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ON COMPLEX-VALUED SOLUTIONS TO A TWO-DIMENSIONAL EIKONAL EQUATION. II. EXISTENCE THEOREMS*

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Abstract. The equation $w_x^2 + w_y^2 + n^2(x,y) = 0$, which arises in generalizations of geometrical optics, is investigated from a theoretical point of view. Here x and y denote rectangular coordinates in the Euclidean plane, and n is real-valued and strictly positive. A framework is set up that involves a Bäcklund transformation relating $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$, second-order partial differential equations in divergence and nondivergence form governing $\mathrm{Re}(w)$, a variational integral, and related free boundary problems, boundary value problems, and viscosity solutions. The present paper is a continuation of a preceding one [R. Magnanini and G. Talenti, *Contemp. Math.* 283, AMS, Providence, RI, 1999, pp. 203–229], where qualitative properties of smooth solutions are offered. Here the existence of the real part of solutions, which need not be smooth, is derived.

Key words. partial differential equations, Bäcklund transformations, convex functionals, minimizers, free boundaries, critical points, variational solutions, viscosity solutions

AMS subject classifications. Primary, 35J70, 35Q60; Secondary, 49N60

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1. Introduction.

1.1. General. Let x and y denote rectangular coordinates in the Euclidean plane $\mathbb{R}^2$, and let n be a *real-valued* function of x and y. Let n be sufficiently smooth and strictly positive; should the range of x and y be unbounded, let n decay fast enough at infinity. The present paper, its predecessor [26], and forthcoming others are devoted to a tentative theory of the partial differential equation

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + n^2(x,y) = 0, \tag{1.1}$$

all of whose solutions are *complex-valued.*

Versions of (1.1) arise in acoustics and optics. Suppose that a two-dimensional isotropic nondissipative medium is under consideration and that n represents the relevant refractive index. Since (1.1) turns into

$$w_x^2 + w_y^2 = n^2(x,y)$$

on replacing w by $\pm iw$, the solutions to (1.1) whose real part is zero call for processes of classical geometrical optics. (We denote $\sqrt{-1}$ by i throughout and denote differentiations either by $\partial/\partial x$ and $\partial/\partial y$ or by subscripts.) On the other hand, solutions to (1.1) whose real part is different from zero are alleged to account for an optical process that is inherently excluded from geometrical optics—the development of *evanescent waves.* Evanescent waves occur beyond a caustic, on the dark side where the geometric optical rays do not penetrate, or else on the optically thinner side of

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an interface that disconnects two different media and totally reflects a wave incident from the optically denser side. A theory, put forward by Felsen and coworkers some twenty years ago and sometimes called evanescent wave tracking (EWT), claims that features of evanescent waves can be portrayed by retaining the asymptotic expansion

$$\text{electromagnetic field} \sim \exp[-i\nu \cdot (\text{time})] \cdot (\text{amplitude}) \cdot \exp[i\nu \cdot (\text{eikonal})],$$

which lies at the very root of geometrical optics, but allowing the eikonal and the components of the amplitude to take *complex values*; here the amplitude and the eikonal are functions of space coordinates only, and ν, the wave number, tends to infinity. A key to EWT amounts precisely to (1.1) and its three-dimensional analogue. By the way, these same objects appear also in a more exhaustive asymptotic analysis of the electromagnetic field, which leads to uniform expansions near caustics, and in modeling deeper diffraction processes. More information can be found in [4], [5], [10], [11], [14], [15], [18], [20], [21], [24], [25], [23], and in the recent surveys [3] and [6].

1.2. Preparatory results. We warm up by recollecting some material from [26]. Let u and v be *real-valued* functions of x and y, and let

$$w = u + iv$$

be the *complex-valued* function of x and y whose real and imaginary parts are u and v, respectively. w is a solution to (1.1) if and only if u and v obey the following system:

$$\begin{aligned} u_x^2 + u_y^2 - v_x^2 - v_y^2 + n^2 &= 0, \\ u_x v_x + u_y v_y &= 0. \end{aligned} \tag{1.2}$$

u and v obey (1.2) if and only if either

$$u_x = u_y = 0 \quad \text{and} \quad v_x^2 + v_y^2 = n^2$$

or the condition

$$u_x^2 + u_y^2 > 0$$

and the following equations

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \pm\sqrt{1 + \frac{n^2}{u_x^2 + u_y^2}} \begin{bmatrix} -u_y \\ u_x \end{bmatrix}, \tag{1.3}$$

$$\frac{\partial}{\partial x}\left\{\sqrt{1 + \frac{n^2}{u_x^2 + u_y^2}}\, u_x\right\} + \frac{\partial}{\partial y}\left\{\sqrt{1 + \frac{n^2}{u_x^2 + u_y^2}}\, u_y\right\} = 0 \tag{1.4}$$

prevail.

Equations (1.3), which result from algebraic manipulations of (1.2), define a *Bäcklund transformation.* (An account of Bäcklund transformations which fits well into the present context is in [30].) Equation (1.4), which amounts to the integrability of (1.3), is a second-order partial differential equation in *divergence form.* If sufficiently smooth solutions are considered whose gradient is different from 0, (1.4) can be recast in the form

$$\begin{aligned} &[(u_x^2 + u_y^2)^2 + n^2 u_y^2] u_{xx} - 2n^2 u_x u_y u_{xy} + [(u_x^2 + u_y^2)^2 + n^2 u_x^2] u_{yy} \\ &\qquad + n\left(u_x^2 + u_y^2\right)(n_x u_x + n_y u_y) = 0, \end{aligned} \tag{1.5}$$

a *semilinear* second-order partial differential equation with *polynomial nonlinearities.* Equations (1.4) and (1.5) are *elliptic-parabolic* or *degenerate elliptic.* A real-valued solution u to either (1.4) or (1.5) is *elliptic* if $u_x^2 + u_y^2 > 0$; a *degeneracy* occurs at any point where $u_x = u_y = 0$.

It should be stressed that (1.4) and (1.5) are *not* equivalent. First, perfectly smooth solutions to (1.5) exist, whose gradients vanish exclusively in a set of measure 0, and that *do not* satisfy (1.4) in the sense of distributions; they make the left-hand side (l.h.s.) of (1.4) a well-defined distribution which is supported by the set of the critical points but is *not* zero. The identity

$$\text{l.h.s. of (1.5)} = (n^2 + u_x^2 + u_y^2)^{\frac{1}{2}}(u_x^2 + u_y^2)^{\frac{3}{2}} \times \{\text{ l.h.s. of (1.4)}\}$$

gives evidence to such a statement. In the case in which $n \equiv 1$, one of the last mentioned solutions is constructed by selecting a constant C such that $0 < C < 1$ (e.g., $C = 10^{-10}$) and letting

$$\begin{aligned} &\text{domain of } u = \{(x,y) : x^2/(1-C^2) - y^2/C^2 < 1\}, \\ &\sqrt{2} \cdot u(x,y) = (((1-x^2-y^2)^2 + 4y^2)^{1/2} + 1 - x^2 - y^2)^{1/2}; \end{aligned} \tag{1.6}$$

see [26, Proposition 2.2.1]. Second, we shall demonstrate in the present paper that a conventional boundary condition need not determine a solution to (1.4) in the whole of a domain prescribed in advance, whereas the same boundary condition does suit appropriate solutions to (1.5).

The two theorems below, which bring *critical points* into relation with *rays,* express distinctive properties of the equations in hand. Recall the following. A point where the gradient vanishes is qualified as *critical.* A critical point where the Hessian determinant vanishes is qualified as *degenerate.* (The implicit function theorem states that the gradient of a sufficiently smooth real-valued function acts as a diffeomorphism from a neighborhood of a nondegenerate critical point into a neighborhood of the origin. Therefore, any nondegenerate critical point is isolated, and, conversely, all nonisolated critical points are degenerate.) The geodesics belonging to the Riemannian metric

$$n(x,y)\sqrt{(dx)^2 + (dy)^2}, \tag{1.7}$$

i.e., the paths making

$$\int n(x,y)\sqrt{(dx/ds)^2 + (dy/ds)^2}\, ds$$

either stationary or a minimum, are nicknamed *rays* and are characterized by the differential equation

$$(\text{gradient of } \log n) \cdot (\text{principal normal}) = 1. \tag{1.8}$$

THEOREM 1.1. *Assume n is strictly positive and w is a smooth solution to* (1.1). *If the gradient of* $\mathrm{Re}(w)$ *vanishes at some point, then the same gradient vanishes everywhere on a ray passing through that point.*

THEOREM 1.2. *Suppose n is smooth and strictly positive. Suppose u is smooth and real-valued and satisfies either* (1.4) *or* (1.5) *in every open subset of its domain where $u_x^2 + u_y^2 > 0$. We make the following assertions:*

(i) *Any critical point of u is degenerate.*

(ii) *If* $u_x = u_y = 0$ *and* $u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 > 0$ *at some point, then* $u_x = u_y = 0$ *everywhere on a smooth curve passing through that point.*

(iii) *If* $u_x = u_y = 0$ *and* $u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 > 0$ *at every point of a smooth curve, then this curve is a ray.*

Theorem 1.1 makes arguments from [14] rigorous. It also offers a proof of the following statement, which plays a role in the so-called *theory of complex rays* and was alleged in [11, section 3.2]. Let w be a solution to (1.1); if a point obeys the principle of locality, i.e., is a critical point of $\mathrm{Re}(w)$, then the phase path crossing that point, i.e., the level line of $\mathrm{Re}(w)$ containing the point in question, is a ray.

Theorem 1.2 basically shows that (1.5), unlike more conventional second-order partial differential equations, prevents its solutions from having *isolated critical points.* The degeneracy at critical points is a feature of (1.5) that causes critical points to cluster.

Another relevant feature is the architecture of (1.5), which exhibits geometric ingredients. If critical points are ignored and h is defined by either

$$h = -\left(u_x^2 + u_y^2\right)^{-3/2}\left(u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}\right)$$

or

$$h = -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right),$$

then (1.5) reads both

$$|\nabla u|\ \Delta u - n^2\left\{h - \nabla \log n \cdot \frac{\nabla u}{|\nabla u|}\right\} = 0$$

and

$$\left(\frac{u_x}{|\nabla u|}\frac{\partial}{\partial x} + \frac{u_y}{|\nabla u|}\frac{\partial}{\partial y}\right)\log\sqrt{n^2 + |\nabla u|^2} = h.$$

(We denote the divergence operator by div and the gradient operator by ∇. We denote the length of a vector by vertical bars and the scalar product of two vectors by either a dot or parentheses. For instance, we let

$$|\nabla u| = \sqrt{u_x^2 + u_y^2} \quad \text{and} \quad \nabla u \cdot \nabla v = (\nabla u, \nabla v) = u_x v_x + u_y v_y$$

in case that u and v are real-valued. As usual,

$$\Delta = \partial^2/\partial x^2 + \partial^2/\partial^2 y,$$

the Laplace operator.) Observe the following. First, the principal normal to the level lines of u is

$$(1/h)\frac{\nabla u}{|\nabla u|};$$

in other words, the value of h at any point (x, y) is a *signed curvature* at (x, y) of the *level line* of u crossing (x, y). Second, the value of

$$\nabla \log n \cdot \frac{\nabla u}{|\nabla u|}$$

at (x, y) equals a *signed curvature* at (x, y) of the *ray* which is tangent at (x, y) to a level line of u. Third,

$$\frac{u_x}{|\nabla u|}\frac{\partial}{\partial x} + \frac{u_y}{|\nabla u|}\frac{\partial}{\partial y}$$

is a directional derivative along the *lines of steepest descent* of u. (The first statement follows from Frenet's formulas; the second statement is a consequence of the differential equation (1.8), which characterizes rays; the third one amounts to saying that the lines of steepest descent are the trajectories of the gradient.)

1.3. Background. The present paper rests upon a background that we fix now. We borrow terminology from [1], [28], [29], [31], [33], [34], and the theory of distributions and offer apropos details in the next paragraphs.

Equation (1.4) is reminiscent of the *Euler–Lagrange equation* of a variational integral. Let

$$\Omega = \text{some open nonempty subset of } \mathbb{R}^2, \tag{1.9}$$

let a real function f be defined by

$$f(\rho) = \frac{1}{2}[\rho\sqrt{\rho^2+1} + \log(\rho + \sqrt{\rho^2+1})] \tag{1.10}$$

for every nonnegative ρ, and let a functional J be defined by

$$J(u) = \int_\Omega f\left(\frac{|\nabla u|}{n}\right) n^2 dxdy \tag{1.11}$$

for every u from some set of nice real-valued functions of x and y.

Observe that, if the Riemannian metric (1.7) is in force, the expressions

$$\frac{|\nabla u|}{n} \quad \text{and} \quad n^2 dxdy,$$

which appear in (1.11), equal the Riemannian length of the covariant derivative of u and the Riemannian area element, respectively. As will be clear presently, the right-hand side (r.h.s.) of (1.11) would become the Riemannian area of the graph of u if f were replaced by its derivative f'.

Equation (1.10) gives $f(0) = 0$,

$$f'(\rho) = \sqrt{\rho^2+1},$$

and

$$f''(\rho) = \rho/f'(\rho)$$

for every nonnegative ρ; moreover, $f''' = (f')^{-3}$. We infer that f is nonnegative, vanishes only at 0, and is strictly increasing and strictly convex—a good *Young function.* Therefore, functional J is *strictly convex,* provided a convex domain is supplied to it.

Roughly, a domain that fits J well consists of real-valued functions defined in Ω whose first-order derivatives are square-integrable in Ω. In fact, the formula

$$2f(\rho) = \inf\{\lambda + \rho^2 \cdot \coth\lambda : \lambda > 0\},$$

which holds for every nonnegative ρ and follows from (1.10), implies either

$$2J(u) \le \lambda \cdot \int n^2 dxdy + \coth\lambda \cdot \int |\nabla u|^2 dxdy$$

for every u and every positive λ or

$$J(u) \le \int n^2 dxdy \times f\left(\sqrt{\frac{\int |\nabla u|^2 dxdy}{\int n^2 dxdy}}\right)$$

for every u. Moreover, an appropriate analysis shows that

$$\sup\left\{\frac{f(\rho_1)-f(\rho_2)}{\rho_1-\rho_2} : 0 \le \rho_1 < \rho_2, \rho_1^2+\rho_2^2 = 2M^2\right\} = f'(M),$$

provided M is positive; hence

$$\begin{aligned}|J(u_1)-J(u_2)|^2 \le &\int |\nabla u_1 - \nabla u_2|^2 dxdy \\ &\times \left\{\int n^2 dxdy + \frac{1}{2}\int |\nabla u_1|^2 dxdy + \frac{1}{2}\int |\nabla u_2|^2 dxdy\right\}\end{aligned}$$

for every u_1 and u_2.

As a working hypothesis, we propose any member of the domain of J to additionally obey a *boundary condition*, e.g., to take prescribed values on the boundary, $\partial\Omega$, of Ω. (On occasion, ∂ denotes either differentiation or the operation which results in the boundary of a point set.) Formal definitions follow.

(i) $W^{1,2}(\Omega)$ = completion of $C^\infty(\Omega)$ under the norm defined by

$$\|u\|^2_{W^{1,2}(\Omega)} = 4\int_\Omega u^2(x^2+y^2+4)^{-2}\, dxdy + \int_\Omega |\nabla u|^2 dxdy.$$

$W_0^{1,2}(\Omega)$ = closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$, i.e., the subset of $W^{1,2}(\Omega)$ consisting of those functions that vanish on $\partial\Omega$ in a generalized sense. (As usual, $C^\infty(\Omega)$ is the set of infinitely differentiable real-valued functions defined in Ω, and $C_0^\infty(\Omega)$ is the subset of $C^\infty(\Omega)$ consisting of those functions that vanish out of a compact subset of Ω.)

(ii) Let j be any given member of $W^{1,2}(\Omega)$; define

$$\text{domain of } J = j + W_0^{1,2}(\Omega), \tag{1.12}$$

i.e., the set of functions u from $W^{1,2}(\Omega)$ such that $u-j$ belongs to $W_0^{1,2}(\Omega)$.

The following assumptions will be made throughout. First, the measure of Ω in Riemannian metric (1.7) is *finite*; i.e.,

$$\int_\Omega n^2 dxdy < \infty. \tag{1.13}$$

Second, Ω is *essentially different* from $\mathbb{R}^2$; i.e.,

$$\text{measure of } (\mathbb{R}^2 \setminus \Omega) > 0. \tag{1.14}$$

Note that Ω is allowed to be either bounded or unbounded. (Relevantly to the present context, Ω may be an *exterior domain,* i.e., an open connected set whose complement is compact.) In the former case, the measure $(x^2+y^2+4)^{-2}dxdy$, appearing

in (i) above, may be virtually replaced by the standard Lebesgue measure $dxdy$; hence $W^{1,2}(\Omega)$ coincides with the collection of functions that are square-integrable in Ω and whose first-order weak derivatives are square-integrable in Ω—a standard Sobolev space. In any case, the measure in question can be thought of as the area element on the two-dimensional unit sphere $\mathbb{S}^2$ parametrized via a stereographic projection; hence $W^{1,2}(\Omega)$ can be identified with a space of standard Sobolev functions defined in an open subset of $\mathbb{S}^2$.

Theorem 2.1 below claims that J does possess a minimum and that the relevant minimizer is unique within the domain specified above.

Since J was born convex, a necessary and sufficient condition for a member of the domain of J to render J a minimum is the Euler–Lagrange equation. J fails to be smoothly differentiable, however. Therefore, the Euler–Lagrange equation of J involves a set-valued subdifferential and must be cast in the form of an inclusion. Details follow.

Let u belong to the domain of J. If φ is any test function, i.e., any member of $W_0^{1,2}(\Omega)$, we have

$$\begin{aligned} J(u+\varphi) - J(u) = &\int_{\{(x,y):\nabla u(x,y)\neq 0\}} \left[f\left(\frac{|\nabla u + \nabla\varphi|}{n}\right) - f\left(\frac{|\nabla u|}{n}\right)\right] n^2 dxdy \\ &+ \int_{\{(x,y):\nabla u(x,y)=0\}} f\left(\frac{|\nabla\varphi|}{n}\right) n^2 dxdy; \end{aligned}$$

moreover,

$$\begin{aligned} t^{-1} &\int_{\{(x,y):\nabla u(x,y)\neq 0\}} \left[f\left(\frac{|\nabla u + t\nabla\varphi|}{n}\right) - f\left(\frac{|\nabla u|}{n}\right)\right] n^2 dxdy \\ &\to \int_{\{(x,y):\nabla u(x,y)\neq 0\}} \frac{n}{|\nabla u|} f'\left(\frac{|\nabla u|}{n}\right) (\nabla u, \nabla\varphi)\, dxdy \end{aligned}$$

as t approaches 0, and

$$\begin{aligned} t^{-1} &\int_{\{(x,y):\nabla u(x,y)=0\}} f\left(\frac{t|\nabla\varphi|}{n}\right) n^2 dxdy \\ &\to f'(0) \cdot \int_{\{(x,y):\nabla u(x,y)=0\}} |\nabla\varphi|\ n\ dxdy \end{aligned}$$

as t approaches 0 through *positive* values. Therefore,

$$\lim_{t\downarrow 0} \left[J(u+t\varphi) - J(u)\right]/t,$$

the *one-sided directional derivative* of J at u with respect to φ, equals

$$\int_{\{(x,y):\nabla u(x,y)\neq 0\}} \sqrt{1+n^2|\nabla u|^{-2}}\,(\nabla u, \nabla\varphi)\, dxdy + \int_{\{(x,y):\nabla u(x,y)=0\}} |\nabla\varphi|\ n\ dxdy.$$

Recall that the *subdifferential* of J, ∂J, may be characterized thusly: (i) $\partial J(u)$ is a *convex set* of distributions; (ii) a distribution T belongs to $\partial J(u)$ if and only if the directional derivative of J at u with respect to φ is greater than or equals $T(\varphi)$ for

every test function φ. Consequently, $\partial J(u)$ is the collection of those distributions T satisfying

$$\int_{\{(x,y):\nabla u(x,y)\neq 0\}} \sqrt{1+n^2|\nabla u|^{-2}}\,(\nabla u, \nabla\varphi)\,dxdy + \int_{\{(x,y):\nabla u(x,y)=0\}} |\nabla\varphi|\ n\ dxdy \geq T(\varphi)$$

for every test function, φ. Such a formula implies that $\partial J(u) \neq \emptyset$, i.e., that J is *everywhere subdifferentiable*, and, moreover, that any member T of $\partial J(u)$ obeys

$$T = -\mathrm{div}\{\sqrt{1+n^2|\nabla u|^{-2}}\nabla u\}$$

in any open set $\mathcal{O}$ contained in Ω and essentially contained in

$$\{(x,y)\in\Omega; \nabla u(x,y)\neq 0\},$$

i.e., satisfying

$$\text{measure of } \mathcal{O}\cap\{(x,y)\in\Omega : \nabla u(x,y)=0\} = 0.$$

We see, in particular, that J is *differentiable* at u if the set of the critical points of u has measure *zero; J fails* to be differentiable at u if the set of the critical points of u has a *positive* measure.

The analysis provided may be summarized in this way. The appropriate *Euler–Lagrange equation* of J reads

$$\partial J(u) \ni 0,$$

an *inclusion* that implies the following: (1.4) holds in the sense of distributions in any open subset of Ω which is essentially contained in $\{(x,y)\in\Omega : \nabla u(x,y)\neq 0\}$.

In other words, a solution u to the Euler–Lagrange equation of J solves a *free boundary problem* for (1.4), the relevant *free boundary* being

$$\Omega\cap\partial\{(x,y)\in\Omega : \nabla u(x,y)\neq 0\}.$$

(Let a manifold $\mathfrak{M}$, a class of nice functions defined in $\mathfrak{M}$, and a differential equation be given. Suppose a member u of the given function class and a subset $\mathfrak{N}$ of $\mathfrak{M}$ are sought such that (i) u solves the given equation in any open subset of $\mathfrak{N}$ or in any open set which is essentially contained in $\mathfrak{N}$; (ii) u obeys special conditions either on $\partial\mathfrak{N}\cap\mathfrak{M}$ or out of $\mathfrak{N}$. It is usual to say that a free boundary problem is in hand. $\partial\mathfrak{N}\cap\mathfrak{M}$, the boundary of $\mathfrak{N}$ relative to $\mathfrak{M}$, is called the free boundary. [16] and [19] are exhaustive references on this matter.)

What is the geometry and the physical meaning of these free boundaries? The results recorded in the present paper, though not equal to a full proof, give evidence to the following statements. The free boundaries in question (i) either are empty or are genuine *curves*—rather than collections of isolated points; and (ii) separate regions where evanescent waves develop from regions where geometrical optics prevails—hence coincide with *caustics.* (Recall that the envelopes of rays are nicknamed caustics, and thus caustics are precisely the contours near and beyond which geometrical optics break down.)

Samples of free boundaries, which affect solutions to (1.4), appear in [26, section 2.4] or can be detected in Figure 1.1.

1.4. Summary of results. We have sketched an existence result that is a key to our investigations; i.e., the *minimizer u of an apposite functional both takes prescribed boundary values and solves a free boundary value problem for* (1.4). The main issues of the present paper, which are detailed in section 2, can be summarized as follows.

Suppose n is differentiable and its first-order derivatives belong to $L^2_{\text{loc}}(\Omega)$. (As usual, $L^2(\Omega)$ is the space of the real-valued functions that are square-integrable in Ω, and $L^2_{\text{loc}}(\Omega)$ is the space of functions φ such that $\varphi \cdot \psi$ belongs to $L^2(\Omega)$ for every ψ from $C_0^\infty(\Omega)$. Occasionally, we will need to replace 2 by some exponent p larger than or equal to 1.)

(i) u is locally twice differentiable in a suitable generalized sense and obeys (1.5) *in the whole* of domain Ω.

(ii) u is a *viscosity solution* to (1.5).

We loosely imitate ideas from [7], [8], [12], and [13, Chapter 10] and mean the following: u_ε approaches u in an appropriate topology as ε approaches zero. Here ε is a strictly positive constant parameter, and u_ε is the twice differentiable real-valued function that obeys a *restored version* of (1.5) and takes the relevant boundary values. Such a version results from adding the extra term

$$\varepsilon \cdot n^2 \left(n^2 + |\nabla u|^2\right) \cdot \Delta u$$

to the l.h.s. of (1.5), i.e., reads

$$\begin{aligned} &\varepsilon \cdot n^2 \left(n^2 + |\nabla u|^2\right) \cdot \Delta u \\ &+ \left\{|\nabla u|^4 + n^2 u_y^2\right\} \cdot u_{xx} - 2n^2 u_x u_y \cdot u_{xy} + \left\{|\nabla u|^4 + n^2 u_x^2\right\} \cdot u_{yy} \\ &+ n|\nabla u|^2 (\nabla n \cdot \nabla u) = 0. \end{aligned} \tag{1.15}$$

Observe that (1.15) is *uniformly elliptic* and that its leading part balances the first-order terms properly; in other words, the injection of viscosity cures degeneracy. In fact, if a_{11}, a_{12}, and a_{22} denote the coefficients of u_{xx}, u_{xy}, and u_{yy} in (1.15) and ρ and ω are defined by

$$|\nabla u| = n\rho, \quad u_x : \cos\omega = u_y : \sin\omega,$$

then

$$\begin{aligned} &\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \\ &= n^4 \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \rho^4 + \varepsilon(1+\rho^2) & 0 \\ 0 & (1+\rho^2)(\varepsilon+\rho^2) \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}. \end{aligned} \tag{1.16}$$

Therefore, the eigenvalues of $[a_{ij}]$ obey

$$\frac{\text{smaller eigenvalue}}{\text{larger eigenvalue}} \geq \sqrt{\varepsilon} \cdot (2 + \sqrt{\varepsilon})(1 + \sqrt{\varepsilon})^{-2},$$

and we have

$$\frac{|\text{first-order term}|}{\text{larger eigenvalue}} \leq (1 + \sqrt{\varepsilon})^{-2} \times \frac{|\nabla n|}{n} \times \text{the first power of } |\nabla u|.$$

Viscosity solutions are focused on in section 5, where we show that (i) a viscosity solution to (1.5) is uniquely determined by its boundary values; (ii) a smooth solution to the same equation need not do the same—therefore, a smooth solution to (1.5) need not be a viscosity solution.

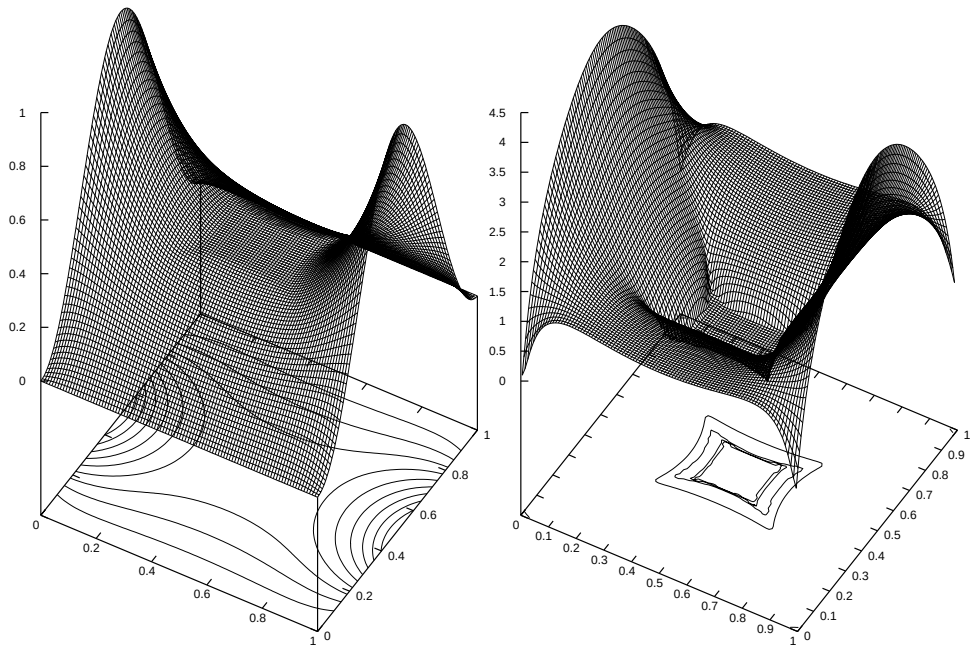

FIG. 1.1. *Typical plots of u and $|\nabla u|$. Here u is a viscosity solution to* (1.5).

1.5. Future developments. Viscosity solutions to (1.5) can be computed efficiently either by finite difference methods or by finite element methods. Details and relevant codes will appear elsewhere.

By way of an example, let u be the viscosity solution that obeys (1.5) in the domain

$$]0,1[\times]0,1[$$

and satisfies the following boundary conditions:

$$\begin{aligned} u(x,0) = u(x,1) = 0 \qquad & \text{if } \ 0 \le x \le 1, \\ u(0,y) = u(1,y) = [\sin(\pi y)]^2 \ \ & \text{if } \ 0 \le y \le 1. \end{aligned}$$

Figure 1.1 shows plots of u and $|\nabla u|$, respectively. There, u is approximated by the solution to (1.15) that takes the boundary values in hand, $\varepsilon = 10^{-8}$, finite differences are used, and a 200×200 uniform grid is involved. Note a peculiarity—the solution in question develops caustics, i.e., an inner plateau.

In part three of our work, which will be assembled in a future paper, we will show how the present results, Bäcklund trasformations, and suitable extra ingredients supply solutions to either (1.1) or (1.2) and guarantee their uniqueness.

The referees pointed out that Theorem 9.3 from [9] should be referenced here. Such a theorem claims that if Ω is any open subset of $\mathbb{R}^2$, φ is any Lipschitz continuous map from Ω into $\mathbb{R}^2$, and n is real-valued and continuous, then system (1.2) admits solutions that are Lipschitz continuous in Ω and equal to φ on $\partial\Omega$.

This theorem departs from our point of view for a couple of reasons. First, we are interested in tractable solutions, i.e., smooth enough, unique, and actually computable. Second, we do not address system (1.2) in the present paper. Treating (1.2)

by the present methods cannot be done in few words and deserves further investigation.

2. Main results. Let J be defined by (1.9), (1.10), (1.11), and (1.12). Assume conditions (1.13) and (1.14).

Let ε be a parameter satisfying

$$0 < \varepsilon \leq 1/2.$$

Let a real function f_ε be defined by

$$f_\varepsilon(\rho) = \int_0^\rho t \left(\frac{1+t^2}{\varepsilon + t^2} \right)^{\frac{1}{2(1-\varepsilon)}} dt \tag{2.1}$$

for every nonnegative ρ; let a functional J_ε be defined by

$$\text{domain of } J_\varepsilon = \text{domain of } J,$$

$$J_\varepsilon(u) = \int_\Omega f_\varepsilon \left(\frac{|\nabla u|}{n} \right) n^2 dxdy. \tag{2.2}$$

THEOREM 2.1. *Functional J achieves a minimum and has a unique minimizer.*

THEOREM 2.2. (i) *Functional J_ε achieves a minimum and has a unique minimizer.*

(ii) *Let u and u_ε denote the minimizer of J and the minimizer of J_ε, respectively; then u_ε converges to u both in $L^2_{\text{loc}}(\Omega)$ and weakly in $W^{1,2}(\Omega)$ as ε approaches* 0.

THEOREM 2.3. *Suppose n is differentiable and the first-order derivatives of n belong to $L^2_{\text{loc}}(\Omega)$; let u and be u_ε be as above. We make the following assertions:*

(i) *u_ε is twice differentiable in the usual generalized sense, the second-order derivatives of u_ε belong to $L^2_{\text{loc}}(\Omega)$, and u_ε obeys* (1.15).

(ii) *u_ε converges to u uniformly on every compact subset of Ω as ε approaches* 0; *∇u_ε converges to ∇u in $L^p_{\text{loc}}(\Omega) \times L^p_{\text{loc}}(\Omega)$ for every p larger than or equal to* 1.

(iii) *u is twice differentiable in a generalized sense and obeys the inequality*

$$\begin{aligned} &\left\{ \int_{\{(x,y):\text{dist}((x,y),\mathbb{R}^2\setminus K)\geq r\}} \frac{|\nabla u|^2}{n^2+|\nabla u|^2} \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \right) \, dxdy \right\}^{\frac{1}{2}} \\ &\leq 6 \left\{ \int_K |\nabla n|^2 \, dxdy \right\}^{\frac{1}{2}} + 2r^{-1} \left\{ \int_K \left(n^2 + |\nabla n|^2 \right) \, dxdy \right\}^{\frac{1}{2}} \end{aligned} \tag{2.3}$$

provided that K is a nice compact subset of Ω and r is a positive number. Moreover, u makes

$$(n^2 + |\nabla u|^2)^{-\frac{3}{2}} \times \{\text{l.h.s. of (1.5)}\}$$

both locally integrable in Ω and equal to 0*; in other words, u obeys* (1.5) *in the whole of Ω.*

3. Proofs of Theorems 2.1 and 2.2.

3.1. An inequality. A proof of Theorem 2.1 relies upon the following lemma.

LEMMA 3.1. *Let Ω obey* (1.9) *and* (1.14), *and let C be any constant such that*

$$C \geq \left\{ \frac{4}{\pi} \int_{\mathbb{R}^2\setminus\Omega} \frac{dxdy}{(x^2+y^2+4)^2} \right\}^{-1}.$$

Then

$$4\int_{\mathbb{R}^2} \varphi^2(x^2+y^2+4)^{-2}\ dxdy \le (C-1)\int_{\mathbb{R}^2} \left(\varphi_x^2+\varphi_y^2\right)\ dxdy$$

provided that φ is smooth enough and real-valued, and

$$\textit{support of } \varphi \subseteq \Omega.$$

Proof. The metric induced by

$$\frac{(dx)^2+(dy)^2}{\left[1+(x^2+y^2)/4\right]^2}$$

makes $\mathbb{R}^2$ a Riemannian manifold $\mathfrak{M}$ that is locally conformal to a unit sphere and enjoys the following properties. First, the Riemannian area element equals

$$\left[1+(x^2+y^2)/4\right]^{-2}\ dxdy.$$

Second, the length of the Riemannian gradient of any smooth scalar field equals

$$\left[1+(x^2+y^2)/4\right] \times \text{length of the Euclidean gradient.}$$

Thus the Riemannian area of $\mathbb{R}^2$ equals 4π, and

$$\text{Riemannian area of } \Omega \le 4\pi(1-1/C);$$

moreover,

$$\int_{\mathbb{R}^2} \varphi^2\left[1+(x^2+y^2)/4\right]^{-2}\ dxdy = \int_{\mathfrak{M}} \varphi^2,$$

and

$$\int_{\mathbb{R}^2} \left(\varphi_x^2+\varphi_y^2\right)\ dxdy = \int_{\mathfrak{M}} |\text{Riemannian gradient of } \varphi|^2.$$

We must show that

$$\int_{\mathfrak{M}} \varphi^2 \le 4(C-1)\int_{\mathfrak{M}} |\text{Riemannian gradient of } \varphi|^2. \tag{3.1}$$

Let μ and φ^* be the distribution function and the decreasing rearrangement of φ, respectively. μ is the map from $[0,\infty[$ into $[0,4\pi]$ such that

$$\mu(t) = \text{Riemannian area of } \{(x,y) : |\varphi(x,y)| > t\}$$

for every nonnegative t. φ^* can be defined as the map from $[0,4\pi]$ into $[0,\infty[$ which is right-continuous, decreasing, and equidistributed with φ, i.e., such that

$$\text{length of } \{s \in [0,4\pi] : \varphi^*(s) > t\} = \mu(t)$$

for every nonnegative t.

We have

$$\int_{\mathfrak{M}} \varphi^2 = \int_0^{4\pi} \left[\varphi^*(s)\right]^2 ds \tag{3.2}$$

since the very definitions of μ and φ^* ensure that both sides equal $\int_0^\infty t^2[-d\mu(t)]$. On the other hand, a version of an important inequality (see, e.g., [2, section 4]) tells us that φ^* is locally absolutely continuous in $]0, 4\pi[$ and satisfies

$$\int_{\mathfrak{M}} |\text{Riemannian gradient of } \varphi|^2 \geq \int_0^{4\pi} s(4\pi - s)\left[-\frac{d\varphi^*}{ds}(s)\right]^2 ds. \tag{3.3}$$

The support of φ^* is an interval whose endpoints are 0 and the Riemannian area of the support of φ. Therefore, our hypotheses yield

$$\text{support of } \varphi^* \subseteq [0, 4\pi(1 - 1/C)].$$

Such an inclusion informs us that φ^* vanishes in a neighborhood of 4π. Thus an integration by parts and a Schwarz inequality give successively

$$\int_0^{4\pi} [\varphi^*(s)]^2 \ ds = 2\int_0^{4\pi} \varphi^*(s)s\left[-\frac{d\varphi^*}{ds}(s)\right] ds$$

and

$$\int_0^{4\pi} [\varphi^*(s)]^2 \ ds \leq 4\int_0^{4\pi} s^2\left[-\frac{d\varphi^*}{ds}(s)\right]^2 ds.$$

The same inclusion also implies that

$$\int_0^{4\pi} s^2\left[-\frac{d\varphi^*}{ds}(s)\right]^2 ds \leq (C-1)\int_0^{4\pi} s(4\pi - s)\left[-\frac{d\varphi^*}{ds}(s)\right]^2 ds.$$

We infer

$$\int_0^{4\pi} [\varphi^*(s)]^2 \ ds \leq 4(C-1)\int_0^{4\pi} s(4\pi - s)\left[-\frac{d\varphi^*}{ds}(s)\right]^2 ds. \tag{3.4}$$

Equation (3.2) and inequalities (3.3) and (3.4) result in (3.1). ∎

3.2. Proof of Theorem 2.1. Uniqueness of the minimizer results from the strict convexity of functional J, while existence follows from the items below via standard arguments of the calculus of variations.

(i) *Boundedness of sublevel sets of J.* The formula

$$f(\rho) = \sup\left\{\rho \cdot \frac{\lambda}{\sinh\lambda} + \rho^2 \cdot \frac{\sinh(2\lambda) - 2\lambda}{4(\sinh\lambda)^2} : \lambda > 0\right\},$$

which holds for every nonnegative ρ and follows from (1.10), gives successively

$$J(u) \geq \frac{\lambda}{\sinh\lambda} \cdot \int |\nabla u| \ n \ dxdy + \frac{\sinh(2\lambda) - 2\lambda}{4(\sinh\lambda)^2} \cdot \int |\nabla u|^2 dxdy$$

for every u and every positive λ and either

$$J(u) \geq \frac{\left(\int |\nabla u| n dxdy\right)^2}{\int |\nabla u|^2 dxdy} \times f\left(\frac{\int |\nabla u|^2 dxdy}{\int |\nabla u| n dxdy}\right)$$

or

$$J(u) \geq \int |\nabla u| n dxdy,$$

or else

$$J(u) \geq \frac{1}{2} \int |\nabla u|^2 dxdy \tag{3.5}$$

for every u.

Lemma 3.1 implies that every u from $j + W_0^{1,2}(\Omega)$ obeys

$$\|u\|_{W^{1,2}(\Omega)} \leq (1+\sqrt{C}) \cdot \|j\|_{W^{1,2}(\Omega)} + \sqrt{C} \cdot \left\{ \int |\nabla u|^2 dxdy \right\}^{1/2}. \tag{3.6}$$

Inequality (3.5) tells us that J is *coercive.* Inequalities (3.5) and (3.6) imply that the sublevel sets of J, i.e., the function classes

$$\{u \in \text{domain of } J : J(u) \leq \text{Constant}\},$$

are all *bounded* in the metric of $W^{1,2}(\Omega)$.

(ii) *Compactness.* The classical Riesz compactness theorem or an oversimplified version of the Rellich–Kondrachov theorem (see, e.g., [1, Chapter V] or [34, section 2.5]) ensures that any sequence which is bounded in $W^{1,2}(\Omega)$ contains some subsequence which converges in $L^2_{\text{loc}}(\Omega)$. The structure of the appropriate dual space (see, e.g., [1, Chapter III] or [34, section 4.3]) ensures that any sequence which is bounded in $W^{1,2}(\Omega)$ and converges in $L^2_{\text{loc}}(\Omega)$ does converge in the weak topology of $W^{1,2}(\Omega)$.

(iii) *Lower semicontinuity of* J. The real function g defined by

$$\begin{aligned} g(\rho) &= 0 && \text{if} \quad 0 \leq \rho \leq 1, \\ &= \tfrac{1}{2}[\rho\sqrt{\rho^2-1} - \log(\rho + \sqrt{\rho^2-1})] && \text{if} \quad \rho > 1 \end{aligned}$$

is the *Young conjugate* of f, i.e., obeys

$$f(\rho) = \sup\{\rho \cdot \lambda - g(\lambda) : \lambda \geq 0\}$$

for every nonnegative ρ. Therefore, either an inspection or a theorem from [29] gives

$$J(u) = \sup \left\{ \int (\nabla u, \varphi)\, dxdy - \int g\left(\frac{|\varphi|}{n}\right) n^2 dxdy : \varphi \in L^2(\Omega) \times L^2(\Omega) \right\} \tag{3.7}$$

for every u.

Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, the former can replace the latter in the preceding formula. Hence an integration by parts gives

$$J(u) = \sup \left\{ -\int u \cdot \text{div } \varphi \; dxdy - \int g\left(\frac{|\varphi|}{n}\right) n^2 dxdy : \varphi \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) \right\} \tag{3.8}$$

for every u.

The supremum of a family of continuous functionals is lower semicontinuous. Thus (3.7) and (3.8) imply that J is *lower semicontinuous* with respect to both the weak topology of $W^{1,2}(\Omega)$ and the topology of $L^2_{\text{loc}}(\Omega)$. $\square$

3.3. Proof of Theorem 2.2. Proposition (i) is a replica of Theorem 2.1 and can be demonstrated similarly. Ad hoc ingredients, such as the convexity and the coerciveness of functional J_ε, are provided by (2.1) and (2.2) and by propositions (i), (ii), and (iii) of Lemma A.1.

Proposition (ii) is straightforward. Since (1.11) and (2.2) give

$$|J(\varphi) - J_\varepsilon(\varphi)| \leq \int_\Omega n^2 \, dxdy \times \sup\{|f(\rho) - f_\varepsilon(\rho)| : 0 \leq \rho < \infty\}$$

for every φ, proposition (iv) of Lemma A.1 implies that

$$\sup\left\{|J(\varphi) - J_\varepsilon(\varphi)| : \varphi \in W^{1,2}(\Omega)\right\} = O(\sqrt{\varepsilon}); \tag{3.9}$$

that is, J_ε converges *uniformly* to J as ε approaches 0.

On the other hand,

$$0 \leq J(u_\varepsilon) - \min J \leq 2 \cdot \sup\left\{|J(\varphi) - J_\varepsilon(\varphi)| : \varphi \in W^{1,2}(\Omega)\right\}. \tag{3.10}$$

Formulas (3.9) and (3.10) imply that

$$\lim_{\varepsilon \to 0} J(u_\varepsilon) = \min J;$$

therefore,

$$\{u_{\varepsilon_k}\}_{k=1,2,3,\ldots}$$

is a *minimizing sequence* relative to functional J whenever $\{\varepsilon_k\}_{k=1,2,3,\ldots}$ obeys $0 < \varepsilon_k \leq 1/2$ for every k and

$$\lim_{k \to \infty} \varepsilon_k = 0.$$

Suppose, by contradiction, that u_ε fails to approach u either in $L^2_{\text{loc}}(\Omega)$ or in the weak topology of $W^{1,2}(\Omega)$ as ε approaches 0. Then a neighborhood of u and a sequence $\{\varepsilon_k\}_{k=1,2,3,\ldots}$ exist such that u_{ε_k} is out of this neighborhood and $0 < \varepsilon_k \leq 1/(2k)$ for every k.

The analysis made in section 3.2, while proving Theorem 2.1, shows that every minimizing sequence relative to J contains a subsequence which converges to a minimizer of J both in $L^2_{\text{loc}}(\Omega)$ and in the weak topology of $W^{1,2}(\Omega)$.

Therefore, a minimizer of J exists which is out of some neighborhood of u and thus is different from u.

This is impossible because J is strictly convex, and a strictly convex functional cannot have two different minimizers. □

4. Proof of Theorem 2.3.

4.1. Proof of proposition (i) of Theorem 2.3. The proof is patterned on conventional arguments of the calculus of variations and consists of the three items below.

(i) *Euler–Lagrange equation of J_ε—weak form.* Proposition (v) of Lemma A.1 tells us that

$$\mathbb{R}^2 \ni (p, q) \mapsto n^2 \cdot f_\varepsilon(n^{-1} \cdot \sqrt{p^2 + q^2})$$

is twice continuously differentiable. If ρ and ω are defined by

$$p = n\rho \cdot \cos\omega \quad \text{and} \quad q = n\rho \cdot \sin\omega,$$

then the gradient of the above function equals

$$f'_\varepsilon(\rho) \cdot \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix},$$

and its Hessian matrix, H, is given by

$$H = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} f''_\varepsilon(\rho) & 0 \\ 0 & f'_\varepsilon(\rho)/\rho \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}. \tag{4.1}$$

Proposition (vi) of Lemma A.1 tells us that the eigenvalues involved obey

$$0 < \text{ eigenvalues } \leq \varepsilon^{-\frac{1}{2(1-\varepsilon)}}.$$

Therefore, Taylor's formula gives

$$J_\varepsilon(u+\varphi) - J_\varepsilon(u) = \int_\Omega \frac{n}{|\nabla u|} \cdot f'_\varepsilon\left(\frac{|\nabla u|}{n}\right) \cdot (\nabla u, \nabla\varphi)\ dxdy + \text{ a remainder},$$
$$0 \leq 2 \cdot (\text{remainder}) \leq \varepsilon^{-\frac{1}{2(1-\varepsilon)}} \int_\Omega |\nabla\varphi|^2\ dxdy$$

provided that u and φ are endowed with square-integrable first-order derivatives. We infer that J_ε is differentiable at every u from its domain, and

$$J'_\varepsilon(u)(\varphi) = \int_\Omega \frac{n}{|\nabla u|} \cdot f'_\varepsilon\left(\frac{|\nabla u|}{n}\right) \cdot (\nabla u, \nabla\varphi)\ dxdy$$

for every φ from $W_0^{1,2}(\Omega)$; in other words,

$$J'_\varepsilon(u) = -\text{div}\left\{n \cdot f'_\varepsilon\left(\frac{|\nabla u|}{n}\right) \cdot \frac{\nabla u}{|\nabla u|}\right\}$$

in the sense of distributions.

The analysis provided shows that the minimizer of J_ε obeys the equation

$$\text{div}\left\{n \cdot f'_\varepsilon\left(\frac{|\nabla u|}{n}\right) \cdot \frac{\nabla u}{|\nabla u|}\right\} = 0 \tag{4.2}$$

in the sense of distributions. Thus the *Euler–Lagrange equation* of functional J_ε amounts precisely to (4.2).

(ii) *Extra regularity of extremals.* Now we resort to the hypothesis made on n and claim that, if u is a distributional solution to (4.2) and

$$\nabla u \in L^2_{\text{loc}}(\Omega) \times L^2_{\text{loc}}(\Omega),$$

then u is twice differentiable and its second-order derivatives are in $L^2_{\text{loc}}(\Omega)$.

A proof of such a claim can be outlined in this way.

Let ρ and ω be defined by

$$p = n\rho \cdot \cos\omega \quad \text{and} \quad q = n\rho \cdot \sin\omega,$$

let H be defined as in (4.1), and let either

$$F = [f'_\varepsilon(\rho) - \rho f''_\varepsilon(\rho)] \cdot \frac{\partial n}{\partial x} \cdot \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}$$

or

$$F = [f'_\varepsilon(\rho) - \rho f''_\varepsilon(\rho)] \cdot \frac{\partial n}{\partial y} \cdot \begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix};$$

consider the partial differential equation

$$\operatorname{div}(H \cdot \nabla v) = \operatorname{div} F. \tag{4.3}$$

Proposition (ii) of Lemma 4.1 tells us that certain constants, depending only upon ε, exist such that

$$0 < \text{ Constant } \leq \text{ eigenvalues of } H \leq \text{ Constant},$$

and

$$|F| \leq \text{ Constant} \cdot |\nabla n|.$$

As a consequence, it can be shown that another constant, depending upon ε, exists such that

$$\begin{aligned} &\int_{\{(x,y):\operatorname{dist}((x,y),\mathbb{R}^2\setminus K)\geq r\}} |\nabla v|^2 \, dxdy \\ &\leq \text{ Constant} \cdot \left[\int_K |\nabla n|^2 \, dxdy + r^{-2} \int_K v^2 \, dxdy\right] \end{aligned} \tag{4.4}$$

provided that v is any distributional solution to (4.3), K is a nice compact subset of Ω, and r is a positive number.

Inequality (4.4), which is sometimes referred to as *Caccioppoli's inequality,* plus an appropriate use of finite differences allow one to conclude that, if either

$$v = \partial u / \partial x$$

or

$$v = \partial u / \partial y,$$

then v actually obeys (4.3) in the sense of distributions and

$$\nabla v \in L^2_{\text{loc}}(\Omega) \times L^2_{\text{loc}}(\Omega).$$

Details can be found, e.g., in [17, section 2.1], [22, sections 4.3 and 4.5], [27, seciton 1.10 and 1.11]. The claim is demonstrated.

(iii) *Euler–Lagrange equation of J_ε—strong form.* An appropriate smoothness and appropriate symbols of relevant ingredients having been established, (4.3) can be recast in the following form:

$$\begin{aligned} &\left[f''_\varepsilon(\rho)(\cos\omega)^2 + \frac{f'_\varepsilon(\rho)}{\rho}(\sin\omega)^2\right] u_{xx} + 2\left[f''_\varepsilon(\rho) - \frac{f'_\varepsilon(\rho)}{\rho}\right] \cos\omega \ \sin\omega \ u_{xy} \\ &+ \left[f''_\varepsilon(\rho)(\sin\omega)^2 + \frac{f'_\varepsilon(\rho)}{\rho}(\cos\omega)^2\right] u_{yy} + \left[\frac{f'_\varepsilon(\rho)}{\rho} - f''_\varepsilon(\rho)\right] \nabla u \cdot \nabla \log n = 0. \end{aligned} \tag{4.5}$$

As observed in the appendix, (2.1) implies (A.1) and (A.2); these equations give

$$\frac{f'_\varepsilon(\rho)}{\rho} \ : \ \big[(1+\rho^2)(\varepsilon+\rho^2)\big] = \left[\frac{f'_\varepsilon(\rho)}{\rho} - f''_\varepsilon(\rho)\right] \ : \ \rho^2 = (1+\rho^2)^{-\frac{1-2\varepsilon}{2(1-\varepsilon)}}(\varepsilon+\rho^2)^{-\frac{3-2\varepsilon}{2(1-\varepsilon)}}$$

for every nonnegative ρ.

Consequently, (4.5) coincides with (1.15). In other words, (1.15) is another form of the Euler–Lagrange equation for J_ε. □

4.2. Two lemmas. A proof of proposition (ii) of Theorem 2.3 relies upon the following lemmas.

LEMMA 4.1. *Suppose A and B are 2×2 real symmetric matrices. Let A be positive definite, and let*

$$\kappa = \frac{\textit{smaller eigenvalue}}{\textit{larger eigenvalue}},$$

a condition number of A. Then

$$\frac{(\mathrm{tr}AB)^2}{\det A} - 2\cdot\det B \geq \kappa\cdot\mathrm{tr}(B^2). \tag{4.6}$$

(Here tr *and* det *stand for trace and determinant, respectively.)*

Proof. Denote the entries of A and B by a_{ij} and b_{ij}, respectively; let

$$\boldsymbol{M} = \frac{1}{a_{11}a_{22}-a_{12}^2}\begin{bmatrix} a_{11}^2 & \sqrt{2}a_{11}a_{12} & a_{12}^2 \\ \sqrt{2}a_{11}a_{12} & a_{11}a_{22}+a_{12}^2 & \sqrt{2}a_{12}a_{22} \\ a_{12}^2 & \sqrt{2}a_{12}a_{22} & a_{22}^2 \end{bmatrix}$$

and

$$\boldsymbol{m} = \begin{bmatrix} b_{11} \\ \sqrt{2}b_{12} \\ b_{22} \end{bmatrix}.$$

We have

$$\frac{(\mathrm{tr}AB)^2}{\det A} - 2\cdot\det B = (\boldsymbol{M}\boldsymbol{m},\boldsymbol{m}), \quad \mathrm{tr}(B^2) = (\boldsymbol{m},\boldsymbol{m}).$$

An inspection shows that the eigenvalues of $\boldsymbol{M}$ are $1/\kappa, 1, \kappa$. Inequality (4.6) follows. □

LEMMA 4.2. *Let a real-valued function t be defined by*

$$t(\rho) = \tan\left(\frac{1}{2}\arctan\rho\right) \tag{4.7}$$

for every nonnegative ρ, and let a mapping T be defined by

$$T\varphi = t\left(\frac{|\nabla\varphi|}{n}\right)\ \nabla\varphi \tag{4.8}$$

for every φ from a space of sufficiently smooth real-valued functions of x and y. Assume $\nabla(T\varphi)$ stands for the Jacobian matrix of $T\varphi$ and

$$|\nabla(T\varphi)| = \sqrt{\mathrm{tr}\,[(\nabla(T\varphi))(\nabla(T\varphi))^T]},$$

a norm for such a matrix.

(i) *If ρ and ω are defined by $|\nabla\varphi| = n\ \rho$, $\varphi_x : \cos\omega = \varphi_y : \sin\omega$, the following equations hold:*

$$\begin{aligned}\nabla(T\varphi) &= -2\left(\sin\left(\frac{1}{2}\arctan\rho\right)\right)^2 \begin{bmatrix} n_x\cos\omega & n_y\cos\omega \\ n_x\sin\omega & n_y\sin\omega \end{bmatrix} \\ &+ \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2}+\frac{1}{2}(t(\rho))^2 \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \\ &\times \frac{|\nabla\varphi|}{\sqrt{n^2+|\nabla\varphi|^2}} \begin{bmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{xy} & \varphi_{yy} \end{bmatrix},\end{aligned} \tag{4.9}$$

$$\begin{aligned} t\left(\frac{|\nabla\varphi|}{n}\right) \begin{bmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{xy} & \varphi_{yy} \end{bmatrix} &= (t(\rho))^2 \begin{bmatrix} n_x\cos\omega & n_y\cos\omega \\ n_x\sin\omega & n_y\sin\omega \end{bmatrix} \\ &+ \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(t(\rho))^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \nabla(T\varphi). \end{aligned} \tag{4.10}$$

(ii) *The following inequalities hold:*

$$|\nabla(T\varphi)| \le |\nabla n| + \frac{|\nabla\varphi|}{\sqrt{n^2+|\nabla\varphi|^2}}\sqrt{\varphi_{xx}^2+2\varphi_{xy}^2+\varphi_{yy}^2}, \tag{4.11}$$

$$\frac{|\nabla\varphi|}{2\sqrt{n^2+|\nabla\varphi|^2}}\sqrt{\varphi_{xx}^2+2\varphi_{xy}^2+\varphi_{yy}^2} \le |\nabla n| + |\nabla(T\varphi)|. \tag{4.12}$$

(iii) *If φ_1 and φ_2 are real-valued and sufficiently smooth, then*

$$|T\varphi_1 - T\varphi_2| \le |\nabla\varphi_1 - \nabla\varphi_2| \tag{4.13}$$

and

$$|\nabla\varphi_1 - \nabla\varphi_2| \le |T\varphi_1 - T\varphi_2|^{\frac{1}{2}} \cdot (4n + |T\varphi_1 - T\varphi_2|)^{\frac{1}{2}}. \tag{4.14}$$

Proof. Equation (4.7) provides us with the properties

$$t(\rho) = \frac{\rho}{1+\sqrt{1+\rho^2}},$$

$$\begin{aligned} t(\rho) &= \frac{\rho}{2\sqrt{1+\rho^2}}\left[1+(t(\rho))^2\right], \quad t(\rho) = \frac{\rho}{2}\left[1-(t(\rho))^2\right], \\ &0 \le t(\rho) < 1, \qquad \frac{\rho}{2\sqrt{1+\rho^2}} \le t(\rho) < \frac{\rho}{\sqrt{1+\rho^2}}, \\ \rho^2 t'(\rho) &= 2\left(\sin\left(\frac{1}{2}\arctan\rho\right)\right)^2, \quad (\rho t(\rho))' = \frac{\rho}{\sqrt{1+\rho^2}}, \end{aligned} \tag{4.15}$$

which hold for every nonnegative ρ and play a role below.

Differentiating both sides of (4.8) gives

$$\nabla(T\varphi) = -\rho^2 t'(\rho) \begin{bmatrix} n_x \cos\omega & n_y \cos\omega \\ n_x \sin\omega & n_y \sin\omega \end{bmatrix}$$

$$+ \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} (\rho t(\rho))' & 0 \\ 0 & t(\rho) \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{xy} & \varphi_{yy} \end{bmatrix}.$$

Equation (4.9) follows because of equations that appear in (4.15).

Inequalities (4.11) and (4.12) are easily derived from (4.9) and (4.10), respectively, via some matrix algebra and inequalities that appear in (4.15).

Suppose $\nabla\varphi_1 \neq \nabla\varphi_2$. Define ρ_1 and ρ_2 by

$$|\nabla\varphi_1| = n\ \rho_1 \quad \text{and} \quad |\nabla\varphi_2| = n\ \rho_2,$$

respectively; let θ be the angle between $\nabla\varphi_1$ and $\nabla\varphi_2$, i.e., be such that

$$0 \le \theta \le \pi, \quad |\nabla\varphi_1||\nabla\varphi_2|\ \cos\theta = (\nabla\varphi_1, \nabla\varphi_2).$$

Equation (4.8) gives successively

$$\frac{|T\varphi_1 - T\varphi_2|^2}{|\nabla\varphi_1 - \nabla\varphi_2|^2} = \frac{(\rho_1 t(\rho_1))^2 + (\rho_2 t(\rho_2))^2 - 2\rho_1\rho_2 t(\rho_1)t(\rho_2)\ \cos\theta}{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\ \cos\theta}$$

and

$$\frac{\partial}{\partial\theta}\frac{|T\varphi_1 - T\varphi_2|^2}{|\nabla\varphi_1 - \nabla\varphi_2|^2} = -2\rho_1\rho_2 \frac{(t(\rho_1) - t(\rho_2))(\rho_1^2 t(\rho_1) - \rho_2^2 t(\rho_2))}{(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\ \cos\theta)^2}\ \sin\theta.$$

We have either $t(\rho_1) \le t(\rho_2)$ and $\rho_1^2 t(\rho_1) \le \rho_2^2 t(\rho_2)$ or $t(\rho_1) > t(\rho_2)$ and $\rho_1^2 t(\rho_1) > \rho_2^2 t(\rho_2)$ since both (4.7) and equations in (4.15) show that t is increasing. We infer successively that

$$\frac{\partial}{\partial\theta}\frac{|T\varphi_1 - T\varphi_2|^2}{|\nabla\varphi_1 - \nabla\varphi_2|^2} \le 0$$

and

$$\frac{\rho_1 t(\rho_1) + \rho_2 t(\rho_2)}{\rho_1 + \rho_2} \le \frac{|T\varphi_1 - T\varphi_2|}{|\nabla\varphi_1 - \nabla\varphi_2|} \le \frac{\rho_1 t(\rho_1) - \rho_2 t(\rho_2)}{\rho_1 - \rho_2}.$$

On the other hand, we have

$$t\left(\frac{\rho_1 + \rho_2}{2}\right) \le \frac{\rho_1 t(\rho_1) + \rho_2 t(\rho_2)}{\rho_1 + \rho_2} \quad \text{and} \quad \frac{\rho_1 t(\rho_1) - \rho_2 t(\rho_2)}{\rho_1 - \rho_2} \le 1$$

since equations in (4.15) show that $0 \le \rho \to \rho t(\rho)$ is convex and contractive. Therefore,

$$t\left(\frac{|\nabla\varphi_1| + |\nabla\varphi_2|}{2n}\right) \le \frac{|T\varphi_1 - T\varphi_2|}{|\nabla\varphi_1 - \nabla\varphi_2|} \le 1.$$

We conclude with (4.13) and the inequality

$$t\left(\frac{|\nabla\varphi_1 - \nabla\varphi_2|}{2n}\right)|\nabla\varphi_1 - \nabla\varphi_2| \le |T\varphi_1 - T\varphi_2|,$$

which leads to (4.14) via algebraic manipulations. $\square$

4.3. Proof of proposition (ii) of Theorem 2.3. Suppose K is a compact subset of Ω whose interior is not empty and whose boundary is sufficiently smooth; let $r > 0$, and define

$$\mathcal{K}(r) = \{(x, y) : \mathrm{dist}((x, y), \mathbb{R}^2 \setminus K) \geq r\}.$$

Let T be as in Lemma 4.2.

The following bounds hold.

Bound 1.

$$\int_{\mathcal{K}(r)} \frac{|\nabla u_\varepsilon|^2}{n^2 + |\nabla u_\varepsilon|^2} \left[\left(\frac{\partial^2 u_\varepsilon}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 u_\varepsilon}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 u_\varepsilon}{\partial y^2} \right)^2 \right] dxdy$$
$$\leq \int_K |\nabla n|^2 \, dxdy + r^{-2} \int_K \left(n^2 + |\nabla u_\varepsilon|^2 \right) \, dxdy. \tag{4.16}$$

Bound 2.

$$\left\{ \int_{\mathcal{K}(r)} |\nabla (T u_\varepsilon)|^2 dxdy \right\}^{\frac{1}{2}}$$
$$\leq 2 \left\{ \int_K |\nabla n|^2 \, dxdy \right\}^{\frac{1}{2}} + r^{-1} \left\{ \int_K \left(n^2 + |\nabla u_\varepsilon|^2 \right) \, dxdy \right\}^{\frac{1}{2}}, \tag{4.17}$$

$$\int_K |T u_\varepsilon|^2 \, dxdy \leq \int_K |\nabla u_\varepsilon|^2 \, dxdy. \tag{4.18}$$

Bound 3. If $p \geq 1$, then

$$\left\{ \int_K |\nabla u_{\varepsilon'} - \nabla u_{\varepsilon''}|^p \, dxdy \right\}^2$$
$$\leq \int_K |T u_{\varepsilon'} - T u_{\varepsilon''}|^p \, dxdy \times \int_K (4n + |T u_{\varepsilon'} - T u_{\varepsilon''}|)^p \, dxdy. \tag{4.19}$$

Bound 4.

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dxdy \leq \text{Constant independent of } \varepsilon. \tag{4.20}$$

Proof of Bound 1. For notational convenience, we temporarily drop the subscript ε and denote u_ε by u in short.

We have shown in proposition (i) of Theorem 2.3 that such a u obeys (1.15). Equation (1.15) implies

$$|a_{11} u_{xx} + 2 a_{12} u_{xy} + a_{22} u_{yy}| \leq n \cdot |\nabla u|^3 \cdot |\nabla n|,$$

where a_{11}, a_{12}, and a_{22} are given by (1.16). Equation (1.16) tells us that, in addition to the inequalities appearing in section 1.4, $[a_{ij}]$ satisfies

$$\frac{\text{smaller eigenvalue}}{\text{larger eigenvalue}} \geq \frac{|\nabla u|^2}{n^2 + |\nabla u|^2}$$

and

$$a_{11}a_{22} - a_{12}^2 \geq n^2 \cdot |\nabla u|^6.$$

Therefore, Lemma 4.1 gives

$$\frac{|\nabla u|^2}{n^2 + |\nabla u|^2}\left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right) \leq 2\left(u_{xy}^2 - u_{xx}u_{yy}\right) + |\nabla n|^2, \tag{4.21}$$

an instance of what is often called *Bernstein's inequality*—see, e.g., [32].

An inspection shows that $u_{xx}u_{yy} - u_{xy}^2$, the Hessian determinant of u, obeys

$$2\left(u_{xy}^2 - u_{xx}u_{yy}\right) = \operatorname{div}\begin{bmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{bmatrix} \cdot \nabla u; \tag{4.22}$$

equivalently,

$$2\left(u_{xy}^2 - u_{xx}u_{yy}\right) dx \wedge dy = d\begin{vmatrix} u_x & u_y \\ du_x & du_y \end{vmatrix}. \tag{4.23}$$

Inequality (4.21) and either (4.22) or (4.23) give

$$\begin{aligned} &\int_{\mathcal{K}(r)} \frac{|\nabla u|^2}{n^2 + |\nabla u|^2}\left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right)\, dxdy \\ &\quad \leq \left\{\int_{\partial\mathcal{K}(r)} \left(n^2 + |\nabla u|^2\right)\sqrt{(dx)^2 + (dy)^2}\right\}^{\frac{1}{2}} \\ &\quad \times \left\{\int_{\partial\mathcal{K}(r)} \frac{|\nabla u|^2}{n^2 + |\nabla u|^2}\left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right)\, \sqrt{(dx)^2 + (dy)^2}\right\}^{\frac{1}{2}} \\ &\quad + \int_{\mathcal{K}(r)} |\nabla n|^2\, dxdy \end{aligned} \tag{4.24}$$

via the Gauss–Green formulas and the Cauchy–Schwarz inequality.

If we define two real-valued functions φ and ψ by

$$\varphi(r) = \int_{\mathcal{K}(r)} \frac{|\nabla u|^2}{n^2 + |\nabla u|^2}\left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right)\, dxdy - \int_K |\nabla n|^2\, dxdy$$

and

$$\psi(r) = \int_{\mathcal{K}(r)} \left(n^2 + |\nabla u|^2\right)\, dxdy,$$

then a version of the coarea formula (see, e.g., [34, section 2.7] and the equation

$$|\nabla \operatorname{dist}((x,y), \mathbb{R}^2 \setminus K)| = 1 \text{ for almost every } (x,y) \in K$$

yield

$$-\varphi'(r) = \int_{\partial\mathcal{K}(r)} \frac{|\nabla u|^2}{n^2 + |\nabla u|^2}\left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right)\, \sqrt{(dx)^2 + (dy)^2}$$

and

$$-\psi'(r) = \int_{\partial\mathcal{K}(r)} \left(n^2 + |\nabla u|^2\right) \ \sqrt{(dx)^2 + (dy)^2}$$

for almost every positive r. Thus (4.24) yields

$$\varphi(r) \leq \sqrt{[-\varphi'(r)]\,[-\psi'(r)]} \tag{4.25}$$

for almost every positive r.

As is easy to see, (4.25) implies

$$\text{positive part of } \varphi(r) \leq \left\{ \int_0^r \frac{dt}{[-\psi'(t)]} \right\}^{-1}$$

for every positive r. Since

$$r^2 \leq \psi(0) \int_0^r \frac{dt}{[-\psi'(t)]},$$

we conclude that

$$\text{positive part of } \varphi(r) \leq r^{-2}\psi(0) \tag{4.26}$$

for every positive r.

The inequality

$$\int_{\mathcal{K}(r)} \frac{|\nabla u|^2}{n^2 + |\nabla u|^2} \left[\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2 \ \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \right] \ dxdy$$
$$\leq \int_K |\nabla n|^2 \ dxdy + r^{-2} \int_K \left(n^2 + |\nabla u|^2\right) \ dxdy$$

follows from (4.26). Bound 1 is demonstrated.

Proof of Bound 2. Inequality (4.17) follows from Bound 1 and proposition (ii) of Lemma 4.2. Inequality (4.18) follows from proposition (iii) of Lemma 4.2.

Proof of Bound 3. Such a bound follows from proposition (iii) of Lemma 4.2 and the Cauchy–Schwarz inequality.

Proof of Bound 4. The inequalities

$$\frac{\rho^2}{2} + \frac{1}{4}\log(1 + 2\rho^2) \leq f_\varepsilon(\rho) \leq f(\rho),$$

which hold for every nonnegative ρ and appear in Lemma A.1, tell us that

$$J_\varepsilon \leq J$$

and

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 dxdy \leq J_\varepsilon(u_\varepsilon).$$

Since

$$J_\varepsilon(u_\varepsilon) = \min J_\varepsilon,$$

we obtain the inequality

$$\frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 dxdy \le \min J,$$

which proves (4.20).

Bound 2, Bound 4, and the Rellich–Kondrachov compactness theorem (see, e.g., [1, Chapter VI] or [34, section 2.5]) ensure that a sequence $\{\varepsilon_k\}_{k=1,2,3,\dots}$ exists such that $0 < \varepsilon_k \le 1/2$ for every k, $\varepsilon_k \to 0$ as $k \to \infty$, and

$$\{Tu_{\varepsilon_k}\}_{k=1,2,3,\dots} \text{ converges in } L^p_{\text{loc}}(\Omega) \times L^p_{\text{loc}}(\Omega)$$

for every p larger than or equal to 1. Consequently, Bound 3 ensures that

$$\{\nabla u_{\varepsilon_k}\}_{k=1,2,3,\dots} \text{ converges in } L^p_{\text{loc}}(\Omega) \times L^p_{\text{loc}}(\Omega)$$

for every p larger than or equal to 1. We now profit by a Sobolev inequality (see, e.g., [1, Chapter V] or [34, section 2.4] and infer that if K is any compact subset of Ω and

$$m_k = (\text{measure of } K)^{-1} \cdot \int_K u_{\varepsilon_k} dxdy,$$

then

$$\{u_{\varepsilon_k} - m_k\}_{k=1,2,3,\dots}$$

converges uniformly in K.

Applying proposition (ii) of Theorem 2.2 leads to the conclusion. ∎

4.4. Proof of proposition (iii) of Theorem 2.3. Proposition (ii) of Theorem 2.3, proposition (iii) of Lemma 4.2, and Bounds 2 and 4 (appearing in the preceding subsection) show that

$$\{\nabla(Tu_\varepsilon)\}_{k=1,2,3,\dots} \text{ converges weakly in } \left(L^2_{\text{loc}}(\Omega)\right)^4 \tag{4.27}$$

as ε approaches 0.

Having (4.27) in hand, we are in a position to resume a former notation, u, and to establish the ultimate properties of u.

(i) *Second-order derivatives.* Previous ingredients, which include proposition (ii) of Theorem 2.3, Bound 2, and (4.27), guarantee that Tu is differentiable and obeys

$$\begin{aligned} &\left\{\int_{\{(x,y):\text{dist}((x,y),\mathbb{R}^2\setminus K)\ge r\}} |\nabla Tu|^2 dxdy\right\}^{\frac{1}{2}} \\ &\le 2\left\{\int_K |\nabla n|^2 dxdy\right\}^{\frac{1}{2}} + r^{-1}\left\{\int_K \left(n^2 + |\nabla u|^2\right) dxdy\right\}^{\frac{1}{2}} \end{aligned} \tag{4.28}$$

if K is a nice compact subset of Ω and $r > 0$. Propositions (i) and (ii) of Lemma 4.2 and inequality (4.28) show that u is twice differentiable and obeys (2.3).

(ii) *Differential equation.* The underlying idea is recasting both (1.5) and (1.15) in a form in which second-order derivatives of u are replaced by the entries of $\nabla(Tu)$ and then letting ε approach zero. Details follow.

Combining proposition (i) of Lemma 4.2 and the identity

$$|\nabla\varphi|^2 \cdot \nabla\left\{n \cdot f'_\varepsilon\left(\frac{|\nabla\varphi|}{n}\right) \cdot \frac{\nabla\varphi}{|\nabla\varphi|}\right\} = n\rho^2[f'_\varepsilon(\rho)/\rho - f''_\varepsilon(\rho)]\begin{bmatrix} n_x\varphi_x & n_y\varphi_x \\ n_x\varphi_y & n_y\varphi_y \end{bmatrix}$$
$$+ \begin{bmatrix} \varphi_x & -\varphi_y \\ \varphi_y & \varphi_x \end{bmatrix}\begin{bmatrix} f''_\varepsilon(\rho) & 0 \\ 0 & f'_\varepsilon(\rho)/\rho \end{bmatrix}\begin{bmatrix} \varphi_x & \varphi_y \\ -\varphi_y & \varphi_x \end{bmatrix}\begin{bmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{xy} & \varphi_{yy} \end{bmatrix}$$

results in

$$\frac{|\nabla\varphi|^3}{\sqrt{n^2+|\nabla\varphi|^2}} \cdot \nabla\left\{n \cdot f'_\varepsilon\left(\frac{|\nabla\varphi|}{n}\right) \cdot \frac{\nabla\varphi}{|\nabla\varphi|}\right\}$$
$$= n\rho^2 t(\rho)\left\{\frac{2f'_\varepsilon(\rho)/\rho}{1+(t(\rho))^2} - f''_\varepsilon(\rho)\right\}\begin{bmatrix} n_x\varphi_x & n_y\varphi_x \\ n_x\varphi_y & n_y\varphi_y \end{bmatrix}$$
$$+ \begin{bmatrix} \varphi_x & -\varphi_y \\ \varphi_y & \varphi_x \end{bmatrix}\begin{bmatrix} f''_\varepsilon(\rho) & 0 \\ 0 & \frac{2f'_\varepsilon(\rho)/\rho}{1+(t(\rho))^2} \end{bmatrix}\begin{bmatrix} \varphi_x & \varphi_y \\ -\varphi_y & \varphi_x \end{bmatrix}\nabla(T\varphi); \tag{4.29}$$

here φ stands for any sufficiently smooth real-valued function, $\rho = |\nabla\varphi| : n$, and t is given by (4.7).

As observed in the proof of Lemma A.1, (2.1) implies

$$\frac{\rho f''_\varepsilon(\rho)}{f'_\varepsilon(\rho)} = 1 - \frac{\rho^2}{(1+\rho^2)(\varepsilon+\rho^2)}$$

for every nonnegative ρ. Therefore,

$$0 < \rho \cdot \left[\frac{\rho f''_\varepsilon(\rho)}{f'_\varepsilon(\rho)} - \frac{\rho^2}{1+\rho^2}\right] \le \frac{\sqrt{\varepsilon}}{2} \tag{4.30}$$

for every nonnegative ρ. In other words, $\rho^2 f''_\varepsilon(\rho)/f'_\varepsilon(\rho)$ converges to $\rho^3/(1+\rho^2)$ *uniformly* with respect to ρ as ε approaches 0.

Mimicking the proof of proposition (iii) of Lemma 4.2 shows that

$$\left|\frac{n}{\sqrt{n^2+|\nabla u_\varepsilon|^2}}\,\nabla u_\varepsilon - \frac{n}{\sqrt{n^2+|\nabla u|^2}}\,\nabla u\right| \le |\nabla u_\varepsilon - \nabla u|. \tag{4.31}$$

Proposition (ii) of Theorem 2.3, (4.27), (4.29), and inequalities (4.30) and (4.31) enable us to conclude that

$$\frac{n^{-4}|\nabla u_\varepsilon|^4}{[1+n^{-2}|\nabla u_\varepsilon|^2]^{3/2}} \cdot \frac{|\nabla u_\varepsilon|}{f'_\varepsilon\left(n^{-1}|\nabla u_\varepsilon|\right)} \cdot \operatorname{div}\left\{n \cdot f'_\varepsilon\left(\frac{|\nabla u_\varepsilon|}{n}\right) \cdot \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right\} \tag{4.32}$$

approaches

$$\frac{|\nabla u|}{n^2+|\nabla u|^2} \cdot \operatorname{tr}\left\{n\rho^2 t(\rho)\left[\frac{2}{1+(t(\rho))^2} - \frac{\rho^2}{1+\rho^2}\right]\begin{bmatrix} n_x u_x & n_y u_x \\ n_x u_y & n_y u_y \end{bmatrix}\right.$$
$$\left. + \begin{bmatrix} u_x & -u_y \\ u_y & u_x \end{bmatrix}\begin{bmatrix} \frac{\rho^2}{1+\rho^2} & 0 \\ 0 & \frac{2}{1+(t(\rho))^2} \end{bmatrix}\begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix}\nabla(Tu)\right\} \tag{4.33}$$

in $L^1_{\mathrm{loc}}(\Omega)$ as ε approaches 0.

If expression (4.33) is named A and

$$U = \text{l.h.s. of (1.5)},$$

then (4.7) and (4.8) cause the following equation to hold:

$$A = |\nabla u|^2 \times (n^2 + |\nabla u|^2)^{-\frac{5}{2}} \times U.$$

Proposition (i) of Theorem 2.3 implies (4.2); hence expression (4.32) is zero. We infer

$$A = 0.$$

Now we let

$$B = (n^2 + |\nabla u|^2)^{-\frac{3}{2}} \times U$$

and claim that

$$B = 0.$$

In fact, if

$$C = \sqrt{2} \cdot |\nabla u| \cdot \sqrt{u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2} + n \cdot |\nabla n|,$$

then inequality (2.3) informs us that C is locally integrable. We have

$$|B| \leq \frac{|\nabla u|}{\sqrt{n^2 + |\nabla u|^2}} \times C$$

because of the Cauchy–Schwarz inequality; moreover,

$$A = \frac{|\nabla u|^2}{n^2 + |\nabla u|^2} \times B.$$

Thus A is locally integrable, irrespective of whether it is zero or not; B is locally integrable too, and the following inequality holds:

$$|B| \leq |A|^{1/3} \cdot |C|^{2/3},$$

which proves the claim.

Equation (1.5) follows. The proof of Theorem 2.3 is complete. □

5. Remarks on viscosity solutions.

THEOREM 5.1. *A viscosity solution to* (1.5) *is uniquely determined by its boundary values. A smooth solution to* (1.5) *need not be uniquely determined by its boundary values.*

Proof. Theorems 2.2 and 2.3 demonstrate the following property: any viscosity solution to (1.5) which takes the relevant boundary values minimizes the functional J. The former assertion results. The latter results via the analysis of an ad hoc example, as shown below.

Suppose $n \equiv 1$ and u is given by (1.6). (For the sake of brevity, we denote the domain of u by Ω.) Arguments from [26, section 2.2] tell us the following. First, u is a smooth solution to (1.5). Second,

$$-\text{div}\left(\sqrt{1 + |\nabla u|^2}\,\frac{\nabla u}{|\nabla u|}\right)$$

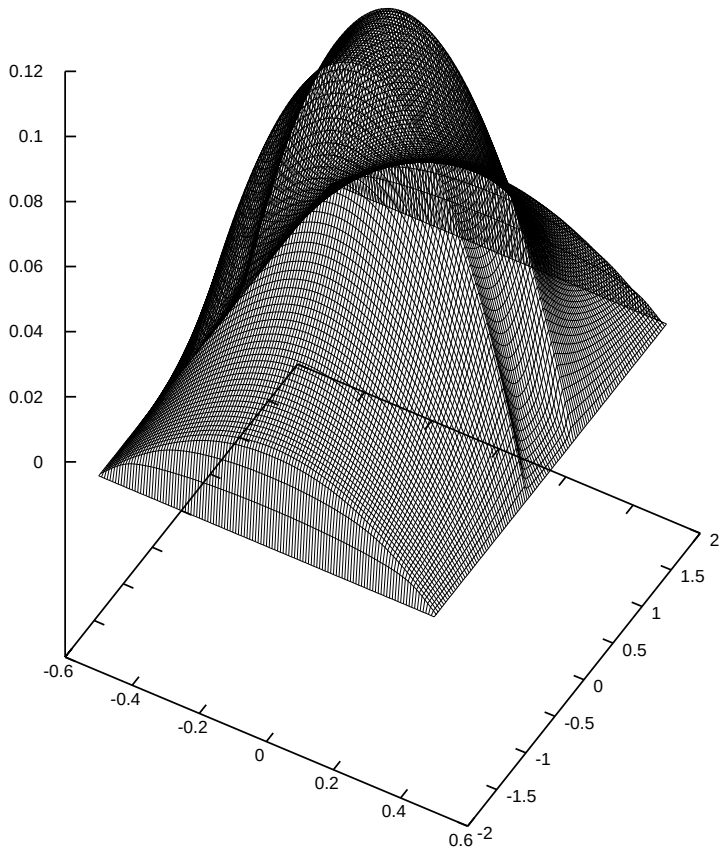

FIG. 5.1. *A plot of the difference between function u given by* (1.6) *and a viscosity solution to* (1.5) *that takes the same boundary values as u.*

equals

$$C_0^\infty(\Omega) \ni \varphi \mapsto 2\int_{-\infty}^{\infty} \varphi(0,y)dy$$

in the sense of distributions. The latter statement informs us that, provided that the domain of J is adjusted as $u + W_0^{1,2}(\Omega)$, the subdifferential of J at u consists of a nonzero measure supported by the y-axis. Therefore, u does not minimize J. It follows that u is not a viscosity solution to (1.5). In other words, u obeys (1.5) but differs from the viscosity solution to (1.5) which is defined in Ω and equals u on $\partial\Omega$. $\square$

Figure 5.1 helps one to visualize the proof above. It displays the difference between the function u given by (1.6) and the viscosity solution to (1.5) that is defined in $]-\frac{1}{2},\frac{1}{2}[\times]-2,2[$ and takes the same values of u on the boundary of such a rectangle. As a matter of fact, such a viscosity solution has been approximated by the solution u_ε to (1.15) with $\varepsilon = 10^{-8}$. Observe the scale in Figure 5.1; we stress that the difference between the u_ε's with $\varepsilon = 10^{-8}$ and $\varepsilon = 10^{-4}$ has order of magnitude 10^{-9}.

Appendix. The following lemma, which analyzes (2.1) closely, is instrumental in proving Theorem 2.2 and proposition (i) of Theorem 2.3.

LEMMA A.1. (i) *f_ε is nonnegative, vanishes only at* 0, *and is strictly increasing and strictly convex.*

(ii)
$$\rho\,\frac{1+\rho^2}{1/2+\rho^2} \le f'_\varepsilon(\rho) \le f'(\rho),$$

and

$$\frac{\rho^2}{2}+\frac{1}{4}\log(1+2\rho^2) \le f_\varepsilon(\rho) \le f(\rho)$$

for every nonnegative ρ.

(iii) *If C_ε is defined by*

$$2C_\varepsilon = \varepsilon^{\frac{1-2\varepsilon}{2(1-\varepsilon)}} + \log(1+\sqrt{\varepsilon}) - \varepsilon - \frac{1}{2}\int_\varepsilon^1 \frac{t^{-\frac{1}{2(1-\varepsilon)}} - \sqrt{t}}{1-t}\,dt,$$

then

$$f_\varepsilon(\rho) = f(\rho) - C_\varepsilon + O(\rho^{-2})$$

as $\rho \to \infty$.

(iv) *f_ε converges uniformly to f on $[0,\infty[$ as ε approaches zero. In effect,*

$$\sup\{|f(\rho)-f_\varepsilon(\rho)| : 0 \le \rho < \infty\} = O(\sqrt{\varepsilon}).$$

(v) *f'_ε has a zero of multiplicity one at* 0. *In effect,*

$$f'_\varepsilon(\rho) = \varepsilon^{-\frac{1}{2(1-\varepsilon)}}\cdot\rho\cdot\left[1-\frac{1}{2\varepsilon}\rho^2+\frac{3+2\varepsilon}{8\varepsilon^2}\rho^4-\frac{15+14\varepsilon+8\varepsilon^2}{48\varepsilon^3}\rho^6+\cdots\right]$$

if $0 \le \rho < \sqrt{\varepsilon}$.

(vi)
$$\frac{f'_\varepsilon(\rho)}{\rho} < \varepsilon^{-\frac{1}{2(1-\varepsilon)}}$$

and

$$f''_\varepsilon(\rho) < \frac{f'_\varepsilon(\rho)}{\rho}$$

if $\rho > 0$;

$$f''_\varepsilon(\rho) \ge \frac{4\varepsilon^{\frac{1-2\varepsilon}{4(1-\varepsilon)}}\,[4+\sqrt{\varepsilon(12+\varepsilon)}+\varepsilon]}{[2+\sqrt{\varepsilon(12+\varepsilon)}+\varepsilon]^{\frac{1-2\varepsilon}{2(1-\varepsilon)}}\,[\sqrt{12+\varepsilon}+3\sqrt{\varepsilon}]^{\frac{3-2\varepsilon}{2(1-\varepsilon)}}}$$

and

$$f'_\varepsilon(\rho) - \rho f''_\varepsilon(\rho) \le \frac{\sqrt{2}\;\varepsilon^{-\frac{\varepsilon}{4(1-\varepsilon)}}\,\left[\sqrt{12+\varepsilon}+\sqrt{\varepsilon}\right]^{\frac{3}{2}}}{[2+\sqrt{\varepsilon(12+\varepsilon)}+\varepsilon]^{\frac{1-2\varepsilon}{2(1-\varepsilon)}}\,[\sqrt{12+\varepsilon}+3\sqrt{\varepsilon}]^{\frac{3-2\varepsilon}{2(1-\varepsilon)}}}$$

if $\rho \ge 0$.

Proof. Equation (2.1) gives successively $f_\varepsilon(0)=0$, and

(A.1)
$$f'_\varepsilon(\rho) = \rho\left(\frac{1+\rho^2}{\varepsilon+\rho^2}\right)^{\frac{1}{2(1-\varepsilon)}},$$

$$f_\varepsilon''(\rho) = (1+\rho^2)^{-\frac{1-2\varepsilon}{2(1-\varepsilon)}}(\varepsilon+\rho^2)^{-\frac{3-2\varepsilon}{2(1-\varepsilon)}}(\rho^4+\varepsilon\rho^2+\varepsilon) \tag{A.2}$$

for every nonnegative ρ. Thus $f_\varepsilon'(\rho)$ equals zero if ρ equals zero and is positive if ρ is positive; $f_\varepsilon''(\rho)$ is positive if ρ is nonnegative. Proposition (i) follows.

Equation (A.1) yields

$$\frac{\partial}{\partial\varepsilon}\log f_\varepsilon'(\rho) = \frac{1}{2(1-\varepsilon)^2}\left[\log\left(1+\frac{1-\varepsilon}{\varepsilon+\rho^2}\right)-\frac{1-\varepsilon}{\varepsilon+\rho^2}\right];$$

hence

$$\frac{\partial}{\partial\varepsilon}f_\varepsilon'(\rho) < 0$$

for every positive ρ. In other words, $\varepsilon \mapsto f_\varepsilon'(\rho)$ *decreases* if $\rho > 0$. Since the range of ε is $]0, 1/2]$, proposition (ii) follows. (Incidentally, one might also show that $\varepsilon \mapsto f_\varepsilon'(\rho)$ is *log-convex* for every positive ρ. Observe also that the difference between the r.h.s. and the l.h.s. of the second inequality in (ii) increases as ρ increases from 0 to ∞, approaches $(1+\log 2)/4$ as ρ approaches ∞, and thus is smaller than $0.423286\ldots$.)

Equation (2.1) reads

$$2f_\varepsilon(\rho) = (1-\varepsilon)\int_\varepsilon^{(\varepsilon+\rho^2)/(1+\rho^2)} \frac{t^{\frac{1}{2(1-\varepsilon)}}}{(1-t)^2}\,dt.$$

Integrations by parts and manipulations give

$$2f_\varepsilon(\rho) = (1+\rho^2)^{\frac{1}{2(1-\varepsilon)}}(\varepsilon+\rho^2)^{\frac{1-2\varepsilon}{2(1-\varepsilon)}} - \varepsilon^{\frac{1-2\varepsilon}{2(1-\varepsilon)}}$$

$$+\log\frac{\sqrt{1+\rho^2}+\sqrt{\varepsilon+\rho^2}}{1+\sqrt{\varepsilon}} + \frac{1}{2}\int_\varepsilon^{(\varepsilon+\rho^2)/(1+\rho^2)} \frac{t^{-\frac{1}{2(1-\varepsilon)}}-t^{-\frac{1}{2}}}{1-t}\,dt$$

for every nonnegative ρ. Proposition (iii) follows.

Proposition (ii) ensures that $f - f_\varepsilon$ is nonnegative and increasing, and proposition (iii) ensures that $f(\rho) - f_\varepsilon(\rho)$ approaches C_ε as $\rho \to \infty$. Hence

$$\sup\{|f(\rho)-f_\varepsilon(\rho)| : 0 \le \rho < \infty\} = C_\varepsilon.$$

Proposition (iv) follows.

Proposition (v) follows from manipulations of (A.1).

Equation (A.1) tells us that $f_\varepsilon'(\rho)/\rho$ decreases strictly from $\varepsilon^{-1/(2(1-\varepsilon))}$ to 1 as ρ increases from 0 to ∞. Equations (A.1) and (A.2) imply that

$$\frac{\rho f_\varepsilon''(\rho)}{f_\varepsilon'(\rho)} = 1 - \frac{\rho^2}{(1+\rho^2)(\varepsilon+\rho^2)}$$

if $\rho > 0$ and

$$f_\varepsilon'''(\rho) = (1+\rho^2)^{-\frac{3-4\varepsilon}{2(1-\varepsilon)}}(\varepsilon+\rho^2)^{-\frac{5-4\varepsilon}{2(1-\varepsilon)}}\rho(\rho^4-\varepsilon\rho^2-3\varepsilon)$$

if $\rho \ge 0$. Therefore, $f_\varepsilon''(\rho)$ is less than $f_\varepsilon'(\rho)/\rho$ if ρ is positive; if $\rho = 0$, then $f_\varepsilon'''(\rho)$ and $f_\varepsilon'(\rho) - \rho f_\varepsilon'''(\rho)$ are an absolute maximum and an absolute minimum, respectively; if

$$\rho = \sqrt{\varepsilon/2 + \sqrt{3\varepsilon+\varepsilon^2/4}},$$

then $f_\varepsilon'''(\rho)$ and $f_\varepsilon'(\rho) - \rho f_\varepsilon'''(\rho)$ are an absolute minimum and an absolute maximum, respectively. Proposition (vi) follows. $\square$

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