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Some nonlocal nonlinear problems in the stationary and evolutionary case

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Introduction

The aim of this Ph.D. thesis is to present new results on existence, multiplicity and qualitative aspects of solutions of problems governed by nonlocal elliptic p–Laplacian operators. The whole work is based on the published articles [77, 78, 79, 50]. Moreover, the paper [78] has been further extended in Chapter 3.

In the whole thesis, we denote with $\mathcal{L}_K$ a general integro–differential nonlocal operator, defined pointwise by

$$\mathcal{L}_K\varphi(x) = -\int_{\mathbb{R}^n} |\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))K(x - y)dy, \tag{I.1}$$

along any function $\varphi \in C_0^\infty(\Omega)$, where Ω is either $\mathbb{R}^n$ or any open bounded domain of $\mathbb{R}^n$, with Lipschitz boundary. The *weight* $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ satisfies some natural restrictions, listed in Chapter 1.

When $K(x) = |x|^{-(n+ps)}$, the operator $-\mathcal{L}_K$ reduces to the more familiar fractional p–Laplacian operator $(-\Delta)_p^s$, which up to a multiplicative constant depending only on n, s and p, is defined by

$$(-\Delta)_p^s\varphi(x) = \int_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x - y|^{n+ps}}dy,$$

along any function $\varphi \in C_0^\infty(\Omega)$, see also the recent monograph [70] by *Molica Bisci*, *Radulescu* and *Servadei*.

Nonlocal and fractional operators arise in a quite natural way in many different applications, such as continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastically stabilization of *Lévy* processes, see for instance [4, 24]. We refer also to [3, 11, 17, 18, 19, 26, 27, 43, 44, 48, 53, 54, 56, 57, 65, 68, 71, 76, 86, 88, 89, 90, 92, 93, 94] and the references therein.

The first question, treated in Chapter 2, is the existence of two nontrivial weak solutions of a one parameter nonlocal eigenvalue problem under homogeneous Dirichlet boundary conditions in bounded domains.

In the paper [5], *Arcoya* and *Carmona* extended to a wide class of functionals the three critical point theorem of *Pucci* and *Serrin* in [81] (see also [80]) and applied it to a one parameter family of functionals J_λ, with $\lambda \in I \subset \mathbb{R}$. Under suitable assumptions, they located an open subinterval of values λ in I for which J_λ possesses at least three critical points. Recently, a slight variant of the main abstract theorem of [5] has been proposed in [36], where the authors considered the problem

$$\text{(I.2)} \qquad \begin{cases} -\operatorname{div}\mathbf{A}(x,\nabla u) = \lambda[a(x)|u|^{p-2}u + f(x,u)] & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{A}$ satisfies some natural structural conditions which are satisfied by the local p–Laplacian operator.

In both papers [5, 36] several interesting applications to quasilinear boundary value problems are given. Taking inspiration from [36], in [77] we establish the existence of two nontrivial weak solutions of a one parameter eigenvalue problem, set in a bounded open domain Ω of $\mathbb{R}^n$, with Lipschitz boundary, under homogeneous Dirichlet boundary conditions. These results are presented in Chapter 2. More precisely, we consider the problem

$$(\mathscr{P}_1) \qquad \begin{cases} -\mathcal{L}_K u = \lambda[a(x)|u|^{p-2}u + f(x,u)] & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The coefficient a *is a positive weight of class* $L^\alpha(\Omega)$, with $\alpha > n/ps$, and the perturbation $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function*, with $f \not\equiv 0$, satisfying some natural assumptions, listed in Section 2.2.

Indeed, we determine precisely the intervals of λ's for which problem $(\mathscr{P}_1)$ admits only the trivial solution and for which $(\mathscr{P}_1)$ has at least two nontrivial solutions. In particular, we study $(\mathscr{P}_1)$ via a slight variant of the *Arcoya* and *Carmona* result in [5], as proved in Theorem 2.1 of [36].

While in [36] the main results were related to a problem driven by an operator whose prototype is the p–Laplacian, in [77] we extend these results to a problem driven by an integro–differential nonlocal operator, whose prototype is the *fractional* p–Laplacian.

Another new and important result of independent interest is Proposition 2.1.1 in Section 2.1, related to the first eigenvalue of the non–perturbed problem

$$\begin{cases} -\mathcal{L}_K u = \lambda\, a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The first eigenvalue λ_1 is defined in Section 2.1, by the infimum of a Rayleigh quotient. In [53] and [65] the authors proved that the infimum is achieved and that $\lambda_1 > 0$, when $a \equiv 1$. The special linear case of the fractional Laplacian and $a \in \mathrm{Lip}(\overline{\Omega})$ is treated in [51]. In [77] we extend the previous papers and prove the result for the general weight a, using a completely different argument.

In Chapter 3, based on the results published in [78], we establish existence and multiplicity of nontrivial non–negative entire weak solutions of a stationary Kirchhoff eigenvalue problem, involving a general nonlocal integro–differential operator. The model under consideration depends on a real parameter λ and involves two superlinear nonlinearities, one of which could be critical or even supercritical.

In recent years stationary Kirchhoff problems have been widely studied. We refer to [3, 32, 45, 46, 63] for problems involving the classical Laplace operator, to [7, 35] for the p–Laplacian case and to [97] for Kirchhoff models with critical nonlinearities. For evolution problems we refer to [6, 8, 15] and the references therein. More recently, following [25] *Fiscella* and *Valdinoci* in [52] proposed a stationary Kirchhoff variational model, in bounded regular domains of $\mathbb{R}^n$, which takes into account the *nonlocal* aspect of the tension arising from nonlocal measurements of the fractional length of the string. In [3, 45, 52] the authors use variational methods, as well as a concentration compactness argument. In [32, 63] variational methods are still used, but the stationary Kirchhoff problems are set in the entire $\mathbb{R}^n$. In [7, 35] the so–called *degenerate* case is covered (see also [8, 15, 97]), that is the main Kirchhoff non–negative non–decreasing function M could be zero at 0, while in [52] only the *non–degenerate* case is covered. Lately, several papers have been devoted to problems involving critical nonlinearities and nonlocal elliptic operators; see [17, 18, 71, 89, 90, 92, 97] in bounded regular domains of $\mathbb{R}^n$ and [11, 63] in all $\mathbb{R}^n$, and the references therein. We refer to [94] for quasilinear Kirchhoff systems involving the fractional p–Laplacian.

In [11], *Autuori* and *Pucci* considered the problem

$$(-\Delta)^s u + a(x)u = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^n, \tag{I.3}$$

which has been further extended by *Pucci* and *Zhang* in [87] and then by *Xiang*, *Zhang* and *Radulescu* in [93].

Inspired by the above articles and the fact that several interesting questions arise from the search of nontrivial non–negative weak solutions, in [78] we deal with existence and multiplicity of nontrivial non–negative entire solutions of a Kirchhoff eigenvalue problem, involving critical nonlinearities and nonlocal elliptic operators, when $p = 2$. In Chapter 3, we further extend the results of [78] and consider the problem

$$(\mathscr{P}_2) \qquad \begin{aligned} &M\left([u]_K^p\right)(-\mathcal{L}_K u) = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \qquad \text{in } \mathbb{R}^n,\\ &[u]_K^p = \iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^p K(x-y)\,dxdy, \end{aligned}$$

where $\lambda \in \mathbb{R}$, $0 < s < 1$, $ps < n$ and $\mathcal{L}_K$ is the integro–differential nonlocal operator defined in (I.1).

More precisely, our model generalizes problem (I.3) proposed in [11], where the fractional Laplacian operator is considered and the Kirchhoff function is $M(\tau) \equiv 1$ for any $\tau \in \mathbb{R}_0^+$, as well as in [87, 93]. Several technical difficulties arise from the Kirchhoff structure of problem $(\mathscr{P}_2)$. Thus, for this reason, we only consider the non–degenerate case.

In Chapter 4, following essentially the results established in [79], we deal with the question of the asymptotic stability of solutions of Kirchhoff systems, governed by the fractional p–Laplacian operator, with an external force and nonlinear damping terms.

Recently, *Cavalcanti*, *Domingos Cavalcanti*, *Jorge Silva* and *Webler* proposed in [28] the following model for the damped wave equation with a degenerate nonlocal weak damping

$$\text{(I.4)} \qquad \begin{cases} u_{tt} - \Delta u + f(u) + M\left(\int_\Omega |\nabla u|^2 dx\right) u_t = 0 & \text{in } \Omega \times (0,\infty),\\ u = 0 & \text{on } \partial\Omega \times (0,\infty),\\ u(\cdot,0) = u_0, \;\; u_t(\cdot,0) = u_1 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, f is a nonlinear source and the Kirchhoff function M corresponds to a nonlocal damping coefficient, since it is multiplied by u_t and not by the Laplacian operator. This kind of nonlocal dissipative effect was introduced by *Lange* and *Perla Menzala* in [62] for the beam equation. The authors considered the model

$$\text{(I.5)} \qquad u_{tt} + \Delta^2 u + M(\|\nabla u(t,\cdot)\|_2^2)u_t = 0,$$

where the function $M : \mathbb{R}_0^+ \to [1,\infty)$ is assumed to be of class C^1 satisfying the condition $M(\tau) \geq \tau + 1$ for all $\tau \geq 0$. The nonlinear term in (I.5) has a dissipative effect, which implies the decay of solutions. The authors remarked that problem (I.5) is closely related to the nonlinear dissipative Schrödinger equation

$$\text{(I.6)} \qquad iw_t + \Delta w + iM(\|\nabla \mathrm{Im}\, w\|_2^2)\mathrm{Re}\, w = 0,$$

where $x \in \mathbb{R}^n$, $t \geq 0$ and $i = \sqrt{-1}$. As a consequence, if w is a smooth solution of (I.6), than the imaginary part $u = \operatorname{Im} w$ of w solves (I.5). Another problem involving this kind of nonlocal dissipative effect was studied by *Cavalcanti*, *Domingos Cavalcanti* and *Ma* in another context, see [29].

Problems, concerning the stability of damped wave models, have been widely studied. We refer to [13, 20, 34, 66, 72, 84, 91, 96] for nonlinear damped equations, to [60] for a plate equation with nonlocal weak damping and to [58, 64, 74, 75] for semilinear damped wave equations.

In [9], *Autuori* and *Pucci* dealt with the question of global and local asymptotic stability, as time tends to infinity, of solutions of dissipative anisotropic Kirchhoff systems, governed by the $p(x)$–Laplacian operator, in the framework of the variable exponent Sobolev spaces. In mechanical engineering, we often encounter structure composed of rigid and elastic components. The flexible parts are of course sensitive of disturbances and inserting an internal dissipation can lead to satisfactory results. Similar considerations motivated the authors to consider the problem

$$\text{(I.7)} \qquad \begin{cases} u_{tt} - M(\mathscr{I}u(t))(\Delta_{p(x)}u + g(t)\Delta_{p(x)}u_t) \\ \quad +\mu|u|^{p(x)-2}u + Q(t,x,u,u_t) + f(t,x,u) = 0 & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}_0^+ \times \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u = (u_1, \ldots, u_N) = u(t,x)$, with $N \geq 1$. The term $\mu|u|^{p(x)-2}u$ represents a perturbation, $\mathscr{I}u(t)$ is the natural associated $p(x)$–Dirichlet energy integral and M is a dissipative Kirchhoff function. Finally, f is an external force, Q is a distribute damping and the function $g \geq 0$ is in $L^1_{\text{loc}}(\mathbb{R}_0^+)$.

In [79], taking inspiration of [9, 84, 85], we generalize the model proposed in [28], considering the following system

$$(\mathscr{P}_3) \quad \begin{cases} u_{tt} + (-\Delta)^s_p u + \mu|u|^{p-2}u + \varrho(t)M([u]^p_{s,\Omega})|u_t|^{p-2}u_t \\ \qquad +Q(t,x,u,u_t) + f(t,x,u) = 0 & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}_0^+ \times (\mathbb{R}^n \setminus \Omega), \end{cases}$$

where $n > ps$ and $u = (u_1, \ldots, u_N) = u(t,x)$ represents the vectorial displacement, with $N \geq 1$. The term $\mu|u|^{p-2}u$, where $\mu \geq 0$, plays the role of a *perturbation*, M is a dissipative Kirchhoff function and $\varrho \geq 0$ is in $L^1_{\text{loc}}(\mathbb{R}_0^+)$.

Problem $(\mathscr{P}_3)$, presented in Chapter 4, is related to (I.7), since the perturbation, the damping term and the external force are structurally similar, but there is a great difference concerning the Kirchhoff function. Indeed, as in [28], in our model the function M corresponds to a nonlocal damping coefficient and it is not multiplied by the fractional p–Laplacian operator. Furthermore, while in [28] the authors considered an equation in the special case $p = 2$, our system involves more general source and damping terms and it is driven by the fractional p–Laplacian. For this reason, we are not able to establish a local exponential stability as in [28], but we obtain local and global stability results in different regions of the potential valley, using completely different arguments than in [28].

Moreover, we also treat the degenerate case of $(\mathscr{P}_3)$, that is, from a physical point of view, when the base tension of the string modeled by the equation is zero. Indeed, in Sections 4.3 and 4.4 we prove the main results of global and local asymptotic stability without assuming the non–degeneracy of the problem $(\mathscr{P}_3)$. This fact represents a generalization of [9], where the non–degeneracy of the problem was assumed in the whole section concerning the local asymptotic stability. However, in some applications, where specified, we have to require non–degeneracy in order to overcome some technical difficulties, due to the Kirchhoff structure of $(\mathscr{P}_3)$.

Finally, as regarding the structural assumptions on M, while in [28] the authors considered the case $M(\tau) = \tau$ for $\tau \geq 0$, here we do not need to assume that M is linear and we do not impose even that M is non–decreasing, as assumed in several other papers in the subject.

Particular attention is devoted to the asymptotic behavior of the solutions in the linear case of $(\mathscr{P}_3)$. In order to simplify the notation we restrict the interval of the time variable to $I = [1, \infty)$ instead of $\mathbb{R}^+_0$.

In Section 4.5, following [84, Section 5], we consider an important special case of $(\mathscr{P}_3)$, that is $p = 2$, $Q(t, x, u, v) = a(t)t^\alpha v$, with a satisfying

$$1/C \leq a(t) \leq C \qquad \text{in } I$$

for some $C > 0$ and $\alpha \in \mathbb{R}$, and $f(t, x, u) = V(t, x)u$, where V is a bounded continuous function in $I \times \Omega$. In other words, we study the asymptotic behavior of the solutions of

$$(\mathscr{P}_{3,\text{lin}}) \qquad \begin{cases} u_{tt} + (-\Delta)^s u + \mu u + \varrho(t)M([u]^2_{s,\Omega})u_t \\ \qquad +a(t)t^\alpha u_t + V(t, x)u = 0 & \text{in } I \times \Omega, \\ u(t, x) = 0 & \text{on } I \times (\mathbb{R}^n \setminus \Omega), \end{cases}$$

where for simplicity we treat only the scalar displacement, that is the case when $N = 1$.

If $|\alpha| \leq 1$ and ϱ is sublinear for t sufficiently large, we get the stability of the solutions of $(\mathscr{P}_{3,\text{lin}})$. A more delicate argument is necessary when either $\alpha < -1$ or $\alpha > 1$. For this reason, we suppose that M is a constant positive function and that $V(t,x) = V(x) > -\mu$ a.e. in Ω when $|\alpha| > 1$. Here the proof techniques rely on an important result of independent interest related to properties of the eigenvalues and the eigenfunctions of the underlying perturbed problem of $(\mathscr{P}_{3,\mathrm{lin}})$, that is of

$$(\mathscr{P}_\lambda) \qquad \begin{cases} (-\Delta)^s u + a_0(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $a_0(x)$ is a bounded non–negative continuous function, with $a_0 > 0$ a.e. in Ω. Indeed, in the main application for $(\mathscr{P}_{3,\mathrm{lin}})$ the function $a_0(x)$ is exactly $V(x) + \mu$.

To this aim in the Appendix of Chapter 4, following the proof of Proposition 3.1 of the monograph [70] by *Molica Bisci*, *Radulescu* and *Servadei*, see also Proposition 9 of [88] by *Servadei* and *Valdinoci*, for a related but different problem, we prove Theorem 4.7.3. In particular, Theorem 4.7.3 gives that the first eigenvalue of $(\mathscr{P}_\lambda)$ is positive and that the eigenfunctions are a basis of the natural solutions space of $(\mathscr{P}_\lambda)$, which is defined in (4.7.1) and in Section 1.2.

Finally, in Chapter 5 we deal with the existence of nontrivial nonnegative solutions of Schrödinger–Hardy systems driven by two possibly different fractional $\wp$–Laplacian operators, via various variational methods, as recently treated in [50]. The main features of the paper [50] are the presence of the Hardy terms and the fact that the nonlinearities do not necessarily satisfy the Ambrosetti–Rabinowitz condition.

The starting point in [50], and so in Chapter 5, is the fractional Schrödinger–Hardy system in $\mathbb{R}^n$

$$(\mathscr{P}_4) \qquad \begin{cases} (-\Delta)^s_m u + a(x)|u|^{m-2}u - \mu\dfrac{|u|^{m-2}u}{|x|^{ms}} & = H_u(x,u,v), \\ (-\Delta)^s_p v + b(x)|v|^{p-2}v - \sigma\,\dfrac{|v|^{p-2}v}{|x|^{ps}} & = H_v(x,u,v), \end{cases}$$

where μ and σ are real parameters, $n > ps$, with $s \in (0,1)$ and

$$1 < m \le p < m^* = \frac{mn}{n-ms}.$$

The nonlinearities H_u and H_v denote the partial derivatives of H with respect to the second variable and the third variable, respectively, and H satisfies assumptions (H_1)–(H_4), given in Section 5.1.

A similar problem was recently studied in [95], without the Hardy terms, that is in the case $\mu = \sigma = 0$. In particular, the authors establish the existence of nontrivial nonnegative solutions of the system

$$\begin{cases} (-\Delta)^s_m u + a(x)|u|^{m-2}u = H_u(x,u,v) & \text{in } \mathbb{R}^n, \\ (-\Delta)^s_p v + b(x)|v|^{p-2}v = H_v(x,u,v) & \text{in } \mathbb{R}^n, \end{cases}$$

for which compactness arguments are easier to get than for $(\mathscr{P}_4)$. We recall that a non–negative solution (u,v) is a vector function with all the components non–negative in $\mathbb{R}^n$. Problems, less general but somehow related to $(\mathscr{P}_4)$, can be found in [1, 33, 41, 47, 49, 95].

In [47], the authors study a fractional problem involving a Hardy potential, subcritical and critical nonlinearities, by variational methods, that is

$$\begin{cases} (-\Delta)^s u - \gamma \dfrac{u}{|x|^{2s}} = \lambda u + \theta f(x,u) + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, γ, λ and θ are real parameters, the function f is a subcritical nonlinearity, while g could be either a critical term or a perturbation. The existence and regularity of a solution is provided in [1] for fractional elliptic problems with a Hardy term and different nonlinearities, even singular. By combining a variational approach and the moving plane method, in [41] the authors prove the existence and qualitative properties of a solution for a fractional problem with a critical nonlinearity and still a Hardy potential. In [54], the authors study a fractional equation in $\mathbb{R}^n$, with three critical Hardy–Sobolev nonlinearities. We refer to [27, 48, 49, 76] for existence results concerning different Kirchhoff–Hardy problems and Hardy–Schrödinger–Kirchhoff equations driven by the fractional Laplacian.

Regarding fractional elliptic systems, besides [95], we mention also the recent paper [33], in which the elliptic system presents only a single fractional Laplace operator and critical concave–convex nonlinearities.

Motivated by the above works, we are interested in the study of nontrivial non–negative solutions of system $(\mathscr{P}_4)$ involving two fractional Laplace operators, but without the *Ambrosetti–Rabinowitz* condition. Actually, the *Ambrosetti–Rabinowitz* condition is quite natural and crucial not only to ensure that the Euler–Lagrange functional associated to variational problems has a mountain pass geometry, but also to guarantee that the Palais–Smale sequence constructed in the mountain pass lemma is bounded. Several authors tried to drop the *Ambrosetti–Rabinowitz* condition since the pioneering work of *Jeanjean* [59]; see, e.g., [30, 31, 69] and the references therein.

The proof of Theorem 5.1.1 is mainly variational. Inspired by [95], we apply the version of the mountain pass theorem given in [42]. For this, we have to show that the Euler–Lagrange functional related to $(\mathscr{P}_4)$ satisfies the *Cerami* compactness condition. Because of the lack of compactness, due to the presence of the Hardy terms, this is more delicate to prove than in [95] and a tricky step in the proof is necessary to overcome this new difficulty.

Moreover, we consider systems including critical nonlinear terms as treated very recently in bounded domains in [33, 39, 43, 44, 56, 57, 68] for fractional systems and in [61] for systems driven by the p–Laplacian operator. That is, we study the system in $\mathbb{R}^n$

$$(\mathscr{P}_5)\quad \begin{cases} (-\Delta)^s_m u + a(x)|u|^{m-2}u - \mu\dfrac{|u|^{m-2}u}{|x|^{ms}} = H_u(x,u,v) + |u|^{m^*-2}u^+ \\ \qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta} + \varphi(x), \\ (-\Delta)^s_p v + b(x)|v|^{p-2}v - \sigma\dfrac{|v|^{p-2}v}{|x|^{ps}} = H_v(x,u,v) + |v|^{p^*_s-2}v^+ \\ \qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1} + \psi(x), \end{cases}$$

where $\theta > 1$, $\vartheta > 1$ with $\theta + \vartheta = m^*$, φ is a non–negative perturbation of class $L^{\mathfrak{m}}(\mathbb{R}^n)$, with $\mathfrak{m}$ the Hölder conjugate of m^*, while ψ is a non–negative perturbation of class $L^{\mathfrak{p}}(\mathbb{R}^n)$, with $\mathfrak{p}$ the Hölder conjugate of p^*, that is

$$\mathfrak{m} = \frac{m'n}{n + m's}, \qquad \mathfrak{p} = \frac{p'n}{n + p's}.$$

Problem $(\mathscr{P}_5)$ is a generalization of (1.1) of [61], since we replace the p–Laplacian operator with two different fractional $\wp$–Laplacian operators. Furthermore, $(\mathscr{P}_5)$ extends problem (1.1) of [56], for which the authors prove the existence of a ground state solution. System (1.1) of [56], treated in the special case when $m = p = 2$, is related to $(\mathscr{P}_5)$, but different, since in it there is a term with the critical exponent, but no Hardy terms and no perturbations are present.

The study of $(\mathscr{P}_5)$ in $\mathbb{R}^n$ becomes more difficult than in [33, 39, 43, 44, 56, 57, 61, 68] and has been treated here by a different alternative method, which is however very simple and direct. Indeed, the existence is obtained by local minimization thanks to the perturbation terms.

Finally, we present radial versions of the main theorems and extend the results of Sections 5.3 and 5.4 when the fractional $\wp$–Laplacian operator is replaced by a more general elliptic nonlocal integro–differential operator of the type (I.1), that is generated by a singular kernel K and satisfying the natural assumptions described by *Caffarelli*, e.g., in [24]. See also [48] and the references therein.

The thesis is organized as follows. In Chapter 1 we present some preliminary definitions and results which will be used in the following. In particular, we list the assumptions for the weight K, we present the construction of the functional spaces and we introduce the Kirchhoff function M. In Chapter 2 we report some results already appeared in [77], concerning the existence of two nontrivial weak solutions of $(\mathscr{P}_1)$. Chapter 3 is based on the paper [78], in which only the linear case $p = 2$ is considered. Here we extend the results of [78] and deal with the existence and the multiplicity of nontrivial non–negative entire solutions of $(\mathscr{P}_2)$, when $p = 2$ is replaced by a general exponent $p \in (1, \infty)$. Chapter 4 contains some new results on the asymptotic stability of solutions of the Kirchhoff system $(\mathscr{P}_3)$, given in the paper [79]. Section 4.5 of this chapter is devoted to problem $(\mathscr{P}_{3,\text{lin}})$. In the Appendix of Chapter 4 we deal with the eigenvalue problem $(\mathscr{P}_\lambda)$. Chapter 5 is based on the paper [50], which deals with the existence of nontrivial non–negative solutions of the Schrödinger–Hardy system $(\mathscr{P}_4)$ driven by two possibly different fractional $\wp$–Laplacian operators, via various variational methods. Moreover, in Section 5.4 we consider system $(\mathscr{P}_5)$, including critical nonlinear terms. Finally, in Chapter 6 we present some open problems arising from the papers listed above, which can be useful for future research.

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Chapter 1

Preliminaries

In this chapter, we present some preliminary definitions and results which will be used in the following. Let us remark that papers [77, 79] deal with problems in Ω, that is an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, while papers [78, 50] deal with problems in the whole space $\mathbb{R}^n$.

As stated in the Introduction, in this thesis we consider some problems driven by the fractional p–Laplacian operator, which up to a multiplicative constant depending only on n, s and p is defined by

$$(-\Delta)_p^s \varphi(x) = \int_{\mathbb{R}^n} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{n+ps}} dy,$$

along any function $\varphi \in C_0^\infty(\Omega)$, where $s \in (0,1)$ and $n > ps$. Denote by p^* the critical Sobolev exponent, that is $p^* = np/(n-ps)$.

Our results can also be generalized, considering the case when the fractional p–Laplacian is replaced by a more general nonlocal integro–differential operator $-\mathcal{L}_K$, which up to a multiplicative constant depending only on n, s and p is defined by

$$-\mathcal{L}_K \varphi(x) = \int_{\mathbb{R}^n} |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))K(x-y)dy,$$

along any function $\varphi \in C_0^\infty(\Omega)$.

Unless otherwise specified, the *weight* $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ satisfies the natural restrictions

(K_1) *there exists* $K_0 > 0$ *such that* $K(x)|x|^{n+ps} \geq K_0$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$*;*

(K_2) $mK \in L^1(\mathbb{R}^n)$*, where* $m(x) = \min\{1, |x|^p\}$*,* $x \in \mathbb{R}^n$*.*

Without loss of generality, we can suppose that K is even, since the odd part of K does not give any contribution in the integral above. Clearly, when $K(x) = |x|^{-(n+ps)}$, the operator $-\mathcal{L}_K$ reduces to the more familiar fractional p–Laplacian operator $(-\Delta)^s_p$.

1.1 Functional spaces in the whole $\mathbb{R}^n$

When we study problems in $\mathbb{R}^n$, we consider the space $D^{s,p}(\mathbb{R}^n)$, which denotes the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the Gagliardo norm

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dxdy \right)^{1/p}.$$

The embedding $D^{s,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is continuous, that is

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C_{p^*}[u]_s, \tag{1.1.1}$$

for all $u \in D^{s,p}(\mathbb{R}^n)$, where $C^p_{p^*} = c(n)\dfrac{s(1-s)}{n-ps}$ by Theorem 1 of [67], see also Theorem 1 of [22].

By (K_2) for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ the function

$$(x,y) \mapsto (\varphi(x)-\varphi(y)) \cdot K(x-y)^{1/p} \in L^p(\mathbb{R}^{2n}).$$

Let $D^{s,p}_K(\mathbb{R}^n)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $[\,\cdot\,]_{s,p,K}$, defined by

$$[u]_{s,p,K} = \left(\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^p K(x-y)\, dxdy, \right)^{1/p}.$$

Denote with $\langle \cdot, \cdot \rangle_{s,p,K}$ the duality product

$$\begin{aligned} \langle u, v\rangle_{s,p,K} = \iint_{\mathbb{R}^{2n}} &|u(x)-u(y)|^{p-2}(u(x)-u(y))\cdot \\ &\times (v(x)-v(y))K(x-y)dxdy. \end{aligned} \tag{1.1.2}$$

Clearly by (K_1) the embedding $D^{s,p}_K(\mathbb{R}^n) \hookrightarrow D^{s,p}(\mathbb{R}^n)$ is continuous, being

$$[u]_{s,p} \le K_0^{-1/p}[u]_{s,p,K} \quad \text{for all } u \in D^{s,p}_K(\mathbb{R}^n). \tag{1.1.3}$$

Hence, by (1.1.1) we obtain

$$\|u\|_{p^*} \le C_{p^*}K_0^{-1/p}[u]_{s,p,K} \quad \text{for all } u \in D^{s,p}_K(\mathbb{R}^n). \tag{1.1.4}$$

1.2 Functional spaces in bounded domains Ω

When the problems are set in a bounded open domain Ω of $\mathbb{R}^n$, the construction of the solutions space is more delicate.

We recall that $D_0^{s,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_\Omega}$, where $\|\cdot\|_\Omega$ is the standard fractional Gagliardo norm, given by

$$\|u\|_\Omega = \left(\iint_{\Omega\times\Omega} |u(x) - u(y)|^p |x-y|^{-(n+ps)} dxdy \right)^{1/p}$$

for all $u \in W_0^{s,p}(\Omega)$. Furthermore, $D^{s,p}(\mathbb{R}^n)$ denotes the fractional Beppo–Levi space, that is the completion of $C_0^\infty(\mathbb{R}^n)$, with respect to the norm

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^p |x-y|^{-(n+ps)} dxdy \right)^{1/p}.$$

Moreover, we recall that by Theorems 1 and 2 of [67] we get

$$\begin{aligned} \|u\|^p_{L^{p^*}(\mathbb{R}^n)} &\le c_{n,p} \frac{s(1-s)}{(n-ps)^{p-1}} [u]^p_{s,p}, \\ \int_{\mathbb{R}^n} |u(x)|^p \frac{dx}{|x|^{ps}} &\le c_{n,p} \frac{s(1-s)}{(n-ps)^p} [u]^p_{s,p} \end{aligned} \tag{1.2.1}$$

for all $u \in D^{s,p}(\mathbb{R}^n)$, where $c_{n,p}$ is a positive constant depending only on n and p. Hence

$$D^{s,p}(\mathbb{R}^n) = \{u \in L^{p^*}(\mathbb{R}^n) : |u(x) - u(y)| \cdot |x-y|^{-(s+n/p)} \in L^p(\mathbb{R}^{2n})\}.$$

Following [55], we put

$$\tilde{D}^{s,p}(\Omega) = \{u \in L^{p^*}(\Omega) : \tilde{u} \in D^{s,p}(\mathbb{R}^n)\},$$

with the norm $[u]_{s,\Omega} = [\tilde{u}]_{s,p}$, where $\tilde{u}$ is the natural extension of u in the entire $\mathbb{R}^n$, with value 0 in $\mathbb{R}^n \setminus \Omega$. Clearly,

$$[u]_{s,\Omega} = \left(\|u\|^p_\Omega + 2\int_\Omega |u(x)|^p dx \int_{\mathbb{R}^n\setminus\Omega} |x-y|^{-(n+ps)} dy \right)^{1/p} \ge \|u\|_\Omega.$$

Since here Ω is regular, an application of Theorem 1.4.2.2 of [55] shows that $\tilde{D}^{s,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{[\,\cdot\,]_{s,\Omega}}$. Finally, since Ω is bounded and regular, by (1.2.1) there exists a constant $c_\Omega > 0$ such that

$$c_\Omega \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \le [\tilde{u}]_{s,p} = [u]_{s,\Omega} \le \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)}$$

for all $u \in \tilde{D}^{s,p}(\Omega)$, and so, using also Corollary 1.4.4.10 of [55], we have the main property

$$\begin{aligned}\tilde{D}^{s,p}(\Omega) &= \{u \in W_0^{s,p}(\Omega) \,:\, u\, d(\cdot, \partial\Omega)^{-s} \in L^p(\Omega)\} \\ &= \{u \in D^{s,p}(\mathbb{R}^n) \,:\, u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\} \\ &= \{u \in W^{s,p}(\Omega) \,:\, \tilde{u} \in W^{s,p}(\mathbb{R}^n)\},\end{aligned}$$

where $d(x, \partial\Omega)$ is the distance from x to the boundary $\partial\Omega$ of Ω.

It is not hard to see that $\tilde{D}^{s,p}(\Omega)$ is a closed subspace of $D^{s,p}(\mathbb{R}^n)$. Hence also $\tilde{D}^{s,p}(\Omega) = \big(\tilde{D}^{s,p}(\Omega), [\,\cdot\,]_{s,\Omega}\big)$ is a reflexive Banach space. For simplicity and abuse of notation, in the following we still denote by u the extension of every function $u \in \tilde{D}^{s,p}(\Omega)$, by setting $u = 0$ in $\mathbb{R}^n \setminus \Omega$.

This construction can be adopted to the vectorial case when we deal with systems, as in Chapter 4, Section 4.2.

1.3 The Kirchhoff function

In both Chapters 3 and 4 we deal with problems where a Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is involved. A typical prototype of Kirchhoff function, due to Kirchhoff in 1883, is given by

$$(1.3.1) \quad M(\tau) = a + b\gamma\tau^{\gamma-1} \quad \textit{with } a = M(0) \geq 0, b \geq 0, \gamma > 1 \textit{ and } a + b > 0.$$

Problems involving Kirchhoff functions are said *degenerate* whether the function M can be zero at some point, that is $\inf_{\tau \in \mathbb{R}_0^+} M(\tau) = 0$, and *non–degenerate* when $M(\tau) > 0$ for any $\tau \in \mathbb{R}_0^+$.

From a physical point of view, as noted in [27], in the large literature on Kirchhoff problems the transverse oscillations of a stretched string, with nonlocal flexural rigidity, depends continuously on the Sobolev deflection norm of u via $M([u]_{s,\Omega}^p)$. In any case, M measures the change of the tension on the string caused by the change of its length during the vibration. The presence of the nonlinear coefficient M is crucial to be considered when the changes in tension during the motion cannot be neglected. In the case of linear string vibrations, the tension is constant that is $M(\tau) = M(0)$, but nonlinear vibrations are more realistic.

In this thesis, we assume that M is non–degenerate in problem $(\mathscr{P}_2)$, while in problem $(\mathscr{P}_3)$ we cover in some cases also the degenerate setting. Furthermore, we remark that in $(\mathscr{P}_2)$ the Kirchhoff function multiplies the nonlocal integro–differential operator, while in $(\mathscr{P}_3)$ it represents a nonlocal damping coefficient.

Chapter 2

Problem $(\mathscr{P}_1)$

In this chapter we establish the existence of two nontrivial weak solutions of the following one parameter eigenvalue problem under homogeneous Dirichlet boundary conditions in an open bounded subset Ω of $\mathbb{R}^n$, with Lipschitz boundary

$$(\mathscr{P}_1) \qquad \begin{cases} -\mathcal{L}_K u = \lambda[a(x)|u|^{p-2}u + f(x,u)] & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $-\mathcal{L}_K$ is an integro–differential nonlocal operator, defined as in the Introduction.

Here we assume $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ satisfying the following assumption:

($\mathcal{K}$) *there exists* $s \in (0,1)$*, with* $n > ps$*, and some suitable numbers* ϵ*,* δ*, with* $0 < \epsilon \leq \delta$*, such that*

$$\epsilon \leq K(x)|x|^{n+ps} \leq \delta \text{ for all } x \in \mathbb{R}^n \setminus \{0\}$$

We endow $\tilde{D}^{s,p}(\Omega)$ with the weighted *Gagliardo* norm

$$[u]_{K,\Omega} = \left(\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^p K(x-y) dx dy \right)^{1/p},$$

equivalent to the norm $[\,\cdot\,]_{s,\Omega}$ by virtue of ($\mathcal{K}$). Indeed, ($\mathcal{K}$) implies at once that the function $mK \in L^1(\mathbb{R}^n)$, where $m(x) = \min\{1, |x|^p\}$, so that in particular $[\varphi]_{K,\Omega} < \infty$ for all $\varphi \in C_0^2(\Omega)$.

Hence, also *the natural solution space* $\tilde{D}^{s,p}(\Omega) = \left(\tilde{D}^{s,p}(\Omega), [\,\cdot\,]_{K,\Omega}\right)$ *of* $(\mathscr{P}_1)$ *is a reflexive Banach space.*

Finally, in $(\mathscr{P}_1)$ we assume that the coefficient a *is a positive weight of class* $L^\alpha(\Omega)$, with $\alpha > n/ps$, and that the perturbation $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function*, with $f \not\equiv 0$, satisfying the main assumption $(\mathscr{F})$ of Section 2.1.

In Section 2.4 we determine precisely the intervals of λ's for which problem $(\mathscr{P}_1)$ admits only the trivial solution and for which $(\mathscr{P}_1)$ has at least two nontrivial solutions. More precisely, we study problem $(\mathscr{P}_1)$ by a slight variant of the *Arcoya* and *Carmona* result in [5], as proved in Theorem 2.1 of [36].

2.1 The weight a and the first eigenvalue

We assume a to be a positive weight of class $L^\alpha(\Omega)$, with $\alpha > n/ps$, and K satisfying $(\mathcal{K})$ in $\mathbb{R}^n \setminus \{0\}$.

Note that by Corollary 7.2 of [40] the embedding $\tilde{D}^{s,p}(\Omega) \hookrightarrow L^{\alpha' p}(\Omega)$ is compact, being $\alpha' p < p^*$ by the assumption that $\alpha > n/ps$. Moreover, the embedding $L^{\alpha' p}(\Omega) \hookrightarrow L^p(\Omega, a)$ is continuous, since $\|u\|_{p,a}^p \le \|a\|_\alpha \|u\|_{\alpha' p}^p$ for all $u \in L^{\alpha' p}(\Omega)$ by Hölder's inequality. Hence,

(2.1.1) *the embedding* $\tilde{D}^{s,p}(\Omega) \hookrightarrow L^p(\Omega, a)$ *is compact.*

Let λ_1 be *the first eigenvalue* of the problem

$$\begin{cases} -\mathcal{L}_K u = \lambda\, a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.1.2}$$

in $\tilde{D}^{s,p}(\Omega)$, that is λ_1 is defined by the Rayleigh quotient

$$\lambda_1 = \inf_{u \in \tilde{D}^{s,p}(\Omega),\ u \neq 0} \frac{\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^p K(x-y)\,dxdy}{\int_\Omega a(x)|u|^p dx}. \tag{2.1.3}$$

By Lemma 2.1 of [53] (see also Theorem 5 of [65] for the fractional p–Laplacian first eigenvalue) *the infimum in* (2.1.3) *is achieved and* $\lambda_1 > 0$, *when* $a \equiv 1$. We refer also to [51] for the special linear case of the fractional Laplacian and $a \in \mathrm{Lip}(\overline{\Omega})$. For sake of completeness we prove the result for the general weight a, using a completely different argument.

Proposition 2.1.1. *The infimum* λ_1 *in* (2.1.3) *is positive and attained at a certain function* $u_1 \in \tilde{D}^{s,p}(\Omega)$, *with* $\|u_1\|_{p,a} = 1$ *and* $[u_1]_{K,\Omega}^p = \lambda_1 > 0$. *Moreover,* u_1 *is a solution of* (2.1.2) *when* $\lambda = \lambda_1$.

Proof. Define the functionals $\mathcal{I}(u) = [u]^p_{K,\Omega}$ and $\mathcal{J}(u) = \|u\|^p_{p,a}$, for any $u \in \tilde{D}^{s,p}(\Omega)$. Let $\lambda_0 = \inf\{\mathcal{I}(u)/\mathcal{J}(u) : u \in \tilde{D}^{s,p}(\Omega) \setminus \{0\},\ \|u\|_{p,a} \le 1\}$. Observe that $\mathcal{I}$ and $\mathcal{J}$ are continuously Fréchet differentiable and convex in $\tilde{D}^{s,p}(\Omega)$. Clearly $\mathcal{I}'(0) = \mathcal{J}'(0) = 0$. Moreover, $\mathcal{J}'(u) = 0$ implies $u = 0$. In particular, $\mathcal{I}$ and $\mathcal{J}$ are weakly lower semi–continuous on $\tilde{D}^{s,p}(\Omega)$. Actually, $\mathcal{J}$ is weakly sequentially continuous on $\tilde{D}^{s,p}(\Omega)$. Indeed, if $(u_k)_k$ and u are in $\tilde{D}^{s,p}(\Omega)$ and $u_k \rightharpoonup u$ in $\tilde{D}^{s,p}(\Omega)$, then $u_k \to u$ in $L^p(\Omega, a)$ by (2.1.1). This implies at once that $\mathcal{J}(u_k) = \|u_k\|^p_{p,a} \to \|u\|^p_{p,a} = \mathcal{J}(u)$, as claimed.

Now, either $W = \{u \in \tilde{D}^{s,p}(\Omega) : \mathcal{J}(u) \le 1\}$ is bounded in $\tilde{D}^{s,p}(\Omega)$, or not. In the first case we are done, while in the latter $\mathcal{I}$ is coercive in W, being coercive in $\tilde{D}^{s,p}(\Omega)$. Therefore, all the assumptions of Theorem 6.3.2 of [21] are fulfilled, being $\tilde{D}^{s,p}(\Omega)$ a reflexive Banach space, so that λ_0 is attained at a point $u_1 \in \tilde{D}^{s,p}(\Omega)$, with $\|u_1\|_{p,a} = 1$. We claim now that $\lambda_0 = \lambda_1$. Indeed,

$$\lambda_1 = \inf_{u \in \tilde{D}^{s,p}(\Omega)\setminus\{0\}} \left[\frac{u}{\|u\|_{p,a}}\right]^p_{K,\Omega} = \inf_{\substack{u \in \tilde{D}^{s,p}(\Omega) \\ \|u\|_{p,a}=1}} [u]^p_{K,\Omega} \ge \inf_{\substack{u \in \tilde{D}^{s,p}(\Omega) \\ 0<\|u\|_{p,a}\le 1}} \frac{[u]^p_{K,\Omega}}{\|u\|^p_{p,a}} = \lambda_0 \ge \lambda_1.$$

In particular, $\lambda_1 = [u_1]^p_{K,\Omega} > 0$ and $\mathcal{I}'(u_1) = \lambda_1 \mathcal{J}'(u_1)$ again by Theorem 6.3.2 of [21]. Hence u_1 is a solution of (2.1.2) when $\lambda = \lambda_1$. □

From the proof of Proposition 2.1.1 it is also evident that

$$\lambda_1 = \inf_{\substack{u \in \tilde{D}^{s,p}(\Omega) \\ \|u\|_{p,a}=1}} [u]^p_{K,\Omega}.$$

Moreover Proposition 2.1.1 gives at once that

$$\lambda_1 \|u\|^p_{p,a} \le [u]^p_{K,\Omega} \quad \text{for every } u \in \tilde{D}^{s,p}(\Omega). \tag{2.1.4}$$

In the following we put $c^p_{p,a} = 1/\lambda_1$.

2.2 The perturbation f

On the perturbation f we assume condition

($\mathscr{F}$) *Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, $f \not\equiv 0$, satisfying the following properties.*

(a) *There exist an exponent* $m \in (1,p)$, *two measurable functions* f_0, f_1 *on* Ω *and an appropriate constant* $C_f > 0$ *such that* $0 \leq f_0(x) \leq C_f a(x)$, $0 \leq f_1(x) \leq C_f a(x)$ *a.e. in* Ω *and*

$$|f(x,u)| \leq f_0(x) + f_1(x)|u|^{m-1} \text{ for a.a. } x \in \Omega \text{ and all } u \in \mathbb{R}.$$

(b) *There exists* $\gamma \in (p, p^*/\alpha')$ *such that* $\limsup_{u \to 0} \dfrac{|f(x,u)|}{a(x)|u|^{\gamma-1}} < \infty$, *uniformly a.e. in* Ω.

(c) $\displaystyle\int_\Omega F(x,u_1(x))dx > 0$, *where* $F(x,u) = \displaystyle\int_0^u f(x,v)dv$ *and* u_1 *is the first normalized eigenfunction given in Proposition 2.1.1.*

Note that, in the more familiar and standard setting in the literature, as e.g. in [51, 53, 65], in which $a \in L^\infty(\Omega)$, the exponent γ in $(\mathcal{F})$–(b) belongs to the open interval (p, p^*). In any case $p < p^*/\alpha'$, since $\alpha > n/ps$.

As shown in [36], it is clear from $(\mathscr{F})$–(a) and (b) that problem $(\mathscr{P}_1)$ admits always the trivial solution since $f(x,0) = 0$ a.e. in Ω, and that *the quantity*

$$S_f = \operatorname*{ess\,sup}_{u \neq 0, x \in \Omega} \frac{|f(x,u)|}{a(x)|u|^{p-1}} \tag{2.2.1}$$

is a finite positive number. In particular,

$$\operatorname*{ess\,sup}_{u \neq 0, x \in \Omega} \frac{|F(x,u)|}{a(x)|u|^p} \leq \frac{S_f}{p} \tag{2.2.2}$$

and *the positive number*

$$\lambda_\star = \frac{\lambda_1}{1 + S_f} \tag{2.2.3}$$

is well defined and positive.

2.3 The energy functional

The main result of the section is proved by using the energy functional J_λ associated to $(\mathscr{P}_1)$, which is given by $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$, where

$$\begin{gathered} \Phi(u) = \frac{1}{p}[u]^p_{K,\Omega}, \quad \Psi(u) = -\mathcal{H}(u), \quad \mathcal{H}(u) = \mathcal{H}_1(u) + \mathcal{H}_2(u), \\ \mathcal{H}_1(u) = \frac{1}{p}\|u\|^p_{p,a}, \quad \mathcal{H}_2(u) = \int_\Omega F(x,u(x))dx. \end{gathered} \tag{2.3.1}$$

It is easy to see that the functional J_λ is well defined in $\tilde{D}^{s,p}(\Omega)$ and of class C^1 in $\tilde{D}^{s,p}(\Omega)$. Furthermore, for all u, $\varphi \in \tilde{D}^{s,p}(\Omega)$,

$$\langle J'_\lambda(u), \varphi\rangle = \iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy \\ - \lambda \int_\Omega \left\{a(x)|u(x)|^{p-2}u(x) + f(x,u(x))\right\}\varphi(x)dx,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\tilde{D}^{s,p}(\Omega)$ and its dual space $\tilde{D}^{-s,p'}(\Omega)$. Therefore, the critical points $u \in \tilde{D}^{s,p}(\Omega)$ of the functional J_λ are exactly the weak solutions of problem $(\mathscr{P}_1)$.

Lemma 2.3.1. *The functional* $\Phi : \tilde{D}^{s,p}(\Omega) \to \mathbb{R}$ *is convex, weakly lower semicontinuous and of class* C^1 *in* $\tilde{D}^{s,p}(\Omega)$.

Moreover, $\Phi' : \tilde{D}^{s,p}(\Omega) \to \tilde{D}^{-s,p'}(\Omega)$ *verifies the* $(\mathscr{S}_+)$ *condition, i.e., for every sequence* $(u_k)_k \subset \tilde{D}^{s,p}(\Omega)$ *such that* $u_k \rightharpoonup u$ *weakly in* $\tilde{D}^{s,p}(\Omega)$ *and*

$$\limsup_{k\to\infty} \langle \Phi'(u_k), u_k - u\rangle \le 0, \tag{2.3.2}$$

then $u_k \to u$ *strongly in* $\tilde{D}^{s,p}(\Omega)$.

Proof. A simple calculation shows that the functional Φ is convex and of class C^1 in $\tilde{D}^{s,p}(\Omega)$. Hence, in particular Φ is weakly lower semicontinuous in $\tilde{D}^{s,p}(\Omega)$, see Corollary 3.9 of [23].

Let $(u_k)_k$ be a sequence in $\tilde{D}^{s,p}(\Omega)$ as in the statement. Therefore, we have $\Phi(u) \le \liminf \Phi(u_k)$, being Φ weakly lower semicontinuous in $\tilde{D}^{s,p}(\Omega)$. Furthermore, the linear functional $\langle \Phi'(u), \cdot\rangle : \tilde{D}^{s,p}(\Omega) \to \mathbb{R}$ is in $\tilde{D}^{-s,p'}(\Omega)$, since $(x,y) \mapsto |u(x)-u(y)|^{p-1}|x-y|^{-(n+ps)/p'} \in L^{p'}(\mathbb{R}^{2n})$, so that also $(x,y) \mapsto |u(x)-u(y)|^{p-1}K(x-y)^{1/p'} \in L^{p'}(\mathbb{R}^{2n})$ by $(\mathcal{K})$. Hence, since $u_k \rightharpoonup u$ in $\tilde{D}^{s,p}(\Omega)$ as $k \to \infty$,

$$\langle \Phi'(u), u_k - u\rangle = o(1) \quad \text{as } k \to \infty. \tag{2.3.3}$$

Therefore, $0 \le \limsup_{k\to\infty} \langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle \le 0$ by convexity and (2.3.2). In other words,

$$\lim_{n\to\infty} \langle \Phi'(u_k) - \Phi'(u), u_k - u\rangle = 0. \tag{2.3.4}$$

Combining (2.3.3) with (2.3.4), we get

$$\lim_{k\to\infty} \langle \Phi'(u_k), u_k - u\rangle = 0. \tag{2.3.5}$$

By the convexity of Φ we have $\Phi(u) + \langle \Phi'(u_k), u_k - u\rangle \geq \Phi(u_k)$ for all k, so that $\Phi(u) \geq \limsup_{k\to\infty} \Phi(u_k)$ by (2.3.5). In conclusion,

$$\Phi(u) = \lim_{k\to\infty} \Phi(u_k). \tag{2.3.6}$$

Furthermore (2.3.4) implies that the sequence

$$\begin{aligned} k \mapsto \mathscr{U}_k(x,y) = \big\{ &|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y)) \\ &-|u(x) - u(y)|^{p-2}(u(x) - u(y))\big\} \cdot \\ &\times (u_k(x) - u(x) - u_k(y) + u(y))K(x-y) \end{aligned}$$

converges to 0 in $L^1(\mathbb{R}^{2n})$.

Fix now a subsequence $(u_{k_j})_j$ of $(u_k)_k$. Hence, up to a further subsequence if necessary, $\mathscr{U}_{k_j}(x,y) \to 0$ a.e. in $\mathbb{R}^{2n}$, and so $u_{k_j}(x) - u_{k_j}(y) \to u(x) - u(y)$ for a.a. $(x,y) \in \mathbb{R}^{2n}$. Indeed, fixing x, $y \in \mathbb{R}^n$, with $x \neq y$ and $\mathscr{U}_{k_j}(x,y) \to 0$, and putting $u_{k_j}(x) - u_{k_j}(y) = \xi_j$ and $u(x) - u(y) = \xi$, we get

$$\big(|\xi_j|^{p-2}\xi_j - |\xi|^{p-2}\xi\big)\cdot(\xi_j - \xi) \to 0, \tag{2.3.7}$$

since $K > 0$ by $(\mathcal{K})$. Hence $(\xi_j)_j$ is bounded in $\mathbb{R}$. Otherwise, up to a subsequence,

$$\big(|\xi_j|^{p-2}\xi_j - |\xi|^{p-2}\xi\big)\cdot(\xi_j - \xi) \sim |\xi_j|^p \to \infty,$$

which is obviously impossible. Therefore, $(\xi_j)_j$ is bounded and possesses a subsequence $(\xi_{j_i})_i$, which converges to some $\eta \in \mathbb{R}$. Thus (2.3.7) implies at once that $\big(|\eta|^{p-2}\eta_j - |\xi|^{p-2}\xi\big)\cdot(\eta - \xi) = 0$ and the strict convexity of $t \mapsto |t|^p$ yields $\eta = \xi$. This also shows that actually the entire sequence $(\xi_j)_j$ converges to ξ.

Consider the sequence $(g_{k_j})_j$ in $L^1(\mathbb{R}^{2n})$ defined pointwise by

$$g_{k_j}(x,y) = \left\{ \frac{1}{2}\big(|u_{k_j}(x) - u_{k_j}(y)|^p + |u(x) - u(y)|^p\big) - \left|\frac{u_{k_j}(x) - u_{k_j}(y) - u(x) + u(y)}{2}\right|^p \right\} K(x-y).$$

By convexity $g_{k_j} \geq 0$ and we have $g_{k_j}(x,y) \to |u(x) - u(y)|^p K(x-y)$ for a.a. $(x,y) \in \mathbb{R}^{2n}$ as $k \to \infty$. Therefore, by the Fatou lemma and (2.3.6) we get that

$$\begin{aligned} p\,\Phi(u) \leq \liminf_{j\to\infty} \iint_{\mathbb{R}^{2n}} g_{k_j}(x,y)dxdy = p\,\Phi(u) \\ - \frac{1}{2^p} \limsup_{j\to\infty} \iint_{\mathbb{R}^{2n}} \big|u_{k_j}(x) - u_{k_j}(y) - u(x) + u(y)\big|^p K(x-y)dxdy. \end{aligned}$$

Hence, $\limsup_{j\to\infty}[u_{k_j}-u]_{K,\Omega} \le 0$, that is $\lim_{j\to\infty}[u_{k_j}-u]_{K,\Omega} = 0$. Since $(u_{k_j})_j$ is an arbitrary subsequence of $(u_k)_k$, this shows that actually the entire sequence $(u_k)_k$ converges strongly to u in $\tilde{D}^{s,p}(\Omega)$, as required. □

If $\Psi(v) < 0$ at some $v \in \tilde{D}^{s,p}(\Omega)$, that is $\Psi^{-1}(I_0)$ is non–empty, where $I_0 = (-\infty, 0) = \mathbb{R}^-$, then *the crucial positive number*

$$\lambda^\star = \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi(u)}{\Psi(u)} \tag{2.3.8}$$

is well defined.

Lemma 2.3.2. *If* $(\mathcal{F})$*–*(a)*,* (b) *and* (c) *hold, then* $\Psi^{-1}(I_0)$ *is non–empty and moreover* $\lambda_\star \le \lambda^\star < \lambda_1$*.*

Proof. By $(\mathcal{F})$–(c) it follows that

$$\mathcal{H}(u_1) > \frac{1}{p}, \quad \text{i.e. } u_1 \in \Psi^{-1}(I_0).$$

Hence, $\lambda^\star$ is well defined. Again by $(\mathcal{F})$–(c) and Proposition 2.1.1

$$\lambda^\star = \inf_{u\in\Psi^{-1}(I_0)} -\frac{\Phi(u)}{\Psi(u)} \le \frac{\Phi(u_1)}{\mathcal{H}(u_1)} = \frac{[u_1]^p_{K,\Omega}}{p\mathcal{H}(u_1)} < [u_1]^p_{K,\Omega} = \lambda_1,$$

as required. Finally, by $(\mathcal{F})$–(a), (b), (2.1.4), (2.2.2) and (2.3.1) we have

$$\frac{\Phi(u)}{|\Psi(u)|} \ge \frac{[u]^p_{K,\Omega}}{(1+S_f)\|u\|^p_{p,a}} \ge \frac{\lambda_1}{1+S_f} = \lambda_\star$$

for all $u \in \tilde{D}^{s,p}(\Omega)$, with $u \ne 0$. Hence, in particular $\lambda^\star \ge \lambda_\star$. □

Lemma 2.3.3. *If* $(\mathcal{F})$*–*(a) *holds, then* $\mathcal{H}_1'$*,* $\mathcal{H}_2'$*,* $\Psi' : \tilde{D}^{s,p}(\Omega) \to \tilde{D}^{-s,p'}(\Omega)$ *are compact and* $\mathcal{H}_1$*,* $\mathcal{H}_2$*,* Ψ *are sequentially weakly continuous in* $\tilde{D}^{s,p}(\Omega)$*.*

Proof. Since $\Psi = -\mathcal{H}$, it is enough to prove the lemma for $\mathcal{H}$. Of course, we have $\mathcal{H}' = \mathcal{H}_1' + \mathcal{H}_2'$, where

$$\langle \mathcal{H}_1'(u), v\rangle = \int_\Omega a(x)|u|^{p-2}uv\,dx \quad \text{and} \quad \langle \mathcal{H}_2'(u), v\rangle = \int_\Omega f(x,u)v\,dx,$$

for all $u, v \in \tilde{D}^{s,p}(\Omega)$. Since $\mathcal{H}_1'$ and $\mathcal{H}_2'$ are continuous, thanks to the reflexivity of $\tilde{D}^{s,p}(\Omega)$ it is sufficient to show that $\mathcal{H}_1'$ and $\mathcal{H}_2'$ are weak–to–strong sequentially continuous, i.e. if $(u_k)_k$, u are in $\tilde{D}^{s,p}(\Omega)$ and $u_k \rightharpoonup u$ in $\tilde{D}^{s,p}(\Omega)$ as $k \to \infty$, then we get that $\|\mathcal{H}_1'(u_k) - \mathcal{H}_1'(u)\|_{\tilde{D}^{-s,p'}(\Omega)} \to 0$ and $\|\mathcal{H}_2'(u_k) - \mathcal{H}_2'(u)\|_{\tilde{D}^{-s,p'}(\Omega)} \to 0$ as $k \to \infty$. To this aim, fix $(u_k)_k \subset \tilde{D}^{s,p}(\Omega)$, with $u_k \rightharpoonup u$ in $\tilde{D}^{s,p}(\Omega)$.

From the fact that $u_k \to u$ in $L^p(\Omega, a)$ by (2.1.1), then $\|u_k\|_{p,a} \to \|u\|_{p,a}$, or equivalently, $\|v_k\|_{p',a} \to \|v\|_{p',a}$, where $v_k = |u_k|^{p-2}u_k$ and similarly $v = |u|^{p-2}u$. We claim that $v_k \to v$ in $L^{p'}(\Omega, a)$. Indeed, fix any subsequence $(v_{k_j})_j$ of $(v_k)_k$. The related subsequence $(u_{k_j})_j$ of $(u_k)_k$ converges in $L^p(\Omega, a)$ and admits a subsequence, say $(u_{k_{j_i}})_i$, converging to u a.e. in Ω. Hence, the corresponding subsequence $(v_{k_{j_i}})_i$ of $(v_{k_j})_j$ converges to v a.e. in Ω. Therefore, being $1 < p' < \infty$, by the Clarkson and Mil'man theorems it follows that $v_{k_{j_i}} \rightharpoonup v$ in $L^{p'}(\Omega, a)$, since the sequence $(\|v_k\|_{p',a})_k$ is bounded, and so by Radon's theorem we get that $v_{k_{j_i}} \to v$ in $L^{p'}(\Omega, a)$, since $\|v_k\|_{p',a} \to \|v\|_{p',a}$. This shows the claim, since the subsequence $(v_{k_j})_j$ of $(v_k)_k$ is arbitrary.

Now, for all $\varphi \in \tilde{D}^{s,p}(\Omega)$, with $[\varphi]_{K,\Omega} = 1$, by Hölder's inequality,

$$
\begin{aligned}
|\langle \mathcal{H}_1'(u_k) - \mathcal{H}_1'(u), \varphi\rangle| &\le \int_\Omega a^{1/p'}|v_k - v| \cdot a^{1/p}|\varphi| dx \le \|v_k - v\|_{p',a}\|\varphi\|_{p,a} \\
&\le c_{p,a}\|v_k - v\|_{p',a},
\end{aligned}
$$

where $c_{p,a}^p = 1/\lambda_1$ is the Sobolev constant for the embedding $\tilde{D}^{s,p}(\Omega) \hookrightarrow L^p(\Omega, a)$ by (2.1.3) and (2.1.4). Therefore, $\|\mathcal{H}_1'(u_n) - \mathcal{H}_1'(u)\|_{\tilde{D}^{-s,p'}(\Omega)} \to 0$ as $k \to \infty$ and $\mathcal{H}_1'$ is compact.

Similarly, $u_k \to u$ in $L^m(\Omega, a)$, since the embedding $\tilde{D}^{s,p}(\Omega) \hookrightarrow L^m(\Omega, a)$ is compact, being $L^p(\Omega, a) \hookrightarrow L^m(\Omega, a)$ continuous, since $1 < m < p$ by assumption $(\mathcal{F})$–(a). Indeed $\|v\|_{m,a} \le \|a\|_1^{1/m-1/p}\|v\|_{p,a}$ for all $v \in L^p(\Omega, a)$ by Hölder's inequality and the fact that $a \in L^\alpha(\Omega) \subset L^1(\Omega)$, Ω is bounded and $\alpha > n/ps > 1$. Clearly, the Nemytskii operator $N_f : L^m(\Omega, a) \to L^{m'}(\Omega, a^{1/(1-m)})$ given by $N_f(u) = f(\cdot, u(\cdot))$ for all $u \in L^m(\Omega, a)$ is well defined thanks to $(\mathcal{F})$–(a). We assert that $N_f(u_k) \to N_f(u)$ in $L^{m'}(\Omega, a^{1/(1-m)})$ as $k \to \infty$. Indeed, fix a subsequence $(u_{k_j})_j$ of $(u_k)_k$. Hence, there exists a subsequence, still denoted by $(u_{k_j})_j$, such that $u_{k_j} \to u$ a.e. in Ω and $|u_{k_j}| \le h$ a.e. in Ω for all $j \in \mathbb{N}$ and some $h \in L^m(\Omega, a)$. In particular, $|N_f(u_{k_j}) - N_f(u)|^{m'}a^{1/(1-m)} \to 0$ a.e. in Ω, being $f(x, \cdot)$ continuous for a.a. $x \in \Omega$. Furthermore, $|N_f(u_{k_j}) - N_f(u)|^{m'}a^{1/(1-m)} \le \kappa a(1 + h^m) \in L^1(\Omega)$, $\kappa = (2C_f)^{m'}2^{m'-1}$, by $(\mathcal{F})$–(a), being $a \in L^\alpha(\Omega) \subset L^1(\Omega)$, since Ω is bounded and $\alpha > n/ps > 1$. This shows the assertion, since $1 < m < p$ by assumption $(\mathcal{F})$–(a). Hence, by the dominated convergence theorem, we have $N_f(u_{k_j}) \to N_f(u)$ in $L^{m'}(\Omega, a^{1/(1-m)})$. Therefore the entire sequence $N_f(u_k) \to N_f(u)$ in $L^{m'}(\Omega, a^{1/(1-m)})$ as $k \to \infty$.

Finally, for all $\varphi \in \tilde{D}^{s,p}(\Omega)$, with $[\varphi]_{K,\Omega} = 1$, we have by Hölder's inequality,

$$\begin{aligned}|\langle \mathcal{H}_2'(u_k) - \mathcal{H}_2'(u), \varphi\rangle| &\le \int_\Omega a^{-1/m}|N_f(u_k) - N_f(u)| \cdot a^{1/m}|\varphi|dx\\ &\le \|N_f(u_k) - N_f(u)\|_{m',a^{1/(1-m)}}\|\varphi\|_{m,a}\\ &\le c_{p,a}\|a\|_1^{1/m-1/p}\|N_f(u_k) - N_f(u)\|_{m',a^{1/(1-m)}},\end{aligned}$$

where $c_{p,a}$ is given in (2.1.4). Thus, $\|\mathcal{H}_2'(u_k) - \mathcal{H}_2'(u)\|_{\tilde{D}^{-s,p'}(\Omega)} \to 0$ as $k \to \infty$, that is $\mathcal{H}_2'$ is compact.

Since by the above steps $\mathcal{H}' = \mathcal{H}_1' + \mathcal{H}_2'$ is compact, then $\mathcal{H}$ is sequentially weakly continuous by [98, Corollary 41.9], being $\tilde{D}^{s,p}(\Omega)$ reflexive. □

Lemma 2.3.4. *Under the assumption* $(\mathcal{F})$–(a) *the energy functional* $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$ *is coercive for every* $\lambda \in (-\infty, \lambda_1)$.

Proof. Fix $\lambda \in (-\infty, \lambda_1)$. Then by (2.1.4) and $(\mathcal{F})$–(a)

$$\begin{aligned}J_\lambda(u) &\ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)[u]_{K,\Omega}^p - |\lambda|\int_\Omega |F(x,u)|dx\\ &\ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)[u]_{K,\Omega}^p - |\lambda|\int_\Omega f_0(x)dx\\ &\quad - |\lambda|\int_\Omega \left[f_0(x) + \frac{f_1(x)}{m}\right]|u|^m dx\\ &\ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)[u]_{K,\Omega}^p - |\lambda|C_1 - |\lambda|\,C_2[u]_{K,\Omega}^m,\end{aligned} \tag{2.3.9}$$

where $C_1 = \|f_0\|_1$, $C_2 = c_{\alpha'm}^m\|f_0 + f_1/m\|_\alpha$ and $c_{\alpha'm}$ denotes the Sobolev constant of the compact embedding $\tilde{D}^{s,p}(\Omega) \hookrightarrow L^{\alpha' q}(\Omega)$, being $\alpha' m < p^*$. Note that $C_1 < \infty$, since $f_0 \in L^\alpha(\Omega) \subset L^1(\Omega)$, by $(\mathcal{F})$–(a), being Ω bounded and $\alpha > n/ps > 1$. This shows the assertion, since $1 < m < p$ by the assumption $(\mathcal{F})$–(a). □

2.4 The main result

In this section we prove an existence theorem for $(\mathscr{P}_1)$ as an application of the principal abstract Theorem 2.1–(ii), Part (a) in [36], which represents the differential version of the *Arcoya* and *Carmona* Theorem 3.4 in [5]. In order to simplify the notation let us introduce the main auxiliary functions

$$
\begin{aligned}
\varphi_1(r) &= \inf_{u\in\Psi^{-1}(I_r)} \frac{\displaystyle\inf_{v\in\Psi^{-1}(r)} \Phi(v)-\Phi(u)}{\Psi(u)-r}, \quad I_r=(-\infty,r),\\
\varphi_2(r) &= \sup_{u\in\Psi^{-1}(I^r)} \frac{\displaystyle\inf_{v\in\Psi^{-1}(r)} \Phi(v)-\Phi(u)}{\Psi(u)-r}, \quad I^r=(r,\infty),
\end{aligned}
\tag{2.4.1}
$$

which are well–defined for all $r \in \big(\inf_{u\in\tilde{D}^{s,p}(\Omega)} \Psi(u), \sup_{u\in\tilde{D}^{s,p}(\Omega)} \Psi(u)\big)$, see [5, 36].

Theorem 2.4.1. *Assume* $(\mathcal{F})$*–*(a) *and* (b)*.*

(*i*) *If* $\lambda \in [0, \lambda_\star)$*, where* $\lambda_\star$ *is defined in* (2.2.3)*, then* $(\mathscr{P}_1)$ *has only the trivial solution.*

(*ii*) *If* f *satisfies also* $(\mathcal{F})$*–*(c)*, then problem* $(\mathscr{P}_1)$ *admits at least two nontrivial solutions for every* $\lambda \in (\lambda^\star, \lambda_1)$*, where* $\lambda^\star < \lambda_1$ *is given in* (2.3.8)*.*

Proof. (*i*) Let $u \in \tilde{D}^{s,p}(\Omega)$ be a nontrivial weak solution of the problem $(\mathscr{P}_1)$, then

$$
\begin{aligned}
\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^{p-2}(u(x)-u(y))\cdot(\varphi(x)-\varphi(y))\cdot K(x-y)dxdy \\
= \lambda \int_\Omega \{a(x)|u|^{p-2}u + f(x,u)\}\varphi\, dx
\end{aligned}
$$

for all $\varphi \in \tilde{D}^{s,p}(\Omega)$. Take $\varphi = u$ and put $\Omega_0 = \{x \in \Omega : u(x) \neq 0\}$, so that

$$
\begin{aligned}
\lambda_1[u]^p_{K,\Omega} &= \lambda_1\lambda\left(\|u\|^p_{p,a} + \int_{\Omega_0} \frac{f(x,u)}{a(x)|u|^{p-1}}\, a(x)|u|^p dx\right) \le \lambda_1\lambda\,(1+S_f)\,\|u\|^p_{p,a}\\
&\le \lambda\,(1+S_f)\,[u]^p_{K,\Omega}
\end{aligned}
$$

by (2.1.4) and (2.2.1). Therefore $\lambda \ge \lambda_\star$ by (2.2.3), as required.

(*ii*) The functional Φ is clearly convex, Φ is also weakly lower semicontinuous in $\tilde{D}^{s,p}(\Omega)$ and Φ' verifies condition $(\mathscr{S}_+)$, as already proved in Lemma 2.3.1. Furthermore, $\Psi' : \tilde{D}^{s,p}(\Omega) \to \tilde{D}^{-s,p'}(\Omega)$ is compact and Ψ is sequentially weakly continuous in $\tilde{D}^{s,p}(\Omega)$ by Lemma 2.3.3. The functional J_λ is coercive for every $\lambda \in I$, where $I = (-\infty, \lambda_1)$, thanks to Lemma 2.3.4.

We claim that $\Psi(\tilde{D}^{s,p}(\Omega)) \supset \mathbb{R}_0^-$. Indeed, $\Psi(0) = 0$ and by $(\mathcal{F})$–(a)

$$\begin{aligned}\Psi(u) &\le -\frac{1}{p}\|u\|_{p,a}^p + \int_\Omega |F(x,u)|dx \le -\frac{1}{p}\|u\|_{p,a}^p + \|f_0\|_1 + 2C_f \int_\Omega a(x)|u|^m dx \\ &\le -\frac{1}{p}\|u\|_{p,a}^p + \|f_0\|_1 + 2C_f\|a\|_1^{(p-m)/p}\|u\|_{p,a}^m,\end{aligned}$$

since $a \in L^\alpha(\Omega) \subset L^1(\Omega)$, being $\alpha > n/ps > 1$ and Ω bounded. Therefore,

$$\lim_{u\in\tilde{D}^{s,p}(\Omega),\ \|u\|_{p,a}\to\infty} \Psi(u) = -\infty,$$

being $m < p$. Hence, the claim follows by the continuity of Ψ.

Thus, $(\inf\Psi, \sup\Psi) \supset \mathbb{R}_0^-$. For every $u \in \Psi^{-1}(I_0)$ we have

$$\varphi_1(r) \le \frac{\Phi(u)}{r - \Psi(u)} \quad \text{for all } r \in (\Psi(u), 0),$$

so that

$$\limsup_{r\to 0^-} \varphi_1(r) \le -\frac{\Phi(u)}{\Psi(u)} \quad \text{for all } u \in \Psi^{-1}(I_0).$$

In other words, by (2.3.8) and (2.4.1)

$$\limsup_{r\to 0^-} \varphi_1(r) \le \varphi_1(0) = \lambda^\star. \tag{2.4.2}$$

From $(\mathcal{F})$–(b) and L'Hôpital rule

$$\limsup_{u\to 0} \frac{|F(x,u)|}{a(x)|u|^\gamma} < \infty \quad \text{uniformly a.e. in } \Omega,$$

so that using also $(\mathcal{F})$–(a) and again (b), that is (2.2.1), it follows the existence of a positive real number $L > 0$ such that

$$|F(x,u)| \le L\, a(x)|u|^\gamma \quad \text{for a.a. } x \in \Omega \text{ and all } u \in \mathbb{R}. \tag{2.4.3}$$

The embedding $\tilde{D}^{s,p}(\Omega) \hookrightarrow L^\gamma(\Omega, a)$ is continuous, since $\gamma \in (p, p^*/\alpha')$ by $(\mathcal{F})$–(b). Indeed, by Hölder's inequality

$$\|u\|_{\gamma,a}^\gamma \le |\Omega|^{1/\wp}\|a\|_\alpha\|u\|_{p^*}^\gamma \le c[u]_{K,\Omega}^\gamma$$

for all $u \in \tilde{D}^{s,p}(\Omega)$, where $c = c_{p^*}^\gamma|\Omega|^{1/\wp}\|a\|_\alpha$ and c_{p^*} is the Sobolev constant of the continuous embedding $\tilde{D}^{s,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $\wp$ is the crucial exponent

$$\wp = \frac{\alpha' p^*}{p^* - \gamma\alpha'} > 1,$$

being $\gamma \in (p, p^*/\alpha')$ by $(\mathcal{F})$–(b). Hence, by (2.4.3)

$$|\Psi(u)| \leq \frac{1}{p\lambda_1}[u]^p_{K,\Omega} + C[u]^\gamma_{K,\Omega}, \tag{2.4.4}$$

for every $u \in \tilde{D}^{s,p}(\Omega)$, where $C = c\,L$. Therefore, given $r < 0$ and $v \in \Psi^{-1}(r)$, we get

$$r = \Psi(v) \geq -\frac{1}{p\lambda_1}[v]^p_{K,\Omega} - C[v]^\gamma_{K,\Omega} = -\frac{1}{\lambda_1}\Phi(v) - \kappa\Phi(v)^{\gamma/p}, \tag{2.4.5}$$

where $\kappa = Cp^{\gamma/p}$. Since the functional Φ is bounded below, coercive and lower semicontinuous on the reflexive Banach space $\tilde{D}^{s,p}(\Omega)$, it is easy to see that Φ is also coercive on the sequentially weakly closed non–empty set $\Psi^{-1}(r)$, see Lemma 2.3.3. Therefore, by Theorem 6.1.1 of [21], there exists $u_r \in \Psi^{-1}(r)$ such that $\Phi(u_r) = \inf_{v\in\Psi^{-1}(r)} \Phi(v)$. Taking $u \equiv 0 \in \Psi^{-1}(I^r)$ in (2.4.1), we have

$$\varphi_2(r) \geq -\frac{1}{r}\inf_{v\in\Psi^{-1}(r)} \Phi(v) = \frac{\Phi(u_r)}{|r|}.$$

Hence (2.4.5), evaluated at $v = u_r$ and divided by $r < 0$, gives

$$1 \leq \frac{1}{\lambda_1}\cdot\frac{\Phi(u_r)}{|r|} + \kappa|r|^{\gamma/p-1}\left(\frac{\Phi(u_r)}{|r|}\right)^{\gamma/p} \leq \frac{\varphi_2(r)}{\lambda_1} + \kappa|r|^{\gamma/p-1}\varphi_2(r)^{\gamma/p}.$$

There are now two possibilities to be considered: either φ_2 is locally bounded at 0^-, so that the above inequality shows at once that

$$\liminf_{r\to 0^-} \varphi_2(r) \geq \lambda_1,$$

being $\gamma > p$ by $(\mathcal{F})$–(b), or $\limsup_{r\to 0^-}\varphi_2(r) = \infty$. In both cases (2.4.2) and Lemma 2.3.2 yield

$$\limsup_{r\to 0^-} \varphi_1(r) \leq \lambda^\star < \lambda_1 \leq \limsup_{r\to 0^-} \varphi_2(r).$$

Hence for all integers $k \geq k^\star = 1 + [2/(\lambda_1 - \lambda^\star)]$ there exists a number $r_k < 0$ so close to zero that $\varphi_1(r_k) < \lambda^\star + 1/k < \lambda_1 - 1/k < \varphi_2(r_k)$. In particular,

$$[\lambda^\star + 1/k, \lambda_1 - 1/k] \subset (\varphi_1(r_k), \varphi_2(r_k)) \cap I = (\varphi_1(r_k), \varphi_2(r_k)) \tag{2.4.6}$$

for all $k \geq k^\star$. Therefore, since all the assumptions of Theorem 2.1–(ii), Part (a), in [36] are satisfied and $u \equiv 0$ is a critical point of J_λ, problem $(\mathscr{P}_1)$

admits at least two nontrivial solutions for all $\lambda \in (\varphi_1(r_k), \varphi_2(r_k))$ and for all $k \geq k^\star$. In conclusion, problem $(\mathscr{P}_1)$ admits at least two nontrivial solutions for all $\lambda \in (\lambda^\star, \lambda_1)$ as required, being

$$(\lambda^\star, \lambda_1) = \bigcup_{k=k^\star}^{\infty} [\lambda^\star + 1/k, \lambda_1 - 1/k] \subset \bigcup_{k=k^\star}^{\infty} (\varphi_1(r_k), \varphi_2(r_k))$$

by (2.4.6). □

Taking inspiration from [36], also in this new setting we can derive an interesting consequence from the main Theorem 2.4.1 for a simpler problem. Let us therefore replace $(\mathcal{F})$–(c) by the next condition much easier to verify.
$(\mathcal{F})$–(c') *Assume there exist* $x_0 \in \Omega$, $t_0 \in \mathbb{R}$ *and* $r_0 > 0$ *so small that the closed ball* $B_0 = \{x \in \mathbb{R}^n : |x - x_0| \leq r_0\}$ *is contained in* Ω *and*

$$\operatorname*{ess\,inf}_{B_0} F(x, |t_0|) = \mu_0 > 0, \qquad \operatorname*{ess\,sup}_{B_0} \max_{|t| \leq |t_0|} |F(x,t)| = M_0 < \infty.$$

Clearly, when f does not depend on x, condition $(\mathcal{F})$–(c') simply reduces to the request that $F(t_0) > 0$ at a point $t_0 \in \mathbb{R}$.

Corollary 2.4.2. *Assume that* $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *satisfies* $(\mathcal{F})$–(a), (b). *Consider the problem*

$$\begin{cases} -\mathcal{L}_K u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.4.7}$$

- (i) *If* $\lambda \in [0, \ell_\star)$, *where* $\ell_\star = \lambda_1 / S_f$, *then* (2.4.7) *has only the trivial solution.*
- (ii) *If furthermore* f *satisfies* $(\mathcal{F})$–(c'), *then there exists* $\ell^\star \geq \ell_\star$ *such that* (2.4.7) *admits at least two nontrivial solutions for all* $\lambda \in (\ell^\star, \infty)$.

Proof. The energy functional J_λ associated to problem (2.4.7) is simply given by $J_\lambda(u) = \Phi(u) + \lambda \Psi_2(u)$, where as before $\Psi_2(u) = -\int_\Omega F(x, u(x))dx$, see (2.3.1). First, note that J_λ is coercive for every $\lambda \in \mathbb{R}$. Indeed, by (2.3.9)

$$J_\lambda(u) \geq \frac{1}{p}[u]_{K,\Omega}^p - |\lambda| \int_\Omega |F(x,u)|dx \geq \frac{1}{p}[u]_{K,\Omega}^p - |\lambda|\, C_1 - |\lambda|\, C_2 [u]_{K,\Omega}^m,$$

where C_1 and C_2 are as in (2.3.9). Hence $J_\lambda(u) \to \infty$ as $[u]_{K,\Omega} \to \infty$, since $1 < m < p$ by $(\mathcal{F})$–(a). In conclusion, here $I = \mathbb{R}$, as claimed.

The part (i) of the statement is proved using the same argument produced for the proof of Theorem 2.4.1–(i), being

$$\lambda_1[u]^p_{K,\Omega} = \lambda_1\lambda\int_\Omega f(x,u)u dx \le \lambda_1\lambda S_f\|u\|^p_{p,a} \le \lambda S_f[u]^p_{K,\Omega}$$

by (2.1.4) and (2.2.1). Thus, if u is a nontrivial weak solution of (2.4.7), then necessarily $\lambda \ge \ell_\star = \lambda_1/S_f$, as required.

In order to prove (ii), we first show that there exists $u_0 \in \tilde{D}^{s,p}(\Omega)$ such that $\Psi_2(u_0) < 0$, so that the crucial number

$$\ell^\star = \varphi_1(0) = \inf_{u\in\Psi_2^{-1}(I_0)} -\frac{\Phi(u)}{\Psi_2(u)}, \quad I_0 = (-\infty, 0) = \mathbb{R}^-,$$

is well defined. Indeed, in this special subcase (2.4.1) simply reduces to

$$\begin{aligned}
\varphi_1(r) &= \inf_{u\in\Psi_2^{-1}(I_r)} \frac{\displaystyle\inf_{v\in\Psi_2^{-1}(r)} \Phi(v)-\Phi(u)}{\Psi_2(u)-r}, \quad I_r = (-\infty, r),\\
\varphi_2(r) &= \sup_{u\in\Psi_2^{-1}(I^r)} \frac{\displaystyle\inf_{v\in\Psi_2^{-1}(r)} \Phi(v)-\Phi(u)}{\Psi_2(u)-r}, \quad I^r = (r, \infty),
\end{aligned} \tag{2.4.8}$$

Clearly $t_0 \ne 0$ in $(\mathcal{F})$–(c'). Now take $\sigma \in (0,1)$ and put

$$B = \{x \in \mathbb{R}^n : |x - x_0| \le \sigma r_0\}, \quad B_1 = \{x \in \mathbb{R}^n : |x - x_0| \le r_1\},$$

where $r_1 = (1+\sigma)r_0/2$. Hence $B \subset B_1 \subset B_0$. Set $v_0(x) = |t_0|\chi_{B_1}(x)$ and denote by ρ_ε the convolution kernel of fixed radius ε, with $0 < \varepsilon < (1-\sigma)r_0/2$. Define

$$u_0(x) = \rho_\varepsilon * v_0(x),$$

so that $u_0(x) = |t_0|$ for all $x \in B$, $0 \le u_0(x) \le |t_0|$ for all $x \in \Omega$, $u_0 \in C_0^\infty(\Omega)$ and $\operatorname{supp} u_0 \subset B_0$. Therefore, $u_0 \in \tilde{D}^{s,p}(\Omega)$ by $(\mathcal{K})$. From $(\mathcal{F})$–(c') we also have

$$\begin{aligned}
\Psi_2(u_0) &= -\int_B F(x,|t_0|)dx - \int_{B_0\setminus B} F(x, u_0(x))\,dx \le M_0\int_{B_0\setminus B} dx - \mu_0\int_B dx\\
&\le \omega_N r_0^n\left[M_0(1-\sigma^n) - \mu_0\sigma^n\right].
\end{aligned}$$

Hence, for $\sigma \in (0,1)$ so large that $\sigma^n > M_0/(\mu_0 + M_0)$, we have $\Psi_2(u_0) < 0$, as claimed.

Furthermore, by (2.1.4), (2.2.2) and (2.3.1), we have for all $u \in \tilde{D}^{s,p}(\Omega)$, with $u \not\equiv 0$,

$$\frac{\Phi(u)}{|\Psi_2(u)|} \geq \frac{[u]^p_{K,\Omega}}{S_f \|u\|^p_{p,a}} \geq \frac{\lambda_1}{S_f} = \ell_\star.$$

Thus, $\ell^\star \geq \ell_\star$.

In particular, for φ_1 given now by (2.4.8) and for all $u \in \Psi_2^{-1}(I_0)$, we get

$$\varphi_1(r) \leq \frac{\Phi(u)}{r - \Psi_2(u)} \quad \text{for all } r \in (\Psi_2(u), 0).$$

Therefore,

$$\limsup_{r \to 0^-} \varphi_1(r) \leq \varphi_1(0) = \ell^\star,$$

which is the analog of (2.4.2).

Also in this setting (2.4.3) holds and (2.4.4) simply reduces to

$$|\Psi_2(u)| \leq C[u]^\gamma_{K,\Omega}.$$

Taken $r < 0$ and $v \in \Psi_2^{-1}(r)$, we obtain

$$r = \Psi_2(v) \geq -C[v]^\gamma_{K,\Omega} \geq -C\left(p\Phi(v)\right)^{\gamma/p}.$$

Therefore, by (2.4.8), since $u \equiv 0 \in \Psi_2^{-1}(I^r)$,

$$\varphi_2(r) \geq \frac{1}{|r|} \inf_{v \in \Psi_2^{-1}(r)} \Phi(v) \geq \kappa |r|^{p/\gamma - 1},$$

where $\kappa = C^{-p/\gamma}/p$. This implies that $\lim_{r \to 0^-} \varphi_2(r) = \infty$, being $\gamma > p$ by assumption $(\mathcal{F})$–(b).

In conclusion, we have proved that

$$\limsup_{r \to 0^-} \varphi_1(r) \leq \varphi_1(0) = \ell^\star < \lim_{r \to 0^-} \varphi_2(r) = \infty.$$

This shows that for all integers $k \geq k^\star = 2 + [\ell^\star]$ there exists $r_k < 0$ so close to zero that $\varphi_1(r_k) < \ell^\star + 1/k < k < \varphi_2(r_k)$. Hence, all the assumptions of Theorem 2.1–(ii), Part (a) are satisfied and, being $u \equiv 0$ a critical point of J_λ and $I = \mathbb{R}$, problem (2.4.7) admits at least two nontrivial solutions for all

$$\lambda \in \bigcup_{k=k^\star}^{\infty} (\varphi_1(r_k), \varphi_2(r_k)) \supset \bigcup_{k=k^\star}^{\infty} [\ell^\star + 1/k, k] = (\ell^\star, \infty),$$

as stated. □

Chapter 3

Problem $(\mathscr{P}_2)$

In this chapter, inspired by [11] and the fact that several interesting questions arise from the search of nontrivial non–negative (weak) solutions, we deal with existence and multiplicity of nontrivial non–negative entire solutions of a Kirchhoff eigenvalue problem, involving critical nonlinearities and nonlocal elliptic operators. More precisely, we consider the problem

$$(\mathscr{P}_2) \qquad \begin{aligned} &M\left([u]_K^p\right)(-\mathcal{L}_K u) = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \qquad \text{in } \mathbb{R}^n, \\ &[u]_K^p = \iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^p K(x-y)\,dxdy, \end{aligned}$$

where $\lambda \in \mathbb{R}$, $0 < s < 1$, $ps < n$ and $\mathcal{L}_K$ is an integro–differential nonlocal operator, defined as in the Introduction.

3.1 Notations and main results

The nonlinear terms in $(\mathscr{P}_2)$ are related to the main elliptic part by the request that

$$p < q < \min\{r, p^*\}, \tag{3.1.1}$$

where $p^* = np/(n-ps)$ is the critical Sobolev exponent for $W^{s,p}(\mathbb{R}^n)$. The *weight w verifies*

$$w \in L^{\wp}(\mathbb{R}^n) \cap L^{\sigma}_{\text{loc}}(\mathbb{R}^n), \quad \text{with } \wp = p^*/(p^*-q), \quad \sigma > \wp, \tag{3.1.2}$$

while h is a positive weight of class $L^1_{\text{loc}}(\mathbb{R}^n)$. Finally, *$h$ and w are related by the condition*

$$\int_{\mathbb{R}^n} \left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)} dx = H \in \mathbb{R}^+. \tag{3.1.3}$$

The *Kirchhoff function* $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *verifies* the following condition.

($\mathcal{M}$) *M is an increasing and continuous function, with $M(\tau) > 0$ for $\tau \geq 0$ and $\mathscr{M}(\tau) = \int_0^\tau M(s)ds$.*

In this chapter we cover *only* the non–degenerate case, as in [52]. From now on we put $M(0) = m_0$ and recall that $m_0 > 0$ by ($\mathcal{M}$). We refer to [35, 6, 7] for further details and references.

In ($\mathscr{P}_2$) the Kirchhoff function M, which represents the elastic tension term, depends on the Gagliardo fractional norm $[\,\cdot\,]_K$ arising from general kernel K and generating nonlocal operators $\mathcal{L}_K$. It is clear from symmetry properties that if u is a solution of ($\mathscr{P}_2$) also $-u$ is a solution of ($\mathscr{P}_2$). The main result of the chapter is

Theorem 3.1.1. *Under the above assumptions there exists $\overline{\lambda} > 0$ such that problem ($\mathscr{P}_2$) admits at least two nontrivial non–negative entire solutions for all $\lambda > \overline{\lambda}$, one of which is a global minimizer of the underlying functional J_λ of ($\mathscr{P}_2$) and the latter independent solution u_λ is a Mountain Pass critical point of J_λ. In particular, $\|u_\lambda\| \to 0$ as $\lambda \to \infty$, where $\|\cdot\|$ is the natural solution space norm of ($\mathscr{P}_2$). Moreover, there exist λ^* and λ^{**}, satisfying $0 < \lambda^* \leq \lambda^{**} \leq \overline{\lambda}$ and such that*

(i) problem ($\mathscr{P}_2$) possesses only the trivial solution if $\lambda < \lambda^*$;

(ii) problem ($\mathscr{P}_2$) admits a nontrivial <u>non–negative</u> entire solution if and only if $\lambda \geq \lambda^{**}$.

In Section 3.2 we define the main solution space Y and give some preliminary results, from which we derive (i) of Theorem 3.1.1. In Section 3.4 we prove the existence of $\overline{\lambda} > 0$ such that for all $\lambda > \overline{\lambda}$ problem ($\mathscr{P}_2$) admits a first nontrivial non–negative entire solution and then, thanks to a modified version of the Mountain Pass Theorem established in [12], we construct a second independent nontrivial non–negative entire solution u_λ of ($\mathscr{P}_2$). We end Section 3.4 by proving the asymptotic property for u_λ stated in Theorem 3.1.1. Finally in Section 3.5 we prove part (ii) of Theorem 3.1.1.

3.2 Solution space and preliminaries

Denote with Y the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\| = \left([u]_K^p + \|u\|_{r,h}^p\right)^{1/p}, \text{ where } \|u\|_{r,h}^r = \int_{\mathbb{R}^n} h(x)|u|^r dx.$$

The embedding

$$Y \hookrightarrow D^{s,p}_K(\mathbb{R}^n) \qquad \textit{is continuous,} \tag{3.2.1}$$

with $[u]_K \le \|u\|$ for all $u \in Y$. In particular, by (1.1.1) and (1.1.3),

$$Y \hookrightarrow D^s_K(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n).$$

Moreover, for all $R > 0$ and $\nu \in [1, p^*)$ the embedding

$$D^{s,p}_K(\mathbb{R}^n) \hookrightarrow\hookrightarrow L^\nu(B_R) \tag{3.2.2}$$

is compact. Indeed, $D^{s,p}_K(\mathbb{R}^n) \hookrightarrow D^{s,p}(\mathbb{R}^n) \hookrightarrow W^{s,p}(B_R)$ by (1.1.3) and the embedding $W^{s,p}(B_R) \hookrightarrow\hookrightarrow L^\nu(B_R)$ is compact for all $\nu \in [1, p^*)$ by Corollary 7.2 of [40].

Proposition 3.2.1. *The Banach space $(Y, \|\cdot\|)$ is reflexive.*

Proof. We proceed as in the proof of Proposition A.11 in [12]. The product space $Z = D^{s,p}_K(\mathbb{R}^n) \times L^r(\mathbb{R}^n, h)$, endowed with the norm $\|u\|_Z = [u]_K + \|u\|_{r,h}$, is a reflexive Banach space by Theorem 1.22–(ii) of [2], since both $D^{s,p}_K(\mathbb{R}^n)$ and $L^r(\mathbb{R}^n, h)$ are uniformly convex Banach spaces (see also Proposition A.6 in [12]).

The operator $T : (Y, \|\cdot\|_Z) \to (Z, \|\cdot\|_Z)$, $T(u) = (u, u)$, is well defined, linear and isometric. Therefore, $T(Y)$ is a closed subspace of the reflexive space Z, and so $T(Y)$ is reflexive by Theorem 1.21–(ii) of [2]. Consequently, $(Y, \|\cdot\|_Z)$ is reflexive, being isomorphic to a reflexive Banach space. Finally, we conclude that also $(Y, \|\cdot\|)$ is reflexive, because reflexivity is preserved under equivalent norms, being $\|u\| \le \|u\|_Z \le 2\|u\|$ for all $u \in Y$. □

The next proposition is given for functions in Y, but of course continues to be valid also in the main fractional weighted Sobolev space $D^{s,p}_K(\mathbb{R}^n)$.

Proposition 3.2.2. *If $(u_k)_k$, $u \in Y$ and $u_k \rightharpoonup u$ in Y, then, up to a subsequence, $u_k \to u$ a.e. in $\mathbb{R}^n$.*

Proof. Let $(u_k)_k$ and u be as in the statement. Then, $u_k \to u$ as $k \to \infty$ in $L^\nu(B_R)$ for all $R > 0$ and $\nu \in [1, p^*)$ by (3.2.1) and (3.2.2). In particular, in correspondence to $R = 1$ there exists a subsequence $(u_{1,k})_k$ of $(u_k)_k$ such that $u_{1,k} \to u$ a.e. in B_1. Clearly $u_{1,k} \rightharpoonup u$ in Y and so, in correspondence to $R = 2$, there exists a subsequence $(u_{2,k})_k$ of $(u_{1,k})_k$ such that $u_{2,k} \to u$ a.e. in B_2, and so on. The diagonal subsequence $(u_{k,k})_k$ of $(u_k)_k$, constructed by induction, converges to u a.e. in $\mathbb{R}^n$ as $k \to \infty$. □

We also have the following main embedding result.

Lemma 3.2.3. *The embedding* $D^{s,p}_K(\mathbb{R}^n) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^n, w)$ *is compact, with*

$$\|u\|_{q,w} \le \mathfrak{C}_w[u]_K \quad \textit{for all } u \in D^{s,p}_K(\mathbb{R}^n), \tag{3.2.3}$$

and $\mathfrak{C}_w = C_{p^*}\|w\|_{\wp}^{1/q} K_0^{-1/p} > 0$. *Furthermore, also the embedding*

$$Y \hookrightarrow\hookrightarrow L^q(\mathbb{R}^n, w)$$

is compact.

Proof. By (3.1.2), (1.1.1), (1.1.3) and Hölder's inequality, for all $u \in D^{s,p}_K(\mathbb{R}^n)$,

$$\begin{aligned}\|u\|_{q,w} &\le \left(\int_{\mathbb{R}^n} w(x)^{\wp} dx\right)^{1/\wp q} \cdot \left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{1/p^*} \le C_{p^*}\|w\|_{\wp}^{1/q}[u]_{s,p} \\ &\le C_{p^*}\|w\|_{\wp}^{1/q} K_0^{-1/p}[u]_K,\end{aligned}$$

that is (3.2.3) holds.

Let us now show that $\|u_k - u\|_{q,w} \to 0$ as $k \to \infty$ whenever $u_k \rightharpoonup u$ in $D^{s,p}_K(\mathbb{R}^n)$. By Hölder's inequality,

$$\int_{\mathbb{R}^n \setminus B_R} w(x)|u_k - u|^q dx \le \mathfrak{M}\left(\int_{\mathbb{R}^n \setminus B_R} w(x)^{\wp} dx\right)^{1/\wp} = o(1)$$

as $R \to \infty$, being $w \in L^{\wp}(\mathbb{R}^n)$ by (3.1.2) and $\mathfrak{M} = \sup_k \|u_k - u\|_{p^*}^q < \infty$. For all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ so large that $\sup_k \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}} w(x)|u_k - u|^q dx < \varepsilon/2$.

Moreover, by (3.1.2), Hölder's inequality and (3.2.2) we have

$$\int_{B_{R_\varepsilon}} w(x)|u_k - u|^q \, dx \le \|w\|_{L^\sigma(B_{R_\varepsilon})}\|u_k - u\|^q_{L^{\sigma' q}(B_{R_\varepsilon})} = o(1)$$

as $k \to \infty$, since $\sigma' q < p^*$. Hence, there exists $k_\varepsilon > 0$ such that

$$\int_{B_{R_\varepsilon}} w(x)|u_k - u|^q \, dx < \varepsilon/2$$

for all $k \ge k_\varepsilon$. In conclusion, for all $k \ge k_\varepsilon$

$$\|u_k - u\|^q_{q,w} = \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}} w(x)|u_k - u|^q \, dx + \int_{B_{R_\varepsilon}} w(x)|u_k - u|^q \, dx < \varepsilon,$$

as required.

The last part of the lemma follows at once by (3.2.1). $\square$

An *entire (weak) solution* u of $(\mathscr{P}_2)$ is a function in Y such that

$$M([u]_K^p)\langle u,\varphi\rangle_K = \lambda \int_{\mathbb{R}^n} w(x)|u|^{q-2}u\varphi dx - \int_{\mathbb{R}^n} h(x)|u|^{r-2}u\varphi dx, \tag{3.2.4}$$

for all $\varphi \in Y$, where $\langle\cdot,\cdot\rangle_K$ is given in (1.1.2).

Lemma 3.2.4. *If $\lambda \in \mathbb{R}$ and $u = u_\lambda \in Y \setminus \{0\}$ satisfies*

$$M([u]_K^p)[u]_K^p + \|u\|_{r,h}^r = \lambda\|u\|_{q,w}^q, \tag{3.2.5}$$

then $\lambda > 0$ and

$$\kappa_1\lambda^{1/(p-q)} \le \|u_\lambda\|_{q,w} \le \kappa_2\lambda^{r/p(r-q)}, \tag{3.2.6}$$

where κ_1 and κ_2 are positive constants independent of λ and u_λ.

Proof. Let $u \in Y\setminus\{0\}$ and $\lambda \in \mathbb{R}$ satisfy (3.2.5). By (3.2.3), $(\mathcal{M})$ and (3.2.5)

$$\|u\|_{q,w}^p \le \mathfrak{C}_w^p[u]_K^p \le \frac{\mathfrak{C}_w^p}{m_0}M([u]_K^p)[u]_K^p \le \lambda\frac{\mathfrak{C}_w^p}{m_0}\|u\|_{q,w}^q. \tag{3.2.7}$$

Hence, $\lambda > 0$, being $u \ne 0$. Moreover, $\lambda\|u\|_{q,w}^{q-p} \ge m_0/\mathfrak{C}_w^p$, that is

$$\|u\|_{q,w} \ge \kappa_1\lambda^{1/(p-q)}, \quad \text{with } \kappa_1 = \left(m_0/\mathfrak{C}_w^p\right)^{1/(q-p)}.$$

In other words, the first part of (3.2.6) holds true. By Young's inequality,

$$ab \le \frac{a^\alpha}{\alpha} + \frac{b^\beta}{\beta},$$

with $a = h(x)^{q/r}|u|^q \ge 0$, $b = \lambda w(x)h(x)^{-q/r} \ge 0$, $\alpha = r/q > 1$ and $\beta = r/(r-q) > 1$, we find

$$\lambda w(x)|u|^q \le \frac{q}{r}h(x)|u|^r + \frac{r-q}{r}\left(\frac{\lambda w(x)}{h(x)^{q/r}}\right)^{r/(r-q)}.$$

Integration over $\mathbb{R}^n$, $(\mathcal{M})$ and (3.2.5) give

$$m_0[u]_K^p \le M([u]_K^p)[u]_K^p \le \frac{q-r}{r}\|u\|_{r,h}^r + \frac{r-q}{r}H\,\lambda^{r/(r-q)} \le \frac{r-q}{r}H\,\lambda^{r/(r-q)},$$

being $q < r$. Since $u \not\equiv 0$ by assumption, the last inequality and (3.2.7) yield the second part of (3.2.6), with $\kappa_2 = [(r-q)\mathfrak{C}_w^p H/m_0 r]^{1/p}$. This completes the proof. □

If $(\mathscr{P}_2)$ admits a *nontrivial* entire solution $u \in Y$, then $\lambda > 0$ by Lemma 3.2.4, and actually $\lambda \geq \lambda_0$ by (3.2.6), where

$$\lambda_0 = (\kappa_1/\kappa_2)^{p(r-q)(q-p)/q(r-p)} > 0.$$

In the following we denote problem $(\mathscr{P}_2)$ with the notation $(\mathscr{P}_2^\lambda)$, when an explicit reference to the specific value λ is necessary.

Define

$$\lambda^* = \sup\{\lambda > 0 : (\mathscr{P}_2^\mu) \textit{ admits only the trivial entire solution for all } \mu < \lambda\}.$$

Clearly $\lambda^* \geq \lambda_0 > 0$. Theorem 3.1.1–$(i)$ follows directly from the definition of λ^*.

3.3 The energy functional

For the proof of Theorem 3.1.1 we use variational arguments since the entire solutions of $(\mathscr{P}_2)$ are exactly the critical points of the natural underlying energy functional J_λ associated to $(\mathscr{P}_2^\lambda)$, that is

$$J_\lambda(u) = \frac{1}{p}\mathscr{M}([u]_K^p) - \frac{\lambda}{q}\|u\|_{q,w}^q + \frac{1}{r}\|u\|_{r,h}^r, \quad u \in Y, \tag{3.3.1}$$

where $\mathscr{M}$ is defined in $(\mathcal{M})$. Clearly, J_λ is Gâteaux–differentiable in Y and for all u, $\varphi \in Y$

$$\begin{aligned} \langle J_\lambda'(u), \varphi\rangle = M([u]_K^p)\langle u, \varphi\rangle_K - \lambda \int_{\mathbb{R}^n} w(x)|u|^{q-2}u\varphi dx \\ + \int_{\mathbb{R}^n} h(x)|u|^{r-2}u\varphi dx, \end{aligned} \tag{3.3.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between Y and its dual space Y'. Thanks to the results of Section 3.2 *from now on we assume that* $\lambda > 0$, without loss of generality.

Lemma 3.3.1. *The functional $J_\lambda : Y \to \mathbb{R}$ is bounded below and coercive in Y. In particular, any sequence $(u_k)_k$ in Y such that $(J_\lambda(u_k))_k$ is bounded admits a weakly convergent subsequence in Y.*

Proof. Let us consider the following elementary inequality: for every k_1, $k_2 > 0$ and $0 < \alpha < \beta$

$$k_1|t|^\alpha - k_2|t|^\beta \le C_{\alpha\beta} k_1 \left(\frac{k_1}{k_2}\right)^{\alpha/(\beta-\alpha)} \quad \text{for all } t \in \mathbb{R}, \tag{3.3.3}$$

where $C_{\alpha\beta} > 0$ is a constant depending only on α and β. Taking $k_1 = \lambda w(x)/q$, $k_2 = h(x)/pr$, $\alpha = q$, $\beta = r$ and $t = u(x)$ in (3.3.3), for all $x \in \mathbb{R}^n$ we have

$$\frac{\lambda}{q}\, w(x)|u(x)|^q - \frac{h(x)}{pr}\,|u(x)|^r \le \mathcal{C}\,\lambda^{r/(r-q)}\left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)},$$

where $\mathcal{C} = C_{qr}\,[pr/q]^{q/(r-q)}/q$. Integrating the above inequality over $\mathbb{R}^n$, we get by (3.1.3)

$$\frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{pr}\|u\|_{r,h}^r \le C_\lambda,$$

where $C_\lambda = \mathcal{C}H\lambda^{r/(r-q)} > 0$ by Lemma 3.2.4.

Therefore, by $(\mathcal{M})$ for all $u \in Y$

$$\begin{aligned} J_\lambda(u) &\ge \frac{1}{p}m_0[u]_K^p - \frac{\lambda}{q}\|u\|_{q,w}^q + \frac{1}{r}\|u\|_{r,h}^r \\ &= \frac{1}{p}m_0[u]_K^p - \left[\frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{pr}\|u\|_{r,h}^r\right] - \frac{1}{pr}\|u\|_{r,h}^r + \frac{1}{r}\|u\|_{r,h}^r \\ &\ge \frac{1}{p}m_0[u]_K^p - C_\lambda + \frac{1}{pr}\|u\|_{r,h}^r \\ &\ge \frac{1}{p}m_0[u]_K^p + \frac{1}{pr}\left(\|u\|_{r,h}^p - 1\right) - C_\lambda \\ &\ge \frac{\min\{m_0, r^{-1}\}}{p}\|u\|^p - C_\lambda - \frac{1}{pr}. \end{aligned} \tag{3.3.4}$$

Hence, J_λ is bounded below and coercive in Y. The last part of the lemma follows at once by the coercivity of J_λ and the reflexivity of the space Y, proved in Proposition 3.2.1. □

For any $(x, u) \in \mathbb{R}^n \times \mathbb{R}$ put

$$f(x,u) = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u, \tag{3.3.5}$$

so that

$$F(x,u) = \int_0^u f(x,v)dv = \frac{\lambda}{q}\,w(x)|u|^q - h(x)\frac{|u|^r}{r}. \tag{3.3.6}$$

Lemma 3.3.2. *For any fixed $u \in Y$ the functional $\mathcal{F}_u : Y \to \mathbb{R}$, defined by*

$$\mathcal{F}_u(v) = \int_{\mathbb{R}^n} f(x, u(x))v(x)dx,$$

is in Y'. In particular, if $v_k \rightharpoonup v$ in Y then $\mathcal{F}_u(v_k) \to \mathcal{F}_u(v)$ as $k \to \infty$.

Proof. Fix $u \in Y$. Clearly $\mathcal{F}_u$ is linear. Moreover, using (3.2.3), we get for all $v \in Y$

$$\begin{aligned} |\mathcal{F}_u(v)| &\le \lambda \int_{\mathbb{R}^n} w(x)|u|^{q-1}|v|\,dx + \int_{\mathbb{R}^n} h(x)|u|^{r-1}|v|dx \\ &\le \lambda \|u\|_{q,w}^{q-1}\|v\|_{q,w} + \|u\|_{r,h}^{r-1}\|v\|_{r,h} \le \sqrt{2}\big(\lambda\,\mathfrak{C}_w\|u\|_{q,w}^{q-1} + \|u\|_{r,h}^{r-1}\big)\|v\|, \end{aligned}$$

and so $\mathcal{F}_u$ is continuous in Y. □

Lemma 3.3.3. *The functional $J_\lambda : Y \to \mathbb{R}$ is of class $C^1(Y)$ and J_λ is sequentially weakly lower semicontinuous in Y, that is if $u_n \rightharpoonup u$ in Y, then*

$$J_\lambda(u) \le \liminf_{k\to\infty} J_\lambda(u_k). \tag{3.3.7}$$

Moreover, J_λ attains its infimum $e = e_\lambda$ in Y, which is an entire solution of $(\mathscr{P}_2^\lambda)$.

Proof. A simple calculation shows that $\frac{1}{p}\mathscr{M}([u]_K^p)$ is convex in Y, since $\mathscr{M}$ is convex and monotone non–decreasing in $\mathbb{R}_0^+$ by $(\mathcal{M})$ and of class $C^1(Y)$. Therefore, $\frac{1}{p}\mathscr{M}([u]_K^p)$ is sequentially weakly lower semicontinuous in Y by Corollary 3.9 of [23], so that

$$\mathscr{M}([u]_K^p) \le \liminf_{k\to\infty} \mathscr{M}([u_k]_K^p) \tag{3.3.8}$$

along any sequence $(u_k)_k$, with $u_k \rightharpoonup u$ in Y.

Denote with Φ_w the functional $u \mapsto \|u\|_{q,w}^q/q$. By Lemma 3.2.3 and Theorem 3.10 of [23] we also have that Φ_w is weakly continuous, so that in particular Φ_w is continuous in Y. Furthermore, Φ_w is Gâteaux–differentiable in Y and for all u, $\varphi \in Y$

$$\langle \Phi_w'(u), \varphi\rangle = \int_{\mathbb{R}^n} w(x)|u|^{q-2}u\varphi\,dx.$$

Now, let $(u_k)_k$, $u \in Y$ be such that $u_k \rightharpoonup u$ in Y and fix $\varphi \in Y$, with $\|\varphi\| = 1$. By Lemma 3.2.3 and Proposition A.8–(ii) of [12], it follows that $v_k = |u_k|^{q-2}u_k \to v = |u|^{q-2}u$ in $L^{q'}(\mathbb{R}^n, w)$. Therefore, by (3.2.3) we have

$$|\langle \Phi_w'(u_k) - \Phi_w'(u), \varphi\rangle| \le \|v_k - v\|_{q',w}\|\varphi\|_{q,w} \le \mathfrak{C}_w\|v_k - v\|_{q',w}.$$

Hence, $\|\Phi'_w(u_k)-\Phi'_w(u)\|_{Y'} \le \mathfrak{C}_w\|v_k-v\|_{q',w}$, that is $\Phi'_w(u_k)\to\Phi'_w(u)$ in Y'. Thus, Φ_w is of class $C^1(Y)$ and as $k\to\infty$

$$\int_{\mathbb{R}^n} w(x)|u_k|^{q-2}u_k\varphi\,dx \to \int_{\mathbb{R}^n} w(x)|u|^{q-2}u\varphi\,dx \tag{3.3.9}$$

for all $\varphi\in Y$.

Finally, it remains to show that also the functional $u\mapsto\|u\|_{r,h}^r/r$, denoted by Φ_h, is of class $C^1(Y)$. The continuity of Φ_h follows from the continuity of the embedding $Y\hookrightarrow L^r(\mathbb{R}^n,h)$. Hence Φ_h is weakly lower semicontinuous in Y again by Corollary 3.9 of [23]. On the other hand, Φ_h is Gâteaux–differentiable in Y and for all u, $\varphi\in Y$

$$\langle\Phi'_h(u),\varphi\rangle = \int_{\mathbb{R}^n} h(x)|u|^{r-2}u\varphi\,dx.$$

Let $(u_k)_k$, $u\in Y$ be such that $u_k\to u$ in Y. Then, $u_k\to u$ in $L^r(\mathbb{R}^n,h)$, and so $v_k=|u_k|^{r-2}u_k\to v=|u|^{r-2}u$ in $L^{r'}(\mathbb{R}^n,h)$ by Proposition A.8–(ii) of [12]. Therefore,

$$\|\Phi'_h(u_k)-\Phi'_h(u)\|_{Y'} \le \sup_{\substack{\varphi\in Y\\ \|\varphi\|=1}} \|v_k-v\|_{r',h}\cdot\|\varphi\|_{r,h} \le \|v_k-v\|_{r',h}=o(1)$$

as $k\to\infty$. This gives the C^1 regularity of Φ_h.

Suppose now that $u_k\rightharpoonup u$ in Y. Fix a subsequence $(v_{k_j})_j$ of the sequence $k\mapsto v_k=|u_k|^{r-2}u_k$. Of course $u_{k_j}\rightharpoonup u$ in Y and by Proposition 3.2.2 there exists a further subsequence $(u_{k_{j_i}})_i$ such that $u_{k_{j_i}}\to u$ a.e. in $\mathbb{R}^n$. Thus $v_{k_{j_i}}\to v=|u|^{r-2}u$ a.e. in $\mathbb{R}^n$. On the other hand, $(v_{k_{j_i}})_i$ is bounded in $L^{r'}(\mathbb{R}^n,h)$, since $\|v_{k_{j_i}}\|_{r',h}^{r'}=\|u_{k_{j_i}}\|_{r,h}^r$ and $(u_{k_{j_i}})_i$ is bounded in $L^r(\mathbb{R}^n,h)$. Therefore, $v_{k_{j_i}}\rightharpoonup v$ in $L^{r'}(\mathbb{R}^n,h)$ by Proposition A.8–(i) of [12]. In conclusion, due to the arbitrariness of $(v_{k_j})_i$, the entire sequence $v_k\rightharpoonup v$ in $L^{r'}(\mathbb{R}^n,h)$ as $k\to\infty$. Hence, in particular for all $\varphi\in Y$

$$\int_{\mathbb{R}^n} h(x)|u_k|^{r-2}u_k\varphi\,dx \to \int_{\mathbb{R}^n} h(x)|u|^{r-2}u\varphi\,dx \tag{3.3.10}$$

as $k\to\infty$.

For the second part of the lemma, let $(u_k)_k$, $u\in Y$ be such that $u_k\rightharpoonup u$ in Y. The definition of J_λ and (3.3.6) give

$$J_\lambda(u)-J_\lambda(u_k)=\frac{1}{p}\left[\mathscr{M}([u]_K^p)-\mathscr{M}([u_k]_K^p)\right]+\int_{\mathbb{R}^n}[F(x,u_k)-F(x,u)]dx.$$

Hence, by (3.3.8)

$$\limsup_{k\to\infty}\,[J_\lambda(u)-J_\lambda(u_k)]\le \limsup_{k\to\infty}\int_{\mathbb{R}^n}[F(x,u_k)-F(x,u)]dx. \tag{3.3.11}$$

By (3.3.5) and (3.3.6), for all $t\in[0,1]$,

$$\begin{aligned} F_u(x,u+t(u_k-u)) &= f(x,u+t(u_k-u)) \\ &= f(x,u)+(u_k-u)\int_0^t f_u(x,u+\tau(u_k-u))d\tau, \end{aligned} \tag{3.3.12}$$

where clearly $f_u(x,z)=\lambda(q-1)w(x)|z|^{q-2}-h(x)(r-1)|z|^{r-2}$. Multiplying (3.3.12) by u_k-u and integrating over $[0,1]$, we obtain

$$\begin{aligned} F(x,u_k)-F(x,u) &= f(x,u)(u_k-u) \\ &\quad + (u_k-u)^2\int_0^1\left(\int_0^t f_u(x,u+\tau(u_k-u))d\tau\right)dt. \end{aligned} \tag{3.3.13}$$

By (3.3.3), with $t=z$, $k_1=\lambda w(x)(q-1)$, $k_2=h(x)(r-1)$, $\alpha=q-2>0$ and $\beta=r-2>0$, we get

$$f_u(x,z)\le 2C_1 w(x)^{2/q}\left[\frac{w(x)^{r/q}}{h(x)}\right]^{(q-2)/(r-q)}, \tag{3.3.14}$$

where C_1 is a positive constant, depending only on q, r and λ. Consequently, (3.3.13) yields

$$\begin{aligned} \int_{\mathbb{R}^n}[F(x,u_k)-F(x,u)]dx \le \int_{\mathbb{R}^n} f(x,u)(u_k-u)dx \\ + C_1H^{(q-2)/q}\|u_k-u\|_{q,w}^2, \end{aligned} \tag{3.3.15}$$

by Hölder's inequality and (3.1.3). Now, Lemma 3.3.2 gives

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} f(x,u)(u_k-u)\,dx=0, \tag{3.3.16}$$

and Lemma 3.2.3 implies

$$\lim_{k\to\infty}\|u_k-u\|_{q,w}=0. \tag{3.3.17}$$

Combining (3.3.15)–(3.3.17) with (3.3.11) we get the claim (3.3.7).

Finally, Corollary 3.23 of [23] yields the existence of a global minimizer $e=e_\lambda$ of J_λ in Y for each $\lambda>0$ and e is therefore an entire solution of $(\mathscr{P}_2)$. □

3.4 Existence of two solutions

The number

$$\overline{\lambda} = \inf_{\substack{u\in Y\\ \|u\|_{q,w}=1}} \left\{ \frac{q}{p}\mathscr{M}([u]_K^p) + \frac{q}{r}\|u\|_{r,h}^r \right\}$$

is positive. Indeed, for all $u \in Y$ with $\|u\|_{q,w} = 1$, by Hölder's inequality and (3.1.3), we have

$$1 = \|u\|_{q,w}^q = \int_{\mathbb{R}^n} \frac{w(x)}{h(x)^{q/r}}\, h(x)^{q/r}|u|^q dx \le H^{(r-q)/r}\|u\|_{r,h}^q.$$

Consequently, we get

$$\frac{q}{p}\mathscr{M}([u]_K^p) + \frac{q}{r}\|u\|_{r,h}^r \ge \frac{m_0 q}{p}[u]_K^p + \frac{q}{r}H^{(q-r)/q} \ge \frac{m_0 q}{p\mathfrak{C}_w^p} + \frac{q}{r}H^{(q-r)/q}.$$

In other words, $\overline{\lambda} \ge m_0 q/p\mathfrak{C}_w^p + qH^{(q-r)/q}/r > 0$, as stated.

Lemma 3.4.1. *For all $\lambda > \overline{\lambda}$ there exists a global nontrivial non–negative minimizer $e \in Y$ of J_λ with negative energy, that is $J_\lambda(e) < 0$. In particular, e is a nontrivial non–negative entire solution of $(\mathscr{P}_2^\lambda)$.*

Proof. By Lemma 3.3.3 for each $\lambda > 0$ there exists a global minimizer $e = e_\lambda \in Y$ of J_λ, that is

$$J_\lambda(e) = \inf_{v\in Y} J_\lambda(v).$$

We prove that $e \not\equiv 0$ whenever $\lambda > \overline{\lambda}$, showing that $J_\lambda(e) < 0$.

Let $\lambda > \overline{\lambda}$. Then there exists a function $v \in Y$, with $\|v\|_{q,w} = 1$, such that

$$\lambda\|v\|_{q,w}^q = \lambda > \frac{q}{p}\mathscr{M}([v]_K^p) + \frac{q}{r}\|v\|_{r,h}^r,$$

that is

$$J_\lambda(v) = \frac{1}{p}\mathscr{M}([v]_K^p) - \frac{\lambda}{q}\|v\|_{q,w}^q + \frac{1}{r}\|v\|_{r,h}^r < 0.$$

In particular, $J_\lambda(e) \le J_\lambda(v) < 0$, as required.

Hence, for any $\lambda > \overline{\lambda}$ equation $(\mathscr{P}_2^\lambda)$ has a nontrivial entire solution $e \in Y$ such that $J_\lambda(e) < 0$. Finally, we may assume $e \ge 0$ in $\mathbb{R}^n$. Indeed, $|e| \in Y$, being $\big|\,|e(x)| - |e(y)|\,\big| \le |e(x) - e(y)|$ for all $x, y \in \mathbb{R}^n$. Moreover, $J_\lambda(|e|) \le J_\lambda(e)$, since $[|u|]_K \le [u]_K$ for all $u \in Y$ and so $\mathscr{M}([|e|]_K^p) \le \mathscr{M}([e]_K^p)$ by $(\mathcal{M})$. This gives $J_\lambda(e) = J_\lambda(|e|)$, due to the minimality of e. $\square$

Proposition 3.4.2. *Non–negative entire solutions of* $(\mathscr{P}_2)$ *are exactly the critical points of the* $C^1(Y)$ *functional*

$$\mathscr{J}_\lambda(u) = \frac{1}{p}\mathscr{M}([u]_K^p) - \frac{\lambda}{q}\|u^+\|_{q,w}^q + \frac{1}{r}\|u\|_{r,h}^r, \quad u \in Y.$$

Proof. It is apparent from the proof of Lemma 3.3.3 that also $\mathscr{J}_\lambda$ is of class $C^1(Y)$. Along any non–negative function $u \in Y$ we have $\mathscr{J}_\lambda(u) = J_\lambda(u)$, so that non–negative entire solutions of $(\mathscr{P}_2)$ are critical points of $\mathscr{J}_\lambda$. To see the vice versa, first observe that $|u^+(x) - u^+(y)| \le |u(x) - u(y)|$ and $|u^-(x) - u^-(y)| \le |u(x) - u(y)|$ for all x, $y \in \mathbb{R}^n$, so that both u^+ and $u^- \in Y$ for all $u \in Y$. Furthermore, for all $u \in Y$

$$\langle u, -u^-\rangle_K = \iint_{\mathbb{R}^{2n}} \big(u^+(x)u^-(y) + u^-(x)u^+(y) + |u^-(x) - u^-(y)|^2\big)\cdot \\ \times |u(x) - u(y)|^{p-2}K(x-y)dxdy \ge [u^-]_K^2.$$

Finally, if $u \in Y$ is a critical point of $\mathscr{J}_\lambda$, then, taking $\varphi = -u^- \in Y$ as test function, we get by $(\mathcal{M})$

$$0 = M([u]_K^p)\langle u, -u^-\rangle_K + \|u^-\|_{r,h}^r \ge m_0[u^-]_K^p + \|u^-\|_{r,h}^r \ge 0,$$

in other words $u^- = 0$ in Y, that is the critical point u of $\mathscr{J}_\lambda$ is non–negative in $\mathbb{R}^n$. □

By Lemma 3.4.1 and Proposition 3.4.2 the global nontrivial non–negative minimizer $e \in Y$ of J_λ is also a critical point of $\mathscr{J}_\lambda$ and $\mathscr{J}_\lambda(e) = J_\lambda(e) < 0$.

Lemma 3.4.3. *For any* $v \in Y \setminus \{0\}$ *and* $\lambda > 0$ *there exist* ϱ*, depending on* v *and* λ*, with* $\varrho \in (0, [v]_K)$*, and* $\alpha = \alpha(\varrho, \lambda) > 0$ *such that* $\mathscr{J}_\lambda(u) \ge \alpha$ *for all* $u \in Y$*, with* $[u]_K = \varrho$*. Furthermore, also* J_λ *verifies the Mountain Pass geometry stated above.*

Proof. Let u be in Y, with $[u]_K = \varrho$. By $(\mathcal{M})$ and (3.2.3)

$$\mathscr{J}_\lambda(u) \ge J_\lambda(u) \ge \frac{m_0}{p}[u]_K^p - \frac{\lambda}{q}\mathfrak{C}_w^q[u]_K^q \ge \left(\frac{m_0}{p} - \frac{\lambda}{q}\mathfrak{C}_w^q[u]_K^{q-p}\right)[u]_K^p.$$

Therefore, it is enough to take ϱ, with $0 < \varrho < \min\big\{(m_0 q/p\lambda\mathfrak{C}_w^q)^{1/(q-p)}, [v]_K\big\}$ and the number $\alpha = \big(m_0/p - \lambda\mathfrak{C}_w^q\varrho^{q-p}/q\big)\varrho^p > 0$ satisfies the assertion. The last part of the lemma follows now at once. □

The proof of Lemma 3.4.3 in particular shows that for all $\lambda > 0$ the trivial solution $u = 0$ is a strict local minimum of both J_λ and $\mathscr{J}_\lambda$ in Y. Indeed, fix a positive number ϱ, with $\varrho < (m_0 q / p\lambda \mathfrak{C}_w^q)^{1/(q-2)}$. Then for all $u \in Y$, with $0 < \|u\| \le \varrho$, by $(\mathcal{M})$ and (3.2.3)

$$\mathscr{J}_\lambda(u) \ge J_\lambda(u) \ge \left(\frac{m_0}{p} - \frac{\lambda}{q} \mathfrak{C}_w^q \varrho^{q-p} \right) [u]_K^p > 0,$$

as stated.

Proof of the first part of Theorem 3.1.1. We recall that $\mathscr{J}_\lambda$ is of class $C^1(Y)$ by Proposition 3.4.2. Moreover, by Lemmas 3.4.1 and 3.4.3 and Theorem A.3 of [12], for all $\lambda > \overline{\lambda}$ there exists a sequence $(u_k)_k$ in Y such that for all k

$$\begin{gathered} c_\lambda \le \mathscr{J}_\lambda(u_k) \le c_\lambda + \frac{1}{k^2} \quad \text{and} \quad \|\mathscr{J}_\lambda'(u_k)\|_{Y'} \le \frac{2}{k}, \\ \text{where } c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathscr{J}_\lambda(\gamma(t)) \text{ and} \\ \Gamma = \{\gamma \in C([0,1]; Y) : \ \gamma(0) = 0, \ \gamma(1) = e\}. \end{gathered} \tag{3.4.1}$$

By Lemma 3.3.1 the sequence $(u_k)_k$ is bounded in Y. By Propositions 3.2.1, 3.2.2, Lemma 3.2.3 and the fact that $L^q(\mathbb{R}^n, w)$ and $L^r(\mathbb{R}^n, h)$ are uniformly convex Banach spaces in virtue of Proposition A.6 of [12], it is possible to extract a subsequence, still relabeled $(u_k)_k$, satisfying

$$\begin{gathered} u_k \rightharpoonup u \quad \text{in } \ Y, \qquad u_k \rightharpoonup u \quad \text{in } \ L^r(\mathbb{R}^n, h), \qquad [u_k]_K \to \ell, \\ u_k \to u \quad \text{in } \ L^q(\mathbb{R}^n, w), \qquad u_k^+ \to u^+ \quad \text{in } \ L^q(\mathbb{R}^n, w), \end{gathered} \tag{3.4.2}$$

for some $u \in Y$ and some $\ell \in \mathbb{R}_0^+$, since $|u_k^+(x) - u^+(x)| \le |u_k(x) - u(x)|$ for all $x \in \mathbb{R}^n$. In particular, by $(\mathcal{M})$

$$M([u_k]_K^p) \to M(\ell^p) > 0 \qquad \text{as } k \to \infty. \tag{3.4.3}$$

We claim that $\|u_k - u\| \to 0$ in Y. Clearly, $\langle \mathscr{J}_\lambda'(u_k) - \mathscr{J}_\lambda'(u), u_k - u \rangle \to 0$ as $k \to \infty$, since $u_k \rightharpoonup u$ in Y and $\mathscr{J}_\lambda'(u_k) \to 0$ in Y' as $k \to \infty$ by (3.4.1) and (3.4.2). Hence, as $k \to \infty$

$$\begin{aligned} o(1) = \langle \mathscr{J}_\lambda'(u_k) - \mathscr{J}_\lambda'(u), u_k - u \rangle &= M([u_k]_K^p)[u_k]_K^p + M([u]_K^p)[u]_K^p \\ &\quad - \langle u_k, u \rangle_K \left[M([u_k]_K^p) + M([u]_K^p) \right] \\ &\quad - \int_{\mathbb{R}^n} \big[g(x, u_k) - g(x, u) \big] (u_k - u) dx, \end{aligned} \tag{3.4.4}$$

where $g(x,z) = \lambda w(x)(z^+)^{q-1} - h(x)|z|^{r-2}z$ for $(x,z) \in \mathbb{R}^{n+1}$. Thus, using the notation (3.3.5) and putting $\mathscr{I}_k = \langle u_k, u\rangle_K \,[M([u_k]_K^p) + M([u]_K^p)]$, we get by (3.3.14), Hölder's inequality and (3.1.3)

$$\begin{aligned}
M([u_k]_K^2)&[u_k]_K^2 + M([u]_K^2)[u]_K^2\\
&= \mathscr{I}_k + \int_{\mathbb{R}^n} \big[g(x,u_k) - g(x,u)\big](u_k - u)\,dx + o(1)\\
&= \mathscr{I}_k + \int_{\mathbb{R}^n} (u_k - u)^2 dx \int_0^1 g_u(x, u + t(u_k - u))dt + o(1)\\
&\le \mathscr{I}_k + \int_{\mathbb{R}^n} (u_k - u)^2 dx \int_0^1 f_u(x, u + t(u_k - u))dt + o(1)\\
&\le \mathscr{I}_k + 2C_1 H^{(q-2)/q}\|u_k - u\|_{q,w}^2 + o(1).
\end{aligned}$$

Passing now to the limit as $k \to \infty$, we have

$$M(\ell^p)\ell^p + M([u]_K^p)[u]_K^p \le [u]_K^p\big[M(\ell^p) + M([u]_K^p)\big],$$

that is $\ell \le [u]_K$ by $(\mathcal{M})$ and (3.4.3). In other words,

$$[u]_K \le \lim_{k\to\infty}[u_k]_K = \ell \le [u]_K,$$

which implies at once that

$$[u_k - u]_K \to 0, \tag{3.4.5}$$

since $u_k \rightharpoonup u$ in $Y \hookrightarrow D_K^s(\mathbb{R}^n)$ and $D_K^s(\mathbb{R}^n)$ is a uniformly convex Banach space. Thus, (3.4.4) and (3.4.5) yield that

$$0 \le \int_{\mathbb{R}^n} h(x)(|u_k|^{r-2}u_k - |u|^{r-2}u)(u_k - u)\,dx \to 0$$

as $k \to \infty$. Hence,

$$\|u_k - u\|_{r,h}^r \le k_r \int_{\mathbb{R}^n} h(x)(|u_k|^{r-2}u_k - |u|^{r-2}u)(u_k - u)\,dx \to 0, \tag{3.4.6}$$

thanks to Simon's inequality $|\xi - \xi_0|^r \le k_r(|\xi|^{r-2}\xi - |\xi_0|^{r-2}\xi_0)\cdot(\xi - \xi_0)$ valid for all ξ, $\xi_0 \in \mathbb{R}$, being $r > 2$. Therefore, $\|u_k - u\|_{r,h} \to 0$ as $k \to \infty$. Combining this fact with (3.4.5) we obtain that $\|u_k - u\| \to 0$, that is $u_k \to u$ in Y as $k \to \infty$.

We next prove that u is a second independent nontrivial non–negative entire solution of problem $(\mathscr{P}_2^\lambda)$. Clearly, for any $\varphi \in Y$

$$\langle \mathscr{J}'_\lambda(u_k), \varphi\rangle = M([u_k]_K^p)\langle u_k, \varphi\rangle_K - \int_{\mathbb{R}^n} g(x, u_k)\varphi\, dx, \tag{3.4.7}$$

with g defined above. Now, by (3.3.9) and (3.4.2), as $k \to \infty$

$$\int_{\mathbb{R}^n} w(x)(u_k^+)^{q-1}\varphi\, dx \to \int_{\mathbb{R}^n} w(x)(u^+)^{q-1}\varphi\, dx$$

for all $\varphi \in Y$. Hence, since $u_k \to u$ in Y, passing to the limit as $k \to \infty$ in (3.4.7) and using also (3.3.10) and (3.4.5), we have for all $\varphi \in Y$

$$M([u]_K^p)\langle u, \varphi\rangle_{s,p} = \lambda \int_{\mathbb{R}^n} w(x)(u^+)^{q-1}\varphi\, dx - \int_{\mathbb{R}^n} h(x)|u|^{r-2}u\varphi\, dx,$$

since $\langle \mathscr{J}'_\lambda(u_k), \varphi\rangle \to 0$ as $k \to \infty$ for all $\varphi \in Y$ by (3.4.1).

Finally, $\mathscr{J}_\lambda(u) = c_\lambda = \lim_{k\to\infty} \mathscr{J}_\lambda(u_k)$, being $\mathscr{J}_\lambda \in C^1(Y)$ by Proposition 3.4.2. Therefore, u is a second nontrivial independent critical point for $\mathscr{J}_\lambda$, being $\mathscr{J}_\lambda(u) = c_\lambda > 0 > \mathscr{J}_\lambda(e)$, that is u is a second nontrivial non–negative entire solution of $(\mathscr{P}_2^\lambda)$. □

From Proposition 3.4.2 it is apparent that the second nontrivial non–negative entire solution $u = u_\lambda \in Y$, constructed in the proof above, is a critical point of J_λ, with $J_\lambda(u) = \mathscr{J}_\lambda(u) = c_\lambda > 0 > \mathscr{J}_\lambda(e) = J_\lambda(e)$. We next prove an important property of the asymptotic behavior as $\lambda \to \infty$ of c_λ.

Proposition 3.4.4. *Under the assumptions of Theorem 3.1.1*

$$\lim_{\lambda\to\infty} c_\lambda = 0,$$

where c_λ is the level in (3.4.1) of the Mountain Pass solution u_λ of $(\mathscr{P}_2^\lambda)$, constructed in the proof of Theorem 3.1.1.

Proof. Let $\lambda > \overline{\lambda}$ and let $e \in Y$ be the function given in Lemma 3.4.1. Since J_λ satisfies the Mountain Pass geometry of Lemma 3.4.3 and the path $t \mapsto te$, $t \in [0, 1]$, is in Γ defined in (3.4.1), it follows that there exists $t_\lambda \in (0, 1)$ such that $J_\lambda(t_\lambda e) = \max_{t\in[0,1]} J_\lambda(te)$, being $c_\lambda > 0$. Hence, $\langle J'_\lambda(t_\lambda e), e\rangle = 0$. Thus, $\langle J'_\lambda(t_\lambda e), t_\lambda e\rangle = 0$ and by (3.3.2)

$$M([t_\lambda e]_K^p)[t_\lambda e]_K^p = \lambda t_\lambda^q \|e\|_{q,w}^q - t_\lambda^r \|e\|_{r,h}^r. \tag{3.4.8}$$

Let $(\lambda_k)_k$ be a sequence, with $\lambda_k \to \infty$. We suppose that $\lambda_k > \overline{\lambda}$ for any $k \in \mathbb{N}$, without loss of generality. Thus, there exists $t \geq 0$ and a subsequence $(t_{k_j})_j$ of $(t_{\lambda_k})_k$ such that $t_{k_j} \to t$ as $j \to \infty$. Clearly $t = 0$. Otherwise, (3.4.8) implies

$$M([te]_K^p)[te]_K^p + \|te\|_{r,h}^r = \|te\|_{q,w}^q \lim_{j\to\infty} \lambda_{k_j} = \infty,$$

which gives an obvious contradiction. In particular, the entire sequence $(t_{\lambda_k})_k$ converges to 0. This shows that

$$\lim_{\lambda\to\infty} t_\lambda = 0,$$

being $(\lambda_k)_k$, with $\lambda_k \to \infty$, arbitrary. In conclusion, as $\lambda \to \infty$

$$\begin{aligned} 0 < c_\lambda \leq \max_{t\in[0,1]} J_\lambda(te) = J_\lambda(t_\lambda e) &= \frac{1}{p}\mathscr{M}([t_\lambda e]_K^p) - \frac{\lambda}{q} t_\lambda^q \|e\|_{q,w}^q + \frac{1}{r} t_\lambda^r \|e\|_{r,h}^r \\ &\leq \frac{1}{p}\mathscr{M}(t_\lambda^p [e]_K^2) + \frac{\|e\|_{r,h}^r}{r}\, t_\lambda^r \to 0, \end{aligned}$$

since of course $\mathscr{M}(\tau) \to 0$ as $\tau \to 0^+$. This completes the proof. □

Proposition 3.4.5. *Under the assumptions of Theorem 3.1.1*

$$\lim_{\lambda\to\infty} \|u_\lambda\| = 0,$$

where u_λ is the Mountain Pass solution of $(\mathscr{P}_2^\lambda)$, constructed in the proof of Theorem 3.1.1.

Proof. Using the notation of the statement, it is apparent that

$$\limsup_{\lambda\to\infty} [u_\lambda]_K < \infty \quad \text{and} \quad \limsup_{\lambda\to\infty} \|u_\lambda\|_{r,h} < \infty. \tag{3.4.9}$$

Otherwise from (3.3.4) and Proposition 3.4.4 we would get an easy contradiction. Now, fix a sequence $(\lambda_k)_k$, with $\lambda_k \to \infty$ as $k \to \infty$, and let $k \mapsto u_k = u_{\lambda_k}$ be the corresponding Mountain Pass sequence of solutions of $(\mathscr{P}_2^{\lambda_k})$. Hence, there exists a subsequence $(u_{k_j})_j$ of $(u_k)_k$, a function $u \in Y$ and a number $\ell \in \mathbb{R}_0^+$ such that $[u_{k_j}]_K \to \ell$ and

$$u_{k_j} \rightharpoonup u \text{ in } D_K^{s,p}(\mathbb{R}^n), \quad u_{k_j} \to u \text{ in } L^q(\mathbb{R}^n, w), \quad u_{k_j} \rightharpoonup u \text{ in } L^r(\mathbb{R}^n, h)$$

as $j \to \infty$, by Proposition 3.2.1 and Lemma 3.2.3. Assume by contradiction that $u \neq 0$. Then, (3.2.5) holds along any solution u_{k_j}, so that

$$M(\ell^p)\ell^p + \limsup_{j\to\infty} \|u_{k_j}\|_{r,h}^r = \|u\|_{q,w}^q \lim_{j\to\infty} \lambda_{k_j},$$

which contradicts (3.4.9). Therefore, $u = 0$ as stated and the *entire* sequence $(u_k)_k$ satisfies (3.4.2), with $u = 0$.

By (3.2.4) for all $k \in \mathbb{N}$ and all $\varphi \in Y$

$$M([u_k]_K^p)\langle u_k, \varphi\rangle_K + \int_{\mathbb{R}^n} h(x)|u_k|^{r-2}u_k\varphi dx = \lambda_k \int_{\mathbb{R}^n} w(x)|u_k|^{q-2}u_k\varphi dx.$$

Thus, by (3.3.10) the left–hand side approaches zero as $k \to \infty$, since $u_k \rightharpoonup 0$ in Y. Hence also the right–hand side should tend to zero as $k \to \infty$. In particular, by (3.3.9)

$$\lim_{k\to\infty} \lambda_k \|u_k\|_{q,w}^q = 0. \tag{3.4.10}$$

Therefore, $[u_k]_K \to \ell = 0$ by (3.2.7), that is $u_k \to 0$ in $D_K^{s,p}(\mathbb{R}^n)$, by (3.4.2) and the fact that $D_K^{s,p}(\mathbb{R}^n)$ is a uniformly convex Banach space. Moreover, (3.2.5) and (3.4.10) imply at once that $u_k \to 0$ in $L^r(\mathbb{R}^n, h)$. In conclusion, $u_k \to 0$ in Y. Since the sequence $(\lambda_k)_k$, with $\lambda_k \to \infty$, is arbitrary, this completes the proof. □

3.5 Existence of non–negative solutions

Put

$$\lambda^{**} = \inf \left\{ \lambda > 0 : (\mathscr{P}_2^\lambda) \textit{ admits a nontrivial non–negative entire solution} \right\}.$$

Lemma 3.4.1 assures that this definition is meaningful and that $\overline{\lambda} \geq \lambda^{**} \geq \lambda^*$, where λ^* is introduced in Section 3.2 thanks to Lemma 3.2.4.

Theorem 3.5.1. *For any $\lambda > \lambda^{**}$ problem $(\mathscr{P}_2^\lambda)$ admits a nontrivial non–negative entire solution $u_\lambda \in Y$.*

Proof. Fix $\lambda > \lambda^{**}$. By definition of λ^{**} there exists $\mu \in (\lambda^{**}, \lambda)$ such that J_μ has a nontrivial critical point $u_\mu \in Y$, with $u_\mu \geq 0$ in $\mathbb{R}^n$. Of course, u_μ is a subsolution for $(\mathscr{P}_2^\lambda)$. Consider the following minimization problem

$$\inf_{v\in\mathcal{C}} J_\lambda(v), \quad \mathcal{C} = \{v \in Y : v \geq u_\mu\}.$$

Clearly $\mathcal{C}$ is closed and convex by Proposition 3.2.2, and in turn also weakly closed in Y. Moreover, by Lemmas 3.3.1 and 3.3.3, Theorem 6.1.1 of [21] can be applied in Y and so in the weakly closed set $\mathcal{C}$. Hence, J_λ attains its infimum in $\mathcal{C}$, i.e. there exists $u_\lambda \geq u_\mu$ such that $J_\lambda(u_\lambda) = \inf_{v\in\mathcal{C}} J_\lambda(v)$.

We claim that u_λ is a solution of $(\mathscr{P}_2^\lambda)$, which is clearly non–negative. Indeed, take $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$. Put

$$\varphi_\varepsilon = \max\{0, u_\mu - u_\lambda - \varepsilon\varphi\} \geq 0 \quad \text{and} \quad v_\varepsilon = u_\lambda + \varepsilon\varphi + \varphi_\varepsilon,$$

so that $v_\varepsilon \in \mathcal{C}$. Of course $0 \leq \langle J_\lambda'(u_\lambda), v_\varepsilon - u_\lambda\rangle = \varepsilon\langle J_\lambda'(u_\lambda), \varphi\rangle + \langle J_\lambda'(u_\lambda), \varphi_\varepsilon\rangle$, and in turn

$$\langle J_\lambda'(u_\lambda), \varphi\rangle \geq -\frac{1}{\varepsilon}\langle J_\lambda'(u_\lambda), \varphi_\varepsilon\rangle. \tag{3.5.1}$$

Define $\Omega_\varepsilon = \{x \in \mathbb{R}^n : u_\lambda(x) + \varepsilon\varphi(x) \leq u_\mu(x) < u_\lambda(x)\}$, so that Ω_ε is a subset of $\operatorname{supp}\varphi$. Since u_μ is a subsolution of $(\mathscr{P}_2^\lambda)$ and $\varphi_\varepsilon \geq 0$, then $\langle J_\lambda'(u_\mu), \varphi_\varepsilon\rangle \leq 0$. In particular,

$$\langle J_\lambda'(u_\lambda), \varphi_\varepsilon\rangle = \langle J_\lambda'(u_\mu), \varphi_\varepsilon\rangle + \langle J_\lambda'(u_\lambda) - J_\lambda'(u_\mu), \varphi_\varepsilon\rangle \leq \langle J_\lambda'(u_\lambda) - J_\lambda'(u_\mu), \varphi_\varepsilon\rangle.$$

Using the notation of (3.3.5), we get

$$\begin{aligned}\left|\int_{\Omega_\varepsilon}[f(x,u_\lambda) - f(x,u_\mu)]\cdot[-u(x) - \varepsilon\varphi(x)]dx\right| \\ \leq \varepsilon\int_{\Omega_\varepsilon}|f(x,u_\lambda) - f(x,u_\mu)|\cdot|\varphi(x)|dx,\end{aligned}$$

since $0 \leq -u - \varepsilon\varphi = u_\mu - u_\lambda + \varepsilon|\varphi| < \varepsilon|\varphi|$ in Ω_ε. Therefore,

$$\begin{aligned}\langle J_\lambda'(u_\lambda), \varphi_\varepsilon\rangle \leq \langle M([u_\lambda]_K^p)u_\lambda - M([u_\mu]_K^p)u_\mu, \varphi_\varepsilon\rangle_K \\ + \varepsilon\int_{\Omega_\varepsilon}|f(x,u_\lambda) - f(x,u_\mu)|\cdot|\varphi(x)|dx\end{aligned}$$

By convexity of $\frac{1}{p}\mathscr{M}([u]_K^p)$ in Y we have

$$\begin{aligned}\frac{1}{p}\mathscr{M}([u_\mu]_K^p) &\geq \frac{1}{p}\mathscr{M}([u_\lambda]_K^p) + \langle M([u_\lambda]_K^p)u_\lambda, u_\mu - u_\lambda\rangle_K \\ &\geq \frac{1}{p}\mathscr{M}([u_\mu]_K^p) + \langle M([u_\mu]_K^p)u_\mu, u_\lambda - u_\mu\rangle_K \\ &\quad + \langle M([u_\lambda]_K^p)u_\lambda, u_\mu - u_\lambda\rangle_K,\end{aligned}$$

so that $\langle M([u_\lambda]_K^p)u_\lambda - M([u_\mu]_K^p)u_\mu, u_\mu - u_\lambda\rangle_K \leq 0$. Hence,

$$\langle J_\lambda'(u_\lambda), \varphi_\varepsilon\rangle \leq \varepsilon\left(\int_{\Omega_\varepsilon}\psi(x)dx + \iint_{\mathbb{R}^{2n}}\mathscr{U}(x,y)dxdy\right),$$

where $\psi(x) = |f(x,u_\lambda) - f(x,u_\mu)| \cdot |\varphi|$ and

$$\begin{aligned}\mathscr{U}(x,y) = \big[&M([u_\lambda]_K^p)|u_\lambda(x)-u_\lambda(y)|^{p-2}\big(u_\lambda(x)-u_\lambda(y)\big)\\ &- M([u_\mu]_K^p)|u_\mu(x)-u_\mu(y)|^{p-2}\big(u_\mu(x)-u_\mu(y)\big)\big]\cdot\\ &\times[\varphi(x)-\varphi(y)]\cdot K(x-y).\end{aligned}$$

Now

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} \mathscr{U}(x,y)dxdy &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\mathscr{U}(x,y)dxdy + \iint_{\Omega_\varepsilon\times(\mathbb{R}^n\setminus\Omega_\varepsilon)}\mathscr{U}(x,y)dxdy\\ &\qquad + \iint_{(\mathbb{R}^n\setminus\Omega_\varepsilon)\times\Omega_\varepsilon}\mathscr{U}(x,y)dxdy\\ &\le \iint_{\Omega_\varepsilon\times\mathbb{R}^n}|\mathscr{U}(x,y)|\,dxdy + \iint_{\mathbb{R}^n\times\Omega_\varepsilon}|\mathscr{U}(x,y)|\,dxdy.\end{aligned}$$

Thus,

$$\begin{aligned}\langle J_\lambda'(u_\lambda),\varphi_\varepsilon\rangle \le \varepsilon\Bigg(&\int_{\Omega_\varepsilon}\psi(x)dx + \iint_{\Omega_\varepsilon\times\mathbb{R}^n}|\mathscr{U}(x,y)|\,dxdy\\ &+\iint_{\mathbb{R}^n\times\Omega_\varepsilon}|\mathscr{U}(x,y)|\,dxdy\Bigg) = \varepsilon\mathscr{I}_\varepsilon.\end{aligned} \tag{3.5.2}$$

We claim that ψ is in $L^1(\operatorname{supp}\varphi)$. Indeed, $|f(x,u_\lambda)-f(x,u_\mu)|$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$, being

$$|f(x,u_\lambda)-f(x,u_\mu)| \le \lambda w(x)\left(u_\lambda^{q-1}+u_\mu^{q-1}\right) + h(x)\left(u_\lambda^{r-1}+u_\mu^{r-1}\right).$$

In fact, by Hölder's inequality and (3.1.2), we obtain

$$\int_{\operatorname{supp}\varphi} w(x)u_\lambda^{q-1}dx \le |\operatorname{supp}\varphi|^{1/2^*}\|w\|_\wp\|u_\lambda\|_{2^*}^{q-1} = C_1, \tag{3.5.3}$$

and $C_1 = C_1(\operatorname{supp}\varphi)$. Finally, since $h \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $u_\lambda \in L^r(\mathbb{R}^n,h)$, then

$$\int_{\operatorname{supp}\varphi} h(x)u_\lambda^{r-1}dx \le \left(\int_{\operatorname{supp}\varphi} h(x)dx\right)^{1/r}\|u_\lambda\|_{r,h}^{r-1} = C_2, \tag{3.5.4}$$

with $C_2 = C_2(\operatorname{supp}\varphi)$. The estimates (3.5.3) and (3.5.4) hold also for u_μ. The claim is so proved.

We next show that

$$\lim_{\varepsilon\to0^+}\mathscr{I}_\varepsilon = 0. \tag{3.5.5}$$

Indeed, $\int_{\Omega_\varepsilon}\psi(x)dx=o(1)$, since $|\Omega_\varepsilon|\to 0$ as $\varepsilon\to 0^+$, $\Omega_\varepsilon\subset\operatorname{supp}\varphi$ and by the fact that $\psi\in L^1(\operatorname{supp}\varphi)$.

Now, $\mathscr{U}(x,y)\in L^1(\mathbb{R}^{2n})$, being $Y\hookrightarrow D^{s,p}_K(\mathbb{R}^n)$ by (3.2.1). Therefore for all $\eta>0$ there exists R_η so large that

$$\iint_{(\operatorname{supp}\varphi)\times(\mathbb{R}^n\setminus B_{R_\eta})}|\mathscr{U}(x,y)|\,dxdy<\eta/2,\qquad \iint_{(\mathbb{R}^n\setminus B_{R_\eta})\times(\operatorname{supp}\varphi)}|\mathscr{U}(x,y)|\,dxdy<\eta/2.$$

Since $|\Omega_\varepsilon\times B_{R_\eta}|=|B_{R_\eta}\times\Omega_\varepsilon|\to 0$ as $\varepsilon\to 0^+$ and $\mathscr{U}\in L^1(\mathbb{R}^{2n})$, it follows that there exist $\delta_\eta>0$ and $\varepsilon_\eta>0$ such that for all $\varepsilon\in(0,\varepsilon_\eta]$

$$|\Omega_\varepsilon\times B_{R_\eta}|<\delta_\eta,$$
$$\iint_{\Omega_\varepsilon\times B_{R_\eta}}|\mathscr{U}(x,y)|\,dxdy<\eta/2\ \text{ and }\ \iint_{B_{R_\eta}\times\Omega_\varepsilon}|\mathscr{U}(x,y)|\,dxdy<\eta/2.$$

Therefore, for all $\varepsilon\in(0,\varepsilon_\eta]$

$$\iint_{\Omega_\varepsilon\times\mathbb{R}^n}|\mathscr{U}(x,y)|\,dxdy<\eta\quad\text{and}\quad\iint_{\mathbb{R}^n\times\Omega_\varepsilon}|\mathscr{U}(x,y)|\,dxdy<\eta,$$

being $\Omega_\varepsilon\subset\operatorname{supp}\varphi$. Hence (3.5.5) holds.

In conclusion, by (3.5.1), (3.5.2) and (3.5.5) we get $\langle J'_\lambda(u_\lambda),\varphi_\varepsilon\rangle\le o(\varepsilon)$ as $\varepsilon\to 0^+$, so that by (3.5.1) it follows that $\langle J'_\lambda(u_\lambda),\varphi\rangle\ge o(1)$ as $\varepsilon\to 0^+$. Therefore, $\langle J'_\lambda(u_\lambda),\varphi\rangle\ge 0$ for all $\varphi\in C_0^\infty(\mathbb{R}^N)$, that is $\langle J'_\lambda(u_\lambda),\varphi\rangle=0$ for all $\varphi\in C_0^\infty(\mathbb{R}^n)$. Since $Y=\overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|}$, we obtain that u_λ is a nontrivial non–negative solution of $(\mathscr{P}_2^\lambda)$. □

Theorem 3.5.2. *Problem* $(\mathscr{P}_2^{\lambda^{**}})$ *admits a nontrivial non–negative entire solution in* Y.

Proof. Let $(\lambda_k)_k$ be a strictly decreasing sequence converging to λ^{**} and $u_k\in Y$ be a nontrivial non–negative entire solution of $(\mathscr{P}_2^{\lambda_k})$. By (3.2.4) we get for all $\varphi\in Y$

$$M([u_k]_K^p)\langle u_k,\varphi\rangle_K=\int_{\mathbb{R}^n}f_n(x,u_k)\varphi\,dx,\tag{3.5.6}$$

where $k\mapsto f_n(x,u_k)=\lambda_n w(x)|u_k|^{q-2}u_k-h(x)|u_k|^{r-2}u_k$, similarly as defined in (3.3.5). By (3.2.4)–(3.2.6) and the monotonicity of $(\lambda_k)_k$, we obtain

$$\begin{aligned}m_0[u_k]_K^p+\|u_k\|_{r,h}^r&\le M([u_k]_K^p)[u_k]_K^p+\|u_k\|_{r,h}^r\\&=\lambda_k\|u_k\|_{q,w}^q\le\kappa_2^q\lambda_k^{1+rq/p(r-q)}\le\kappa_2^q\lambda_1^{1+rq/p(r-q)}.\end{aligned}$$

Therefore $([u_k]_K)_k$ and $(\|u_k\|_{r,h})_k$ are bounded, and in turn also $(\|u_k\|)_k$ is bounded. By Propositions 3.2.1, 3.2.2, Lemma 3.2.3 and the fact that $L^q(\mathbb{R}^n, w)$ and $L^r(\mathbb{R}^n, h)$ are uniformly convex Banach spaces in virtue of Proposition A.6 of [12], it is possible to extract a subsequence, still relabeled $(u_k)_k$, satisfying

$$\begin{gathered} u_k \rightharpoonup u \quad \text{in } Y, \qquad u_k \rightharpoonup u \quad \text{in } L^r(\mathbb{R}^n, h), \qquad [u_k]_K \to \ell, \\ u_k \to u \quad \text{in } L^q(\mathbb{R}^n, w), \qquad u_k \to u \quad \text{a.e. in } \mathbb{R}^n, \end{gathered} \tag{3.5.7}$$

for some $u \in Y$ and some $\ell \in \mathbb{R}_0^+$. In particular, by $(\mathcal{M})$

$$M([u_k]_K^p) \to M(\ell^p) > 0 \quad \text{as } k \to \infty.$$

Furthermore, $u \geq 0$ a.e. in $\mathbb{R}^n$ and we claim that u is the solution we are looking for.

To this aim, we first show that $[u]_K = \ell$. Since u_k is a nontrivial non–negative entire solution of $(\mathscr{P}_{\lambda_k})$, it follows that $\langle J'_{\lambda_k}(u_k), \varphi\rangle = 0$ for all $\varphi \in Y$ and for all $k \in \mathbb{N}$. In particular, taking $\varphi = u_k - u$, we obtain

$$\begin{aligned} 0 = \langle J'_{\lambda_k}(u_k), u_k - u\rangle &= M([u_k]_K^p)\left([u_k]_K^p - \langle u_k, u\rangle_K\right) \\ &- \lambda_k\left[\|u_k\|_{q,w}^q - \int_{\mathbb{R}^n} w(x)|u_k|^{q-2}u_k u dx\right] \\ &+ \|u_k\|_{r,h}^r - \int_{\mathbb{R}^n} h(x)|u_k|^{r-2}u_k u dx. \end{aligned} \tag{3.5.8}$$

Clearly $\langle u_k, u\rangle_K \to [u]_K^p$ and $\int_{\mathbb{R}^n} w(x)|u_k|^{q-2}u_k u\, dx \to \|u\|_{q,w}^q$ as $k \to \infty$ by (3.5.7). Thus, passing to the inferior limit in (3.5.8) and using also (3.3.10), we get

$$M(\ell^p)\left(\ell^p - [u]_K^p\right) + \left(\liminf_{k\to\infty}\|u_k\|_{r,h}^r - \|u\|_{r,h}^r\right) = 0. \tag{3.5.9}$$

Now, $[u]_K \leq \liminf_{k\to\infty}[u_k]_K \leq \ell$ and $\|u\|_{r,h}^r \leq \liminf_{k\to\infty}\|u_k\|_{r,h}^r$, so that the two addends in (3.5.9) vanish, being both non–negative. In particular, this yields that $[u]_K = \ell$, since $M(\ell^p) > 0$ by $(\mathcal{M})$. Therefore, passing to the limit in (3.5.6) as $k \to \infty$, we get by (3.3.9) and (3.3.10)

$$M([u]_K^p)\langle u, \varphi\rangle_K = \lambda^{**}\int_{\mathbb{R}^n} w(x)|u|^{q-2}u\varphi dx - \int_{\mathbb{R}^n} h(x)|u|^{r-2}u\varphi dx$$

for all $\varphi \in Y$. Hence u is a non–negative entire solution of $(\mathscr{P}_2^{\lambda^{**}})$.

We finally claim that $u \not\equiv 0$. Indeed, (3.2.6) applied to each u_k implies that $\|u_k\|_{q,w} \geq \kappa_1 \lambda_k^{1/(p-q)}$, so that by (3.5.7)

$$\|u\|_{q,w} = \lim_{k\to\infty} \|u_k\|_{q,w} \geq \kappa_1 (\lambda^{**})^{1/(p-q)} > 0,$$

since $\lambda_k \searrow \lambda^{**}$ and $\lambda^{**} > 0$. This shows the claim and completes the proof. □

Proof of Theorem 3.1.1–(*ii*). The existence of $\lambda^{**} \geq \lambda^*$ follows directly from Lemma 3.4.1, as already noted. The definition of λ^{**}, Theorems 3.5.1 and 3.5.2 show at once the validity of (*ii*) of Theorem 3.1.1. □

Of course the nontrivial non–negative entire solutions constructed in Theorems 4.3.6 and 4.3.9 are also critical points of $\mathscr{J}_\lambda$.

Chapter 4

Problem $(\mathscr{P}_3)$

In this chapter we deal with the question of the asymptotic stability of solutions of the following Kirchhoff system, governed by the fractional p–Laplacian operator, with an external force and nonlinear damping terms

$$(\mathscr{P}_3) \quad \begin{cases} u_{tt} + (-\Delta)_p^s u + \mu|u|^{p-2}u + \varrho(t)M([u]_{s,\Omega}^p)|u_t|^{p-2}u_t \\ \qquad +Q(t,x,u,u_t) + f(t,x,u) = 0 & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}_0^+ \times (\mathbb{R}^n \setminus \Omega), \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$, with $n > ps$, and $u = (u_1, \ldots, u_N) = u(t,x)$ represents the vectorial displacement, with $N \geq 1$. The term $\mu|u|^{p-2}u$, where $\mu \geq 0$, plays the role of a *perturbation*, M is a dissipative Kirchhoff function and $\varrho \geq 0$ is in $L^1_{\text{loc}}(\mathbb{R}_0^+)$.

Problem $(\mathscr{P}_3)$ generalizes the model proposed in [28], where the Laplacian operator is considered. Moreover, in problem $(\mathscr{P}_3)$ the source function f depends also on t and x, the Kirchhoff function is multiplied by the function $\varrho(t)$ and we add a perturbation and a nonlinear damping term.

The *Kirchhoff function* $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *verifies* the following condition.

$(\mathcal{M})$ *M is a continuous function, with $M(\tau) \geq 0$ for $\tau \geq 0$.*

Here we consider also the degenerate case, that is, from a physical point of view, when the base tension of the string modeled by the equation is zero. However, in some results of Sections 4.3 and 4.4, where specified, we need the non–degeneracy of the problem in order to overcome some technical difficulties due to the Kirchhoff structure of the problem. Sometimes the Kirchhoff function M is assumed Lipschitz continuous, but not always monotone, as in [38], even if the model proposed by Kirchhoff is clearly monotone. In $(\mathcal{M})$ we do not request the monotonicity of the function, as assumed in other papers such as [9, 10, 14], where M is taken of the standard form (1.3.1).

4.1 Structural setting

Throughout the chapter we assume

$$Q \in C(\mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N), \qquad f \in C(\mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N).$$

The function Q, representing a *nonlinear damping*, is always assumed to verify

$$(Q(t,x,u,v),v) \geq 0 \quad \text{for all arguments } t,x,u,v, \tag{4.1.1}$$

where $(\cdot,\cdot)$ is the inner product of $\mathbb{R}^N$.

The *external force* f is assumed to be derivable from a potential F, that is

$$f(t,x,u) = \partial_u F(t,x,u), \tag{4.1.2}$$

where $F \in C^1(\mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N \to \mathbb{R}_0^+)$ and $F(t,x,0) = 0$. Moreover, we allow $(f(t,x,u),u)$ to take negative values, that is

$$(f(t,x,u),u) \geq -\lambda |u|^p \qquad \text{in } \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N, \tag{4.1.3}$$

for some $\lambda \in [0,\lambda_1)$, where λ_1 denotes the first eigenvalue of the scalar fractional p–Laplacian in Ω, with zero Dirichlet boundary conditions, defined in Section 4.2.

Here we need and assume the condition $p^* \geq 2$, that is

$$2n/(n+2s) \leq p < n/s, \tag{4.1.4}$$

in order to get the standard embeddings.

Following [84], we also deal with a special case of $(\mathscr{P}_3)$, which occurs when $p=2$, $Q(t,x,u,v) = a(t)t^\alpha v$, with a satisfying

$$1/C \leq a(t) \leq C \qquad \text{in } \mathbb{R}_0^+,$$

for some $C > 0$ and $\alpha \in \mathbb{R}$, and $f(t, x, u) = V(t, x)u$, where V is a bounded continuous function in $I \times \Omega$. In other words, we study the solutions of

$$(\mathscr{P}_{3,\text{lin}}) \qquad \begin{cases} u_{tt} + (-\Delta)^s u + \mu u + \varrho(t) M([u]^2_{s,\Omega}) u_t \\ \qquad +a(t)t^\alpha u_t + V(t,x)u = 0 & \text{in } I \times \Omega, \\ u(t,x) = 0 & \text{on } I \times (\mathbb{R}^n \setminus \Omega), \end{cases}$$

where $I = [1, \infty)$ and for simplicity we treat only the case $N = 1$.

The behavior of solutions as $t \to \infty$ depends crucially on the parameter α. We show in Section 4.5 that if $|\alpha| \le 1$, then the rest field is asymptotically stable. On the other hand, when $\alpha < -1$ there exist oscillatory solutions that do not approach zero when $t \to \infty$, whereas if $\alpha > 1$ there exist solutions that approach nonzero functions ψ as $t \to \infty$. In this section, we also need an important result related to the eigenvalues and eigenfunctions of a perturbed problem driven by the fractional Laplacian. This result is proved in the Appendix, following [70] and [88].

In Section 4.2 we introduce the natural solution space for $(\mathscr{P}_3)$, we give the definition of a *strong solution* and we make the assumptions needed for f and Q. In Section 4.3 we prove some auxiliary lemmas and the main result of [79], concerning the global stability of the solutions, and in Section 4.4 we deal with local asymptotic stability for problem $(\mathscr{P}_3)$, also in the special case when the auxiliary function k and ϱ are related. Section 4.5 is dedicated to the linear case while in Section 4.6 we extend the results of Sections 4.3 and 4.4 when the fractional p–Laplacian operator is replaced by a more general elliptic nonlocal integro–differential operator. In particular, we consider the problem

$$(\mathscr{P}_{3,K}) \quad \begin{cases} u_{tt} - \mathcal{L}_K u + \varrho(t) M([u]^p_{K,\Omega}) |u_t|^{p-2} u_t + \mu |u|^{p-2} u \\ \qquad +Q(t,x,u,u_t) + f(t,x,u) = 0 & \text{in } \mathbb{R}^+_0 \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}^+_0 \times (\mathbb{R}^n \setminus \Omega), \end{cases}$$

where $\mathcal{L}_K$ is an integro–differential nonlocal operator and $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ is a positive *weight*, satisfying the natural restriction listed in Section 4.6. The last section is an appendix, where we give the detailed proof of the results related to eigenvalues and eigenfunctions of a perturbed problem involving the fractional Laplacian.

4.2 Preliminaries

We denote with

$$\langle \varphi, \psi \rangle = \int_\Omega (\varphi, \psi) dx$$

the elementary bracket pairing in $\Omega \subset \mathbb{R}^n$, provided that $(\varphi, \psi) \in L^1(\Omega)$. We consider $L^\rho(\Omega; \mathbb{R}^N)$ where $\rho > 1$, endowed with the natural norm

$$\|\varphi\|_\rho = \left(\int_\Omega |\varphi|^\rho \right)^{1/\rho}.$$

Following Section 1.1, we introduce the natural solution space, adapting the construction to the vectorial case. We recall that $D_0^{s,p}(\Omega; \mathbb{R}^N) = \overline{C_0^\infty(\Omega; \mathbb{R}^N)}^{\|\cdot\|_\Omega}$, where $\|\cdot\|_\Omega$ is the standard fractional Gagliardo norm, given by

$$\|u\|_\Omega = \left(\iint_{\Omega \times \Omega} |u(x) - u(y)|^p |x - y|^{-(n+ps)} dxdy \right)^{1/p}$$

for all $u \in W_0^{s,p}(\Omega; \mathbb{R}^N)$. Furthermore, $D^{s,p}(\mathbb{R}^n; \mathbb{R}^N)$ denotes the fractional Beppo–Levi space, that is the completion of $C_0^\infty(\mathbb{R}^n; \mathbb{R}^N)$, with respect to the norm

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^p |x - y|^{-(n+ps)} dxdy \right)^{1/p}.$$

Moreover, by Theorems 1 and 2 of [67]

$$\begin{aligned} \|u\|^p_{L^{p^*}(\mathbb{R}^n;\mathbb{R}^N)} &\le c_{n,p} \frac{s(1-s)}{(n-ps)^{p-1}} [u]^p_{s,p}, \\ \int_{\mathbb{R}^n} |u(x)|^p \frac{dx}{|x|^{ps}} &\le c_{n,p} \frac{s(1-s)}{(n-ps)^p} [u]^p_{s,p} \end{aligned} \tag{4.2.1}$$

for all $u \in D^{s,p}(\mathbb{R}^n; \mathbb{R}^N)$, where $c_{n,p}$ is a positive constant depending only on n and p. Hence

$$D^{s,p}(\mathbb{R}^n; \mathbb{R}^N) = \{ u \in L^{p^*}(\mathbb{R}^n; \mathbb{R}^N) : |u(x) - u(y)| \cdot |x - y|^{-(s+n/p)} \in L^p(\mathbb{R}^{2n}; \mathbb{R}^N) \}.$$

Following [55], we put

$$\tilde{D}^{s,p}(\Omega; \mathbb{R}^N) = \{ u \in L^{p^*}(\Omega; \mathbb{R}^N) : \tilde{u} \in D^{s,p}(\mathbb{R}^n; \mathbb{R}^N) \},$$

with the norm $[u]_{s,\Omega} = [\tilde{u}]_{s,p}$, where $\tilde{u}$ is the natural extension of u in the entire $\mathbb{R}^n$, with value 0 in $\mathbb{R}^n \setminus \Omega$. Clearly,

$$[u]_{s,\Omega} = \left(\|u\|_{\Omega}^p + 2 \int_{\Omega} |u(x)|^p dx \int_{\mathbb{R}^n \setminus \Omega} |x-y|^{-(n+ps)} dy \right)^{1/p} \geq \|u\|_{\Omega}.$$

Since here Ω is regular, an application of Theorem 1.4.2.2 of [55] shows that $\tilde{D}^{s,p}(\Omega;\mathbb{R}^N) = \overline{C_0^\infty(\Omega;\mathbb{R}^N)}^{[\,\cdot\,]_{s,\Omega}}$. Finally, since Ω is bounded and regular, by (4.2.1) there exists a constant $c_\Omega > 0$ such that

$$c_\Omega \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n;\mathbb{R}^N)} \leq \|\tilde{u}\|_{s,p} = [u]_{s,p} \leq \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n;\mathbb{R}^N)}$$

for all $u \in \tilde{D}^{s,p}(\Omega;\mathbb{R}^N)$, and so, using also Corollary 1.4.4.10 of [55], we have the main property

$$\begin{aligned}\tilde{D}^{s,p}(\Omega;\mathbb{R}^N) &= \{u \in W_0^{s,p}(\Omega;\mathbb{R}^N) \,:\, u\, d(\cdot,\partial\Omega)^{-s} \in L^p(\Omega;\mathbb{R}^N)\} \\ &= \{u \in D^{s,p}(\mathbb{R}^n;\mathbb{R}^N) \,:\, u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\} \\ &= \{u \in W^{s,p}(\Omega;\mathbb{R}^N) \,:\, \tilde{u} \in W^{s,p}(\mathbb{R}^n;\mathbb{R}^N)\},\end{aligned}$$

where $d(x,\partial\Omega)$ is the distance from x to the boundary $\partial\Omega$ of Ω.

For simplicity and abuse of notation, in the following we still denote by u the extension of every function $u \in \tilde{D}^{s,p}(\Omega;\mathbb{R}^N)$, by setting $u = 0$ in $\mathbb{R}^n \setminus \Omega$. Moreover, we put

$$\langle \varphi, \psi \rangle_{s,\Omega} = \iint_{\mathbb{R}^{2n}} |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))|x-y|^{-(n+ps)} dxdy,$$

for all $\varphi, \psi \in \tilde{D}^{s,p}(\Omega;\mathbb{R}^N)$.

Let λ_1 be *the first eigenvalue of the scalar problem*

$$\begin{cases} (-\Delta)_p^s u = \lambda\, |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.2.2}$$

in $\tilde{D}^{s,p}(\Omega)$, that is λ_1 is defined by the Rayleigh quotient

$$\lambda_1 = \inf_{u \in \tilde{D}^{s,p}(\Omega),\ u \neq 0} \frac{\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^p |x-y|^{-(n+ps)} dxdy}{\int_\Omega |u|^p dx}. \tag{4.2.3}$$

By Theorem 5 of [65] *the infimum in* (4.2.3) *is achieved and* $\lambda_1 > 0$.

Now define

$$X' = C(\mathbb{R}_0^+ \to \tilde{D}^{s,p}(\Omega;\mathbb{R}^N)) \cap C^1(\mathbb{R}_0^+ \to L^2(\Omega;\mathbb{R}^N))$$

and

$$X = \left\{\phi \in X' \,:\, E\phi \text{ is locally bounded on } \mathbb{R}_0^+\right\},$$

where $E\phi$ is the *total energy of the field* ϕ, that is

$$E\phi = E\phi(t) = \frac{1}{2}\|\phi_t\|_2^2 + \mathscr{A}\phi(t) + \mathscr{F}\phi(t),$$

where

$$\mathscr{A}\phi(t) = \frac{1}{p}[\phi]_{s,p}^p + \frac{\mu}{p}\|\phi\|_p^p$$

and $\mathscr{F}\phi$, the *potential energy of the field*, is given by

$$\mathscr{F}\phi = \mathscr{F}\phi(t) = \int_\Omega F(t,x,\phi(t,x))dx.$$

In writing $E\phi$ and $\mathscr{F}\phi$ we make the tacit agreement that $F(t,\cdot,\phi(t,\cdot))$ is of class $L^1(\Omega)$ for all $t \in \mathbb{R}_0^+$, so $\mathscr{F}\phi$ is *well–defined*.

Our motivation for introducing the set X is that a solution of $(\mathscr{P}_3)$ should, whatever else, be sought in a function space for which the total energy is well–defined and bounded on any finite interval, and X has just this property.

The definition of X moreover applies without reference to the external force condition (4.1.3), so that the definition of solution given below applies equally whether f satisfies (4.1.3) or not. Of course f must be derivable from a potential as in (4.1.2).

A *strong solution* of $(\mathscr{P}_3)$ is a function $u \in X$ satisfying the following two conditions:

(A) *Distribution identity*

$$\langle u_t, \phi\rangle]_0^t = \int_0^t \left\{\langle u_t, \phi_t\rangle - \langle u,\phi\rangle_{s,p} - \langle \mu|u|^{p-2}u,\phi\rangle - \langle \mathcal{D}u,\phi\rangle\right\} d\tau$$
$$\mathcal{D}u = \varrho(\cdot)M([u]_{s,\Omega}^p)|u_t|^{p-2}u_t + Q(\cdot,\cdot,u,u_t) + f(\cdot,\cdot,u) : \mathbb{R}_0^+ \times \Omega \to \mathbb{R}^N$$

for all $t \in \mathbb{R}_0^+$ and $\phi \in X$.

(B) *Conservation law*

(i) $\mathscr{D}u = \varrho(t)M([u]_{s,\Omega}^p)\|u_t\|_p^p + \langle Q(t,\cdot,u,u_t),u_t\rangle - \mathscr{F}_t u \quad \in L^1_{\rm loc}(\mathbb{R}_0^+)$,

(ii) $\mathscr{F}_t u \le 0, \qquad t \mapsto Eu(t) + \int_0^t \mathscr{D}u(\tau)d\tau \quad$ is non–increasing in $\mathbb{R}_0^+$.

We emphasize that *condition* (B) *is an essential attribute of solution.* Indeed, standard existence theorems for $(\mathscr{P}_3)$ in the literature always yield solutions satisfying *both* (A) and (B) in the stronger form in which the function in (B)–(ii) *is assumed to be constant.* On the other hand (A) alone does not imply (B), even if the integrability condition (B)–(i) is assumed a priori. Conditions (B)–(ii) and (4.1.1) imply, however, that *Eu is non–increasing in $\mathbb{R}_0^+$.*

A remaining issue is to determine a category of functions f and Q for which the preceding definition is meaningful. In particular, it must be shown that

$$\langle f(t,\cdot,u),\phi(t,\cdot)\rangle,\ \langle Q(t,\cdot,u,u_t),\phi(t,\cdot)\rangle \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+), \tag{4.2.4}$$

so that the right–hand side integral in identity (A) will be well–defined.

To obtain (4.2.4) observe first that if $u,\phi\in X$, then

$$u,\phi \in C(\mathbb{R}_0^+ \to L^{p^*}(\Omega;\mathbb{R}^N)). \tag{4.2.5}$$

We make the following natural hypotheses on f and Q, in the principal case $n\ge 2$.

(H) *Conditions* (4.1.2) *and* (4.1.3) *hold and there exists an exponent $q\ge p$ such that*

$$(a) \qquad |f(t,x,u)| \le \mathrm{const.}\,(1+|u|^{q-1})$$

for all $(t,x,u)\in\mathbb{R}_0^+\times\Omega\times\mathbb{R}^N$.

Moreover, if $q>p^*$, f verifies (a) and

$$(b)\quad (f(t,x,u),u) \ge \kappa_1|u|^q - \kappa_2|u|^{1/q} - \kappa_3|u|^{p^*} \quad \textit{for all } (t,x,u)\in\mathbb{R}_0^+\times\Omega\times\mathbb{R}^N$$

for appropriate constants $\kappa_1>0$, $\kappa_2,\kappa_3\ge 0$.

When $f\equiv 0$ then (H)–(a) holds for any fixed $q\in[p,p^*]$, so that (H)–(b) is unnecessary.

(AS) *Condition* (4.1.1) *holds and there are exponents m, r satisfying*

$$2\le m<r\le\nu,\qquad \nu=\max\{q,p^*\},$$

where m' and r' are the Hölder conjugates of m and r, and non–negative continuous functions $d_1=d_1(t,x)$, $d_2=d_2(t,x)$, such that for all arguments t, x, u, v,

$$(a)\qquad \begin{aligned} |Q(t,x,u,v)| &\le d_1(t,x)^{1/m}(Q(t,x,u,v),v)^{1/m'} \\ &\quad + d_2(t,x)^{1/r}(Q(t,x,u,v),v)^{1/r'} \end{aligned}$$

and the following functions δ_1 *and* δ_2 *are well–defined*

$$\delta_1(t) = \|d_1(t,\cdot)\|_{\nu/(\nu-m)}, \qquad \delta_2(t) = \begin{cases} \|d_2(t,\cdot)\|_{\nu/(\nu-r)}, & \textit{if } r < \nu, \\ \|d_2(t,\cdot)\|_\infty, & \textit{if } r = \nu. \end{cases}$$

Moreover, there are functions $\sigma = \sigma(t)$, $\omega = \omega(\tau)$, $\tau = |v|$, *such that*

$$(b) \qquad (Q(t,x,u,v),v) \geq \sigma(t)\omega(|v|) \quad \textit{for all arguments } t,\ x,\ u,\ v,$$

where $\omega \in C(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$ *is such that*

$$\omega(0) = 0, \qquad \omega(\tau) > 0 \quad \textit{for } 0 < \tau < 1, \qquad \omega(\tau) = \tau^2 \quad \textit{for } \tau \geq 1,$$

while $\sigma \geq 0$ *and* $\sigma^{1-\wp} \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+)$ *for some exponent* $\wp > 1$.

The conditions (4.2.4) are consequences of the assumptions (H) and (AS). Indeed, by (H)–(a) for all u, $\phi \in X$

$$|\langle f(t,\cdot,u),\phi(t,\cdot)\rangle| \leq \mathrm{const.}(\|\phi\|_1 + \|u\|_q^{q-1}\cdot\|\phi\|_q),$$

so that $\langle f(t,\cdot,u),\phi(t,\cdot)\rangle$ is locally bounded on $\mathbb{R}_0^+$, whenever $u,\phi \in X$, since $\|\cdot\|_q$ of any function of X is locally bounded in $\mathbb{R}_0^+$ either by the Sobolev embedding theorem when $1 < q \leq p^*$ or by the assumption (H)–(b) when $q > p^*$. Indeed, in the latter case for all $u \in X$

$$\begin{aligned} F(t,x,u) &= \int_0^1 (f(t,x,\tau u),u)d\tau \\ &\geq \int_0^1 (\kappa_1|u|^q\tau^{q-1} - \kappa_2|u|^{1/q}\tau^{-1/q'} - \kappa_3|u|^{p^*}\tau^{p^*-1})d\tau \\ &= \frac{\kappa_1}{q}|u|^q - q\kappa_2|u|^{1/q} - \frac{\kappa_3}{p^*}|u|^{p^*}, \end{aligned}$$

and, since $\kappa_1 > 0$, we then have

$$(4.2.6) \qquad \|u(t,\cdot)\|_q^q \leq \frac{q}{\kappa_1}\left(\mathscr{F}u(t) + q\kappa_2|\Omega|^{1/q'}\|u(t,\cdot)\|_1^{1/q} + \frac{\kappa_3}{p^*}\|u(t,\cdot)\|_{p^*}^{p^*}\right).$$

Therefore $\|u\|_q \in L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+)$, since $\mathscr{F}u$ is locally bounded by the definition of X and the fact that $u \in \tilde{D}^{s,p}(\Omega;\mathbb{R}^N)$. This completes the proof of the claim in the case $q > p^*$.

Moreover

$$(4.2.7) \qquad \langle Q(t,\cdot,u,u_t),\phi(t,\cdot)\rangle \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+),$$

since δ_1, $\delta_2 \in L^1_{\rm loc}(\mathbb{R}^+_0)$ by (AS)–(a) and $\mathscr{D}u \in L^1_{\rm loc}(\mathbb{R}^+_0)$ by (B)–(i), see Section 2 of [85]. Thus (4.2.4) holds and so the *distribution identity* (A) is well–defined.

The main results are proved *under the additional assumption that*

$$\mathscr{F}_t u \le 0 \qquad \textit{in} \quad \mathbb{R}^+_0, \tag{4.2.8}$$

along a global solution $u \in X$. This request is trivially automatic, whenever either f does not depend on t, or $F_t \le 0$ in $\mathbb{R}^+_0 \times \Omega \times \mathbb{R}^N$, as well as in many other cases, and for simplicity, we assume it in the definition of solution.

4.3 Global asymptotic stability for $(\mathscr{P}_3)$

In this section, we present some auxiliary lemmas and the proof of the main theorem.

Theorem 4.3.1. *Let* (H) *and* (AS) *hold. Suppose there exists a function* k *satisfying either*

$$k \in CBV(\mathbb{R}^+_0 \to \mathbb{R}^+_0) \quad \textit{and} \quad k \notin L^1(\mathbb{R}^+_0) \quad \textit{or} \tag{4.3.1}$$

$$k \in W^{1,1}_{\rm loc}(\mathbb{R}^+_0 \to \mathbb{R}^+_0), \quad k \not\equiv 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\displaystyle\int_0^t |k'(\tau)|d\tau}{\displaystyle\int_0^t k(\tau)d\tau} = 0. \tag{4.3.2}$$

Assume finally

$$\liminf_{t\to\infty} \left[\mathcal{A}(k(t)) + \mathcal{C}(k(t))\right] \Big/ \int_0^t k d\tau < \infty\,, \tag{4.3.3}$$

where

$$\begin{gathered} \mathcal{A}(k(t)) = \mathcal{B}(k(t)) + \left(\int_0^t \sigma^{1-\wp} k^{\wp}\, d\tau\right)^{1/\wp}, \\ \mathcal{B}(k(t)) = \left(\int_0^t \delta_1 k^m\, d\tau\right)^{1/m} + \left(\int_0^t \delta_2 k^r\, d\tau\right)^{1/r} \end{gathered} \tag{4.3.4}$$

and $\mathcal{C}(k(t))$ *is defined in Lemma 4.3.4. Then along any strong solution* u *of* $(\mathscr{P}_3)$ *we have*

$$\lim_{t\to\infty} Eu(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} (\|u_t\|_2 + [u]_{s,\Omega}) = 0. \tag{4.3.5}$$

The integral condition (4.3.3) prevents the damping term Q being either too small (*underdamping*) or too large (*overdamping*) as $t \to \infty$ and was introduced by *Pucci* and *Serrin* in [85], see also [83].

When $N = 1$, or Q is tame, that is

($\mathscr{T}$) Q is *tame*, if *there exists* $\kappa \geq 1$ *such that*

$$|Q(t, x, u, v)| \cdot |v| \leq \kappa (Q(t, x, u, v), v) \qquad \text{(automatic if } N = 1\text{)},$$

then condition (AS)–(a) is equivalent to

$$|Q(t, x, u, v)| \leq \text{const.}\{d_1(t, x)|v|^{m-1} + d_2(t, x)|v|^{r-1}\} \tag{4.3.6}$$

(this can be proved exactly as in Remark 1 of Section 5 in [85]).

Before proving Theorem 4.3.1 we give three preliminary lemmas under conditions (AS)–(a) and (H) which make the definition of strong solution meaningful.

Lemma 4.3.2. *Let u be a strong solution of $(\mathscr{P}_3)$. Then the non–increasing energy function Eu verifies in $\mathbb{R}_0^+$*

$$Eu \geq \frac{1}{2}\|u_t\|_2^2 + \frac{\mu}{p}\|u\|_p^p + \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)[u]_{s,\Omega}^p \geq 0. \tag{4.3.7}$$

Moreover

$$\begin{gathered} \|u\|_2,\ \|u_t\|_2,\ [u]_{s,\Omega},\ \|u\|_q,\ \|u\|_{p^*},\ M([u]_{s,\Omega}^p) \in L^\infty(\mathbb{R}_0^+), \\ \mathscr{D}u = \langle Q(t, x, u, u_t), u_t\rangle + \varrho(t) M([u]_{s,\Omega}^p)\|u_t\|_p^p - \mathscr{F}_t u \in L^1(\mathbb{R}_0^+). \end{gathered} \tag{4.3.8}$$

Furthermore, if

$$\inf_{\mathbb{R}_0^+} M(t) > 0 \tag{4.3.9}$$

then $\varrho\|u_t\|_p^p \in L^1(\mathbb{R}_0^+)$.

Proof. By (4.1.3) we get $F(t, x, u) \geq -\lambda|u|^p/p$, so $\mathscr{F}u(t) \geq -\lambda\|u(t, \cdot)\|_p^p/p$. Clearly, if $u \in \tilde{D}^{s,p}(\Omega; \mathbb{R}^N)$, then $|u| \in \tilde{D}^{s,p}(\Omega)$. Moreover, we have $\lambda_1\|u\|_p^p = \lambda_1\||u|\|_p^p \leq [|u|]_{s,\Omega}^p \leq [u]_{s,\Omega}^p$ for all $u \in \tilde{D}^{s,p}(\Omega; \mathbb{R}^N)$, by (4.2.3) and elementary inequalities. Hence by the definition of E and (4.2.3) we have

$$\begin{aligned} Eu &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}[u]_{s,\Omega}^p + \frac{\mu}{p}\|u\|_p^p + \mathscr{F}u(t) \\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}[u]_{s,\Omega}^p + \frac{\mu}{p}\|u\|_p^p - \frac{\lambda}{p}\|u\|_p^p \\ &\geq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}[u]_{s,\Omega}^p + \frac{\mu}{p}\|u\|_p^p - \frac{\lambda}{p\lambda_1}[u]_{s,\Omega}^p \\ &= \frac{1}{2}\|u_t\|_2^2 + \frac{\mu}{p}\|u\|_p^p + \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)[u]_{s,\Omega}^p \geq 0. \end{aligned}$$

Since $0 \leq \lambda < \lambda_1$, we get (4.3.7).

In order to prove conditions (4.3.8), first note that Eu is bounded above by $Eu(0)$. We immediately have that $\|u_t\|_2$, $\|u\|_p$ and $[u]_{s,\Omega} \in L^\infty_{\text{loc}}(\mathbb{R}_0^+)$. Since the space $\tilde{D}^{s,p}(\Omega;\mathbb{R}^N)$ is continuously embedded in $L^2(\Omega;\mathbb{R}^N)$ by (4.1.4), we have also $\|u\|_2 \in L^\infty_{\text{loc}}(\mathbb{R}_0^+)$. Furthermore, when $p \leq q \leq p^*$, since the embeddings $\tilde{D}^{s,p}(\Omega;\mathbb{R}^N) \hookrightarrow L^q(\Omega;\mathbb{R}^N)$ and $\tilde{D}^{s,p}(\Omega;\mathbb{R}^N) \hookrightarrow L^{p^*}(\Omega;\mathbb{R}^N)$ are continuous, also $\|u\|_q$, $\|u\|_{p^*} \in L^\infty(\mathbb{R}_0^+)$.

If $q > p^*$, using (H)–(b) we get (4.2.6) since $u \in X$. Therefore also $\|u\|_q$ is of class $L^\infty(\mathbb{R}_0^+)$, since $\mathscr{F}u$ is bounded above – actually also below by (4.1.3) – being $Eu(t) \leq Eu(0)$ and $\|u_t\|_2$, $[u]_{s,\Omega} \in L^\infty(\mathbb{R}_0^+)$, so that also $M([u]_{s,\Omega}^p)$ is in $L^\infty(\mathbb{R}_0^+)$. This completes the proof of (4.3.8).

Moreover, $\mathscr{D}u \in L^1(\mathbb{R}_0^+)$ is a consequence of (B): indeed $Eu \geq 0$ by (4.3.7) and $\mathscr{D}u \geq 0$, giving

$$\begin{aligned} 0 \leq \int_0^t \mathscr{D}u(\tau)d\tau &= \int_0^t \left\{\langle Q(\tau,\cdot,u,u_t),u_t\rangle + \varrho(t)M([u]_{s,\Omega}^p)\|u_t\|_p^p - \mathscr{F}_t u\right\} d\tau \\ &\leq Eu(0) - Eu(t) \leq Eu(0). \end{aligned}$$

Finally, if $\inf_{\mathbb{R}_0^+} M(t) > 0$, put m_0 the corresponding positive value. We have

$$m_0 \int_0^t \varrho(\tau)\|u_t\|_p^p d\tau \leq \int_0^t \varrho(\tau)M([u]_{s,\Omega}^p)\|u_t\|_p^p d\tau < \infty$$

for all $t > 0$, so $\varrho\|u_t\|_p^p \in L^1(\mathbb{R}_0^+)$, as claimed. □

By (B)–(ii) and Lemma 4.3.2 it is clear that there exists $l \geq 0$ such that

$$\lim_{t\to\infty} Eu(t) = l. \tag{4.3.10}$$

Lemma 4.3.3. *Let u be a strong solution of* $(\mathscr{P}_3)$ *and suppose that* $l > 0$ *in* (4.3.10)*. Then there exists a constant* $\alpha = \alpha(l) > 0$ *such that on* $\mathbb{R}_0^+$

$$\mathscr{L}u = \|u_t\|_2^2 + [u]_{s,\Omega}^p + \mu\|u\|_p^p + \langle f(t,\cdot,u), u\rangle \geq \alpha. \tag{4.3.11}$$

Proof. The proof relies on the principal ideas used for proving [14, Lemma 3.3] and [85, Lemma 3.4]. Since $Eu(t) \geq l$ for all $t \in \mathbb{R}_0^+$ it follows that

$$\|u_t\|_2^2 + [u]_{s,\Omega}^p + \mu\|u\|_p^p \geq \eta(l - \mathscr{F}u) \qquad \text{on} \quad \mathbb{R}_0^+, \tag{4.3.12}$$

where $\eta = \min\{2, p\} > 1$. Let

$$J_1 = \{t \in \mathbb{R}_0^+ : \mathscr{F}u(t) \leq l/\eta'\}, \qquad J_2 = \{t \in \mathbb{R}_0^+ : \mathscr{F}u(t) > l/\eta'\}. \tag{4.3.13}$$

For $t \in J_1$

$$\|u_t\|_2^2 + [u]_{s,\Omega}^p + \mu\|u\|_p^p \geq \eta\left(l - \frac{l}{\eta'}\right) = l. \tag{4.3.14}$$

Using (4.1.3) and (4.3.14), we find that in J_1

$$\begin{aligned}
\mathscr{L}u &= \|u_t\|_2^2 + [u]_{s,\Omega}^p + \mu\|u\|_p^p + \langle f(t,\cdot,u), u\rangle \\
&\geq \|u_t\|_2^2 + [u]_{s,\Omega}^p + \mu\|u\|_p^p - \lambda\|u\|_p^p \\
&\geq \|u_t\|_2^2 + [u]_{s,\Omega}^p + \mu\|u\|_p^p - \frac{\lambda}{\lambda_1}[u]_{s,\Omega}^p \geq \left(1 - \frac{\lambda}{\lambda_1}\right) l.
\end{aligned}$$

Before dividing the proof into two parts, we observe that

$$|\mathscr{F}u| \leq C_1(\|u\|_1 + \|u\|_q^q) \tag{4.3.15}$$

by (H)–(a), see [85]. Now we denote with ξ_ρ the Sobolev constant of the continuous embedding $\tilde{D}^{s,p}(\Omega;\mathbb{R}^N) \hookrightarrow L^\rho(\Omega;\mathbb{R}^N)$, for all $1 \leq \rho \leq p^*$, that is,

$$\|u\|_\rho \leq \xi_\rho [u]_{s,\Omega}, \tag{4.3.16}$$

where $\xi_\rho = \xi_{p^*}|\Omega|^{1/\rho - 1/p^*}$ and depends on n, p, s, ρ, $|\Omega|$.
Case 1. $q \leq p^*$. By (4.3.15) and (4.3.16) we have

$$|\mathscr{F}u| \leq C([u]_{s,\Omega} + [u]_{s,\Omega}^q) \tag{4.3.17}$$

consequently in J_2

$$\frac{l}{\eta'} < \mathscr{F}u(t) \leq 2C \begin{cases} [u(t,\cdot)]_{s,\Omega}, & \text{if } [u(t,\cdot)]_{s,\Omega} \leq 1, \\ [u(t,\cdot)]_{s,\Omega}^q, & \text{if } [u(t,\cdot)]_{s,\Omega} > 1, \end{cases} \tag{4.3.18}$$

for an appropriate constant $C > 0$, depending on C_1 given in (4.3.15), ξ_1, ξ_q given in (4.3.16) and q. Hence

$$[u(t,\cdot)]_{s,\Omega} \geq \min\left\{\frac{l}{2C\eta'}, \left(\frac{l}{2C\eta'}\right)^{1/q}\right\} = C_2(l) > 0 \tag{4.3.19}$$

and in J_2

$$\mathscr{L}u \geq \left(1 - \frac{\lambda}{\lambda_1}\right) C_2^p(l).$$

Therefore, if $J_2 \neq \emptyset$, (4.3.11) holds with

$$\alpha = \alpha(l) = \left(1 - \frac{\lambda}{\lambda_1}\right) \min\{l, C_2^p(l)\} > 0.$$

Case 2. $q > p^*$. Using (4.3.15), (H)–(b) and Hölder's inequality, we have for $t \in J_2$

$$\begin{aligned}\frac{l}{\eta'} &< \mathscr{F}u(t) \leq C_1(\|u(t,\cdot)\|_1 + \|u(t,\cdot)\|_q^q)\\ &\leq c_0\big(\langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle + \kappa_1\|u(t,\cdot)\|_1\\ &\qquad + \kappa_2|\Omega|^{1/q'}\|u(t,\cdot)\|_1^{1/q} + \kappa_3\|u(t,\cdot)\|_{p^*}^{p^*}\big),\end{aligned}$$

where $c_0 = C_1/\kappa_1$, since $\kappa_1 > 0$. But $\|u\|_1 \leq \xi_1[u]_{s,\Omega}$ by (4.3.16) and Hölder's inequality. Therefore, by (4.3.16)

$$\langle f(t,\cdot,u), u\rangle + c_1[u]_{s,\Omega} + c_2[u]_{s,\Omega}^{1/q} + c_3[u]_{s,\Omega}^{p^*} > l/c_0\eta',$$

where $c_1 = \kappa_1\xi_1$, $c_2 = \kappa_2|\Omega|^{1/q'}\xi_1^{1/q} \geq 0$ and $c_3 = \kappa_3\xi_{p^*}^{p^*} \geq 0$. Hence, if $\langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle \geq 0$, for $t \in J_2$ we have

$$\text{either} \quad \langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle \geq l/2c_0\eta' \quad \text{or} \quad [u(t,\cdot)]_{s,\Omega} \geq c_4, \tag{4.3.20}$$

where $c_4 = c_4(l, c_0, \eta) > 0$ is an appropriate constant, arising when

$$c_1[u]_{s,\Omega} + c_2[u]_{s,\Omega}^{1/q} + c_3[u]_{s,\Omega}^{p^*} \geq l/2c_0\eta'.$$

On the other hand, if $t \in J_2$ and $\langle f(t,\cdot,u(t,\cdot)), u(t,\cdot)\rangle < 0$, then we get $[u(t,\cdot)]_{s,\Omega} \geq c_5$, where $c_5 \geq c_4$ is a suitable number arising from

$$c_1[u]_{s,\Omega} + c_2[u]_{s,\Omega}^{1/q} + c_3[u]_{s,\Omega}^{p^*} > l/c_0\eta'.$$

By (4.1.3) the conclusion (4.3.11) holds, with

$$\alpha = \min\{(1 - \lambda/\lambda_1)l,\ (1 - \lambda/\lambda_1)c_5^p,\ c_4^p,\ l/2c_0\eta'\} > 0,$$

since $l > 0$, $\lambda \in [0, \lambda_1)$, $c_0 > 0$ and $c_5 \geq c_4 > 0$. This completes the proof. $\square$

Lemma 4.3.4. *For all $t \geq T \geq 0$ we have*

$$\int_T^t \varrho(\tau)k(\tau)M([u]_{s,\Omega}^p)\left|\langle |u_t|^{p-2}u_t, u\rangle\right| d\tau \leq \varepsilon_3(T)\mathcal{C}(k(t)), \tag{4.3.21}$$

where $\varepsilon_3(T) = \mathcal{K}\left(\int_T^\infty \mathscr{K}_\varrho(t)dt\right)^{1/p'} \to 0$ *as* $T \to \infty$*, the Kirchhoff damped function* $\mathscr{K}_\varrho$ *is defined by*

$$\mathscr{K}_\varrho u = \varrho M([u]_{s,\Omega}^p)\|u_t\|_p^p, \qquad \mathcal{K} = \sup_{t\in\mathbb{R}_0^+}\left(\|u(t,\cdot)\|_p \cdot M([u(t,\cdot)]_{s,\Omega}^p)^{1/p}\right)$$

and

$$\mathcal{C}(k(t)) = \left(\int_T^t \varrho(\tau)k^p(\tau)d\tau\right)^{1/p}. \tag{4.3.22}$$

Proof. By (4.3.8) clearly $\mathcal{K} < \infty$. Hence by integration from T to t and use of Hölder's inequality twice, we obtain

$$\begin{aligned}\int_T^t &\varrho(\tau)k(\tau)M([u(\tau,\cdot)]_{s,\Omega}^p)\left|\langle |u_t|^{p-2}u_t, u\rangle\right| d\tau \\ &\leq \mathcal{K}\int_T^t \varrho(\tau)k(\tau)M([u(\tau,\cdot)]_{s,\Omega}^p)^{1/p'}\||u_t|^{p-1}(\tau,\cdot)\|_{p'}d\tau \\ &\leq \varepsilon_3(T)\left(\int_T^t \varrho(\tau)k^p(\tau)d\tau\right)^{1/p},\end{aligned}$$

where $\varepsilon_3(T) \to 0$ as $T \to \infty$ by Lemma 4.3.2. □

Proof. Following the main ideas of the proof of [14, Theorem 3.1] and [85, Theorem 1], first we treat case (4.3.1) in the simpler situation in which k is not only $CBV(\mathbb{R}_0^+)$, but also of class $C^1(\mathbb{R}_0^+)$. Suppose, for contradiction that $l > 0$ in (4.3.10). Define a Lyapunov function by

$$V(t) = k(t)\langle u, u_t\rangle = \langle u_t, \phi\rangle, \qquad \phi = k(t)u.$$

Since $k \in C^1(\mathbb{R}_0^+)$ and $\phi_t = k'u + ku_t$, it is clear that $\phi \in X$. Hence, by the distribution identity (A) in Section 4.2, we have for any $t \geq T \geq 0$

$$\begin{aligned}V(\tau)\Big]_T^t = &\int_T^t \Big\{k'\langle u, u_t\rangle + 2k\|u_t\|_2^2 - k\left[\|u_t\|_2^2\right. \\ &\left.+[u]_{s,\Omega}^p + \mu\|u\|_p^p + \langle f(\tau,\cdot,u), u\rangle\right]\Big\}\, d\tau \\ &-\int_T^t k\langle Q(\tau,\cdot,u,u_t), u\rangle d\tau \\ &-\int_T^t k\varrho M([u]_{s,\Omega}^p)\langle |u_t|^{p-2}u_t, u\rangle d\tau.\end{aligned} \tag{4.3.23}$$

We now estimate the right hand side of (4.3.23). First

$$\sup_{\mathbb{R}_0^+} |\langle u(t,\cdot), u_t(t,\cdot)\rangle| \le \sup_{\mathbb{R}_0^+} \|u(t,\cdot)\|_2 \cdot \|u_t(t,\cdot)\|_2 = U < \infty \tag{4.3.24}$$

by (4.3.8) of Lemma 4.3.2. Now, using Lemma 4.3.3

$$-\int_T^t k\left\{\|u_t\|_2^2 + [u]_{s,\Omega}^p + \mu\|u\|_p^p + \langle f(\tau,\cdot,u),u\rangle\right\} d\tau \le -\alpha\int_T^t k d\tau, \tag{4.3.25}$$

and by Lemmas 3.2 and 3.3 of [85]

$$-\int_T^t k\langle Q(\tau,\cdot,u,u_t),u\rangle d\tau \le \varepsilon_1(T)\mathcal{B}(k(t)), \tag{4.3.26}$$

$$\int_T^t k\|u_t\|_2^2 d\tau \le \theta\int_T^t k d\tau + \varepsilon_2(T)C(\theta)\left(\int_0^t \sigma^{1-\wp}k^{\wp}d\tau\right)^{1/\wp}, \tag{4.3.27}$$

where $C(\theta) = \omega_\theta^{1/\wp'}$, $\omega_\theta = \sup\{\tau^2/\omega(\tau) : \tau \ge \sqrt{\theta/|\Omega|}\}$,

$$\begin{aligned}\varepsilon_1(T) = \left(\sup_{\mathbb{R}_0^+}\|u(t,\cdot)\|_\nu\right)\cdot\Bigg[&\left(\int_T^\infty \langle Q(\tau,\cdot,u,u_t),u_t\rangle d\tau\right)^{1/m'} \\ &+ \left(\int_T^\infty \langle Q(\tau,\cdot,u,u_t),u_t\rangle d\tau\right)^{1/r'}\Bigg],\end{aligned} \tag{4.3.28}$$

and

$$\varepsilon_2(T) = \left(\sup_{\mathbb{R}_0^+}\|u_t(t,\cdot)\|_2^{2/\wp}\right)\cdot\left(\int_T^\infty \langle Q(\tau,\cdot,u,u_t),u_t\rangle d\tau\right)^{1/\wp'}, \tag{4.3.29}$$

with $\varepsilon_1(T) = o(1)$ and $\varepsilon_2(T) = o(1)$ as $T \to \infty$ by (4.3.8) of Lemma 4.3.2. Now, applying (4.3.24)–(4.3.27) and (4.3.21), from (4.3.23) we obtain

$$\begin{aligned} V(\tau)\big]_T^t \le U\int_T^t |k'|d\tau + 2\theta\int_T^t k d\tau \\ + 2\varepsilon_2(T)C(\theta)\left(\int_0^t \sigma^{1-\wp}k^{\wp}d\tau\right)^{1/\wp} \\ - \alpha\int_T^t k d\tau + \varepsilon_3(T)\mathcal{C}(k(t)) + \varepsilon_1(T)\mathcal{B}(k(t)),\end{aligned} \tag{4.3.30}$$

where $\varepsilon_1(T)$ is defined in (4.3.28), $\varepsilon_2(T)$ in (4.3.29), $\varepsilon_3(T)$ in (4.3.21), $\mathcal{B}(k)$ in (4.3.4) and $\mathcal{C}(k)$ in (4.3.22). By (4.3.3) there is a sequence $t_i \nearrow \infty$ and a number $\ell > 0$ such that

$$\mathcal{A}(k(t_i)) + \mathcal{C}(k(t_i)) \le \ell \int_0^{t_i} k d\tau. \tag{4.3.31}$$

Choose $\theta = \theta(l) = \alpha/4$ and fix $T > 0$ so large that

$$\varepsilon(T)[2C(\theta) + 1]\ell \le \alpha/4, \tag{4.3.32}$$

being $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_2(T), \varepsilon_3(T)\} = o(1)$ as $T \to \infty$. Consequently, for $t_i \ge T$,

$$V(t_i) \le U \int_T^{t_i} |k'| d\tau + S(T) - \frac{\alpha}{4} \int_T^{t_i} k d\tau, \tag{4.3.33}$$

where $S(T) = V(T) + \varepsilon(T)[2C(\theta) + 1]\ell \int_0^T k d\tau$. Thus by (4.3.1) we get

$$\lim_{i \to \infty} V(t_i) = -\infty, \tag{4.3.34}$$

since $k' \in L^1(\mathbb{R}_0^+)$ being $k \in CBV(\mathbb{R}_0^+)$. On the other hand, by (4.3.24) and recalling that k is bounded,

$$|V(t)| \le \left(\sup_{\mathbb{R}_0^+} k \right) \|u(t, \cdot)\|_2 \cdot \|u_t(t, \cdot)\|_2 \le \left(\sup_{\mathbb{R}_0^+} k \right) U \tag{4.3.35}$$

for all $t \in \mathbb{R}_0^+$. This contradiction completes the first part of the proof.

The proof of the general case, $k \in CBV(\mathbb{R}_0^+)$ but not $k \in C^1(\mathbb{R}_0^+)$, proceeds as in [9, Theorem 3.3]. Let $\overline{k} \in C^1(\mathbb{R}_0^+)$ and $G \subset \mathbb{R}_0^+$ be an open subset such that

$$(i) \quad 2k \ge \overline{k} \ge \begin{cases} k & \text{in } \mathbb{R}_0^+ \setminus G \\ 0 & \text{in } G \end{cases}; \qquad (ii) \quad \operatorname{Var} \overline{k} \le 2 \operatorname{Var} k;$$

$$(iii) \quad \int_G k ds \le 1.$$

Clearly $\overline{k} \in CBV(\mathbb{R}_0^+)$ by (ii). We next prove that $\overline{k}$ satisfies both (4.3.1) and (4.3.3). Note that since $k \notin L^1(\mathbb{R}_0^+)$ it is possible to find a value T_1 such that

$$\int_0^{T_1} k d\tau \ge 2. \tag{4.3.36}$$

Considering $t \geq T_1$, by (i), (ii) and (4.3.36) we obtain

$$\int_0^t \overline{k} d\tau \geq \int_{[0,t]\setminus G} k d\tau \geq \int_0^t k d\tau - \int_G k d\tau \geq \int_0^t k d\tau - 1 \geq \frac{1}{2}\int_0^t k d\tau. \tag{4.3.37}$$

Hence $\overline{k}$ satisfies (4.3.1). Moreover, by (i) and (4.3.37) for all $t \geq T_1$

$$\left[\mathcal{A}(\overline{k}(t)) + \mathcal{C}(\overline{k}(t))\right] \int_0^t k d\tau \leq 4\left[\mathcal{A}(k(t)) + \mathcal{C}(k(t))\right] \int_0^t \overline{k} d\tau,$$

where $k \mapsto \mathcal{A}(k)$ is defined in (4.3.4) and $k \mapsto \mathcal{C}(k)$ in (4.3.22). This shows that $\overline{k}$ also satisfies (4.3.3).

The general case is therefore reduced to the situation when k is smooth, and the proof is complete under the condition (4.3.1).

Suppose now that k verifies (4.3.2). We still obtain $\phi_t = k'u + ku_t$, so that $\phi \in X$. Clearly Lemmas 4.3.2 and 4.3.3 continue to hold, as well as Lemmas 3.2 and 3.3 of [85], that is (4.3.26) and (4.3.27) are available. From now on we follow the proof of the previous case until obtaining (4.3.33). By the definition of V we now get

$$|V(t_i)| \leq Uk(t_i) \leq U\left\{k(0) + \int_0^{t_i} |k'| d\tau\right\},$$

so that by (4.3.33)

$$\frac{\alpha}{4}\int_0^{t_i} k d\tau \leq 2U \int_0^{t_i} |k'| d\tau + S(T) + Uk(0). \tag{4.3.38}$$

First note that $k \notin L^1(\mathbb{R}_0^+)$ by (4.3.2). Now, dividing (4.3.38) by $\int_0^{t_i} k d\tau$, we contradict (4.3.2) as $i \to \infty$.

In conclusion, in both cases, we have proved that $Eu(t)$ approaches zero as $t \to \infty$. Finally, by (4.3.7), it follows that (4.3.5) holds. □

We now give other weaker versions of Theorem 4.3.1 when k and ϱ are related, only considering the case $p \geq 2$. These facts were first noted in Section 7 of the celebrated article [84] in the simpler case of strongly damped systems.

We first give a stability result when k and ϱ are related by the condition

$$k(t) \leq \text{ const. } \varrho(t) \qquad \text{for } t \text{ sufficiently large.} \tag{4.3.39}$$

The importance of Theorem 4.3.6 relies on the fact that condition (4.3.39) allows to remove the assumption (AS)–(b) on the distributed damping Q, which forces a control from below for Q, while the natural growth condition for Q, that is (AS)–(a), continues to be required. Before proving these theorems, in Lemma 4.3.5 below we establish an estimate for the norm $\|u_t(t,\cdot)\|_2$ in terms of $\|u_t(t,\cdot)\|_p$.

From now on, we only consider the case $p \geq 2$, without further mentioning.

Lemma 4.3.5. *For all $\theta > 0$ there exists a number $\Lambda(\theta) \geq 0$ such that*

$$\|u_t(t,\cdot)\|_2^2 \leq \theta + \Lambda(\theta)\|u_t(t,\cdot)\|_p^p \quad \text{for all } t \in \mathbb{R}_0^+. \tag{4.3.40}$$

Proof. Fix $t \in \mathbb{R}_0^+$ and $\theta > 0$. Define

$$\Omega_1 = \Omega_1(t) = \{x \in \Omega : |u_t(t,x)| \leq \sqrt{\theta/|\Omega|}\},$$
$$\Omega_2 = \Omega_2(t) = \{x \in \Omega : |u_t(t,x)| \geq \sqrt{\theta/|\Omega|}\}.$$

Clearly,

$$\|u_t(t,\cdot)\|_2^2 = \left(\int_{\Omega_1} + \int_{\Omega_2}\right)|u_t(t,x)|^2 dx \quad \text{and} \quad \int_{\Omega_1}|u_t(t,x)|^2 dx \leq \theta. \tag{4.3.41}$$

We immediately have that

$$\int_{\Omega_2}|u_t(t,x)|^2 dx = \int_{\Omega_2}\frac{|u_t(t,x)|^2}{|u_t(t,x)|^p}|u_t(t,x)|^p dx \leq \left(\frac{\theta}{|\Omega|}\right)^{(2-p)/2}\|u_t(t,\cdot)\|_p^p,$$

so (4.3.40) holds, with $\Lambda(\theta) = (\theta/|\Omega|)^{(2-p)/2}$. □

Theorem 4.3.6. *If condition (4.3.9) holds and there exists a function k, satisfying either (4.3.1) or (4.3.2), the relation (4.3.39) and*

$$\liminf_{t\to\infty}\,[\mathcal{B}(k(t)) + \mathcal{C}(k(t))] \Big/ \int_0^t k\,d\tau < \infty\,, \tag{4.3.42}$$

where $k \mapsto \mathcal{B}(k)$ is defined in (4.3.4) and $k \mapsto \mathcal{C}(k)$ is defined in (4.3.22), then along any solution of $(\mathscr{P}_3)$ property (4.3.5) holds.

Proof. We once more proceed by contradiction assuming $l > 0$ in (4.3.10) and distinguish two cases.

If k satisfies (4.3.1), we follow the proof of the same case of Theorem 4.3.1 until the derivation of (4.3.23). Now, Lemmas 4.3.2–4.3.3 and Lemma 3.2 of [85] are still valid, so that (4.3.24)–(4.3.26) are available. Moreover, Lemma 4.3.4 and (4.3.8) give (4.3.21). While (4.3.27) no longer holds and Lemma 3.3 of [85] must be replaced by the following argument; cf. [84, Lemma 7.3].

Take $\theta = \alpha/4$ in (4.3.40), so that $\Lambda(\theta) = \Lambda(\alpha)$, combining (4.3.39) with (4.3.40), for T sufficiently large, in replacement of (4.3.27) we have for all $t \geq T$

$$\begin{aligned} 2\int_T^t k\|u_t(\tau,\cdot)\|_2^2 d\tau &\leq \frac{\alpha}{2}\int_T^t k d\tau + \tilde{\Lambda}(\alpha)\int_T^t \varrho(\tau)\|u_t(\tau,\cdot)\|_p^p\, d\tau \\ &\leq \frac{\alpha}{2}\int_T^t k d\tau + \varepsilon_4(T), \end{aligned} \tag{4.3.43}$$

where $\tilde{\Lambda}(\alpha)$ depends on $\Lambda(\alpha)$ and on the constant given in (4.3.39), while

$$\varepsilon_4(T) = \tilde{\Lambda}(\alpha)\int_T^\infty \varrho(\tau)\|u_t(\tau,\cdot)\|_p^p\, d\tau = o(1) \tag{4.3.44}$$

as $T \to \infty$ by Lemma 4.3.2, since (4.3.9) holds. Therefore, instead of (4.3.30) we get

$$\begin{aligned} V(\tau)\big]_T^t \leq U&\int_T^t |k'| d\tau + \varepsilon_4(T) \\ &- \frac{\alpha}{2}\int_T^t k d\tau + \varepsilon_3(T)\mathcal{C}(k(t)) + \varepsilon_1(T)\mathcal{B}(k(t)), \end{aligned} \tag{4.3.45}$$

where $\varepsilon_1(T)$ is defined in (4.3.28) and $\varepsilon_3(T)$ in (4.3.21). Thus, we obtain

$$V(t_i) \leq U\int_T^{t_i} |k'| d\tau + S(T) - \frac{\alpha}{4}\int_T^{t_i} k d\tau, \tag{4.3.46}$$

where now $t_i \nearrow \infty$ and $\ell > 0$ are taken so that

$$\mathcal{B}(k(t_i)) + \mathcal{C}(k(t_i)) \leq \ell\int_0^{t_i} k d\tau \tag{4.3.47}$$

by (4.3.42), $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_3(T), \varepsilon_4(T)\} \leq \alpha/4\ell$ for T even larger, if necessary, and $S(T) = V(T) + \varepsilon(T)\left\{1 + \ell\int_0^T k d\tau\right\}$. Hence, (4.3.1) implies (4.3.34), which gives the required contradiction in virtue of (4.3.35). When $k \in CBV(\mathbb{R}_0^+) \setminus C^1(\mathbb{R}_0^+)$ the proof can proceed as in Theorem 4.3.1.

If k satisfies (4.3.2), arguing as in Theorem 4.3.1 and using (4.3.43) in place of (4.3.27), and (4.3.47) in place of (4.3.31), we get again a contradiction. □

Corollary 4.3.7. *Let (4.3.9) hold and δ_1, $\delta_2 \in L^\infty(\mathbb{R}_0^+)$. Suppose that ϱ is either of class $CBV(\mathbb{R}_0^+) \setminus L^1(\mathbb{R}_0^+)$, or that $\varrho \in W^{1,1}_{\rm loc}(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+)$, $\varrho \not\equiv 0$ and $|\varrho'(t)| = o(\varrho(t))$ as $t \to \infty$. Then (4.3.5) holds.*

Proof. It is sufficient to apply Theorem 4.3.6, with $k = \varrho$. Indeed, in the case $\varrho \in CBV(\mathbb{R}_0^+) \setminus L^1(\mathbb{R}_0^+)$, conditions (4.3.1) and (4.3.39) hold at once. On the other hand, if $\varrho \in W^{1,1}_{\rm loc}(\mathbb{R}_0^+)$, $|\varrho'(t)| = o(\varrho(t))$ as $t \to \infty$ and $\varrho \not\equiv 0$, then (4.3.2), (4.3.39) are satisfied and $\varrho \notin L^1(\mathbb{R}_0^+)$, as in the previous case. Otherwise, if $\varrho \in L^1(\mathbb{R}_0^+)$, by (4.3.2) it follows that $\int_0^\infty |\varrho'(\tau)| d\tau = 0$, and so $\varrho \equiv$ const. and in turn $\varrho \equiv 0$ since $\varrho \in L^1(\mathbb{R}_0^+)$ by assumption of contradiction, which is impossible, being $\varrho \not\equiv 0$. Moreover, since ϱ is of class $CBV(\mathbb{R}_0^+) \setminus L^1(\mathbb{R}_0^+)$, or $\varrho \in W^{1,1}_{\rm loc}(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+)$, it follows that ϱ is bounded and so $\sup_{\mathbb{R}_0^+} \varrho \le \varrho_0 < \infty$. Therefore, since $m < r$ and applying Young's inequality, we get

$$
\begin{aligned}
\left(\int_0^t \delta_2 \varrho^r d\tau\right)^{1/r} &= \left(\int_0^t \delta_2 \varrho^{m+(r-m)} d\tau\right)^{1/r} \\
&\le \varrho_0^{(r-m)/r} \left(\int_0^t \delta_2 \varrho^m d\tau\right)^{1/r} \\
&\le \frac{r-m}{r}\varrho_0 + \frac{m}{r}\left(\int_0^t \delta_2 \varrho^m d\tau\right)^{1/m} \\
&\le \text{ const. } \left(\int_0^t \delta_2 \varrho^m d\tau\right)^{1/m}.
\end{aligned}
\tag{4.3.48}
$$

We repeat the same argument with m and p. First suppose that $p < m$. Arguing as in (4.3.48), we immediately obtain

$$
\begin{aligned}
&\left(\int_0^t \delta_1 \varrho^m d\tau\right)^{1/m} + \left(\int_0^t \varrho^{p+1} d\tau\right)^{1/p} \\
&\qquad \le \text{const.} \left[\left(\int_0^t \delta_1 \varrho^p d\tau\right)^{1/p} + \left(\int_0^t \varrho^{p+1} d\tau\right)^{1/p}\right] \\
&\qquad \le \text{const.} \left[\delta_1^{1/p} \rho_0^{1/p'} \left(\int_0^t \varrho d\tau\right)^{1/p} + \varrho_0 \left(\int_0^t \varrho d\tau\right)^{1/p}\right]
\end{aligned}
\tag{4.3.49}
$$

$$\leq \text{const.}\left(\int_0^t \varrho d\tau\right)^{1/p}.$$

Suppose now that $m < p$. Arguing as before, we get

$$\begin{aligned}
&\left(\int_0^t \delta_1\varrho^m d\tau\right)^{1/m} + \left(\int_0^t \varrho^{p+1} d\tau\right)^{1/p} \\
&\qquad \leq \text{const.}\left[\left(\int_0^t \delta_1\varrho^m d\tau\right)^{1/m} + \left(\int_0^t \varrho^{m+1} d\tau\right)^{1/m}\right] \\
&\qquad \leq \text{const.}\left[\delta_1^{1/m}\rho_0^{1/m'}\left(\int_0^t \varrho d\tau\right)^{1/m} + \varrho_0\left(\int_0^t \varrho d\tau\right)^{1/m}\right] \\
&\qquad \leq \text{const.}\left(\int_0^t \varrho d\tau\right)^{1/m}.
\end{aligned} \tag{4.3.50}$$

Combining (4.3.48)–(4.3.50), we obtain

$$0 \leq \{\mathcal{B}(k(t)) + \mathcal{C}(k(t))\} \Big/ \int_0^t k d\tau \leq \text{const.}\left(\int_0^t \varrho\, d\tau\right)^{-(1-1/\min\{p,m\})} \to 0,$$

as $t \to \infty$, being $\min\{p, m\} > 1$ and $\varrho \notin L^1(\mathbb{R}_0^+)$ in both cases. Hence, also condition (4.3.42) holds. $\square$

We provide now a global stability for $(\mathscr{P}_3)$ when (4.3.39) is replaced by

$$|k'| \leq \text{ const. } \varrho^{1/p} k^{1/p'} \qquad \text{a.e. in } \mathbb{R}_0^+, \tag{4.3.51}$$

assuming the entire condition (AS).

Theorem 4.3.8. *Let (4.3.9) and (AS)–(b) hold. If there exists a function $k \in [W^{1,1}_{\text{loc}}(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+)] \setminus L^1(\mathbb{R}_0^+)$, satisfying (4.3.51) and (4.3.3), then along any solution u of $(\mathscr{P}_3)$ property (4.3.5) holds.*

Proof. Assume by contradiction that $l > 0$ in (4.3.10). We proceed as in the proof of the first case in Theorem 4.3.1, but estimating the term $\int_T^t k'\langle u, u_t\rangle d\tau$ as in [84]. Since $p \geq 2$, we immediately have that

$$\|u_t\|_2 \leq |\Omega|^{1/2-1/p}\|u_t\|_p. \tag{4.3.52}$$

Moreover, $\|u\|_2 \in L^\infty(\mathbb{R}_0^+)$ by Lemma 4.3.2, and in turn, using (4.3.51), we obtain for a.a. $t \in \mathbb{R}_0^+$

$$\begin{aligned}
|k'\langle u(t,\cdot), u_t(t,\cdot)\rangle| &\leq \text{const.}\, \varrho^{1/p} k^{1/p'}\|u(t,\cdot)\|_2 \cdot \|u_t(t,\cdot)\|_2 \\
&\leq \text{const.}\, \varrho^{1/p} k^{1/p'}\,\|u_t(t,\cdot)\|_p.
\end{aligned} \tag{4.3.53}$$

By (4.3.52), (4.3.53) and Hölder's inequality we get

$$\int_T^t |k'\langle u, u_t\rangle| d\tau \leq \text{const.} \left(\int_T^t k d\tau\right)^{1/p'} \left(\int_T^t \varrho \|u_t(t,\cdot)\|_p^p \, d\tau\right)^{1/p}.$$

It follows immediately that

$$\int_T^t |k'\langle u, u_t\rangle| d\tau \leq \varepsilon_5(T)\left(1+\int_T^t k d\tau\right), \tag{4.3.54}$$

where

$$\varepsilon_5(T) = \text{const.} \left(\int_T^\infty \varrho \|u_t(\tau,\cdot)\|_p^p \, dt\right)^{1/p} = o(1)$$

as $T \to \infty$ by Lemma 4.3.2, since (4.3.9) holds. Hence (4.3.30) becomes

$$\begin{aligned} V(\tau)\big]_T^t \leq \varepsilon_5(T)\left(1+\int_T^t k d\tau\right) + 2\theta \int_T^t k d\tau \\ + 2\varepsilon_2(T)C(\theta)\left(\int_0^t \sigma^{1-\wp} k^{\wp} d\tau\right)^{1/\wp} - \alpha \int_T^t k d\tau \\ + \varepsilon_3(T)\mathcal{C}(k(t)) + \varepsilon_1(T)\mathcal{B}(k(t)), \end{aligned} \tag{4.3.55}$$

where again the dominating term is $-\alpha \int_T^t k d\tau$, and $\varepsilon_1(T)$ is defined in (4.3.28), $\varepsilon_2(T)$ in (4.3.29) and $\varepsilon_3(T)$ in (4.3.21). Call ℓ the number verifying (4.3.31) and take $\theta = \theta(l) = 3\alpha/16$. Furthermore, fix $T > 0$ even larger, if necessary, so that $\varepsilon(T) \leq 3\alpha/8\,[1+\ell+2C(\theta)\ell]$, where now we define $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_2(T), \varepsilon_3(T), \varepsilon_5(T)\} = o(1)$ as $T \to \infty$. Proceeding as in Theorem 4.3.1, in place of (4.3.33), we obtain, after some calculations,

$$V(t_i) \leq S(T) - \frac{\alpha}{4}\int_T^{t_i} k d\tau, \tag{4.3.56}$$

where $S(T) = V(T) + \varepsilon(T)\{1 + [1+2C(\theta)]\ell \int_0^T k d\tau\}$. Since $k \notin L^1(\mathbb{R}_0^+)$, by (4.3.56) we get (4.3.34), which gives a contradiction in virtue of (4.3.24) and of the fact that k is bounded. This contradiction produce at once (4.3.5) as in the proof of Theorem 4.3.6. □

Combining the proof techniques of the main Theorems 4.3.6 and 4.3.8, we obtain another result of independent interest in which only condition (AS)–(a) is required on the damping term Q. More precisely, we have the following theorem.

Theorem 4.3.9. *If* (4.3.9) *holds and there exists a function*

$$k \in [W^{1,1}_{\text{loc}}(\mathbb{R}^+_0) \cap L^\infty(\mathbb{R}^+_0)] \setminus L^1(\mathbb{R}^+_0),$$

satisfying (4.3.39), (4.3.51) *and* (4.3.42), *then any solution* u *of* $(\mathscr{P}_3)$ *has the property* (4.3.5).

Proof. As in the proof of Theorem 4.3.8 we proceed by contradiction assuming $l > 0$ and observe that the technique used in Theorem 4.3.1 for the regular case when k satisfies (4.3.1) can be adopted in a similar way. To do this we only estimate $\int_T^t k(\tau)\|u_t(\tau,\cdot)\|_2^2 d\tau$ in the same way of the proof of the Theorem 4.3.6, while the term $\int_T^t k'(\tau)\langle u(\tau,\cdot), u_t(\tau,\cdot)\rangle d\tau$ in the same way of the proof of the Theorem 4.3.8. More precisely, we get once more (4.3.43), (4.3.53) and (4.3.54). Therefore, (4.3.55) becomes

$$\begin{aligned} V(\tau)]_T^t \le \varepsilon_5(T)\left(1 + \int_T^t k d\tau\right) + \varepsilon_4(T) - \frac{\alpha}{2}\int_T^t k d\tau \\ + \varepsilon_3(T)\mathcal{C}(k(t)) + \varepsilon_1(T)\mathcal{B}(k(t)), \end{aligned} \tag{4.3.57}$$

where $\varepsilon_1(T)$ is defined in (4.3.28), $\varepsilon_3(T)$ in (4.3.21), $\varepsilon_4(T)$ in (4.3.44) and $\varepsilon_5(T)$ in (4.3.54). Denote by ℓ the number verifying (4.3.47). Of course we have that $\varepsilon(T) = \max\{\varepsilon_1(T), \varepsilon_3(T), \varepsilon_4(T), \varepsilon_5(T)\} = o(1)$ as $T \to \infty$, hence fix $T > 0$ so large that $\varepsilon(T) \le \alpha/4(1+\ell)$. Proceeding as in the proofs of Theorems 4.3.6 and 4.3.8, we obtain (4.3.56) where now $S(T) = V(T) + \varepsilon(T)\{2 + \ell\int_0^T k d\tau\}$. Since $k \notin L^1(\mathbb{R}^+_0)$ by (4.3.56), we get (4.3.34) which gives a contradiction using (4.3.24).

Hence (4.3.5) follows at once, as always. □

Remark: The importance of Theorems 4.3.6 and 4.3.9 relies mainly on the fact that the stability of global solutions of $(\mathscr{P}_3)$ is established without requiring any lower bound for the external damping Q, that is *without assuming* (AS)–(b). This is possible thanks to assumption (4.3.39).

Corollary 4.3.10. *Let* (4.3.9) *hold. Suppose that*

$$\varrho \in [W^{1,1}_{\text{loc}}(\mathbb{R}^+_0) \cap L^\infty(\mathbb{R}^+_0)] \setminus L^1(\mathbb{R}^+_0), \qquad |\varrho'(t)| = O(\varrho(t)) \text{ as } t \to \infty,$$

and that δ_1, $\delta_2 \in L^\infty(\mathbb{R}^+_0)$. *Then* (4.3.5) *holds.*

Proof. It is sufficient to apply Theorem 4.3.9, with $k = \varrho$, since, by the same main arguments used in the proof of Corollary 4.3.7, now ϱ trivially verifies all the structural assumptions of Theorem 4.3.9. □

4.4 Local asymptotic stability for $(\mathscr{P}_3)$

In this section, we show some results concerning the local asymptotic stability. Theorem 4.4.6 provides a local stability result under a growth condition on f, assumed only for u sufficiently small. Following [9], the purpose is reached thanks to a deep qualitative analysis of the geometry of the problem, which allows us to get stability, provided that the initial data belong to an appropriate region $\tilde{\Sigma}_0$ in the phase plane, see Figure 4.1.

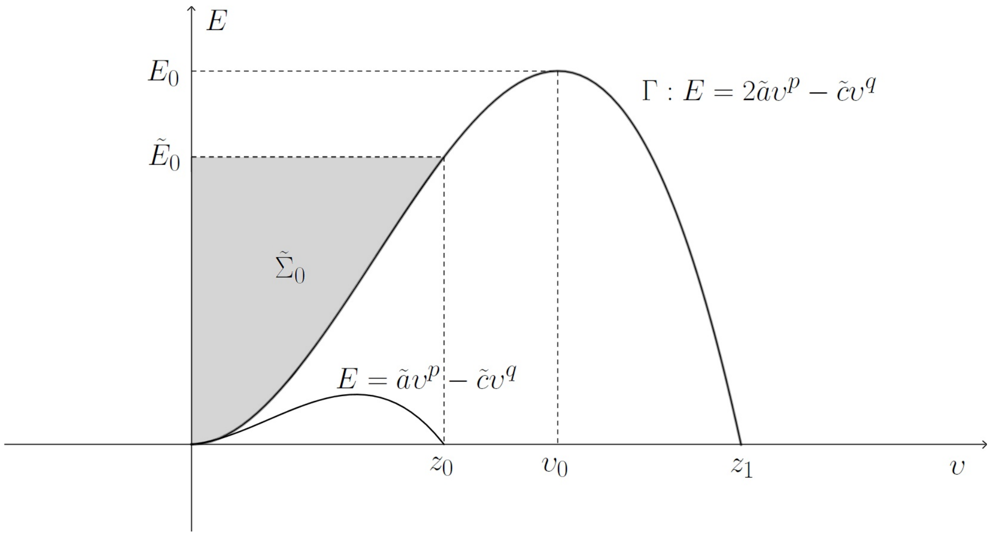

Figure 4.1: The phase plane (v, E)

In particular, if the couple $(\|u(0,\cdot)\|_q, Eu(0)) \in \tilde{\Sigma}_0$ then $(\|u(t,\cdot)\|_q, Eu(t))$ belongs to the region $\tilde{\Sigma}_0$ for all $t \in \mathbb{R}_0^+$, along any global solution u of $(\mathscr{P}_3)$. This means that is the trajectory of $(\|u(t,\cdot)\|_q, Eu(t))$ lies in the same region as the time grows up, once the initial values $(\|u(0,\cdot)\|_q, Eu(0))$ belong to it. Here we do not need the non–degeneracy of the problem, assumed in [9] for the local stability in order to overcome some technical difficulties due to the *Kirchhoff structure* of the problem. For further general comments we refer to the paper [73].

From here on, we assume (AS)–(a) and the assumption (H) on the function f, with the exception that condition (4.1.3) is replaced by the following

$$\liminf_{u\to 0} \frac{(f(t,x,u),u)}{|u|^p} \geq -\bar{\lambda}, \qquad \text{for some } \bar{\lambda} \in [0,\lambda_1), \tag{4.4.1}$$

while (H)–(a) is assumed with the further restriction that $p < q < p^*$ and (b) is dropped.

Note that in the proof of the next three lemmas we do not use the assumption (4.2.8).

Lemma 4.4.1. *There exist* $\lambda \in (\overline{\lambda}, \lambda_1)$ *and* $c > 0$ *such that*

$$(f(t,x,u),u) \geq -\lambda|u|^p - c|u|^q \tag{4.4.2}$$

for all $(t,x,u) \in \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N$*. Moreover, if* u *is a solution of* $(\mathscr{P}_3)$*, then*

$$\begin{aligned}
\mathscr{F}u(t) &\geq -\frac{\lambda}{p}\|u(t,\cdot)\|_p^p - \frac{c}{q}\|u(t,\cdot)\|_q^q \\
Eu(t) &\geq \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{1}{2p}\left(1 - \frac{\lambda}{\lambda_1}\right)[u(t,\cdot)]_{s,\Omega}^p \\
&\qquad + \tilde{a}\|u(t,\cdot)\|_q^p - \tilde{c}\|u(t,\cdot)\|_q^q, \\
\tilde{a} &= \frac{1}{2p\xi_q^p}\left(1 - \frac{\lambda}{\lambda_1}\right) > 0, \quad \tilde{c} = \frac{c}{q}
\end{aligned} \tag{4.4.3}$$

for all $t \in \mathbb{R}_0^+$*, where* ξ_q *is given in* (4.3.16)*.*

Proof. Fix $\lambda \in (\overline{\lambda}, \lambda_1)$. By (4.4.1) we have $(f(t,x,u),u) \geq -\lambda|u|^p$ for all $(t,x,u) \in \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N$, with $|u| < \delta$, provided that $\delta \in (0,1]$ is sufficiently small. On the other hand, (H)–(a) and $|u| \geq \delta$ imply

$$(f(t,x,u),u) \geq -\kappa(|u|^{1-q} + 1)|u|^q \geq -\kappa(\delta^{1-q} + 1)|u|^q,$$

so that, putting $c = \kappa(\delta^{1-q} + 1)$, we have $(f(t,x,u),u) \geq -c|u|^q$ for all $(t,x,u) \in \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^N$ such that $|u| \geq \delta$. Hence (4.4.2) holds, with c as large as we wish. Integrating (4.4.2), we obtain at once the first part of (4.4.3) along the solution u.

As already noted in the proof of Lemma 4.3.2, clearly $\lambda_1\|u\|_p^p \leq [|u|]_{s,\Omega}^p \leq [u]_{s,\Omega}^p$ for all $u \in \tilde{D}^{s,p}(\Omega;\mathbb{R}^N)$, by (4.2.3). Hence, by the definitions of Eu and of λ_1, we have in $\mathbb{R}_0^+$

$$\begin{aligned}
Eu(t) &\geq \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{1}{p}[u(t,\cdot)]_{s,\Omega}^p + \mathscr{F}u(t) \\
&\geq \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{1}{2p}\left(1 - \frac{\lambda}{\lambda_1}\right)[u(t,\cdot)]_{s,\Omega}^p \\
&\qquad + \frac{1}{2p\xi_q^p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u(t,\cdot)\|_q^p - \frac{c}{q}\|u(t,\cdot)\|_q^q.
\end{aligned}$$

In particular, the second inequality of (4.4.3) follows at once by application of (4.3.16). □

Since $p < q$ by (H)–(a), the set

$$\tilde{\Sigma}_0 = \{(\upsilon, E) \in \mathbb{R}^2 \,:\, 0 \le \upsilon < z_0,\ \Gamma(\upsilon) \le E < \tilde{E}_0\},$$

where $\Gamma(\upsilon) = 2\tilde{a}\upsilon^p - \tilde{c}\upsilon^q$,

$$(4.4.4) \qquad z_0 = \left(\frac{\tilde{a}}{c}\right)^{1/(q-p)} \quad \text{and} \quad \tilde{E}_0 = \tilde{a}\left(2 - \frac{1}{q}\right) z_0^p,$$

with $\tilde{a}$ and $\tilde{c}$ given in (4.4.3), see Figure 4.1, is well defined. Without loss of generality, we also assume that $\tilde{a}/c \le 1$, by taking c sufficiently large, if necessary. *Here and in the rest of the section u is a fixed solution of* $(\mathscr{P}_3)$ *and* $\upsilon(t) = \|u(t,\cdot)\|_q$.

Lemma 4.4.2. *If* $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$, *then for all* $t \in \mathbb{R}_0^+$

$$(4.4.5) \quad (\upsilon(t), Eu(t)) \in \tilde{\Sigma}_0 \ \text{ and } \ 2Eu(t) \ge \|u_t(t,\cdot)\|_2^2 + \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)[u(t,\cdot)]_{s,\Omega}^p.$$

Proof. By (4.4.3) and again (4.3.16), we have

$$\begin{aligned} Eu(t) &\ge \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{1}{2p\xi_q^p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u(t,\cdot)\|_q^p + \tilde{a}\|u(t,\cdot)\|_q^p - \tilde{c}\|u(t,\cdot)\|_q^q \\ &\ge 2\tilde{a}\upsilon(t)^p - \tilde{c}\upsilon(t)^q. \end{aligned}$$

Now, if there would exist t such that $\upsilon(t) = z_0$, then we would get

$$2\tilde{a}z_0^p - \tilde{c}z_0^q = \tilde{E}_0 > Eu(0) \ge Eu(t) \ge 2\tilde{a}z_0^p - \tilde{c}z_0^q,$$

which is impossible. Therefore $\upsilon(t) \ne z_0$ for all $t \in \mathbb{R}_0^+$. Hence by the continuity of υ we have $\upsilon(\mathbb{R}_0^+) \subset [0, z_0)$, being $\upsilon(0) < z_0$. Consequently, we have proved that along any solution $u \in X$

$$(4.4.6) \qquad \tilde{E}_0 > Eu(0) \ge Eu(t) \ge 2\tilde{a}\upsilon(t)^p - \tilde{c}\upsilon(t)^q \ge 0 \quad \text{for all } t \in \mathbb{R}_0^+,$$

since $0 \le \upsilon(t) < z_0 \le 1$ for all $t \in \mathbb{R}_0^+$. Hence $(\upsilon(t), Eu(t)) \in \tilde{\Sigma}_0$ and (4.4.3) gives

$$(4.4.7) \quad Eu(t) \ge \frac{1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{1}{2p}\left(1 - \frac{\lambda}{\lambda_1}\right)[u(t,\cdot)]_{s,\Omega}^p + \tilde{a}\upsilon(t)^p - \tilde{c}\upsilon(t)^q.$$

Since $\tilde{a}\upsilon^p - \tilde{c}\upsilon^q \ge 0$ in $[0, z_0)$ we get (4.4.5) at once. □

It is possible to obtain similar results as in Lemmas 4.4.1 and 4.4.2, in terms of $[u(t,\cdot)]_{s,\Omega}$, rather than $\|u(t,\cdot)\|_q$.

Lemma 4.4.3. *If $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$, then for all $t \in \mathbb{R}_0^+$*

$$[u(t,\cdot)]_{s,\Omega}^p + \langle f(t,\cdot,u), u\rangle \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)[u(t,\cdot)]_{s,\Omega}^p.$$

Proof. First of all

$$\langle f(t,\cdot,u), u\rangle \geq -\lambda\|u(t,\cdot)\|_p^p - c\|u(t,\cdot)\|_q^q$$

by (4.4.2), so that, by (4.3.16), we get

$$\begin{aligned}
[u(t,\cdot)]_{s,\Omega}^p + \langle f(t,\cdot,u), u\rangle &\geq [u(t,\cdot)]_{s,\Omega}^p - \frac{\lambda}{\lambda_1}[u(t,\cdot)]_{s,\Omega}^p - c\|u(t,\cdot)\|_q^q \\
&\geq \left(1 - \frac{\lambda}{\lambda_1}\right)[u(t,\cdot)]_{s,\Omega}^p - c\|u(t,\cdot)\|_q^q \\
&\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)[u(t,\cdot)]_{s,\Omega}^p + \frac{1}{2\xi_q^p}\left(1 - \frac{\lambda}{\lambda_1}\right)\upsilon(t)^p \\
&\quad - c\upsilon(t)^q \\
&\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)[u(t,\cdot)]_{s,\Omega}^p + \tilde{a}\upsilon(t)^p - c\upsilon(t)^q,
\end{aligned} \tag{4.4.8}$$

where $\tilde{a}$ is defined in (4.4.3). The quantity $\tilde{a}\upsilon(t)^p - c\upsilon(t)^q \geq 0$ for all $t \in \mathbb{R}_0^+$, since $\upsilon(\mathbb{R}_0^+) \subset [0, z_0)$. Thus, from (4.4.8) the lemma is proved. □

Lemma 4.4.4. *If $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$, then (4.3.7) and (4.3.8) continue to hold.*

Proof. The fact that $\|u_t\|_2$, $[u]_{s,\Omega}$ and $M([u]_{s,\Omega}^p)$ are in $L^\infty(\mathbb{R}_0^+)$ now follows at once by (4.4.5); moreover $\|u\|_2 \in L^\infty(\mathbb{R}_0^+)$ by (4.1.4). The latter part of $(4.3.7)_1$ is a consequence of the continuity of the Sobolev embeddings $\tilde{D}^{s,p}(\Omega;\mathbb{R}^N) \hookrightarrow L^q(\Omega;\mathbb{R}^N)$ and $\tilde{D}^{s,p}(\Omega;\mathbb{R}^N) \hookrightarrow L^{p^*}(\Omega;\mathbb{R}^N)$, being in particular $p < q \leq p^*$ by (H)–(a). Property (4.3.8) can be proved exactly as in Lemma 4.3.2. □

Also in the proof of the next lemma we do not use that (4.2.8) holds.

Lemma 4.4.5. *If the limit l is positive in (4.3.10), then there exists a positive number $\alpha = \alpha(l)$ such that (4.3.11) is true.*

Proof. Let us denote by $\mathscr{L}u$ the same operator introduced in (4.3.11). By Lemma 4.4.3, we get in $\mathbb{R}_0^+$

$$\begin{aligned}\mathscr{L}u(t) &\geq \|u_t(t,\cdot)\|_2^2 + \mu\|u(t,\cdot)\|_p^p + \frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right)[u(t,\cdot)]_{s,\Omega}^p \\ &\geq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right)\left[\|u_t(t,\cdot)\|_2^2 + \mu\|u(t,\cdot)\|_p^p + [u(t,\cdot)]_{s,\Omega}^p\right].\end{aligned} \tag{4.4.9}$$

As in the proof of Lemma 4.3.3 we have (4.3.12) and now divide $\mathbb{R}_0^+$ into the sets J_1 and J_2 given in (4.3.13). Hence, by (4.3.14), in J_1 we obtain

$$\mathscr{L}u(t) \geq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right) l. \tag{4.4.10}$$

Condition (4.3.15) is still valid and in turn we get once more (4.3.19) in J_2. Moreover, by (4.4.9) for all $t \in J_2$

$$\mathscr{L}u(t) \geq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right) C_2^p(l), \tag{4.4.11}$$

where $C_2(l)$ is defined in (4.3.19). Hence, combining (4.4.10) with (4.4.11), we obtain (4.3.11) with

$$\alpha = \alpha(l) = \frac{1}{2}\left(1-\frac{\lambda}{\lambda_0}\right)\begin{cases}\min\{l, C_2^p(l)\} & \text{if } J_2 \neq \emptyset,\\ l, & \text{if } J_2 = \emptyset.\end{cases}$$

This completes the proof. □

The next theorem is our main local asymptotic stability result for $(\mathscr{P}_3)$.

Theorem 4.4.6. *Let also* (AS)–(b) *hold and* k *be an auxiliary function satisfying* (4.3.3) *and either* (4.3.1) *or* (4.3.2)*. If the initial data* $[u(0,\cdot)]_{s,\Omega}$, $\|u_t(0,\cdot)\|_2$ *are sufficiently small, then* (4.3.5) *continues to hold.*

Proof. Let us first prove that if the initial data $\|u_t(0,\cdot)\|_2$ and $[u(0,\cdot)]_{s,\Omega}$ are sufficiently small, then $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$. Indeed, the continuity of the embedding $\tilde{D}^{s,p}(\Omega;\mathbb{R}^N) \hookrightarrow L^q(\Omega;\mathbb{R}^N)$ implies that

$$\upsilon(0) = \|u(0,\cdot)\|_q < z_0 \leq 1$$

if $[u(0,\cdot)]_{s,\Omega}$ is small enough. On the other hand, the definition of Eu, (4.3.17) and (4.3.19) give

$$Eu(0) \leq \frac{1}{2}\|u_t(0,\cdot)\|_2^2 + \left(\frac{\mu}{p\lambda_1} + \frac{1}{p} + 2C\right)[u(0,\cdot)]_{s,\Omega}.$$

This shows that $Eu(0) < \tilde{E}_0$ for sufficiently small initial data. Finally, since $0 \le \upsilon(0) < z_0$, it follows that $\tilde{a}\upsilon(0)^p - \tilde{c}\upsilon(0)^q \ge 0$ and so $Eu(0) \ge 0$ by (4.4.3).

Now we prove that $Eu(t) \to 0$ as $t \to \infty$ and proceed by contradiction, assuming $l > 0$ in (4.3.10) and showing that all the necessary estimates used in the proof of Theorem 4.3.1 are still valid in this setting, provided that the couple $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$. We consider separately the cases when k verifies (4.3.1) and (4.3.2).

Let us first suppose $k \in CBV(\mathbb{R}_0^+) \cap C^1(\mathbb{R}_0^+)$. Define the Lyapunov function $V(t) = k(t)\langle u, u_t\rangle$ as in Theorem 4.3.1, so that (4.3.23) is still true. Of course, Lemmas 3.2 and 3.3 of [85] give respectively (4.3.26) and (4.3.27) as before. Inequality (4.3.25) is a consequence of Lemma 4.4.5, while (4.3.24) is now valid by Lemma 4.4.4 and also by the request that $(\upsilon(0), Eu(0))$ is in $\tilde{\Sigma}_0$. Hence we obtain (4.3.30) and the proof can now follow word by word the proof of Theorem 4.3.1 in the regular case, as well as in the general case $k \in CBV(\mathbb{R}_0^+) \setminus C^1(\mathbb{R}_0^+)$.

If k verifies (4.3.2), the proof is the same as in Theorem 4.3.1, with the only exception that Lemmas 4.4.1–4.4.5 are used in place of Lemmas 4.3.2 and 4.3.3. □

Also for the local asymptotic stability we consider cases in which k and ϱ are related and investigate the effects on the stability of the problem. From now on we suppose $p \ge 2$, so Lemma 4.3.5 is valid and (4.3.40) is available once more. Here we also need the *non–degeneracy* of the problem, so we assume that (4.3.9) holds, without further mentioning.

Theorem 4.4.7. *If there exists an auxiliary function k, satisfying* (4.3.39), (4.3.42) *and either* (4.3.1) *or* (4.3.2), *and if the initial data $[u(0,\cdot)]_{s,\Omega}$ and $\|u_t(0,\cdot)\|_2$ are sufficiently small, then* (4.3.5) *continues to hold.*

Proof. As shown in Theorem 4.4.6, if the initial data $[u(0,\cdot)]_{s,\Omega}$ and $\|u_t(0,\cdot)\|_2$ are sufficiently small, then $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$. As usually, if $l = 0$ in (4.3.10), then the second condition in (4.3.5) follows at once. Hence, let us assume by contradiction that $l > 0$ in (4.3.10).

We start by proving the theorem in the case when k verifies (4.3.1). Suppose first that $k \in CBV(\mathbb{R}_0^+) \cap C^1(\mathbb{R}_0^+)$. Define $V(t) = k(t)\langle u_t, u\rangle$ as always and get (4.3.23). Condition (4.3.24) is valid thanks to Lemma 4.4.4 and to the fact that $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$. Moreover, (4.3.25) derives from Lemma 4.4.5 and (4.3.26) from Lemma 3.2 of [85]. On the contrary, (4.3.27) is no more available, so that we need to estimate the term $\int_T^t k\|u_t(\tau,\cdot)\|_2^2 d\tau$ as in the proof of Theorem 4.3.6. Taking $\theta = \alpha/4$ in (4.3.40), we obtain (4.3.43)–(4.3.44). Hence, (4.3.45) follows at once. Therefore, by (4.3.42) we derive (4.3.47) and from now on the proof can be completed as in Theorem 4.3.6.

Finally, if k satisfies condition (4.3.2), the proof is the same as for Theorem 4.3.6. □

Corollary 4.4.8. *Suppose that δ_1, $\delta_2 \in L^\infty(\mathbb{R}_0^+)$, and either that ϱ is of class $CBV(\mathbb{R}_0^+) \setminus L^1(\mathbb{R}_0^+)$, or that ϱ is in $W^{1,1}_{\rm loc}(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+)$, $\varrho \not\equiv 0$ and $|\varrho'(t)| = o(\varrho(t))$ as $t \to \infty$. If the initial data $[u(0,\cdot)]_{s,\Omega}$, $\|u_t(0,\cdot)\|_2$ are sufficiently small, then (4.3.5) holds.*

Proof. It is sufficient to apply Theorem 4.4.7, with $k = \varrho$, since ϱ satisfies the structural assumptions as shown in the proof of Corollary 4.3.7. □

Theorem 4.4.9. *Let (AS)–(b) hold and let k be an auxiliary function of class $[W^{1,1}_{\rm loc}(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+)] \setminus L^1(\mathbb{R}_0^+)$, satisfying (4.3.3) and (4.3.51). If the initial data $[u(0,\cdot)]_{s,\Omega}$, $\|u_t(0,\cdot)\|_2$ are sufficiently small, then (4.3.5) continues to hold.*

Proof. We remind that, as shown in the proof of Theorem 4.4.6, if $[u(0,\cdot)]_{s,\Omega}$ and $\|u_t(0,\cdot)\|_2$ are sufficiently small, then $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$. Hence, let $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$. We follow the strategy used in the proof of the regular case given in Theorem 4.4.6, with the difference that the term $\int_T^t k'\langle u, u_t\rangle d\tau$ in (4.3.23) is treated as in Theorem 4.3.8, so we obtain again the relations (4.3.53)–(4.3.54). Therefore, we get (4.3.55), with $\alpha = \alpha(l) > 0$ given by Lemma 4.4.5, and consequently (4.3.56). Hence, we find the required contradiction exactly as in the proof of Theorem 4.3.8. □

Theorem 4.4.10. *If there exists an auxiliary function*

$$k \in [W^{1,1}_{\rm loc}(\mathbb{R}_0^+) \cap L^\infty(\mathbb{R}_0^+)] \setminus L^1(\mathbb{R}_0^+),$$

satisfying (4.3.39), (4.3.51) and (4.3.42), and the initial data $[u(0,\cdot)]_{s,\Omega}$, $\|u_t(0,\cdot)\|_2$ are sufficiently small, then (4.3.5) continues to hold.

Proof. Again, as shown in Theorem 4.4.6, if the initial data $[u(0,\cdot)]_{s,\Omega}$ and $\|u_t(0,\cdot)\|_2$ are sufficiently small, then $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$. Hence, consider $(\upsilon(0), Eu(0)) \in \tilde{\Sigma}_0$. We use the technique adopted for the proof of the regular case in Theorem 4.4.6, but we estimate the term $\int_T^t k'\langle u, u_t\rangle d\tau$ in (4.3.23) in the same manner of Theorem 4.4.9. Indeed, (4.3.52)–(4.3.54) are still valid, so that we obtain once more (4.3.57), where now α derives from Lemma 4.4.5. As in the proof of Theorem 4.3.9, let ℓ be the number satisfying (4.3.47), in virtue of (4.3.42). We get once more (4.3.56) and the proof can be completed as in Theorem 4.3.9. □

Corollary 4.4.11. *Suppose that ϱ is of class $[W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+_0)\cap L^\infty(\mathbb{R}^+_0)]\setminus L^1(\mathbb{R}^+_0)$, $|\varrho'(t)| = O(\varrho(t))$ as $t\to\infty$ and that δ_1, $\delta_2 \in L^\infty(\mathbb{R}^+_0)$. If the initial data $[u(0,\cdot)]_{s,\Omega}$, $\|u_t(0,\cdot)\|_2$ are sufficiently small, then (4.3.5) holds.*

Proof. It is sufficient to apply Theorem 4.4.10, with $k=\varrho$, since, by the same main arguments used in the proof of Corollary 4.4.8 and Theorem 4.4.9, now ϱ trivially verifies all the structural assumptions of Theorem 4.4.10. □

4.5 The linear case

In this section, following [84, Section 5], we consider an important special case of $(\mathscr{P}_3)$, that is $p=2$, $Q(t,x,u,v)=a(t)t^\alpha v$, with a satisfying

$$1/C \le a(t) \le C \qquad \text{in } \mathbb{R}^+_0 \tag{4.5.1}$$

for some $C>0$ and $\alpha\in\mathbb{R}$, and $f(t,x,u)=V(t,x)u$, where V is a bounded continuous function in $I\times\Omega$. In other words, here we study the solutions of

$$(\mathscr{P}_{3,\mathrm{lin}})\qquad \begin{cases} u_{tt} + (-\Delta)^s u + \mu u + \varrho(t) M([u]^2_{s,\Omega})u_t & \\ \qquad +a(t)t^\alpha u_t + V(t,x)u = 0 & \text{in } I\times\Omega,\\ u(t,x)=0 & \text{on } I\times(\mathbb{R}^n\setminus\Omega), \end{cases}$$

where $I=[1,\infty)$ and for simplicity we treat only the case $N=1$.

We immediately observe that (H)–(a) is verified with $q=2$, being V bounded. Thus (H)–(b) is dropped. Furthermore, condition (4.3.6) is verified, with $m=2$, $r\in(2,2^*]$,

$$d_1(t,x) = \frac{1}{\sigma(t)} = Ct^{|\alpha|} \tag{4.5.2}$$

and $d_2(t,x)=0$. Indeed,

$$|Q(t,x,u,v)| = a(t)t^\alpha|v| \le Ct^{|\alpha|}|v|, \tag{4.5.3}$$

since $t \geq 1$ and (4.5.1) holds. Moreover, again by (4.5.1), (4.5.2) and the fact that $t \in I$, we get

$$Q(t,x,u,v) \cdot v = a(t)t^{\alpha}v^2 \geq \frac{v^2}{Ct^{|\alpha|}}, \tag{4.5.4}$$

that is (AS)–(b), where we can take $\wp = 2$.

Theorem 4.5.1. *Let the assumptions listed in this section hold. If*

$$\varrho(t) \leq \text{const.}\, t \qquad \textit{for } t \textit{ sufficiently large} \tag{4.5.5}$$

and $|\alpha| \leq 1$, *then all the solutions of problem* $(\mathscr{P}_{3,\mathrm{lin}})$ *have property* (4.3.5).

Proof. We next prove that in this setting all the hypotheses of Theorem 4.3.1 still hold. Take $k \equiv 1$, so (4.3.1) is verified. Moreover, recalling that $\delta_1(t) = \|d_1(t,\cdot)\|_{2^*/(2^*-2)} = \|d_1(t,\cdot)\|_{n/2s}$ we get

$$\begin{aligned}
\mathcal{A}(1)+\mathcal{C}(1) &= \left(\int_0^t \delta_1 d\tau\right)^{1/2} + \left(\int_0^t \frac{1}{\sigma}\, d\tau\right)^{1/2} + \left(\int_0^t \varrho(\tau) d\tau\right)^{1/2} \\
&\leq \left(\int_0^t C\tau^{|\alpha|}|\Omega|^{2s/n} d\tau\right)^{1/2} + \left(\int_0^t C\tau^{|\alpha|} d\tau\right)^{1/2} \\
&\quad + \left(\int_0^T \varrho(\tau) d\tau + \int_T^t \tau d\tau\right)^{1/2} \\
&\leq \text{const.}\ \left(t^{(|\alpha|+1)/2} + 1 + t\right).
\end{aligned} \tag{4.5.6}$$

Hence, by (4.5.5) and (4.5.6) condition (4.3.3) reduces to

$$\liminf_{t\to\infty} \frac{\mathcal{A}(1)+\mathcal{C}(1)}{t} \leq \text{const.} \lim_{t\to\infty} \left(t^{(|\alpha|-1)/2} + \frac{1}{t} + 1\right) < \infty, \tag{4.5.7}$$

being $|\alpha| \leq 1$. Therefore, we can apply Theorem 4.3.1 and property (4.3.5) holds at once. □

When $|\alpha| > 1$, we cannot apply Theorem 4.5.1. If $|\alpha| > 1$, we consider the special case

$$(\hat{\mathscr{P}}_{3,\mathrm{lin}}) \qquad \begin{cases} u_{tt} + (-\Delta)^s u + \mu u + \varrho(t)u_t & \\ \qquad + a(t)t^{\alpha}u_t + V(x)u = 0 & \text{in } I \times \Omega, \\ u(t,x) = 0 & \text{on } I \times (\mathbb{R}^n \setminus \Omega), \end{cases}$$

of $(\mathscr{P}_{3,\text{lin}})$, that is the case in which $M(\tau) \equiv m_0 > 0$, $V(t,x) = V(x) \geq -\mu$, $V(x) > -\mu$ a.e. in Ω and $\varrho \in L^1(I)$ denotes the function $m_0\varrho$.

Now consider solutions of $(\hat{\mathscr{P}}_{3,\text{lin}})$ having the separated form

$$u(t,x) = w(t)e(x), \tag{4.5.8}$$

where $e = e_k$ is an eigenfunction of $(-\Delta)^s + V(x) + \mu$ in Ω, with Dirichlet boundary conditions, that is $e = 0$ in $\mathbb{R}^n \setminus \Omega$. The corresponding eigenvalue λ_k is positive, by Theorem 4.7.3 in the Appendix, applied here with $a_0(x) = V(x) + \mu$. An easy calculation shows that w is a solution of the ordinary differential equation

$$w'' + [a(t)t^\alpha + \varrho(t)]w' + \lambda w = 0, \qquad t \in I. \tag{4.5.9}$$

We recall now Theorem 5.1′ of [83], which provides necessary conditions for global asymptotic stability of the rest state $u \equiv 0$ for the quasi–variational ordinary differential system

$$(\nabla G(u,u'))' - \nabla_u G(u,u') + f(t,u) + Q(t,u,u') = 0, \qquad J = [R,\infty), \tag{4.5.10}$$

where $G = G(u,v)$, ∇ denotes the gradient operator with respect to the variable v and $f(t,u) = \nabla_u F(t,u)$. We suppose that

$$G \in C^1(\mathbb{R}^N \times R^N;\mathbb{R}), \qquad F \in C^1(J \times \mathbb{R}^N;\mathbb{R}), \qquad Q \in C(J \times \mathbb{R}^N \times \mathbb{R}^N;\mathbb{R}^N)$$

and that $G(u,0) = F(u,0) = 0$. Put $H(u,v) = (\nabla G(u,v),v) - G(u,v)$, so in particular $H(u,0) = 0$ for all u.

Theorem 4.5.2 (Theorem 5.1′ of [83])**.** *Assume $N = 1$ and suppose that $H(0,v)$ is strictly increasing for $v > 0$ and strictly decreasing for $v < 0$. Let u be a solution of (4.5.10) on J such that $u(t) \to 0$ as $t \to \infty$. Suppose that for every $t \in J$ and for all u, v sufficiently small*

$$H(u,v) > 0, \qquad v \neq 0, \tag{4.5.11}$$

$$F(t,u) \geq 0, \qquad 0 \leq F_t(t,u) \leq \psi(t) \qquad \textit{with } \psi \in L^1(J), \tag{4.5.12}$$

$$0 \leq (Q(t,u,v),v) \leq \hat{\delta}(t)H(u,v), \tag{4.5.13}$$

where

$$\hat{\delta} \in L^1(J). \tag{4.5.14}$$

Then $u \equiv 0$ in J.

Observe now that, as in [84], equation (4.5.9) satisfies hypotheses (4.5.11)–(4.5.14), with $J = I$, $H(u,v) = v^2/2$, $F(t,u) = \lambda u^2/2$ and $\hat{\delta} = 2[a(t)t^\alpha + \varrho(t)]$. Therefore the only solution of (4.5.9) that approaches zero at infinity is the trivial solution $w \equiv 0$.

This being shown, it is easy to argue that all nontrivial solutions of (4.5.9) are oscillatory, with amplitude approaching a nonzero limit as $t \to \infty$. It may be noted that the behavior of solutions of (4.5.9) when $\alpha < -1$ is then essentially the same as for the wave equation itself in a bounded domain with zero boundary data.

When $\alpha > 1$, solutions of ($\hat{\mathscr{P}}_{3,\mathrm{lin}}$) again do not in general approach zero as $t \to \infty$, though their behavior is quite different from the case $\alpha < -1$. We say that a function

$$\psi = \psi(x) \in Y = \operatorname{span}\{e_k\}_{k=1}^\infty$$

is *attainable* if there exists a solution u of ($\hat{\mathscr{P}}_{3,\mathrm{lin}}$) such that

$$\lim_{t\to\infty} \|u(t,\cdot) - \psi\|_2 = 0. \tag{4.5.15}$$

Theorem 4.5.3. *Every function $\psi \in Y$ is attainable for problem ($\hat{\mathscr{P}}_{3,\mathrm{lin}}$) and the set of attainable functions is dense in $L^2(\Omega)$.*

Proof. The proof is based on [84, Theorem 5.1]. We first show that every eigenfunction e_k of $(-\Delta)^s + V(x) + \mu$ in Ω, with eigenvalue $\lambda_k > 0$ by Theorem 4.7.3 of the Appendix, is attainable. For this purpose, consider the function

$$u_k(t,x) = w_k(t)e_k(x),$$

which satisfies ($\hat{\mathscr{P}}_{3,\mathrm{lin}}$) if and only if w_k is a solution of (4.5.9), with $\lambda = \lambda_k$. Moreover, we have

$$\frac{1}{a(t)t^\alpha + \varrho(t)} \in L^1(I),$$

since $\alpha > 1$ and a verifies (4.5.1). Hence, by [82, Theorem 4.4] it follows that the set of attainable limits at ∞ of solutions of (4.5.9) is dense in $\mathbb{R}$. On the other hand, since (4.5.9) is linear, the set of attainable limits for (4.5.9) must in fact be all of $\mathbb{R}$. Hence for an appropriate solution $w_k \neq 0$ we get

$$\lim_{t\to\infty} \|u_k(t,\cdot) - e_k\|_2 = 0.$$

Finally, again from the linearity of $(\hat{\mathscr{P}}_{3,\mathrm{lin}})$, we obtain (4.5.15) for every $\psi \in Y$. Indeed, given $\psi \in Y$, there exist $e_{k_1}, \dots, e_{k_j}$ and real coefficients $\beta_1, \dots, \beta_j$ such that $\psi = \sum_{i=1}^{j} \beta_i e_{k_i}$. Consider the function

$$u(t,x) = \sum_{i=1}^{j} \beta_i u_{k_i}(t,x) = \sum_{i=1}^{j} \beta_i w_{k_i}(t) e_{k_i}(x),$$

where the functions w_{k_i} are the appropriate solutions such that

$$\lim_{t\to\infty} \|u_{k_i}(t,\cdot) - e_{k_i}\|_2 = 0 \quad \text{for any } i = 1, \dots, j.$$

Since $(\hat{\mathscr{P}}_{3,\mathrm{lin}})$ is linear, we get that also u is a solution. Moreover, we have

$$\|u(t,\cdot) - \psi\|_2 = \left\| \sum_{i=1}^{j} \beta_i u_{k_i}(t,\cdot) - \sum_{i=1}^{j} \beta_i e_{k_i} \right\|_2 \le \sum_{i=1}^{j} \beta_i \|u_{k_i}(t,\cdot) - e_{k_i}\|_2 \to 0$$

as $t \to \infty$. Hence, (4.5.15) holds for every $\psi \in Y$, as claimed. □

4.6 Asymptotic stability for $(\mathscr{P}_{3,K})$

In this section, we extend the results of Sections 4.3 and 4.4 when the fractional p–Laplacian operator is replaced by a more general elliptic nonlocal integro–differential operator.

Hence, we consider the problem $(\mathscr{P}_{3,K})$, governed by the operator $-\mathcal{L}_K$, which up to a multiplicative constant depending only on n, s and p is defined by

$$-\mathcal{L}_K \varphi(x) = \int_{\mathbb{R}^n} |\varphi(x) - \varphi(y)|^{p-2} [\varphi(x) - \varphi(y)] K(x-y) dy,$$

along any function $\varphi \in C_0^\infty(\Omega)$.

The *weight* $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ satisfies the natural restrictions listed in Chapter 2.

From now on we endow $\tilde{D}^{s,p}(\Omega;\mathbb{R}^N)$ with the weighted *Gagliardo* norm

$$[u]_{K,\Omega} = \left(\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^p K(x-y) dx dy \right)^{1/p}.$$

By (K_2), we get

$$[u]_{s,\Omega} \le K_0^{-1/p} [u]_{K,\Omega},$$

so that (4.3.16) is still valid.

Let $\lambda_{1,K}$ be *the first eigenvalue* of the scalar problem

$$\begin{cases} -\mathcal{L}_K u = \lambda\, |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.6.1}$$

in $\tilde{D}^{s,p}(\Omega)$, that is $\lambda_{1,K}$ is defined by the Rayleigh quotient

$$\lambda_{1,K} = \inf_{u\in\tilde{D}^{s,p}(\Omega),\ u\neq 0} \frac{\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^p K(x-y)dxdy}{\int_\Omega |u|^p dx}. \tag{4.6.2}$$

By Lemma 2.1 of [53] *the infimum in* (4.6.2) *is achieved and* $\lambda_{1,K} > 0$.

In this setting, we can extend the results of Sections 4.3 and 4.4, where the proofs proceed in the same way, up to the replacement of the appropriate norm and the first eigenvalue.

4.7 Appendix

In this section, following [70] and [88], we present an important result of independent interest, concerning the problem

$$\begin{cases} (-\Delta)^s u + a_0(x) u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{$\mathscr{P}_\lambda$}$$

where $a_0(x)$ is a bounded non–negative continuous function, with $a_0 > 0$ a.e. in Ω.

Denote by E the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_E = \left([u]^2_{s,\Omega} + \|u\|^2_{2,a_0}\right)^{1/2}, \tag{4.7.1}$$

where $\|\cdot\|_{2,a_0}$ is the natural norm in the weighted Lebesgue space $L^2(\Omega, a_0)$, that is

$$\|u\|_{2,a_0} = \left(\int_\Omega a_0(x)|u(x)|^2 dx\right)^{1/2}.$$

Notice that E is a Hilbert space, endowed with the inner product

$$\langle u, v\rangle_E = \langle u, v\rangle_{s,\Omega} + \langle a_0 u, v\rangle, \tag{4.7.2}$$

where in this section $\langle\cdot,\cdot\rangle$ denotes the bracket pairing in $L^2(\Omega)$ and, with abuse of notation, the duality pairing. We consider the weak formulation of the problem and define

$$\mathcal{J}(u) = \frac{1}{2}\|u\|_E^2. \tag{4.7.3}$$

Let us give some preliminary results.

Proposition 4.7.1. *If $X_\star$ is a non–empty weakly closed subspace of E and $\mathcal{M}_\star = \{u \in X_\star : \|u\|_2 = 1\}$, then there exists $u_\star \in \mathcal{M}_\star$ such that*

$$\min_{u\in\mathcal{M}_\star} \mathcal{J}(u) = \mathcal{J}(u_\star) \tag{4.7.4}$$

and

$$\langle \mathcal{J}'(u_\star), \varphi\rangle = \lambda_\star \langle u_\star, \varphi\rangle \text{ for any } \varphi \in X_\star, \text{ where } \lambda_\star = 2\mathcal{J}(u_\star) > 0. \tag{4.7.5}$$

Proof. In order to prove (4.7.4), we use the direct method of minimization. Let us take a minimizing sequence u_j of $\mathcal{J}$ on $\mathcal{M}_\star$, i.e. a sequence $u_j \in \mathcal{M}_\star$ such that

$$\mathcal{J}(u_j) \to \inf_{u\in\mathcal{M}_\star} \mathcal{J}(u) \geq 0 > -\infty \text{ as } j \to \infty. \tag{4.7.6}$$

Then the sequence $\mathcal{J}(u_j)$ is bounded in $\mathbb{R}$, and so, by the definition of $\mathcal{J}$, we get that $\|u_j\|_E$ is also bounded.

Since E is a reflexive space, up to a subsequence still denoted by u_j, we have that u_j converges weakly in E to some $u_\star \in X_\star$, being $X_\star$ weakly closed. The weak convergence gives that

$$\langle \mathcal{J}'(u_j), \varphi\rangle \to \langle \mathcal{J}'(u_\star), \varphi\rangle \qquad \text{for any } \varphi \in E$$

as $j \to \infty$. Moreover, by the boundedness of $\|u_j\|_E$, up to a subsequence we have

$$u_j \to u_\star \quad \text{in } L^2(\mathbb{R}^n)$$

as $j \to \infty$, since

$$\text{the embedding } E \hookrightarrow L^2(\Omega) \text{ is compact.} \tag{4.7.7}$$

Hence, $\|u_\star\|_2 = 1$, that is $u_\star \in \mathcal{M}_\star$. Using the Fatou lemma we deduce that

$$\begin{aligned}
\lim_{j\to\infty} \mathcal{J}(u_j) &= \frac{1}{2}\lim_{j\to\infty}\left([u_j]_{s,\Omega}^2 + \|u_j\|_{2,a_0}^2\right) \\
&\geq \frac{1}{2}\left([u_\star]_{s,\Omega}^2 + \|u_\star\|_{2,a_0}^2\right) = \mathcal{J}(u_\star) \geq \inf_{u\in\mathcal{M}_\star} \mathcal{J}(u),
\end{aligned}$$

so that, by (4.7.6), we get $\mathcal{J}(u_\star) = \inf_{u\in\mathcal{M}_\star} \mathcal{J}(u)$. This gives (4.7.4).

Now we prove (4.7.5). For this, let $\epsilon \in (-1,1)$, $\varphi \in X_\star$, $c_\epsilon = \|u_\star + \epsilon\varphi\|_2$ and $u_\epsilon = (u_\star + \epsilon\varphi)/c_\epsilon$. We observe that $u_\star \in \mathcal{M}_\star$,

$$c_\epsilon^2 = \|u_\star\|_2^2 + 2\epsilon \int_\Omega u_\star(x)\varphi(x)dx + o(\epsilon)$$
$$\text{and } [u_\star + \epsilon\varphi]_{s,\Omega}^2 = [u_\star]_{s,\Omega}^2 + 2\epsilon\langle u_\star, \varphi\rangle + o(\epsilon).$$

Consequently, being $\|u_\star\|_2 = 1$,

$$\begin{aligned}
2\mathcal{J}(u_\epsilon) &= \frac{\|u_\star\|_E^2 + 2\epsilon\langle u_\star, \varphi\rangle_E + o(\epsilon)}{1 + 2\epsilon\int_\Omega u_\star\varphi dx + o(\epsilon)} \\
&= (2\mathcal{J}(u_\star) + 2\epsilon\langle \mathcal{J}'(u_\star), \varphi\rangle + o(\epsilon)) \cdot \left(1 - 2\epsilon \int_\Omega u_\star\varphi dx + o(\epsilon)\right) \\
&= 2\mathcal{J}(u_\star) - 2\epsilon\left(\langle \mathcal{J}'(u_\star), \varphi\rangle - 2\mathcal{J}(u_\star)\int_\Omega u_\star\varphi dx\right) + o(\epsilon).
\end{aligned}$$

Moreover, notice that we have $\mathcal{J}(u_\star) > 0$ because otherwise we would have $u_\star \equiv 0$, but $0 \notin \mathcal{M}_\star$. This and the minimality of $u_\star$ imply (4.7.5). □

Proposition 4.7.2. *If $\lambda \neq \tilde{\lambda}$ are different eigenvalues of problem $(\mathscr{P}_\lambda)$, with eigenfunctions e and $\tilde{e} \in E$, respectively, then*

$$\langle e, \tilde{e}\rangle_E = 0 = \int_\Omega e(x)\tilde{e}(x)dx. \tag{4.7.8}$$

Moreover, if e is an eigenfunction of problem $(\mathscr{P}_\lambda)$ corresponding to an eigenvalue λ, then

$$\|e\|_E^2 = \lambda\|e\|_2^2. \tag{4.7.9}$$

Proof. We may suppose that $e \not\equiv 0$ and $\tilde{e} \not\equiv 0$. We put $g = e/\|e\|_2$ and $\tilde{g} = \tilde{e}/\|\tilde{e}\|_2$, which are eigenfunctions as well and we calculate the weak formulation of problem $(\mathscr{P}_\lambda)$ for g with test function $\tilde{g}$ and vice versa. We obtain

$$\lambda \int_\Omega g(x)\tilde{g}(x) = \langle \mathcal{J}'(g), \tilde{g}\rangle = \langle \mathcal{J}'(\tilde{g}), g\rangle = \tilde{\lambda}\int_\Omega g(x)\tilde{g}(x)dx, \tag{4.7.10}$$

that is

$$(\lambda - \tilde{\lambda})\int_\Omega g(x)\tilde{g}(x)dx = 0.$$

Thus, since $\lambda \neq \tilde{\lambda}$,

$$\int_\Omega g(x)\tilde{g}(x)dx = 0. \tag{4.7.11}$$

By plugging (4.7.11) into (4.7.10), we obtain

$$\langle g, \tilde{g}\rangle_E = \langle \mathcal{J}'(g), \tilde{g}\rangle = 0.$$

This and (4.7.11) complete the proof of (4.7.8). Finally, (4.7.9) can be easily proved by choosing $\varphi = e$ in the weak formulation of $(\mathscr{P}_\lambda)$. □

Theorem 4.7.3. *Consider the problem*

$$(-\Delta)^s u + a_0(x)u = \lambda u. \tag{$\mathscr{P}_\lambda$}$$

(*a*) *Problem* $(\mathscr{P}_\lambda)$ *admits an eigenvalue* λ_1 *which is positive and that can be characterized as follows*

$$\lambda_1 = \min_{u\in E\setminus\{0\}} \frac{\|u\|_E^2}{\int_\Omega |u|^2dx}. \tag{4.7.12}$$

(*b*) *There exists a non–negative function* $e_1 \in E$*, which is an eigenfunction corresponding to the eigenvalue* λ_1*, which attains the minimum in* (4.7.12)*, that is* $\|e_1\|_2 = 1$ *and*

$$\lambda_1 = \|e_1\|_E^2. \tag{4.7.13}$$

(*c*) *The set of the eigenvalues of problem* $(\mathscr{P}_\lambda)$ *consists of a sequence* $(\lambda_k)_{k\in\mathbb{N}}$*, with*

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \lambda_{k+1} \le \ldots \tag{4.7.14}$$

and

$$\lambda_k \to \infty \text{ as } k \to \infty. \tag{4.7.15}$$

Moreover, for any $k \in \mathbb{N}$ *the eigenvalues can be characterized as follows*

$$\lambda_{k+1} = \min_{u\in \mathbb{P}_{k+1}\setminus\{0\}} \frac{\|u\|_E^2}{\int_\Omega |u|^2dx}, \tag{4.7.16}$$

where

$$\mathbb{P}_{k+1} = \{u \in E : \langle u, e_j\rangle_E = 0 \text{ for any } j = 1, \ldots, k\}. \tag{4.7.17}$$

(*d*) *For any* $k \in \mathbb{N}$ *there exists a function* $e_{k+1} \in \mathbb{P}_{k+1}$*, which is an eigenfunction corresponding to* λ_{k+1}*, attaining the minimum in* (4.7.16)*, that is* $\|e_{k+1}\|_2 = 1$ *and*

$$\lambda_{k+1} = \|e_{k+1}\|_E^2. \tag{4.7.18}$$

(*e*) *The sequence* $(e_k)_{k\in\mathbb{N}}$ *of eigenfunctions corresponding to* λ_k *is an orthogonal basis of both* E *and* $\tilde{D}^{s,2}(\Omega)$*.*

Proof. We follow the proof of Proposition 9 of [88].

(*a*) For this, we note that the minimum defining λ_1 exists and that λ_1 is an eigenvalue, thanks to (4.7.4) and (4.7.5), applied here with $X_\star = E$.

(*b*) Again by (4.7.4), the minimum defining λ_1 is attained at some $e_1 \in E$, with $\|e_1\|_2 = 1$. The fact that e_1 is an eigenfunction corresponding to λ_1 and formula (4.7.12) follow from (4.7.5), again with $X_\star = E$.

(*c*) We define λ_{k+1} as in (4.7.16): we notice indeed that the minimum in (4.7.16) exists and it is attained at some $e_{k+1} \in \mathbb{P}_{k+1}$, thanks to (4.7.4) and (4.7.5), applied here with $X_\star = \mathbb{P}_{k+1}$, which, by construction, is weakly closed.

Moreover, since $\mathbb{P}_{k+1} \subseteq \mathbb{P}_k \subseteq E$, we get that

$$0 < \lambda_1 \le \lambda_2 \le \dots \lambda_k \le \lambda_{k+1} \le \dots$$

Also, (4.7.5) with $X_\star = \mathbb{P}_{k+1}$ says that

$$\langle \mathcal{J}'(e_{k+1}), \varphi\rangle = \lambda_{k+1}\langle e_{k+1}, \varphi\rangle \qquad \text{for any } \varphi \in \mathbb{P}_{k+1}. \tag{4.7.19}$$

In order to show that λ_{k+1} is en eigenvalue with eigenfunction e_{k+1}, we need to show that formula (4.7.19) holds for any $\varphi \in E$, not only in $\mathbb{P}_{k+1}$. For this, we argue recursively, assuming that the claim holds for $1, \dots, k$ and proving it for $k+1$. The base of induction is given by the fact that λ_1 is an eigenvalue, as shown in assertion (*a*). We use the direct sum decomposition

$$E = \text{span}\,\{e_1, \dots, e_k\} \oplus (\text{span}\,\{e_1, \dots, e_k\})^\perp = \text{span}\,\{e_1, \dots, e_k\} \oplus \mathbb{P}_{k+1},$$

where the orthogonal $\perp$ is intended with respect to the scalar product of E, namely $\langle\cdot,\cdot\rangle_E$. Thus, given any $\varphi \in E$, we write $\varphi = \varphi_1 + \varphi_2$, with $\varphi_2 \in \mathbb{P}_{k+1}$

and $\varphi_1 = \sum_{i=1}^{k} c_i e_i$, for some $c_1, \dots, c_k \in \mathbb{R}$. Then, from (4.7.19) tested with $\varphi_2 = \varphi - \varphi_1$, we know that

$$\begin{aligned} \langle \mathcal{J}'(e_{k+1}), \varphi\rangle - \lambda_{k+1}\langle e_{k+1}, \varphi\rangle &= \langle \mathcal{J}'(e_{k+1}), \varphi_1\rangle - \lambda_{k+1}\langle e_{k+1}, \varphi_1\rangle \\ &= \sum_{i=1}^{k} c_i \left[\langle \mathcal{J}'(e_{k+1}), e_i\rangle - \lambda_{k+1}\langle e_{k+1}, e_i\rangle\right]. \end{aligned} \tag{4.7.20}$$

Furthermore, testing the weak formulation of problem $(\mathscr{P}_\lambda)$ for e_i against e_{k+1} for $i = 1, \dots, k$, allowed by the inductive assumption, and recalling that $e_{k+1} \in \mathbb{P}_{k+1}$, we see that

$$0 = \langle e_{k+1}, e_i\rangle_E = \langle \mathcal{J}'(e_{k+1}), e_i\rangle = \lambda_{k+1}\langle e_{k+1}, e_i\rangle.$$

Thus, by (4.7.14)

$$\langle \mathcal{J}'(e_{k+1}), e_i\rangle = 0 = \langle e_{k+1}, e_i\rangle$$

for any $i = 1, \dots, k$. By plugging this into (4.7.20), we conclude that (4.7.19) holds true for any $\varphi \in E$, that is λ_{k+1} is an eigenvalue with eigenfunction e_{k+1}.

Now we prove (4.7.15): for this, we start by showing that if k, $h \in \mathbb{N}$, with $k \neq h$, then

$$\langle e_k, e_h\rangle_E = 0 = \int_\Omega e_k(x)e_h(x)dx.$$

Indeed, let $k > h$, hence $k - 1 \geq h$. Therefore,

$$e_k \in \mathbb{P}_k = \left(\operatorname{span}\{e_1, \dots, e_{k-1}\}\right)^\perp \subseteq \left(\operatorname{span}\{e_h\}\right)^\perp,$$

and so $\langle e_k, e_h\rangle_E = 0$. But e_k is an eigenfunction and, using the weak formulation of problem $(\mathscr{P}_\lambda)$ for e_k tested with $\varphi = e_h$, we get

$$0 = \langle e_k, e_h\rangle_E = \langle \mathcal{J}'(e_k), e_h\rangle = \lambda_k \int_\Omega e_k(x)e_h(x)dx,$$

as claimed.

To complete the proof of (4.7.15), suppose by contradiction that $\lambda_k \to c$ for some constant $c \in \mathbb{R}$. Then λ_k is bounded in $\mathbb{R}$. Since $\|e_k\|_E^2 = \lambda_k$ by (4.7.9), we deduce by (4.7.7) that there is a subsequence for which

$$e_{k_j} \to e_\infty \qquad \text{in } L^2(\Omega)$$

as $k_j \to \infty$, for some $e_\infty \in L^2(\Omega)$. In particular, $(e_{k_j})_j$ is a Cauchy sequence in $L^2(\Omega)$. But this is in contradiction with the fact that if e_{k_i} and e_{k_j} are orthogonal in $L^2(\Omega)$ we have

$$\|e_{k_j} - e_{k_i}\|_2^2 = \|e_{k_j}\|_2^2 + \|e_{k_i}\|_2^2 = 2.$$

Now, to complete the proof of (c), we need to show that the sequence of eigenvalues constructed in (4.7.16) exhausts all the eigenvalues of the problem, i.e. that any eigenvalue of problem $(\mathscr{P}_\lambda)$ can be written in the form (4.7.16). We show this by arguing, once more, by contradiction. Let us suppose that there exists an eigenvalue

$$\lambda \notin (\lambda_k)_{k\in\mathbb{N}}\,, \tag{4.7.21}$$

and let $e \in E$ be a normalized eigenfunction relative to λ, so $\|e\|_2 = 1$. Then, by (4.7.9) we have

$$2\mathcal{J}(e) = \|e\|_E^2 = \lambda. \tag{4.7.22}$$

Thus, by the minimality of λ_1 given in (4.7.12) and (4.7.13), we get that

$$\lambda = 2\mathcal{J}(e) \geq 2\mathcal{J}(e_1) = \lambda_1.$$

This, (4.7.21) and (4.7.15) imply that there exists $k \in \mathbb{N}$ such that

$$\lambda_k < \lambda < \lambda_{k+1}. \tag{4.7.23}$$

We claim that $e \notin \mathbb{P}_{k+1}$. Indeed, if $e \in \mathbb{P}_{k+1}$, from (4.7.22) and (4.7.16) we deduce that $\lambda = 2\mathcal{J}(e) \geq \lambda_{k+1}$, which contradicts (4.7.23). As a consequence, there exists $i \in \{1, \dots, k\}$ such that $\langle e, e_i\rangle_E \neq 0$. But this is in contradiction with (4.7.8), which can be easily adapted to our problem. In conclusion, (4.7.21) is false and this completes the proof of (c).

(d) Again using (4.7.4) with $X_\star = \mathbb{P}_{k+1}$, the minimum defining λ_{k+1} is attained at some $e_{k+1} \in \mathbb{P}_{k+1}$. The fact that e_{k+1} is an eigenfunction corresponding to λ_{k+1} has been checked in (c) and in turn (4.7.18) follows from (4.7.5).

(e) The orthogonality has been already showed in (c), so we need to prove that the sequence of eigenfunctions $(e_k)_{k\in\mathbb{N}}$ is a basis for both E and $\tilde{D}^{s,2}(\Omega)$. Let us start to prove that it is a basis for E. First, we show that if $v \in E$ is such that $\langle v, e_k\rangle_E = 0$ for any $k \in \mathbb{N}$, then $v \equiv 0$. To this aim, we argue by contradiction and suppose that there exists a nontrivial $v \in E$ such that

$$\langle v, e_k\rangle_E = 0 \text{ for any } k \in \mathbb{N}. \tag{4.7.24}$$

Then, up to normalization, we can assume that $\|v\|_2 = 1$. Thus, from (4.7.15), there exists $k \in \mathbb{N}$ such that

$$2\mathcal{J}(v) < \lambda_{k+1} = \min_{\substack{u\in\mathbb{P}_{k+1}\\ \|u\|_2=1}} \|u\|_E.$$

Hence, $v \notin \mathbb{P}_{k+1}$ and so there exists $j \in \mathbb{N}$ for which $\langle v, e_j \rangle_E \neq 0$. This contradicts (4.7.24), as claimed.

We use now a standard Fourier analysis technique in order to show that $(e_k)_{k\in\mathbb{N}}$ is a basis for E. We define $\upsilon_i = e_i / \|e_i\|_E$ and, given $g \in E$,

$$g_j = \sum_{i=1}^{j} \langle g, \upsilon_i \rangle_E \upsilon_i.$$

We point out that for any $j \in \mathbb{N}$, g_j belongs to span $\{e_1, \dots, e_j\}$. Let $v_j = g - g_j$. By the orthogonality of $(e_k)_{k\in\mathbb{N}}$ in E, we get

$$\begin{aligned} 0 \leq \|v_j\|_E = \langle v_j, v_j \rangle_E &= \|g\|_E^2 + \|g_j\|_E^2 - 2\langle g, g_j \rangle_E \\ &= \|g\|_E^2 + \langle g_j, g_j \rangle_E^2 - 2\sum_{i=1}^{j} \langle g, \upsilon_i \rangle_E^2 = \|g\|_E^2 - \sum_{i=1}^{j} \langle g, \upsilon_i \rangle_E^2. \end{aligned}$$

Therefore, for any $j \in \mathbb{N}$

$$\sum_{i=1}^{j} \langle g, \upsilon_i \rangle_E^2 \leq \|g\|_E^2$$

and so $\sum_{i=1}^{\infty} \langle g, \upsilon_i \rangle_E^2$ is a convergent series. Thus, if we set

$$\tau_j = \sum_{i=1}^{j} \langle g, \upsilon_i \rangle_E^2,$$

we get that $(\tau_j)_j$ is a Cauchy sequence in $\mathbb{R}$. Moreover, using again the orthogonality of $(e_k)_{k\in\mathbb{N}}$ in E, we see that, if $h > j$,

$$\|v_h - v_j\|_E^2 = \left\| \sum_{i=j+1}^{h} \langle g, \upsilon_i \rangle_E \upsilon_i \right\|_E^2 = \sum_{i=j+1}^{h} \langle g, \upsilon_i \rangle_E^2 = \tau_h - \tau_j.$$

Hence, $(v_j)_j$ is a Cauchy sequence in E. By the completeness of E, it follows that there exists $v \in E$ such that

$$v_j \to v \text{ in } E \text{ as } j \to \infty. \tag{4.7.25}$$

Now, we observe that if $j \geq k$,

$$\langle v_j, \upsilon_k \rangle_E = \langle g, \upsilon_k \rangle_E - \langle g_j, \upsilon_k \rangle_E = \langle g, \upsilon_k \rangle_E - \langle g, \upsilon_k \rangle_E = 0.$$

Hence, by (4.7.25), it easily follows that $\langle v, v_k \rangle_E = 0$ for any $k \in \mathbb{N}$, and so that $v \equiv 0$. Therefore, we get

$$g_j = g - v_j \to g - v = g \text{ in } E \qquad \text{as } j \to \infty.$$

This and the fact that g_j belongs to span $\{e_1, \ldots, e_j\}$ for all $j \in \mathbb{N}$ yield that $(e_k)_{k\in\mathbb{N}}$ is a basis in E.

To complete the proof of (e), we need to show that $(e_k)_{k\in\mathbb{N}}$ is a basis for $\tilde{D}^{s,2}(\Omega)$. For this, take $v \in \tilde{D}^{s,2}(\Omega)$ and let $v_j \in C_0^\infty(\Omega)$ be such that $[v_j - v]_{s,\Omega} \leq 1/j$. Notice that $v_j \in E$. Therefore, since we know that $(e_k)_{k\in\mathbb{N}}$ is a basis for E, there exist $k_j \in \mathbb{N}$ and a function w_j, belonging to span $\left\{e_1, \ldots, e_{k_j}\right\}$, such that

$$\|v_j - w_j\|_E \leq 1/j.$$

In conclusion, we get

$$[v - w_j]_{s,\Omega} \leq [v - v_j]_{s,\Omega} + [v_j - w_j]_{s,\Omega} \leq [v - v_j]_{s,\Omega} + \|v_j - w_j\|_E \leq 2/j.$$

This shows that the sequence $(e_k)_{k\in\mathbb{N}}$ is a basis in $\tilde{D}^{s,2}(\Omega)$, as required. $\square$

Chapter 5

Problems $(\mathscr{P}_4)$ and $(\mathscr{P}_5)$

In this chapter we deal with the existence of nontrivial non–negative solutions of Schrödinger–Hardy systems driven by two possibly different fractional $\wp$–Laplacian operators.

The starting point is the fractional Schrödinger–Hardy system in $\mathbb{R}^n$

$$(\mathscr{P}_4) \qquad \begin{cases} (-\Delta)_m^s u + a(x)|u|^{m-2}u - \mu\dfrac{|u|^{m-2}u}{|x|^{ms}} & = H_u(x,u,v), \\ (-\Delta)_p^s v + b(x)|v|^{p-2}v - \sigma\,\dfrac{|v|^{p-2}v}{|x|^{ps}} & = H_v(x,u,v), \end{cases}$$

where μ and σ are real parameters, $n > ps$, with $s \in (0,1)$ and $1 < m \le p < m^* = mn/(n-ms)$.

5.1 Structural setting and main results

Throughout the chapter we assume that $s \in (0,1)$, $n > ps$ and $1 < m \le p < m^*$, without further mentioning. As noticed in Section 1.1, the embedding $D^{s,\wp}(\mathbb{R}^n) \hookrightarrow L^{\wp^*}(\mathbb{R}^n)$ is continuous. By Theorems 1 and 2 of [67], we know that

$$(5.1.1) \qquad \begin{aligned} &\|u\|_{\wp^*}^{\wp} \le c_{n,\wp}\frac{s(1-s)}{(n-\wp s)^{\wp-1}}[u]_{s,\wp}^{\wp}, \qquad \wp^* = \frac{\wp n}{n-\wp s}, \qquad n > \wp s, \\ &\|u\|_{H_\wp}^{\wp} \le c_{n,\wp}\frac{s(1-s)}{(n-\wp s)^{\wp}}[u]_{s,\wp}^{\wp}, \qquad \|u\|_{H_\wp}^{\wp} = \int_{\mathbb{R}^n}|u(x)|^{\wp}\frac{dx}{|x|^{\wp s}}, \end{aligned}$$

for all $u \in D^{s,\wp}(\mathbb{R}^n)$, where the positive constant $c_{n,\wp}$ depends only on n and $\wp$.

The first result is based on the best fractional Hardy–Sobolev constant, which for all $\wp > 1$ is denoted by $\mathcal{H}_\wp = \mathcal{H}(\wp, n, s)$, and is given by

$$\mathcal{H}_\wp = \inf_{\substack{u \in D^{s,\wp}(\mathbb{R}^n) \\ u \neq 0}} \frac{[u]_{s,\wp}^\wp}{\|u\|_{H_\wp}^\wp}, \tag{5.1.2}$$

where $\|\cdot\|_{H_\wp}$ is defined in (5.1.1). Clearly, $\mathcal{H}_\wp > 0$ thanks to (5.1.1).

The weight functions a and b are of class $\mathscr{V}(\mathbb{R}^n)$. The *family* $\mathscr{V}(\mathbb{R}^n)$ consists of all functions $V \in C(\mathbb{R}^n)$ satisfying

(V_1) *V is bounded from below by a positive constant*;

(V_2) *there exists $\kappa > 0$ such that* $\lim_{|y|\to\infty} \operatorname{meas}(\{x \in B_\kappa(y) : V(x) \le c\}) = 0$ *for any $c > 0$*,

where $B_\kappa(y)$ denotes any open ball of $\mathbb{R}^n$ centered at y and of radius $\kappa > 0$.

The natural solution space for system ($\mathscr{P}_4$) is the real Banach space $W = E_{m,a} \times E_{p,b}$, endowed with the norm $\|(u,v)\| = \|u\|_{E_{m,a}} + \|v\|_{E_{p,b}}$, where

$$E_{m,a} = \left\{ u \in D^{s,m}(\mathbb{R}^n) : \int_{\mathbb{R}^n} a(x)|u(x)|^m dx < \infty \right\},$$
$$E_{p,b} = \left\{ v \in D^{s,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} b(x)|v(x)|^p dx < \infty \right\},$$
$$\|u\|_{E_{m,a}} = \left([u]_{s,m}^m + \|u\|_{m,a}^m\right)^{1/m}, \quad \|v\|_{E_{p,b}} = \left([v]_{s,p}^p + \|v\|_{p,b}^p\right)^{1/p},$$

and $\|\varphi\|_{\wp,V} = \left(\int_{\mathbb{R}^n} V(x)|\varphi|^\wp dx\right)^{1/\wp}$ for all $\wp > 1$, $V \in \mathscr{V}(\mathbb{R}^n)$ and φ is in $L^\wp(\mathbb{R}^n, V)$.

As noted in Lemma 5.2.2, see Lemma 4.1 of [35] for a proof, under the solely condition (V_1), the embeddings

$$W \hookrightarrow W^{s,m}(\mathbb{R}^n) \times W^{s,p}(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$$

are certainly continuous for all $\nu \in [p, m^*]$, being $1 < m \le p < m^*$. Thus, the numbers

$$\lambda_\nu = \inf \left\{ \|u\|_{E_{m,a}}^\nu + \|v\|_{E_{p,b}}^\nu \; : \; \int_{\mathbb{R}^n} |(u,v)|^\nu dx = 1 \right\} \tag{5.1.3}$$

are well defined and strictly positive.

Concerning the nonlinearity H, we first assume

(H_1) *$H : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and admits partial derivatives H_u and H_v of class $C(\mathbb{R}^n \times \mathbb{R}^2)$, $H \geq 0$ in $\mathbb{R}^n \times \mathbb{R}^2$, $H(x, 0, 0) = 0$ in $\mathbb{R}^n$ and $H_u(x, u, v) = 0$ if $x \in \mathbb{R}^n$ and $u \leq 0$, $v \in \mathbb{R}$, while $H_v(x, u, v) = 0$ if $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $v \leq 0$;*

(H_2) *There are an exponent $r \in (p, m^*)$ and a number $\lambda \in [0, \lambda_p)$ such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which the inequality*

$$|H_z(x, z)| \leq (\lambda + \varepsilon)|z|^{p-1} + C_\varepsilon |z|^{r-1}, \quad z = (u, v), \ |z| = \sqrt{u^2 + v^2},$$

holds for all $(x, z) \in \mathbb{R}^n \times \mathbb{R}^2$, where λ_p is introduced in (5.1.3) and $H_z = (H_u, H_v)$;

(H_3) $\displaystyle\lim_{\substack{|z|\to\infty \\ u>0 \vee v>0}} \frac{H(x, z)}{|z|^p} = \infty$, *uniformly in* $\mathbb{R}^n$;

(H_4) *There exist a non–negative function g of class $L^1(\mathbb{R}^n)$ and a constant $C_F \geq 1$ such that*

$$F(x, tu, tv) \leq C_F F(x, u, v) + g(x)$$

for a.e $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}_0^+$, $v \in \mathbb{R}_0^+$ and $t \in (0, 1)$, where

$$F(x, z) = H_z(x, z) \cdot z - pH(x, z).$$

Clearly, when F does not depend on x the function g should be identically zero in (H_4) and $F = F(u, v) \geq 0$ by (H_1). A simple example of function $H = H(u, v)$ verifying (H_1)–(H_4) is given by $H(u, v) = \xi^2 \log(1 + \xi)$, $\xi = \xi(u, v) = \sqrt{(u^+)^2 + (v^+)^2}$, with $2n/(n + 2s) < m \leq p = 2$, $C_F = 1$ and $g = 0$, see Figure 5.1. Thus $2 < m^*$.

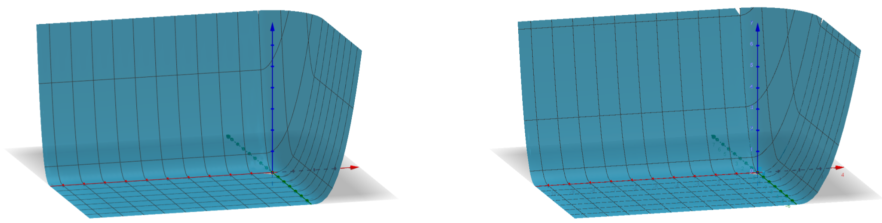

Figure 5.1: The functions $H(u, v) = H(\xi) = \xi^2 \log(1+\xi)$ and $F(u, v) = F(\xi) = \xi^3/(1+\xi)$

Another model function for H satisfying (H_1)–(H_4), when again $1 < m \leq p = 2$ and $\xi = \xi(u,v) = \sqrt{(u^+)^2 + (v^+)^2}$, is given by

$$H(u,v) = \begin{cases} 4\xi^2 \log(1+\xi), & 0 \leq \xi \leq 1, \\ (4\log 2 + 1/2)\xi^2 + \xi - 3/2, & 1 \leq \xi \leq 2, \\ (4\log 2 + 5/8)\xi^2 + \dfrac{\xi^2}{8} \cdot \log(\xi - 1), & \xi \geq 2, \end{cases}$$

(5.1.4)

$$F(u,v) = H_z(u,v)\cdot(u,v) - 2H(u,v) = \begin{cases} \dfrac{4\xi^3}{1+\xi}, & 0 \leq \xi \leq 1, \\ 3 - \xi, & 1 \leq \xi \leq 2, \\ \dfrac{1}{8} \cdot \dfrac{\xi^3}{\xi - 1}, & \xi \geq 2, \end{cases}$$

with $g = 0$ and $C_F = 2$.

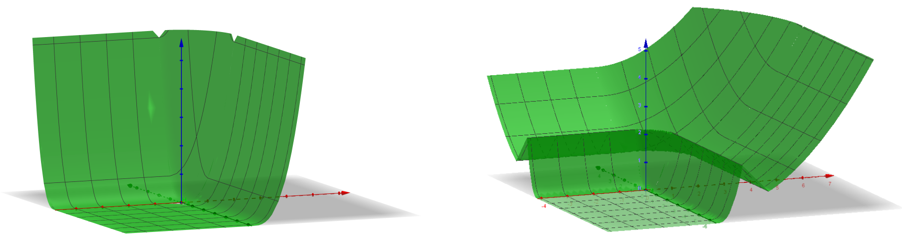

Figure 5.2: The functions $H = H(u,v) = H(\xi)$ and $F = F(u,v) = F(\xi)$ defined in (5.1.4)

A final model function for H satisfying (H_1)–(H_4), when again $1 < m \leq p = 2$ and $\xi = \xi(u,v) = \sqrt{(u^+)^2 + (v^+)^2}$, is given for all $x \in \mathbb{R}^n$ by

$$H(x,u,v) = \phi(x) \cdot \begin{cases} 2\xi^2 \log(1+\xi), & 0 \leq \xi \leq 1, \\ (\log 4 + 7/2)\xi^2 + \xi^2 \log \xi^2 - 7/2, & 1 \leq \xi \leq 3, \\ (2\log 6 + 28/9 - (8\log 2)/27)\xi^2 & \\ \qquad + \dfrac{4\xi^2}{27} \cdot \log(1+\xi), & \xi \geq 3, \end{cases}$$

$$\begin{aligned} F(x,u,v) &= H_z(x,u,v)\cdot(u,v) - 2H(x,u,v) \\ &= \phi(x) \cdot \begin{cases} \dfrac{2\xi^3}{1+\xi}, & 0 \leq \xi \leq 1, \\ 2\xi^2 - 8\xi + 7, & 1 \leq \xi \leq 3, \\ \dfrac{4}{27} \cdot \dfrac{\xi^3}{1+\xi}, & \xi \geq 3, \end{cases} \end{aligned}$$

where ϕ is a non–negative nontrivial bounded continuous function of class $L^1(\mathbb{R}^n)$. In particular, (H_4) holds, with $C_F = 1$ and $g = 2\phi$. In this example, $H(x,z) = \phi(x)\mathfrak{H}(z)$, $F(x,z) = \phi(x)\Phi(z)$ and Φ has a negative minimum. See Figure 5.3 for the graphs of $\mathfrak{H}$ and Φ.

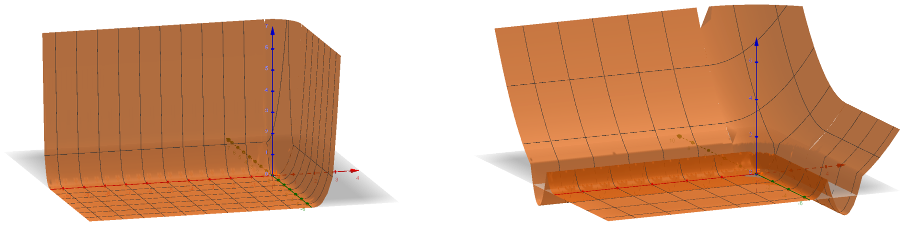

Figure 5.3: The functions $\mathfrak{H} = \mathfrak{H}(u,v) = \mathfrak{H}(\xi)$ and $\Phi = \Phi(u,v) = \Phi(\xi)$

Theorem 5.1.1. *Under the assumptions (H_1)–(H_4) and with a and b satisfying (V_1)–(V_2) system $(\mathscr{P}_4)$ has at least one nontrivial non–negative entire solution $(u,v) \in W$ for any $\mu \in (-\infty, \mathcal{H}_m)$ and for any $\sigma \in (-\infty, \mathcal{H}_p)$ such that*

$$1 - \frac{\mu^+}{\mathcal{H}_m} - \frac{\sigma^+}{\mathcal{H}_p} - 2^{p-1}\frac{\lambda}{\lambda_p} > 0, \tag{5.1.5}$$

being $\lambda \in [0, \lambda_p)$ given in (H_2).

It is interesting to note that when $|H_z(x,z)| = o(|z|^{p-1})$ as $z \to 0$, uniformly in $\mathbb{R}^n$, and there exist an exponent $r \in (p, m^*)$ and a constant $c > 0$ such that

$$|H_z(x,z)| \le c(1 + |z|^{r-1}) \quad \text{for all } (x,z) \in \mathbb{R}^n \times \mathbb{R}^2,$$

then (H_2) holds with $\lambda = 0$. This is the case of the example $H(u,v) = \xi^2 \log(1+\xi)$, $\xi = \xi(u,v) = \sqrt{(u^+)^2 + (v^+)^2}$ and $2n/(n+2s) < m \le p = 2$. In these circumstances, condition (5.1.5) simplifies into the more familiar natural request $\mu < \mathcal{H}_m$ and $\sigma < \mathcal{H}_p$.

The second result is a radial version of Theorem 5.1.1 under the solely condition (V_1) on the coefficients a and b, that is possibly covering the interesting case $a \equiv$Constant> 0 and $b \equiv$Constant> 0.

Theorem 5.1.2. *Assume that $n \ge 2$, that a, b, $H_z(\cdot, z)$ are radial for all $z \in \mathbb{R}^2$. Suppose that (V_1) and (H_1) hold and replace (H_2) by*

$(H_2)'$ *There are numbers q, r, with $p < q < r < m^*$, and $\lambda \in [0, \lambda_q)$ such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which the inequality*

$$|H_z(x,z)| \leq (\lambda + \varepsilon)|z|^{q-1} + C_\varepsilon |z|^{r-1}$$

holds for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^2$.

Then system $(\mathscr{P}_4)$ has at least one nontrivial non–negative entire radial solution $(u,v) \in W$ for any $\mu \in (-\infty, \mathcal{H}_m)$ and for any $\sigma \in (-\infty, \mathcal{H}_p)$ such that

$$1 - \frac{\mu^+}{\mathcal{H}_m} - \frac{\sigma^+}{\mathcal{H}_p} - 2^{p-1}\frac{\lambda}{\lambda_q} > 0. \tag{5.1.6}$$

Recall that $s \in (0,1)$ and $n > 2s$ are fixed. The prototype $H(u,v) = \xi^q \log(1+\xi)$, $\xi = \xi(u,v) = \sqrt{(u^+)^2 + (v^+)^2}$, satisfies condition $(H_2)'$, with $\lambda = 0$, $2n/(n+2s) < m \leq p = 2$, and any $q \in (2, 2+\eta)$, where $r = 2+\eta$ and $\eta \in (0, [m(n+2s) - 2n]/(n - ms))$.

Furthermore, we study the system in $\mathbb{R}^n$

$$(\mathscr{P}_5) \qquad \begin{cases} (-\Delta)^s_m u + a(x)|u|^{m-2}u - \mu\dfrac{|u|^{m-2}u}{|x|^{ms}} = H_u(x,u,v) + |u|^{m^*-2}u^+ \\ \qquad\qquad + \dfrac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta} + \varphi(x), \\ (-\Delta)^s_p v + b(x)|v|^{p-2}v - \sigma\dfrac{|v|^{p-2}v}{|x|^{ps}} = H_v(x,u,v) + |v|^{p^*-2}v^+ \\ \qquad\qquad + \dfrac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1} + \psi(x), \end{cases}$$

where $\theta > 1$, $\vartheta > 1$ with $\theta + \vartheta = m^*$, φ is a non–negative perturbation of class $L^{\mathfrak{m}}(\mathbb{R}^n)$, with $\mathfrak{m}$ the Hölder conjugate of m^*, that is $\mathfrak{m} = m'n/(n+m's)$, while ψ is a non–negative perturbation of class $L^{\mathfrak{p}}(\mathbb{R}^n)$, with $\mathfrak{p}$ the Hölder conjugate of p^*, i.e. $\mathfrak{p} = p'n/(n+p's)$.

Theorem 5.1.3. *Under the assumptions (H_1), (H_2), with a and b satisfying (V_1)–(V_2), there exists a number $\delta > 0$ such that for all non–negative perturbations φ and ψ, with $0 < \|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} < \delta$, system $(\mathscr{P}_5)$ has at least one nontrivial non–negative entire solution $(u,v) \in W$ for any $\mu \in (-\infty, \mathcal{H}_m)$ and for any $\sigma \in (-\infty, \mathcal{H}_p)$ satisfying (5.1.5), provided that either $m < p$ and $\mu \leq 0$, or $m = p$.*

We end with the radial version of the previous result.

Theorem 5.1.4. *Assume that $n \geq 2$, that a, b, $H_z(\cdot, z)$ are radial for all $z \in \mathbb{R}^2$. Suppose that a and b satisfy (V_1), and that (H_1) and $(H_2)'$ hold. Then there exists a number $\delta > 0$ such that for all non–negative perturbations φ and ψ, with $0 < \|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} < \delta$, system $(\mathscr{P}_5)$ has at least one nontrivial non–negative entire radial solution $(u, v) \in W$ for any $\mu \in (-\infty, \mathcal{H}_m)$ and for any $\sigma \in (-\infty, \mathcal{H}_p)$ satisfying (5.1.6), provided that either $m < p$ and $\mu \leq 0$, or $m = p$.*

5.2 Preliminaries

We start the section commenting the structural assumptions used in Theorem 5.1.1. Condition (V_2), which is weaker than the coercivity assumption, $V(x) \to \infty$ as $|x| \to \infty$, was originally discussed by *Bartsch* and *Wang* in [19] to overcome the lack of compactness. Model functions H satisfying (H_1)–(H_4) and $(H_2)'$ are briefly discussed in the Introduction.

We say that the couple $(u, v) \in W$ is an *entire (weak) solution* of $(\mathscr{P}_4)$ if

$$\begin{aligned} \langle u, \Phi\rangle_{E_{m,a}} + \langle v, \Psi\rangle_{E_{p,b}} - \mu\langle u, \Phi\rangle_{H_m} - \sigma\langle v, \Psi\rangle_{H_p} \\ = \int_{\mathbb{R}^n} [H_u(x, u, v)\Phi(x) + H_v(x, u, v)\Psi(x)]dx \end{aligned} \tag{5.2.1}$$

for any $(\Phi, \Psi) \in W$, where

$$\begin{aligned} \langle u, \Phi\rangle_{s,\wp} &= \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^{\wp-2}(u(x) - u(y))(\Phi(x) - \Phi(y))}{|x - y|^{n+\wp s}} dxdy, \\ \langle u, \Phi\rangle_{E_{m,a}} &= \langle u, \Phi\rangle_{s,m} + \int_{\mathbb{R}^n} a(x)|u(x)|^{m-2}u(x)\Phi(x)dx, \\ \langle v, \Psi\rangle_{E_{p,b}} &= \langle v, \Psi\rangle_{s,p} + \int_{\mathbb{R}^n} b(x)|v(x)|^{p-2}v(x)\Psi(x)dx, \\ \langle u, \Phi\rangle_{H_m} &= \int_{\mathbb{R}^n} |u(x)|^{m-2}u(x)\Phi(x)\frac{dx}{|x|^{ms}}, \\ \langle v, \Psi\rangle_{H_p} &= \int_{\mathbb{R}^n} |v(x)|^{p-2}v(x)\Psi(x)\frac{dx}{|x|^{ps}}. \end{aligned}$$

Clearly, the entire (weak) solutions of $(\mathscr{P}_4)$ are exactly the critical points of the Euler–Lagrange functional associated with $(\mathscr{P}_4)$, that is of

$$I_{\mu,\sigma}(u, v) = \frac{1}{m}\|u\|^m_{E_{m,a}} + \frac{1}{p}\|v\|^p_{E_{p,b}} - \frac{\mu}{m}\|u\|^m_{H_m} - \frac{\sigma}{p}\|v\|^p_{H_p} - \int_{\mathbb{R}^n} H(x, u, v)dx.$$

The functional $I_{\mu,\sigma}$ is well defined in W and under conditions (H_1)–(H_2) the functional $I_{\mu,\sigma}$ is of class $C^1(W)$, and for any fixed $(u,v) \in W$

$$\begin{aligned}\langle I'_{\mu,\sigma}(u,v),(\Phi,\Psi)\rangle =&\langle u,\Phi\rangle_{E_{m,a}} + \langle v,\Psi\rangle_{E_{p,b}} - \mu\langle u,\Phi\rangle_{H_m} - \sigma\langle v,\Psi\rangle_{H_p} \\ &- \int_{\mathbb{R}^n} [H_u(x,u,v)\Phi + H_v(x,u,v)\Psi]dx\end{aligned} \tag{5.2.2}$$

for all $(\Phi,\Psi) \in W$.

Let us recall that a functional $J : X \to \mathbb{R}$ of class $C^1(X)$, on a real Banach space $X = (X, \|\cdot\|)$, with its dual space X', is said to satisfy the *Cerami condition* (C) if any *Cerami sequence* associated with J has a strongly convergent subsequence in X. A sequence $(u_k)_k$ in X is called a *Cerami sequence*, if $(J(u_k))_k$ is bounded and $(1 + \|u_k\|) \cdot \|J'(u_k)\|_{X'} \to 0$ as $k \to \infty$.

The celebrated mountain pass theorem of *Ambrosetti* and *Rabinowitz* can be stated also in terms of the existence of Cerami's sequences as a direct consequence of Corollaries 4 and 9 of [42]. We give it in the stronger form as presented in Theorem I of [37] and refer to [37, 42] for further comments. Indeed, this is exactly the version we shall use in order to prove Theorem 5.1.1.

Theorem 5.2.1. *Let X be a real Banach space and let $J \in C^1(X)$ satisfy*

$$\max\{J(0), J(e)\} \le \alpha < \beta \le \inf_{\|u\|=\rho} J(u),$$

for some $\alpha < \beta$, $\rho > 0$ and $e \in X$, with $\|e\| > \rho$. Let $c \ge \beta$ be characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a Cerami sequence $(u_k)_k$ in X such that $J(u_k) \to c \ge \beta$ as $k \to \infty$.

Finally, if the functional J satisfies the Cerami condition (C) at mini–max level c, then c is a critical value of J in X.

We end the section by recalling that $0 < s < 1$, $n > ps$, $1 < m \le p < m^*$, so that some basic results on the fractional Sobolev space W can be derived and used in the next sections.

Lemma 10 of [86] shows that the spaces $E_{m,a} = (E_{m,a}, \|\cdot\|_{E_{m,a}})$ and $E_{p,b} = (E_{p,b}, \|\cdot\|_{E_{p,b}})$ are two separable, reflexive Banach spaces. Hence, $W = (W, \|\cdot\|)$ is a separable and reflexive Banach space by Theorem 1.12 of [2]. Furthermore, combining the results of Lemmas 4.1 and 5.1 of [35] and Theorem 2.1 of [86], we have

Lemma 5.2.2. *Let* (V_1) *hold. Then the embeddings*

$$W \hookrightarrow W^{s,m}(\mathbb{R}^n) \times W^{s,p}(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$$

are continuous if $\nu \in [p, m^*]$, *and*

$$\|(u,v)\|_\nu \le \|u\|_\nu + \|v\|_\nu \le C_\nu \|(u,v)\| \quad \textit{for all } (u,v) \in W, \tag{5.2.3}$$

where C_ν *depends on* ν, n, s, m *and* p. *If in addition also* (V_2) *holds, then the embedding*

$$W \hookrightarrow\hookrightarrow L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$$

is compact when $\nu \in [p, m^*)$.

Finally, if $n \ge 2$, *then the embedding* $W_{\text{rad}} \hookrightarrow\hookrightarrow L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$ *is compact for all* $\nu \in (p, m^*)$, *where*

$$W_{\text{rad}} = \{(u,v) \in W \,:\, u \textit{ and } v \textit{ are radially symmetric with respect to } 0\}.$$

The number C_ν in (5.2.3) is related to λ_ν defined in (5.1.3) by $C_\nu = \lambda_\nu^{-1/\nu}$. According to Proposition A.10 of [12], we have

Lemma 5.2.3. *Let* $\{(u_k, v_k)\}_k \subset W$ *be such that* $(u_k, v_k) \rightharpoonup (u, v)$ *weakly in* W *as* $k \to \infty$. *Then, up to a subsequence,* $(u_k, v_k) \to (u, v)$ *a.e. in* $\mathbb{R}^n$ *as* $k \to \infty$.

5.3 Existence of solutions of $(\mathscr{P}_4)$

To prove Theorem 5.1.1, we shall apply Theorem 5.2.1 to the functional $I_{\mu,\sigma}$ introduced in Section 5.2. In what follows C might denote different constants.

Lemma 5.3.1. *Any Cerami sequence of* $I_{\mu,\sigma}$ *is bounded in* W, *provided that* $\mu < \mathcal{H}_m$ *and* $\sigma < \mathcal{H}_p$.

Proof. Let $\{(u_k, v_k)\}_k$ be a Cerami sequence of $I_{\mu,\sigma}$ in W. Then there exists $C > 0$ independent of k such that

$$\begin{gathered} |I_{\mu,\sigma}(u_k, v_k)| \le C \quad \text{for all } k \quad \text{and} \\ (1 + \|(u_k, v_k)\|) I'_{\mu,\sigma}(u_k, v_k) \to 0 \text{ as } k \to \infty. \end{gathered} \tag{5.3.1}$$

Hence there exists $\varepsilon_k > 0$, with $\varepsilon_k \to 0$, such that

$$|\langle I'_{\mu,\sigma}(u_k, v_k), (\Phi, \Psi)\rangle| \le \frac{\varepsilon_k \|(\Phi, \Psi)\|}{1 + \|(u_k, v_k)\|} \tag{5.3.2}$$

for all $(\Phi, \Psi) \in W$ and all $k \in \mathbb{N}$. Choosing $(\Phi, \Psi) = (u_k, v_k)$ in (5.3.2), we deduce

$$\begin{aligned} \Big| \|u_k\|^m_{E_{m,a}} + \|v_k\|^p_{E_{p,b}} - \mu \|u_k\|^m_{H_m} - \sigma \|v_k\|^p_{H_p} \\ - \int_{\mathbb{R}^n} [H_u(x, u_k, v_k)u_k + H_v(x, u_k, v_k)v_k]\, dx \Big| \\ = |\langle I'_{\mu,\sigma}(u_k, v_k), (u_k, v_k)\rangle| \le \frac{\varepsilon_k \|(u_k, v_k)\|}{1 + \|(u_k, v_k)\|} \le \varepsilon_k \le C. \end{aligned}$$

Hence we have

$$\begin{aligned} -\|u_k\|^m_{E_{m,a}} - \|v_k\|^p_{E_{p,b}} + \mu \|u_k\|^m_{H_m} + \sigma \|v_k\|^p_{H_p} \\ + \int_{\mathbb{R}^n} [H_u(x, u_k, v_k)u_k + H_v(x, u_k, v_k)v_k]\, dx \le C \end{aligned} \tag{5.3.3}$$

We claim that $\{(u_k, v_k)\}_k$ is bounded in W.

Arguing by contradiction, we assume that $\|(u_k, v_k)\| \to \infty$ as $k \to \infty$ and, without loss of generality, that $\|(u_k, v_k)\| \ge 1$ for all k. Set $(X_k, Y_k) = (u_k, v_k)/\|(u_k, v_k)\|$. Of course, $\|(X_k, Y_k)\| = 1$. Then there exists $(X, Y) \in W$ such that, up to a subsequence,

$$\begin{gathered} (X_k, Y_k) \rightharpoonup (X, Y) \text{ in } W, \quad (X_k, Y_k) \to (X, Y) \text{ a.e. in } \mathbb{R}^n, \\ (X_k, Y_k) \to (X, Y) \text{ in } L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n) \end{gathered}$$

for any $\nu \in [p, m^*)$ by Lemmas 5.2.2 and 5.2.3.

Let $X_k^- = \min\{0, X_k\}$ and $Y_k^- = \min\{0, Y_k\}$. Clearly, $\{(X_k^-, Y_k^-)\}_k$ is also bounded in W. Take $(\Phi, \Psi) = (X_k^-, Y_k^-)$ in (5.3.2). Since $\|(u_k, v_k)\| \to \infty$, by (H_1) as $k \to \infty$

$$\begin{aligned} o(1) &= \frac{\langle I'_{\mu,\sigma}(u_k, v_k), (X_k^-, Y_k^-)\rangle}{\|(u_k, v_k)\|^{m-1}} \\ &= \frac{1}{\|(u_k, v_k)\|^m} \Big[\langle u_k, u_k^-\rangle_{s,m} + \langle u_k, u_k^-\rangle_{m,a} + \langle v_k, v_k^-\rangle_{s,p} \\ &\qquad + \langle v_k, v_k^-\rangle_{p,b} - \mu \langle u_k, u_k^-\rangle_{H_m} - \sigma \langle v_k, v_k^-\rangle_{H_p} \Big] \end{aligned}$$

$$
\begin{aligned}
&\quad - \int_{\mathbb{R}^n} \frac{H_u(x,u_k,v_k)u_k^- + H_v(x,u_k,v_k)v_k^-}{\|(u_k,v_k)\|^m} dx \\
&= \frac{1}{\|(u_k,v_k)\|^m}\Big[\langle u_k,u_k^-\rangle_{s,m} + \|u_k^-\|_{m,a}^m \\
&\qquad + \langle v_k,v_k^-\rangle_{s,p} + \|v_k^-\|_{p,b}^p - \mu\|u_k^-\|_{H_m}^m - \sigma\|v_k^-\|_{H_p}^p\Big] \\
&\geq \frac{1}{\|(u_k,v_k)\|^m}\left(\|u_k^-\|_{E_{m,a}}^m + \|v_k^-\|_{E_{p,b}}^p - \mu\|u_k^-\|_{H_m}^m - \sigma\|v_k^-\|_{H_p}^p\right) \\
&\geq \left(1-\frac{\mu^+}{\mathcal{H}_m}\right)\|X_k^-\|_{E_{m,a}}^m + \left(1-\frac{\sigma^+}{\mathcal{H}_p}\right)\frac{\|v_k^-\|_{E_{p,b}}^p}{\|(u_k,v_k)\|^m} \\
&\geq \left(1-\frac{\mu^+}{\mathcal{H}_m}\right)\|X_k^-\|_{E_{m,a}}^m,
\end{aligned}
\tag{5.3.4}
$$

where the first inequality follows from the following elementary inequality valid for all $\wp > 1$

$$
|\xi^- - \eta^-|^\wp \leq |\xi-\eta|^{\wp-2}(\xi-\eta)(\xi^- - \eta^-) \quad \text{for } \xi,\eta\in\mathbb{R}, \tag{5.3.5}
$$

while the second inequality follows from (5.1.1) and the fact that $\mu < \mathcal{H}_m$ and $\sigma < \mathcal{H}_p$. Hence, it follows at once that $\|X_k^-\|_{E_{m,a}} \to 0$ as $k\to\infty$. Similarly, as $k\to\infty$

$$
\begin{aligned}
o(1) &= \frac{\langle I'_{\mu,\sigma}(u_k,v_k),(X_k^-,Y_k^-)\rangle}{\|(u_k,v_k)\|^{p-1}} \\
&\geq \frac{1}{\|(u_k,v_k)\|^p}\left(\|u_k^-\|_{E_{m,a}}^m + \|v_k^-\|_{E_{p,b}}^p - \mu\|u_k^-\|_{H_m}^m - \sigma\|v_k^-\|_{H_p}^p\right) \\
&\geq \left(1-\frac{\mu^+}{\mathcal{H}_m}\right)\frac{\|u_k^-\|_{E_{m,a}}^m}{\|(u_k,v_k)\|^p} + \left(1-\frac{\sigma^+}{\mathcal{H}_p}\right)\|Y_k^-\|_{E_{p,b}}^p \\
&\geq \left(1-\frac{\sigma^+}{\mathcal{H}_p}\right)\|Y_k^-\|_{E_{p,b}}^p.
\end{aligned}
$$

Thus $\|Y_k^-\|_{E_{p,b}} \to 0$ as $k\to\infty$. Therefore, we get

$$
(X_k^-,Y_k^-) \to (0,0) \quad \text{in } W \text{ as } k\to\infty.
$$

This implies that $(X^-,Y^-) = (0,0)$ a.e. in $\mathbb{R}^n$. Hence, $X\geq 0$ and $Y\geq 0$ a.e. in $\mathbb{R}^n$.

Set $\Omega^+ = \{x\in\mathbb{R}^n : \text{either } X(x)>0 \text{ or } Y(x)>0\}$ and

$$
\Omega^0 = \{x\in\mathbb{R}^n : (X(x),Y(x)) = (0,0)\}.
$$

Assume Ω^+ has a positive Lebesgue measure. Since $\|(u_k, v_k)\| \to \infty$ as $k \to \infty$, then

$$|(u_k, v_k)| = \big\|(u_k, v_k)\big\| \cdot |(X_k, Y_k)| \to \infty \quad \text{a.e. in } \Omega^+.$$

Consequently, by (H_3)

$$\lim_{k\to\infty} \frac{H(x, u_k, v_k)}{\|(u_k, v_k)\|^p} = \lim_{k\to\infty} \frac{H(x, u_k, v_k)}{|(u_k, v_k)|^p} \cdot |(X_k, Y_k)|^p = \infty,$$

a.e. in Ω^+. Then the Fatou lemma gives at once that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{H(x, u_k, v_k)}{\|(u_k, v_k)\|^p} dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{H(x, u_k, v_k)|(X_k, Y_k)|^p}{|(u_k, v_k)|^p} dx = \infty. \tag{5.3.6}$$

Now, (5.3.1) yields that

$$\begin{aligned} \int_{\mathbb{R}^n} H(x, u_k, v_k)dx \leq \frac{1}{m}\|u\|^m_{E_{m,a}} + \frac{1}{p}\|v\|^p_{E_{p,b}} - \frac{\mu}{m}\|u\|^m_{H_m} \\ - \frac{\sigma}{p}\|v\|^p_{H_p} + C \end{aligned} \tag{5.3.7}$$

for all $k \in \mathbb{N}$. Dividing by $\|(u_k, v_k)\|^p \geq 1$ for all k, we get forthwith

$$\limsup_{k\to\infty} \int_{\mathbb{R}^n} \frac{H(x, u_k, v_k)}{\|(u_k, v_k)\|^p} dx \leq \frac{2}{m} + \frac{|\mu^-|}{m\mathcal{H}_m} + \frac{|\sigma^-|}{p\mathcal{H}_p} + \frac{C}{\|(u_k, v_k)\|^p},$$

by virtue of (5.1.1), where as before $t^- = \min\{0, t\}$ for any $t \in \mathbb{R}$. This contradicts (5.3.6). In conclusion, Ω^+ has zero measure, that is, $(X, Y) = (0, 0)$ a.e. in $\mathbb{R}^n$.

Let t_k be the smallest value of $t \in [0, 1]$ such that

$$I_{\mu,\sigma}(t_k u_k, t_k v_k) = \max_{0\leq t\leq 1} I_{\mu,\sigma}(tu_k, tv_k).$$

Take $\alpha > 1/2$ and set

$$(U_k, V_k) = (2\alpha)^{1/m}(X_k, Y_k) = (2\alpha)^{1/m} \frac{(u_k, v_k)}{\|(u_k, v_k)\|} \in W.$$

Lemma 5.2.2 implies that $(U_k, V_k) \to (0, 0)$ in $L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$ for any $\nu \in [p, m^*)$. Hence, by (H_1) and (H_2), with $\varepsilon = 1$, we have

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^n} H(x, U_k, V_k)dx &\leq \int_{\mathbb{R}^n} [(\lambda + 1)|(U_k, V_k)|^p + C_1|(U_k, V_k)|^r] \, dx \\ &\leq (\lambda + 1)\|(U_k, V_k)\|^p_p + C_1\|(U_k, V_k)\|^r_r \to 0 \end{aligned} \tag{5.3.8}$$

as $k \to \infty$, since $1 < m \le p < r < m^*$. Thus, we get

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} H(x, U_k, V_k)dx = 0. \tag{5.3.9}$$

Since $\|(u_k, v_k)\| \to \infty$ and $\|(u_k, v_k)\| \ge 1$ for all k, we take k_0 so large that $(2\alpha)^{1/m}/\|(u_k, v_k)\| \in (0, 1)$ for all $k \ge k_0$. Thanks to the facts that $1 < m \le p$, $\alpha > 1/2$ and $\|X_k\|_{E_{m,a}} \le \|X_k\|_{E_{m,a}} + \|Y_k\|_{E_{p,b}} = 1$, we obtain for all $k \ge k_0$

$$\begin{aligned}
I_{\mu,\sigma}(t_k u_k, t_k v_k) &\ge I_{\mu,\sigma}\left((2\alpha)^{1/m} u_k/\|(u_k, v_k)\|,\ (2\alpha)^{1/m} v_k/\|(u_k, v_k)\|\right) \\
&= \frac{2\alpha}{m}\|X_k\|^m_{E_{m,a}} + \frac{(2\alpha)^{p/m}}{p}\|Y_k\|^p_{E_{p,b}} \\
&\quad - \mu\frac{2\alpha}{m}\|X_k\|^m_{H_m} - \sigma\frac{(2\alpha)^{p/m}}{p}\|Y_k\|^p_{H_p} - \int_{\mathbb{R}^n} H(x, U_k, V_k)dx \\
&\ge \frac{2\alpha}{m}\left(1 - \frac{\mu^+}{\mathcal{H}_m}\right)\|X_k\|^m_{E_{m,a}} + \frac{(2\alpha)^{p/m}}{p}\left(1 - \frac{\sigma^+}{\mathcal{H}_p}\right)\|Y_k\|^p_{E_{p,b}} \\
&\quad - \int_{\mathbb{R}^n} H(x, U_k, V_k)dx \\
&\ge 2\frac{\alpha\kappa}{p}\left(\|X_k\|^p_{E_{m,a}} + \|Y_k\|^p_{E_{p,b}}\right) - \int_{\mathbb{R}^n} H(x, U_k, V_k)dx \\
&\ge 2\frac{\alpha\kappa}{p2^{p-1}} - \int_{\mathbb{R}^n} H(x, U_k, V_k)dx,
\end{aligned}$$

where $\kappa = \min\{1 - \mu^+/\mathcal{H}_m,\ 1 - \sigma^+/\mathcal{H}_p\} > 0$.

By (5.3.9), there exists $k_1 \ge k_0$ such that

$$\int_{\mathbb{R}^n} H(x, U_k, V_k)dx \le \frac{\alpha\kappa}{p2^{p-1}} \quad \text{for all } k \ge k_1.$$

Therefore

$$I_{\mu,\sigma}(t_k u_k, t_k v_k) \ge \frac{\alpha\kappa}{p2^{p-1}} \quad \text{for all } k \ge k_1.$$

This, together with the arbitrariness of $\alpha > 1/2$, yields

$$\lim_{k\to\infty} I_{\mu,\sigma}(t_k u_k, t_k v_k) = \infty. \tag{5.3.10}$$

Since $0 \le t_k \le 1$, then (H_4) gives at once that

$$\int_{\mathbb{R}^n} F(x, t_k u_k, t_k v_k)dx \le C_F \int_{\mathbb{R}^n} F(x, u_k, v_k)dx + \int_{\mathbb{R}^n} g(x)dx. \tag{5.3.11}$$

Following [16, Lemma 7.3] and using the fact that

$$I_{\mu,\sigma}(0,0)=0 \quad \text{and} \quad I_{\mu,\sigma}(u_k,v_k)\to c\in\mathbb{R},$$

by (5.3.10) we can assume that $t_k\in(0,1)$ for all k sufficiently large and in turn

$$\begin{aligned} 0 &= t_k\frac{d}{dt}I_{\mu,\sigma}(tu_k,tv_k)\Big|_{t=t_k} = \langle I'_{\mu,\sigma}(t_ku_k,t_kv_k),(t_ku_k,t_kv_k)\rangle \\ &= \|t_ku_k\|^m_{E_{m,a}}-\mu\|t_ku_k\|^m_{H_m}+\|t_kv_k\|^p_{E_{p,b}}-\sigma\|t_kv_k\|^p_{H_p} \\ &\quad -\int_{\mathbb{R}^n}[H_u(x,t_ku_k,t_kv_k)t_ku_k+H_v(x,t_ku_k,t_kv_k)t_kv_k]\,dx. \end{aligned} \tag{5.3.12}$$

Combining (5.3.11) and (5.3.12), we get

$$\begin{aligned} &\|t_ku_k\|^m_{E_{m,a}}-\mu\|t_ku_k\|^m_{H_m}+\|t_kv_k\|^p_{E_{p,b}}-\sigma\|t_kv_k\|^p_{H_p} \\ &\qquad = p\int_{\mathbb{R}^n}H(x,t_ku_k,t_kv_k)dx+\int_{\mathbb{R}^n}F(x,t_ku_k,t_kv_k)dx \\ &\qquad \le p\int_{\mathbb{R}^n}H(x,t_ku_k,t_kv_k)dx+C_F\int_{\mathbb{R}^n}F(x,u_k,v_k)dx \\ &\qquad\quad +\int_{\mathbb{R}^n}g(x)dx \end{aligned}$$

for k sufficiently large. From this, it follows that

$$\begin{aligned} pI_{\mu,\sigma}(t_ku_k,t_kv_k) &= \frac{p}{m}\left(\|t_ku_k\|^m_{E_{m,a}}-\mu\|t_ku_k\|^m_{H_m}\right)+\|t_kv_k\|^p_{E_{p,b}}-\sigma\|t_kv_k\|^p_{H_p} \\ &\quad -p\int_{\mathbb{R}^n}H(x,t_ku_k,t_kv_k)dx \\ &= \left(\frac{p}{m}-1\right)\left(\|t_ku_k\|^m_{E_{m,a}}-\mu\|t_ku_k\|^m_{H_m}\right)+\|t_ku_k\|^m_{E_{m,a}} \\ &\quad -\mu\|t_ku_k\|^m_{H_m}+\|t_kv_k\|^p_{E_{p,b}}-\sigma\|t_kv_k\|^p_{H_p} \\ &\quad -p\int_{\mathbb{R}^n}H(x,t_ku_k,t_kv_k)dx \\ &\le \left(\frac{p}{m}-1\right)\left(\|t_ku_k\|^m_{E_{m,a}}-\mu\|t_ku_k\|^m_{H_m}\right) \\ &\quad +C_F\int_{\mathbb{R}^n}F(x,u_k,v_k)dx+\int_{\mathbb{R}^n}g(x)dx \\ &\le \left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}}-\mu\|u_k\|^m_{H_m}\right)+C_F\int_{\mathbb{R}^n}F(x,u_k,v_k)dx \\ &\quad +\int_{\mathbb{R}^n}g(x)dx, \end{aligned}$$

since $t_k \in (0,1)$ and $\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m} \geq 0$ by (5.1.1) and the fact that $\mu < \mathcal{H}_m$. Thus, (5.3.10) gives in particular that

$$\frac{1}{C_F}\left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m}\right) + \int_{\mathbb{R}^n} F(x,u_k,v_k)dx \to \infty \tag{5.3.13}$$

as $k \to \infty$. On the other hand, (5.3.1) and the definition of F in (H_4) imply that

$$\begin{aligned}
\widetilde{C} \geq pI_{\mu,\sigma}(u_k,v_k) &= \frac{p}{m}\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m}\right) + \|v_k\|^p_{E_{p,b}} - \sigma\|v_k\|^p_{H_p} \\
&\qquad - p\int_{\mathbb{R}^n} H(x,u_k,v_k)dx \\
&= \left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m}\right) + \|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m} + \|v_k\|^p_{E_{p,b}} \\
&\qquad - \sigma\|v_k\|^p_{H_p} - p\int_{\mathbb{R}^n} H(x,u_k,v_k)dx \\
&= \left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m}\right) + \|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m} + \|v_k\|^p_{E_{p,b}} \\
&\qquad - \sigma\|v_k\|^p_{H_p} - \int_{\mathbb{R}^n} \left[H_u(x,u_k,v_k)u_k + H_v(x,u_k,v_k)v_k\right]dx \\
&\qquad + \int_{\mathbb{R}^n} F(x,u_k,v_k)dx \\
&\geq -C + \left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m}\right) + \int_{\mathbb{R}^n} F(x,u_k,v_k)dx
\end{aligned}$$

by (5.3.3). Thus, in particular

$$\begin{aligned}
&\frac{1}{C_F}\left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m}\right) + \int_{\mathbb{R}^n} F(x,u_k,v_k)dx \\
&\qquad \leq \left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m}\right) + \int_{\mathbb{R}^n} F(x,u_k,v_k)dx \leq \text{Const.},
\end{aligned}$$

being $C_F \geq 1$ by (H_4), $1 < m \leq p$, and $\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{H_m} \geq 0$ in virtue of (5.1.1) and the fact that $\mu < \mathcal{H}_m$. This contradicts (5.3.13) and proves the claim.

Therefore, we conclude that $\{(u_k,v_k)\}_k$ is bounded in W. □

Lemma 5.3.2. *The functional $I_{\mu,\sigma}$ satisfies the Cerami condition (C) in W for all $\mu < \mathcal{H}_m$ and for all $\sigma < \mathcal{H}_p$.*

Proof. Let $\{(u_k,v_k)\}_k$ be a Cerami sequence for $I_{\mu,\sigma}$ in W. Then there exists $C > 0$ independent of k such that (5.3.1) holds. Lemma 5.3.1 asserts that

$\{(u_k, v_k)\}_k$ is bounded in W. Hence, up to a subsequence, still denoted by $\{(u_k, v_k)\}_k$, there exists $(u, v) \in W$ such that

$$\begin{gathered}(u_k, v_k) \rightharpoonup (u, v) \text{ in } W,\\ u_k \rightharpoonup u \text{ in } L^m(\mathbb{R}^n, |x|^{-ms}),\ v_k \rightharpoonup v \text{ in } L^p(\mathbb{R}^n, |x|^{-ps}),\\ (u_k, v_k) \to (u, v) \text{ in } L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n),\\ (u_k, v_k) \to (u, v) \text{ a.e. in } \mathbb{R}^{2n},\\ \|u_k - u\|_{E_{m,a}} \to \mathfrak{i},\ \|u_k - u\|_{H_m} \to \mathfrak{j},\\ \|v_k - v\|_{E_{p,b}} \to \mathfrak{k},\ \|v_k - v\|_{H_p} \to \mathfrak{l},\end{gathered} \tag{5.3.14}$$

for any $\nu \in [p, m^*)$ by Lemmas 5.2.2, 5.2.3 and (5.1.1). Clearly, (5.3.14) implies that $|z_k - z| \to 0$ in $L^\nu(\mathbb{R}^n)$ for all $\nu \in [p, m^*)$, where $z_k = (u_k, v_k)$ and $z = (u, v)$.

In particular, the sequence $(\mathcal{U}_k)_k$, defined in $\mathbb{R}^{2n} \setminus \operatorname{Diag}(\mathbb{R}^{2n})$ by

$$(x, y) \mapsto \mathcal{U}_k(x, y) = \frac{|u_k(x) - u_k(y)|^{m-2}[u_k(x) - u_k(y)]}{|x - y|^{(n+ms)/m'}},$$

is bounded in $L^{m'}(\mathbb{R}^{2n})$ as well as $\mathcal{U}_k \to \mathcal{U}$ a.e. in $\mathbb{R}^{2n}$, where

$$\mathcal{U}(x, y) = \frac{|u(x) - u(y)|^{m-2}[u(x) - u(y)]}{|x - y|^{(n+ms)/m'}}.$$

Thus, going if necessary to a further subsequence, we get that $\mathcal{U}_k \rightharpoonup \mathcal{U}$ in $L^{m'}(\mathbb{R}^{2n})$ as $k \to \infty$. Furthermore, $|u_k|^{m-2}u_k \rightharpoonup |u|^{m-2}u$ in $L^{m'}(\mathbb{R}^n, a)$ by Proposition A.8 of [12]. Hence,

$$\langle u_k, \Phi\rangle_{E_{m,a}} \to \langle u, \Phi\rangle_{E_{m,a}} \tag{5.3.15}$$

for any $\Phi \in E_{m,a}$, since $(x, y) \mapsto |\Phi(x) - \Phi(y)| \cdot |x - y|^{-(n+ms)/m} \in L^m(\mathbb{R}^{2n})$ and $\Phi \in L^m(\mathbb{R}^n, a)$. In the same way, (5.3.14) and Proposition A.8 of [12] imply that $|u_k|^{m-2}u_k \rightharpoonup |u|^{m-2}u$ in $L^{m'}(\mathbb{R}^n, |x|^{-ms})$ as $k \to \infty$. Consequently,

$$\langle u_k, \Phi\rangle_{H_m} \to \langle u, \Phi\rangle_{H_m} \tag{5.3.16}$$

for any $\Phi \in E_{m,a}$.

A similar argument shows that the sequence $(\mathcal{V}_k)_k$, defined in $\mathbb{R}^{2n} \setminus \operatorname{Diag}(\mathbb{R}^{2n})$ by

$$(x, y) \mapsto \mathcal{V}_k(x, y) = \frac{|v_k(x) - v_k(y)|^{p-2}[v_k(x) - v_k(y)]}{|x - y|^{(n+ps)/p'}},$$

is bounded in $L^{p'}(\mathbb{R}^{2n})$ as well as $\mathcal{V}_k \to \mathcal{V}$ a.e. in $\mathbb{R}^{2n}$, where

$$\mathcal{V}(x,y) = \frac{|v(x)-v(y)|^{p-2}[v(x)-v(y)]}{|x-y|^{(n+ps)/p'}}.$$

Hence, going if necessary to a further subsequence, we have

$$\langle v_k, \Psi\rangle_{E_{p,b}} \to \langle v, \Psi\rangle_{E_{p,b}}, \qquad \langle v_k, \Psi\rangle_{H_p} \to \langle v, \Psi\rangle_{H_p} \tag{5.3.17}$$

for all $\Psi \in E_{p,b}$.

By (H_2), with $\varepsilon = 1$, and (5.3.14), the Hölder inequality gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big| \big(H_u(x,u_k,v_k) - H_u(x,u,v)\big)(u_k - u) \\
&\qquad\qquad + \big(H_v(x,u_k,v_k) - H_v(x,u,v)\big)(v_k - v)\Big| dx \\
&= \int_{\mathbb{R}^n} |H_z(x,z_k)(z_k - z) - H_z(x,z)(z_k - z)| dx \\
&\le \int_{\mathbb{R}^n} \big[(\lambda+1)(|z_k|^{p-1} + |z|^{p-1})|z_k - z| \\
&\qquad\qquad + C_1(|z_k|^{r-1} + |z|^{r-1})|z_k - z|\big] dx \\
&\le C_\lambda\big(\|z_k - z\|_p + \|z_k - z\|_r\big) \to 0,
\end{aligned} \tag{5.3.18}$$

as $k \to \infty$, for a suitable $C_\lambda > 0$.

Since $\{(u_k, v_k)\}_k$ is bounded in W, by (5.3.1) and (5.3.14)–(5.3.18) we have as $k \to \infty$

$$\begin{aligned}
o(1) &= \langle I'_{\mu,\sigma}(z_k) - I'_{\mu,\sigma}(z), z_k - z\rangle \\
&= \|u_k - u\|^m_{E_{m,a}} - \mu\,\|u_k - u\|^m_{H_m} + \|v_k - v\|^p_{E_{p,b}} - \sigma\,\|v_k - v\|^p_{H_p} + o(1),
\end{aligned}$$

which yields by (5.3.14) the main formula

$$\begin{aligned}
\mathfrak{i}^m + \mathfrak{k}^p &= \lim_{k\to\infty} \|u_k - u\|^m_{E_{m,a}} + \lim_{k\to\infty} \|v_k - v\|^p_{E_{p,b}} \\
&= \mu \lim_{k\to\infty} \|u_k - u\|^m_{H_m} + \sigma \lim_{k\to\infty} \|v_k - v\|^p_{H_p} = \mu\mathfrak{j}^m + \sigma\mathfrak{l}^p.
\end{aligned} \tag{5.3.19}$$

Clearly (5.3.19) gives at once that $(u_k, v_k) \to (u, v)$ in W as $k \to \infty$ when either $\mu^+ + \sigma^+ = 0$ or $\mathfrak{j} + \mathfrak{l} = 0$ and we are done. Let us therefore assume by contradiction that $\mu^+ + \sigma^+ > 0$ and $\mathfrak{j} + \mathfrak{l} > 0$. If either $\mu^+ + \mathfrak{l} = 0$ or $\sigma^+ + \mathfrak{j} = 0$, then either $\mathfrak{j} > 0$ and $\mathfrak{i} = 0$ or $\mathfrak{l} > 0$ and $\mathfrak{k} = 0$ by (5.3.19). Both cases are impossible by (5.1.1). Now, if either $\mu^+ + \mathfrak{j} = 0$ or $\sigma^+ + \mathfrak{l} = 0$, then either $\mathfrak{l} > 0$, $\sigma^+ > 0$ and $\mathfrak{k}^p \le \sigma^+\mathfrak{l}^p < \mathcal{H}_p\mathfrak{l}^p \le \mathfrak{k}^p$ or $\mathfrak{j} > 0$, $\mu^+ > 0$

and $\mathfrak{i}^m \le \mu^+\mathfrak{j}^m < \mathcal{H}_m\mathfrak{j}^m \le \mathfrak{i}^m$ by (5.3.19) and (5.1.1). Both cases give a contradiction. Finally, it remains to consider the case $\mu^+ > 0$, $\sigma^+ > 0$, $\mathfrak{j} > 0$ and $\mathfrak{l} > 0$, for which (5.3.19) and (5.1.1) yield that

$$\mathfrak{i}^m + \mathfrak{k}^p = \mu\mathfrak{j}^m + \sigma\mathfrak{l}^p \le \mu^+\mathfrak{j}^m + \sigma^+\mathfrak{l}^p \le \mathcal{H}_m\mathfrak{j}^m + \mathcal{H}_p\mathfrak{l}^p < \mathfrak{i}^m + \mathfrak{k}^p,$$

which is again the desired contradiction. In conclusion, $\mathfrak{j} + \mathfrak{l} = 0$ in all cases and so $(u_k, v_k) \to (u, v)$ in W as $k \to \infty$ by (5.3.19), as stated. □

Now we are in position to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. We first show that the functional $I_{\mu,\sigma}$ satisfies a mountain pass geometry. Take $\varepsilon > 0$, with $2^p\varepsilon/\lambda_p = \kappa - 2^{p-1}\lambda/\lambda_p$, where $\kappa = \min\{1 - \mu^+/\mathcal{H}_m,\, 1 - \sigma^+/\mathcal{H}_p\} > 0$. This is possible thanks to the main restriction (5.1.5). By (5.1.3), (5.2.3) and (H_2), we have for all $(u, v) \in W$, with $\|(u, v)\| \le 1$,

$$\begin{aligned}
I_{\mu,\sigma}(u,v) &\ge \frac{1}{m}\left(1 - \frac{\mu^+}{\mathcal{H}_m}\right)\|u\|^m_{E_{m,a}} + \frac{1}{p}\left(1 - \frac{\sigma^+}{\mathcal{H}_p}\right)\|v\|^p_{E_{p,b}} \\
&\qquad - \frac{1}{p}\int_{\mathbb{R}^n}(\lambda+\varepsilon)|(u,v)|^p dx - C_\varepsilon \int_{\mathbb{R}^n}|(u,v)|^r dx \\
&\ge \frac{\kappa}{p}\big(\|u\|^p_{E_{m,a}} + \|v\|^p_{E_{p,b}}\big) - \frac{1}{p}(\lambda+\varepsilon)\|(u,v)\|^p_p \\
&\qquad - C_\varepsilon C_r^r\|(u,v)\|^r \\
&\ge \frac{1}{2^{p-1}p}\left(\kappa - 2^{p-1}\frac{\lambda+\varepsilon}{\lambda_p}\right)\|(u,v)\|^p - C_\varepsilon C_r^r\|(u,v)\|^r \\
&= \frac{1}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p} - 2^p p\, C_\varepsilon C_r^r\|(u,v)\|^{r-p}\right)\|(u,v)\|^p.
\end{aligned} \tag{5.3.20}$$

Now fix $\rho \in (0, 1)$ so small that $\kappa - 2^{p-1}\lambda/\lambda_p - 2^p p\, C_\varepsilon C_r^r \rho^{r-p} > 0$. This can be done by (5.1.5). Therefore, for all $(u, v) \in W$, with $\|(u, v)\| = \rho$, we have

$$I_{\mu,\sigma}(u,v) \ge \frac{\rho^p}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p} - 2^p p\, C_\varepsilon C_r^r \rho^{r-p}\right) = \alpha > 0.$$

Let B_1 be the unit ball in $\mathbb{R}^n$ centered at 0 and let u^*, $v^* \in C_0^\infty(B_1)$ be two non–negative nontrivial radial functions, such that $\|\,|(u^*, v^*)|\,\|_{L^p(B_1)} > 0$. Let u_0 and v_0 be the natural extensions of u^* and v^*, respectively, to the entire $\mathbb{R}^n$, defining $u_0(x) = 0$ and $v_0(x) = 0$ in $\mathbb{R}^n \setminus \overline{B_1}$. Clearly, $(u_0, v_0) \in W$, with $\|u_0\|_{E_{m,a}} > 0$ and $\|v_0\|_{E_{p,b}} > 0$.

By (H_3) for any positive constant $A > 0$ there exists $\delta_A > 0$ such that $H(x,u,v) \geq A\,|(u,v)|^p/p$ for all $(x,u,v) \in \mathbb{R}^n \times \mathbb{R}^+_0 \times \mathbb{R}^+_0$, with $u > \delta_A$ and $v > \delta_A$. Clearly,

$$\min_{(x,u,v)\in\overline{B_1}\times[0,\delta_A]^2} \left(H(x,u,v) - \frac{A}{p}\,|(u,v)|^p \right) \in \mathbb{R},$$

so that there exists $C_A \geq 0$ such that

$$H(x,u,v) \geq \frac{A}{p}\,|(u,v)|^p - C_A \quad \text{for all } (x,u,v) \in \overline{B_1} \times \mathbb{R}^+_0 \times \mathbb{R}^+_0.$$

Then, for $t \geq 1$

$$\begin{aligned} I_{\mu,\sigma}(tu_0,tv_0) &= \frac{t^m}{m}\|u_0\|^m_{E_{m,a}} - \frac{\mu t^m}{m}\|u_0\|^m_{H_m} + \frac{t^p}{p}\|v_0\|^p_{E_{p,b}} - \frac{\sigma t^p}{p}\|v_0\|^p_{H_p} \\ &\qquad - \int_{B_1} H(x,tu_0,tv_0)dx \\ &\leq \frac{t^p}{m}\big(\|u_0\|^m_{E_{m,a}} + |\mu^-|\cdot\|u_0\|^m_{H_m} + \|v_0\|^p_{E_{p,b}} + |\sigma^-|\cdot\|v_0\|^p_{H_p} \\ &\qquad - A\|\,|(u_0,v_0)|\,\|^p_p\big) + C_A|B_1|, \end{aligned}$$

where $\tau^- = \min\{0,\tau\}$ for all $\tau \in \mathbb{R}$. Choosing A so large that

$$0 < \|u_0\|^m_{E_{m,a}} + |\mu^-|\cdot\|u_0\|^m_{H_m} + \|v_0\|^p_{E_{p,b}} + |\sigma^-|\cdot\|v_0\|^p_{H_p} < A\,\|\,|(u_0,v_0)|\,\|^p_p,$$

we get $I_{\mu,\sigma}(tu_0,tv_0) \to -\infty$ as $t \to \infty$. Thus, there exists $(\omega,w) = (t_0u_0,t_0v_0) \in W$ such that $\|(\omega,w)\| \geq 2 > \rho$ and $I_{\mu,\sigma}(\omega,w) < 0$.

Therefore, we have proved that $I_{\mu,\sigma}$ satisfies a mountain pass geometry. Combining this fact with Lemma 5.3.2, an application of Theorem 5.2.1 gives the existence of $(u,v) \in W$, with $(u,v) \neq (0,0)$, satisfying

$$\begin{aligned} \langle u,\Phi\rangle_{E_{m,a}} + \langle v,\Psi\rangle_{E_{p,b}} - \mu\langle u,\Phi\rangle_{H_m} - \sigma\langle v,\Psi\rangle_{H_p} \\ = \int_{\mathbb{R}^n} \big[H_u(x,u,v)\Phi + H_v(x,u,v)\Psi\big]dx \end{aligned}$$

for all $(\Phi,\Psi) \in W$. Taking $\Phi = u^- = \min\{0,u\}$ and $\Psi = v^- = \min\{0,v\}$, we have by (H_1), (5.1.5) and (5.3.5)

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \big[H_u(x,u,v)u^- + H_v(x,u,v)v^-\big]dx \\ &\geq \left(1 - \frac{\mu^+}{\mathcal{H}_m}\right)\|u^-\|^m_{E_{m,a}} + \left(1 - \frac{\sigma^+}{\mathcal{H}_p}\right)\|v^-\|^p_{E_{p,b}} \geq 0. \end{aligned}$$

In conclusion, $u^- = 0$ and $v^- = 0$ a.e. in $\mathbb{R}^n$, that is, $u \geq 0$ and $v \geq 0$ a.e. in $\mathbb{R}^n$. This completes the proof. □

Let us sketch

Proof of Theorem 5.1.2. The proof of Lemma 5.3.1 goes without essential changes, by replacing (5.3.8) by

$$0 \le \int_{\mathbb{R}^n} H(x, U_k, V_k)dx \le (\lambda + 1)\|(U_k, V_k)\|_q^q + C_1\|(U_k, V_k)\|_r^r \to 0$$

as $k \to \infty$, thanks to (H_1) and $(H_2)'$, with $\varepsilon = 1$, since $(U_k, V_k) \to (0, 0)$ in $L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$ for any $\nu \in (p, m^*)$ by Lemma 5.2.2.

Similarly, the proof of Lemma 5.3.2 is almost unchanged, since it is enough to request in (5.3.14) that the exponent $\nu \in (p, m^*)$, so that the main property (5.3.18) is now a direct consequence of $(H_2)'$, which gives at once

$$\begin{aligned}\int_{\mathbb{R}^n} \big|\big(H_u(x, u_k, v_k) - &H_u(x, u, v)\big)(u_k - u) \\ &+ \big(H_v(x, u_k, v_k) - H_v(x, u, v)\big)(v_k - v)\big|dx \\ &\le C\big(\|z_k - z\|_q + \|z_k - z\|_r\big) \to 0,\end{aligned}$$

as $k \to \infty$, by virtue of Lemma 5.2.2.

Concerning the mountain pass geometry, the proof is the same as for Theorem 5.1.1, with the only exception that $\varepsilon > 0$ in $(H_2)'$ is taken so that $2^p\varepsilon/\lambda_q = \kappa - 2^{p-1}\lambda/\lambda_q$ and (5.3.20) is now replaced by

$$\begin{aligned} I_{\mu,\sigma}(u, v) \ge &\frac{1}{m}\left(1 - \frac{\mu^+}{\mathcal{H}_m}\right)\|u\|_{E_{m,a}}^m + \frac{1}{p}\left(1 - \frac{\sigma^+}{\mathcal{H}_p}\right)\|v\|_{E_{p,b}}^p \\ &- \frac{1}{p}\int_{\mathbb{R}^n} (\lambda + \varepsilon)|(u, v)|^q dx - C_\varepsilon \int_{\mathbb{R}^n} |(u, v)|^r dx \\ \ge &\frac{\kappa}{2^{p-1}p}\|(u, v)\|^p - \frac{1}{p} \cdot \frac{\lambda + \varepsilon}{\lambda_q}\|(u, v)\|^q - C_\varepsilon C_r^r\|(u, v)\|^r \\ \ge &\frac{1}{2^{p-1}p}\left(\kappa - 2^{p-1}\frac{\lambda + \varepsilon}{\lambda_q}\right)\|(u, v)\|^p - C_\varepsilon C_r^r\|(u, v)\|^r \\ = &\frac{1}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_q} - 2^p p\, C_\varepsilon C_r^r\|(u, v)\|^{r-p}\right)\|(u, v)\|^p, \end{aligned}$$

which holds for all $(u, v) \in W$, with $\|(u, v)\| \le 1$, thanks to (5.1.3), (5.2.3) and the fact that $p < q < r$. The rest of the proof is unchanged.

Therefore, the functional $I_{\mu,\sigma}\big|_{W_{\rm rad}}$ admits a nontrivial non–negative critical point $(u, v) \in W_{\rm rad}$, that is

$$\langle I'_{\mu,\sigma}(u, v), (\Phi, \Psi)\rangle = 0 \quad \text{for all } (\Phi, \Psi) \in W_{\rm rad}$$

and then (u, v) is a critical point of $I_{\mu,\sigma}$ in the entire space W by the principle of symmetric criticality of Palais, see Lemma 5.4 of [35], since

$$(I_{\mu,\sigma} \circ a)(u, v) = I_{\mu,\sigma}(u, v) \quad \text{for all } a \in G,$$

where $SO(n) = \{A \in \mathbb{R}^{n\times n} : A^t A = \mathbb{I}_n \text{ and } \det A = 1\}$ and

$$G = \{a : W \to W \ : \ a(u, v) = (u, v) \circ A, \ A \in SO(n)\}. \tag{5.3.21}$$

This completes the proof. □

5.4 Existence of solutions of $(\mathscr{P}_5)$

System $(\mathscr{P}_5)$ has a variational structure and the underlying functional is $J_{\mu,\sigma} : W \to W$ given by

$$\begin{aligned} J_{\mu,\sigma}(u, v) = {} & \frac{1}{m}\|u\|^m_{E_{m,a}} + \frac{1}{p}\|v\|^p_{E_{p,b}} - \frac{\mu}{m}\|u\|^m_{H_m} - \frac{\sigma}{p}\|v\|^p_{H_p} - \int_{\mathbb{R}^n} H(x, u, v)dx \\ & - \frac{1}{m^*}\|u^+\|^{m^*}_{m^*} - \frac{1}{p^*}\|v^+\|^{p^*}_{p^*} - \frac{1}{m^*}\int_{\mathbb{R}^n}(u^+)^\theta(v^+)^\vartheta dx \\ & - \int_{\mathbb{R}^n}\varphi(x)udx - \int_{\mathbb{R}^n}\psi(x)vdx. \end{aligned}$$

By Lemma 5.2.2 and the choice of θ and ϑ, $J_{\mu,\sigma}$ is well–defined and of class $C^1(W)$, with

$$\begin{aligned} \langle J'_{\mu,\sigma}(u, v), (\Phi, \Psi)\rangle = {} & \langle u, \Phi\rangle_{E_{m,a}} + \langle v, \Psi\rangle_{E_{p,b}} - \mu\langle u, \Phi\rangle_{H_m} - \sigma\langle v, \Psi\rangle_{H_p} \\ & - \int_{\mathbb{R}^n}[H_u(x, u, v)\Phi + H_v(x, u, v)\Psi]dx - \langle u^+, \Phi\rangle_{m^*} - \langle v^+, \Psi\rangle_{p^*} \\ & - \int_{\mathbb{R}^n}\left[\frac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^\vartheta\Phi + \frac{\vartheta}{m^*}(u^+)^\theta(v^+)^{\vartheta-1}\Psi\right]dx \\ & - \int_{\mathbb{R}^n}\phi(x)\Phi dx - \int_{\mathbb{R}^n}\psi(x)\Psi dx, \end{aligned}$$

for any $(\Phi, \Psi) \in W$, where

$$\begin{aligned} \langle u^+, \Phi\rangle_{m^*} &= \int_{\mathbb{R}^n}|u(x)|^{m^*-2}u^+(x)\Phi(x)dx, \\ \langle v^+, \Psi\rangle_{p^*} &= \int_{\mathbb{R}^n}|v(x)|^{p^*-2}v^+(x)\Psi(x)dx. \end{aligned}$$

We first prove that $(\mathscr{P}_5)$ presents a suitable geometry for existence of local minima provided that the perturbations φ and ψ are sufficiently small in their norms, as shown in [35] for general equations in a different framework.

Lemma 5.4.1. *Under* (5.1.5) *there exist numbers* α*,* ρ *and* $\delta > 0$ *such that* $J_{\mu,\sigma}(u,v) \geq \alpha$ *for all* $(u,v) \in W$*, with* $\|(u,v)\| = \rho$*, and for all* $\varphi \in L^{\mathfrak{m}}(\mathbb{R}^n)$ *and* $\psi \in L^{\mathfrak{p}}(\mathbb{R}^n)$*, with* $\|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} \leq \delta$*, provided that* (5.1.5) *holds.*

Proof. Take $\varepsilon > 0$, with $2^p\varepsilon/\lambda_p = \kappa - 2^{p-1}\lambda/\lambda_p$. This is possible thanks to the main restriction (5.1.5). By (5.1.3), (5.2.3), (H_2) and the Hölder inequality, we have for all $(u,v) \in W$, with $\|(u,v)\| \leq 1$,

$$\begin{aligned}
J_{\mu,\sigma}(u,v) \geq \frac{1}{2^{p-1}p}&\left(\kappa - 2^{p-1}\frac{\lambda+\varepsilon}{\lambda_p}\right)\|(u,v)\|^p - C_\varepsilon C_r^r\|(u,v)\|^r \\
&- \frac{C_{m^*}^{m^*}}{m^*}\|(u,0)\|^{m^*} - \frac{C_{p^*}^{p^*}}{p^*}\|(0,v)\|^{p^*} - \frac{1}{m^*}\|u\|_{m^*}^{\theta}\|v\|_{m^*}^{\vartheta} \\
&- C_{m^*}\|\varphi\|_{\mathfrak{m}}\|(u,0)\| - C_{p^*}\|\psi\|_{\mathfrak{p}}\|(0,v)\| \\
\geq &\left[\frac{1}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p}\right)\|(u,v)\|^{p-1} - C_\varepsilon C_r^r\|(u,v)\|^{r-1}\right. \\
&- \frac{C_{m^*}^{m^*}}{m^*}\|(u,v)\|^{m^*-1} - \frac{C_{p^*}^{p^*}}{p^*}\|(u,v)\|^{p^*-1} - \frac{C_{m^*}^{m^*}}{m^*}\|(u,v)\|^{m^*-1} \\
&\left. - C_{m^*}\|\varphi\|_{\mathfrak{m}} - C_{p^*}\|\psi\|_{\mathfrak{p}}\right]\|(u,v)\|,
\end{aligned}$$

since $\|u\|_{m^*}^{\theta}\|v\|_{m^*}^{\vartheta} \leq C_{m^*}^{m^*}\|u\|_{E_{m,a}}^{\theta}\|v\|_{E_{p,b}}^{\vartheta} \leq C_{m^*}^{m^*}\|(u,v)\|^{m^*}$, being $\theta + \vartheta = m^*$. Define for all $t \in [0,1]$

$$\eta_{\mu,\sigma}(t) = \frac{1}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p}\right)t^{p-1} - C_\varepsilon C_r^r t^{r-1} - 2\frac{C_{m^*}}{m^*}t^{m^*-1} - \frac{C_{p^*}}{p^*}t^{p^*-1}.$$

There exists $\rho \in (0,1)$ such that $\max_{t\in[0,1]}\eta_{\mu,\sigma}(t) = \eta_{\mu,\sigma}(\rho) > 0$, since $1 < m \leq p < r < m^*$. Taking $\delta = \eta_{\mu,\sigma}(\rho)/2(C_{m^*} + C_{p^*})$ we obtain $J_{\mu,\sigma}(u,v) \geq \alpha = \rho\eta_{\mu,\sigma}(\rho)/2$ for all $(u,v) \in W$, with $\|(u,v)\| = \rho$, and for all $\varphi \in L^{\mathfrak{m}}(\mathbb{R}^n)$ and $\psi \in L^{\mathfrak{p}}(\mathbb{R}^n)$, with $\|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} \leq \delta$. □

Lemma 5.4.2. *Let* ρ *be given as in Lemma* 5.4.1*. Set*

$$m_{\mu,\sigma} = \inf\left\{J_{\mu,\sigma}(u,v):\ (u,v) \in \overline{B}_\rho\right\},$$

where $\overline{B}_\rho = \{(u,v) \in W :\ \|(u,v)\| \leq \rho\}$*. Then* $m_{\mu,\sigma} < 0$ *for all non–negative perturbations* $\varphi \in L^{\mathfrak{m}}(\mathbb{R}^n)$ *and* $\psi \in L^{\mathfrak{p}}(\mathbb{R}^n)$*, with* $\|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} > 0$*.*

Proof. Fix $\varphi \in L^{\mathfrak{m}}(\mathbb{R}^n)$ and $\psi \in L^{\mathfrak{p}}(\mathbb{R}^n)$, with $\|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} > 0$. We claim that there exists a non–negative function $h \in C_0^\infty(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} h(x)(\varphi + \psi)dx > 0. \tag{5.4.1}$$

Since $\varphi \in L^{\mathfrak{m}}(\mathbb{R}^n)$ and $\psi \in L^{\mathfrak{p}}(\mathbb{R}^n)$, with $\|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} > 0$, the functions

$$\widetilde{\varphi}(x) = \begin{cases} \varphi(x)^{\mathfrak{m}-1}, & \text{if } \varphi(x) \neq 0 \\ 0, & \text{if } \varphi(x) = 0 \end{cases} \in L^{m^*}(\mathbb{R}^n),$$
$$\widetilde{\psi}(x) = \begin{cases} \psi(x)^{\mathfrak{p}-1}, & \text{if } \psi(x) \neq 0 \\ 0, & \text{if } \psi(x) = 0 \end{cases} \in L^{p^*}(\mathbb{R}^n).$$

Then, there exist two sequences $(\varphi_k)_k$ and $(\psi_k)_k$ in $C_0^\infty(\mathbb{R}^n)$ such that $\varphi_k \to \widetilde{\varphi}$ strongly in $L^{m^*}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$, while $\psi_k \to \widetilde{\psi}$ strongly in $L^{p^*}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$, since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{m^*}(\mathbb{R}^n)$ and in $L^{p^*}(\mathbb{R}^n)$. For k_0 and k_1 in $\mathbb{N}$ large enough we have

$$\varphi_{k_0},\ \psi_{k_1} \geq 0 \text{ a.e. in } \mathbb{R}^n,$$
$$\|\varphi_{k_0} - \widetilde{\varphi}\|_{m^*} \leq \frac{1}{2}\|\varphi\|_{\mathfrak{m}}^{\mathfrak{m}-1}, \quad \|\psi_{k_1} - \widetilde{\psi}\|_{p^*} \leq \frac{1}{2}\|\psi\|_{\mathfrak{p}}^{\mathfrak{p}-1}.$$

Put $h = \varphi_{k_0} + \psi_{k_1}$. Clearly, $h \in C_0^\infty(\mathbb{R}^n)$, $h \geq 0$ a.e. in $\mathbb{R}^n$, and $(h, h) \in W$. Furthermore, the Hölder inequality yields

$$\begin{aligned} \int_{\mathbb{R}^n} h(x)\big(\varphi + \psi\big)dx &\geq \int_{\mathbb{R}^n} \varphi_{k_0}\varphi dx + \int_{\mathbb{R}^n} \psi_{k_1}\psi dx \\ &\geq -\|\varphi_{k_0} - \widetilde{\varphi}\|_{m^*}\|\varphi\|_{\mathfrak{m}} + \|\varphi\|_{\mathfrak{m}}^{\mathfrak{m}} - \|\psi_{k_1} - \widetilde{\psi}\|_{p^*}\|\psi\|_{\mathfrak{p}} + \|\psi\|_{\mathfrak{p}}^{\mathfrak{p}} \\ &\geq \frac{1}{2}\|\varphi\|_{\mathfrak{m}}^{\mathfrak{m}} + \frac{1}{2}\|\psi\|_{\mathfrak{p}}^{\mathfrak{p}} > 0, \end{aligned}$$

by assumption. The claim (5.4.1) is so proved.

By (5.4.1) and (H_1)

$$\begin{aligned} J_{\mu,\sigma}(th, th) \leq \frac{t^m}{m}\|h\|_{E_{m,a}}^m + \frac{t^p}{p}\|h\|_{E_{p,b}}^p - \frac{\mu t^m}{m}\|h\|_{H_m}^m - \frac{\sigma t^p}{p}\|h\|_{H_p}^p - 2\frac{t^{m^*}}{m^*}\|h\|_{m^*}^{m^*} \\ - \frac{t^{p^*}}{p^*}\|h\|_{p^*}^{p^*} - t\int_{\mathbb{R}^n} h(x)\big[\varphi(x) + \psi(x)\big]dx < 0, \end{aligned}$$

provided that for $t \in (0, 1)$ is sufficiently small. $\square$

Proof of Theorem 5.1.3. Fix $\mu \in (-\infty, \mathcal{H}_m)$ and $\sigma \in (-\infty, \mathcal{H}_p)$ so that (5.1.5) holds. First note that if (u, v) is a solution of $(\mathscr{P}_5)$, then

$$\begin{aligned}
\langle u, \Phi\rangle_{E_{m,a}} - \mu\langle u, \Phi\rangle_{H_m} + \langle v, \Psi\rangle_{E_{p,b}} - \sigma\langle \Psi\rangle_{H_p} \\
= \int_{\mathbb{R}^n} [H_u(x,u,v)\Phi + H_v(x,u,v)\Psi]dx + \langle u^+, \Phi\rangle_{m^*} + \langle v^+, \Psi\rangle_{p^*} \\
+ \int_{\mathbb{R}^n} \left[\frac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta}\Phi + \frac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1}\Psi\right] dx \\
+ \int_{\mathbb{R}^n} \varphi(x)\Phi dx + \int_{\mathbb{R}^n} \psi(x)\Psi dx
\end{aligned}$$

for all $(\Phi, \Psi) \in W$. Taking $\Phi = u^- = \min\{0, u\}$ and $\Psi = v^- = \min\{0, v\}$, by (H_1), (5.1.5) and (5.3.5) we get

$$\begin{aligned}
0 &\geq \int_{\mathbb{R}^n} \varphi(x)u^- dx + \int_{\mathbb{R}^n} \psi(x)v^- dx \\
&= \int_{\mathbb{R}^n} [H_u(x,u,v)u^- + H_v(x,u,v)v^-]dx + \langle u^+, u^-\rangle_{m^*} \\
&\quad + \langle v^+, v^-\rangle_{p^*} + \int_{\mathbb{R}^n} \left[\frac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta}u^- + \frac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1}v^-\right] dx \\
&\quad + \int_{\mathbb{R}^n} \varphi(x)u^- dx + \int_{\mathbb{R}^n} \psi(x)v^- dx \\
&\geq \left(1 - \frac{\mu^+}{\mathcal{H}_m}\right)\|u^-\|^m_{E_{m,a}} + \left(1 - \frac{\sigma^+}{\mathcal{H}_p}\right)\|v^-\|^p_{E_{p,b}} \geq 0.
\end{aligned}$$

In conclusion, $u^- = 0$ and $v^- = 0$ a.e. in $\mathbb{R}^n$, that is, $u \geq 0$ and $v \geq 0$ a.e. in $\mathbb{R}^n$. In other words, any solution of $(\mathscr{P}_5)$ is non–negative, component by component.

Take $\varphi \in L^{\mathfrak{m}}(\mathbb{R}^n)$ and $\psi \in L^{\mathfrak{p}}(\mathbb{R}^n)$, with $0 < \|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} \leq \delta$, where $\delta > 0$ is the number determined in Lemma 5.4.1. By Lemmas 5.4.1, 5.4.2 and the Ekeland variational principle, in $\overline{B}_\rho$ there exists a sequence $\{(u_k, v_k)\}_k$ in B_ρ such that

$$\begin{aligned}
m_{\mu,\sigma} \leq J_{\mu,\sigma}(u_k, v_k) \leq m_{\mu,\sigma} + \frac{1}{k} \quad \text{and} \\
J_{\mu,\sigma}(u,v) \geq J_{\mu,\sigma}(u_k, v_k) - \frac{1}{k}\|(u - u_k, v - v_k)\|
\end{aligned} \tag{5.4.2}$$

for all $k \in \mathbb{N}$ and for any $(u, v) \in \overline{B}_\rho$. Fixed $k \in \mathbb{N}$, for all $(\omega, w) \in S_W$, where $S_W = \{(\omega, w) \in W : \|(\omega, w)\| = 1\}$, and for all $\varepsilon > 0$ so small that

$(u_k+\varepsilon\,\omega, v_k+\varepsilon w)\in\overline{B}_\rho$, we have

$$J_{\mu,\sigma}(u_k+\varepsilon\,\omega, v_k+\varepsilon w)-J_{\mu,\sigma}(u_k,v_k)\geq-\frac{\varepsilon}{k}$$

by (5.4.2). Since $J_{\mu,\sigma}$ is Gâteaux differentiable in W, we get

$$\langle J'_{\mu,\sigma}(u_k,v_k),(\omega,w)\rangle=\lim_{\varepsilon\to0}\frac{J_{\mu,\sigma}(u_k+\varepsilon\,\omega, v_k+\varepsilon w)-J_{\mu,\sigma}(u_k,v_k)}{\varepsilon}\geq-\frac{1}{k}$$

for all $(\omega,w)\in S_W$. Hence

$$\left|\langle J'_{\mu,\sigma}(u_k,v_k),(\omega,w)\rangle\right|\leq\frac{1}{k},$$

since $(\omega,w)\in S_W$ is arbitrary. Consequently, $J'_{\mu,\sigma}(u_k,v_k)\to0$ in W' as $k\to\infty$ and clearly, up to a subsequence, the bounded sequence $\{(u_k,v_k)\}_k$ weakly converges to some $(u,v)\in\overline{B}_\rho$ and has the following properties

$$\begin{gathered}(u_k,v_k)\rightharpoonup(u,v)\text{ in }W,\qquad(u_k,v_k)\to(u,v)\text{ a.e. in }\mathbb{R}^n,\\(u_k,v_k)\to(u,v)\text{ in }L^\nu(\mathbb{R}^n)\times L^\nu(\mathbb{R}^n),\\u_k\rightharpoonup u\text{ in }L^m(\mathbb{R}^n,|x|^{ms}),\qquad v_k\rightharpoonup v\text{ in }L^p(\mathbb{R}^n,|x|^{ps}),\\\|u_k\|_{E_{m,a}}\to\mathfrak{u},\ \|v_k\|_{E_{p,b}}\to\mathfrak{v},\qquad\|u_k\|_{H_m}\to\mathfrak{h},\ \|v_k\|_{H_p}\to\mathfrak{k},\\\|u_k^+\|_{m^*}\to\mathfrak{i},\quad\|v_k^+\|_{p^*}\to\mathfrak{j},\\u_k^+\rightharpoonup u^+\text{ in }L^{m^*}(\mathbb{R}^n),\qquad v_k^+\rightharpoonup v^+\text{ in }L^{p^*}(\mathbb{R}^n),\\(u_k^+)^{\theta-1}(v_k^+)^{\vartheta}\rightharpoonup(u^+)^{\theta-1}(v^+)^{\vartheta}\text{ in }L^{m^*/(m^*-1)}(\mathbb{R}^n),\\(u_k^+)^{\theta}(v_k^+)^{\vartheta-1}\rightharpoonup(u^+)^{\theta}(v^+)^{\vartheta-1}\text{ in }L^{m^*/(m^*-1)}(\mathbb{R}^n),\end{gathered}\tag{5.4.3}$$

for any $\nu\in[p,m^*)$ by Lemmas 5.2.2, 5.2.3 and (5.1.1). In particular,

$$\begin{aligned}\lim_{k\to\infty}\int_{\mathbb{R}^n}(u_k^+)^{\theta-1}(v_k^+)^{\vartheta}u^+dx&=\int_{\mathbb{R}^n}(u^+)^{\theta}(v^+)^{\vartheta}dx,\\\lim_{k\to\infty}\int_{\mathbb{R}^n}(u_k^+)^{\theta}(v_k^+)^{\vartheta-1}v^+dx&=\int_{\mathbb{R}^n}(u^+)^{\theta}(v^+)^{\vartheta}dx,\end{aligned}\tag{5.4.4}$$

since $(u^+,v^+)\in W$. While, the Fatou lemma gives

$$\int_{\mathbb{R}^n}(u^+)^{\theta}(v^+)^{\vartheta}dx\leq\liminf_{k\to\infty}\int_{\mathbb{R}^n}(u_k^+)^{\theta}(v_k^+)^{\vartheta}dx.\tag{5.4.5}$$

Furthermore, by (H_2), (5.4.3) and the Lebesgue dominated convergence theorem we have

$$\begin{aligned}&\lim_{k\to\infty}\int_{\mathbb{R}^n}[H_u(x,u_k,v_k)u+H_v(x,u_k,v_k)v]dx\\&\qquad=\int_{\mathbb{R}^n}[H_u(x,u,v)u+H_v(x,u,v)v]dx\end{aligned}\tag{5.4.6}$$

and similarly

$$\tag{5.4.7}\begin{aligned}\lim_{k\to\infty}\int_{\mathbb{R}^n}[H_u(x,u_k,v_k)u_k+H_v(x,u_k,v_k)v_k]dx\\ =\int_{\mathbb{R}^n}[H_u(x,u,v)u+H_v(x,u,v)v]dx,\\ \lim_{k\to\infty}\int_{\mathbb{R}^n}H(x,u_k,v_k)dx=\int_{\mathbb{R}^n}H(x,u,v)dx.\end{aligned}$$

Moreover,

$$\tag{5.4.8}\begin{aligned}\lim_{k\to\infty}\int_{\mathbb{R}^n}\varphi(x)u_kdx=\int_{\mathbb{R}^n}\varphi(x)udx,\\ \lim_{k\to\infty}\int_{\mathbb{R}^n}\psi(x)v_kdx=\int_{\mathbb{R}^n}\psi(x)vdx\end{aligned}$$

by (5.4.3), being $\varphi\in L^{\mathfrak{m}}(\mathbb{R}^n)$ and $\psi\in L^{\mathfrak{p}}(\mathbb{R}^n)$.

Let us prove that (u,v), given in (5.4.3), is actually in B_ρ, so that (u,v) is a critical point of $J_{\mu,\sigma}$ at level $m_{\mu,\sigma}<0$. In other words, (u,v) is a nontrivial solution of $(\mathscr{P}_5)$. Clearly, $J_{\mu,\sigma}(u,v)\geq m_{\mu,\sigma}$, since $(u,v)\in\overline{B}_\rho$ by (5.4.3). Moreover, (5.4.3), (5.4.4) and (5.4.6) yield as $k\to\infty$

$$\tag{5.4.9}\begin{aligned}0&=\langle J'_{\mu,\sigma}(u_k,v_k),(u,v)\rangle+o(1)\\ &=\langle u_k,u\rangle_{E_{m,a}}-\mu\langle u_k,u\rangle_{H_m}+\langle v_k,v\rangle_{E_{p,b}}-\sigma\langle v_k,v\rangle_{H_p}\\ &\quad-\int_{\mathbb{R}^n}[H_u(x,u_k,v_k)u+H_v(x,u_k,v_k)v]dx\\ &\quad-\int_{\mathbb{R}^n}\left[\frac{\theta}{m^*}(u_k^+)^{\theta-1}(v_k^+)^{\vartheta}u^++\frac{\vartheta}{m^*}(u_k^+)^{\theta}(v_k^+)^{\vartheta-1}v^+\right]dx\\ &\quad-\langle u_k^+,u\rangle_{m^*}-\langle v_k^+,v\rangle_{p^*}-\int_{\mathbb{R}^n}\varphi(x)udx-\int_{\mathbb{R}^n}\psi(x)vdx+o(1)\\ &=\|u\|_{E_{m,a}}^m-\mu\|u\|_{H_m}^m+\|v\|_{E_{p,b}}^p-\sigma\|v\|_{H_p}^p\\ &\quad-\int_{\mathbb{R}^n}[H_u(x,u,v)u+H_v(x,u,v)v]dx-\|u^+\|_{m^*}^{m^*}-\|v^+\|_{p^*}^{p^*}\\ &\quad-\int_{\mathbb{R}^n}(u^+)^\theta(v^+)^\vartheta dx-\int_{\mathbb{R}^n}\varphi(x)udx-\int_{\mathbb{R}^n}\psi(x)vdx.\end{aligned}$$

Now we divide the proof into two cases.

Case $m<p$ and $\mu\leq 0$. Multiplying the expression (5.4.9) by $1/p$ and subtracting it below, by (5.4.3), (5.4.5) and (5.4.7)–(5.4.8), we find as $k\to\infty$

$$m_{\mu,\sigma}\leq J_{\mu,\sigma}(u,v)=\frac{1}{m}\|u\|_{E_{m,a}}^m-\frac{\mu}{m}\|u\|_{H_m}^m+\frac{1}{p}\|v\|_{E_{p,b}}^p-\frac{\sigma}{p}\|v\|_{H_p}^p$$

$$
\begin{aligned}
&\quad - \int_{\mathbb{R}^n} H(x,u,v)dx - \frac{1}{m^*}\|u^+\|_{m^*}^{m^*} - \frac{1}{p^*}\|v^+\|_{p^*}^{p^*} \\
&\quad - \frac{1}{m^*}\int_{\mathbb{R}^n} (u^+)^\theta (v^+)^\vartheta dx - \int_{\mathbb{R}^n} \varphi(x) u dx - \int_{\mathbb{R}^n} \psi(x) v dx \\
&= \left(\frac{1}{m} - \frac{1}{p}\right)\left(\|u\|_{E_{m,a}}^m + |\mu|\cdot\|u\|_{H_m}^m\right) \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^n} [H_u(x,u,v)u + H_v(x,u,v)v]dx - \int_{\mathbb{R}^n} H(x,u,v)dx \\
&\quad + \left(\frac{1}{p} - \frac{1}{m^*}\right)\|u^+\|_{m^*}^{m^*} + \left(\frac{1}{p} - \frac{1}{p^*}\right)\|v^+\|_{p^*}^{p^*} \\
&\quad + \left(\frac{1}{p} - \frac{1}{m^*}\right)\int_{\mathbb{R}^n} (u^+)^\theta (v^+)^\vartheta dx \\
&\quad - \left(1 - \frac{1}{p}\right)\int_{\mathbb{R}^n} \varphi(x) u dx - \left(1 - \frac{1}{p}\right)\int_{\mathbb{R}^n} \psi(x) v dx \\
&\le \left(\frac{1}{m} - \frac{1}{p}\right)\left(\|u_k\|_{E_{m,a}}^m + |\mu|\cdot\|u_k\|_{H_m}^m\right) \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^n} [H_u(x,u_k,v_k)u_k + H_v(x,u_k,v_k)v_k]dx - \int_{\mathbb{R}^n} H(x,u_k,v_k)dx \\
&\quad + \left(\frac{1}{p} - \frac{1}{m^*}\right)\|u_k^+\|_{m^*}^{m^*} + \left(\frac{1}{p} - \frac{1}{p^*}\right)\|v_k^+\|_{p^*}^{p^*} \\
&\quad + \left(\frac{1}{p} - \frac{1}{m^*}\right)\int_{\mathbb{R}^n} (u_k^+)^\theta (v_k^+)^\vartheta dx - \left(1 - \frac{1}{p}\right)\int_{\mathbb{R}^n} \varphi(x) u_k dx \\
&\quad - \left(1 - \frac{1}{p}\right)\int_{\mathbb{R}^n} \psi(x) v_k dx + o(1) \\
&\le J_{\mu,\sigma}(u_k,v_k) - \frac{1}{p}\langle J'_{\mu,\sigma}(u_k,v_k),(u_k,v_k)\rangle + o(1) = m_{\mu,\sigma},
\end{aligned}
$$

since $\|u\|_{E_{m,a}} \le \mathfrak{u}$, $\|u\|_{H_m} \le \mathfrak{h}$, $\|v\|_{E_{p,b}} \le \mathfrak{v}$, $\|v\|_{H_p} \le \mathfrak{k}$ and $1 < m < p < m^*$.
Case $m = p$. Again we multiply the expression (5.4.9) by $1/p$ and, subtracting it below, we obtain that as $k \to \infty$

$$
\begin{aligned}
m_{\mu,\sigma} \le J_{\mu,\sigma}(u,v) &= \frac{1}{p}\|u\|_{E_{p,a}}^p - \frac{\mu}{p}\|u\|_{H_p}^p + \frac{1}{p}\|v\|_{E_{p,b}}^p - \frac{\sigma}{p}\|v\|_{H_p}^p \\
&\quad - \int_{\mathbb{R}^n} H(x,u,v)dx - \frac{1}{p^*}\|u^+\|_{p^*}^{p^*} - \frac{1}{p^*}\|v^+\|_{p^*}^{p^*} \\
&\quad - \frac{1}{p^*}\int_{\mathbb{R}^n} (u^+)^\theta (v^+)^\vartheta dx - \int_{\mathbb{R}^n} \varphi(x) u dx - \int_{\mathbb{R}^n} \psi(x) v dx \\
\le \frac{1}{p}\int_{\mathbb{R}^n} [H_u(x,u_k,v_k)u_k &+ H_v(x,u_k,v_k)v_k]dx - \int_{\mathbb{R}^n} H(x,u_k,v_k)dx
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{1}{p}-\frac{1}{p^*}\right)\left(\|u_k^+\|_{p^*}^{p^*}+\|v_k^+\|_{p^*}^{p^*}\right)-\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^n}\varphi(x)u_k dx\\
&+\left(\frac{1}{p}-\frac{1}{p^*}\right)\int_{\mathbb{R}^n}(u_k^+)^\theta(v_k^+)^\vartheta dx-\left(1-\frac{1}{p}\right)\int_{\mathbb{R}^n}\psi(x)v_k dx+o(1)\\
=\, & J_{\mu,\sigma}(u_k,v_k)-\frac{1}{p}\langle J'_{\mu,\sigma}(u_k,v_k),(u_k,v_k)\rangle+o(1)=m_{\mu,\sigma},
\end{aligned}
$$

using once more (5.4.3), (5.4.5) and (5.4.7)–(5.4.8).

In conclusion, in both cases, (u,v) is a minimizer of $J_{\mu,\sigma}$ in $\overline{B}_\rho$ and $J_{\mu,\sigma}(u,v)=m_{\mu,\sigma}<0<\alpha\le J_{\mu,\sigma}(\omega,w)$ for all $(\omega,w)\in\partial B_\rho$ by Lemma 5.4.1. Thus in turn $(u,v)\in B_\rho$, so that $J'_{\mu,\sigma}(u,v)=0$ and this implies that (u,v) is a nontrivial solution of $(\mathscr{P}_5)$. □

Let us sketch

Proof of Theorem 5.1.4. The argument is essentially the same as in the proof of Theorem 5.1.2, where now the proof of Lemma 5.4.1 goes almost without changes, taking ε such that $2^p\varepsilon/\lambda_q=\kappa-2^{p-1}\lambda/\lambda_q$ by (5.1.6). Lemma 5.4.2 holds in the same way as for Theorem 5.1.3. Furthermore, as in the proof of Theorem 5.1.3, now (5.4.3) holds for all $\nu\in(p,m^*)$. Thus, (5.4.6)–(5.4.7) continue to hold by $(H_2)'$ and the rest of the proof is unchanged.

Therefore, the functional $J_{\mu,\sigma}|_{W_{\rm rad}}$ admits a nontrivial non–negative critical point $(u,v)\in W_{\rm rad}$, that is

$$
\langle J'_{\mu,\sigma}(u,v),(\Phi,\Psi)\rangle=0 \quad \text{for all } (\Phi,\Psi)\in W_{\rm rad}
$$

and then (u,v) is a critical point of $J_{\mu,\sigma}$ in the entire W by the principle of symmetric criticality of Palais, see Lemma 5.4 of [35], since

$$
(J_{\mu,\sigma}\circ a)(\omega,w)=J_{\mu,\sigma}(\omega,w) \quad \text{for all } (\omega,w)\in W \text{ and } a\in G,
$$

where G is defined in (5.3.21). □

5.5 Existence of solutions for $(\mathscr{P}_{4,K})$ and $(\mathscr{P}_{5,K})$

In this section, we extend the results of Sections 5.3 and 5.4 when the fractional $\wp$–Laplacian operators are replaced by more general elliptic nonlocal integro–differential operators.

Hence, we consider in $\mathbb{R}^n$ problems

$$(5.5.1)\qquad \begin{cases} -\mathcal{L}_{K_m}u + a(x)|u|^{m-2}u - \mu\dfrac{|u|^{m-2}u}{|x|^{ms}} = H_u(x,u,v),\\ -\mathcal{L}_{K_p}v + b(x)|v|^{p-2}v - \sigma\dfrac{|v|^{p-2}v}{|x|^{ps}} = H_v(x,u,v),\end{cases}$$

and

$$(5.5.2)\qquad \begin{cases} -\mathcal{L}_{K_m}u + a(x)|u|^{m-2}u - \mu\dfrac{|u|^{m-2}u}{|x|^{ms}} = H_u(x,u,v) + |u|^{m^*-2}u^+\\ \qquad\qquad + \dfrac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta} + \varphi(x),\\ -\mathcal{L}_{K_p}v + b(x)|v|^{p-2}v - \sigma\dfrac{|v|^{p-2}v}{|x|^{ps}} = H_v(x,u,v) + |v|^{p^*-2}v^+\\ \qquad\qquad + \dfrac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1} + \psi(x),\end{cases}$$

governed by the operator $-\mathcal{L}_{K_\wp}$, which up to a multiplicative constant depending only on n, s and $\wp$ is defined for all $x \in \mathbb{R}^n$ by

$$-\mathcal{L}_{K_\wp}\varphi(x) = \int_{\mathbb{R}^n} |\varphi(x) - \varphi(y)|^{\wp-2}[\varphi(x) - \varphi(y)]K_\wp(x-y)dy,$$

along any function $\varphi \in C_0^\infty(\mathbb{R}^n)$.

The *weight* $K_\wp : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$, $\wp > 1$, satisfies the natural restrictions listed in Chapter 1.

Let us denote by $D^{s,\wp}_{K_\wp}(\mathbb{R}^n)$ the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to

$$[u]_{s,K_\wp} = \left(\iint_{\mathbb{R}^{2n}} |u(x) - u(y)|^\wp K_\wp(x-y)dxdy\right)^{1/\wp},$$

which is well–defined by (K_2). The embedding $D^{s,\wp}_{K_\wp}(\mathbb{R}^n) \hookrightarrow D^{s,\wp}(\mathbb{R}^n)$ is continuous, since $[u]_{s,\wp} \le K_0^{-1/\wp}[u]_{s,K_\wp}$ for all $u \in D^{s,\wp}_{K_\wp}(\mathbb{R}^n)$ by (K_1). Thus (5.1.1) is still valid.

The natural solution space for (5.5.1) is $W_K = E_{K_m,a} \times E_{K_p,b}$, where

$$E_{K_m,a} = \left\{u \in D^{s,m}_{K_m}(\mathbb{R}^n) : \int_{\mathbb{R}^n} a(x)|u(x)|^m dx < \infty\right\},$$

$$E_{K_p,b} = \left\{v \in D^{s,p}_{K_p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} b(x)|v(x)|^p dx < \infty\right\},$$

endowed with the norm $\|(u,v)\| = \|u\|_{E_{K_m,a}} + \|v\|_{E_{K_p,b}}$, with

$$\|u\|_{E_{K_m,a}} = \left([u]^m_{s,K_m} + \|u\|^m_{m,a}\right)^{1/m}, \quad \|v\|_{E_{K_p,b}} = \left([v]^p_{s,K_p} + \|v\|^p_{p,b}\right)^{1/p}.$$

Under the solely condition (V_1), we have that the embeddings

$$W_K \hookrightarrow W^{s,m}_{K_m}(\mathbb{R}^n) \times W^{s,p}_{K_p}(\mathbb{R}^n) \hookrightarrow L^\nu(\mathbb{R}^n) \times L^\nu(\mathbb{R}^n)$$

are certainly continuous for all $\nu \in [p, m^*]$, being $1 < m \le p < m^*$, again by Lemma 5.2.2. Thus, the numbers

$$\lambda_\nu = \inf\left\{\|u\|^\nu_{E_{K_m,a}} + \|v\|^\nu_{E_{K_p,b}} \ : \ \int_{\mathbb{R}^n} |(u,v)|^\nu dx = 1\right\} \tag{5.5.3}$$

are well defined and strictly positive.

This being shown, the proofs in Sections 5.3 and 5.4 can proceed in the same way, up to the replacement of the appropriate norms. Thus we obtain the following results, recalling that λ_p in (H_2) and λ_q in $(H_2)'$ are now defined by (5.5.3).

Theorem 5.5.1. *Under the assumptions (K_1)–(K_2) on K_m and K_p, (H_1)–(H_4) and (V_1)–(V_2) on a and b, system (5.5.1) has at least one nontrivial non–negative entire solution $(u,v) \in W_K$ for any $\mu \in (-\infty, \mathcal{H}_m)$ and for any $\sigma \in (-\infty, \mathcal{H}_p)$ verifying (5.1.5).*

Theorem 5.5.2. *Assume that $n \ge 2$, that a, b, K_m, K_p, $H_z(\cdot, z)$ are radial for all $z \in \mathbb{R}^2$. Suppose that a and b satisfy (V_1), that K_m and K_p verify (K_1)–(K_2) and that (H_1) holds, while (H_2) is replaced by $(H_2)'$. Then system (5.5.1) has at least one nontrivial non–negative entire radial solution $(u,v) \in W_K$ for any $\mu \in (-\infty, \mathcal{H}_m)$ and for any $\sigma \in (-\infty, \mathcal{H}_p)$ verifying (5.1.6).*

Theorem 5.5.3. *Under the assumptions (H_1)–(H_2), (K_1)–(K_2) on K_m and K_p, and (V_1)–(V_2) on a and b, there exists a number $\delta > 0$ such that for all non–negative perturbations φ and ψ, with $0 < \|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} < \delta$, system (5.5.2) has at least one nontrivial non–negative entire solution $(u,v) \in W_K$ for any $\mu \in (-\infty, \mathcal{H}_m)$ and for any $\sigma \in (-\infty, \mathcal{H}_p)$ satisfying (5.1.5), provided that either $m < p$ and $\mu \le 0$, or $m = p$.*

Theorem 5.5.4. *Assume that* $n \geq 2$*, that* a*,* b*,* K_m*,* K_p*,* $H_z(\cdot, z)$ *are radial for all* $z \in \mathbb{R}^2$*. Suppose that* a *and* b *satisfy* (V_1)*, that* K_m *and* K_p *verify* (K_1)*–*(K_2) *and that* (H_1) *and* $(H_2)'$ *hold. Then there exists a number* $\delta > 0$ *such that all for non–negative perturbations* φ *and* ψ*, with* $0 < \|\varphi\|_{\mathfrak{m}} + \|\psi\|_{\mathfrak{p}} < \delta$*, system* (5.5.2) *has at least one nontrivial non–negative entire radial solution* $(u, v) \in W_K$ *for any* $\mu \in (-\infty, \mathcal{H}_m)$ *and for any* $\sigma \in (-\infty, \mathcal{H}_p)$ *satisfying* (5.1.6)*, provided that either* $m < p$ *and* $\mu \leq 0$*, or* $m = p$*.*

Chapter 6

Conclusions and open problems

In this chapter we present some open problems arising from the papers [77, 78, 79, 50], which can be useful for future research. Since the proposed arguments in this thesis are quite varied, we divide this chapter in sections, each one related to a particular problem.

6.1 Problem $(\mathscr{P}_1)$

We recall that in Chapter 2 we deal with the following problem

$$(\mathscr{P}_1) \qquad \begin{cases} -\mathcal{L}_K u = \lambda[a(x)|u|^{p-2}u + f(x,u)] & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

The main result is Theorem 2.4.1, where we establish for which values of the parameter λ problem $(\mathscr{P}_1)$ admits only the trivial solution or at least two solutions.

In particular, under the assumptions $(\mathcal{F})$–(a) and (b), we get that

(i) problem $(\mathscr{P}_1)$ has only the trivial solution if $\lambda \in [0, \lambda_\star)$, where $\lambda_\star$ is defined in (2.2.3);

(ii) if f satisfies also $(\mathcal{F})$–(c), then problem $(\mathscr{P}_1)$ admits at least two non-trivial solutions for every $\lambda \in (\lambda^\star, \lambda_1)$, where $\lambda^\star < \lambda_1$ is given in (2.3.8).

An interesting open question is the relation between the crucial values $\lambda_\star$ and $\lambda^\star$. Indeed, now we are not able to establish the existence or non–existence of solutions in the interval $[\lambda_\star, \lambda^\star]$.

The same question arises for the simpler problem

$$\begin{cases} \mathcal{L}_K u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

under condition $(\mathcal{F})$–(c'). In this case, we have no information if $\lambda \in [\ell_\star, \ell^\star]$, so this situation can be investigated.

6.2 Problem $(\mathscr{P}_2)$

In Chapter 3 we study problem

$$(\mathscr{P}_2) \qquad \begin{aligned} &M\left([u]_K^p\right)(-\mathcal{L}_K u) = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \qquad \text{in } \mathbb{R}^n, \\ &[u]_K^p = \iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^p K(x-y)\,dxdy. \end{aligned}$$

In the main result, that is Theorem 3.1.1, we claim that there exists some crucial values λ^*, λ^{**} and $\overline{\lambda}$, with $0 < \lambda^* \leq \lambda^{**} \leq \overline{\lambda}$ such that

(i) problem $(\mathscr{P}_2)$ possesses only the trivial solution if $\lambda < \lambda^*$;

(ii) problem $(\mathscr{P}_2)$ admits a nontrivial non–negative entire solution if and only if $\lambda \geq \lambda^{**}$.

(iii) problem $(\mathscr{P}_2)$ admits at least two nontrivial non–negative entire solutions for all $\lambda > \overline{\lambda}$

Also in this case, we can investigate if under suitable assumptions it is possible that $\lambda^* = \lambda^{**}$.

Another open question related to this problem is extending the results to the degenerate case. In fact, in [78] we cover only the non–degenerate case, in order to overcome some technical difficulties due to the Kirchhoff structure of the problem.

6.3 Problem $(\mathscr{P}_3)$

Chapter 4 presents some results on the asymptotic stability of solutions for problem

$$(\mathscr{P}_3) \qquad \begin{cases} u_{tt} + (-\Delta)_p^s u + \mu|u|^{p-2}u + \varrho(t)M([u]_{s,\Omega}^p)|u_t|^{p-2}u_t \\ \qquad +Q(t,x,u,u_t) + f(t,x,u) = 0 & \text{in } \mathbb{R}_0^+ \times \Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}_0^+ \times (\mathbb{R}^n \setminus \Omega). \end{cases}$$

In the paper [79], we do not need in general the non–degeneracy of the problem, but we have to assume it in some applications, when ϱ and the auxiliary function k of Theorem 4.3.1 are related. It can be interesting to see if there are other relations between k and ϱ which allow us to cover also the degenerate case.

Furthermore, in Section 4.5 we study the linear case of $(\mathscr{P}_3)$ and in the second part of the section we consider the Kirchhoff function as a constant. Hence, in this setting, we can investigate if the results can be extended under a more general condition on the function M.

6.4 Problems $(\mathscr{P}_4)$ and $(\mathscr{P}_5)$

The starting point in Chapter 5 is the fractional Schrödinger–Hardy system in $\mathbb{R}^n$

$$(\mathscr{P}_4) \qquad \begin{cases} (-\Delta)^s_m u + a(x)|u|^{m-2}u - \mu \dfrac{|u|^{m-2}u}{|x|^{ms}} &= H_u(x,u,v), \\ (-\Delta)^s_p v + b(x)|v|^{p-2}v - \sigma \dfrac{|v|^{p-2}v}{|x|^{ps}} &= H_v(x,u,v), \end{cases}$$

where μ and σ are real parameters, $n > ps$, with $s \in (0,1)$ and $1 < m \leq p < m^* = mn/(n-ms)$.

In $(\mathscr{P}_4)$, we tried to add a Kirchhoff function multiplying the fractional $\wp$–Laplacian operator. However, the complexity of the system and the Kirchhoff structure of the problem did not allow us to get the desired contradiction in the proof of the key Lemma 5.3.1. For these reasons, an interesting open problem is the study of the related Kirchhoff version of the original fractional Schrödinger–Hardy system $(\mathscr{P}_4)$, that is

$$(\mathscr{P}_4)' \quad \begin{cases} M(\|u\|^m_{E_{m,a}})\big[(-\Delta)^s_m u + a(x)|u|^{m-2}u\big] - \mu \dfrac{|u|^{m-2}u}{|x|^{ms}} &= H_u(x,u,v), \\ M(\|v\|^p_{E_{p,b}})\big[(-\Delta)^s_p v + b(x)|v|^{p-2}v\big] - \sigma \dfrac{|v|^{p-2}v}{|x|^{ps}} &= H_v(x,u,v), \end{cases}$$

The same open problem $(\mathscr{P}_5)'$ could be considered also for $(\mathscr{P}_5)$.

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