A generalized Korn inequality and strong unique continuation for the Reissner–Mindlin plate system

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Abstract

We prove constructive estimates for elastic plates modeled by the Reissner–Mindlin theory and made by general anisotropic material. Namely, we obtain a generalized Korn inequality which allows to derive quantitative stability and global H^2 regularity for the Neumann problem. Moreover, in case of isotropic material, we derive an interior three spheres inequality with optimal exponent from which the strong unique continuation property follows.
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1. Introduction

In the present paper we consider elastic plates modeled by the Reissner–Mindlin theory. This theory was developed for moderately thick plates, that is for plates whose thickness is of the order of one tenth of the planar dimensions of the middle surface [1], [2]. Our aim is to give a rigorous, thorough and self-contained presentation of mathematical results concerning the Neumann problem, a boundary value problem which poses interesting features which, at our knowledge, have not yet been pointed out in the literature.

Throughout the paper we consider an elastic plate $\Omega \times \left[-\frac{h}{2}, \frac{h}{2} \right]$, where $\Omega \subset \mathbb{R}^2$ is the middle surface and h is the constant thickness of the plate. A transversal force field $\overline{Q}$ and a couple field $\overline{M}$ are applied at the boundary of the plate. According to the Reissner–Mindlin model, at any point $x = (x_1, x_2)$ of Ω we denote by $w = w(x)$ and $\omega_\alpha(x)$, $\alpha = 1, 2$, the infinitesimal transversal displacement at x and the infinitesimal rigid rotation of the transversal material fiber through x, respectively. Therefore, the pair (φ, w), with $(\varphi_1 = \omega_2, \varphi_2 = -\omega_1)$, satisfies the following Neumann boundary value problem

$$\begin{cases} \operatorname{div}\left(S(\varphi + \nabla w) \right) = 0 & \text{in } \Omega, \quad \text{(a)} \\ \operatorname{div}\left(\mathbb{P} \nabla \varphi \right) - S(\varphi + \nabla w) = 0, & \text{in } \Omega, \quad \text{(b)} \\ \left(S(\varphi + \nabla w) \right) \cdot n = \overline{Q}, & \text{on } \partial\Omega, \quad \text{(c)} \\ \left(\mathbb{P} \nabla \varphi \right) n = \overline{M}, & \text{on } \partial\Omega, \quad \text{(d)} \end{cases} \tag{1.1}$$

where $\mathbb{P}$ and S are the fourth-order bending tensor and the shearing matrix of the plate, respectively. The vector n denotes the outer unit normal to Ω.

The weak formulation of (1.1)(a)–(1.1)(d) consists in determining $(\varphi, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ satisfying

$$a((\varphi, w), (\psi, v)) = \int_{\partial\Omega} \overline{Q} v + \overline{M} \cdot \psi, \quad \forall \psi \in H^1(\Omega, \mathbb{R}^2), \forall v \in H^1(\Omega), \tag{1.2}$$

where

$$a((\varphi, w), (\psi, v)) = \int_\Omega \mathbb{P} \nabla \varphi \cdot \nabla \psi + \int_\Omega S(\varphi + \nabla w) \cdot (\psi + \nabla v). \tag{1.3}$$

The coercivity of the bilinear form $a(\cdot, \cdot)$ in the subspace

$$\mathcal{H} = \left\{ (\psi, v) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega) \mid \int_\Omega \psi = 0, \int_\Omega v = 0 \right\}$$

with respect to the norm induced by $H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ is not standard. To prove this property – in other terms, the equivalence of the standard norm in $\mathcal{H}$ with the norm induced by the energy functional – we derive the following generalized Korn-type inequality

$$\| \nabla \varphi \|_{L^2(\Omega)} \leq C \left(\| \widehat{\nabla} \varphi \|_{L^2(\Omega)} + \| \varphi + \nabla w \|_{L^2(\Omega)} \right), \forall \varphi \in H^1(\Omega, \mathbb{R}^2), \forall w \in H^1(\Omega, \mathbb{R}), \tag{1.4}$$

where $\widehat{\nabla}$ denotes the symmetric part of the gradient and the constant C is constructively determined in terms of the parameters describing the geometrical properties of the Lipschitz domain Ω. Inequality (1.4) allows to solve the Neumann problem and provides a quantitative stability estimate in the H^1 norm.

Assuming Lipschitz continuous coefficients and $C^{1,1}$ regularity of the boundary, we prove global H^2 regularity estimates. For the proof, which is mainly based on the regularity theory developed by Agmon [3] and Campanato [4], a key role is played by quantitative Poincaré inequalities for functions vanishing on a portion of the boundary, derived in [5].

Finally, in case of isotropic material, we adapt arguments in [6] to H^2 solutions of the plate system (1.1)(a)–(1.1)(b), obtaining a three spheres inequality with optimal exponent and, as a standard consequence, we derive the strong unique continuation property.

Let us notice that the constructive character of all the estimates derived in the present paper is crucial for possible applications to inverse problems associated to the Neumann problem (1.1)(a)–(1.1)(d). As a future direction of research, we plan to use such results to treat inverse problems concerning the determination of defects, such as elastic inclusions, in isotropic elastic plates modeled by the Reissner–Mindlin model. The interested reader can refer, for instance, to [7] for applications to the determination of inclusions in a thin plate described by the Kirchhoff-Love model, which involves a single scalar fourth order elliptic equation.

The paper is organized as follows. In section 2 we collect the notation and in section 3 we present a self-contained derivation of the mechanical model for general anisotropic material. Section 4 contains the proof of the generalized Korn-type inequality (1.4), which is the key ingredient used in section 5 to study the Neumann problem. In section 6 we derive H^2 global regularity estimates. In section 7 we state and prove the three spheres inequality. Finally, section 8 is an Appendix where we have postponed some technical estimates about regularity up to the boundary.

2. Notation

Let $P = (x_1(P), x_2(P))$ be a point of $\mathbb{R}^2$. We shall denote by $B_r(P)$ the disk in $\mathbb{R}^2$ of radius r and center P and by $R_{a,b}(P)$ the rectangle $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a, \ |x_2 - x_2(P)| < b\}$. To simplify the notation, we shall denote $B_r = B_r(O)$, $R_{a,b} = R_{a,b}(O)$.

Definition 2.1. ($C^{k,1}$ regularity) Let Ω be a bounded domain in $\mathbb{R}^2$. Given $k \in \mathbb{N}$, we say that a portion S of $\partial\Omega$ is of *class $C^{k,1}$ with constants* ρ_0, $M_0 > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap R_{\frac{\rho_0}{M_0},\rho_0} = \{x = (x_1, x_2) \in R_{\frac{\rho_0}{M_0},\rho_0} \mid x_2 > \psi(x_1)\},$$

where ψ is a $C^{k,1}$ function on $\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)$ satisfying

$$\psi(0) = 0, \quad \psi'(0) = 0 \ \text{when } k \geq 1,$$

$$\|\psi\|_{C^{k,1}\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)} \leq M_0\rho_0.$$

When $k = 0$ we also say that S is of *Lipschitz class with constants* ρ_0, M_0.

Remark 2.2. We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous with the L^∞ norm and coincide with the standard definition when the dimensional parameter equals one. For instance, the norm appearing above in case $k = 1$ is meant as follows

$$\|\psi\|_{C^{1,1}\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)} = \|\psi\|_{L^\infty\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)} + \rho_0 \|\nabla\psi\|_{L^\infty\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)} + \rho_0^2 \|\nabla^2\psi\|_{L^\infty\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)}.$$

Similarly, we shall set

$$\|u\|_{L^2(\Omega)} = \rho_0^{-1} \left(\int_\Omega u^2 \right)^{\frac{1}{2}},$$

$$\|u\|_{H^1(\Omega)} = \rho_0^{-1} \left(\int_\Omega u^2 + \rho_0^2 \int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}},$$

and so on for boundary and trace norms such as $\|\cdot\|_{L^2(\partial\Omega)}$, $\|\cdot\|_{H^{\frac{1}{2}}(\partial\Omega)}$, $\|\cdot\|_{H^{-\frac{1}{2}}(\partial\Omega)}$.

Given a bounded domain Ω in $\mathbb{R}^2$ such that $\partial\Omega$ is of class $C^{k,1}$, with $k \geq 1$, we consider as positive the orientation of the boundary induced by the outer unit normal n in the following sense. Given a point $P \in \partial\Omega$, let us denote by $\tau = \tau(P)$ the unit tangent at the boundary in P obtained by applying to n a counterclockwise rotation of angle $\frac{\pi}{2}$, that is $\tau = e_3 \times n$, where $\times$ denotes the vector product in $\mathbb{R}^3$, $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$ and $e_3 = e_1 \times e_2$.

We denote by $\mathbb{M}^2$ the space of 2×2 real valued matrices and by $\mathcal{L}(X, Y)$ the space of bounded linear operators between Banach spaces X and Y.

For every 2×2 matrices A, B and for every $\mathbb{L} \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$, we use the following notation:

$$(\mathbb{L}A)_{ij} = L_{ijkl} A_{kl}, \tag{2.1}$$

$$A \cdot B = A_{ij} B_{ij}, \quad |A| = (A \cdot A)^{\frac{1}{2}}. \tag{2.2}$$

Notice that here and in the sequel summation over repeated indexes is implied.

3. The Reissner–Mindlin plate model

The Reissner–Mindlin plate is a classical model for plates having moderate thickness [1], [2]. The Reissner–Mindlin plate theory can be rigorously deduced from the three-dimensional linear elasticity using arguments of Γ-convergence of the energy functional, as it was shown in [8]. Our aim in this section is more modest, namely, we simply derive the boundary value problem governing the statical equilibrium of an elastic Reissner–Mindlin plate under Neumann boundary conditions following the engineering approach of the Theory of Structures. This allows us to introduce some notation useful in the sequel and to make the presentation of the physical problem complete.

Let us consider a plate $\Omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ with middle surface represented by a bounded domain Ω in $\mathbb{R}^2$ having uniform thickness h and boundary $\partial \Omega$ of class $C^{1,1}$. In this section we adopt the convention that Greek indexes assume the values 1, 2, whereas Latin indexes run from 1 to 3.

We follow the direct approach to define the infinitesimal deformation of the plate. In particular, we restrict ourselves to the case in which the points $x = (x_1, x_2)$ of the middle surface Ω are subject to transversal displacement $w(x_1, x_2)e_3$, and any transversal material fiber $\{x\} \times \left[-\frac{h}{2}, \frac{h}{2}\right]$, $x \in \Omega$, undergoes an infinitesimal rigid rotation $\omega(x)$, with $\omega(x) \cdot e_3 = 0$. In this section we shall be concerned exclusively with regular functions on their domain of definition. The above kinematical assumptions imply that the displacement field present in the plate is given by the following three-dimensional vector field:

$$u(x, x_3) = w(x)e_3 + x_3\varphi(x), \quad x \in \overline{\Omega}, \; |x_3| \le \frac{h}{2}, \tag{3.1}$$

where

$$\varphi(x) = \omega(x) \times e_3, \quad x \in \overline{\Omega}. \tag{3.2}$$

By (3.1) and (3.2), the associated infinitesimal strain tensor $E[u] \in \mathbb{M}^3$ takes the form

$$E[u](x, x_3) \equiv (\nabla u)^{sym}(x, x_3) = x_3(\nabla_x \varphi(x))^{sym} + (\gamma(x) \otimes e_3)^{sym}, \tag{3.3}$$

where $\nabla_x(\cdot) = \frac{\partial}{\partial x_\alpha}(\cdot)e_\alpha$ is the surface gradient operator, $\nabla^{sym}(\cdot) = \frac{1}{2}(\nabla(\cdot) + \nabla^T(\cdot))$, and

$$\gamma(x) = \varphi(x) + \nabla_x w(x). \tag{3.4}$$

Within the approximation of the theory of infinitesimal deformations, γ expresses the angular deviation between the transversal material fiber at x and the normal direction to the deformed middle surface of the plate at x.

The classical deduction of the mechanical model of a thin plate follows essentially from integration over the thickness of the corresponding three-dimensional quantities. In particular, taking advantage of the infinitesimal deformation assumption, we can refer the independent variables to the initial undeformed configuration of the plate.

Let us introduce an arbitrary portion $\Omega' \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ of plate, where $\Omega' \subset\subset \Omega$ is a subdomain of Ω with regular boundary. Consider the material fiber $\{x\} \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ for $x \in \partial\Omega'$ and denote by $t(x, x_3, e_\alpha) \in \mathbb{R}^3$, $|x_3| \le \frac{h}{2}$, the *traction vector* acting on a plane containing the direction of the fiber and orthogonal to the direction e_α. By Cauchy's Lemma we have $t(x, x_3, e_\alpha) = T(x, x_3)e_\alpha$, where $T(x, x_3) \in \mathbb{M}^3$ is the (symmetric) Cauchy stress tensor at the point (x, x_3). Denote by n the unit outer normal vector to $\partial\Omega'$ such that $n \cdot e_3 = 0$. To simplify the notation, it is convenient to consider n as a two-dimensional vector belonging to the plane $x_3 = 0$ containing the middle surface Ω of the plate. By the classical Stress Principle for plates, we postulate that the two complementary parts Ω' and $\Omega \setminus \Omega'$ interact with one another through a field of force vectors $R = R(x, n) \in \mathbb{R}^3$ and couple vectors $M = M(x, n) \in \mathbb{R}^3$ assigned per unit length at $x \in \partial\Omega'$. Denoting by

$$R(x, e_\alpha) = \int\limits_{-h/2}^{h/2} t(x, x_3, e_\alpha)dx_3 \tag{3.5}$$

the force vector (per unit length) acting on a direction orthogonal to e_α and passing through $x \in \partial\Omega'$, the contact force $R(x, n)$ can be expressed as

$$R(x, n) = T^\Omega(x)n, \quad x \in \partial\Omega', \tag{3.6}$$

where the *surface force tensor* $T^\Omega(x) \in \mathbb{M}^{3\times2}$ is given by

$$T^\Omega(x) = R(x, e_\alpha) \otimes e_\alpha, \quad \text{in } \Omega. \tag{3.7}$$

Let $P = I - e_3 \otimes e_3$ be the projection of $\mathbb{R}^3$ along the direction e_3. T^Ω is decomposed additively by P in its *membranal* and *shearing* component

$$T^\Omega = PT^\Omega + (I - P)T^\Omega \equiv T^{\Omega(m)} + T^{\Omega(s)}, \tag{3.8}$$

where, following the standard nomenclature in plate theory, the components $T_{\alpha\beta}^{\Omega(m)}$ $(= T_{\beta\alpha}^{\Omega(m)})$, $\alpha, \beta = 1, 2$, are called the *membrane forces* and the components $T_{3\beta}^{\Omega(s)}$, $\beta = 1, 2$, are the *shear forces* (also denoted as $T_{3\beta}^{\Omega(s)} = Q_\beta$). The assumption of infinitesimal deformations and the hypothesis of vanishing in-plane displacements of the middle surface of the plate allow us to take

$$T^{\Omega(m)} = 0, \quad \text{in } \Omega. \tag{3.9}$$

Denote by

$$M(x, e_\alpha) = \int_{-h/2}^{h/2} x_3 e_3 \times t(x, x_3, e_\alpha)dx_3, \quad \alpha = 1, 2, \tag{3.10}$$

the contact couple acting at $x \in \partial\Omega'$ on a direction orthogonal to e_α passing through x. Note that $M(x, e_\alpha) \cdot e_3 = 0$ by definition, that is $M(x, e_\alpha)$ actually is a two-dimensional couple field belonging to the middle plane of the plate. Analogously to (3.6), we have

$$M(x, n) = M^\Omega(x)n, \quad x \in \partial\Omega', \tag{3.11}$$

where the *surface couple tensor* $M^\Omega(x) \in \mathbb{M}^{3\times2}$ has the expression

$$M^\Omega(x) = M(x, e_\alpha) \otimes e_\alpha. \tag{3.12}$$

A direct calculation shows that

$$M(x, e_\alpha) = e_3 \times e_\beta M_{\beta\alpha}(x), \tag{3.13}$$

where

$$M_{\beta\alpha}(x) = \int_{-h/2}^{h/2} x_3 T_{\beta\alpha}(x, x_3)dx_3, \quad \alpha, \beta = 1, 2, \tag{3.14}$$

are the *bending moments* (for $\alpha = \beta$) and the *twisting moments* (for $\alpha \neq \beta$) of the plate at x (per unit length).

Denote by $q(x)e_3$ the external transversal force per unit area acting in Ω. The statical equilibrium of the plate is satisfied if and only if the following two equations are simultaneously satisfied:

$$\begin{cases} \displaystyle\int_{\partial\Omega'} T^{\Omega}n\,ds + \int_{\Omega'} qe_3 dx = 0, & \text{(a)} \\[4mm] \displaystyle\int_{\partial\Omega'} \left((x-x_0) \times T^{\Omega}n + M^{\Omega}n\right) ds + \int_{\Omega'}(x-x_0) \times qe_3 dx = 0, & \text{(b)} \end{cases} \tag{3.15}$$

for every subdomain $\Omega' \subseteq \Omega$, where x_0 is a fixed point. By applying the Divergence Theorem in Ω' and by the arbitrariness of Ω' we deduce

$$\begin{cases} \operatorname{div}_x T^{\Omega(s)} + qe_3 = 0, & \text{in } \Omega, \quad \text{(a)} \\[2mm] \operatorname{div}_x M^{\Omega} + (T^{\Omega(s)})^T e_3 \times e_3 = 0, & \text{in } \Omega. \quad \text{(b)} \end{cases} \tag{3.16}$$

Consider the case in which the boundary of the plate $\partial\Omega$ is subjected simultaneously to a couple field $\overline{M}^*$, $\overline{M}^* \cdot e_3 = 0$, and a transversal force field $\overline{Q}e_3$. Local equilibrium considerations on points of $\partial\Omega$ yield the following boundary conditions:

$$\begin{cases} M^{\Omega}n = \overline{M}^*, & \text{on } \partial\Omega, \quad \text{(a)} \\[2mm] T^{\Omega(s)}n = \overline{Q}e_3, & \text{on } \partial\Omega, \quad \text{(b)} \end{cases} \tag{3.17}$$

where n is the unit outer normal to $\partial\Omega$. In cartesian components, the equilibrium equations (3.16)(a)–(3.17)(b) take the form

$$\begin{cases} M_{\alpha\beta,\beta} - Q_\alpha = 0, & \text{in } \Omega, \alpha = 1, 2, \quad \text{(a)} \\[2mm] Q_{\alpha,\alpha} + q = 0, & \text{in } \Omega, \quad \text{(b)} \\[2mm] M_{\alpha\beta}n_\beta = \overline{M}_\alpha, & \text{on } \partial\Omega, \quad \text{(c)} \\[2mm] Q_\alpha n_\alpha = \overline{Q}, & \text{on } \partial\Omega, \quad \text{(d)} \end{cases} \tag{3.18}$$

where we have defined $\overline{M}_1 = \overline{M}_2^*$ and $\overline{M}_2 = -\overline{M}_1^*$.

To complete the formulation of the equilibrium problem, we need to introduce the constitutive equation of the material. We limit ourselves to the Reissner–Mindlin theory and we choose to regard the kinematical assumptions $E_{33}[u] = 0$ as internal constraint, that is we restrict the possible deformations of the points of the plate to those whose infinitesimal strain tensor belongs to the set

$$\mathcal{M} = \{E \in \mathbb{M}^{3\times3} | E = E^T, E \cdot A = 0, \text{ for } A = e_3 \otimes e_3\}. \tag{3.19}$$

Therefore, by the *Generalized Principle of Determinism* [9], the Cauchy stress tensor T at any point (x, x_3) of the plate is additively decomposed in an *active* (symmetric) part T_A and in a *reactive* (symmetric) part T_R:

$$T = T_A + T_R, \tag{3.20}$$

where T_R does not work in any admissible motion, e.g., $T_R \in \mathcal{M}^\perp$. Consistently with the Principle, the active stress T_A belongs to $\mathcal{M}$ and, in cartesian coordinates, we have

$$T_A = T_{A\alpha\beta}e_\alpha \otimes e_\beta + T_{A\alpha 3}e_\alpha \otimes e_3 + T_{A3\alpha}e_3 \otimes e_\alpha, \quad \alpha, \beta = 1, 2, \tag{3.21}$$

$$T_R = T_{R33}e_3 \otimes e_3. \tag{3.22}$$

In linear theory, on assuming the reference configuration unstressed, the active stress in a point (x, x_3) of the plate, $x \in \overline{\Omega}$ and $|x_3| \leq h/2$, is given by a linear mapping from $\mathcal{M}$ into itself by means of the fourth order *elasticity tensor* $\mathbb{C}_\mathcal{M} \in \mathcal{L}(\mathbb{M}^3, \mathbb{M}^3)$:

$$T_A = \mathbb{C}_\mathcal{M} E[u]. \tag{3.23}$$

We assume that $\mathbb{C}_\mathcal{M}$ is constant over the thickness of the plate and satisfies the minor and major symmetry conditions expressed in cartesian coordinates as (we drop the subscript $\mathcal{M}$)

$$C_{ijrs} = C_{jirs} = C_{ijsr} = C_{rsij}, \quad i, j, r, s = 1, 2, 3, \quad \text{in } \Omega. \tag{3.24}$$

Using (3.20) and recalling (3.9), we have:

$$T^\Omega = T_A^{\Omega(s)}, \quad M^\Omega = M_A^\Omega, \tag{3.25}$$

that is, both the shear forces and the moments have active nature. By (3.23), after integration over the thickness, the surface force tensor and the surface couple tensor are given by

$$T^\Omega(x) = h\mathbb{C}(x)(\gamma \otimes e_3)^{sym}, \quad \text{in } \Omega, \tag{3.26}$$

$$M^\Omega(x) = \frac{h^3}{12}\mathcal{E}\mathbb{C}(x)(\nabla_x\varphi(x))^{sym}, \quad \text{in } \Omega, \tag{3.27}$$

where $\mathcal{E} \in \mathbb{M}^3$ is the unique skew-symmetric matrix such that $\mathcal{E}a = e_3 \times a$ for every $a \in \mathbb{R}^3$. The constitutive equations (3.26), (3.27) can be written in more expressive way in terms of the cartesian components of shear forces and bending-twisting moments, namely

$$Q_\alpha = S_{\alpha\beta}(x)(\varphi_\beta + w,_\beta), \quad \alpha = 1, 2, \tag{3.28}$$

$$M_{\alpha\beta} = P_{\alpha\beta\gamma\delta}(x)\varphi_{\gamma,\delta}, \quad \alpha, \beta = 1, 2, \tag{3.29}$$

where the *plate shearing matrix* $S \in \mathbb{M}^2$ and the *plate bending tensor* $\mathbb{P} \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$ are given by

$$S_{\alpha\beta}(x) = hC_{3\alpha 3\beta}(x), \quad \alpha, \beta = 1, 2, \tag{3.30}$$

$$P_{\alpha\beta\gamma\delta}(x) = \frac{h^3}{12}C_{\alpha\beta\gamma\delta}(x), \quad \alpha, \beta, \gamma, \delta = 1, 2. \tag{3.31}$$

From the symmetry assumptions (3.24) on the elastic tensor $\mathbb{C}$ it follows that the shearing matrix S is symmetric and the bending tensor $\mathbb{P}$ satisfies the minor and major symmetry conditions, namely (in cartesian coordinates)

$$S_{\alpha\beta} = S_{\beta\alpha}, \quad \alpha, \beta = 1, 2, \text{ in } \Omega, \tag{3.32}$$

$$P_{\alpha\beta\gamma\delta} = P_{\beta\alpha\gamma\delta} = P_{\alpha\beta\delta\gamma} = P_{\gamma\delta\alpha\beta}, \quad \alpha, \beta, \gamma, \delta = 1, 2, \text{ in } \Omega. \tag{3.33}$$

We recall that the symmetry conditions (3.33) are equivalent to

$$\mathbb{P}A = \mathbb{P}\widehat{A}, \quad \mathbb{P}A \text{ is symmetric}, \quad \mathbb{P}A \cdot B = \mathbb{P}B \cdot A, \tag{3.34}$$

for every 2×2 matrices A, B, where, here and in the sequel, we denote for brevity $\widehat{A} = A^{sym}$.

On S and $\mathbb{P}$ we also make the following assumptions.

I) *Regularity (boundedness)*

$$S \in L^{\infty}(\Omega, \mathcal{L}(\mathbb{M}^2)), \tag{3.35}$$

$$\mathbb{P} \in L^{\infty}(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)). \tag{3.36}$$

II) *Ellipticity (strong convexity)* There exist two positive constants σ_0, σ_1 such that

$$h\sigma_0 |v|^2 \le Sv \cdot v \le h\sigma_1 |v|^2, \quad \text{a.e. in } \Omega, \tag{3.37}$$

for every $v \in \mathbb{R}^2$, and there exist two positive constants ξ_0, ξ_1 such that

$$\frac{h^3}{12}\xi_0 |\widehat{A}|^2 \le \mathbb{P}A \cdot A \le \frac{h^3}{12}\xi_1 |\widehat{A}|^2, \quad \text{a.e. in } \Omega, \tag{3.38}$$

for every 2×2 matrix A.

Finally, under the above notation and in view of (3.28)–(3.29), the problem (3.18)(a)–(3.18)(d) for $q \equiv 0$ in Ω takes the form (1.1)(a)–(1.1)(d), namely (in cartesian components)

$$\begin{cases} (P_{\alpha\beta\gamma\delta}\varphi_{\gamma,\delta})_{,\beta} - S_{\alpha\beta}(\varphi_\beta + w_{,\beta}) = 0, & \text{in } \Omega, \quad \text{(a)} \\ (S_{\alpha\beta}(\varphi_\beta + w_{,\beta}))_{,\alpha} = 0, & \text{in } \Omega, \quad \text{(b)} \\ (P_{\alpha\beta\gamma\delta}\varphi_{\gamma,\delta})n_\beta = \overline{M}_\alpha, & \text{on } \partial\Omega, \quad \text{(c)} \\ S_{\alpha\beta}(\varphi_\beta + w_{,\beta})n_\alpha = \overline{Q}, & \text{on } \partial\Omega. \quad \text{(d)} \end{cases} \tag{3.39}$$

4. A generalized Korn inequality

Throughout this section, Ω will be a bounded domain in $\mathbb{R}^2$, with boundary of Lipschitz class with constants ρ_0, M_0, satisfying

$$\text{diam}(\Omega) \le M_1 \rho_0, \tag{4.1}$$

$$B_{s_0\rho_0}(x_0) \subset \Omega, \tag{4.2}$$

for some $s_0 > 0$ and $x_0 \in \Omega$. For any $E \subset \Omega$, we shall denote by

$$x_E = \frac{1}{|E|} \int_E x, \tag{4.3}$$

$$v_E = \frac{1}{|E|} \int_E v, \tag{4.4}$$

the center of mass of E and the integral mean of a function v with values in $\mathbb{R}^n$, $n \geq 1$, respectively.

In order to prove the generalized Korn inequality of Theorem 4.3, let us recall the constructive Poincaré and classical Korn inequalities.

Proposition 4.1 (*Poincaré inequalities*). *There exists a positive constant C_P only depending on M_0 and M_1, such that for every $u \in H^1(\Omega, \mathbb{R}^n)$, $n = 1, 2$,*

$$\|u - u_\Omega\|_{L^2(\Omega)} \leq C_P \rho_0 \|\nabla u\|_{L^2(\Omega)}, \tag{4.5}$$

$$\|u - u_E\|_{H^1(\Omega)} \leq \left(1 + \left(\frac{|\Omega|}{|E|}\right)^{\frac{1}{2}}\right) \sqrt{1 + C_P^2}\, \rho_0 \|\nabla u\|_{L^2(\Omega)}. \tag{4.6}$$

See for instance [5, Example 3.5] and also [10] for a quantitative evaluation of the constant C_P.

Proposition 4.2 (*Korn inequalities*). *There exists a positive constant C_K only depending on M_0 and M_1, such that for every $u \in H^1(\Omega, \mathbb{R}^2)$,*

$$\left\|\nabla u - \frac{1}{2}(\nabla u - \nabla^T u)_\Omega\right\|_{L^2(\Omega)} \leq C_K \|\widehat{\nabla} u\|_{L^2(\Omega)}, \tag{4.7}$$

$$\left\|u - u_E - \frac{1}{2}(\nabla u - \nabla^T u)_E (x - x_E)\right\|_{H^1(\Omega)} \leq C_{E,\Omega} C_K \sqrt{1 + C_P^2}\, \rho_0 \|\widehat{\nabla} u\|_{L^2(\Omega)}, \tag{4.8}$$

where

$$C_{E,\Omega} = 1 + \left(2\frac{|\Omega|}{|E|}\left(1 + M_1^2\right)\right)^{\frac{1}{2}}. \tag{4.9}$$

See the fundamental paper by Friedrichs [11] on second Korn inequality and also [5, Example 5.3] for a proof of (4.8)–(4.9).

Notice that, when $E = B_{s_0\rho_0}(x_0)$, $C_{E,\Omega} \leq 1 + \sqrt{2}\left(1 + M_1^2\right)^{\frac{1}{2}} \frac{M_1}{s_0}$.

The following generalized Korn-type inequality is useful for the study of the Reissner–Mindlin plate system.

Theorem 4.3 *(Generalized second Korn inequality). There exists a positive constant C only depending on M_0, M_1 and s_0, such that, for every $\varphi \in H^1(\Omega, \mathbb{R}^2)$ and for every $w \in H^1(\Omega, \mathbb{R})$,*

$$\|\nabla \varphi\|_{L^2(\Omega)} \leq C \left(\|\widehat{\nabla} \varphi\|_{L^2(\Omega)} + \frac{1}{\rho_0} \|\varphi + \nabla w\|_{L^2(\Omega)} \right). \tag{4.10}$$

Proof. We may assume, with no loss of generality, that $\int_{B_{s_0 \rho_0}(x_0)} \varphi = 0$. Let

$$\mathcal{S} = \left\{ \nabla w \mid w \in H^1(\Omega), \int_\Omega w = 0 \right\} \subset L^2(\Omega, \mathbb{R}^2).$$

$\mathcal{S}$ is a closed subspace of $L^2(\Omega, \mathbb{R}^2)$. In fact, let $\nabla w_n \in \mathcal{S}$ and $F \in L^2(\Omega, \mathbb{R}^2)$ such that $\nabla w_n \to F$ in $L^2(\Omega, \mathbb{R}^2)$. By the Poincaré inequality (4.5), w_n is a Cauchy sequence in $H^1(\Omega)$, so that there exists $w \in H^1(\Omega)$ such that $w_n \to w$ in $H^1(\Omega)$. Therefore $F = \nabla w \in \mathcal{S}$. By the projection theorem, for every $\varphi \in L^2(\Omega, \mathbb{R}^2)$, there exists a unique $\nabla \overline{w} \in \mathcal{S}$ such that

$$\|\varphi - \nabla \overline{w}\|_{L^2(\Omega)} = \min_{\nabla w \in \mathcal{S}} \|\varphi - \nabla w\|_{L^2(\Omega)} = \min_{\nabla w \in \mathcal{S}} \|\varphi + \nabla w\|_{L^2(\Omega)}. \tag{4.11}$$

Moreover, $\nabla \overline{w}$ is characterized by the condition

$$\varphi - \nabla \overline{w} \perp \nabla w \text{ in } L^2(\Omega), \text{ for every } \nabla w \in S. \tag{4.12}$$

Let us consider the infinitesimal rigid displacement

$$r = \frac{1}{2} (\nabla \varphi - \nabla^T \varphi)_{B_{s_0 \rho_0}(x_0)} (x - x_0) := W(x - x_0), \tag{4.13}$$

where

$$W = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix},$$

that is

$$r = (\alpha(x - x_0)_2, -\alpha(x - x_0)_1).$$

Let us distinguish two cases:
i) $\Omega = B_{s_0 \rho_0}(x_0)$,
ii) $B_{s_0 \rho_0}(x_0) \subsetneq \Omega$.

Case i). Let us see that, when one takes $\varphi = r$ in (4.11), with r given by (4.13), then its projection into $\mathcal{S}$ is

$$\nabla \overline{w} = 0, \tag{4.14}$$

that is, by the equivalent condition (4.12), $r \perp \nabla w$ in $L^2(\Omega)$, for every $\nabla w \in \mathcal{S}$. In fact

$$\int\limits_{B_{s_0\rho_0}(x_0)} r \cdot \nabla w = \int\limits_{B_{s_0\rho_0}(x_0)} \alpha(x-x_0)_2 w_{x_1} - \alpha(x-x_0)_1 w_{x_2} =$$

$$= \alpha \int\limits_{\partial B_{s_0\rho_0}(x_0)} w\left((x-x_0)_2 v_1 - (x-x_0)_1 v_2\right). \quad (4.15)$$

Since $v = \frac{x-x_0}{s_0\rho_0}$, we have

$$(x-x_0)_2 v_1 - (x-x_0)_1 v_2 = (x-x_0)_2 \frac{(x-x_0)_1}{s_0\rho_0} - (x-x_0)_1 \frac{(x-x_0)_2}{s_0\rho_0} = 0,$$

so that

$$\int\limits_{B_{s_0\rho_0}(x_0)} r \cdot \nabla w = 0, \qquad \text{for every } \nabla w \in S. \quad (4.16)$$

Therefore, by (4.11) and (4.14),

$$\|r\|_{L^2(\Omega)} \le \|r + \nabla w\|_{L^2(\Omega)}, \qquad \text{for every } \nabla w \in S. \quad (4.17)$$

By the definition of r and recalling that $\Omega = B_{s_0\rho_0}(x_0)$, it follows trivially that

$$\|r\|_{L^2(\Omega)}^2 = \frac{\pi}{2}\alpha^2 s_0^4 \rho_0^2 = \frac{s_0^2 \rho_0^2}{4}\|\nabla r\|_{L^2(\Omega)}^2. \quad (4.18)$$

By the Korn inequality (4.8), by (4.17) and (4.18), we have

$$\|\nabla\varphi\|_{L^2(\Omega)} \le \|\nabla(\varphi - r)\|_{L^2(\Omega)} + \|\nabla r\|_{L^2(\Omega)} =$$

$$= \|\nabla(\varphi - r)\|_{L^2(\Omega)} + \frac{2}{s_0\rho_0}\|r\|_{L^2(\Omega)} \le \|\nabla(\varphi - r)\|_{L^2(\Omega)} + \frac{2}{s_0\rho_0}\|r + \nabla w\|_{L^2(\Omega)} \le$$

$$\le \|\nabla(\varphi - r)\|_{L^2(\Omega)} + \frac{2}{s_0\rho_0}\|\varphi + \nabla w\|_{L^2(\Omega)} + \frac{2}{s_0\rho_0}\|\varphi - r\|_{L^2(\Omega)} \le$$

$$\le C\left(\|\widehat{\nabla}\varphi\|_{L^2(\Omega)} + \frac{1}{\rho_0}\|\varphi + \nabla w\|_{L^2(\Omega)}\right), \quad (4.19)$$

with C only depending on M_0, M_1 and s_0.

Case ii). Let r be the infinitesimal rigid displacement given by (4.13). By (4.12), its projection $\nabla\overline{w}$ into S satisfies

$$\int\limits_{\Omega} r \cdot \nabla w = \int\limits_{\Omega} \nabla\overline{w} \cdot \nabla w, \text{ for every } \nabla w \in S. \quad (4.20)$$

Choosing, in particular, $w = \overline{w}$ in (4.20), and by the same arguments used to prove (4.16), we have

$$\int_{\Omega} |\nabla \overline{w}|^2 = \int_{\Omega} r \cdot \nabla \overline{w} = \int_{B_{\frac{s_0 \rho_0}{2}}(x_0)} r \cdot \nabla \overline{w} + \int_{\Omega \setminus B_{\frac{s_0 \rho_0}{2}}(x_0)} r \cdot \nabla \overline{w} = \int_{\Omega \setminus B_{\frac{s_0 \rho_0}{2}}(x_0)} r \cdot \nabla \overline{w}, \quad (4.21)$$

so that, by Hölder inequality,

$$\|\nabla \overline{w}\|_{L^2(\Omega)} \leq \|r\|_{L^2(\Omega \setminus B_{\frac{s_0 \rho_0}{2}}(x_0))}. \quad (4.22)$$

By a direct computation, we have

$$\frac{\int_{\Omega \setminus B_{\frac{s_0 \rho_0}{2}}(x_0)} |r|^2}{\int_{\Omega} |r|^2} = 1 - \frac{\int_{B_{\frac{s_0 \rho_0}{2}}(x_0)} |r|^2}{\int_{\Omega} |r|^2} \leq 1 - \frac{\int_{B_{\frac{s_0 \rho_0}{2}}(x_0)} |r|^2}{\int_{B_{s_0 \rho_0}(x_0)} |r|^2} = \frac{15}{16}, \quad (4.23)$$

and, by (4.22) and (4.23),

$$\|\nabla \overline{w}\|_{L^2(\Omega)} \leq \frac{\sqrt{15}}{4} \|r\|_{L^2(\Omega)}. \quad (4.24)$$

Therefore

$$\|r - \nabla \overline{w}\|_{L^2(\Omega)} \geq \|r\|_{L^2(\Omega)} - \|\nabla \overline{w}\|_{L^2(\Omega)} \geq \left(1 - \frac{\sqrt{15}}{4}\right) \|r\|_{L^2(\Omega)}. \quad (4.25)$$

From (4.11) and (4.25), it follows that

$$\|r\|_{L^2(\Omega)} \leq \frac{4}{4 - \sqrt{15}} \|r + \nabla w\|_{L^2(\Omega)}, \qquad \text{for every } w \in H^1(\Omega). \quad (4.26)$$

Now, $\nabla r = W$, $|\nabla r|^2 = 2\alpha^2$, so that

$$\int_{\Omega} |\nabla r|^2 \leq 8\alpha^2 \pi M_1^2 \rho_0^2. \quad (4.27)$$

Since $|W(x - x_0)|^2 = \alpha^2 |x - x_0|^2$, by (4.27), we have

$$\int_{\Omega} |r|^2 = \alpha^2 \int_{\Omega} |x - x_0|^2 \geq \frac{\pi}{2} \alpha^2 s_0^4 \rho_0^4 \geq \left(\frac{s_0^2}{4M_1}\right)^2 \rho_0^2 \int_{\Omega} |\nabla r|^2. \quad (4.28)$$

By (4.8), (4.26) and (4.28),

$$\|\nabla\varphi\|_{L^2(\Omega)} \leq \|\nabla(\varphi - r)\|_{L^2(\Omega)} + \|\nabla r\|_{L^2(\Omega)} \leq$$

$$\leq C\left(\|\nabla(\varphi - r)\|_{L^2(\Omega)} + \frac{1}{\rho_0}\|r\|_{L^2(\Omega)}\right) \leq C\left(\|\nabla(\varphi - r)\|_{L^2(\Omega)} + \frac{1}{\rho_0}\|r + \nabla w\|_{L^2(\Omega)}\right) \leq$$

$$\leq C\left(\|\nabla(\varphi - r)\|_{L^2(\Omega)} + \frac{1}{\rho_0}\|\varphi + \nabla w\|_{L^2(\Omega)} + \frac{1}{\rho_0}\|\varphi - r\|_{L^2(\Omega)}\right) \leq$$

$$\leq C\left(\|\widehat{\nabla}\varphi\|_{L^2(\Omega)} + \frac{1}{\rho_0}\|\varphi + \nabla w\|_{L^2(\Omega)}\right), \quad (4.29)$$

with C only depending on M_0, M_1 and s_0.

Notice that a more accurate estimate can be obtained by replacing $B_{\frac{s_0\rho_0}{2}}(x_0)$ with $B_{s_0\rho_0}(x_0)$ in (4.21) and in what follows, obtaining

$$\|\nabla\overline{w}\|_{L^2(\Omega)} \leq \sqrt{\gamma}\|r\|_{L^2(\Omega)}, \quad (4.30)$$

where the constant γ,

$$\gamma = \frac{\int_{\Omega\setminus B_{s_0\rho_0}(x_0)} |r|^2}{\int_\Omega |r|^2} = 1 - \frac{\frac{\pi}{2}s_0^4\rho_0^4}{\int_\Omega |x - x_0|^2} < 1 \quad (4.31)$$

can be easily estimated in terms of the geometry of Ω. $\quad\square$

Remark 4.4. Let us notice that, choosing in particular $w \equiv 0$ in (4.10), it follows that there exists a positive constant C only depending on M_0, M_1 and s_0, such that for every $u \in H^1(\Omega, \mathbb{R}^2)$,

$$\|u\|_{H^1(\Omega)} \leq C(\rho_0\|\widehat{\nabla}u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}). \quad (4.32)$$

The above inequality was first proved by Gobert in [12] by using the theory of singular integrals, a different proof for regular domains being presented by Duvaut and Lions in [13].

5. The Neumann problem

Let us consider a plate $\Omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ with middle surface represented by a bounded domain Ω in $\mathbb{R}^2$ having uniform thickness h, subject to a transversal force field $\overline{Q}$ and to a couple field $\overline{M}$ acting on its boundary. Under the kinematic assumptions of Reissner–Mindlin's theory, the pair (φ, w), with $\varphi = (\varphi_1, \varphi_2)$, where φ_α, $\alpha = 1, 2$, are expressed in terms of the infinitesimal rigid rotation field ω by (3.2) and w is the transversal displacement, satisfies the equilibrium problem (1.1)(a)–(1.1)(d). The shearing matrix $S \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2))$ and the bending tensor $\mathbb{P} \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$, introduced in section 3, are assumed to satisfy the symmetry conditions (3.32), (3.33) and the ellipticity conditions (3.37), (3.38), respectively.

Summing up the weak formulation of equations (1.1)(a) and (1.1)(b), one derives the following *weak formulation* of the equilibrium problem (1.1)(a)–(1.1)(d):

A pair $(\varphi, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ is a weak solution to (1.1)(a)–(1.1)(d) *if for every* $\psi \in H^1(\Omega, \mathbb{R}^2)$ *and for every* $v \in H^1(\Omega)$,

$$\int_\Omega \mathbb{P}\nabla\varphi \cdot \nabla\psi + \int_\Omega S(\varphi + \nabla w) \cdot (\psi + \nabla v) = \int_{\partial\Omega} \overline{Q}v + \overline{M} \cdot \psi. \tag{5.1}$$

Choosing $\psi \equiv 0$, $v \equiv 1$, in (5.1), we have

$$\int_{\partial\Omega} \overline{Q} = 0. \tag{5.2}$$

Inserting $\psi \equiv -b$, $v = b \cdot x$ in (5.1), we have

$$\int_{\partial\Omega} b \cdot (\overline{Q}x - \overline{M}) = 0, \qquad \text{for every } b \in \mathbb{R}^2,$$

so that

$$\int_{\partial\Omega} \overline{Q}x - \overline{M} = 0. \tag{5.3}$$

We refer to (5.2)–(5.3) as the *compatibility conditions* for the equilibrium problem.

Remark 5.1. Given a solution (φ, w) to the equilibrium problem (1.1)(a)–(1.1)(d), then all its solutions are given by

$$\varphi^* = \varphi - b, \quad w^* = w + b \cdot x + a, \qquad \forall a \in \mathbb{R}, \forall b \in \mathbb{R}^2. \tag{5.4}$$

It is obvious that any (φ^*, w^*) given by (5.4) is a solution. Viceversa, given two solutions (φ, w), (φ^*, w^*), by subtracting their weak formulations one has

$$\int_\Omega \mathbb{P}\nabla(\varphi - \varphi^*) \cdot \nabla\psi + \int_\Omega S((\varphi - \varphi^*) + \nabla(w - w^*)) \cdot (\psi + \nabla v) = 0,$$

$$\forall v \in H^1(\Omega), \forall \psi \in H^1(\Omega, \mathbb{R}^2).$$

Choosing $\psi = \varphi - \varphi^*$, $v = w - w^*$, and by the ellipticity conditions (3.37), (3.38), we have

$$0 = \int_\Omega \mathbb{P}\nabla(\varphi - \varphi^*) \cdot \nabla(\varphi - \varphi^*) + \int_\Omega S((\varphi - \varphi^*) + \nabla(w - w^*)) \cdot ((\varphi - \varphi^*) + \nabla(w - w^*)) \geq$$

$$\geq \frac{h^3}{12}\xi_0 \int_\Omega |\widehat{\nabla}(\varphi - \varphi^*)|^2 + h\sigma_0 \int_\Omega |(\varphi - \varphi^*) + \nabla(w - w^*)|^2. \tag{5.5}$$

From the generalized Korn inequality (4.10) it follows that $\nabla(\varphi - \varphi^*) = 0$, so that there exists $b \in \mathbb{R}^2$ such that $\varphi^* = \varphi - b$. By the above inequality we also have that $\nabla(w^* - w) = \varphi - \varphi^* = b$, and therefore there exists $a \in \mathbb{R}$ such that $w^* = w + b \cdot x + a$.

An alternative proof of $\nabla(\varphi - \varphi^*) = 0$, that better enlightens the mathematical aspects of the Reissner–Mindlin model, is based on a qualitative argument which avoids the use of (4.10). Precisely, from (5.5), one has that $\widehat{\nabla}(\varphi - \varphi^*) = 0$ and $\nabla(w - w^*) = \varphi^* - \varphi$. Therefore $\varphi - \varphi^* = Wx + b$ for some skew symmetric matrix $W = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$ and some constant $b \in \mathbb{R}^2$ and $\nabla(w^* - w) = Wx + b \in \mathbb{C}^\infty$. Hence we can compute $(w^* - w)_{x_1 x_2} = (\alpha x_2 + b_1)_{x_2} = \alpha$, $(w^* - w)_{x_2 x_1} = (-\alpha x_1 + b_2)_{x_1} = -\alpha$ and, by the Schwarz theorem, $\alpha = 0$, so that $\varphi - \varphi^* = b$.

Proposition 5.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of Lipschitz class with constants ρ_0, M_0, satisfying (4.1)–(4.2). Let the second order tensor $S \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2))$ and the forth order tensor $\mathbb{P} \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$ satisfy the symmetry conditions (3.32), (3.33) and the ellipticity conditions (3.37), (3.38), respectively. Let $\overline{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$ and $\overline{Q} \in H^{-\frac{1}{2}}(\partial\Omega)$ satisfy the compatibility conditions (5.2)–(5.3) respectively. Problem (1.1)(a)–(1.1)(d) admits a unique solution $(\varphi, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ normalized by the conditions*

$$\int_\Omega \varphi = 0, \qquad \int_\Omega w = 0. \tag{5.6}$$

Moreover

$$\|\varphi\|_{H^1(\Omega)} + \frac{1}{\rho_0}\|w\|_{H^1(\Omega)} \leq \frac{C}{\rho_0^2}\left(\|\overline{M}\|_{H^{-\frac{1}{2}}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right), \tag{5.7}$$

with C only depending on M_0, M_1, s_0, ξ_0, σ_0, $\frac{\rho_0}{h}$.

Proof. Let us consider the linear space

$$\mathcal{H} = \left\{ (\psi, v) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega) \mid \int_\Omega \psi = 0, \int_\Omega v = 0 \right\}, \tag{5.8}$$

which is a Banach space equipped with the norm

$$\|(\psi, v)\|_{\mathcal{H}} = \|\psi\|_{H^1(\Omega)} + \frac{1}{\rho_0}\|v\|_{H^1(\Omega)}. \tag{5.9}$$

The symmetric bilinear form

$$a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$$

$$a((\varphi, w), (\psi, v)) = \int_\Omega \mathbb{P}\nabla\varphi \cdot \nabla\psi + S(\varphi + \nabla w) \cdot (\psi + \nabla v), \tag{5.10}$$

is continuous in $\mathcal{H} \times \mathcal{H}$. Let us see that it is also coercive. By the ellipticity conditions (3.37), (3.38),

$$a((\varphi, w), (\varphi, w)) \geq \frac{h^3}{12}\xi_0 \int_{\Omega} |\widehat{\nabla}\varphi|^2 + h\sigma_0 \int_{\Omega} |\varphi + \nabla w|^2 \geq$$

$$\geq h^3 \min\left\{\frac{\xi_0}{12}, \sigma_0\left(\frac{\rho_0}{h}\right)^2\right\} \left(\int_{\Omega} |\widehat{\nabla}\varphi|^2 + \frac{1}{\rho_0^2}\int_{\Omega} |\varphi + \nabla w|^2\right). \quad (5.11)$$

On the other hand, from Poincaré and Korn inequalities (4.5) and (4.10), and by the trivial estimate $\|\nabla w\|_{L^2(\Omega)} \leq \|\varphi + \nabla w\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}$, one has

$$\|(\varphi, w)\|_{\mathcal{H}} \leq C\left(\rho_0\|\widehat{\nabla}\varphi\|_{L^2(\Omega)} + \|\varphi + \nabla w\|_{L^2(\Omega)}\right), \quad (5.12)$$

with C only depending on M_0, M_1 and s_0.

From (5.11)–(5.12), one has

$$a((\varphi, w), (\varphi, w)) \geq C\rho_0^3 \|(\varphi, w)\|_{\mathcal{H}}^2, \quad (5.13)$$

where C only depends on M_0, M_1, s_0, ξ_0, σ_0, $\frac{\rho_0}{h}$.

Therefore the bilinear form (5.10) is a scalar product inducing an equivalent norm in $\mathcal{H}$, which we denote by $|||\cdot|||$.

The linear functional

$$F : \mathcal{H} \to \mathbb{R}$$

$$F(\psi, v) = \int_{\partial\Omega} \widehat{Q}v + \widehat{M} \cdot \psi$$

is bounded and, by (5.13), it satisfies

$$|F(\psi, v)| \leq C\rho_0 \left(\|\overline{M}\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|\psi\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\overline{Q}\|_{H^{-\frac{1}{2}}(\partial\Omega)} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)}\right) \leq$$

$$\leq C\rho_0 \left(\|\overline{M}\|_{H^{-\frac{1}{2}}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right) \|(\psi, v)\|_{\mathcal{H}} \leq$$

$$\leq C\rho_0^{-\frac{1}{2}} \left(\|\overline{M}\|_{H^{-\frac{1}{2}}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right) |||(\psi, v)|||, \quad (5.14)$$

so that

$$|||F|||_* \leq C\rho_0^{-\frac{1}{2}} \left(\|\overline{M}\|_{H^{-\frac{1}{2}}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right), \quad (5.15)$$

with C only depending on M_0, M_1, s_0, ξ_0, σ_0, $\frac{\rho_0}{h}$. By the Riesz representation theorem, there exists a unique $(\varphi, w) \in \mathcal{H}$ such that $a((\varphi, w), (\psi, v)) = F(\psi, v)$ for every $(\psi, v) \in \mathcal{H}$, that is (5.1) holds for every $(\psi, v) \in \mathcal{H}$. Moreover

$$|||(\varphi, w)||| = |||F|||_*. \quad (5.16)$$

Let us prove (5.1) for every $(\psi, v) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$. Given any $\psi \in H^1(\Omega, \mathbb{R}^2)$ and any $v \in H^1(\Omega)$, let

$$\widetilde{\psi} = \psi - \psi_\Omega, \qquad \widetilde{v} = v + \psi_\Omega \cdot (x - x_\Omega) - v_\Omega.$$

We have that $\widetilde{\psi} + \nabla \widetilde{v} = \psi + \nabla v$. Hence, by the compatibility conditions (5.2)–(5.3),

$$\int_\Omega \mathbb{P}\nabla\varphi \cdot \nabla\psi + S(\varphi + \nabla w) \cdot (\psi + \nabla v) = \int_{\partial\Omega} \overline{M} \cdot \widetilde{\psi} + \overline{Q}\widetilde{v} = \int_{\partial\Omega} (\overline{M} \cdot \psi + \overline{Q}v) -$$

$$- \psi_\Omega \cdot \int_{\partial\Omega} (\overline{M} - \overline{Q}x) - v_\Omega \int_{\partial\Omega} \overline{Q} - \left(\psi_\Omega \cdot x_\Omega \int_{\partial\Omega} \overline{Q} \right) = \int_{\partial\Omega} \overline{M} \cdot \psi + \overline{Q}v. \quad (5.17)$$

Finally, (5.7) follows from (5.13), (5.15) and (5.16). $\quad \square$

6. H^2 regularity

Our main result is the following global regularity theorem.

Theorem 6.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of class $C^{1,1}$, with constants ρ_0, M_0, satisfying (4.1), (4.2). Let $S \in C^{0,1}(\overline{\Omega}, \mathcal{L}(\mathbb{M}^2))$ and $\mathbb{P} \in C^{0,1}(\overline{\Omega}, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$ satisfy the symmetry conditions (3.32), (3.33) and the ellipticity conditions (3.37), (3.38). Let $\overline{M} \in H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$ and $\overline{Q} \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy the compatibility conditions (5.2), (5.3), respectively. Then, the weak solution $(\varphi, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ of the problem (1.1)(a)–(1.1)(d), normalized by the conditions (5.6), is such that $(\varphi, w) \in H^2(\Omega, \mathbb{R}^2) \times H^2(\Omega)$ and*

$$\|\varphi\|_{H^2(\Omega)} + \frac{1}{\rho_0}\|w\|_{H^2(\Omega)} \leq \frac{C}{\rho_0^2}\left(\|\overline{M}\|_{H^{\frac{1}{2}}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{\frac{1}{2}}(\partial\Omega)} \right), \quad (6.1)$$

where the constant $C > 0$ only depends on M_0, M_1, s_0, ξ_0, σ_0, $\frac{\rho_0}{h}$, $\|S\|_{C^{0,1}(\overline{\Omega})}$ and $\|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})}$.

The proof of the theorem is mainly based on the approach to regularity for second order elliptic systems adopted, for instance, in [3] and [4]. For the sake of completeness, the main steps of the proof are recalled in the sequel.

Let us introduce the following notation. Let

$$B_\sigma^+ = \{(y_1, y_2) \in \mathbb{R}^2 | \ y_1^2 + y_2^2 < \sigma^2, \ y_2 > 0\} \quad (6.2)$$

be the hemidisk of radius σ, $\sigma > 0$, and let

$$\Gamma_\sigma = \{(y_1, y_2) \in \mathbb{R}^2 | -\sigma \leq y_1 \leq \sigma, \ y_2 = 0\} \quad (6.3)$$

and

$$\Gamma_\sigma^+ = \partial B_\sigma^+ \setminus \Gamma_\sigma \quad (6.4)$$

be the flat and the curved portion of the boundary ∂B_σ^+, respectively. Moreover, let

$$H^1_{\Gamma_\sigma^+}(B_\sigma^+) = \{g \in H^1(B_\sigma^+) \mid g = 0 \text{ on } \Gamma_\sigma^+\}. \tag{6.5}$$

Without loss of generality, hereinafter we will assume $\rho_0 = 1$. Moreover, the dependence of the constants C on the plate thickness h will be not explicitly indicated in the estimates below.

By the regularity of $\partial\Omega$, we can construct a finite collection of open sets $\Omega_0, \{\Omega_j\}_{j=1}^N$ such that $\Omega = \Omega_0 \cup \left(\bigcup_{j=1}^N \mathcal{T}_{(j)}^{-1}(B_{\frac{\sigma}{2}}^+)\right)$, $\Omega_0 \subset \Omega_{\delta_0}$, where $\Omega_{\delta_0} = \{x \in \Omega \mid dist(x, \partial\Omega) > \delta_0\}$, $\delta_0 > 0$ only depends on M_0, and N only depends on M_0, M_1. Here, $\mathcal{T}_{(j)}$, $j = 1, ..., N$, is a homeomorphism of $C^{1,1}$ class which maps Ω_j into B_σ, $\Omega_j \cap \Omega$ into B_σ^+, $\overline{\Omega}_j \cap \partial\Omega$ into Γ_σ, and $\partial\Omega_j \cap \Omega$ into Γ_σ^+.

The estimate of $(\|\varphi\|_{H^2(\Omega_0)} + \|w\|_{H^2(\Omega_0)})$ is a consequence of the following local interior regularity result, whose proof can be obtained, for example, by adapting the arguments illustrated in [4].

Theorem 6.2. *Let us denote by B_σ the open ball in $\mathbb{R}^2$ centered at the origin and with radius σ, $\sigma > 0$. Let $(\varphi, w) \in H^1(B_\sigma, \mathbb{R}^2) \times H^1(B_\sigma)$ be such that*

$$a((\varphi, w), (\psi, v)) = 0, \quad \text{for every } (\psi, v) \in H^1(B_\sigma, \mathbb{R}^2) \times H^1(B_\sigma), \tag{6.6}$$

where

$$a((\varphi, w), (\psi, v)) = \int_{B_\sigma} \mathbb{P}\nabla\varphi \cdot \nabla\psi + \int_{B_\sigma} S(\varphi + \nabla w) \cdot (\psi + \nabla v), \tag{6.7}$$

with $\mathbb{P} \in C^{0,1}(\overline{B}_\sigma, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$, $S \in C^{0,1}(\overline{B}_\sigma, \mathcal{L}(\mathbb{M}^2))$ satisfying the symmetry conditions (3.32), (3.33) and the ellipticity conditions (3.37), (3.38). Then, $(\varphi, w) \in H^2(B_\sigma, \mathbb{R}^2) \times H^2(B_\sigma)$ and we have

$$\|\varphi\|_{H^2(B_{\frac{\sigma}{2}})} + \|w\|_{H^2(B_{\frac{\sigma}{2}})} \leq C \left(\|\varphi\|_{H^1(B_\sigma)} + \|w\|_{H^1(B_\sigma)}\right), \tag{6.8}$$

where the constant $C > 0$ only depends on ξ_0, σ_0, $\|S\|_{C^{0,1}(\overline{B}_\sigma)}$ and $\|\mathbb{P}\|_{C^{0,1}(\overline{B}_\sigma)}$.

In order to complete the proof of the regularity estimate, let us control the quantity $(\|\varphi\|_{H^2(\Omega_j \cap \Omega)} + \|w\|_{H^2(\Omega_j \cap \Omega)})$ for every $j \in \{1, ..., N\}$.

For every $v \in H^1_{\partial\Omega_j \cap \Omega}(\Omega_j \cap \Omega)$ and for every $\psi \in H^1_{\partial\Omega_j \cap \Omega}(\Omega_j \cap \Omega, \mathbb{R}^2)$, the solution (φ, w) to (1.1)(a)–(1.1)(d) satisfies the weak formulation

$$\int_{\Omega_j \cap \Omega} \mathbb{P}(x)\nabla_x\varphi \cdot \nabla_x\psi \, dx + \int_{\Omega_j \cap \Omega} S(x)(\varphi + \nabla_x w) \cdot (\psi + \nabla_x v) \, dx =$$

$$= \int_{\Omega_j \cap \partial\Omega} (\overline{Q}v + \overline{M} \cdot \psi) ds_x. \tag{6.9}$$

Let us introduce the change of variables

$$y = \mathcal{T}_{(j)}(x), \quad y \in B_\sigma^+, \tag{6.10}$$

$$x = \mathcal{T}_{(j)}^{-1}(y), \quad x \in \Omega_j \cap \Omega, \tag{6.11}$$

and let us define

$$\widetilde{w}(y) = w(\mathcal{T}_{(j)}^{-1}(y)), \quad \widetilde{\varphi}_r(y) = \varphi_r(\mathcal{T}_{(j)}^{-1}(y)), \ r = 1, 2, \tag{6.12}$$

$$\widetilde{v}(y) = v(\mathcal{T}_{(j)}^{-1}(y)), \quad \widetilde{\psi}_r(y) = \psi_r(\mathcal{T}_{(j)}^{-1}(y)), \ r = 1, 2. \tag{6.13}$$

Then, the pair $(\widetilde{\varphi}, \widetilde{w}) \in H^1(B_\sigma^+, \mathbb{R}^2) \times H^1(B_\sigma^+)$ satisfies

$$\widetilde{a}_+((\widetilde{\varphi}, \widetilde{w}), (\widetilde{\psi}, \widetilde{v})) = \widetilde{F}_+(\widetilde{\psi}, \widetilde{v}), \ \text{ for every } (\widetilde{\psi}, \widetilde{v}) \in H^1_{\Gamma_\sigma^+}(B_\sigma^+, \mathbb{R}^2) \times H^1_{\Gamma_\sigma^+}(B_\sigma^+), \tag{6.14}$$

where

$$\widetilde{a}_+((\widetilde{\varphi}, \widetilde{w}), (\widetilde{\psi}, \widetilde{v})) =$$

$$= \int_{B_\sigma^+} \widetilde{\mathbb{P}}(y) \nabla_y \widetilde{\varphi} \cdot \nabla_y \widetilde{\psi} dy + \int_{B_\sigma^+} \widetilde{S}(y)(\widetilde{\varphi} + L^T \nabla_y \widetilde{w}) \cdot (\widetilde{\psi} + L^T \nabla_y \widetilde{v}) dy, \tag{6.15}$$

$$\widetilde{F}_+(\widetilde{\psi}, \widetilde{v}) = \int_{\Gamma_\sigma} (\widetilde{Q}\widetilde{v} + \widetilde{\mathcal{M}} \cdot \widetilde{\psi}) ds_y, \tag{6.16}$$

with

$$(L)_{ks} = L_{ks} = \frac{\partial \mathcal{T}_k}{\partial x_s}, \quad k, s = 1, 2, \tag{6.17}$$

$$\iota = |\det L|, \quad \iota^* = \sqrt{\left(\frac{\partial \mathcal{T}^{-1}(y)}{\partial y}\right)^T \frac{\partial \mathcal{T}^{-1}(y)}{\partial y} \Big|_{y_1, y_2 = 0}}, \tag{6.18}$$

$$(\widetilde{\mathbb{P}}(y))_{ilrk} = \widetilde{P}_{ilrk}(y) = \sum_{j,s=1}^{2} P_{ijrs}(\mathcal{T}^{-1}(y)) L_{ks} L_{lj} \iota^{-1}, \quad i, l, r, k = 1, 2, \tag{6.19}$$

$$\widetilde{S}(y) = S(\mathcal{T}^{-1}(y)) \iota^{-1}, \tag{6.20}$$

$$\widetilde{Q}(y) = \overline{Q}(\mathcal{T}^{-1}(y)) \iota^*, \quad \widetilde{\mathcal{M}}(y) = \overline{M}(\mathcal{T}^{-1}(y)) \iota^*. \tag{6.21}$$

Since $L \in C^{0,1}(\Omega_j \cap \Omega, \mathbb{M}^2)$ is nonsingular and there exist two constants c_1, c_2, only depending on M_0, such that $0 < c_1 \le \iota \le c_2, c_1 \le \iota^* \le c_2$ in Ω_j, the fourth order tensor $\widetilde{\mathbb{P}}$ in (6.19) has the following properties:

i) (major symmetry) for every 2×2 matrices A and B we have

$$\widetilde{\mathbb{P}}A \cdot B = A \cdot \widetilde{\mathbb{P}}B; \tag{6.22}$$

ii) (strong ellipticity) there exists a constant κ_0, $\kappa_0 > 0$ and κ_0 only depending on M_0 and ξ_0, such that for every pair of vectors $a, b \in \mathbb{R}^2$ and for every $y \in \overline{B}_\sigma^+$ we have

$$\widetilde{\mathbb{P}}(y)(a \otimes b) \cdot (a \otimes b) \geq \kappa_0 |a|^2 |b|^2; \tag{6.23}$$

iii) (regularity) $\widetilde{\mathbb{P}} \in C^{0,1}(\overline{B}_\sigma^+, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$.

The matrix $\widetilde{S}$ defined in (6.20) is symmetric and there exists a constant χ_0, $\chi_0 > 0$ only depending on σ_0 and M_0, such that for every vector $a \in \mathbb{R}^2$ and for every $y \in \overline{B}_\sigma^+$ we have

$$\widetilde{S}(y)a \cdot a \geq \chi_0 |a|^2. \tag{6.24}$$

Moreover, $\widetilde{S} \in C^{0,1}(\overline{B}_\sigma^+, \mathbb{M}^2)$.

We now use the regularity up to the flat boundary of the hemidisk B_1^+ stated in the next theorem, whose proof is postponed in the Appendix.

Theorem 6.3. *Under the above notation and assumptions, let $(\widetilde{\varphi}, \widetilde{w}) \in H^1(B_\sigma^+, \mathbb{R}^2) \times H^1(B_\sigma^+)$ defined in (6.12) be the solution to (6.14). Then $(\widetilde{\varphi}, \widetilde{w}) \in H^2(B_{\frac{\sigma}{2}}^+, \mathbb{R}^2) \times H^2(B_{\frac{\sigma}{2}}^+)$ and we have*

$$\|\widetilde{\varphi}\|_{H^2(B_{\frac{\sigma}{2}}^+)} + \|\widetilde{w}\|_{H^2(B_{\frac{\sigma}{2}}^+)}$$

$$\leq \frac{C}{\rho_0^2} \left(\|\widetilde{\mathcal{Q}}\|_{H^{\frac{1}{2}}(\Gamma_\sigma)} + \|\widetilde{\mathcal{M}}\|_{H^{\frac{1}{2}}(\Gamma_\sigma)} + \rho_0^2 \left(\|\widetilde{\varphi}\|_{H^1(B_\sigma^+)} + \|\widetilde{w}\|_{H^1(B_\sigma^+)} \right) \right), \tag{6.25}$$

where the constant $C > 0$ only depends on M_0, ξ_0, σ_0, $\|S\|_{C^{0,1}(\overline{\Omega})}$ and $\|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})}$.

Recalling that $\Omega = \Omega_0 \cup \left(\bigcup_{j=1}^N \mathcal{T}_{(j)}^{-1}(B_{\frac{\sigma}{2}}^+) \right)$, the estimate (6.1) follows by applying the inverse mapping $\mathcal{T}_{(j)}^{-1}$ to (6.25), $j = 1, ..., N$, and by using the interior estimate (6.8).

7. Three spheres inequality and strong unique continuation

Our approach to derive a three spheres inequality for the solutions to the Reissner–Mindlin system consists in the derivation of a new system of four scalar differential equations in Laplacian principal part from (1.1)(a)–(1.1)(b). This can be made following the approach developed by Lin, Nakamura and Wang in [6], which is based on the introduction of a new independent variable, see (7.18) below. In order to perform this reduction, it is essential to assume that both the plate bending tensor $\mathbb{P}$ and the plate shearing matrix S are isotropic.

In the present section we assume that Ω is a bounded domain in $\mathbb{R}^2$ of Lipschitz class with constants ρ_0, M_0 and we assume that the plate is isotropic with Lamé parameters λ, μ. We assume that $\lambda, \mu \in C^{0,1}(\overline{\Omega})$ and that, for given positive constants $\alpha_0, \alpha_1, \gamma_0$, they satisfy the following conditions

$$\mu(x) \ge \alpha_0, \quad 2\mu(x) + 3\lambda(x) \ge \gamma_0, \tag{7.1}$$

and

$$\|\lambda\|_{C^{0,1}(\overline{\Omega})} + \|\mu\|_{C^{0,1}(\overline{\Omega})} \le \alpha_1. \tag{7.2}$$

We assume that the *plate shearing matrix* has the form $S I_2$ where $S \in C^{0,1}(\overline{\Omega})$ is the real valued function defined by

$$S = \frac{Eh}{2(1+\nu)}, \tag{7.3}$$

where

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}, \quad \nu = \frac{\lambda}{2(\mu+\lambda)} \tag{7.4}$$

and we assume that *plate bending tensor* $\mathbb{P}$ has the following form

$$\mathbb{P}A = B\left[(1-\nu)\widehat{A} + \nu tr(A)I_2\right], \quad \text{for every } 2 \times 2 \quad \text{matrix } A, \tag{7.5}$$

where

$$B = \frac{Eh^3}{12(1-\nu^2)}. \tag{7.6}$$

By (7.1) and (7.2) and noticing that $S = h\mu$, we have that

$$h\sigma_0 \le S, \quad \text{in } \Omega, \quad \|S\|_{C^{0,1}(\overline{\Omega})} \le h\sigma_1 \tag{7.7}$$

and

$$\frac{h^3}{12}\xi_0|\widehat{A}|^2 \le \mathbb{P}A \cdot A \le \frac{h^3}{12}\xi_1|\widehat{A}|^2, \quad \text{in } \Omega, \tag{7.8}$$

for every 2×2 matrix A, where

$$\sigma_0 = \alpha_0, \quad \sigma_1 = \alpha_1, \quad \xi_0 = \min\{2\alpha_0, \gamma_0\}, \quad \xi_1 = 2\alpha_1. \tag{7.9}$$

Theorem 7.1. *Under the above hypotheses on Ω, S and $\mathbb{P}$, let $(\varphi, w) \in H^2_{loc}(\Omega, \mathbb{R}^2) \times H^2_{loc}(\Omega)$ be a solution of the system*

$$\begin{cases} \text{div}\,(S(\varphi + \nabla w)) = 0, & \text{in } \Omega, \quad (a) \\ \text{div}\,(\mathbb{P}\nabla\varphi) - S(\varphi + \nabla w) = 0, & \text{in } \Omega. \quad (b) \end{cases} \tag{7.10}$$

Let $\bar{x} \in \Omega$ and $R_1 > 0$ be such that $B_{R_1}(\bar{x}) \subset \Omega$. Then there exists $\theta \in (0, 1)$, θ depending on $\alpha_0, \alpha_1, \gamma_0, \frac{\rho_0}{h}$ only, such that if $0 < R_3 < R_2 < R_1$ and $\frac{R_3}{R_1} \le \frac{R_2}{R_1} \le \theta$ then we have

$$\int_{B_{R_2}(\bar{x})} |V|^2 \le C \left(\int_{B_{R_3}(\bar{x})} |V|^2 \right)^{\tau} \left(\int_{B_{R_1}(\bar{x})} |V|^2 \right)^{1-\tau} \tag{7.11}$$

where

$$|V|^2 = |\varphi|^2 + \frac{1}{\rho_0^2}|w|^2, \tag{7.12}$$

$\tau \in (0, 1)$ *depends on* α_0, α_1, γ_0, $\frac{R_3}{R_1}$, $\frac{R_2}{R_1}$, $\frac{\rho_0}{h}$ *only and* C *depends on* α_0, α_1, γ_0, $\frac{R_2}{R_1}$, $\frac{\rho_0}{h}$ *only. In addition, keeping* R_2, R_1 *fixed, we have*

$$\tau = \mathcal{O}\left(|\log R_3|^{-1} \right), \quad as\ R_3 \to 0. \tag{7.13}$$

Proof. It is not restrictive to assume that $\bar{x} = 0 \in \Omega$. In order to prove (7.11), first we introduce an auxiliary unknown which allows us to obtain a new system of equations with the Laplace operator as the principal part, then we obtain (7.11) by applying [6, Theorem 1.1]. By (7.3) and (7.5) we have

$$\text{div}\,(\mathbb{P}\nabla\varphi) - S(\varphi + \nabla w) =$$

$$= \frac{h^3}{12} \left[\text{div}\,\left(\mu \left(\nabla\varphi + \nabla^T\varphi \right) \right) + \nabla \left(\frac{2\lambda\mu}{2\mu + \lambda} \text{div}\,\varphi \right) - \frac{12\mu}{h^2}(\varphi + \nabla w) \right]. \tag{7.14}$$

Now we denote

$$a = \frac{2\mu + 3\lambda}{4(\lambda + \mu)}, \quad b = \frac{4(\lambda + \mu)}{2\mu + \lambda}, \tag{7.15}$$

$$G = \left(\nabla\varphi + \nabla^T\varphi \right) \frac{\nabla\mu}{\mu} - \left[\frac{\nabla\mu}{\mu} + \frac{\mu(2\mu + 3\lambda)}{2\mu + \lambda} \nabla \left(\frac{1}{\mu} \right) \right] \text{div}\,\varphi$$

and

$$v = b\,\text{div}\,\varphi. \tag{7.16}$$

By (7.14) we have

$$\text{div}\,(\mathbb{P}\nabla\varphi) - S(\varphi + \nabla w) = \frac{h^3\mu}{12} \left[\Delta\varphi + \nabla(av) + G - \frac{12}{h^2}(\varphi + \nabla w) \right],$$

therefore equation (7.10)(b) is equivalent to the equation

$$\Delta\varphi + \nabla(av) + G - \frac{12}{h^2}(\varphi + \nabla w) = 0. \tag{7.17}$$

Now, noticing that (7.15) gives $a + \frac{1}{b} = 1$, we have

$$\text{div}\,(\Delta\varphi + \nabla(av)) = \Delta\left(\frac{v}{b}\right) + \Delta\,(av) = \Delta\left(\left(a + \frac{1}{b}\right)v\right) = \Delta v. \quad (7.18)$$

Now we apply the divergence operator to both the sides of (7.17) and by (7.18) we get

$$\Delta v + \text{div}\,G - \frac{12}{h^2}\text{div}\,(\varphi + \nabla w) = 0. \quad (7.19)$$

Finally, observing that by equation (7.10)(a) we have

$$\text{div}\,(\varphi + \nabla w) = \text{div}\,\left(\frac{1}{S}S(\varphi + \nabla w)\right) = \nabla\left(\frac{1}{S}\right) \cdot S(\varphi + \nabla w),$$

by (7.19) we obtain

$$\Delta v + \text{div}\,G - \frac{12}{h^2}\nabla\left(\frac{1}{S}\right) \cdot S(\varphi + \nabla w) = 0. \quad (7.20)$$

On the other side by (7.16) we have

$$\text{div}\,(S(\varphi + \nabla w)) = S\Delta w + \frac{S}{b}v + \nabla S \cdot \varphi + \nabla S \cdot \nabla w, \quad (7.21)$$

therefore, by (7.21), (7.10)(a), (7.3) and (7.4), we have

$$\Delta w + \frac{2\mu + \lambda}{4(\lambda + \mu)}v + \frac{\nabla S}{S} \cdot \varphi + \frac{\nabla S}{S} \cdot \nabla w = 0. \quad (7.22)$$

Now, in order to satisfy the homogeneity of norms we define

$$\widetilde{w} = w, \quad \widetilde{\varphi} = \rho_0\varphi, \quad \widetilde{v} = \rho_0^2 v$$

and

$$\widetilde{G} = \rho_0 G = \left(\nabla\widetilde{\varphi} + \nabla^T\widetilde{\varphi}\right)\frac{\nabla\mu}{\mu} - \left[\frac{\nabla\mu}{\mu} + \frac{\mu(2\mu + 3\lambda)}{2\mu + \lambda}\nabla\left(\frac{1}{\mu}\right)\right]\text{div}\,\widetilde{\varphi}.$$

By (7.17), (7.20), (7.22), we have that $\widetilde{w}, \widetilde{\varphi}, \widetilde{v}$ satisfy the system

$$\begin{cases} \Delta\widetilde{w} + \frac{2\mu+\lambda}{4\rho_0^2(\lambda+\mu)}\widetilde{v} + \frac{\nabla S}{\rho_0 S} \cdot \widetilde{\varphi} + \frac{\nabla S}{S} \cdot \nabla\widetilde{w} = 0, & \text{in } \Omega, \\ \Delta\widetilde{\varphi} + \nabla(\frac{a}{\rho_0}\widetilde{v}) + \widetilde{G} - \frac{12}{h^2}(\widetilde{\varphi} + \rho_0\nabla\widetilde{w}) = 0, & \text{in } \Omega, \\ \Delta\widetilde{v} + \rho_0\text{div}\,\widetilde{G} - \frac{12}{h^2}\rho_0\nabla\left(\frac{1}{S}\right) \cdot S(\widetilde{\varphi} + \rho_0\nabla\widetilde{w}) = 0, & \text{in } \Omega. \end{cases} \quad (7.23)$$

The above system has the same form of system (1.5) of [6]. As a matter of fact, as soon as we introduce the following notation

$$u = (\widetilde{w}, \widetilde{\varphi}),$$

$$P_1(x, \partial)\widetilde{v} = \begin{pmatrix} \frac{2\mu+\lambda}{4\rho_0^2(\lambda+\mu)}\widetilde{v} \\ \nabla(\frac{a}{\rho_0}\widetilde{v}) \end{pmatrix}, \quad P_2(x, \partial)u = \begin{pmatrix} \frac{\nabla S}{\rho_0 S} \cdot \widetilde{\varphi} + \frac{\nabla S}{S} \cdot \nabla\widetilde{w} \\ \widetilde{G} - \frac{12}{h^2}(\widetilde{\varphi} + \rho_0\nabla\widetilde{w}), \end{pmatrix}$$

$$Q_1(x, \partial)\widetilde{v} = 0, \quad Q_2(x, \partial)u = -\frac{12}{h^2}\rho_0\nabla\left(\frac{1}{S}\right) \cdot S(\widetilde{\varphi} + \rho_0\nabla\widetilde{w}),$$

system (7.23) is equivalent to

$$\begin{cases} \Delta u + P_1(x, \partial)\widetilde{v} + P_2(x, \partial)u = 0, & \text{in } \Omega, \\ \Delta\widetilde{v} + Q_1(x, \partial)\widetilde{v} + Q_2(x, \partial)u + \rho_0\text{div}\,\widetilde{G} = 0, & \text{in } \Omega. \end{cases} \tag{7.24}$$

Notice that, likewise to [6], $P_j(x, \partial)$ and $Q_j(x, \partial)$, $j = 1, 2$, are first order operators with L^∞ coefficients. In addition, although $\widetilde{G}$ is slightly different from the term G of [6], the proof of Theorem 1.1 (after the scaling $x \to R_1 x$) of such a paper can be used step by step to derive (7.11). $\quad\square$

Corollary 7.2. *Assume that S, $\mathbb{P}$ and Ω satisfy the same hypotheses of 7.1, let $x_0 \in \Omega$ and let $(\varphi, w) \in H^2_{loc}(\Omega, \mathbb{R}^2) \times H^2_{loc}(\Omega)$ be a solution of the system (7.10)(a)–(7.10)(b) such that*

$$\|\varphi\|_{L^2(B_r(\bar{x}))} + \frac{1}{\rho_0}\|w\|_{L^2(B_r(\bar{x}))} = \mathcal{O}\left(r^N\right), \quad \text{as } r \to 0, \quad \forall N \in \mathbb{N} \tag{7.25}$$

then $\varphi \equiv 0$, $w \equiv 0$ in Ω.

Proof. It is standard consequence of the inequality (7.11) and of the connectedness of Ω. For more details see [7, Corollary 6.4]. $\quad\square$

8. Appendix

In this appendix we sketch a proof of Theorem 6.3.

Without loss of generality, we can assume $\sigma = 1$. Our proof consists of two main steps. As first step, we estimate the partial derivatives $\frac{\partial}{\partial y_1}\nabla\widetilde{\varphi}, \frac{\partial}{\partial y_1}\nabla\widetilde{w}$ along the direction e_1 parallel to the flat boundary Γ_1 of B_1^+. The second step will concern with the estimate of the partial derivatives $\frac{\partial}{\partial y_2}\nabla\widetilde{\varphi}, \frac{\partial}{\partial y_2}\nabla\widetilde{w}$ along the direction orthogonal to the flat boundary Γ_1.

First step. (Estimate of the tangential derivatives)

Let $\vartheta \in C_0^\infty(\mathbb{R}^2)$ be a function such that $0 \leq \vartheta(y) \leq 1$ in $\mathbb{R}^2$, $\vartheta \equiv 1$ in B_ρ, $\vartheta \equiv 0$ in $\mathbb{R}^2 \setminus B_\eta$, and $|\nabla^k\vartheta| \leq C$, $k = 1, 2$, where $\rho = \frac{3}{4}$, $\eta = \frac{7}{8}$ and $C > 0$ is an absolute constant.

For every functions $\widetilde{\psi} \in H^1_{\Gamma_1^+}(B_1^+, \mathbb{R}^2)$, $\widetilde{v} \in H^1_{\Gamma_1^+}(B_1^+)$, we still denote by $\widetilde{\psi} \in H^1(\mathbb{R}^2_+, \mathbb{R}^2)$, $\widetilde{v} \in H^1(\mathbb{R}^2_+)$ their corresponding extensions to $\mathbb{R}^2_+$ obtained by taking $\widetilde{\psi} = 0$, $\widetilde{v} = 0$ in $\mathbb{R}^2_+ \setminus B_1^+$.

Given a real number $s \in \mathbb{R} \setminus \{0\}$, the difference operator in direction y_1 of any function f is defined as

$$(\tau_{1,s} f)(y) = \frac{f(y + s e_1) - f(y)}{s}. \tag{8.1}$$

In the sequel we shall assume $|s| \leq \frac{1}{16}$. We note that if $\widetilde{\varphi} \in H^1(B_1^+, \mathbb{R}^2)$, $\widetilde{w} \in H^1(B_1^+)$, then $\tau_{1,s}(\vartheta \widetilde{\varphi}) \in H^1_{\Gamma_1^+}(B_1^+, \mathbb{R}^2)$ and $\tau_{1,s}(\vartheta \widetilde{w}) \in H^1_{\Gamma_1^+}(B_1^+)$.

We start by evaluating the bilinear form $\widetilde{a}_+((\cdot, \cdot), (\cdot, \cdot))$ defined in (6.15) with $\widetilde{\varphi}$, $\widetilde{w}$ replaced by $\tau_{1,s}(\vartheta \widetilde{\varphi})$, $\tau_{1,s}(\vartheta \widetilde{w})$, respectively. Next, we elaborate the expression of $\widetilde{a}_+((\cdot, \cdot), (\cdot, \cdot))$ and, by integration by parts, we move the difference operator in direction y_1 from the functions $\vartheta \widetilde{\varphi}$, $\vartheta \widetilde{w}$ to the functions $\widetilde{\psi}$, $\widetilde{v}$. After these calculations, we can write

$$\widetilde{a}_+((\tau_{1,s}(\vartheta \widetilde{\varphi}), \tau_{1,s}(\vartheta \widetilde{w})), (\widetilde{\psi}, \widetilde{v})) = -\widetilde{a}_+((\widetilde{\varphi}, \widetilde{w}), (\vartheta \tau_{1,-s} \widetilde{\psi}, \vartheta \tau_{1,-s} \widetilde{v})) + \widetilde{r}, \tag{8.2}$$

where the remainder $\widetilde{r}$ can be estimated as follows

$$|\widetilde{r}| \leq C \left(\|\widetilde{\varphi}\|_{H^1(B_1^+)} + \|\widetilde{w}\|_{H^1(B_1^+)} \right) \left(\|\nabla \widetilde{\psi}\|_{L^2(B_1^+)} + \|\nabla \widetilde{v}\|_{L^2(B_1^+)} \right), \tag{8.3}$$

where the constant $C > 0$ depends on M_0, $\|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})}$ and $\|S\|_{C^{0,1}(\overline{\Omega})}$ only. It should be noticed that a constructive Poincaré inequality for functions belonging to $H^1(B_1^+)$ and vanishing on the portion Γ_1^+ of the boundary of B_1^+ has been used in obtaining (8.3), see, for example, [10].

Since $\widetilde{\psi} \in H^1_{\Gamma_1^+}(B_1^+, \mathbb{R}^2)$, $\widetilde{v} \in H^1_{\Gamma_1^+}(B_1^+)$, the functions $\vartheta \tau_{1,-s} \widetilde{\psi}$, $\vartheta \tau_{1,-s} \widetilde{v}$ are test functions in the weak formulation (6.14), so that the opposite of the first term on the right hand side of (8.2) can be written as

$$\widetilde{a}_+((\widetilde{\varphi}, \widetilde{w}), (\vartheta \tau_{1,-s} \widetilde{\psi}, \vartheta \tau_{1,-s} \widetilde{v})) = \widetilde{F}_+(\vartheta \tau_{1,-s} \widetilde{\psi}, \vartheta \tau_{1,-s} \widetilde{v}) \tag{8.4}$$

and, by using trace inequalities, we have

$$|\widetilde{F}_+(\vartheta \tau_{1,-s} \widetilde{\psi}, \vartheta \tau_{1,-s} \widetilde{v})| \leq C \left(\|\widetilde{\mathcal{Q}}\|_{H^{\frac{1}{2}}(\Gamma_1)} \cdot \|\nabla \widetilde{v}\|_{L^2(B_1^+)} + \|\widetilde{\mathcal{M}}\|_{H^{\frac{1}{2}}(\Gamma_1)} \cdot \|\nabla \widetilde{\psi}\|_{L^2(B_1^+)} \right), \tag{8.5}$$

where $C > 0$ only depends on M_0. By (8.2)–(8.5) we have

$$\widetilde{a}_+((\tau_{1,s}(\vartheta \widetilde{\varphi}), \tau_{1,s}(\vartheta \widetilde{w})), (\widetilde{\psi}, \widetilde{v})) \leq$$

$$\leq C \left(\|\widetilde{\mathcal{Q}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\mathcal{M}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\varphi}\|_{H^1(B_1^+)} + \|\widetilde{w}\|_{H^1(B_1^+)} \right) \cdot$$

$$\cdot \left(\|\nabla \widetilde{v}\|_{L^2(B_1^+)} + \|\nabla \widetilde{\psi}\|_{L^2(B_1^+)} \right), \tag{8.6}$$

for every $(\widetilde{\psi}, \widetilde{v}) \in H^1_{\Gamma_1^+}(B_1^+, \mathbb{R}^2) \times H^1_{\Gamma_1^+}(B_1^+)$, where $C > 0$ only depends on M_0, $\|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})}$ and $\|S\|_{C^{0,1}(\overline{\Omega})}$.

We choose in (8.6) the test functions

$$\widetilde{\psi} = \tau_{1,s}(\vartheta\widetilde{\varphi}), \quad \widetilde{v} = \tau_{1,s}(\vartheta\widetilde{w}). \tag{8.7}$$

The next step consists in estimating from below the quadratic form $\widetilde{a}_+((\cdot,\cdot),(\cdot,\cdot))$. To perform this estimate, we write

$$\widetilde{a}_+((\tau_{1,s}(\vartheta\widetilde{\varphi}), \tau_{1,s}(\vartheta\widetilde{w})), (\tau_{1,s}(\vartheta\widetilde{\varphi}), \tau_{1,s}(\vartheta\widetilde{w})) = \widetilde{a}_+^{\widetilde{\mathbb{P}}}(\tau_{1,s}(\vartheta\widetilde{\varphi})) + \widetilde{a}_+^{\widetilde{S}}(\tau_{1,s}(\vartheta\widetilde{\varphi}), \tau_{1,s}(\vartheta\widetilde{w})), \tag{8.8}$$

where

$$\widetilde{a}_+^{\widetilde{\mathbb{P}}}(\tau_{1,s}(\vartheta\widetilde{\varphi})) = \int_{B_1^+} \widetilde{\mathbb{P}}\nabla(\tau_{1,s}(\vartheta\widetilde{\varphi})) \cdot \nabla(\tau_{1,s}(\vartheta\widetilde{\varphi})), \tag{8.9}$$

$$\widetilde{a}_+^{\widetilde{S}}(\tau_{1,s}(\vartheta\widetilde{\varphi}), \tau_{1,s}(\vartheta\widetilde{w})) =$$
$$= \int_{B_1^+} \widetilde{S}\left(\tau_{1,s}(\vartheta\widetilde{\varphi}) + L^T\nabla(\tau_{1,s}(\vartheta\widetilde{w}))\right) \cdot \left(\tau_{1,s}(\vartheta\widetilde{\varphi}) + L^T\nabla(\tau_{1,s}(\vartheta\widetilde{w}))\right). \tag{8.10}$$

By (6.24), the matrix $\widetilde{S}$ is definite positive, and then $\widetilde{a}_+^{\widetilde{S}}(\cdot,\cdot)$ can be easily estimated from below as follows

$$\widetilde{a}_+^{\widetilde{S}}(\tau_{1,s}(\vartheta\widetilde{\varphi}), \tau_{1,s}(\vartheta\widetilde{w})) \geq C \int_{B_1^+} |\tau_{1,s}(\vartheta\widetilde{\varphi}) + L^T\nabla(\tau_{1,s}(\vartheta\widetilde{w}))|^2, \tag{8.11}$$

where $C > 0$ only depends on M_0 and σ_0.

The fourth order tensor $\widetilde{\mathbb{P}}$ neither has the minor symmetries nor is strongly convex. Then, in order to estimate from below $\widetilde{a}_+^{\widetilde{\mathbb{P}}}(\tau_{1,s}(\vartheta\widetilde{\varphi}))$, we found convenient apply the inverse transformation $\mathcal{T}_{(j)}^{-1}$ (see (6.11)) and use the strong convexity of the tensor $\mathbb{P}$. To simplify the notation, let $\widetilde{f} \equiv \tau_{1,s}(\vartheta\widetilde{\varphi}) \in H^1_{\Gamma_1^+}(B_1^+, \mathbb{R}^2)$. We have

$$\widetilde{a}_+^{\widetilde{\mathbb{P}}}(\widetilde{f}) = \int_{B_1^+} \widetilde{\mathbb{P}}(y)\nabla_y\widetilde{f} \cdot \nabla_y\widetilde{f}\,dy = \int_{\Omega_j\cap\Omega} \mathbb{P}(x)\nabla_x f \cdot \nabla_x f\,dx \geq C \int_{\Omega_j\cap\Omega} |\widehat{\nabla}_x f|^2 dx, \tag{8.12}$$

where $f(x) = \widetilde{f}(\mathcal{T}_{(j)}(x))$ and $C > 0$ is a constant only depending on ξ_0. By Korn's inequality on $H^1_{\Gamma_1^+}(B_1^+, \mathbb{R}^2)$ (see, for example, Theorem 5.7 in [5]) and by the change of variables $y = \mathcal{T}_{(j)}(x)$, we have

$$\int_{\Omega_j\cap\Omega} |\widehat{\nabla}_x f|^2 dx \geq C \int_{\Omega_j\cap\Omega} |\nabla_x f|^2 dx = \int_{B_1^+} |\nabla_y\widetilde{f}L|^2\iota^{-1}dy \geq C' \int_{B_1^+} |\nabla_y\widetilde{f}|^2 dy, \tag{8.13}$$

where $C' > 0$ only depends on M_0, and in the last step we have taken into account that the matrix L is nonsingular. Then, by (8.12) and (8.13), we have

$$\widetilde{a}_+^{\widetilde{\mathbb{P}}}(\tau_{1,s}(\vartheta\widetilde{\varphi})) \geq C \int\limits_{B_1^+} |\nabla(\tau_{1,s}(\vartheta\widetilde{\varphi}))|^2, \tag{8.14}$$

where $C > 0$ only depends on M_0 and ξ_0. Now, by inserting the estimates (8.11) and (8.14) in (8.6), with $\widetilde{\psi}, \widetilde{v}$ as in (8.7), and by Poincaré's inequality in $H_{\Gamma_1^+}^1(B_1^+)$, we have

$$\|\nabla(\tau_{1,s}(\vartheta\widetilde{\varphi}))\|_{L^2(B_1^+)} + \|\tau_{1,s}(\vartheta\widetilde{\varphi}) + L^T\nabla(\tau_{1,s}(\vartheta\widetilde{w}))\|_{L^2(B_1^+)} \leq$$

$$\leq C \left(\|\widetilde{\mathcal{Q}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\mathcal{M}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\varphi}\|_{H^1(B_1^+)} + \|\widetilde{w}\|_{H^1(B_1^+)} \right) \tag{8.15}$$

where $C > 0$ only depends on M_0, ξ_0, σ_0, $\|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})}$ and $\|S\|_{C^{0,1}(\overline{\Omega})}$. Taking the limit as $s \to 0$ and by the definition of the function ϑ, we have

$$\left\| \frac{\partial}{\partial y_1}\nabla\widetilde{\varphi} \right\|_{L^2(B_\rho^+)} + \left\| \frac{\partial\widetilde{\varphi}}{\partial y_1} + L^T\frac{\partial}{\partial y_1}\nabla\widetilde{w} \right\|_{L^2(B_\rho^+)} \leq$$

$$\leq C \left(\|\widetilde{\mathcal{Q}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\mathcal{M}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\varphi}\|_{H^1(B_1^+)} + \|\widetilde{w}\|_{H^1(B_1^+)} \right) \tag{8.16}$$

where $C > 0$ only depends on M_0, ξ_0, σ_0, $\|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})}$ and $\|S\|_{C^{0,1}(\overline{\Omega})}$. Therefore, the tangential derivatives $\frac{\partial}{\partial y_1}\nabla\widetilde{\varphi}$, $\frac{\partial}{\partial y_1}\nabla\widetilde{w}$ exist and belong to $L^2(B_\rho^+)$.

Second step. (Estimate of the normal derivatives)

To obtain an analogous estimate of the normal derivatives $\frac{\partial}{\partial y_2}\nabla\widetilde{\varphi}$, $\frac{\partial}{\partial y_2}\nabla\widetilde{w}$ we need to prove the following two facts:

$$\left| \int\limits_{B_\rho^+} \frac{\partial\widetilde{\varphi}_r}{\partial y_2}\frac{\partial\widetilde{\psi}}{\partial y_2} \right| \leq C\|\widetilde{\psi}\|_{L^2(B_\rho^+)}, \quad \text{for every } \widetilde{\psi} \in C_0^\infty(B_\rho^+), \ r = 1, 2, \tag{8.17}$$

$$\left| \int\limits_{B_\rho^+} \frac{\partial\widetilde{w}}{\partial y_2}\frac{\partial\widetilde{v}}{\partial y_2} \right| \leq C\|\widetilde{v}\|_{L^2(B_\rho^+)}, \quad \text{for every } \widetilde{v} \in C_0^\infty(B_\rho^+), \tag{8.18}$$

for some constant $C > 0$ depending only on the data. Since

$$\widetilde{a}_+((\widetilde{\varphi}, \widetilde{w}), (\widetilde{\psi}, \widetilde{v})) = 0, \quad \text{for every } (\widetilde{\psi}, \widetilde{v}) \in C_0^\infty(B_\rho^+, \mathbb{R}^2) \times C_0^\infty(B_\rho^+), \tag{8.19}$$

by integration by parts we have

$$\int_{B_\rho^+} \mathcal{P}_{ir}\widetilde{\varphi}_{r,2}\widetilde{\psi}_{i,2} + \int_{B_\rho^+} \mathcal{S}_{22}\widetilde{w}_{,2}\,\widetilde{v}_{,2} = \int_{B_\rho^+} \sum_{\substack{i,j,r,s=1 \\ (j,s)\neq(2,2)}}^{2} (\widetilde{P}_{ijrs}\widetilde{\varphi}_{r,s})_{,j}\,\widetilde{\psi}_i -$$

$$-\int_{B_\rho^+}\left(\widetilde{S}\widetilde{\varphi}\cdot\widetilde{\psi} - (\widetilde{S}_{ij}\widetilde{\varphi}_j(L^T)_{ik})_{,k}\,\widetilde{v} + \widetilde{S}(L^T\nabla\widetilde{w})\cdot\widetilde{\psi} - \sum_{\substack{i,j=1 \\ (i,j)\neq(2,2)}}^{2} ((L\widetilde{S}L^T)_{ij}\widetilde{w}_{,j})_{,i}\,\widetilde{v} \right), \quad (8.20)$$

for every $(\widetilde{\psi},\widetilde{v}) \in C_0^\infty(B_\rho^+,\mathbb{R}^2) \times C_0^\infty(B_\rho^+)$, where

$$\mathcal{P}_{ir} = \widetilde{P}_{i2r2}, \ i,r = 1,2, \quad \mathcal{S}_{22} = (L\widetilde{S}L^T)_{22}. \quad (8.21)$$

By the properties (6.22)–(6.23) of $\widetilde{\mathbb{P}}$ and the definite positiveness of $\widetilde{S}$ (see (6.24)), the matrix $(\mathcal{P}_{ir})_{i,r=1,2}$ is symmetric and positive definite and $\mathcal{S}_{22} > 0$.

Let $\widetilde{v} = 0$ in (8.20). Then, by using estimate (8.16) we have

$$\left| \int_{B_\rho^+} \mathcal{P}_{ir}\widetilde{\varphi}_{r,2}\widetilde{\psi}_{i,2} \right| \le$$

$$\le C\left(\|\widetilde{\mathcal{Q}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\mathcal{M}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\varphi}\|_{H^1(B_1^+)} + \|\widetilde{w}\|_{H^1(B_1^+)} \right) \|\widetilde{\psi}\|_{L^2(B_\rho^+)}, \quad (8.22)$$

for every $\widetilde{\psi} \in C_0^\infty(B_\rho^+)$, where the constant $C > 0$ only depends on M_0, ξ_0, σ_0, $\|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})}$ and $\|S\|_{C^{0,1}(\overline{\Omega})}$. This inequality implies the existence in $L^2(B_\rho^+)$ of the derivative $\frac{\partial}{\partial y_2}\left(\sum_{r=1}^{2} \mathcal{P}_{ir}\widetilde{\varphi}_{r,2} \right)$, $i = 1, 2$. Then, it is easy to see that this condition implies $\frac{\partial^2\widetilde{\varphi}_r}{\partial y_2^2} \in L^2(B_\rho^+)$, $r = 1, 2$.

Similarly, choosing $\widetilde{\psi} = 0$ in (8.20) we have

$$\left| \int_{B_\rho^+} \mathcal{S}_{22}\widetilde{w}_{,2}\,\widetilde{v}_{,2} \right| \le$$

$$\le C\left(\|\widetilde{\mathcal{Q}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\mathcal{M}}\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|\widetilde{\varphi}\|_{H^1(B_1^+)} + \|\widetilde{w}\|_{H^1(B_1^+)} \right) \|\widetilde{v}\|_{L^2(B_\rho^+)}, \quad (8.23)$$

for every $\widetilde{v} \in C_0^\infty(B_\rho^+)$, where the constant $C > 0$ only depends on M_0, ξ_0, σ_0, $\|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})}$ and $\|S\|_{C^{0,1}(\overline{\Omega})}$. As before, this condition implies the existence in $L^2(B_\rho^+)$ of $\frac{\partial^2\widetilde{w}}{\partial y_2^2}$.

Finally, from (8.22) and (8.23), the L^2-norm of $\frac{\partial^2\widetilde{\varphi}_r}{\partial y_2^2}$, $r = 1, 2$, and $\frac{\partial^2\widetilde{w}}{\partial y_2^2}$ can be estimated in terms of known quantities, and the proof of Theorem 6.3 is complete.

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