochsendorf, 2010), minimum ring thickness and extremal values of thrusts induced by vertical point loads at keystone or at haunches, or by relative displacements of abutments are investigated for a set of pointed arches with changing geometry. To this end, a linear elastic analysis is applied in parallel with a graphical method supported by a dedicated software tool, and results are verified through a wide experimental campaign on dry concrete block models.



(a) Littler bricks and thicker joints (b) Stonework with a unique carved for brickwork voussoir



(c) Stonework with two mirrored voussoirs

Fig. 2.9 Constructive solutions for keystone in pointed arches, (Ungewitter, 1890)

In (Aita et al., 2011), a system formed by a vertical wall supported by a pointed arch is compared with similar systems made of circular and elliptical arches. Strain and stress distributions at collapse are evaluated using 1-D non-linear elastic analysis under the assumption of a perfect elastic-plastic constitutive relation. Load bearing capacities are compared against those obtained through the graphical method of the stability area. The method, which takes into account finite compressive strength of the material and finite friction among blocks, is previously specialized for the case of pointed arches in (Aita et al., 2004).

According to (Aita et al., 2011, 2004; Block et al., 2006; Romano and Ochsendorf, 2010), pointed arches withstand greater thrusts and greater abutment displacements than comparable arches with the circular profile. Despite the broad diffusion of pointed arches in the built cultural heritage of seismic prone areas, no work in the literature investigates the effect of the pointed profile on the dynamical behaviour.

Regarding non-circular profiles in general, e.g. onion shaped, ogee, four-centred and semi-elliptical, these are investigated through FEM analyses in (El-Mahdy, 2014) focusing on the problem of in-plane buckling, and in (Pouraminian et al., 2014) with the aim of evaluating the response for specific time histories.

2.5 Summary

The dynamics of a rigid block and related impact issues have attracted a long time attention of researchers in the field of earthquake engineering. This seemingly basic model turns out to be not simple at all to be handled. Investigations on the rocking arch, on the contrary, pivots mainly on the analytical model proposed by Oppenheim in the early nineties and the response of pointed arches to vertical loading has received minor attention.

Basic conclusions can be summarised as follows:

- Systematic trend in the response of the rocking block can be effectively tackled only in a probabilistic sense.
- Modelling multiple impacts with friction even for the simple rocking block model can be tough and results of research are constantly updated.
- The rocking problem, requiring integration of motion equations, is computationally expensive and not actually implemented among practitioners.

- The equivalence between a SDOF oscillator and a rocking block is inconsistent, methods based on this assumption, making use of elastic spectra will always be inherently limited. They should be put aside in favour of rocking spectra based methods, which can provide a clear idea of the kinematic characteristics of ground motions and related implications on the response of rigid structures.
- The study of the circular arch bearing horizontal loads can be handled through force- or displacement-based methods, which are useful for practitioners for their simplicity, but highly conservative and unsuccessful in working on the dynamic scale effect, even when high safety coefficient are assumed.
- The main dynamical model for the investigation of the rocking arch limits the qualitative response to perfect hinging mechanism.
- For circular rocking arches, failure for pulse type accelerations always occurs after an impact, given the assumption of conservation of momenta, the scale effect yield littler arches to be more vulnerable of similar bigger ones.
- Pointed arches can be thinner than circular and transmits lower horizontal thrusts to abutments, but the phenomenon is not related directly with Sharpness.
- Pointed arches can sustain larger abutment displacements in comparison with circular profiles and greater superimposed loads, especially when placed at crown.
- Removing the assumption of infinite compressive strength enables a more conservative estimation of the value of the collapse load.

Chapter 3

Analytical modelling of pointed arches forced by horizontal actions

3.1 Overview

This chapter reports a parametric analysis on the response to horizontal actions of pointed arches made of two circular arcs. The analysis considers variations of arch slenderness and sharpness that result from different positions of centres of circular arcs.

First, the arch is considered as a rigid macro-block system, and limit analysis with the kinematic approach is exploited to determine the acceleration necessary to initiate motion through Non-Linear Programming optimisation.

Then, dynamic response of pointed arches for rectangular shaped and harmonic inputs are investigated transforming arch mechanisms into four-bar linkages and integrating linearised a non linear forms of motion equations.

Last, a sensitivity analysis validates the procedure predicting the failure of circular arches as a particular case of pointed.

3.2 Geometrical model

The pointed arch considered here is the simplest way to conceive a discontinuous curvature, it is characterised by a shape obtained by two arcs of circumference, with symmetric centres on the springing plane (Figure 3.1). Eccentricity, e, measures the

distance between the symmetry axis and the centre of each of the circumferences, and it determines how sharp the arch is with respect to a circular profile. Sharpness, $Sh = e/R_c$, is the ratio of the eccentricity over the corresponding round arch radius, and slenderness is $Sd = t/R_p$, where t is ring thickness and $R_p = e + R_c$ is radius of pointed arch. The analytical description of this geometry model considers two piecewise continuous curves, which define an arch of thickness t, eccentricity e, spanning a $(2R_c-t)$ length, as showed in Figure 3.1.



Fig. 3.1 Geometrical model of the pointed arch. Shape parameters: R_p , mean radius of the pointed arch; e, eccentricity; t, thickness; R_c , radius of the corresponding circular arch. Parameters of the analytical model: R_e and R_i , extrados and intrados radii; $E(\beta)$, extrados profile; $I(\beta)$, intrados profile; O, principal reference system origin; O_L and O_R local reference systems origins. β_n , δ_n and γ_n angles identifying hinge positions and plane that separates two blocks; ρ_j polar distance identifying the centre of mass, G_j ; K surface of half keystone defined by ring thickness varying from t_v to t_r

Given a polar reference (ρ, θ) centred at O, the arch profile is defined by four circumference arcs, two are centred at $O_R = e, 0$ and two are centred at $O_L = e, \pi$ for the left and the right sides respectively. Then, $R_e = R_p + t/2$ and $R_i = R_p - t/2$ represent extrados and intrados radii expressed as function of the pointed arch radius $R_p = R_c + e$, Figure 3.1. Equations 3.1 and 3.2 define the intrados $I(\beta)$ and $E(\beta)$ extrados curves of the pointed arch:

$$I(\beta) = \begin{cases} \sqrt{\cos^2(\beta)e^2 - e^2 + R_i} - e\cos(\beta) & 0 \le \beta \le \frac{\pi}{2} \\ e\cos(\beta) + \sqrt{\cos^2(\beta)e^2 - e^2 + R_i} & \frac{\pi}{2} < \beta \le \pi \end{cases}$$
(3.1)

$$E(\beta) = \begin{cases} \sqrt{\cos^2(\beta)e^2 - e^2 + R_e} - e\cos(\beta) & 0 \le \beta \le \frac{\pi}{2} \\ e\cos(\beta) + \sqrt{\cos^2(\beta)e^2 - e^2 + R_e} & \frac{\pi}{2} < \beta \le \pi \end{cases}$$
(3.2)

Any point $P_n = (\rho_n, \beta_n)$ along $I(\beta)$ or $E(\beta)$ can become the location of a hinge. Two subsequent alternate (intrados/extrados) hinges comprise a block, whose central angle γ , i.e. angle comprised between the directions identified by the block hinges and points O_R or O_L , see Figure 3.1, is conveniently referred to local reference systems centred at O_R or O_L since the lying planes of the blocks are radial as an intrinsic characteristic of standard constructive techniques. Horizontal shifting between local and global reference systems is accounted for when the location of a hinge $P_n = (\rho_n, \beta_n)$ for $n = 1, \ldots, 4$ is to be related to block dimensions – radial sectors, through the δ_n angles, differently defined for odd and even hinges. Let $\gamma_n = \beta_n - \delta_n$ be the slope of the plane that separates two subsequent blocks in the local reference system, then for the case displayed in 3.1, first hinge is place at extrados and expressions of δ_n angles are:

$$\beta_n(\delta_n) = \cos^{-1}\left(\frac{-e^2 + R_e^2 + E(\beta_n)^2}{2R_e E(\beta_n)}\right) \qquad n = 1,3$$
(3.3)

$$\beta_n(\delta_n) = \cos^{-1}\left(\frac{-e^2 + R_i^2 + I(\beta_n)^2}{2R_i I(\beta_n)}\right) \qquad n = 2, 4$$
(3.4)

where, R_e and R_i are extrados and intrados radii and e is the eccentricity of the arch. Same expressions with inverted values for *n*-th indexes are valid when first hinge is placed at intrados. Angle δ_n varies with β , is null at the springing plane and maximum at the keystone. The keystone embodies a discontinuity in the profile and a variation in the thickness of the arch between the last radial slope before keystone, i.e. $t = t_r$, and the vertical section at keystone, t_v . Half of the keystone is a portion of a circular sector and its area, K, comprised by t_r and t_v is given by:

$$K = \frac{1}{2}R_e^2 \left(\delta_{max} - \sin^{-1}\left(\frac{e}{R_e}\right)\right) - \frac{1}{2}R_e R_i \sin\left(\delta_{max} - \sin^{-1}\left(\frac{e}{R_e}\right)\right)$$
(3.5)

where $\delta_{max} = \delta_n(\pi/2)$ is the value of δ at the keystone, Figure 3.1.

3.3 The pointed rocking arch

3.3.1 Onset of motion for equivalent static actions

An arch mechanism activates when a sufficient number of cracks appear, such an instance is investigated for a set of horizontal forces, proportional to self-weight and affected by a multiplier, using the assumptions of limit analysis. In the limit condition of equilibrium, i.e. when the thrust line becomes tangent to the arch profile on four points, the arch transforms in a one degree of freedom mechanism, consisting of three blocks and four alternate (intrados – extrados) hinges, (Clemente, 1998; Heyman, 1969; Sinopoli et al., 1998). Load configuration causing the occurrence of the mechanism is indeed the ultimate load and the related thrust line and mechanism configuration are the only possible.

Referring to the model reported in Figure 3.1, the mass of the *j*-th block $(j=1, \ldots, 3)$ is derived by subtracting the mass of a block computed from the springing plane to the *n*-th+1 hinge $(n=1,\ldots,4)$ to the mass of a block computed from the springing plane to the *n*-th hinge. When the keystone is included in a block, the mass is the sum of the mass of circular sectors and the mass corresponding to a 2K surface, i.e. the complete keystone.

The multiplier of inertial loads is found adopting the uniqueness theorem of limit analysis, (Heyman, 1966), through minimization of the equation of virtual work. The value of the multiplier so computed is not affected by any a-priori assumption about block discretisation since the arch profile is treated as piecewise continuous and hinges can occur anywhere along the profile. In the limit condition of equilibrium, virtual work done by inertial forces equals virtual work done by vertical forces so that a function $\Lambda(\beta)$, dependent on the geometry of macro-blocks can be set:

$$\Lambda(\boldsymbol{\beta}) = \rho \quad \frac{\sum_{i=1}^{3} m_i(\boldsymbol{\beta}) \eta_i(\boldsymbol{\beta})}{\sum_{i=1}^{3} m_i(\boldsymbol{\beta}) \zeta_i(\boldsymbol{\beta})} \qquad \Lambda : \mathbb{R}^4 \to \mathbb{R}$$
(3.6)

$$\eta_{G1} = \vartheta(X_1 - x_1)$$

$$\eta_{G2} = \vartheta \frac{x_2 - x_1}{x_{2a} - x_2} (X_2 - x_{2a})$$

$$\eta_{G3} = \vartheta \frac{x_2 - x_1}{x_{2a} - x_2} (X_2 - x_{2a}) \frac{x_3 - X_{2a}}{x_4 - x_3} (X_3 - x_4)$$
(3.7)

$$\zeta_{G1} = \vartheta(Y_1 - y_1)$$

$$\zeta_{G2} = \vartheta \frac{y_2 - y_1}{y_{2a} - y_2} (Y_2 - y_{2a})$$

$$\zeta_{G3} = \vartheta \frac{y_2 - y_1}{y_{2a} - y_2} (Y_2 - y_{2a}) \frac{y_3 - y_{2a}}{y_4 - y_3} (Y_3 - y_4)$$
(3.8)

In Equation 3.6, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4), \beta \in \mathbb{R}^4$, ρ is the mass density, m_i is the mass of the *i*-th block and η_i and ζ_i are the *i*-th vertical and horizontal displacements respectively, expressed in terms of virtual rotation ϑ of the first body by virtue of rigid kinematics. In Equations 3.7 and 3.8, x_i or y_i stands for abscissa or ordinate of the hinge indicated in lower case, and X_i or Y_i indicates abscissa or ordinate of the centre of gravity of the block indicated in lower case. The lower case 2a is referred to the position of absolute centre of rotation of second macro-block, identified by the intersection of line through first and second hinge and line through fourth and third hinge. The minimum value of the function $\Lambda(\beta)$ that leads to a compatible collapse mechanism is stated by the following non-linear optimization problem:

$$\lambda = \min \left(\Lambda \left(\boldsymbol{\beta} \right) \right)$$

s.t. $\beta_{i-1} < \beta_i < \beta_{i+1}$
 $0 \le \beta_i \le \pi$
 $\beta_i < \beta_j; \quad (i < j)$
 $\beta_2 < \pi/2$
 $\eta_3 \ge \pi/2$
 $\eta_1 \le 0$
 $\eta_3 \ge 0$
(3.9)

Constraints to the optimization were derived by compatibility conditions of mechanism. In particular, the constraints that the virtual displacements of the first block were negative with respect to the reference system of Figure 3.1, implies the clockwise rotation of first block, which causes positive work for both vertical and horizontal forces. The minimization of the objective function, Equation 3.9, was carried out in Mathematica, (Wolfram Research, 2016), exploiting the method proposed by Nelder and Mead (1965), which is based on the use of simplexes exploiting function values without using any approximate gradient. A simplex $S \in \mathbb{R}^n$ is defined as the convex hull of n + 1 vertices, thus a simplex in \mathbb{R}^2 is triangle and a simplex in \mathbb{R}^2 is a tetrahedron.

To minimize a function in *n*-variables, let n + 1 points $P_0, P_1, \ldots, P_{n+1} \in \mathbb{R}^n$ define the vertices of the starting simplex and $y_j := f(P_j)$ for $j = 0, \ldots, n$ with the related function values at the vertices. At each iteration, points are ordered so that

$$y_1 = f(P_j) \le y_2 \le \ldots \le y_{n+1}$$

The worst point, P_{n+1} , is substituted with a new point using reflection, contraction or expansion operations. Set c the centroid of each iteration simplex, reflection of P_{n+1} is given by

$$P^* = (1+\alpha)c - \alpha P_{n+1}; \quad \alpha = \frac{[P^*c]}{[P_{n+1}c]}; \quad \alpha > 0$$

where, α is the reflection coefficient and square brackets stand for distance.

Then, if $y_l \leq y^* \leq y_{n+1}$, P^* replaces P_{n+1} , a new simplex is generated and a new iteration starts. If $y^* \leq y_1$, it means that the reflection found a minimum, and then P^* has to be expanded, through the following expression:

$$P^{**}=\gamma P^*+(1-\gamma)c;\quad \gamma=[P^{**}c]/[P^*c];\quad \gamma>1$$

where, γ is expansion coefficient.

If $y^{**} < y_1$, P_{n+1} is replaced by P^{**} , a new simplex forms and a new iteration starts. If $y^{**} > y_1$ expansion failed, and P^* has to replace P_{n+1} . If, during reflection operation $y^* > y_i$ $i \neq n+1$, then the simplex has to be contracted and the contracted point is:

$$P^{**} = \mu P_{n+1} + (1-\mu)c; \quad \mu = [P^{**}c]/[P_{n+1}c]; \quad 0 \le \mu \le 1$$

where, μ is the contraction coefficient. Then, P^{**} is accepted unless it is worse than the better between P* and Pn+1. If contraction fails, all points are substituted with $\frac{(P_i+P_1)}{2}$, for $i = 1, \ldots, n+1$.

Results of minimization routines exposed in the following, values of reflection, expansion and contraction coefficients are, $\alpha = 0.95$, $\gamma = 2$ and $\mu = 0.95$.

For each arch considered in this work, 10 sets of trial points have been chosen in the 4-dimensional space of $\Lambda(\beta)$. Each minimization encountered 10 starting simplexes randomly and simultaneously generated enabling a comparison among results of ten optimizations. In Appendix A the implemented lines in Mathematica are reported.

Results

A set of 48 arches has been considered in this analysis, with span ranging 5m, 10m and 20m, sharpness ratio, $Sh = e/R_c$, equal to 0.2, 0.6 and 1, and slenderness ratio, $Sd = t/R_p$, equal to 0.1, 0.15, 0.2 and 0.25. Minimizations have been carried out and results, displayed Figure 3.2, are plotted in terms of collapse load multiplier as function of Sharpness for different values of Slenderness, trend of multiplier of loads for the corresponding circular arch, i.e. Sh = 0, is displayed with filled pointers.



Fig. 3.2 Multiplier of loads for varying thickness for circular and pointed arches

Thicker arches are more shape dependant in their vulnerability to horizontal loads than those with slender shapes.

Figure 3.3 reports the angles identifying the positions of the four collapse hinges as a function of the slenderness for three values of sharpness. For each value of slenderness, the vertical distance between pointers in each plot makes explicit the radial width of each block and the total width of the embrace angle. Each plot also shows the trend of the multiplier of loads (dotted line). Grey continuous lines represent the position of hinges for circular arches, deduced from results of Clemente (1998). As a general trend, it is clear how last hinge does not break away from the springing plane, as occurs also for circular arches, Clemente (1998), and the first hinge occurs at a location between 0 and 0.42r.

Concerning Sh = 0.2, Figure 3.3a, for increasing thickness, the distance between the second and the third hinge, i.e. the width of the second block, is almost invariant with mean value 0.928r and comparable with the second block of a circular arch, which ranges between 0.8r and 0.93r, as reported by Clemente (1998).

For Sh = 0.2 and 0.6, Figures 3.3a and 3.3b, the slope of the multiplier grows with the increase in the angle of embrace, with the sole exception of the case Sd = 0.12. Thus, as opposed to the case of circular arches, the increase of the embrace angle does not directly affect the vulnerability of a pointed arch.

For Sh = 0.6 and 1, Figures 3.3b and 3.3c, and for high values of ring thickness (Sd = 0.25), the load multiplier increases with the decrease in the angular width of the second block. Indeed, a smaller second block yields a lower mass at the highest point, which in turn decreases the vulnerability.

For Sh = 1 (Figure 3.3c), the effect of the reduction of the second block width adds to the effect of the reduction of the embrace angle, resulting in the highest increase rate of the multiplier.

3.3.2 Dynamic response for acceleration inputs

Overview

Dynamic response for rectangular shaped and harmonic acceleration inputs have been investigated by transforming the mechanisms as derived from Section 3.3.1, in four-bar linkages. To effectively highlight the central role played by slenderness and sharpness avoiding the influence of hinges positions, a set of arches with geometrical variations but a common position of hinges has been considered as well.

Failure domains (acceleration – time and acceleration – frequency) are determined and compared with those of circular arches and a sensitivity analysis of the response of



Fig. 3.3 Hinge positions, β_n , versus Sd; trend of load multiplier, λ , in dotted lines, and position of hinges for circular arches, (Clemente, 1998), in grey continuous lines. Circles, triangles and squares represent the position of the first, second and third hinge respectively



Fig. 3.4 Four-bar linkage applied to pointed arch: l_{-} , length of links connecting hinges; θ_{-} slope of the links; r_{-} , massless bars connecting hinges to centres of mass of each macro-block; ψ_{-} slopes of r_{-} with respect to l_{-} ; F_{i} and F_{iv} , reactions exerted by impulsive forces assumed on the corner of new hinges forming after an impact has occurred. Arabic numbering of lower cases refers to configuration before impact, Roman numbering to after impact

circular and pointed arches on coefficient of restitution is illustrated for both rectangular shaped and harmonic inputs.

The four-bar mechanism, (Erdman and Sandor, 1997), is a closed chain linkage that can be usefully exploited to investigate the response of a four-hinge mechanism of an arch.

The scheme of Figure 3.4, represents the three rotating blocks as rigid links. Links connecting hinges of lengths l_1 , l_2 , l_3 are assumed to have no weight and the masses of blocks, m_1 , m_2 , m_3 , are concentrated at block centres of mass and connected to the pivot points 1, 2, and 4 through fixed rigid links of length r_1 , r_2 , r_3 . Hinges 1 and 4 are joined one to another and pinned at ground.

Motion of the four-bar linkage is a single degree of freedom, thus one independent generalized coordinate governs motion. Specifically, rotation of any of the links can be taken as the generalised coordinate, (Arnol'd, 2013; Meirovitch, 1975). Here, rotation of link l_1 , expressed by $\varphi[t] = \theta_1[t] - \theta_{1,0}$ is considered as generalised coordinate, where, $\theta_1[t]$ is the slope of the first link at generic instant and $\theta_{1,0}$ is the slope at equilibrium position.

The rotations of the other two blocks are evaluated through direct trigonometric relations. Specifically, the transmission angle relates the rotation of the coupler link, l_2 , to the rotation of the follower link, l_3 , omitting the dependence on time, it yields:

$$\boldsymbol{k}\left[\boldsymbol{\theta}_{1}\right] = \cos^{-1}\left(\frac{(l_{1}l_{4})\cos\left(\boldsymbol{\theta}_{1}-\boldsymbol{\theta}_{4}\right)}{l_{2}l_{3}} + \frac{-l_{1}^{2}+l_{2}^{2}+l_{3}^{2}-l_{4}^{2}}{2l_{2}l_{3}}\right)$$
(3.10)

$$\boldsymbol{\theta_2}\left[\boldsymbol{\theta_1}\right] = \theta_4 - \tan^{-1} \frac{l_1 \sin\left(\boldsymbol{\theta_1} - \theta_4\right)}{l_4 - l_1 \cos\left(\boldsymbol{\theta_1} - \theta_4\right)} + \tan^{-1} \frac{l_3 \sin\left(k\left[\boldsymbol{\theta_1}\right]\right)}{l_2 - l_3 \cos\left(k\left[\boldsymbol{\theta_1}\right]\right)}$$
(3.11)

$$\boldsymbol{\theta}_{\mathbf{3}}\left[\boldsymbol{\theta}_{\mathbf{1}}\right] = \theta_{4} - \tan^{-1}\left(\frac{l_{2}\sin\left(k\left[\boldsymbol{\theta}_{\mathbf{1}}\right]\right)}{l_{3} - l_{2}\cos\left(k\left[\boldsymbol{\theta}_{\mathbf{1}}\right]\right)}\right) - \tan^{-1}\left(\frac{l_{1}\sin\left(\boldsymbol{\theta}_{\mathbf{1}} - \theta_{4}\right)}{l_{4} - l_{1}\cos\left(\boldsymbol{\theta}_{\mathbf{1}} - \theta_{4}\right)}\right) + \pi \quad (3.12)$$

Given the independent parameter of the system, motion equation can be derived from Hamilton's principle, (Meirovitch, 1975), and omitting the dependence on time, it yields:

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{T}}{\partial \dot{\varphi}} \right) - \frac{\partial \mathcal{T}}{\partial \varphi} + \frac{\partial \mathcal{V}}{\partial \varphi} = \mathcal{Q}$$
(3.13)

where \mathcal{T} and \mathcal{V} are kinetic and potential energy respectively and \mathcal{Q} is the generalised force due to non-conservative forces:

$$\mathcal{T}[\boldsymbol{\theta_{1}}] = \frac{1}{2} \left(m_{1} \left(r_{1} \dot{\boldsymbol{\theta_{1}}} \right)^{2} + m_{2} \left(r_{2} \dot{\boldsymbol{\theta_{2}}} \left[\boldsymbol{\theta_{1}} \right] \right)^{2} + m_{2} \left(l_{1} \dot{\boldsymbol{\theta_{1}}} \right)^{2} + m_{3} \left(r_{3} \dot{\boldsymbol{\theta_{3}}} \left[\boldsymbol{\theta_{1}} \right] \right)^{2} \right) + l_{1} m_{2} r_{2} \dot{\boldsymbol{\theta_{2}}} \left[\boldsymbol{\theta_{1}} \right] \dot{\boldsymbol{\theta_{1}}} \cos \left(\boldsymbol{\theta_{1}} - \boldsymbol{\theta_{2}} \left[\boldsymbol{\theta_{1}} \right] - \psi_{2} \right) \quad (3.14) + \frac{1}{2} \left(I_{0,1} \dot{\boldsymbol{\theta_{1}}}^{2} + I_{0,2} \dot{\boldsymbol{\theta_{2}}} \left[\boldsymbol{\theta_{1}} \right]^{2} + I_{0,3} \dot{\boldsymbol{\theta_{3}}} \left[\boldsymbol{\theta_{1}} \right]^{2} \right) \frac{\mathcal{V}[\boldsymbol{\theta_{1}}]}{g} = m_{1} r_{1} \sin \left(\boldsymbol{\theta_{1}} + \psi_{1} \right) + m_{2} l_{1} \sin \left(\boldsymbol{\theta_{1}} \right) + m_{2} r_{2} \sin \left(\boldsymbol{\theta_{2}} \left[\boldsymbol{\theta_{1}} \right] + \psi_{2} \right)$$
(3.15)

$$+m_3r_3\sin\left(\boldsymbol{\theta_3}\left[\boldsymbol{\theta_1}\right]+\psi_3\right)$$

$$\frac{\mathcal{Q}[\boldsymbol{\theta}_{1}]}{\ddot{u}_{g}} = m_{1}r_{1}\sin\left(\boldsymbol{\theta}_{1}+\psi_{1}\right)+m_{2}l_{1}\sin\left(\boldsymbol{\theta}_{1}\right)-m_{2}r_{2}\dot{\boldsymbol{\theta}_{2}}\left[\boldsymbol{\theta}_{1}\right]\sin\left(\boldsymbol{\theta}_{2}\left[\boldsymbol{\theta}_{1}\right]+\psi_{2}\right) +m_{3}r_{3}\dot{\boldsymbol{\theta}_{3}}\left[\boldsymbol{\theta}_{1}\right]\sin\left(\boldsymbol{\theta}_{3}\left[\boldsymbol{\theta}_{1}\right]+\psi_{3}\right)$$
(3.16)

where g and \ddot{u}_g are gravity and horizontal ground accelerations, respectively and all other symbols are referred to geometrical and inertial characteristics of the model showed in Figure 2.6.

Problem statement given by Equation 3.13 can be rearranged as:

$$\begin{cases} M \left[\boldsymbol{\varphi} \right] \ddot{\boldsymbol{\varphi}} + L \left[\boldsymbol{\varphi} \right] \dot{\boldsymbol{\varphi}}^{2} + F \left[\boldsymbol{\varphi} \right] g - P \left[\boldsymbol{\varphi} \right] \ddot{u}_{g} = 0 \\ \varphi_{0} = \boldsymbol{\varphi} \left[0 \right] = 0 \\ \dot{\varphi_{0}} = \dot{\boldsymbol{\varphi}} \left[0 \right] = 0 \end{cases}$$
(3.17)

where $M[\varphi]$, $L[\varphi]$, $F[\varphi]$ and $P[\varphi]$ are non-linear functions in $\varphi[t]$, whose explicit forms for the case of a circular arch can be found in (De Santis and de Felice, 2014; DeJong and Dimitrakopoulos, 2014; Oppenheim, 1992).

If the motion in the neighbourhood of the equilibrium position is evaluated, Equation 3.17 can be linearised taking into account constant values of the coefficients, and the general integral assumes the following closed form:

$$\boldsymbol{\varphi}\left[t\right] = \log\left(\cos\left(\frac{\sqrt{L\left[\boldsymbol{\varphi}_{0}\right]\left(F\left[\boldsymbol{\varphi}_{0}\right]\ g - P\left[\boldsymbol{\varphi}_{0}\right]\ \ddot{u}_{g}\right)t}}{M\left[\boldsymbol{\varphi}_{0}\right]}\right)\right)$$
(3.18)

In the framework of linearised motion, thus for small angle oscillations, the acceleration necessary to activate motion, a_g , is the ratio of the work performed by horizontal forces over the generalized gravitational force that the structure experiences at $\varphi_0 = \varphi[0]$, $a_g = P[\varphi_0]/F[\varphi_0]$, (Oppenheim, 1992). This value coincides with the value computed through limit analysis as $a_g = \lambda g$, where λ is the multiplier of loads and g is gravity acceleration. In addition, a suitably high coefficient of friction has to be assumed to prevent the arch from sliding, (Shenton III, 1996).

To assess stability of the arch mechanism one can use potential energy of the system as function of the driver link rotation as proposed by Oppenheim (1992) and Housner (1963). For the arch mechanism shown in Figure 3.4, an acceleration coming from right to left leads clockwise rotation of driver link, l_1 , potential energy at the beginning of motion is $\mathcal{V}_0 = \mathcal{V}[\varphi_0]$. Potential energy of the system increases as motion evolves, since the position of centre of gravity of the whole system heightens. This condition corresponds to negative work done by self-weights. Indeed, self-weight tends to stabilise the arch until a specific deformed geometry. Potential energy reaches a maximum for a critical displaced configuration. From that configuration on, self-weights start to ease the mechanism evolution, actually making positive work, and potential energy decreases, since position of centre of gravity of the whole system commence lowering.

The zero point of work done by vertical forces represent a maximum for potential energy of the system $\mathcal{V}[\varphi[\tau]] = \mathcal{V}_{max}$ for which a critical value of driver link rotation can be associated $\varphi_{cr} = \varphi[\tau]$.

Thus, an acceleration pulse, $\ddot{u}_g[t]$, lasting a time interval τ , brings the arch to collapse when it produces sufficient velocity to displace the system in a non-recovery configuration (Housner, 1963). In other words, the variation of the work done by inertial forces of the system (i.e. a positive work) has to equal the difference in potential energy necessary to pass from the initial configuration \mathcal{V}_0 to peak value \mathcal{V}_{max} . Omitting time dependence of the variable, the collapse condition stated in (De Santis and de Felice, 2014) for circular arches is still valid for pointed arches:

$$\int_{0}^{\tau} \ddot{u}_{g} \left(-m_{1}r_{1}\sin\left(\psi_{1}+\varphi\right)\dot{\varphi}+m_{2}\left(-l_{1}\sin\left(\varphi\right)\dot{\varphi}-r_{2}\sin\left(\psi_{2}+\varphi_{2}\left[\varphi\right]\right)\dot{\varphi}_{2}\left[\varphi\right]\right) -m_{3}r_{3}\sin\left(\psi_{3}+\varphi_{3}\left[\varphi\right]\right)\dot{\varphi}_{3}\left[\varphi\right]\right)dt = \mathcal{V}_{max}-\mathcal{V}_{0}$$

$$(3.19)$$

where, the expression of $\varphi[t]$ is given by 3.18.

Integration of Equation 3.17 can be applied since explicit forms of non-linear coefficients of $\varphi[t]$ appearing in Equation 3.17, originally derived for circular arches, turn out to be directly applicable also for pointed arches, if a determined shape of the mechanism is assumed *a priori*, i.e. fixed position of hinges and inertial characteristics of blocks, and given the impossibility of sub-mechanism activation within each macroblock. In particular, inertial characteristics of first and last macro-blocks are the same as for a circular arch, i.e. portions of circular sectors, while, given the profile discontinuity, definition of inertial characteristics of second block required a computation by part.

The response to rectangular shaped and sinusoidal pulse signals for varying geometrical parameters have been investigated for a set of 48 pointed arches. A half of the considered set has collapse layouts as deduced from minimization procedure, i.e. position of hinges changes, see Section 3.3.1; the second half has the same geometrical characteristics of the first one but with fixed hinge positions. Results are presented in terms of failure domains plotted in an input-duration input-intensity space.

Rectangular shaped pulse

The response of pointed arches to rectangular shaped acceleration has been investigated through two different inputs types and collapse domains so obtained have been compared. The first input type, pulse Type A , is an acceleration pulse expressed by:

$$a(t) = a \quad \text{for } t \in [0, t_i] \tag{3.20}$$

Input Type A has been associated to an approximation of motion near equilibrium position. The linearised version of Equation 3.17 and related general integral in closed form of the type expressed by Equation 3.18 were employed.

Collapse is associated with the balance between the energy necessary to pass from equilibrium position to critical displaced configuration, to which the maximum of potential energy and critical rotation are associated to, and the work done by horizontal forces induced by a given constant acceleration, a, lasting a time interval, t_i . Since the energy necessary to pass from initial configuration to the critical is independent from input, failure domains were built evaluating repeatedly Equation (3.19) decreasing the acceleration level for a fixed duration of the pulse; related routine to evaluate failure domains is reported in Appendix C.

Failure domains acceleration over duration are reported in Figure 3.5 for increasing value of Slenderness and three values of Sharpness. For each curve, the space over each curve identifies a collapse. The asymptotic value of acceleration for long time durations identifies the static multiplier of loads. The vertical band on the left of each curve, corresponding to shorter and stronger pulses, identifies a hinging-with-no-collapse area.

The comparison among domains for pulse type A, Figure 3.5, underlines that increasing sharpness brings a wider hinging-with-no-collapse area for slender profiles (Sd = 0.125 and 0.15). Instead, for thicker profiles (Sd = 0.2 and 0.25), with the increase in sharpness, upper parts of the curves tend to superpose. This behaviour is affected by the change in the mechanism layout, as made evident in Figure 3.2: when thickness increases, the width of the second block generally tends to shrink, polar inertia diminishes and rotational velocity increases. Pointed arches in Figure 3.5c confirm the fundamental role played by the second block. The curve representing the thickest profile (Sd = 0.25) tends to the same behaviour of those of more slender arches (Sd = 0.15 and 0.2) for input durations lower than 0.3s.



Fig. 3.5 Failure domains input duration - acceleration for Pulse Type A for three Sh values and increasing Sd

The second input type, pulse type B, is arranged as a pulse of intensity a and duration t_i followed by a pulse of half the intensity with inverted sign and twice the duration:

$$a(t) = \begin{cases} -a & \text{for } t \in [0, t_i] \\ a/2 & \text{for } t \in (t_i, 3t_i] \end{cases}$$
(3.21)

For pulse type B the non-linear form of Equation 3.17 has been considered and collapse occurs when rotation becomes strictly increasing. This condition implies that the rotation can reach the threshold expressed by φ_{cr} without experiencing collapse if motion rotation diagram is concave downward, i.e. decreasing first derivative (rotational velocities) and negative second derivative (rotational accelerations). Indeed, failure for Pulse type B can occur before or after change in the sign of rotational velocity, thus before or after impact.

Removing the assumption of little displacements affects Equation 3.17 that becomes a non-linear ODE, the solution has to be numerically approximated and computing time sensibly heightens. While for tangent approximation failure is reasonable only for rotations lower that the critical, the non-linear formulation permits the evaluation of a complete cycle of motion. In particular, if the arch recovers from rotation in one direction, it returns in the undisplaced configuration; the first half cycle of motion ends, an impact occurs at each hinge interface and rocking motion starts in the opposite direction.

The impact model envisages friction coefficient is sufficiently high to prevent macroblocks from sliding or other mixed mode impacts, position of the system after impact is the same than before impact, velocity before and after impact changes immediately, duration of the impact is short and impulsive forces are large with respect to others.

At impact instant, hinges 1, 2, 3 and 4 close, four mirrored hinges open at same interfaces but on the opposite side of the section and impulsive forces are assumed to be placed at the new hinges positions, i.e. i, ii, iii, iv (Figure 3.4). Internal impulsive forces at hinges ii and iii auto-equilibrate each other. For first and last hinges, unknown external reactions to the impulsive forces arise, namely F_i and F_{iv} . Thus, the four components of the reactions to the external impulsive forces, $F_{i,x}, F_{i,y}, F_{iv,x}$ and $F_{iv,y}$; and the coefficient of restitution, COR_{im} , are the unknowns of the impact problem.

To determine the unknowns one can simultaneously solve a system of five equations expressing the balance before and after impact of linear momentum, angular momentum about origin and about second and third hinges considering the contribution of impulses. Specifically, for linear momenta and impulses before and after impact we get:

$$\int F_{x,i}dt - \int F_{x,iv}dt - P_{x,a} = -P_{x,b}$$
(3.22)

$$\int F_{y,i}dt + \int F_{y,iv}dt - P_{y,a} = -P_{y,b}$$
(3.23)

where, $F_{i,x}$, $F_{i,y}$, $F_{iv,x}$ and $F_{iv,y}$ are the x and y components of F_i and F_{iv} , and $P_{x,a}$, $P_{x,b}$, $P_{y,a}$ and $P_{y,b}$ are the x and y components of linear momenta before and after impact, i.e. subscript b and a respectively.

Then, the balance of angular impulses and momentum about origin O before and after impact is expressed by:

$$-\int F_{x,i}dt \ y_i + \int F_{y,i}dt \ x_i + \int F_{x,iv}dt \ y_{iv} + \int F_{y,iv}dt \ x_{iv} - L_{O,a} = -L_{O,b}$$
(3.24)

where, x_i , y_i , x_{iv} , y_{iv} are the x and y components of the position vector identifying hinge i and iv, and $L_{O,a}$, $L_{O,b}$ are the angular momenta about origin for the whole arch before and after impact, with same subscript convention as before. Finally, the expressions of angular momenta and impulses about hinges *ii* and *iii* before and after impact are:

$$\int F_{x,i}dt \, (y_i - y_{ii}) - \int F_{y,i}dt \, (x_i - x_{ii}) - L_{ii,a} = -L_{ii,b}$$
(3.25)

$$-\int F_{x,iv}dt(y_{iv} - y_{iii}) + \int F_{y,iv}dt(x_{iv} - x_{iii}) - L_{iii,a} = -L_{iii,b}$$
(3.26)

where, x_i , x_{ii} , y_i , y_{ii} , x_{iii} , x_{iv} , y_{iii} , y_{iv} are the x and y components of the position vector identifying hinge *ii* and *iii*, and $L_{ii,a}$, $L_{ii,b}$, $L_{iii,a}$, $L_{iii,b}$ are the angular momenta about hinge *ii* and *iii* for the portion of arch on the left of hinge *ii* and on the right of hinge *iii*, before and after impact, with same subscript convention as before.

In particular, Equations 3.22-3.26 in terms of rotational velocity of first link at impact instant, $\dot{\varphi}_1$, become:

$$m_{1}r_{1,y}\dot{\varphi}_{i} - m_{2}(-l_{i,y}\dot{\varphi}_{i} - r_{ii,y}\dot{\varphi}_{ii}) + m_{3}r_{iii,y}\dot{\varphi}_{iii} + \int F_{x,i}dt - \int F_{x,iv}dt = m_{1}r_{1,y}\dot{\varphi}_{1} - m_{2}(-l_{1,y}\dot{\varphi}_{1} - r_{2,y}\dot{\varphi}_{2}) + m_{3}r_{3,y}\dot{\varphi}_{3}$$
(3.27)

$$-m_{1}r_{1,x}\dot{\varphi}_{i} - m_{2}(-l_{i,x}\dot{\varphi}_{i} - r_{ii,x}\dot{\varphi}_{ii}) + m_{3}r_{iii,x}\dot{\varphi}_{iii} + \int F_{y,i}dt - \int F_{y,iv}dt = -m_{1}r_{1,x}\dot{\varphi}_{1} - m_{2}(l_{1,x}\dot{\varphi}_{1} + r_{2,x}\dot{\varphi}_{2}) - m_{3}r_{3,x}\dot{\varphi}_{3}$$
(3.28)

$$m_{1}(-r_{i,x}^{2} - r_{i,y}^{2} + r_{i,x}x_{i} + r_{i,y}y_{i})\dot{\varphi}_{i}$$

$$+m_{2}\Big((-l_{i,x}r_{ii,x} - l_{i,y}r_{ii,y} + l_{i,x}x_{iii} + l_{i,y}y_{iii})\dot{\varphi}_{i} + (-r_{ii,x}^{2} - r_{ii,y}^{2} + r_{ii,x}x_{iii} + r_{ii,y}y_{iii})\dot{\varphi}_{ii})$$

$$+m_{3}(-r_{iii,x}^{2} - r_{iii,y}^{2} + r_{iii,x}x_{iv}r_{iii,y}y_{iv})\dot{\varphi}_{iii}$$

$$-\int F_{x,i}dt y_{i} + \int F_{y,i}dt x_{i} + \int F_{x,iv}dt y_{iv} + \int F_{y,iv}dt x_{iv} =$$

$$m_{1}(-r_{1,x}^{2} - r_{1,y}^{2} + r_{1,x}x_{1} + r_{1,y}y_{1})\dot{\varphi}_{1}$$

$$+m_{2}\Big((-l_{1,x}r_{2,x} - l_{1,y}r_{2,y} + l_{1,x}x_{2} + l_{1,y}y_{2})\dot{\varphi}_{1} + (-r_{2,x}^{2} - r_{2,y}^{2} + r_{2,x}x_{2} + r_{2,y}y_{2})\dot{\varphi}_{2}\Big)$$

$$+m_{3}(-r_{3,x}^{2} - r_{3,y}^{2} + r_{3,x}x_{4}r_{3,y}y_{4})\dot{\varphi}_{3}$$

$$(3.29)$$

$$m_{1}(l_{i,x}r_{i,x} - r_{i,x}^{2} + l_{i,y}r_{i,y} - r_{i,y}^{2})\dot{\varphi}_{i} + \int F_{x,i}dt (-y_{i} + y_{ii}) - \int F_{y,i}dt (-x_{i} + x_{ii}) = m_{1}(-r_{1,x}^{2} - r_{1,y}^{2} - r_{1,x}x_{1} + r_{i,x}x_{i} - r_{1,y}y_{1} + r_{i,y}y_{i})\dot{\varphi}_{1}$$
(3.30)

$$m_{3}(l_{iii,x}r_{iii,x} - r_{iii,x}^{2} + l_{iii,y}r_{iii,y} - r_{iii,y}^{2})\dot{\varphi}_{i}ii + \int F_{x,iv}dt (y_{iv} - y_{iii}) - \int F_{y,iv}dt (x_{iv} - x_{iii}) = m_{3}(-r_{3,x}^{2} - r_{3,y}^{2} - r_{3,x}x_{4} + r_{3,x}x_{iv} - r_{3,y}y_{4} + r_{3,y}y_{iv})\dot{\varphi}_{1}$$

$$(3.31)$$

Equations 3.27 - 3.31 refer to the geometrical model showed in Figure 3.4, thus, as a general convention, coefficients related to situation before impact are identified with Arabic numbering appearing in the first lower cases, e.g. $r_{1,-}$ and $l_{1,-}$, while after impact configuration is described by Roman numbering, e.g. $r_{i,-}$ and $l_{i,-}$.

When a second lower case appears, it refers to a Cartesian component of a distance or a vector, e.g. $r_{3,x}$ is the *x*-component of the mass-radius of the third bar before impact, $F_{iv,y}$ is the *y*-component of the impulsive force at the fourth hinge after impact.

Sh	Sd			
	0.125	0.15	0.2	0.25
0.2	0.979	0.959	0.945	0.859
0.6	0.956	0.930	0.910	0.937
1	0.965	0.906	0.874	0.762

Table 3.1 Values of the Coefficient of Restitution, COR_{im} , for three values of Sh and Sd and collapse hinge layout as illustrated in Figure 3.2

Then, *m* refers to mass of first, second or third macro-block depending on lower case. $\dot{\varphi}_{-}$ indicates rotational velocities with subscript convention as explained before.

For geometrical dimensions, $l_{-,-}$ refers to length components of link, $r_{-,-}$ refers to length components of radii connecting hinges to the concentrated masses. Also, xand y, appearing as normal cases, refer to components of position vector of the hinge indicated in the lower case, e.g. y_4 is the y position of hinge four before impact, x_{iii} is the x position of the third hinge after impact. Finally, $F_{-,-}$ represents unknown impulsive force exerted at hinge interfaces, with components and hinge indicated in the lower cases.

The coefficient of restitution so evaluated is affected by sharpness, slenderness and position of hinges, but not by the scale of the arch. Table 1 shows values of coefficient of restitution, COR_{im} , for a set of 12 arches. Sharpness ratio ranges from 0.2 to 1 and Slenderness ratio from 0.125 to 0.25. For the represented set of arches, positions of hinges are those of Figure 3.2. It is worth noting how the coefficient reduces for any increase in ring thickness, but the medium sharp profile (Sh = 0.6) exhibit the least decreasing trend and the lowest starting value (0.956).

To evaluate complete motion, numerical integration of Equation 3.17 has been carried out in Wolfram's Mathematica exploiting an implicit linear multi-step method, in particular a back differentiation formula method (BDF). BDF method provide robust stability in case of a rapid change in the size of integration steps. Specifically, maximum step size was limited to 1×10^{-4} and relative error to 1×10^{-7} .

To detect impact instant, a numerical root finding was executed on motion equation, constraining the search interval between the values of time of the first two roots of velocity equation. Then, finding velocity at impact instant is straightforward. The evaluation continues with the second integration on the mirrored geometry and suitable initial conditions. A flow chart of the computing loop to solve motion problem stated by Eq.3.17 is reported in Appendix B.

Figure 3.6 shows failure domains for pulse type B for four values of slenderness and three values of sharpness. Failure for pulse type B always occurs during second half cycle of motion for circular arches, (De Lorenzis et al., 2007), instead medium sharp profiles can exhibit a mixed mode failure. In particular in Figure 3.6b, for the series defined by squared pointers, Sd = 0.15-B, failure during second half cycle is depicted by empty pointers while black filled pointers stands for failure during first half cycle of motion. The changing shape of the domains for pulse type B is affected by the differences in the restitution coefficients caused by different mechanism layouts. For a given geometry and restitution coefficient, as shown in Figure 3.6d, increasing the scale of the arch increases the relative distance between the two related domains.

Comparing failure domains for Pulse Type A and B, Figures 3.7 and 3.8, it can be noted how linearising equation of motion offers, for all considered profiles, a more conservative assessment. The scatter between evaluations is relevant for sharp and stocky profiles, as shown in Figures 3.8c and 3.8b. Slender profiles, Figure 3.7a, instead, given the general higher vulnerability, exhibit failure domains built on non-linear form of EOM closer to those built on the linearised version. However, increasing sharpness, Figures 3.7b and 3.7c, brings lower vulnerability levels and an increased scatter.

Harmonic pulse

Harmonic input shape has been considered as forcing function for both the set of arches with minimised position of hinges and with fixed position:

$$\ddot{\boldsymbol{u}}_{\boldsymbol{g}}[t] = \boldsymbol{a}_p \sin(\omega_p[t] + \psi) \quad t \in [2\pi/\omega_p]$$
(3.32)

where, the phase angle $\psi = \sin^{-1}(a_g/a_p)$, with $a_g = \lambda g$ the collapse acceleration. Failure domains a_p/a_g over ω_p/ω_0 are showed in Figure 3.9 for the least thickness value and varying sharpness, Figure 3.9a, and for the highest value of sharpness and varying thickness, Figure 3.9b. The normalised values in abscissa account for ω_0 that is the frequency parameter of the system Arnol'd (2013), evaluated on an equivalent linearised system, DeJong and Dimitrakopoulos (2014), as:

$$\omega_0^2 = \frac{\frac{\partial^2 V}{\partial \varphi^2}|_{\varphi=\varphi[0]}}{\sum_{k=1}^3 m_k \frac{\partial^2 r_{k,x}}{\partial \varphi^2}}$$
(3.33)



Fig. 3.6 Failure domains *input duration* - *acceleration* for Pulse Type B, duration of first interval of the input is considered, for three Sh values and increasing Sd; and for fixed Sh and Sd values and increasing R_c . Black filled pointers indicate failure without impact.



Fig. 3.7 Comparison among failure domains for Pulse Type A and B for slender of three *Sh* values. Black filled pointers indicate failure without impact.


Fig. 3.8 Comparison among failure domains for Pulse Type A and B for stocky profiles of three Sh values

where, the numerator represents the constant valued coefficients of the quadratic form obtained with the second-order Taylor's series expansion about the equilibrium point of potential energy and the denominator represents constant valued coefficients of rotational velocity appearing in kinetic energy.

Failure for sine pulse can occur after an impact, i.e. during second half cycle of motions, or directly during first half cycle of motion. For a given frequency, failure with impact occurs for an intensity lower than that necessary to overturn the arch without any impact. Among the two failure modes, similarly to the rocking block, a temporary recovery interval can be detected for a set of accelerations that can vary depending on the given frequency. Results show that the shape of the failure domain is similar to the rocking spectrum of a block, Zhang and Makris (2001), but left shifted. In fact, for a fixed frequency ratio, failure during second half cycle of motion, i.e. after impact, occurs for sinusoidal amplitudes lower than that necessary for failure during first half cycle of motion, i.e. without impact, which is identified by upper curves. A recovery interval is bounded by the upper curve and the upper limit of the region of failure after impact.

To represent effectively this behaviour, recovery intervals in Figure 3.9 are coloured in grey, this enables to observe that for a given frequency ratio the grey region and the vulnerability both reduce for increasing values of sharpness, Figure 3.9a.

As expected, comparing curves with the lowest thickness ratios, Figure 3.9a with continuous series of Figure 3.9b, the sharpest profile is the least vulnerable, but the shape of the rocking spectrum sensibly changes. For the sharpest profile Sh = 1, failure domains represented in Figure 3.9b show that for low and medium thickness ratios (i.e. continuous and dotted series), the temporary recovery interval nearly vanishes, no grey region coloured, and acceleration levels connected to failure with impact and without impact substantially coincide at high frequency ratios.

For high thickness profiles (dashed series Figure 3.9b), instead, the recovery interval is clearly distinguishable by the grey region, and the vulnerability level lowers sensibly in comparison with medium thickness profile (Sd = 0.15, dotted series, Figure 3.9b), which indeed turn out to be the most vulnerable.

This seemingly inconsistent behaviour is actually influenced by the value of the coefficient of restitution, which keeps straight decreasing for increasing thickness, see Table 3.1, and by position of hinges, which differ sensibly between different thickness of the same sharpness.



Fig. 3.9 Failure domains for sinusoidal input and minimised layout

Therefore, a set of arches with common positions of hinges is exploited to isolate the effect of changing geometry (Sharpness and Slenderness) on the dynamic response. In particular, the Oppenheim's circular arch layout, (Oppenheim, 1992), has been assumed as mechanism layout ($\beta_1 = 0.196r$, $\beta_2 = 1.374r$, $\beta_3 = 2.159r$, $\beta_4 = 2.945r$) to generate four-bar linkages with features influenced only by geometrical aspects. Values of load multiplier and trends of failure domain boundaries reported in the following paragraphs are not intended to be interpreted as they are, in fact, the aim is comparing results rather than assessing safety of a specific profile. Regarding load multipliers, the increase is directly related with thickness and sharpness, Figure 3.10, but for the sharpest profile, the static acceleration decreases when slenderness passes from 0.2 to 0.25 (rhomboidal pointers), reaching nearly the same value of Sh = 0.2.

Figure 3.11 shows failure domains for pulse Type A for the set of arches with common hinges. In the short input durations, curves tend to superpose also for low sharpness values, comparing to Figure 3.5. The increase in vulnerability with the increase in ring thickness for Sh = 0.6 and 1, Figures 3.11c and 3.11d, is more pronounced than that reported in Figures 3.5b and 3.5c.

As for pulse Type B, Figure 3.12, the mixed mode failure, depicted with black filled pointers, occurs for durations greater than 0.11s for Sh = 0.2, Figure 3.12b, instead of Sh = 0.6 in the case of minimised hinge position, Figure 3.6b and invests the majority of the considered durations. In addition, relative distance among failure domains for



Fig. 3.10 Load multipliers for Oppenheim's circular arch, (Oppenheim, 1992), i.e. Sh = 0, and for pointed arches with same mechanism layout

pulse Type B increases evidently with the increase in sharpness and for Sh = 1, Figure 3.12d, Sd = 0.2 becomes the safest profile.

For sinusoidal input, Figure 3.13a shows that, for all sharpness ratios considered, a small recovery interval is found and that the interval tends to shrink for higher sharpness ratios, highlighting how the dominant failure mode is that after first impact. For the sharpest profile, Figure 3.13b, the recovery interval reduces to a strip that nearly vanishes with the increase in thickness. Thus, the recovery interval is mainly influenced by mechanism shape rather than sharpness or thickness themselves, accordingly with Zhang and Makris (2001). Even though the coefficient of restitution quickly reduces with increasing thickness, the sharper and the stockier is the profile the more is vulnerable. This is due to a direct increase in inertial quantities without the related adaptation of hinges layout. Comparing Figures 3.9 and 3.13, it is clear that sharpness mainly influences levels of activation acceleration, while slenderness can affect the collapse mode. Since increasing the span of the arch does not affect the shape of the domain but only extremal values, variations of the domains for variation of R_c are not plotted.

Sensitivity analysis

In this section, a validation study is proposed to demonstrate that the implemented procedure can predict failure of circular arches as a particular case of pointed, i.e.



Fig. 3.11 Fixed mechanism layout. Failure domains *input duration* - *acceleration* for Pulse Type A for three Sh values and increasing Sd



Fig. 3.12 Fixed mechanism layout. Failure domains *input duration - acceleration* for Pulse Type B, duration of first interval of the input is considered, for four Sh ratios and increasing Sd. Black filled pointers indicate failure without impact



Fig. 3.13 Fixed mechanism layout. Failure domains for sinusoidal input

sharpness ratio Sh = 0. Thus, a comparison among rotation time histories of three pointed arches and a circular of same span and ring thickness for different input shapes is reported.

Time history of driver link rotation is the output of the integration procedure from which an immediate visualisation of arch response is possible. During the first half cycle of motion the arch can recovery and undergo impact or fail directly. In case of recovery – convex shape of time history – impact instant and velocity at impact instant have to be detected. Figure 3.14 shows time histories of a circular arch and three pointed arches varying input shape and Coefficients of Restitution, COR_{im} . Harmonic sine pulses and rectangular shaped inputs were considered in the sensitivity analysis, Figure 3.14a. They are:

$$\ddot{u}_g = -a\sin(\omega t + \chi) \quad t \in [0, \frac{2\pi - \chi}{\omega}]$$
(3.34)

$$\ddot{u}_r = \begin{cases} -a & t \in [0, \frac{\pi - \chi}{\omega}] \\ a & 2 & t \in (\frac{\pi - \chi}{\omega}, \frac{3\pi - \chi}{\omega}] \end{cases}$$
(3.35)

where, $\chi = \sin^{-1}(a_g/a)$, a_g is the collapse acceleration (as evaluated through limit analysis, Section 3.3.1) specific for each arch, a is the acceleration amplitude, which is

related to collapse accelerations of each arch, in particular $a = 2.25a_g$, to highlight the role played by arch sharpness and restitution coefficient. Finally, $0.5s = 2\pi/\omega$ is the fixed value of sine pulse period. Duration of first part of inputs are the same to ease a direct comparison, Figure 3.14a.

Figure 3.14 represents motion rotations normalised to critical rotation values of a circular arch and three pointed arches. Sharpness ranges from 0 to 1, ring thickness and span are fixed (1m and 10m respectively). Time histories for rectangular shaped inputs are grey plots, and black plots are those related to sine pulse inputs. Restitution coefficients vary from the value deduced from the impact model (COR_{im}, continuous plots), specific for each arch, to fixed values of 0.9 (dashed plots) and 0.8 (dot-dashed plots). Values of COR_{im} and collapse accelerations for each arch are reported in the caption of Figure 3.14.

It is worth underlining that explicit values of collapse accelerations, reported in the caption of Figure 3.14, decrease with the increase in sharpness since fixed values of thickness and span make slenderness ratio, Sd, decrease with the increase in eccentricity. In particular, Sh = 0, Sh = 0.2, Sh = 0.6 and Sh = 1 have Sd = 0.2, Sd = 0.166, Sd = 0.125 and Sd = 0.1 respectively. Motion rotations for rectangular shaped inputs are similar before and after impact. For sine pulse input, circular arch fails for both coefficient of restitution as evaluated through the impact model, COR_{im} , and for COR = 0.9 during second half cycle of motion. For rectangular shaped input, circular arch fails only for COR_{im} .

Further considerations are necessary to qualify the possibility of comparing the behaviour of circular and pointed arches. Indeed, given the radial symmetry of circular profiles, it is common practice to consider the failure mechanism symmetrically with respect to the angle of embrace, (De Lorenzis et al., 2007; De Santis and de Felice, 2014; DeJong et al., 2008), thus independently from the absolute position of the first and the last hinge. This approach turns out to operate a rotation of the real mechanism such that the line identifying the bisector of the embrace angle becomes a vertical line, as shown in Figure 3.15.

In light of this, comparisons between pointed and circular arches require particular attention. Figure Figure 3.15 shows the differences in the collapse domains for pulse Type A for the same mechanism considered in its original position, filled pointers, and rotated with respect to the embrace angle, empty pointers for arches with same span (Rc=10m) and same thickness to radius ratios (Sd = 0.125). In particular, for sharpness equal 0.1 the multiplier increases nearly by two for a rotation of 0.21r, while



(d) Sh = 0.6, $COR_{im} = 0.961$, $a_g = 0.231$ g (e) Sh = 1, $COR_{im} = 0.968$, $a_g = 0.231$ g

Fig. 3.14 Sensitivity analysis results for different inputs



(a) Mechanism in the original posi-(b) Failure domains. Rotated laytions and rotated symmetrically with out (empty pointers), original posirespect to the angle of embrace tion otherwise

Fig. 3.15 Failure domains for Pulse Type A for a circular and a pointed arch with different mechanism orientation

for sharpness equal 0.6 the multiplier growths more than by half for a 0.11r rotation. For the circular arch, the rotation of the mechanism is 0.26r and the multiplier increase is the highest. The comparison between the four-bar linkages of the circular arch and the two pointed arches in the rotated configuration, highlights that, as expected, pointed arches result in higher static multipliers, and their dynamic behaviour is less sensitive to short duration actions.

3.4 Summary

Main conclusions, partially reported in (Misseri and Rovero, 2017), can be summarised as follows:

- The position of the first hinge is affected by geometry, but last hinge is always on the springing plane.
- Variation in the embrace angle has not a direct effect on load multiplier.
- Vulnerability of sharp profiles is nearly insensitive to variation in thickness for short duration inputs.

- Linearised form of motion equation enables a more rapid and conservative assessment.
- Change of failure mode from second half cycle to first can occur when the non linear form of motion equation is employed, but this behaviour is deeply affected by the impact model adopted.
- The medium sharp profile exploits the increase in ring thickness better than others.
- Representing the hinging layout symmetrically with respect to the angle of embrace bring to non-conservative estimations.

Chapter 4

Experimental evaluations on reduced-scale block models

4.1 Specimens

The geometrical model described in Section 3.2 is here considered as piecewise linear profile, thus as a system of discrete linearised blocks with the same geometrical proportions. In this framework, static and dynamic tests carried out on a set of reduced scaled profiles are reported in Section 4.2 and 4.3 respectively.

4.2 Tilt tests

4.2.1 Test apparatus

Tilt test evaluates the equivalent horizontal static force that a ground acceleration can cause to a structure. The test apparatus requires inclining the laying plane of a structure continuously. Relating the slope of the table with its Arctangent one can easily deduce the load multiplier.

The apparatus of the tilt test consists of two hinged wooden tables (0.8 m x 0.2 m x 0.02 m) on which the arch rest on. The tables are connected through a threaded rod that can spin on a nut fixed in the thickness of the upper wooden table, so that spinning the rod produce a continuous sloping in the upper table. Abrasive paper



Fig. 4.1 Setting up tilt test

glued on the upper table prevent from sliding phenomena at the abutments. Figure 4.1 shows basic steps carried out to set up tests.

The experimental campaign addressed 11 reduced scale models of arches made of dry blocks of autoclaved aerated concrete (AAC), which can be easily cut into precise blocks with an circular saw with wood blade, Figure 4.3a.

Each specimen comprises 16 voussoir blocks and a crown block, Figure 4.2; scaffolding was realised from extruded polystyrene panels cut with a hot wire XPS cutter table, Figure 4.1b. Table 4.1 synthesize block characteristics for each specimen. Specimens showed in Figure 4.2 have been accommodated so as to represent a stone pointed arch as represented in Figure 2.9b, thus blocks adjacent to the key stone were glued together.

4.2.2 Results

Each test has been repeated three times and results in terms of load multipliers are showed in Figure 4.4a and in Table 4.1, from which one can directly note that values for thick profiles increase with a lower rate than thin ones, independently from sharpness and substantially superpose for the highest slenderness ratio. Values of load multipliers



(c) Sh = 1

Fig. 4.2 Tested specimens

 Table 4.1 Geometrical characteristics of reduced scale models and results of tilt tests

	Geometric Characteristics				Tilt test results		
Specimen ID	R_i	R_e	e	t	Δv	st dev	CoV
			[mm]		110.	50.407.	001
Sh02-Sd015	222	258	40	36	0.133	0.027	0.206
Sh02-Sd020	216	264	40	48	0.268	0.027	0.092
Sh02-Sd025	210	270	40	60	0.401	0.026	0.067
Sh06-Sd010	304	336	120	32	0.072	0.014	0.200
Sh06-Sd015	296	344	120	48	0.218	0.032	0.162
Sh06-Sd020	288	352	120	64	0.337	0.039	0.129
Sh06-Sd025	280	360	120	80	0.433	0.013	0.029
Sh1-Sd010	380	420	200	40	0.114	0.021	0.189
Sh1-Sd015	370	430	200	60	0.256	0.021	0.089
Sh1-Sd020	360	440	200	80	0.327	0.007	0.022
Sh1-Sd025	350	450	200	100	0.405	0.022	0.052



(a) Block cutting (b) Block refinement

Fig. 4.3 Preparation of models

together with those of the minimization procedure carried out in Section 3.3.1 are reported in Figure 4.4.

As a general trend, for shallow profiles a proper rocking mechanism has occurred, Figures 4.5 and 4.6, dashed line is consistent with experimental values. For sharp and thick profiles, sliding has occurred instead of rocking, as showed in Figure 4.7d, with relevant scatter between experimental results and numerical evaluations, Figure 4.4d. Thus, load multipliers evaluated through a macro-block approach for pure hinging mechanisms always overestimate capacity and cannot tackle the sliding phenomenon.

Figure 4.8, similarly to Figure 3.3 reports the angles identifying the positions of the four collapse hinges as a function of the slenderness for three values of sharpness. Empty pointers refer to the sliding interface and couples of dashed lines indicate the angular with occupied by the crown block.

For Sh = 0.2, a global rocking mechanism activated for all considered profiles, 4.5. However, for the least thick profile, 4.5a, a partial sliding occurred on the third interface on the left, and for Sd = 0.25, 4.5c, partial sliding was recorded on keystone interface.

Regarding position of hinges for the shallowest profile, Figure 4.8a, last hinge never detached from abutment; with the increase in thickness, embrace angle, i.e. angle subtended by the mechanism, widens, hinge on right hand side passes from second to last interface, Figure 4.5b to last, Figure 4.5c. The angle subtended by the second macro-block also widens, upper right hand side hinge lowers from Sh02Sd020 to Sh02Sd025, Figures 4.5b and 4.5c.



Fig. 4.4 Results of tilt tests and comparison between multiplier of loads as deduced from tilt tests, λ_{exp} , and evaluated through procedure of Section 3.3.1, λ_{min} , for different *Sh* and *Sd* values. Error bars indicate ranging outcomes of repeated tests



(a) Sd = 0.10





Fig. 4.6 Results of tilt test for Sh = 0.6







Similarly, for Sh = 0.6, global rocking mechanism apparently activated for all considered profiles apart from the thickest, Figure 4.6d that underwent sliding. Moreover, conversely from the continuous model a shrinking trend of the amplitude of the second block cannot be noticed nor an increase in the embrace angle with the increase in thickness, compare Figure 4.8b and 3.3b. Last hinge always occur at abutment. The sliding for the thickest profile occurred at first block free to move apart.

The sharpest profile exhibited the highest scatter between predictions of the numerical model and experimental results both in terms of values of load multipliers and as mechanism layout. Indeed, positions of first and last hinges never change and amplitude of second rocking macro-block tend to a slight shrink, distance between triangles and squares series Figure 4.8c, differently from what predicted by the numerical model, Figure 3.3c.

For the thickest and sharper profiles, i.e. Sh06Sd025 and Sh1Sd025, a different mechanism activated. Sliding occurred on the first joint free to move and two hinges opened at abutments. Figures 4.7d and 4.6d. Thus, a three-interfaces mechanism actually occurred. Accordingly, series indicating the position of third hinge in Figure 4.8, square pointers, are truncated at Sd = 0.20 for Sh = 0.6 and 1, and position of second hinge is displayed as an empty pointer to highlight it as a sliding-hinge.

4.3 Shake table tests

4.3.1 Test apparatus

The set of 11 reduced scale models was tested on one DOF shake table reproducing sine pulse inputs with frequency ranging from 2Hz to 8Hz. The shake table is constituted by an alloy ball rail table (TKK 15-155 Al) actuated through a Mannesmann Rexroth servo drive MAC 115 (nominal speed 1000 mms^{-1}) in conjunction with kinetic drive controller and a modular, microprocessor-based positioning control module, CLM, all distributed by Indramat GmbH.

A 1200x800 mm sized wooden table was mounted on the ball rail table to get enough space to dispose specimens and related protective panels, and on the wooden table a strip of abrasive paper was glued to avoid sliding of specimens. An accelerometer was placed at the level of the table, Figure 4.9.



Fig. 4.8 Position of hinges for tilt tests. Circles, triangles and squares represent the position of the first, second and third hinge respectively. Empty triangular pointers represent sliding joints and dashed lines indicate the angular width occupied by the crown block



Fig. 4.9 Setting up shake table test

Due to limitations of maximum velocity that the shake table attains, for some specimens it was not possible to reach failure for high frequencies. Two geometrical configurations were tested: with blocks adjacent to the crown block glued, like in the tilt tests, and separated.

Expected threshold values of intensity - frequency were evaluated through integration of motion equations of specimens-related four bar linkages. Then, inputs have been scaled for each frequency until reaching failure during tests.

4.3.2 Results

Figures 4.10, 4.11 and 4.12 show results of shake table tests for both configurations, named Free and Glued, together with thresholds values as expected by the related four bar linkages, named Numerical.

As a general trend, for slow inputs, 2Hz and 4Hz, a better accordance between numerical predictions and experimental results was found compared to values evaluated through the numerical procedure for high frequency inputs. In addition to this, for all thinner profiles, predictions for 6Hz, 8Hz and 10Hz are less distant than for thicker profiles, and numerical outcomes always offer a conservative estimation, Figures 4.10a, 4.11a and 4.12a. For thicker profiles and high frequency range, predictions of the numerical model are sensibly lower than experimental results, since twisting between adjacent blocks occurred during tests inducing energy dissipation and delaying failure, Figures 4.10c, 4.11d and 4.12d, which could not be tackled by the four bar linkage model.



Fig. 4.10 Results of shake table tests for Sh = 0.2



Fig. 4.11 Results of shake table tests for Sh = 0.6



Fig. 4.12 Results of shake table tests for Sh = 1



(a) Sh02Sd015 - 2Hz

(b) Sh02Sd020 - 2Hz

Fig. 4.13 Dynamic tests for Sh = 0.2



(a) Sh02Sd025 - 4Hz, 2^{nd} half cycle (b) Sh02Sd025 - 10Hz, 2^{nd} half cycle

Fig. 4.14 Dynamic tests for Sh = 0.2

Concerning quality of the response, shallow profiles always failed for rocking mechanisms, during second half cycle of motion, thus after that an impact had occurred. For Sh02Sd015 and Sd02Sd020, Figure 4.13, failure occurred for rocking, with no twisting among blocks. Upper hinge slightly moves from right to left "following" the successive one for lower velocities, overpassing crown.

A "moving" hinge is recorded also for the thickest profile of Sh = 0.2 for frequencies higher than 2Hz. In particular, for 4Hz to 8Hz hinge moves in the upper part of the profile and twisting among blocks increases, Figure 4.14a. For 10Hz failure mode becomes more chaotic, but still a rocking mechanism is recognisable, Figure 4.14b.

Specimen Sh06Sd010 failed for rocking during second cycle of motion for all considered frequencies, with slight twisting among upper blocks and substantially fixed position of hinges, Figure 4.15.



Fig. 4.15 Sh06Sd010 - 2Hz, 2nd half cycle

Increasing thickness of medium sharp profiles, Sh06Sd015, and for 2Hz frequency, rocking behaviour and failure during second half cycle of motion kept clear; external hinges slightly moved downward until finding the collapse position, Figure 4.16a.

Instead, for 8Hz and 10Hz during second half cycle of motion, sliding of the right hand half of the arch occurred on forst free joint adjacent to crown block, Figure 4.16c. For 10Hz, twisting among upper blocks becomes more evident and failure quality is more chaotic, Figure 4.16d.

For Sh06Sd020 and 4Hz frequency, Figure 4.17, rocking activated during first half cycle of motion, i.e. upper left hinge is inward, 4.17a. Motion evolves with sliding on crown joint on left side, same of the first hinge, Figures 4.17b, then failure occurred for mixed sliding-hinging during second half cycle of motion on upper blocks on right hand side of crown, Figure 4.17c. Increasing frequency the sliding depth during the first half cycle of motion increases as well, but collapse remained rocking driven.

For Sh06Sd025, similar collapse mode was recorded, sliding on the left of crown block during first half cycle of motion, Figure 4.18a, and mixed hinging sliding on the right side of crown block during second half cycle of motion, Figure 4.18b.

For Sh1Sd010 at 2Hz and 4Hz, rocking mechanism and failure during second half cycle of motion, Figure 4.19a. For higher frequencies, the mixed mode described for Sh06Sd020 and Sh06Sd025 occured also of Sh1Sd010, Figure 4.19b.

The sharpest profile with Sd = 0.15 at 2Hz failed for rocking on the right side of the arch, thus during second half cycle of motion, after initial partial sliding and mixed sliding hinging of right hand side of the arch, Figure 4.20a.



(c) Sh06-Sd015 - 8Hz, 2^{nd} half cycle (d) Sh06-Sd015 - 10Hz, 2^{nd} half cycle Fig. 4.16 Dynamic tests for Sh = 0.6 and low thickness

The same profile at 4Hz and 6Hz failed in a different manner. Sliding of the crown block occurred during the first half cycle of motion, Figure 4.20b. Then, during second half cycle, the arch recover from rocking failure triggered on the right hand side of the profile, Figure 4.20c, and collapsed for sliding of the crown block on the left hand side of the profile, Figure 4.20d.

For 8Hz and 10Hz failure occurred for direct sliding, during the first part of motion, rocking mechanism not even forms, Figure 4.21a.

For Sh1Sd020 at 6Hz or lower, failure occurred during second half cycle of motion after initial sliding and mixed hinging-sliding during second half of motion, Figures 4.21b and 4.21c. Increasing frequency upper blocks tend to bounce among each other and for 8Hz and 10Hz frequencies, failure occurs directly for sliding during first half cycle of motion with increased twisting for 10Hz, Figure 4.21d.

For Sh1Sd025, at all frequencies tested failure occurred during second half cycle of motion according to the mixed sliding - hinging mode after initial sliding, Figure 4.22.



(a) 4Hz, 1st half cycle rocking





(c) 4Hz, 2^{nd} half cycle rocking

Fig. 4.17 Dynamic tests for Sh = 0.6 and Sd = 0.20



(a) 4Hz, 1st half cycle(b) 4Hz, 2nd half cycle

Fig. 4.18 Dynamic tests for Sh = 0.6 and Sd = 0.25



(a) Sh1Sd010 - 2Hz, 2^{nd} half cycle

(b) Sh1Sd010 - 8Hz, 2^{nd} half cycle

Fig. 4.19 Dynamic tests for Sh = 1 and low thickness



Fig. 4.20 Sh1Sd015 failure for low frequency range



(c) Sh1Sd020 - 6Hz, 2nd half cycle(d) Sh1Sd020 - 8Hz direct sliding

Fig. 4.21 Dynamic tests for Sh = 1 and medium to high thickness





(b) Sh1Sd025 - 2Hz, 2nd half cycle



4.4 Summary

Results from tilt tests can be effectively summarised as follows:

- Load multipliers deduced from tilt tests tend to superpose for high thickness ratios.
- Compared with experimental results, outcomes of four hinge mechanisms always overestimate capacity and cannot tackle the sliding phenomenon, this is remarkably relevant for high pointedness and high thickness.
- Partial sliding was recorded for all tested profiles, connected with inherent limited friction reached with AAC blocks, but for sharp and thick profiles, the sliding mechanism is global, driven by three interfaces.

Regarding shake table tests:

- For slow sine pulses good accordance was found between four-bar linkage predictions and experimental values but capacity overestimation of four-bar linkage predictions is also recorded.
- For higher frequencies and thin profiles the scatter between numerical and experimental values is lower than for thicker, but numerical outcomes always offer a safe estimation.
- For high frequency and thickness, twisting among upper blocks induces energy dissipation and a relevant increase in the response compared to predictions of the four-bar linkage.
- Collapse mode is deeply influenced by thickness and input frequency, turning out to occur either during second half cycle of motion for rocking or mixed rocking-sliding or for direct sliding during first part of motion.
- For the least sharp profile, failure for harmonic pulse happens during second half cycle of motion independently from the frequency, although for high thickness profiles, an incipient sliding of the crown block and a change in hinges position is recorded.
- For medium sharp profiles, failure happens during second half cycle of motion. However for increasing thickness and frequency, twisting phenomena in the upper

part of the arch become relevant and non-failing sliding often occur after hinging, the position of hinges tend to move

• The sharpest profile exhibit a second cycle failure for low thickness and frequency ranges, while the increase in thickness induce twisting among upper blocks even for medium low frequencies.

Chapter 5

Analytical and numerical interpretation of tests results

In this chapter, results of analytical and numerical modelling of experimental tests are proposed. In particular, load multipliers have been re-evaluated in the framework of limit analysis with specific constraints to the macro-block objective function, Equation 3.9, and through a simplified kinematic model. Moreover, through the commercial code 3DEC, failure domains for equivalent static inputs and acceleration records of shake table tests have been built, focusing on the role played by friction coefficient.

5.1 A simplified rocking-sliding model for tilt test outcomes

Load multipliers as evaluated through tilt tests in some cases sensibly differ from numerical predictions of the macro element model for high thickness and pointedness. This can be due to different causes:

- Inherent imperfections of blocks can induce early collapse
- The friction coefficient reached is not sufficient to prevent blocks from a slight initial sliding among each other
- Rocking mechanism had not activated at all, but response is driven by a different kinematics

In particular considering the lowest Sd value for Sh = 0.6 and 1, load multipliers deduced from tilt tests are lower than those predicted by the minimization model. In this regard, it is worth underlying that given the physical dimensions of the thickness of these two specimens, i.e. Sh06Sd010 and Sh1Sd010, imperfections due to block cutting and refining surely affect the quality of interlocking two blocks plausibly causing an early loss of contact.

For medium values of Sd, i.e. 0.15 and 0.2, although providing overestimations, load multipliers evaluated through the minimization procedure keep at fixed distance from experimental values, Figure 4.4b. Thus, the evolution trend of load multipliers is tackled.

For Sd = 0.25, the shallow profile Sh = 0.2, Figure 4.4b, show a linear trend of increase in the value of load multiplier consistent with values of more slender profiles of same Sh. For higher Sh values and high Sd, Figure 4.4c and 4.4d, a piecewise trend can be recorded in the increase of load multipliers. Specifically, load multipliers for higher Sd values, increase less than those with low Sd values. This is clear for Sh = 1 and slightly observable for Sh = 0.6.

First two issues are related to physical circumstances of this set of tests, even though representative and repeatable; and can be overcome considering, for example, a reduced value for ring thickness. For this reason, multiplier of loads have been evaluated through the macro-block approach considering a value of ring thickness equal to 80% of the actual one, similarly to DeJong (2009), modifying the constraints to the objective function assuming a lower threshold for the minimum distance between two adjacent blocks equal to block spacing, named ϕ . Thus, Equation 3.9 becomes:
$$\lambda = \min \left(\Lambda \left(\boldsymbol{\beta} \right) \right)$$

s.t. $\beta_{i-1} < \beta_i < \beta_{i+1}$
 $0 \le \beta_i \le \pi$
 $\beta_i < \beta_j; \quad (i < j)$
 $\beta_j - \beta_i \ge \phi; \quad (i < j)$
 $\beta_2 < \pi/2$
 $\beta_3 \ge \pi/2$
 $\eta_1 \le 0$
 $\zeta_1 \le 0$
 $\eta_3 \ge 0$
(5.1)

Figure 5.1 shows multipliers of load for tested profiles considering whole or reduced thickness, superposed to experimental results. Although decreasing thickness improves quality of assessment for Sh = 0.2 profiles decreasing overestimation of the minimization procedure, it is not completely effective.

In particular, for low Sd = 0.10, the scatter is still visible, confirming that those tests suffered from errors of different nature that could be removed for example pretreating contact surfaces with a coating film, choosing different material or increasing global scale of specimens. For higher values Sd and Sh = 0.2, estimations become conservative, Figure 4.4b. For medium and high sharpness profiles estimations are non-conservative or fluctuating between experimental values and the general piecewise decreasing trend for Sh = 1 values is still not tackled.

As video frames of tests show, Figures 4.6d and 4.7d, instead of a four hinge, a global three interface mechanism occurred activating two cylindrical hinges at abutment level and a slider on the first free-to-move section, namely the one next to crown block.

This layout has been represented through a two bar mechanism, l_1 and l_2 , with hinges at abutments and a slider in correspondence of the crown interface, S, left or right depending on action direction, Figure 5.2, assuming a slider oriented perpendicular to ground. Each bar, representing an arch portion, is equipped with a fixed massless link, r_1 and r_2 , Figure 5.2, connecting pivoting points to the centre of gravity of each macro-element, where the actual arch masses, m_1 and m_2 , are lumped.



Fig. 5.1 Load multipliers evaluated through minimization procedure, Section 3.3.1, for whole ring thickness, λ_{1t} , and 80% ring thickness, $\lambda_{0.8t}$, superposed to experimental values, λ_{exp}

The two-bar model synthesises a more complex response that involves minor hinging phenomena among blocks from abutment level to the sliding crown. In other words, the transition between the cylindrical hinge and the sliding interface is possible through a series of minor hinging occurrences represented through a rigid rotation. However, the final aim was to define a simple 1-DOF mechanism able to tackle the overall response.

Given that the condition for a mechanism to initiate is that the centres of absolute and relative rotations of trunks are aligned, if the slider was placed following the actual slope of the crown interface, one of the two cylindrical hinges would have to move upward consistently, resulting in an a-priori defined mechanism layout depending on sharpness.

Instead, inclining the slider perpendicularly to the ground permit to place the cylindrical hinges at abutment level that is a safe assumption, for all profiles. In so doing, the geometrical parameters of the arch affect the value of the multiplier of loads according with dimension and inclination of radii and links and enabling a direct comparison among profiles.



Fig. 5.2 Two bar model for the sliding pointed arch

The concepts of limit analysis can be applied to this kinematic chain as well. In particular, the sum of the work done by self-weights for vertical displacements and the work done for horizontal displacements by inertial forces proportional to self-weights, according to a multiplier, is zero in the limit equilibrium condition:

$$g(m_1\eta_1 + m_2\eta_2) + \lambda g(m_1\zeta_1 + m_2\zeta_2) = 0$$
(5.2)

where, η_1 , η_2 , ζ_1 and ζ_2 are virtual vertical and horizontal displacements of weights. The position of the slider is conservatively placed at crown interface on outer side:

$$S = (R_e \cos(\pi - \gamma_a) + e; R_e \sin(\pi - \gamma_a))$$
(5.3)

where, $\gamma_a = \frac{\pi}{2} - \sin^{-1}\left(\frac{e}{R_i}\right)$ and simple geometrical relations define rotation of one bar with respect to another, which are concordant:

$$\varphi_2\left(\varphi_1\right) = \varphi_1 \frac{l_1 \cos \alpha_1}{l_2 \cos \alpha_2} \tag{5.4}$$

Figure 5.3 shows results of the two-bar model compared to experimental results and previous evaluations. The difficulties in predicting load multipliers of experimental tests carried out on the specimens Sh06Sd010 and Sh1Sd010, i.e. lowest Sd values, is confirmed. From the joined comparison it is clear how for low and medium Sh values, Figure 5.3a and 5.3b, the two bar model offer always a conservative estimation of the load multiplier. In addition, for Sh = 0.2 and increasing Sd values, the assessment becomes over conservative and nearly insensitive to the increasing trend recorded in the experimental load multipliers, meaning that, instead of the two-bar chain, the rocking mechanism activated and the four-bar linkage, in the reduced thickness formulation, describes the collapse mode more effectively.

Conversely for high Sh and Sd values, Figure 5.3c, the two bar model is more effective than the reduced thickness model in predicting the experimental threshold but also in tackling the general trend of the increase in the multiplier value for increasing values of Sd.

5.2 The DE method for the interpretation of static and dynamics tests

Discrete Element Modelling (DEM) is inherently effective in the representation of the discontinuous nature of masonry and permits to carry out equivalent static and dynamic analyses in the framework of finite displacements. Application of DEM to masonry has greatly increased in recent years, however, as mentioned in Section 2.2.2. In this study, the final aim of modelling tested specimens as discrete elements is to highlight the relevance of the role played by sliding phenomena for pointed arches and



Fig. 5.3 Load multipliers evaluated through minimization procedure, Section 3.3.1, for whole ring thickness, λ_{1t} , and 80% ring thickness, $\lambda_{0.8t}$, superposed to experimental values, λ_{exp} and evaluations of the two bar model

to compare the scatter with predictions offered by the analytical method exploited in Section 3.3.2.

To this end, different analyses have been carried out through the commercial DEM code 3Dec. In particular, all the specimens have been modelled through an equivalent static analysis and a linear dynamic analysis to evaluate load multipliers and natural frequencies. Moreover, specimens with Sd = 0.15 and 0.20 were subjected also to the sine pulses of shake table tests. All the dynamic analyses carried out in 3Dec and reported here after have been performed at the Engineering Department of Cambridge University.

The DEM code 3Dec permit to model blocks as rigid or deformable medium and interface law through a system of spring and dashpot. Consistently with assumptions made throughout this study, voussoirs of arches were modelled as a system of rigid blocks with frictional joints, i.e. nil values for tensile strength, cohesion coefficient and dilatancy angle. Then, joint properties are specified through axial and shear stiffness and friction angle, set at 30° among blocks and at 45° for joints between last block and abutments, where not differently specified.

Following the approach proposed and validated in (DeJong, 2009), elasticity of block material is lumped at joints. It is worth noting that joint stiffness assigned has no counterpart in the analytical model, thus it has to be considered as a penalty stiffness. Values need to be refined so that they take the largest values to allow joints the minimum of deformation necessary to compare results with other models. Expressions for stiffness of a rectangular block are:

$$k_N = \frac{EA}{L} \quad k_S = \frac{5G}{6} \tag{5.5}$$

where, E and G are elastic constants of autoclaved aerated concrete (AAC) and assuming voussoirs as equivalent rectangular blocks of dimensions $t = R_e - R_i$, L = (B-b)/2 + b and transverse section A = wt, being B and b outer and inner voussoir length respectively, t, ring thickness and w, width of arch equal to 100mm.

Critical issue in DEM is also damping, which is represented in 3Dec through the Rayleigh model:

$$\boldsymbol{C} = \alpha_R \boldsymbol{M} + \beta_R \boldsymbol{K} \tag{5.6}$$

where, C, M and K are respectively damping, mass and stiffness matrices. Then, α_R and β_R are mass-proportional and stiffness-proportional damping constants, which

Specimen ID	k_N	ξ at ω_r	ω_r	β_R
1	[Pa/m]	3 .	[rad/s]	, 10
Sh02-Sd015	4.8×10^{8}	1	305	3.28×10^{-3}
Sh02-Sd020	4.82×10^8	1	472.7	2.12×10^{-3}
Sh02-Sd025	4.83×10^8	1	665.3	1.5×10^{-3}
Sh06-Sd010	4.3×10^8	1	204.4	4.89×10^{-3}
Sh06-Sd015	4.34×10^8	1	383	2.61×10^{-3}
Sh06-Sd020	4.38×10^8	1	601.9	1.66×10^{-3}
Sh06-Sd025	4.44×10^8	1	859.8	1.16×10^{-3}
Sh1-Sd010	3.9×10^8	1	240	4.17×10^{-3}
Sh1-Sd015	4×10^8	1	574.8	1.74×10^{-3}
Sh1-Sd020	4.1×10^8	1	726.4	1.38×10^{-3}
Sh1-Sd025	4.2×10^8	1	1054.3	9.49×10^{-4}

Table 5.1 Employed stiffness and stiffness-proportional constant for Rayleigh damping in3Dec analyses

relate pulse, ω to the critical damping ratio, ξ , through the following relation:

$$\xi(\omega) = \frac{1}{2} \left(\frac{\alpha_R}{\omega} + \beta_R \omega\right) \tag{5.7}$$

Specified damping is necessary to avoid high frequency vibrations without ending up with over damped blocks which would cause non-conservative estimations, DeJong (2009). Here critical damping have been set at the frequency necessary for rotation impact of block constituting arch, thought as rectangle as mentioned before, DeJong (2009). Pulse for rotational impact is:

$$\omega_r = \sqrt{\frac{k_N (R_e - R_i)^2}{J_o}} \tag{5.8}$$

Where, J_o is polar inertia with respect to the pivot point of block. Table 5.1 summarizes values of axial stiffness and stiffness-proportional constant adopted for analyses reported hereafter.

In 3Dec, solution for gravity acceleration only represents the closer approximation to the equilibrium problem for self-weights, though solved integrating equations of motion. To represent tilt tests, a dedicated routine has been written in 3Dec environment through the embedded programming code, FISH, to increase gradually x-component of gravity acceleration in loops. Acceleration increment has been set to 0.05m/s^2 each 4000 cycles, with initial time step evaluated by the program ranging between 6.19×10^{-6} s

and 8.45×10^{-6} s depending on block dimensions, thus with an acceleration rate ranging between 1.24 and 1.48 mm/s.

It is worth underlying that the time step mentioned, for "static" analyses in 3Dec, is purely computational and does not have an actual time-history meaning, it is accommodated on how many cycles the program is asked to find solution.

To evaluate x-component acceleration necessary to cause collapse it is essential to control the stability of the solution, which means identifying the step number for which the out-of-balance force has an unacceptable increase, here evaluated as 1%000 global weight of the model.

One of the scope of these analyses, in addition to determine the pure value of the load multiplier, was to determine the nature of the collapse, i.e. rocking or sliding. To this end, a control on the increase of the work done by frictional forces have been considered and the diagram of friction work over step number has been superposed to the one of out-of-balance force versus step.

Slight changes or clear instability of the OOB force in relation with sudden increase or substantial stability of the work done by friction forces became clear through the exploitation of a logarithmic axis for ordinates. In so doing, the trend of the friction work diagram tells if sliding is increasing or not at a specific step for which the outof balance force reached the desired threshold. Figure 5.4 shows the trend of the superposed diagrams for failure driven by sliding or rocking. Abscissa axis reports the step number, while ordinate axis does not show any unit for the inconsistent nature of the dimensions of superposed diagrams.

Figure 5.5 reports values of multiplier of loads as deduced from 3Dec analyses superposed to experimental values and numerical estimation of the minimization procedure, Section 3.3.1. Friction angles considered are related to friction angle typical of AAC blocks and 45°, resulting in a friction coefficient equal to 1.

Moreover, a sensitivity analysis have been considered for each arch model increasing values of friction coefficient among blocks until finding the threshold between sliding and rocking. In so doing a failure-mode domain for equivalent static input could be assembled. Figure 5.6 plots for each Sharpness ratio considered, a curve which separates rocking from sliding with the increase in the friction coefficient for increasing values of slenderness, the space under each curve represent a sliding failure and upper subspace identify rocking failure. For Sh = 1, hinging mechanism can hardly be activated unless



Fig. 5.4 Determination of acceleration necessary for collapse in 3Dec

it is assumed a friction coefficient greater than one, which can be thought as a slightly cohesive behaviour in the interface.

It is worth underlying that outcomes of Figure 5.6 are sensitive to joint stiffness considered; however, the aim of this representation is to highlight the different nature of the response depending on the geometry of the profile for values of joint stiffness, Table 5.1, refined to get threshold acceleration levels, Figure 5.5. As a general trend rocking mechanism can be naturally activated for most plausible thickness values and low sharpness, consistently with outcomes of experimental tests. For high Sharpness, sliding becomes the governing failure mode.

To address the dynamic response of specimens under sine pulse of the shake table test, an evaluation of the natural frequencies of oscillation of the tested models has been carried out. Natural frequency analyses in 3Dec are implemented assuming that the behaviour of system interfaces is elastic and that the kinematic variables are the 6 degrees of freedom for each block, namely 3 translations and three rotations. Then, the assumed diagonal mass matrix is assembled with three entries for block mass and three moments of inertia in the global reference system, which are generally not coincident with the principal moments of the block.



Fig. 5.5 Load multipliers evaluated through 3Dec superposed to experimental values, Section 4.2.1, and values deduced from minimization procedure, Section 3.3.1



Fig. 5.6 Rocking-sliding threshold for three Sharpness for increasing ring thickness



Fig. 5.7 Determination of natural frequencies of tested specimens in 3Dec

The global stiffness matrix, which relates the forces and moments applied to the blocks by their neighbours with the block displacements and rotations, is assembled assuming small displacements. For each of the two blocks in contact, possible displacements and rotations determine 12 configurations, for each of whom axial and shear forces are calculated depending on stiffness and contact area. The forces and moments that result at the centroid of each of the 2 blocks provide the columns of the contact stiffness matrix for each of the 12 configurations. Adding the elementary sub-contact matrices leads to the stiffness matrix of the contact between the 2 blocks. The global stiffness matrix is obtained by assembling all the contact matrices.

Then the evaluation of the natural frequencies is straightforward, although the ordering of may not be exactly ascending. For each model, the first eighty frequencies have been evaluated and compared. For the sake of clarity, values have been reported in Figure 5.7, merging same sharpness

Comparing natural frequencies of models, it is possible to identify a set of low frequencies that clearly involve the whole structure of the arch and a set of higher frequency values. This last group of natural frequencies involve vibrations among single or multiple blocks, instead of the complete structure, and explains well some differences in the response of the specimens, as it will be reported hereafter.

In particular, for increasing values of Sd, the set of higher frequencies drops until reaching for Sd = 0.2 and 0.25, Figures 5.7c and 5.7d, the upper dashed line, which indicates the beginning of the frequency band of input pulses. Moreover, for the highest Sd value, Figure 5.7d, the total number of the sub-vibration modes augments.

It is also worth noting that for Sh = 0.2 and 0.6 a second sub-vibration set, i.e. short plateau of the highest frequencies recorded between the 30th and the 40th modes, is clear but for Sh = 1 involves only few modes for low Sd values and vanishes for the highest Sd value, Figure 5.7d.

Regarding representation of shake table tests in 3DEc, records from the accelerometer placed on the shake table during tests have been exploited as input time-history. Figures 5.8 and 5.9 shows results of predictions of 3Dec models for Slenderness values equal to 0.15 and 0.2 and frequencies between 2Hz and 10Hz, superposed to values deduced through the four-bar linkage models and experimental ones.

Outcomes from 3Dec agree with experimental tests better than the four-bar linkage model in determining the increase trends of load multipliers, although predicted values are not conservative estimations. Nevertheless, the DEM modelling has demonstrated



Fig. 5.8 Failure for experimental sine pulse input. Comparison among shake table tests, Section 4.3, values deduced from minimization procedure, Section 3.3.1, and evaluated through dynamic analyses in 3Dec for increasing Sh and Sd = 0.15



Fig. 5.9 Failure for experimental sine pulse input. Comparison among shake table tests, Section 4.3, values deduced from minimization procedure, Section 3.3.1, and evaluated through dynamic analyses in 3Dec for increasing Sh and Sd = 0.20

effective in detecting the activation of those sub-vibrations modes detected during tests and confirmed by natural frequency analyses for thick and sharp profiles when subjected to higher frequency inputs, Figures 5.8b, 5.9a and 5.9c. Energy dissipation provided by the mutual rocking among blocks indeed triggers a sudden increase in the capacity not properly tackled by the four bar linkage model.

Moreover, through crossed comparison of series, it is seen how the increase in thickness affects the response differently for different sharpness. In particular, for Sh = 0.2 and Sd = 0.15, Figure 5.8a, was fully activated and four hinge mechanisms with failure during second half cycle of motion were recorded during tests, Figure 4.13. 3Dec results are more conservative than the four-bar model in the estimations for low frequencies but become non-conservative for higher frequencies.

For medium sharp profiles instead, Figure 5.8b, evaluations of 3Dec are consistent with actual behaviour of specimens, compare Figures 4.16d and 5.11, although estimated failure thresholds are higher than the experimental ones. In particular, mechanism layouts superpose and collapse occur for rocking, after block twisting during second half cycle of motion, such a phenomenon is noticeable looking at the displaced configuration of blocks with respect to initial geometry.

For the sharpest profile, DEM and four-bar failure threshold are conservative, however the experimental response mode, i.e. remarkable sub-vibration and failure for direct sliding, Figure 4.20 and 4.21a, is tackled by DEM modelling, Figure 5.12, while rocking response, although providing same conservativeness, can only provide a lacking interpretation of mode of failure.

Increasing Sd enhanced the sub-vibration phenomenon for all Sh during experimental tests, Figure 4.13b, 4.17 and 4.21, and 3Dec estimations become consistent with the trend, Figure 5.9, although keep overestimating in terms of expected thresholds.

5.3 Summary

Analytical and numerical modelling of tests suggested further considerations:

• The two bar model proposed is a very simple scheme, independent from friction coefficient and completely effective in representing the three-interface mechanism when occurs.



Fig. 5.10 Sh02Sd015-8Hz, 3Dec modelling of the response, hinging during 1^{st} half cycle of motion and successive hinge moving



Fig. 5.11 Sh06Sd015-8Hz, 3Dec modelling of the response, hinging during 1^{st} half cycle of motion and successive block twisting during 2^{nd} half cycle of motion



Fig. 5.12 Sh1Sd015-8Hz, 3Dec modelling of the response, partial sliding and chaotic block twisting during 2^{nd} half cycle of motion

- Reducing ring dimensions to take in to account defects embodies a partial safety factor, can become scale dependent, but enable a safe assessment.
- Numerical evaluations of load multipliers through 3DEC offered good estimations and the possibility to distinguish those profiles apparently failed for rocking but where partial sliding sensibly reduced capacity.
- Systematic analyses in 3DEC enabled to define the likelihood of sliding or rocking for different sharpness and increasing thickness
- Inputting velocity records in 3Dec made possible a comparison in dynamic sense among plausible thickness.
- Dynamic comparison of experimental results with DEM and numerical predictions revealed a singular behaviour for medium sharp profiles compared to others, i.e. effectiveness in energy dissipation also for thin profiles and velocity accommodation to keep rocking.

Chapter 6

A case study: The Noh-Gunbad Mosque in Balkh, Afghanistan

6.1 Overiview

The Noh-Gunbad Mosque in Balkh is certainly the most ancient mosque in Afghanistan, and one of the oldest monuments of all Islamic world; the few still standing parts are constantly exposed to weathering degradation and severe seismic risk, Figure 6.1. Such extraordinary building has an enormous importance due to the magnificent gypsum decoration and it is now the subject of a first intervention of reinforcement in order safeguard its valuable testimony.



Fig. 6.1 General view of the mosque by Pugachenkova (1968)

The Noh Gunbad mosque is currently in a state of extreme weakness, having already lost all its domes and large part of the arches system. The remaining parts required a strong strengthening action and in particular, the two still standing big arches. The intervention, carried out in a difficult situation, attempted to consolidate the system, close to collapse due to non-balanced thrusts, trying to achieve an improvement also from a seismic viewpoint.

The "Noh Gunbad", which in Dari language means "nine domes", is built a few kilometers from the city of Balkh in the North of Afghanistan, during the last decades of II HG (VIII A.D.) Century under the Governor Fazl the Barmakids, according to recent studies by Adle (2011).

The building, named by local people Hâji-Piyâda in memory of an holy man buried nearby, has a perimeter of twenty meters and is closed on three sides, Figure 6.2; inside, fifteen big arches used to support the domes, Figures 6.3 and 6.4. The whole system relied on a system of columns partly isolated.

All the internal surface of the monument is covered by a decoration made of carved gypsum and representing different stylized beautiful drawings (once coloured too), Figure 6.7.

The perimeter walls are made by three layers of different materials: the external part by rammed earth, the middle layer by adobe (useful for the implementation of niches and details of architectonic need), finally the inner part consisting of columns, supporting arches and domes, made by baked bricks, Figure 6.8. Gypsum mortar was employed for columns while mud mortar for arches and domes.

An historic earthquake occurred few decades after completion of the Mosque (819AD-203HG), and probably caused the partial collapse of the mosque. The thick layer of debris hid and somehow protected half the height of the columns.

In the 60s, the monument was made known to the international community by the studies of Chirvani (1969); Golombek (1969); Pugachenkova (1968) and, Adle (2011), Figure 6.5, who agree in addressing this site as a unique example of early Islamic architecture influenced by the Sassanid culture.

In the seventies the still standing parts of the mosque were covered with a metal roof to face weathering processes, Figure 6.6; more recently, a brickwork pillar was added to support the most damaged arch, Figure 6.4a.

Few years after the end of the Taliban regime, the French Archaeological Delegation in Afghanistan (DAFA) set off preliminary measures to defend the Mosque from further deterioration and loss. Together with Associazione Giovanni Secco Suardo and World Monument Fund, DAFA arranged a team of experts, involving University of Florence



Fig. 6.2 Plan of the Mosque



Fig. 6.3 View of the still standing arches



(a) Front view of the West arch



(b) Front view of the South arch

Fig. 6.4 Views of the still standing arches of the Mosque $\,$



Fig. 6.5 Historical images of the mosque by Pugachenkova $\left(1968\right)$



Fig. 6.6 Metal roof to protect the site work



Fig. 6.7 Detail of the gypsum decoration

and French restores, to define a conservation plan. In 2011, Aga Kahn Trust for Culture (AKTC) foundation supported the first phase of consolidation works related to the standing arch system, which is now completed.

6.2 Static consistency of the arched system of the Mosque

An intrinsic weakness connected to the original arched system consists in the absence of a containment wall on the façade (East side used as entrance side), Figure 6.2. Indeed, the thrust lines for vertical loads of arches (today collapsed) perpendicular to the East side were so close to the external perimeter of columns that any ground settlement or dynamic action made the thrusts shifting outside, activating a mechanism and a "domino" effect for the whole vaulted system. Therefore, the only still standing portions are two archways that are today in very bad conditions, after twelve centuries of life, when the strengthening and conservation care emerges.

As consequence of the collapse of the domes the whole system is quite deformed: the three columns are tilted (most inclined the North one) and the archway in front of Mihrab shows two great cracks close to the haunches while the key brickwork partially collapsed. A clear mechanism, constituted by four hinges, appears complete due to the northern column fracture, located at the level of the debris (now the walking surface). The archway stability became worse – as the large cracks at the haunches, Figure



Fig. $6.8\,$ View of the perimeter wall



Fig. 6.9 Detail of the deep crack in the west arch ring

6.9, and the falling of the key clearly attest – because of a sudden motion during last decade, Figure 6.10.

Thus, a rough brick pillar was placed in the middle of the span in order to prevent a final collapse, Figure 6.4a, such new configuration could be explained by the intervention of a seismic action. The second, orthogonal, archway shows a similar situation (as regards location and shape of the fractures) even if with a lower level of damage. The whole system suffered a complex motion, asymmetric, also characterized by some torsional effects. The weakness of the arch brickwork (owing to the key texture and the mud mortar) certainly played an important role together with the loss of the original thrust control, but, on another side, it can be at least noted the positive effect of the presence of debris.



Fig. 6.10 Detail of the arrangement of bricks at crown

The two orthogonal arches standing on three columns constitutes the outstanding part of the complex. The geometry of the Noh Gunbad pointed arches is a consequence of the existence of a double centre of curvature. The semi arch can be equated to an incomplete quarter circle, whose centre is shifted, with respect to the vertical symmetry axis, on the opposite side on the diameter.

The arches can be considered as two pointed barrel vaults, width is around 1.50 meters and a 65 cm thick ring arranged in two and half layer of squared baked bricks, side 23 cm, and mud mortar. Adobe brickwork and mud mortar constitute the masonry superimposed to arch ring, necessary for thrust balance and regularization for the connection to system of domes.

Concerning the vertical bearing structure, the brickwork of columns has an annular arrangement in baked bricks, Figure 6.11, average dimensions 5x10x28 cm bonded with gypsum mortar.

The overall quality of the constructive technique, which constitutes a significant testimony of the local culture in the transition between Sassanid and Abbasid age, is not uniform and exhibits some limitations, e.g. in arch implementation. Indeed, the difficulty in executing the crowning for pointed arches without making use of a specifically carved element, as happens for Gothic stone arches, means a weak crown. In this case, the geometrical discontinuity is coarsely implemented with littler elements and thicker joints, showing an arrangement with horizontal layers instead of a proper radial layout. Consequently, the arch key becomes the most vulnerable area of the ring, affecting the whole response.



Fig. 6.11 Detail of the arrangement of bricks in the columns

6.3 Physical-mineralogical and mechanical characterization of materials

Mechanical, physical and mineralogical tests were carried out on most of materials on a limited number of samples because of the obvious difficulty of transfer (achieved thanks to the availability of DAFA and Italian Air Force) at the Laboratory of Department of Architecture (Florence University) and at ICVBC-CNR (Florence, Italy).

The samples were identified and numbered and they refer respectively to: 1. column brick, 2. thin brick 'b type' (belonging to arch or vault system); 3. thin brick 'a-type'(belonging to arch or vault system); 4. adobe brick; 5. column mortar; 6. little piece of column brick; 7. adobe mortar.



Fig. 6.12 Specimens cut from sample 3

Sample	Sample	Number of	Bulk Density	Compressive Strength
nr.	ref.	specimens	$[\mathrm{kg}\mathrm{m}^{-3}]$	[MPa]
1	Column brick	4	1396	10.46
2	Brick B	5	1515	21.26
3	Brick A	8	1562	25.05
4	Adobe brick	4	1703	2.16

 Table 6.1 Results of mechanical tests on the Noh-Gunbad specimens

The samples 1, 2, 3, and 4 were subjected to compressive tests while physical and mineralogical analysis on samples 2, 4, 5, 6, and 7 were performed at the Institute for the Conservation and Promotion of Cultural Heritage (C.N.R. of Florence), Figure 6.12.

The samples were cut and regularized in order to subject them to compressive tests. The results were to be simply indicative because of the small number of the specimens, not of standardized sizes, of remarkable, not homogeneous density. Nevertheless, these tests are very important in order to acquire information on the mosque's original materials and structures and for the purpose of designing a structural consolidation. The regularized specimens were numbered, measured, weighed, and finally subjected to a mono-axial compressive test through a hydraulic press, controlled by a load cell with a capacity of 20 kN and connected to a TDS data recorder (that made it possible to draw the load-displacements diagrams).

From test results showed in Table 6.1, it is worth noting that brick type A and B (belonging to arches and vaults) exhibit high compressive strengths, comparable to contemporary bricks. Their density is not too high, and quite similar. On the other hand, samples of column bricks show around half of compressive strength of brick types A and B, and also a lower density. Adobe bricks exhibit, obviously, the lowest compressive strength (five or ten times lower in comparison with column or arch bricks). Moreover, their mixture is not very homogeneous; indeed, the material

Minoral	Sample ref.					
winnerai	Brick B	Adobe A	Mortar	Col. Brick	Adobe B	
Quartz B	17	22	tr	12	23	
Feldspar	4	6	tr	2	10	
Calcite*	17	29	23	-	29	
Gypsum	-	-	Х	tr	tr	
Gehelinite	Х	-	-	Х	-	
Clay Minerals	60	43	-	8	38	

Table 6.2 Mineralogical composition of the Noh-Gunbad specimens. *Data obtained throughcalcimetry test

Table 6.3 Clay composition of Adobe bricks of the Noh-Gunbad mosque

Minoral	Sample ref.			
minerai	Adobe A	Adobe B		
Kaolinite	30	40		
Illite	40	35		
Chlorite	15	15		
Illite-smectite	15	10		

shows hollows and discontinuities. However, the average strength value is within a range expected for raw old bricks.

X ray diffraction tests and calcimetry tests were carried out to determine the principal mineralogical composition and the amount of calcite and of all samples, and the composition of clay minerals on adobe samples. Table 6.2 and 6.3 summarize test results.

Regarding the physical analysis, it could be observed that column mortar is a fundamentally gypsum mortar. However, it was impossible to carry out compression tests on this mortar because of the scarcity of the sample. In short, we can deduce that there was a particular, intentional hierarchy in the structural design of the mosque. The different mechanical engagement of arch and column bricks is perfectly corresponding with the different width of the structural cross-sections (the column cross section is very wide). Adobe bricks were used in walls with only slight structural purpose and their mechanical performance is to carry their own load. In the adobe case, the dominant problem is connected with physical decay and with the loss of connection (and sometimes verticality) of the brickwork. On the contrary, these last issues do not concern the column brickwork.

Mochanism ID	λ	hinge position [rad]			
Mechanishi 1D		β_1	β_2	β_3	β_4
Ring	0.421	0.009	0.787	2.096	3.138
Half-height abutm	0.361	-0.698	0.865	1.655	4.127
Full-height abutm	0.239	-0.987	0.654	1.633	4.361

Table 6.4 Multiplier of loads and mechanism layout for the arches of Noh-Gunbad mosque

6.4 Vulnerability assessment

The minimization of load multiplier, λ , was carried out with the procedure exposed in Section 3.3.1. Values of the minimum multiplier of loads and related hinges positions for mechanisms involving arch ring, and arch ring and abutments (in the actual configuration, i.e. half height, and original configuration, i.e. full height), are reported in Table 6.4, mechanisms layout are reported in Figure 6.13



Fig. 6.13 Mechanisms involving arch ring or half and full height of abutments of the arches of Noh-Gunbad mosque

Transforming the mechanisms so evaluated into four-bar linkages, similarly to the procedure described in Section 3.3.2, it was possible to define failure domains for rectangular shaped pulse Type A, Equation 3.20, under the linearised form of the distinctive EOM, Equation 3.17, considering the collapse condition related to the maximum of potential energy, Equation 3.19.

Also, employing the non-linear form of Equation 3.17, and the aforementioned assumption for the impact model, Equations 3.27,3.28, 3.29,3.30 and 3.31, failure domains for pulse Type B, Equation 3.21, were evaluated as well. Values of the coefficients of restitutions range between 0.972 for the ring mechanisms to 0.954 and 0.968 for the full and half height mechanisms respectively.



Fig. 6.14 Failure domains for rectangular shaped pulse Type A and B for the Noh-Gunbad Mosque

Figure , shows the comparison among failure domains for Pulse Type A and B for the three mechanisms identified for the Noh-Gunbad mosque. As expected, evaluations of the linearised form of EOM are more conservative than those that consider failure during second half cycle of motion, which anyway precedes the failure without impact for the considered layouts.

6.5 Seismic Hazard and safety assessment

6.5.1 Afghan seismicity

Afghanistan is located in a geologically active part of the world where the northwardmoving Indian and Arabian plates are colliding with the southern part of the Eurasian plate. This collision is responsible for the world's highest mountains and causes slips on major faults that generate large, often devastating earthquakes.

Inner Afghanistan is relatively stable area, coinciding with southern part of the Eurasian plate, while the most relevant tectonic activities are located at its boundaries, Figure 4.

West and east of the south boundary of the Eurasian Plate there are the Arabian and the Indian Plates respectively, both causing subduction under Eurasia at rates of nearly 22mm/yr and 40 mm/yr respectively as reported in Ambraseys and Bilham



(a) Eurasian, Arabian and In-(b) Major faulting of Afghanistan and dian plates layout surrounding area

Fig. 6.15 Afghan seismicity, Ambraseys and Bilham (2003)

(2003) thanks to GPS data recently collected, and presented in Sella et al. (2002). The south contact contour between these three plates is located nearly in the middle of the Oman Gulf. Inner the south Eurasian boundary, southwester Pakistan and southeaster Iran, a wide deformation belt is characterised by the presence of many secondary north-dipping faults and related folds leaning eastward, Figure 6.15. The direction of the Indian plate convergence is nearly normal to the eastern trending contact boundary of the Eurasian plate, therefore reverse faults and related folds are produced much more than strike-slip faults. All this affects most of the south area of Afghanistan.

As regards to the lateral Afghan borders, east and west plate boundaries produce north-north-east movement and north-north-west movement respectively. The oblique nature of the slip movement generates a trans-compressive convergence. The western border of the Eurasian plate interacting with the Arabian is completely within Iran and thus outside the area of our interest.

The east boundary of the Eurasian plate has generated instead a wide area of deformation involving also Afghan territories. Near this contact line, thrust faults can be found as a sign of the upper crustal strain. Starting from the west contact line and proceeding eastern towards Afghan territories, a north-north-east trending belt of sinistral strike slip faults are found until the Chaman fault Wheeler et al. (2005).

This 900 km long fault has been the site of infrequent, moderate to large earthquakes, Kazmi and Jan (1997). The history of ruptures and the size in the offsets measured on this fault may indicate that a significant part of the movement among the Indian and the Eurasian plates occur along this important tectonic boundary. In this area, also some minor reverse faults were found that strike east and north east dipping northerly,



Fig. 6.16 Location of main activities in Afghanistan recently recorded by USGS, Ambraseys and Bilham (2003)
Kazmi and Jan (1997); Quittmeyer and Jacob (1979). The presence of such relevant plate boundaries causes two depth ranges of activity in Afghan seismicity: crustal and mantle seismicity, Wheeler et al. (2005).

In fact, deep activities are associated with the subduction of Oceanic crust and can be found in the Makran area, south of Afghanistan and in the Hindu Kush region in the north of the country at the Pakistan border. Here most events are located at intermediate depth and cause both considerable damage and wide area of shaking sensation. As regards for other areas, seismicity involves more shallow layers, ruptures are located in the range of 30 km depth at most and lower crust is aseismic, Ambraseys and Bilham (2003), Figure 5. The complex of the deformation belt so described acts on the rim of the Eurasian promontory which constitutes part of Afghanistan, while, the interior of the promontory complex, i.e. in the central and western part of Afghanistan is much less active, Figure 6.16.

Afghanistan earthquakes have been assigned to very different locations, magnitudes and depths by different authors in recent year. Even if the states surrounding Afghanistan have quite complete records, Afghanistan have only few historical registration thanks to Persian documents for earlier ages, and some French and British reports due to diplomat activities in the area. Conversely, recent instrumental data are available from the end of the last century thanks to Russian and Indian stations. Most of the pre-historic earthquakes were reported in relation with trade routes, remained unchanged from ancient era. This instance affected the quality of records, since the central desert Afghanistan is rich neither in water nor in urban centres. Thus, historical documentation demonstrate the difficulties in retrieving macro seismic information outside Kabul, which has historically been the major centre of the country.

Added difficulties in historic records are found with reference to north-western Afghanistan where both crustal and mantle activities are present, Quittmeyer and Jacob (1979). In fact, superficial ruptures can cause severe damage to things and people, however, if they occur in remote areas is highly improbable that historic records have ever considered them. In contrast, deeper activities cause slight damage on surface, but the energy release is much higher, shaking effects are felt further away, and thus the event would probably be reported. Especially for the area of our interest, this fact certainly has affected historic record. Strong historical events recorded in the area of our interest can be dated to 819, 1410 and 1911.

Thanks to results reported in Boyd et al. (2007) a preliminary hazard map of Afghanistan and related hazard curves, Figure 6.17, reveal a 0.375g expected acceler-

ation for 10% probability of exceedance in 50 years and 0.75g for 2% probability of exceedance in 50 for 0.2s-period buildings.



Fig. 6.17 Preliminary hazard curves for the Mazar-e Sharif area, Boyd et al. (2007). The solid black line is the seismic hazard curve resulting from a combination of all sources. The dashed-dot curves reflect contributions to seismic hazard using the ground-motion relation of Ambraseys et al. (1996). Solid curves represent Western United States ground-motion relations. The red curves are the contribution to seismic hazard from fault sources that are characteristic, and the green curves are for Gutenberg-Richter. The blue curves represent contributions from background seismicity less than 50-km depth, while the cyan curves represent contributions from seismicity between 50- and 100-km depth (solid) and 100- and 250-km depth (dashed-dot)

6.5.2 Arches safety assessment

In the case of the Noh Gumbad mosque, masonry arches behaved as a composition of rigid blocks. Local mechanisms of overturning are caused by both out of plane or in-plane actions. It is thus necessary to define different damage states: the mechanism activation (i.e. damage limit state and the corresponding acceleration threshold) and ultimate condition (i.e. the collapse limit state and its corresponding displacement capacity Lagomarsino and Resemini (2009). Since no design code in Afghanistan refer to safety assessment of unreinforced masonry building, nor any prescription on monumental building, methodologies reported in Ministero Infrastrutture e Trasporti (2008a,b); Presidenza del Consiglio dei Ministri (2011) of Italian Design Codes specific for monumental masonry were exploited here. Once the multiplier of load is evaluated, a comparison between demand and capacity acceleration can be performed. According to Ministero Infrastrutture e Trasporti (2008a,b); Presidenza del Consiglio dei Ministri (2011), the capacity acceleration of a rocking mechanism can be evaluated as:



(a) FRP strips on arch extrados

(b) Strengthened arch system

Fig. 6.18 Strengthening interventions on the arches of Noh-Gunbad mosque, Abassi et al. (2016)

$$a_0^* = \frac{\lambda(\sum_{i=1}^n P_i \delta_{x,i})^2}{\sum_{i=1}^n P_i \delta_{x,i}^2}$$
(6.1)

Where, λ is the kinematic multiplier, P_i is the i-th dead or live load and $\delta_{x,i}$ is the virtual horizontal displacement of the application point of the i-th load P_i . Then, this acceleration can be compared to the requested acceleration, see Section 6.5.1.

The main action of strengthening of the arch system was carried out, subsequently an accurate consolidation of the whole damaged masonry brickwork, by the application of FRP strips on the arches extrados, Figure 6.18a. The detailed description of such intervention, carried out by MBrace system (BASF), with all technical specifications, can be found in Abassi et al. (2016).

The safety assessment has been considered for the unreinforced and reinforced configuration with FRP strips. The capacity of the FRP strips are ultimate tensile strength ftu= 1964 MPa and equivalent thickness t = 0.047mm for strengthening a strip 680 mm width. The multiplier of loads have been evaluated taking into account the contribution offered by the stabilizing work offered by the reinforcement. Specifically it is assumed that the hinges position which minimizes the load producing the first displacement of the structure do not change during its further displacements. Results are reported in Table 6.5.

	PGA	λ_{UR}	$a^*_{0,UR}$	Safety index UR	λ_{FRP}	$a_{0,FRP}^{*}$	Safety index FRP
	$[ms^{-2}]$		$[ms^{-2}]$			$[ms^{-2}]$	
Ring		0.43	3.66	0.98	0.58	5.95	1.61
Half-height	3.67	0.41	3.33	0.91	0.66	5.43	1.47
Full-height		0.23	2.88	0.78	0.31	4.01	1.01

Table 6.5 Safety assessment for different mechanism layouts in reinforced and unreinforcedconfigurations of the arches of Noh-Gunbad mosque

6.6 Conclusions

The vulnerability of the Noh Gunbad Mosque in Balkh, Afghanistan has been presented in its featuring elements and intrinsic weaknesses. The Mosque has been object of extensive investigations and experimental testing aimed at defining proper strengthening interventions. Considering the difficulties in defining a reliable local seismic hazard, a preliminary safety assessment has shown the capability of composite material intervention, which offer tangible capacity improvement against seismic actions without adding nearly any inertial mass. In this framework, limit analysis is demonstrated to be an effective tool, even though conservative, to evaluate the response behaviour of masonry structures.

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Chapter 7

Concluding remarks and future research

7.1 Main findings

The main objective of this research is to uncover driving features in the response of pointed arches under horizontal actions, Section 1.3. To get a deep insight, static and dynamic approaches have never been separated, methods for the assessment of masonry arches have been created or exploited and improved. The main findings can be listed as follows:

- Onset of motion: Four hinge mechanisms for pointed arches differ from those of circular profiles and are deeply affected by geometrical Sharpness. Parameters directly influencing vulnerability of circular arches, like the embrace angle, do not have a similar effect on pointed, Section 3.3.1.
- *Motion around equilibrium position*: Safety assessment in this framework is more rapid and conservative, although not able to tackle few relevant aspects. Ring thickness increase for high pointedness does not lower vulnerability for short duration inputs, Section 3.3.2
- Non-linear form of motion: Safety assessment requires more computational effort and permitted to uncover the change in failure mode from second half cycle to first for medium sharp and thick profiles, although this behaviour is phenomenon deeply affected by the impact model adopted, Section 3.3.2.

- *Geometrical effectiveness*: The medium sharp profile, known in architecture practice as the pointed third, exploits the increase in ring thickness better than other profiles. Low sharp profiles, widely spread in seismic prone areas of east and middle est countries, benefit from not being perfectly circular, i.e. reduced thrusts at abutments, without recording same sliding problems of high pointedness, Sections 3.3.1, 4.2 and 4.3.
- *Misleading practice*: Representing the hinging layout symmetrically with respect to the angle of embrace brings to non-conservative estimations, often employed for circular arches, Section 3.3.2.
- *Tilt tests*: Load multipliers deduced from tilt tests tend to superpose for high thickness ratios. Indeed, partial sliding was recorded for all tested profiles, connected with inherent limited friction reached with AAC blocks, but for sharp and thick profiles, the sliding mechanism is global, driven by three interfaces, Section 4.2.1.
- Shake table tests: Collapse mode is deeply influenced by thickness and input frequency, turning out to occur either during second half cycle of motion for rocking or mixed rocking-sliding or for direct sliding during first part of motion, Section 4.3.
- *Tilt tests modelling*: Compared with experimental results, predictions of four hinge mechanisms always overestimates capacity and cannot tackle the sliding phenomenon, this is remarkably relevant for high pointedness and high thickness. Reducing ring dimensions to take in to account defects embodies a partial safety factor, can become scale dependant, but enable a more conservative safety assessment. The two bar model proposed is a very simple scheme, independent from friction coefficient and completely effective in representing the three-interface mechanism, Section 5.1. Numerical evaluations of load multipliers through 3DEC offered good estimations and the possibility to distinguish those profiles apparently failed for rocking, but actually underwent partial sliding, which decreased sensibly expected capacity. Systematic analyses in 3DEC enabled to define the likelihood of sliding or rocking for different sharpness and increasing thickness, Section 5.2.
- Shake table tests modelling: Four-bar linkage predictions for slow sine pulses, though qualitatively satisfying, overestimate capacity. For higher frequencies and thin profiles the scatter between numerical and experimental values is lower than for thicker, and four bar linkage predictions always offer a safe estimation.

Comparison of experimental results with 3Dec and numerical predictions revealed a singular behaviour for medium sharp profiles compared to others, i.e. effectiveness in energy dissipation also for thin profiles and velocity accommodation to keep rocking, Section 5.2.

7.2 Contributions

Considering main findings reported above, this dissertation addressed the objectives listed in Section 1.3. In particular, this research provided a deep insight in the behaviour of pointed arches under horizontal loads, highlighted main differences with respect to circular profiles and set up specifically designed tools. In particular:

- *Non linear optimization*: The actual profile (non-linear piecewise continuous) has been effectively represented through a non linear analytical model, applicable also to circular arches, which does not require any a priori discretization in blocks.
- *Rocking Spectra*: Non linear form of motion have been systemically integrated and rocking spectra for pointed arches have been provided as a useful tool to assess safety for main pulse driven inputs.
- *Two-bar model*: A very simple though effective kinematic model is proposed to represent the sliding mechanism independently from friction coefficient.
- Enhanced application of Discrete Element Modelling: Tilt tests have been modelled considering actual friction coefficients of tests and a sensitivity analysis permitted to define possibility of rocking or sliding for different profiles.

7.3 Future research

Based on achieved results, further objectives can be identified, among these:

- Assessment of the role played by live loading and infill, so beneficial for the statics of pointed arches
- Analysis for full time histories to evaluate energy maximizing inputs and probability of collapse for given intensities.

- Extend analyses to three dimensional pointed structures like rib vaults or groin vaults generated by increasing-height barrel vaults
- Implementing a dynamic version of the two-bar model and including non conservative contributions

Few final considerations seem necessary. The possibility to assess the robustness of a numerical model with respect to another was possible only thanks to a closed loop approach, in a plan-do-check-act framework, indeed only the experimental campaign permitted to highlight and refine the commonly adopted numerical tools. Hence, further experimental tests on reduced scale models made of rigid blocks will be carried out.

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Appendix A

Minimization code

```
"featuring characteristics of the arch"
RC = 0
gm = 0
p = 0
Sd = 0
RI = (p + RC) - (1/2)*Sd*(p + RC)
RE = (1/2) * Sd*(p + RC) + (p + RC)
\[CurlyTheta] = ArcSin[p/RI]
RE1 = (1/2)*(Sqrt[2]*Sqrt[p^2*Cos[2*gm1] - p^2 + 2*RE^2] -
    2*p*Cos[gm1])
RE2 = (1/2)*(Sqrt[2]*Sqrt[p^2*Cos[2*gm2] - p^2 + 2*RE^2] -
    2*p*Cos[gm2])
RE3 = (1/2)*(Sqrt[2]*Sqrt[p^2*Cos[2*gm3] - p^2 + 2*RE^2] +
    2*p*Cos[gm3])
RE4 = (1/2)*(Sqrt[2]*Sqrt[p^2*Cos[2*gm4] - p^2 + 2*RE^2] +
    2*p*Cos[gm4])
RI1 = (1/2)*(Sqrt[2]*Sqrt[p^2*Cos[2*gm1] - p^2 + 2*RI^2] -
    2*p*Cos[gm1])
RI2 = (1/2)*(Sqrt[2]*Sqrt[p^2*Cos[2*gm2] - p^2 + 2*RI^2] -
    2*p*Cos[gm2])
RI3 = (1/2)*(Sqrt[2]*Sqrt[p^2*Cos[2*gm3] - p^2 + 2*RI^2] +
    2*p*Cos[gm3])
RI4 = (1/2)*(Sqrt[2]*Sqrt[p^2*Cos[2*gm4] - p^2 + 2*RI^2] +
    2*p*Cos[gm4])
```

```
RIpi = Sqrt[2*RI<sup>2</sup> - 2*p<sup>2</sup>]/Sqrt[2]
REpi = Sqrt[2*RE<sup>2</sup> - 2*p<sup>2</sup>]/Sqrt[2]
Rb = (1/2)*(Sqrt[2]*Sqrt[p^2*Cos[2*(Pi/2 - ki)] - p^2 + 2*RE^2] -
    2*p*Sin[ki])
"angles identifying blocks in the global and local coordinate systems"
gma = Pi/2 - [CurlyTheta]
dt1 = ArcCos[(-p<sup>2</sup> + RE<sup>2</sup> + RE1<sup>2</sup>)/(2*(RE*RE1))]
dt2 = ArcCos[(-p<sup>2</sup> + RI<sup>2</sup> + RI2<sup>2</sup>)/(2*(RI*RI2))]
dt3 = ArcCos[(-p<sup>2</sup> + RE<sup>2</sup> + RE3<sup>2</sup>)/(2*(RE*RE3))]
dt4 = ArcCos[(-p<sup>2</sup> + RI<sup>2</sup> + RI<sup>4</sup>)/(2*(RI*RI4))]
bt3 = ArcCos[(p<sup>2</sup> + RE<sup>2</sup> - RE3<sup>2</sup>)/(2*p*RE)]
agm3b = -bt3 - ([CurlyTheta] + Pi/2) + Pi
bt4 = ArcCos[(p<sup>2</sup> + RI<sup>2</sup> - RI4<sup>2</sup>)/(2*p*RI)]
agm4b = -bt4 - ([CurlyTheta] + Pi/2) + Pi
"Weigth of blocks"
Pgma = (1/2)*gm*gma*(RE^2 - RI^2)
Pt = (1/2)*gm*((RE - RI)*(REpi - RIpi)*Sin[\[CurlyTheta]])
Ppi = Pgma + Pt
Pgm3b = (1/2)*agm3b*gm*(RE^2 - RI^2)
Pgm4b = (1/2)*agm4b*gm*(RE^2 - RI^2)
Pgm1 = (1/2)*gm*(gm1 - dt1)*(RE^2 - RI^2)
Pgm2 = (1/2)*gm*(gm2 - dt2)*(RE^2 - RI^2)
Pgm3 = Pgm3b + Ppi + Pt
Pgm4 = Pgm4b + Ppi + Pt
W1 = Pgm2 - Pgm1
W2 = Pgm3 - Pgm2
W3 = Pgm4b - Pgm3b
"centre of gravitiy of macroblocks"
yt = (1/3)*Rb*Sin[Pi/2 - ki] + REpi/3 + RIpi/3
xts = (1/3)*(-Rb)*Cos[Pi/2 - ki]
xtd = (1/3) * Rb * Cos [Pi/2 - ki]
xggma = (4*Sin[gma/2]*Cos[gma/2]*(RE^3 - RI^3))/(3*
      gma*(RE<sup>2</sup> - RI<sup>2</sup>)) - p
yggma = (4*Sin[gma/2]^2*(RE^3 - RI^3))/(3*gma*(RE^2 - RI^2))
xggm3b = (4*Sin[agm3b/2]*(RE^3 - RI^3)*
```

```
Cos[(agm3b/2 + \[CurlyTheta]) + Pi/2])/(3*agm3b*(RE^2 - RI^2)) + p
yggm3b = (4*Sin[agm3b/2]*Cos[agm3b/2]*(RE^3 - RI^3)*
     Cos[\[CurlyTheta]])/(3*agm3b*(RE<sup>2</sup> - RI<sup>2</sup>)) -
     (4*Sin[agm3b/2]^2*(RE^3 - RI^3)*Sin[\[CurlyTheta]])/(3*
     agm3b*(RE<sup>2</sup> - RI<sup>2</sup>))
x3B = (Pgm3b*xggm3b + Pt*xtd + Pt*xts)/(Pgm3b + Pt + Pt)
y3B = (Pgm3b*yggm3b + Pt*yt + Pt*yt)/(Pgm3b + Pt + Pt)
xggm4b = (4*Sin[agm4b/2]*(RE^3 - RI^3)*
     Cos[(agm4b/2 + \[CurlyTheta]) + Pi/2])/(3*agm4b*(RE^2 - RI^2)) + p
yggm4b = (4*Sin[agm4b/2]*Cos[agm4b/2]*(RE^3 - RI^3)*
     Cos[\[CurlyTheta]])/(3*agm4b*(RE<sup>2</sup> - RI<sup>2</sup>)) -
     (4*Sin[agm4b/2]<sup>2</sup>*(RE<sup>3</sup> - RI<sup>3</sup>)*Sin[\[CurlyTheta]])/(3*
     agm4b*(RE<sup>2</sup> - RI<sup>2</sup>))
x4B = (Pgm4b*xggm4b + Pt*xtd + Pt*xts)/(Pgm4b + Pt + Pt)
y4B = (Pgm4b*yggm4b + Pt*yt + Pt*yt)/(Pgm4b + Pt + Pt)
xg1 = (4*(RE<sup>3</sup> - RI<sup>3</sup>)*Sin[(gm1 - dt1)/2]*
     Cos[(gm1 - dt1)/2])/(3*(gm1 - dt1)*(RE<sup>2</sup> - RI<sup>2</sup>)) - p
yg1 = (4*(RE^3 - RI^3)*
    Sin[(gm1 - dt1)/2]^2)/(3*(gm1 - dt1)*(RE^2 - RI^2))
xg2 = (4*(RE<sup>3</sup> - RI<sup>3</sup>)*Sin[(gm2 - dt2)/2]*
     Cos[(gm2 - dt2)/2])/(3*(gm2 - dt2)*(RE<sup>2</sup> - RI<sup>2</sup>)) - p
yg2 = (4*(RE^3 - RI^3)*
    Sin[(gm2 - dt2)/2]^2)/(3*(gm2 - dt2)*(RE^2 - RI^2))
xg3 = (Pgma*xggma + x3B*(Pgm3b + 2*Pt))/(Pgma + (Pgm3b + 2*Pt))
yg3 = (Pgma*yggma + y3B*(Pgm3b + 2*Pt))/(Pgma + (Pgm3b + 2*Pt))
xg4 = (Pgma*xggma + x4B*(Pgm4b + 2*Pt))/(Pgma + (Pgm4b + 2*Pt))
yg4 = (Pgma*yggma + y4B*(Pgm4b + 2*Pt))/(Pgma + (Pgm4b + 2*Pt))
XG1 = (Pgm2*xg2 - Pgm1*xg1)/W1
YG1 = (Pgm2*yg2 - Pgm1*yg1)/W1
XG2 = (Pgm3*xg3 - Pgm2*xg2)/W2
YG2 = (Pgm3*yg3 - Pgm2*yg2)/W2
XG3 = (Pgm4b*xggm4b - Pgm3b*xggm3b)/(Pgm4b - Pgm3b)
YG3 = (Pgm4b*yggm4b - Pgm3b*yggm3b)/(Pgm4b - Pgm3b)
"hinge postion"
x1 = RE1 * Cos[gm1]
x2 = RI2*Cos[gm2]
```

```
x3 = RE3*Cos[gm3]
x4 = RI4 * Cos[gm4]
y1 = RE1 * Sin[gm1]
y2 = RI2*Sin[gm2]
y3 = RE3*Sin[gm3]
y4 = RI4 * Sin[gm4]
x2a = (x1*(x3*(y2 - y4) + x4*(y3 - y2)) +
            x^2*(x^3*(y^4 - y^1) + x^4*(y^1 - y^3)))/((x^1 - x^2)*(y^3 - y^4) +
            x3*(y2 - y1) + x4*(y1 - y2))
y_{2a} = (x_{1*y_{2*y_{3}}} - x_{1*y_{2*y_{4}}} + x_{2*y_{1*}}(y_{4} - y_{3}) - x_{3*y_{1*y_{4}}} + x_{3*y_{2*y_{4}}} + x_{3*y_{2}} + x_{3*y_{2}} + x_{3*y_{2}} + x_{3*y_{2}} + x_{3*y_{2}} + x_{3*y_{4}} + x_{3*y_{
            x4*y3*(y1 - y2))/((x1 - x2)*(y3 - y4) + x3*(y2 - y1) +
            x4*(y1 - y2))
\[Eta]g1 = XG1 - x1
Eta]g2 = ((x2 - x1)*(XG2 - x2a))/(x2a - x2)
[Eta]g3 = ((x2 - x1)*(x3 - x2a)*(XG3 - x4))/((x2a - x2)*(x4 - x3))
dtg1 = YG1 - y1
dtg2 = ((y2 - y1)*(YG2 - y2a))/(y2a - y2)
dtg3 = ((y2 - y1)*(y3 - y2a)*(YG3 - y4))/((y2a - y2)*(y4 - y3))
LStab = \langle Eta ]g1*W1 + \langle Eta ]g2*W2 + \langle Eta ]g3*W3
LRib = dtg1*W1 + dtg2*W2 + dtg3*W3
\[Lambda] = LStab/LRib
Table[NMinimize[{\[Lambda], Pi > gm4, Pi > gm3, Pi/2 > gm1,
         gm4 > gm3, gm3 > gm2, gm2 > gm1, gm4 > 0, gm3 > 0, gm2 > 0,
         gm1 >= 0, gm2 < Pi/2,
                     [Eta]g3 \ge 0, dtg3 \ge 0, {gm1, gm2, gm3, gm4},
      Method -> {"NelderMead", "RandomSeed" -> i}], {i, 10}]
```

Appendix B

Flow chart of the computing loop to solve motion problem



Appendix C

Rectangular shaped acceleration input

```
Ecoll = V[\[Alpha]1] - V[\[Theta]cr]
FailDom = \{\{0.11, 10\}\};
agmin = -0.626994 g;
a = -9.81;
While[a <= agmin,
 {bmin = Last[FailDom][[1]];
  bmax = 5;
  While[bmax >=
    bmin, {NEcoll = Re[NIntegrate[ColFun[t], {t, 0, bmin}]],
    Print[NEcoll], Print[bmin], Print[a "a"],
    If [Ecoll - 0.2 \leq NEcoll \leq Ecoll + 0.2]
     a {Print[{bmin, a} "punto dominio"],
       FailDom = Append[FailDom, {bmin, -a}], Break[]}],
    bmin = bmin + 0.005 bmin\}],
  a = a - 0.1 a}]; Print[FailDom]; ListLinePlot[FailDom,
 AxesOrigin -> {0, 0}, PlotMarkers -> Automatic]
```