On the power law asymptotics for a model of dielectric breakdown

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ON THE POWER LAW ASYMPTOTICS

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Abstract: In this paper we analize a model of dielectric breakdown by adapting a technique which was recently introduced in [6].

1. Introduction. In [6] a general homogenization result was proved for variational functionals with constraints on the gradient. Here we restrict our attention to a model of dielectric breakdown introduced in [8]. The results of [6] do not apply here since boundary conditions are not of the same type: homogeneous Dirichlet condition in [6] versus periodic boundary condition in the present note. However the technique used in [6] is adapted and in particular the use of a monotone approximation for the proof of the $\Gamma - \liminf$ estimate. Here we choose to use L^p (or power law) approximation instead of the Moreau-Yosida approximation. This is appropriate in view of the rescaling properties of L^p approximation. The idea of L^p-approximations is also contained in [8] althought applied in a different way and with the aim of obtaining some optimal bound.

Let us describe the problem. The behaviour of a body subject to a given electric field ∇u can be modelled in various ways. A common way to do it is (see [8] for more references) to associate a suitable convex set $K(x)$ at each point of the body and to distinguish the behaviour of the body with respect to a given electric field ∇u considering if the condition $\nabla u(x) \in K(x)$ a.e. is satisfied or not. If $\nabla u(x) \in K(x)$ a.e. is satisfied then the body behaves as an insulator, if $\nabla u(x) \in K(x)$ a.e. is not satisfied then the dielectric breakdown occurs and the body starts to conduct. Therefore a pointwise constraint on the gradient play the main role. In [8] the authors propose to replace the pointwise constraint with a less degenerate supremal functional associated to the convex set K_{eff} of all average electric fields ξ for which there exists u such that $\xi = \frac{1}{|\Omega|} \int \nabla u dx$ and $\nabla u(x) \in K(x)$ almost everywhere. Then they obtain optimal bounds for K_{eff}. Here we would like to find K_{eff} using the Γ-convergence and homogenization theory. To avoid confusion (following the example of [8]) we will call K^{hom} the convex set that play the role of K_{eff} in the model. The body will be represented by the unitary n-cube

$I^n := [0,1]^n$ and K^{hom} will be defined by

$$\xi \in K^{hom} \Leftrightarrow u \in W^{1,\infty}_{\sharp}(I^n) \;\; s.t. \;\; \nabla u(x) + \xi \in K(x) \; a.e..$$

Throughout the paper we will always follow the convention that a function in $W^{1,\infty}$ is identified with its continuous representant. We denote by χ_A the characteristic function of the set A (i.e. $\chi_A(x) = 0$ if $x \in A$, $\chi_A(x) = \infty$ otherwise). We will prove the following homogenization result:

Theorem 1.1 *The family of functionals* $(\chi_{\{\nabla u(x) \in K(\frac{x}{\varepsilon}) \; a.e.\}})_\varepsilon$ Γ*-converges to* $\chi_{\{\nabla u(x) \in K^{hom} \; a.e.\}}$ *on the space* $W^{1,\infty}_{\#}(I^n)$ *equipped with the uniform convergence of continuous functions as* $\varepsilon \to 0$.

Remark. As the functionals we consider have sublevel sets which are (up to addition of constant functions) compact in $C^0(I^n)$ endowed with the sup norm topology (this is a consequence of Ascoli-Arzela theorem), the Γ-limit result also holds with the L^1 norm.

To be short and simplify the notations we will denote $\chi_{\{\nabla u(x) \in K(\frac{x}{\varepsilon}) \; a.e.\}}$ by χ^ε and
$\chi_{\{\nabla u(x) \in K^{hom} \; a.e.\}}$ by χ^{hom}. The result we prove is already known (see [5] for example) but the proof we give here is simpler than the others in litterature. We split the proof in two parts: first we prove the $\Gamma - \liminf$ inequality by approximating χ^ε by standard energies F_p^ε with p-growth, then we show the $\Gamma - \limsup$ inequality using a constructive method introduced in [6].

2. $\Gamma - liminf$ inequality.

In this section we prove the following theorem:

Theorem 2.1. *Let* $(u_\varepsilon)_\varepsilon$ *be a family in* $W^{1,\infty}_{\sharp}(I^n)$ *which converges uniformly to* u *then* $\liminf_{\varepsilon \to 0} \chi^\varepsilon(u_\varepsilon) \geq \chi^{hom}(u)$.

The above theorem states that $\Gamma - \liminf_\varepsilon \chi^\varepsilon \geq \chi^{hom}$. Our proof of theorem 2.1 is based on the L^p approximation: for any positive ε we introduce the approximating energies F_p^ε defined on $C^0(I^n)$ by:

$$F_p^\varepsilon(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega (\phi(\frac{x}{\varepsilon}, Du(x))^p - 1)dx & if \; u \in W^{1,p}_{\sharp}(I^n), \\ +\infty & otherwise, \end{cases}$$

where $n < p$ and $\phi(x,\cdot)$ denotes the gauge function of $K(x)$ (i.e. $\phi(x,\cdot)$ is the only non-negative homogeneous convex function such that $\phi(x,\xi) \leq 1 \Leftrightarrow \xi \in K(x)$).

The L^p approximation scheme is monotone thanks to the following lemma:

Lemma 2.2. *For all numbers* $0 \leq a$ *the function* $p \mapsto \frac{1}{p}a^p - \frac{1}{p}$ *is non-decreasing in* p *and* $\lim_{p\to\infty} \frac{1}{p}a^p - \frac{1}{p}$ *is equal to* 0 *if* $a \leq 1$ *and to* $+\infty$ *if* $1 < a$.

Proof: The lemma is straightforward in the case $a = 0$. When $a > 0$, then the convexity of the exponential function yields

$$\forall p > q > 1 \qquad \exp\left(\frac{p}{q}\ln(a^q) + (1-\frac{p}{q})\ln(1)\right) \le \frac{p}{q}a^q + \left(1 - \frac{p}{q}\right)$$

from which the lemma follows easily. $\square$

As a direct consequence we get the following

Proposition 2.3. *For any positive ε the family $(F_p^\varepsilon)_p$ $\Gamma-$converges in $C^0(I^n)$ to χ^ε non-decreasingly when $p \to \infty$.*

The previous proposition is in some sense the non homogeneous, anisotropic version of the well known theorem on the limit of the $p-laplacian$ for $p \to \infty$.

For fixed p, the functionals F_p^ε satisfy the p-growth conditions

$$\frac{1}{p}\int_{I^n}\left(\left(\frac{|Du|(x)}{R}\right)^p - 1\right)dx \le F_p^\varepsilon(u) \le \frac{1}{p}\int_{I^n}\left(\left(\frac{|Du|(x)}{r}\right)^p - 1\right)dx.$$

Then the following classical homogenization result holds (see chapter 14 of [4])

$$(\Gamma - \lim_{\varepsilon\to 0} F_p^\varepsilon)(u) = \int_{I^n}\phi_p(Du(x))dx,$$

where the integrand ϕ_p is given by the formula:

$$\phi_p(\xi) = \inf_{v \in W^{1,p}_\sharp} \frac{1}{p}\int_{I^n}(\phi(x, \xi + Dv)^p - 1)dx.$$

Proof (of theorem 2.1). Let $u_\varepsilon \to u$ as in the statement. Using proposition 2.2 and what precedes we get

$$\forall p \qquad \liminf_{\varepsilon\to 0}\chi^\varepsilon(u_\varepsilon) \ge \liminf_{\varepsilon\to 0} F_p^\varepsilon(u_\varepsilon) \ge F_p(u).$$

Then taking the limit in p and applying lemma 2.2 yield

$$\liminf_{\varepsilon\to 0}\chi^\varepsilon(u_\varepsilon) \ge \lim_{p\to\infty} F_p(u) = \chi^{hom}(u).$$

$\square$

3. $\Gamma - limsup$ inequality.

In this section we prove the following result:

Theorem 3.1. *Let $u \in W^{1,\infty}_\#(I^n)$ be such that $\chi^{hom}(u) = 0$, then there exists a family $(u_\varepsilon)_\varepsilon \subset W^{1,\infty}_\#(I^n)$ converging to u uniformly and such that $\chi^\varepsilon(u_\varepsilon) = 0$.*

It follows from the above result and theorem 2.1 that $\Gamma - \limsup_\varepsilon \chi^\varepsilon \le \chi^{hom}$.

Proof. Step 1: Let $u \in W^{1,\infty}_\#(I^n)$ be piecewise affine and such that $\chi^{hom}(u) = 0$. We claim that there exists a family $(u_\varepsilon)_\varepsilon$ in $W^{1,\infty}_\#(I^n)$ converging uniformly to u and such that $\chi^\varepsilon(u_\varepsilon) = 0$ for any $\varepsilon > 0$.

Since u is piecewise affine, there exists a family $(\omega_i)_{1\le i\le k}$ of disjoint open sets such that $\overline{I^n} = \cup_{1\le i\le n}\overline{\omega_i}$ and $u(x) = \alpha_i + \xi_i \cdot x$ for any x in $\overline{\omega_i}$. For any z in $\{-1,0,1\}^n$, we define $u^{i,z}$ on $\mathbb{R}^n$ by $u^{i,z} : x \mapsto \alpha_i + \xi_i \cdot (x-z)$. By periodicity of u, one has $u(x) = u^{i,z}(x)$ for any x in $z + I^n$. Then u can be written as a finite combination of min and max of the functions $u^{i,z}$ on the set $[-1,2]^n$: we shall write formally this combination $u = c(u^{i,z} : z \in \{-1,0,1\}^n, 1 \leq i \leq k)$.

For any i in $\{1,\ldots,k\}$, ξ_i belongs to K^{hom} so that there exists a function w_i in $W^{1,\infty}_{\#}(I^n)$ such that $\xi_i + \nabla w_i(x)$ belongs to $K(x)$ a.e. in I^n. For any $\varepsilon > 0$ and $z \in \{-1,0,1\}^n$, we define the function $u^{i,z}_\varepsilon$ on $\mathbb{R}^n$ by $u^{i,z}_\varepsilon : x \mapsto u^{i,z}(x) + \varepsilon w_i(\frac{x}{\varepsilon})$. We then set $\tilde{u}_\varepsilon := c(u^{i,z}_\varepsilon : z \in \{-1,0,1\}^n, 1 \leq i \leq k)$ for any $\varepsilon > 0$, i.e. $\tilde{u}_\varepsilon$ is defined through the same formal combination of min and max as u. We remark that for two points x, y in I^n for which there exists $z \in \{0,1\}^n$ such that $x = y + z$, the active functions in the combination $c(u^{i,z} : z \in \{-1,0,1\}^n, 1 \leq i \leq k)$ that define $u(x)$ and $u(y)$ are the same up to the translation of vector z. The same holds true for $\tilde{u}_\varepsilon$, so that the restriction u_ε of $\tilde{u}_\varepsilon$ to I^n is a function of $W^{1,\infty}_{\#}(I^n)$. We now notice that of course one has $\chi^\varepsilon(u_\varepsilon) = 0$ for any $\varepsilon > 0$, and it is clear that the family $(u_\varepsilon)_\varepsilon$ converges uniformly to u on I^n.

Step 2: Let $u \in C^\infty_{\#}(I^n)$ be such that $\chi^{hom}(u) = 0$. We claim that there exists a sequence $(u_n)_n$ of functions in $W^{1,\infty}_{\#}(I^n)$ which are piecewise affine, satisfy $\chi^{hom}(u_n) = 0$ for any $n \in \mathbb{N}$ and converge uniformly to u on I^n.

Since u is smooth, K^{hom} is closed and $\chi^{hom}(u) = 0$ then $\nabla u(x) \in K^{hom}$ for any x in I^n. Moreover thanks to the smoothness of u there exists a sequence of functions $(u_n)_n$ which are periodic , piecewise affine and such that $u_n \to u$ strong in $W^{1,\infty}$. As K is a convex set of nonempty interior it is possible to choose u_n so that it satisfies $\chi^{hom}(u_n) = 0$.

Step 3: Let now u in $W^{1,\infty}_{\#}(I^n)$ be such that $\chi^{hom}(u) = 0$. We claim that there exists a family $(u_\varepsilon)_\varepsilon$ in $W^{1,\infty}_{\#}(I^n)$ converging uniformly to u and such that $\chi^\varepsilon(u_\varepsilon) = 0$ for any $\varepsilon > 0$. To prove this we fix a positive real number η and show that there exists a function u_η in $W^{1,\infty}_{\#}(I^n)$ which is piecewise affine and satisfies $\chi^{hom}(u_\eta) = 0$ as well as $\|u - u_\eta\|_\infty \leq \eta$. Since this is true for any $\eta > 0$, step 1 and a standard diagonal argument prove the claim.

Now fix $\eta > 0$. Let $\gamma \in C^\infty_c(\mathbb{R}^n, \mathbb{R}_+)$ be a standard mollifier (i.e. with $\int_{\mathbb{R}^n} \gamma(x)dx = 1$) and $(\gamma_\delta)_{\delta>0}$ the associated approximation of unity (i.e. $\gamma_\delta(\cdot) := \delta^n \gamma(\cdot/\delta)$). For any $\delta > 0$, we set $v_\delta = \gamma_\delta * u$ (where u is defined on $\mathbb{R}^n$ by periodicity), then the function v_δ belongs to $W^{1,\infty}_{\#}(I^n)$ and we conclude from the identity

$$\forall x \in \mathbb{R}^N \qquad Dv_\delta(x) = \int_{\mathbb{R}^N} \gamma_\delta(x-y) Du(y) dy$$

that $Dv_\delta(x)$ belongs to K^{hom} for any x in I^n. As a consequence, for any positive δ the function v_δ belongs to $C^\infty(I^n)$ and is such that $\chi^{hom}(v_\delta) = 0$. Since the family $(v_\delta)_\delta$ converges uniformly to u on I^n, the existence of the desired function u_η follows easily from step 2 applied to v_δ for $\delta > 0$ small enough. □

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