Stationary critical points of the heat flow in the plane

STATIONARY CRITICAL POINTS OF THE HEAT FLOW IN THE PLANE

ROLANDO MAGNANINI AND SHIGERU SAKAGUCHI

May 10, 2000

ABSTRACT. In the previous papers [MS 1, 2], we considered stationary critical points of solutions of the initial-boundary value problems for the heat equation on bounded domains in $\mathbb{R}^N$, $N \geqq 2$.

In [MS 1], we showed that a solution u has a stationary critical point O if and only if u satisfies some balance law with respect to O for any time. Furthermore, we proved necessary and sufficient conditions relating the symmetry of the domain to the initial data u_0; in this way, we gave a characterization of the ball in $\mathbb{R}^N$ ([MS 1]) and of centrosymmetric domains ([MS 2]).

In the present paper, we consider a rotation A_d by an angle $2\pi/d, d \geqq 2$, for planar domains and we give some necessary and some sufficient conditions on u_0 which relate to domains invariant under A_d. We also establish some conjectures.

1. Introduction. In this paper we consider stationary critical points of the heat flow particularly in the plane. Let us recall known results for the heat flow in $\mathbb{R}^N$ $(N \geqq 2)$. Let $B_\delta(0)$ be an open ball in $\mathbb{R}^N$ centered at the origin with radius $\delta > 0$, and let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $\overline{B_\delta(0)} \subset \Omega$. We consider the following initial-boundary value problem:

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x) & x \in \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where ν denotes the exterior normal unit vector to the boundary $\partial\Omega$, and α is a constant with $0 \leqq \alpha \leqq 1$. Let Φ, Φ_c be families of initial data u_0 defined by

$$\Phi = \{u_0 \in C_0^\infty(B_\delta(0)) \ : \ \int_{S^{N-1}} \omega u_0(r\omega)d\omega = 0 \text{ for any } r \in [0,\delta) \ \}, \tag{1.2}$$

$$\Phi_c = \{u_0 \in C_0^\infty(B_\delta(0)) \ : \ u_0(x) = u_0(-x) \ \text{ for any } x \in B_\delta(0) \ \}. \tag{1.3}$$

This research was partially supported by a Grant-in-Aid for Scientific Research (C) (♯ 10640175) of Japan Society for the Promotion of Science. The first author was supported also by the italian MURST.

1991 *Mathematics Subject Classification.* 35K05, 35K20, 35J05, 35J25, 35B38, 35B40.

Key words and phrases. heat equation, Poisson's equation, stationary critical point, initial-boundary value problem, homogeneous Dirichlet condition, rotational symmetry, planar domain.

Here in (1.2) $\omega = (\omega_1, ..., \omega_N)$ is a vector in the standard (N-1)-dimensional unit sphere S^{N-1} in $\mathbb{R}^N$, and $d\omega$ is the volume element of S^{N-1}. Note that $\Phi_c \subset \Phi$. In the previous papers [MS 1, 2] we have proved the following results.

Theorem 1.1 [MS 1]. *For any $u_0 \in \Phi$ the corresponding solution u of* (1.1) *always satisfies $\nabla u(0,t) = 0$ for any $t > 0$, if and only if $\Omega = B_R(0)$ for some $R > \delta$.*

Theorem 1.2 [MS 2]. *For any $u_0 \in \Phi_c$ the corresponding solution u of* (1.1) *always satisfies $\nabla u(0,t) = 0$ for any $t > 0$, if and only if Ω is centrosymmetric with respect to the origin.*

In [MS 2, Section 6, p. 711] we posed the question:

Does there exist another subset of Φ which relates to some symmetry of the domain Ω?

Theorems 1.1 and 1.2 mean that Φ relates to radial symmetry of Ω, and Φ_c does to centrosymmetry. In the present paper, we consider the action of the rotation A_d of angle $2\pi/d$ on planar domains and we introduce the family Φ_d of initial data, compactly supported in $B_\delta(0)$, and invariant under the action of A_d.

In Section 2, we prove that domain invariance under A_d relates to Φ_d, if $d = 2$ or 3 (Theorem 2.3), while Φ_4 characterizes centrosymmetric domains (Theorem 2.5). Also, we prove that there exists another family of initial data characterizing centrosymmetric domains (Theorem 2.6). This family consists of functions symmetric with respect to both coordinates axes.

In Section 3, we prove related symmetry theorems (Theorems 3.1 and 3.2) for Poisson's equation.

In Section 4, we give a relation between subfamilies of Φ and essential symmetry subgroups of the orthogonal group $O(N)$, whose relationship to stationary critical points was studied by Chamberland and Siegel [CS] (see Proposition 4.1). Finally, we pose three conjectures.

2. Symmetry results: the parabolic case. It is convenient to use the complex variable $z = x + iy \in \mathbb{C}$. Let Ω be a bounded and simply connected domain in $\mathbb{C}$ with boundary $\partial\Omega$, and let $\overline{B_\delta(0)} \subset \Omega$. Consider the following initial-Dirichlet problem for the heat equation:

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0,\infty), \\ u(z,0) = u_0(z) & z \in \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{2.1}$$

Let $A = A_d \in O(2)$ be the matrix of rotation of angle $2\pi/d$ with an integer $d \geqq 2$. We will use the convenient identification $A = e^{2\pi i/d}$. Then, A^d is the identical transformation and $\{1, A, ..., A^{d-1}\}$ is an essential symmetry group as in Chamberland and Siegel [CS] (see Section 4 of the present paper for its definition). We introduce the family Φ_d of initial data u_0 by

$$\Phi_d = \{ \ u_0 \in C_0^\infty(B_\delta(0)) \ : \ u_0(z) = u_0(Az) \ \text{ for any } z \in B_\delta(0) \ \}. \tag{2.2}$$

Here note that $\Phi_d \subset \Phi$. Our starting point is the following theorem that generalizes a result in [CS].

Theorem 2.1. *Consider the initial-Dirichlet problem* (2.1). *If* Ω *is invariant under the action of* $A = A_d$*, then the solution* u *satisfies* $\nabla u(0,t) = 0$ *for any* $t \in (0,\infty)$ *and for any initial data* $u_0 \in \Phi_d$.

Proof. Take any $u_0 \in \Phi_d$. Then, it follows from the uniqueness of the solution of (2.1) that $u(z,t) = u(Az,t)$ for any $t \in (0,\infty)$. This implies that $\nabla u(0,t) = 0$ for any $t \in (0,\infty)$. $\square$

In the sequel, we will show that the converse of Theorem 2.1 is true for $d = 2$ and 3 but not for $d = 4$. In preparation to these results, we will prove the following fact.

Theorem 2.2. *Let* D *be the unit disk in* $\mathbb{C}$ *centered at the origin. If the solution* u *of the initial-Dirichlet problem* (2.1) *satisfies* $\nabla u(0,t) = 0$ *for any* $t \in (0,\infty)$ *and for any initial data* $u_0 \in \Phi_d$*, then there exists a conformal mapping* $f : \Omega \to D$ *with* $f(0) = 0$ *and such that for any* $z \in B_\delta(0) \setminus \{0\}$

$$\sum_{k=0}^{d-1} f(A^k z) = 0 \quad \text{and} \quad \sum_{k=0}^{d-1} \frac{1}{f(A^k z)} = 0. \tag{2.3}$$

Proof. For any $\psi \in C_0^\infty(B_\delta(0))$, put $u_0(z) = \sum_{k=0}^{d-1} \psi(A^k z)$; then $u_0 \in \Phi_d$. Let $K = K(z,\zeta,t)$ be the Green's function for problem (2.1). We have

$$u(z,t) = \int_{B_\delta(0)} K(z,\zeta,t) u_0(\zeta)\, d\xi d\eta, \tag{2.4}$$

where $\zeta = \xi + i\eta$ and $d\xi d\eta$ denotes the area element.

By the assumption, for any $\psi \in C_0^\infty(B_\delta(0))$ and for any $t > 0$ we get

$$\begin{aligned} 0 &= \nabla u(0,t) \\ &= \int_{B_\delta(0)} \nabla_z K(0,\zeta,t) \sum_{k=0}^{d-1} \psi(A^k \zeta)\, d\xi d\eta \\ &= \int_{B_\delta(0)} \left\{ \sum_{k=0}^{d-1} \nabla_z K(0, A^k\zeta, t) \right\} \psi(\zeta)\, d\xi d\eta. \end{aligned}$$

Since ψ is arbitrary, we get

$$\sum_{k=0}^{d-1} \nabla_z K(0, A^k \zeta, t) = 0 \quad \text{for any } (\zeta, t) \in B_\delta(0) \times (0,\infty). \tag{2.5}$$

Recall that the Green's function $G = G(z,\zeta)$ of $-\Delta$ under the homogeneous Dirichlet boundary condition is given by

$$G(z,\zeta) = \int_0^\infty K(z,\zeta,t)\, dt.$$

Hence integrating (2.5) with respect to t from 0 to ∞ yields

$$\sum_{k=0}^{d-1} \nabla_z G(0, A^k \zeta) = 0 \quad \text{for any } \zeta \in B_\delta(0) \setminus \{0\}. \tag{2.6}$$

Since Ω is simply connected, it follows from the Riemann mapping theorem that there exists a conformal mapping $f : \Omega \to D$ with $f(0) = 0$. Then we have an explicit representation of the Green's function G for Ω (see [N, Exercise 1, p. 182] for example) :

$$G(z, \zeta) = -\frac{1}{2\pi} \log \left| \frac{f(z) - f(\zeta)}{1 - \overline{f(\zeta)} f(z)} \right|, \tag{2.7}$$

where $\overline{f(\zeta)}$ is the complex conjugate of $f(\zeta)$. The latter formula gives

$$2\pi \nabla_z G(0, \zeta) = 4\pi \frac{\partial}{\partial \overline{z}} G(0, \zeta) = \overline{f'(0)} \left(-f(\zeta) + \frac{f(\zeta)}{|f(\zeta)|^2} \right), \tag{2.8}$$

where $\frac{\partial}{\partial \overline{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$. Then from (2.6) and (2.8), it follows that

$$\sum_{k=0}^{d-1} \left[-f(A^k z) + \frac{f(A^k z)}{|f(A^k z)|^2} \right] = 0, \tag{2.9}$$

or

$$\sum_{k=0}^{d-1} f(A^k z) = \overline{\sum_{k=0}^{d-1} \left(\frac{1}{f(A^k z)} \right)} \quad \text{for any } z \in B_\delta(0) \setminus \{0\}. \tag{2.10}$$

Observe that (2.10) shows that each side of (2.10) is both holomorphic and anti-holomorphic in $B_\delta(0) \setminus \{0\}$. Therefore both sides must be a constant and equal zero since $f(0) = 0$, that is, (2.3) must hold. □

Theorem 2.3. *Let $d = 2$ or 3. If the solution u of the initial-Dirichlet problem (2.1) satisfies $\nabla u(0, t) = 0$ for any $t \in (0, \infty)$ and for any initial data $u_0 \in \Phi_d$, then Ω is invariant under the action of $A = A_d$.*

Proof. Since the case $d = 2$ is contained in Theorem 2 in [MS 2], let us suppose that $d = 3$. By eliminating $f(A^2 z)$ from (2.3) and by rearranging the terms, we have:

$$\frac{f(Az)}{f(z)} + \frac{f(z)}{f(Az)} + 1 = 0.$$

Hence, the function $h(z) = \frac{f(Az)}{f(z)}$ satisfies the quadratic equation $h^2(z) + h(z) + 1 = 0$ whose roots are A and $\overline{A}$. Since $h(z) \to A$ as $z \to 0$, we obtain that $h(z) = A$, and hence

$$f(Az) = A f(z).$$

This implies that $g(Aw) = Ag(w)$ for any $w \in f(B_\delta(0))$, where $g : D \to \Omega$ is the inverse mapping of f. Therefore, since g is holomorphic in D, we conclude that

$$g(Aw) \equiv Ag(w) \quad \text{in } D.$$

This means that $\Omega = g(D)$ is invariant under the action of A. □

Remark 2.4. If d is not a prime number, Theorem 2.3 does not hold. Indeed, suppose that d is not a prime number. Then, there exists a prime number p dividing d, and $\Phi_d \subset \Phi_p$. Let Ω be invariant under the action of A_p and not invariant under the action of A_d. Here, the solution u of (2.1) satisfies $\nabla u(0,t) = 0$ for any $t \in (0,\infty)$ and for any initial data $u_0 \in \Phi_p$, then also for any initial data $u_0 \in \Phi_d$.

Related to Remark 2.4, when $d = 4$, we have

Theorem 2.5. *Let $d = 4$. Consider the initial-Dirichlet problem* (2.1)*. Then, the solution u satisfies $\nabla u(0,t) = 0$ for any $t \in (0,\infty)$ and for any initial data $u_0 \in \Phi_4$, if and only if Ω is invariant under the action of A_2 (that is, Ω is centrosymmetric with respect to the origin).*

Proof. It suffices to show "only if"part. Suppose that $\nabla u(0,t) = 0$ for any $t \in (0,\infty)$ and for any initial data $u_0 \in \Phi_4$. Then, Theorem 2.2 implies that for any $z \in B_\delta(0) \setminus \{0\}$

$$\begin{cases} f(z) + f(iz) + f(i^2 z) + f(i^3 z) = 0, \\ \dfrac{1}{f(z)} + \dfrac{1}{f(iz)} + \dfrac{1}{f(i^2 z)} + \dfrac{1}{f(i^3 z)} = 0, \end{cases} \tag{2.11}$$

where $f : \Omega \to D$ is a conformal mapping with $f(0) = 0$ and D is the unit disk in $\mathbb{C}$ centered at the origin. Then, by using (2.11) we get for any $z \in B_\delta(0) \setminus \{0\}$

$$(f(z) + f(iz)) \left(\frac{1}{f(z)f(iz)} - \frac{1}{f(i^2 z)f(i^3 z)} \right) = 0. \tag{2.12}$$

Since $(f(z) + f(iz))'|_{z=0} = (1+i)f'(0) \neq 0$, it follows from (2.12) that

$$f(z)f(iz) = f(-z)f(-iz) \quad \text{for any } z \in B_\delta(0). \tag{2.13}$$

Replacing z by $-iz$ yields

$$f(-iz)f(z) = f(iz)f(-z) \quad \text{for any } z \in B_\delta(0). \tag{2.14}$$

Combining (2.13) with (2.14) gives

$$(f(z))^2 f(iz)f(-iz) = (f(-z))^2 f(iz)f(-iz) \quad \text{for any } z \in B_\delta(0),$$

and hence

$$(f(z) - f(-z))\,(f(z) + f(-z)) = 0 \quad \text{for any } z \in B_\delta(0). \tag{2.15}$$

Since $(f(z) - f(-z))'|_{z=0} = 2f'(0) \neq 0$, we conclude that

$$f(-z) = -f(z) \quad \text{for any } z \in B_\delta(0). \tag{2.16}$$

By the same argument as in the proof of Theorem 2.3, we show that

$$g(-w) \equiv -g(w) \quad \text{in } D$$

for the inverse mapping $g : D \to \Omega$ of f, and we conclude that $\Omega = g(D)$ is invariant under the action of A_2. $\Box$

Theorem 2.5 means that Φ_4 characterizes centrosymmetric domains. Furthermore, let us show that there exists another family of initial data characterizing centrosymmetric domains. Precisely, let $\mathfrak{R} \in O(2)$ be the matrix of reflection with respect to the real axis. Then, $\mathfrak{R}z = \overline{z}$, and $\{\pm I, \pm\mathfrak{R}\}$ is an essential symmetry group, where I denotes the identity matrix. We introduce the family $\Phi_{\mathfrak{R}}$ of initial data u_0 by

$$\Phi_{\mathfrak{R}} = \{\ u_0 \in C_0^\infty(B_\delta(0))\ :\ u_0(z) = u_0(\overline{z}) = u_0(-z)\ \text{ for any } z \in B_\delta(0)\ \}. \quad (2.17)$$

Here note that $\Phi_{\mathfrak{R}} \subsetneqq \Phi_2 (= \Phi_c)$, $\Phi_{\mathfrak{R}} \not\subset \Phi_4$, and $\Phi_4 \not\subset \Phi_{\mathfrak{R}}$. For each $u_0 \in C_0^\infty(B_\delta(0))$, u_0 belongs to $\Phi_{\mathfrak{R}}$ if and only if u_0 is symmetric with respect to both real and imaginary axes. Then another characterization of centrosymmetric domains is

Theorem 2.6. *Consider the initial-Dirichlet problem* (2.1). *Then, the solution u satisfies $\nabla u(0, t) = 0$ for any $t \in (0, \infty)$ and for any initial data $u_0 \in \Phi_{\mathfrak{R}}$, if and only if Ω is invariant under the action of A_2 (that is, Ω is centrosymmetric with respect to the origin).*

Proof. It suffices to show "only if" part. Suppose that $\nabla u(0, t) = 0$ for any $t \in (0, \infty)$ and for any initial data $u_0 \in \Phi_{\mathfrak{R}}$. For any $\psi \in C_0^\infty(B_\delta(0))$, put

$$u_0(z) = \psi(z) + \psi(-z) + \psi(\overline{z}) + \psi(-\overline{z}). \quad (2.18)$$

Then $u_0 \in \Phi_{\mathfrak{R}}$. Proceeding as in the proof of Theorem 2.2 yields that for any $\zeta \in B_\delta(0) \setminus \{0\}$

$$\nabla_z G(0, \zeta) + \nabla_z G(0, -\zeta) + \nabla_z G(0, \overline{\zeta}) + \nabla_z G(0, -\overline{\zeta}) = 0, \quad (2.19)$$

and hence for any $z \in B_\delta(0) \setminus \{0\}$

$$f(z) + f(-z) + f(\overline{z}) + f(-\overline{z}) = \overline{\left(\frac{1}{f(z)} + \frac{1}{f(-z)} + \frac{1}{f(\overline{z})} + \frac{1}{f(-\overline{z})}\right)}, \quad (2.20)$$

where $f : \Omega \to D$ is a conformal mapping with $f(0) = 0$. By rearranging the terms, we have for any $z \in B_\delta(0) \setminus \{0\}$

$$f(z) + f(-z) - \overline{\left(\frac{1}{f(\overline{z})} + \frac{1}{f(-\overline{z})}\right)} = \overline{\left(\frac{1}{f(z)} + \frac{1}{f(-z)}\right)} - (f(\overline{z}) + f(-\overline{z})). \quad (2.21)$$

Observe that (2.21) shows that each side of (2.21) is both holomorphic and anti-holomorphic in $B_\delta(0) \setminus \{0\}$. Therefore both sides must be a constant, say c_0.

Let us show that $c_0 = 0$. Indeed, since f is conformal and $f(0) = 0$, f is expanded into the Taylor series about the origin in $B_\delta(0)$:

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{with } a_1 \neq 0. \quad (2.22)$$

Then $1/f(z)$ is expanded into the Laurent series about the origin in $B_\delta(0) \setminus \{0\}$:

$$\frac{1}{f(z)} = \frac{1}{a_1 z} + \sum_{n=0}^{\infty} b_n z^n. \quad (2.23)$$

Hence we have in $B_\delta(0) \setminus \{0\}$

$$\frac{1}{f(z)} + \frac{1}{f(-z)} = 2\sum_{k=0}^{\infty} b_{2k} z^{2k}. \tag{2.24}$$

Then, since $f(0) = 0$, the left hand side of (2.21) at $z = 0$ equals $-2\overline{b_0}$, and the right hand side of (2.21) at $z = 0$ equals $2\overline{b_0}$. Thus, $\overline{b_0} = 0$ and hence $c_0 = 0$.

Therefore, we conclude that

$$f(z) + f(-z) = \frac{1}{\overline{f(\overline{z})}} + \frac{1}{\overline{f(-\overline{z})}} \tag{2.25}$$

and

$$\overline{f(\overline{z})} + \overline{f(-\overline{z})} = \frac{1}{f(z)} + \frac{1}{f(-z)} \tag{2.26}$$

for any $z \in B_\delta(0) \setminus \{0\}$. Combining (2.25) and (2.26) and rearranging the terms, give

$$[f(z) + f(-z)] \left[1 - \frac{1}{f(z)f(-z)\overline{f(\overline{z})f(-\overline{z})}}\right] = 0 \tag{2.27}$$

for any $z \in B_\delta(0) \setminus \{0\}$. Since $f(0) = 0$, (2.27) implies that

$$f(-z) = -f(z) \text{ for any } z \in B_\delta(0).$$

By the same argument as in the proof of Theorem 2.3, we show that

$$g(-w) \equiv -g(w) \text{ in } D$$

for the inverse mapping $g : D \to \Omega$ of f, and we conclude that $\Omega = g(D)$ is invariant under the action of A_2. $\square$

3. Symmetry results: the elliptic case. Let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with $\overline{B_\delta(0)} \subset \Omega$, and let $A = A_d \in O(2)$ be the matrix of rotation of angle $2\pi/d$ with an integer $d \geqq 2$. Consider the boundary value problem:

$$-\Delta u = \varphi_\zeta \text{ in } \Omega, \quad \text{and } u = 0 \text{ on } \partial\Omega, \tag{3.1}$$

where $\varphi_\zeta = \varphi_\zeta(z) = \sum_{j=0}^{d-1} \delta_{\zeta_j}(z)$, $\zeta \in \partial B_\delta(0)$, $\zeta_j = A^j\zeta$, and $\delta_\zeta(z)$ denotes a Dirac measure at ζ. Denote by $G(z, \zeta)$ the positive Green's function. Then we have $u(z) = \sum_{j=0}^{d-1} G(z, \zeta_j)$. Also, we use multiindices:

$$\gamma = (\alpha, \beta) \in (\mathbb{N} \cup \{0\})^2, \ |\gamma| = \alpha + \beta, \ \partial_z^\gamma = \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial y}\right)^\beta.$$

Theorem 3.1. *Consider the boundary value problem* (3.1)*. Then, the solution* u *of* (3.1) *satisfies* $\partial_z^\gamma u(0) = 0$ *for any* γ *with* $1 \leqq |\gamma| \leqq d-1$ *and for any* $\zeta \in \partial B_\delta(0)$*, if and only if* Ω *is invariant under the action of* $A = A_d$*.*

Proof. Suppose that Ω is invariant under the action of $A = A_d$. Take any $\zeta \in \partial B_\delta(0)$. Then, by the uniqueness of the solution of the boundary value problem (3.1), the solution u satisfies $u(z) = u(Az)$ for any $z \in \Omega$. Then, the origin must be a critical point of u. Recall that critical points of non-constant harmonic functions are isolated and each critical point has finite integral multiplicity (see [W] or [A]). Observe that Ω is simply connected, u is harmonic in Ω except d points $\{\zeta_0, \cdots, \zeta_{d-1}\}$, $u = +\infty$ at such points, and $u = 0$ on $\partial\Omega$.

We claim that the origin is the only critical point of u in $\Omega \setminus \{\zeta_0, \cdots, \zeta_{d-1}\}$. Indeed, if there were another critical point, then, by using the rotational symmetry, we could infer that u would have a total of at least $d+1$ critical points, which contradicts Theorem 1.1 in [AM]. Hence, by the same theorem, the origin has exact multiplicity $d-1$, which implies that

$$\partial_z^\gamma u(0) = 0 \quad \text{for any } \gamma \text{ with } 1 \leqq |\gamma| \leqq d-1.$$

Conversely, suppose that this holds for any $\zeta \in \partial B_\delta(0)$. Let

$$\Gamma(z,\zeta) = -\frac{1}{2\pi}\log|z-\zeta| \quad \text{and} \quad G(z,\zeta) = \Gamma(z,\zeta) + H(z,\zeta). \tag{3.2}$$

Then H is smooth in $\Omega \times \Omega$ and satisfies

$$H(z,\zeta) = H(\zeta,z) \quad \text{together with} \quad \Delta_z H = 0. \tag{3.3}$$

Let us write

$$\begin{aligned} u(z) &= \sum_{j=0}^{d-1}\Gamma(z,\zeta_j) + \sum_{j=0}^{d-1}H(z,\zeta_j) \\ &\equiv g(z) + h(z). \end{aligned} \tag{3.4}$$

Since we know that

$$\partial_z^\gamma g(0) = 0 \quad \text{for any } \gamma \text{ with } 1 \leqq |\gamma| \leqq d-1 \text{ and for any } \zeta \in \overline{B_\delta(0)} \setminus \{0\}, \tag{3.5}$$

by the assumption we see that

$$\partial_z^\gamma h(0) = 0 \quad \text{for any } \gamma \text{ with } 1 \leqq |\gamma| \leqq d-1 \text{ and for any } \zeta \in \partial B_\delta(0). \tag{3.6}$$

Since any derivative $\partial_z^\gamma h(0)$ in (3.6) is harmonic in ζ, by the maximum principle we see that

$$\partial_z^\gamma h(0) = 0 \quad \text{for any } \gamma \text{ with } 1 \leqq |\gamma| \leqq d-1 \text{ and for any } \zeta \in \overline{B_\delta(0)}. \tag{3.7}$$

Therefore we get by (3.5) and (3.7)

$$\partial_z^\gamma u(0) = 0 \quad \text{for any } \gamma \text{ with } 1 \leqq |\gamma| \leqq d-1 \text{ and for any } \zeta \in \overline{B_\delta(0)} \setminus \{0\}. \tag{3.8}$$

Let

$$\Omega_j = \{ \zeta \ : \ A^j\zeta \in \Omega\} \ \text{ for each } j = 0, 1, ..., d-1. \tag{3.9}$$

Let C be the connected component of $\bigcap_{j=0}^{d-1} \Omega_j$ containing the origin. Then, (3.8) and the analyticity in ζ imply that

$$\partial_z^\gamma u(0) = 0 \ \text{ for any } \gamma \text{ with } 1 \leqq |\gamma| \leqq d-1 \text{ and for any } \zeta \in \overline{C} \setminus \{0\}. \tag{3.10}$$

On the other hand, by the boundary condition:

$$G(z, \zeta) = 0 \ \text{ for any } \zeta \in \partial\Omega$$

we have

$$\partial_z^\gamma G(0, \zeta) = 0 \ \text{ for any } \zeta \in \partial\Omega \text{ and for any } \gamma. \tag{3.11}$$

Therefore, we get for any γ and for any $j = 0, ..., d-1$

$$\partial_z^\gamma G(0, A^j\zeta) = 0 \ \text{ for any } \zeta \in \partial\Omega_j. \tag{3.12}$$

Now, it suffices to show that $\partial C \subset \bigcap_{j=0}^{d-1} \partial\Omega_j$. Let $\zeta^* \in \partial C$. Since $\partial C \subset \bigcup_{j=0}^{d-1} \partial\Omega_j$, there exists a j_0 such that $\zeta^* \in \partial\Omega_{j_0}$. Then it suffices to show that $\zeta^* \in \partial\Omega_j$ for any $j = 0, ..., d-1$. Suppose that $\zeta^* \in \Omega_\ell$ for some $\ell \neq j_0$. Then, by putting

$$v(z) = \sum_{\substack{0 \leq j \leq d-1 \\ j \neq j_0}} G(z, \zeta_j^*), \tag{3.13}$$

from (3.10) and (3.12) we see that v satisfies

$$\partial_z^\gamma v(0) = 0 \ \text{ for any } \gamma \text{ with } 1 \leqq |\gamma| \leqq d-1. \tag{3.14}$$

Since v is not constant, this is a contradiction. Indeed, observe that Ω is simply connected, v is harmonic in Ω except at most $d-1$ points, $v = +\infty$ at such points, and $v = 0$ on $\partial\Omega$. Therefore, with the help of a result of [AM, Theorem 1.1, pp. 567-568], we see that the multiplicity of the origin is at most $d-2$. This contradicts (3.14). $\square$

For any $\varphi \in \Phi_d$, we consider the boundary value problem:

$$-\Delta u = \varphi \ \text{ in } \Omega, \quad \text{ and } \ u = 0 \ \text{ on } \partial\Omega. \tag{3.15}$$

In view of the proofs of Theorems 2.2, 2.3, 2.5, and 2.6, we have

Theorem 3.2. *Consider the boundary value problem* (3.15)*. Then, the following hold:*

(i) *Suppose that* $d = 2$ *or* 3*. Then the solution* u *of* (3.15) *satisfies* $\nabla u(0) = 0$ *for any* $\varphi \in \Phi_d$*, if and only if* Ω *is invariant under the action of* A_d*.*

(ii) *The solution* u *of* (3.15) *satisfies* $\nabla u(0) = 0$ *for any* $\varphi \in \Phi_4$*, if and only if* Ω *is invariant under the action of* A_2*.*

(iii) *The solution* u *of* (3.15) *satisfies* $\nabla u(0) = 0$ *for any* $\varphi \in \Phi_{\mathfrak{R}}$*, if and only if* Ω *is invariant under the action of* A_2*.*

Remark 3.3. When $d = 3$, it follows from the proof of Theorem 2.3 that the solution u of (3.1) satisfies $\nabla u(0) = 0$ for any $y \in \partial B_\delta(0)$, if and only if Ω is invariant under the action of A_3. Namely, when $d = 3$ in Theorem 3.1, we can replace the condition that $\partial_z^\gamma u(0) = 0$ for any γ with $1 \leqq |\gamma| \leqq 2$ by $\nabla u(0) = 0$.

4. Conjectures. There is a relation between the families of initial data and the essential symmetry groups, which were introduced by Chamberland and Siegel [CS]. Let $N \geqq 2$ and let $B_\delta(0)$ be a ball in $\mathbb{R}^N$ centered at the origin with radius $\delta > 0$. Consider a subgroup G of the orthogonal group $O(N)$. Let us introduce the family $\Phi(G)$ of initial data u_0 by

$$\Phi(G) = \{\ u_0 \in C_0^\infty(B_\delta(0))\ :\ u_0(x) = u_0(gx)\ \text{ for any } (x, g) \in B_\delta(0) \times G\ \}. \tag{4.1}$$

Recall that G is said to be *essential*, if for any $x \neq 0$ there exists $g \in G$ satisfying $gx \neq x$. Then we have:

Proposition 4.1. *G is essential, if and only if* $\Phi(G) \subset \Phi$.

Proof. Suppose that G is essential. Let us show that $\Phi(G) \subset \Phi$. Take any $u_0 \in \Phi(G)$. Consider the Cauchy problem for the heat equation:

$$\partial_t u = \Delta u \ \text{ in } \mathbb{R}^N \times (0, \infty), \quad \text{ and } \ u(x, 0) = u_0(x) \ \text{ in } \mathbb{R}^N. \tag{4.2}$$

Take any $g \in G$. Since $u_0 \in \Phi(G)$, by the uniqueness of the solution of (4.2) we have that $u(x, t) \equiv u(gx, t)$ in $\mathbb{R}^N \times (0, \infty)$. This implies that

$$\nabla u(0, t) = g \nabla u(0, t) \ \text{ for any } g \in G \text{ and } t > 0. \tag{4.3}$$

Since G is essential, we get

$$\nabla u(0, t) = 0 \ \text{ for any } t > 0.$$

Hence, by [MS 1, Theorem 1, p. 239]

$$\int_{S^{N-1}} \omega u_0(r\omega) d\omega = 0 \ \text{ for any } r \in [0, \delta). \tag{4.4}$$

Namely, $u_0 \in \Phi$ and then $\Phi(G) \subset \Phi$.

Suppose that G is not essential. Let us show that $\Phi(G) \not\subset \Phi$. By the definition there exists a unit vector $z \in \mathbb{R}^N$ satisfying

$$gz = z \ \text{ for any } g \in G. \tag{4.5}$$

Take an orthogonal transformation $\tau \in O(N)$ satisfying $\tau e_1 = z$ where $e_1 = (1, 0, ..., 0) \in \mathbb{R}^N$. Then, denoting by τ^* the transposed matrix of τ yields

$$\tau^* g \tau e_1 = e_1 \ \text{ for any } g \in G. \tag{4.6}$$

Hence for any $g \in G$ we have

$$\tau^* g \tau = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_{N2} & \dots & h_{NN} \end{pmatrix}$$

for some $h = [h_{ij}] \in O(N-1)$. Let us introduce the change of variables: $x = \tau y$. Consider the function

$$u_0(x) = \eta(y_1)\psi(y_2, ..., y_N) \quad (:= \eta(y_1)\psi(\hat{y})),$$

where $\eta \in C_0^\infty(\mathbb{R})$, $\psi \in C_0^\infty(\mathbb{R}^{N-1})$, and ψ is a $(N-1)$-dimensional *radially symmetric* and nonnegative function. Then, by choosing the supports of η and ψ sufficiently small, we get $u_0 \in \Phi(G)$. Furthermore, we may assume that for some $0 < \varepsilon < \rho < \frac{1}{2}\delta$

$$\begin{cases} \text{supp } \eta = [-\rho, \rho], \quad \text{supp } \psi = \hat{B} = \{\hat{y} = (y_2, ..., y_N) : |\hat{y}|^2 \equiv y_2^2 + \cdots + y_N^2 < \varepsilon^2\}, \\ \eta(-y_1) = -\eta(y_1) \text{ and } \eta(y_1) > 0 \text{ for any } y_1 \in (0, \rho). \end{cases}$$

On the other hand, by the change of variables: $x = \tau y$ we have

$$\int_{S^{N-1}} \omega u_0(r\omega) d\omega = \tau \left(\int_{S^{N-1}} \omega' \eta(r\omega_1') \psi(r\omega_2', ..., r\omega_N') d\omega' \right).$$

By choosing $r \in (\varepsilon, \rho)$, we compute

$$\begin{aligned} & r^{N-1} \int_{S^{N-1}} \omega_1' \eta(r\omega_1') \psi(r\omega_2', ..., r\omega_N') d\omega' \\ &= \int_{\partial B_r(0)} \frac{y_1}{r} \eta(y_1) \psi(y_2, ..., y_N) d\sigma_y \\ &= 2 \int_{\hat{B}} \frac{\sqrt{r^2 - |\hat{y}|^2}}{r} \eta(\sqrt{r^2 - |\hat{y}|^2}) \psi(\hat{y}) \sqrt{1 + \frac{|\hat{y}|^2}{r^2 - |\hat{y}|^2}} d\hat{y} \\ &= 2 \int_{\hat{B}} \eta(\sqrt{r^2 - |\hat{y}|^2}) \psi(\hat{y}) d\hat{y} > 0. \end{aligned}$$

Here we used the fact that $\eta(-y_1) \equiv -\eta(y_1)$. This computation shows that this function u_0 does not belong to Φ. Therefore we get that $\Phi(G) \not\subset \Phi$. The proof is completed. $\square$

Remark 4.2. In the case $N = 2$, let G_d be the cyclic group generated by the rotation of the angle $2\pi/d$ with an integer $d \geqq 2$. Then G_d is essential. We notice that d is prime if and only if G_d is the only subgroup of G_d which is essential.

Let us pose three conjectures. The first one is related to Theorem 2.3 and Remark 2.4.

Conjecture 4.3. *Let $d \geqq 2$ be an arbitrary prime number and let $A = A_d$ be the matrix of rotation of angle $2\pi/d$. Consider the initial-Dirichlet problem* (2.1)*. Then, the solution u satisfies $\nabla u(0, t) = 0$ for any $t \in (0, \infty)$ and for any initial data $u_0 \in \Phi_d$, if and only if Ω is invariant under the action of A_d.*

More general conjectures related to Theorems 2.5 and 2.6 are the following:

Conjecture 4.4. *Let $d \geqq 2$ be an arbitrary integer and let $A = A_d$ be the matrix of rotation of angle $2\pi/d$. Consider the initial-Dirichlet problem* (2.1)*. Then, the solution u satisfies $\nabla u(0, t) = 0$ for any $t \in (0, \infty)$ and for any initial data $u_0 \in \Phi_d$, if and only if there exists a prime number p dividing d such that Ω is invariant under the action of A_p.*

Conjecture 4.5. *Let G be an essential subgroup of $O(N)$. Consider the initial-boundary value problem (1.1) for the heat equation. Then, the solution u satisfies $\nabla u(0,t) = 0$ for any $t \in (0,\infty)$ and for any initial data $u_0 \in \Phi(G)$, if and only if there exists an essential subgroup H of G such that Ω is invariant under the action of H.*

Acknowledgments. The authors would like to thank Professor Makoto Sakai for informing them that Theorem 2.5 holds.

References

[A] G. Alessandrini, *Critical points of solutions of elliptic equations in two variables*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 14 (1987), pp. 229 – 256.

[AM] G. Alessandrini and R. Magnanini, *The index of isolated critical points and solutions of elliptic equations in the plane*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19 (1992), pp. 567 – 589.

[CS] M. Chamberland and D. Siegel, *Convex domains with stationary hot spots*, Math. Meth. Appl. Sci. 20 (1997), pp. 1163 – 1169.

[MS 1] R. Magnanini and S. Sakaguchi, *The spatial critical points not moving along the heat flow*, J. Analyse Math. 71 (1997), pp. 237 – 261.

[MS 2] R. Magnanini and S. Sakaguchi, *Spatial critical points not moving along the heat flow II : The centrosymmetric case*, Math. Z. 230 (1999), pp. 695 – 712, Corrigendum, 232 (1999), p. 389.

[N] Z. Nehari, Conformal Mappings, McGraw-Hill, 1952, Dover Publications, Inc., New York 1975 (republication).

[W] J. L. Walsh, The Location of Critical Points of Analytic and Harmonic Functions, American Mathematical Society, New York, 1950.

Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy
e-mail: magnanin@math.unifi.it

and

Department of Mathematical Sciences, Faculty of Science, Ehime University, 2-5 Bunkyo-cho, Matsuyama-shi, Ehime 790-8577 Japan.
e-mail: sakaguch@dpc.ehime-u.ac.jp