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RESEARCH ARTICLE

Strict Nash equilibria in non-atomic games with strict single crossing in players (or types) and actions

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Abstract In this paper, we study games where the space of players (or types, if the game is one of incomplete information) is atomless and payoff functions satisfy the property of strict single crossing in players (types) and actions. Under an additional assumption of quasisupermodularity in actions of payoff functions and mild assumptions on the player (type) space—partially ordered and with sets of uncomparable players (types) having negligible size—and on the action space—lattice, second countable and satisfying a separation property with respect to the ordering of actions—we prove that every Nash equilibrium is essentially strict. Furthermore, we show how our result can be applied to incomplete information games, obtaining the existence of an evolutionary stable strategy, and to population games with heterogeneous players.

Keywords Single crossing · Strict Nash · Pure Nash · Monotone Nash · Incomplete information · ESS

JEL Classification C72

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1 Introduction

Strict Nash equilibrium is a solution concept that possesses desirable features.[1] In this paper, we identify a class of games where every pure-strategy Nash equilibrium is essentially strict. Only pure-strategy Nash equilibria are considered in the paper. Equilibria in mixed strategies might also be considered, but then a proper definition of mixed strategies should be carefully provided, tackling the difficulty of modeling the independence of a continuum of players. We refer the interested reader to Khan et al. (2015) for a possible solution. More precisely, we consider games with an atomless space of players (or types, if the game if of incomplete information), and action sets that are second countable and satisfy a mild separation property.[2,3] In addition, we restrict attention to games where the payoff functions satisfy the strict single crossing property (Milgrom and Shannon 1994) in players (types) and actions. We are aware that this is a severe restriction. However, from the one hand, we relax such an assumption to some extent by considering, first, partial orders on the action sets together with quasisupermodular utility functions and, second, partial orders on the player (type) sets together with a comparability property that limits the numerosity of uncomparable players (types). On the other hand, we think that the strict single crossing property is less demanding when we come to applied models, where instead the possibility to work with action spaces such as the real line (or its intervals) is usually appreciated.

Our main contribution is the identification of conditions that guarantee that every Nash equilibrium is essentially strict (Theorem 1). However, the same conditions do not guarantee that a Nash equilibrium actually exists. To obtain existence of essentially strict Nash equilibria, one can apply our result together with one of the many equilibrium existence theorems that the literature provides. Actually, we follow this line in Sect. 4, where in Sects. 4.1 and 4.2 we provide applications of our main result to incomplete information games and large games. In particular, we show the existence of an evolutionarily stable strategy in a general class of incomplete information games, and strict Nash equilibrium in a class of population games with heterogenous players.

The paper is organized as follows. In Sect. 2, we introduce the assumptions. In Sect. 3, we state our main result. We conclude with Sect. 4, where we provide a discussion, first showing how to combine our main result with existence theorems and then commenting on the assumptions and the findings. The "Appendix" collects one technical lemma (Lemma 1), its proof, and the proof of Proposition 2.

2 Assumptions

Let us consider a non-atomic game $\Gamma = \langle I, \{(T_i, \mathcal{T}_i, \tau_i)\}_{i\in I}, \{A_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$, where:

[1] When working with a finite set of actions, strict Nash equilibria have been proven to be evolutionary stable (see, e.g., Crawford 1990) and asymptotically stable (see, e.g., Ritzberger and Weibull 1995).

[2] Second countability implies a cardinality less than or equal to the cardinality of the continuum.

[3] The separation property that we assume ensures that every two actions that can be strictly ordered can also be separated by a third action not greater than the largest of the two.

- I is a finite set of player groups or institutions;[4]
- for all $i \in I$, $(T_i, \mathcal{T}_i, \tau_i)$ is an atomless probability space with T_i set of players for group/institution i, $\mathcal{T}_i$ σ-algebra and τ_i probability measure;[5]
- for all $i \in I$, A_i is the set of actions for players in group i;
- for all $i \in I$, $u_i : T_i \times F \to \mathbb{R}$ is the utility function for all players of group i, where $F = \prod_{j \in I} \prod_{t \in T_j} A_j$.

We call $f \in F$ a profile of actions, since it maps, for all $i \in I$, every player $t \in T_i$ into an action $f_t \in A_i$.[6]

We denote with f_{-t} the restriction of f to $F_{-t} = \prod_{j \in N} \prod_{t' \in T_j, t' \neq t} A_j$, and we call it a profile of actions by players other than t.[7] We write $u_i(t, a, f_{-t})$ to indicate the utility accruing to player $t \in T_i$ if she chooses action $a \in A_i$ and faces a profile of actions f_{-t}.

We now introduce assumptions on $\{(T_i, \mathcal{T}_i, \tau_i)\}_{i \in I}$ (collected in AT), on $\{A_i\}_{i \in I}$ (collected in AA), and on $\{u_i\}_{i \in I}$ (collected in AU).

Assumption (AT). For all $i \in I$:

AT1 $(T_i, \leq_i^T)$ is a partial order;

AT2 for every $T' \subseteq T_i$ such that there do not exist $t, t' \in T'$ with either $t \leq_i^T t'$ or $t' \leq_i^T t$, we have that (1) $T' \in \mathcal{T}_i$, and (2) $\tau_i(T') = 0$.

Assumption AT2 provides a bound on the cardinality of sets of uncomparable players, basically requiring for each T_i that any subset of players such that every pair is uncomparable has negligible size. Indeed, the possibility that some players are not comparable is left open by AT1, since the order may not be total. We observe that AT2 is trivially satisfied when $(T_i, \leq_i^T)$ is a linear order. More interestingly, AT2 allows us to consider other cases that might be of interest in applications. For instance, think of T_i as made of a finite or countably infinite number of populations, where comparability is within each population, but not across populations. This is not allowed if T_i is a linear order, while it is compatible with our assumption. Moreover, AT2 is satisfied if, for every $i \in I$, T_i is made by a subset of a multidimensional Euclidean space, as shown in the Proof of Proposition 2.

Assumption (AA). For all $i \in I$:

AA1 $(A_i, \leq_i^A)$ is a lattice, i.e., for every two actions $a, a' \in A_i$, there exists the least upper bound $a \vee a'$, and the greatest lower bound $a \wedge a'$;

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4 Here we follow the labeling proposed by Khan and Sun (2002), which allows to encompass both games with many players and games with incomplete information.

5 For games with incomplete information, the set I of groups/institutions has to be interpreted as the set of players, while the set of players T_i has to be interpreted as the set of types for player $i \in I$.

6 We note that, under this definition of F as uncountable cross product of action sets, measurability issues can emerge. These issues cannot be settled without imposing further structure, that is however unnecessary for our main result. Therefore, we choose to take care of measurability only in the applications of Sect. 4.

7 In case of incomplete information games (where i is intepreted as a player and T_i as her set of types), player i has already known her type t when computing expected utility. So, it is redundant to consider the actions that would be taken by types in $T_i \backslash \{t\}$, and hence, we have to require that $u_i(t, f)$ is constant over the actions chosen by types $t' \in T_i \backslash \{t\}$.

AA2 $(A_i, \mathcal{S}_i)$ is a topological space;
AA3 $(A_i, \mathcal{S}_i)$ is second countable, i.e., there exists a countable base for topology $\mathcal{S}_i$;
AA4 $(A_i, \mathcal{S}_i)$ is such that for every two actions $a, a' \in A_i$, with $a <_i^A a'$, there exists an open set $S \in \mathcal{S}_i$ such that $a' \in S$ and $a'' \notin S$ for every $a'' \leq_i^A a$.

Beyond imposing a lattice structure (AA1) and a topological structure (AA2) on the action space, AA contains two further topological properties: AA3, which is a standard assumption that imposes a bound on the topological size of the space, and AA4, which is about order separation with respect to the lattice structure and turns out to be a strengthening of the axiom of separation T0.[8]

Assumption (AU). For all $f \in F$, $i \in I$, $t, t' \in T_i$, and $a, a' \in A_i$:

AU1 u_i is quasisupermodular in actions, i.e., $u_i(t, a, f_{-t}) \geq u_i(t, a \wedge a', f_{-t})$ implies $u_i(t, a \vee a', f_{-t}) \geq u_i(t, a', f_{-t})$, and $u_i(t, a, f_{-t}) > u_i(t, a \wedge a', f_{-t})$ implies $u_i(t, a \vee a', f_{-t}) > u_i(t, a', f_{-t})$;
AU2 u_i satisfies strict single crossing in players and actions, i.e., for all $t <_i^T t'$ and $a <_i^A a'$, we have that $u_i(t, a', f_{-t}) \geq u_i(t, a, f_{-t})$ implies $u_i(t', a', f_{-t'}) > u_i(t', a, f_{-t'})$.

Assumption AU1 is always satisfied when A_i is a total order, while it implies a sort of complementarity in own actions when A_i is a partial order, as for instance when $A_i = [0, 1]^k$ for some $k > 1$. Assumption AU2, instead, introduces a sort of complementarity between actions and players.[9]

Finally, we present some further definitions. A profile of actions $f \in F$ is said to be (essentially) a *Nash equilibrium in pure strategies*, or simply a *Nash equilibrium*, if, for all $i \in I$, for τ_i-almost all $t \in T_i$, we have that $u_i(t, f_t, f_{-t}) \geq u_i(t, a, f_{-t})$ for all $a \in A_t$. A Nash equilibrium f is said to be *essentially strict* if, for all $i \in I$, for τ_i-almost all $t \in T_i$, we have that $u_i(t, f_t, f_{-t}) > u_i(t, a, f_{-t})$ for $a \neq f_t$ such that $a_i \in A_i$, while it is said to be *monotone* if, for all $i \in I$, for all $t, t' \in T_i$, we have that $t' >_i^T t$ implies $f_{t'} \geq_i^A f_t$.

3 Main result

We are ready to state our main result.

Theorem 1 *Let Γ be a game that satisfies AT, AA, and AU. Then, every Nash equilibrium of Γ is essentially strict and monotone.*

[8] T0 requires that any two distinct points in a set are topologically distinguishable, i.e., the sets of neighborhoods of the two points differ one from the other.

[9] We note that AU2 is slightly different from the standard definition of strict single crossing property since the profile of opponents' actions, which is a third argument of function u in addition to t and f_t, is not exactly the same in f_{-t} and $f_{-t'}$: Indeed, the behavior of players different from t and t' is the same, while the behavior of t is considered in f_{-t} but not in $f_{-t'}$, and the behavior of t' is considered in $f_{-t'}$ but not in f_{-t}. This difference disappears if, for instance, we assume individual negligibility (see discussion at the end of Sect. 3) or if we constrain players to care only about actions of groups/institutions different from theirs (as it happens, e.g., in games with incomplete information).

Proof We first show that every Nash equilibrium is essentially strict. Let $R_{i,t}(f)$ denote the set of best replies to f for player $t \in T_i$, namely $R_{i,t}(f) = \{a \in A_i : u_i(t, a, f_{-t}) \geq u(t, a', f_{-t})$ for all $a' \in A_i\}$. By Lemma 1 (see "Appendix 1"), we know that, for all $i \in I$, the set $\{t \in T_i : ||R_{i,t}(f)|| > 1\}$ is a countable union of sets having measure zero. Since the countable union of zero-measure sets has measure zero, we can conclude that $\tau_i(\{t \in T_i : ||R_{i,t}(f)|| > 1\}) = 0$ for all $i \in I$. This, together with the observation that when f is a Nash equilibrium we have $||R_{i,t}(f)|| > 0$ for τ_i-almost all $t \in T_i$ and for all $i \in I$, implies that $u_i(t, f_t, f_{-t}) > u_i(t, a, f_{-t})$ for τ_i-almost all $t \in T_i$ and for all $i \in I$.

We now show that every Nash equilibrium is monotone. Suppose that $t' >_i^T t$, $a \in R_{i,t}(f)$, $a' \in R_{i,t'}(f)$ and, ad absurdum, $a \not\leq_i^A a'$. Since $a \in R_t(f)$, we have that $u(t, a, f_{-t}) \geq u(t, a \wedge a', f_{-t})$, but then $u(t, a \vee a', f_{-t}) \geq u(t, a', f_{-t})$ by quasisupermodularity in actions, and $u(t', a \vee a', f_{-t}) > u(t', a', f_{-t})$ by strict single crossing property in players and actions and $a \vee a' \neq a'$, which in turn comes from $a \not\leq_i^A a'$. We simply observe that $u(t', a \vee a', f_{-t}) > u(t', a', f_{-t})$ is in contradiction with $a' \in R_{i,t}(f)$. □

The fact that f is essentially strict follows from $(T_i, \mathcal{T}_i, \tau_i)$ being atomless for all $i \in I$ and from the set of weakly best responders being countable. Then, a straightforward application of the property of strict single crossing in players and actions allows establishing the monotonicity between players and actions in Nash equilibria—this result following basically from Theorem 4' of Milgrom and Shannon (1994).

Let us conclude with a remark on players' negligibility. In Theorem 1, utility depends on the actions of each single player $t \in T_i$, $i \in I$. We did this in order to state our findings in a setting which allows for a general form of utility functions. However, we note that when we have an atomless space of players, it may be reasonable to impose that any single player $j \neq t$ is negligible in terms of t's utility. This assumption is particularly reasonable if one also assumes continuity of the utility function (see the discussion in Khan and Sun 2002, Section 2). To introduce negligibility in our framework, it suffices to impose that, for all $i \in I$, the utility function u_i is such that whenever $f, f' \in F$ agree on a set of measure one according to τ_i, we have that $u_i(t, f) = u_i(t, f')$ for every $t \in T_i$, such that $f_t = f_{t'}$. We observe that such kind of players' negligibility is implied in the applications of Sect. 4.

4 Discussion

The celebrated result in Harsanyi (1973) says that independently perturbing the payoffs of a finite normal form game produces an incomplete information game with a continuum of types where all equilibria are essentially pure and essentially strict [10] and that for any regular equilibrium of the original game and any sequence of perturbed games

[10] Note that strict Nash equilibria are called strong Nash equilibria in Harsanyi (1973).

converging to the original one, there is a sequence of essentially pure and essentially strict equilibria converging to the regular equilibrium.[11,12]

For Theorem 1 to have some bite, it needs to be coupled with a result guaranteeing the existence of a pure-strategy Nash equilibrium. The literatures on incomplete information games and large games have provided several of such existence results. We first discuss some known existence results in non-atomic games. Then, in Sects. 4.1 and 4.2, we illustrate how our contribution can be used to shed light on the strictness of Nash equilibria in applications to incomplete information games and large games, respectively. Unless otherwise specified, any topological space in this section is understood to be equipped with its Borel σ-algebra, and the measurability is defined based on it. Finally, in Sect. 4.3, we comment on the assumptions used in the paper, arguing in favor of their tightness.

The use of single crossing properties is not new in the literature on games with many player types. Athey (2001) analyzes games of incomplete information where each agent has private information about her own type, and the types are drawn from an atomless joint probability distribution. The main result establishes the existence of pure Nash equilibria under an assumption called single crossing condition for games of incomplete information, which is a weak version of the single crossing property in Milgrom and Shannon (1994).[13] In Sect. 4.3, we argue that such a property is not a sensible generalization for our purposes.

In a finite-player incomplete information game with diffused information, if in addition players' information is independent (instead of assuming an order structure), the existence of a pure Nash equilibrium can be established similarly to the one in a large game (with a non-atomic space of players). It is now well recognized (see Khan et al. 2006) that the purification principle due to Dvoretzky et al. (1951) guarantees the existence of pure Nash equilibria in non-atomic games[14] when the action space is finite as, for example, in large games like Schmeidler (1973), or in games with diffused information as in Radner and Rosenthal (1982) and Milgrom and Weber (1985) (see Khan and Sun 2002, for a survey on games with many players).[15,16] Existence of pure Nash equilibria does not extend, however, to general games. For action spaces that

[11] See also Dubey et al. (1980) for a related use of strict equilibria in large games.

[12] The work of Harsanyi (1973) has been extended by a series of contributions providing more general conditions for the existence of pure equilibria, but disregarding the issue of approachability and the existence of strict equilibria (see Morris 2008 and references there in).

[13] Reny and Zamir (2004) prove the existence of pure- strategy Nash equilibria under a slightly weaker condition. McAdams (2003) further extends the analysis to multidimensional type spaces and action spaces, while Reny (2011) extends it to more general partially ordered type spaces and action spaces.

[14] Interest in games with many players has recently spanned across different settings (see, e.g., Alós-Ferrer and Ritzberger 2013, for extensive form games and Balbus et al. 2013, for games with differential information), and different notions of equilibrium (see, e.g., Correa and Torres-Martínez 2014, can exists when the make for essential equilibria).

[15] Mas-Colell (1984) deals with the issue of Schmeidler (1973) using a different approach based on distributions rather than measurable functions. See Khan et al. (2013b) for a recent discussion of related issues.

[16] Approximated versions of the result in Schmeidler (1973) have been given for a large but finite number of players (Rashid 1983; Carmona 2004, 2008).

are countable and compact, conditions for the existence of pure Nash equilibrium are given in Khan and Sun (1995) and then generalized in Yu and Zhang (2007). When the action space is an uncountable compact metric space, saturated probability spaces can be used to guarantee the existence of a pure-strategy Nash equilibrium, as shown in Keisler and Sun (2009) and Khan et al. (2013a).[17]

4.1 An application to incomplete information games

We now show how Theorem 1 can be used to shed light on the strictness of a Nash equilibrium in a Bayesian setting. We use the setup given by McAdams (2003),[18] which is a generalization of the one in Athey (2001). More precisely, we consider the incomplete information game $\Gamma^I = \langle I, ([0,1]^h, \phi), A, \{u_i\}_{i \in I} \rangle$, where:

- I is the set of players with cardinality $||I|| = n \in \mathbb{N}$;
- for all $i \in I$, $([0,1]^h, \phi)$ describes the h-dimensional common type space, with $\phi : \mathbb{R}^{nh} \to \mathbb{R}_{++}$ the positive and bounded joint density on type profiles;
- for all $i \in I$, $A \subset \mathbb{R}^k$ is the set of actions for types of player i,[19] with A being either a finite sublattice with respect to the product order or $[0,1]^k$;
- for all $i \in I$, $u_i^I(t_i, a_i, \boldsymbol{\alpha_{-i}}) = \int_{[0,1]^{h(n-1)}} U_i(a_i, \boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}))\phi(\mathbf{t}_{-i}|t_i)\mathrm{d}\mathbf{t}_{-i}$ is the utility function for all types of i, where $\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i})$ is the vector of others' actions as a function of their type, $\mathbf{t}_{-i}$ is the vector of others' types, $\phi(\mathbf{t}_{-i}|t_i)$ is the conditional density of $\mathbf{t}_{-i}$ given t_i, and U_i is bounded, Lebesgue measurable and, if $A = [0,1]^k$, also continuous in $\mathbf{a} \in A^n$.

In Γ^I a strategy for player i can be described by function $\alpha_i : [0,1]^h \to A$. So, we can say that a strategy profile $(\alpha_1, \ldots, \alpha_n)$ is a *Nash equilibrium* of game Γ^I if it induces a profile of actions such that for all $i \in I$, for all $t \in [0,1]^h$, $u_i(t, \alpha_i(t), \boldsymbol{\alpha_{-i}}) \geq u_i(t, a, \boldsymbol{\alpha_{-i}})$ for all $a \in A$.

By construction, Γ^I satisfies AA and AT. So, if Γ^I also satisfies AU, then by virtue of our Theorem 1 every Nash equilibrium of Γ^I is essentially strict, and monotone in types and actions. Moreover, existence of a Nash equilibrium follows from Theorem 1 in McAdams (2003) that can be applied since AU2 implies the single crossing condition—which is required by the Theorem.

Perhaps more interestingly, we can use the setup of incomplete information games to show what Theorem 1 can say from the perspective of evolutionary game theory.[20] Indeed, although the notion of evolutionarily stable strategy remains a prominent solution concept in evolutionary game theory, its use has some shortcomings when

[17] See Carmona and Podczeck (2009) for a discussion on the relationship between alternative formalizations of non-atomic games and existence results, with a focus on large games. See also Fu and Yu (2015) for a discussion of the connection between the class of large games and the class of finite-player Bayesian games.

[18] McAdams (2006) applies and extends this setup to prove existence of pure Nash equilibria in multiunit auctions.

[19] As noted by McAdams (2003), the assumptions of a common support for types and a common set for actions are just for notational simplicity and can be safely removed.

[20] Evolution in the context of Bayesian games is analyzed in Ely and Sandholm (2005) and Sandholm (2007).

continuous strategy spaces are employed.[21] If an order structure is imposed on types, our Theorem 1 can allow to tackle the issue. This follows a seminal idea in Riley (1979), where incomplete information and a form of the strict single crossing property are used to show existence of an evolutionarily stable strategy in the "war of attrition".

For this purpose, we restrict attention to a game Γ^I that is symmetric, i.e., we focus on game $\Gamma^{IS} = \langle I, ([0,1]^h, \phi), A, u \rangle$. We also provide some further useful notation and definitions.

The following expression denotes ex-ante utility for a player choosing strategy α when all other players choose strategy α':

$$V(\alpha, \alpha') = \int_{[0,1]^h} \left(\int_{[0,1]^{h(n-1)}} U(\alpha(t), \boldsymbol{\alpha}'_{-i}(\mathbf{t}_{-i}))\phi(\mathbf{t}_{-i}|t)\mathrm{d}\mathbf{t}_{-i} \right) \phi_i(t)\mathrm{d}t,$$

where $\phi_i(t)$ is the marginal density function of types for player i.

Given two strategies α, α', we define $D(\alpha, \alpha')$ as the set of types that pick different actions in α and α', i.e., $D(\alpha, \alpha') = \{t \in [0,1]^h : \alpha(t) \neq \alpha'(t)\}$.

The following definition adapts the standard definition of evolutionarily stable strategy to our setup. A strategy α is an *evolutionarily stable strategy* (henceforth, ESS) if and only if there exists $\epsilon > 0$ such that, for all α' such that $\int_{D(\alpha,\alpha')} \phi_i > 0$:

$$(1-\epsilon)V(\alpha, \alpha) + \epsilon V(\alpha, \alpha') > (1-\epsilon)V(\alpha', \alpha) + \epsilon V(\alpha', \alpha').$$

Basically, the above definition requires that a strategy performs strictly better than any invading strategy that differs non-negligibly from the incumbent strategy.

While an evolutionarily stable strategy may not exist in general, we are able to prove the following result (see "Appendix 2" for the proof).

Proposition 2 *Suppose Γ^{IS} satisfies AU. Then, (1) every pure-strategy Nash equilibrium is an evolutionarily stable strategy, and (2) an evolutionarily stable strategy exists.*

We observe that our Proposition 2 is not implied by the Harsanyi's purification theorem, which applies only to games with a finite number of strategies for each player, while we allow for continuous strategies as well.

4.2 An application to large games

A pure Nash equilibrium is not necessarily a strict Nash equilibrium, so our Theorem 1 can be usefully employed to establish Nash strictness in games where this is a desirable property (e.g., in games where the local stability of a Nash equilibrium is a crucial property). Below, we provide an example of such applicability.

Consider the following game, which is an instance of the class of games considered in Khan et al. (2013a) (see discussion at p. 1130), and that represents a slight generalization of a static population game (see Sandholm 2010, for a formal definition

[21] Alternative notions of evolutionary stability have been proposed in the literature (Vickers and Cannings 1987; Bomze and Pötscher 1989; Oechssler and Riedel 2001, 2002).

of population games). There is a large population of heterogeneous players whose characteristics consist of both an individual payoff structure and an ordered numerical trait, with a player's payoff depending on own action and societal summary of actions traits. In particular, a player's payoff depends on her own action and type as well as the sum of the traits of the players choosing each action.[22] Formally, consider the game $\Gamma^P = \langle([\underline{t}, \bar{t}], \phi), (B, \beta), A, u^P\rangle$ where:

- there is a unit-mass population of players distributed over $[\underline{t}, \bar{t}]$ according to the positive and bounded probability density ϕ;
- $B = \{b_1, \ldots, b_n\}$ is a finite and totally ordered set of traits, with $\beta : [\underline{t}, \bar{t}] \to B$ a measurable function that assigns each player to a trait;
- $A = \{1, \ldots, m\}$ is a finite and totally ordered set of actions, common to all players;
- $u^P(t, a, \boldsymbol{\alpha}) = U(t, a, (\sigma_{11}, \ldots, \sigma_{mn}))$ is agents' utility function, which we assume to be measurable in t and continuous in $(\sigma_{11}, \ldots, \sigma_{mn})$, and where $\boldsymbol{\alpha} : [\underline{t}, \bar{t}] \to A$ is a measurable function representing the actions chosen by every player in the population, and $\sigma_{jk} = \int_{(\alpha,\beta)^{-1}(j,b_k)} \phi t$ measures the amount of players with trait b_k who play action $j \in A$.

We observe that if, in addition to AA and AT which are satisfied by construction, Γ^P also satisfies AU, then Theorem 1 implies that every Nash equilibrium of Γ^P is essentially strict, and monotone in players and actions.[23] So, we know that all Nash equilibria of Γ^P are locally stable with respect to dynamics typically applied in population games (see e.g., Sandholm 2015).

We think that considering the heterogeneity of characteristics in a population is a natural addition to population games. Also, assuming the strict single crossing property in players and actions appears to us, at least in some cases, a reasonable hypothesis. Think of this variant of a congestion game, where the trait is the length of the car possessed, and the congestion along a route depends on the overall length of cars in that route. If a longer route is preferred by the owner of some car, then it means that the shorter route has heavier traffic. Hence, it is reasonable to assume that the owners of longer cars prefer a fortiori the shorter route, since a larger car typically performs relatively worse under heavy traffic.

4.3 Discussion of assumptions

Negligibility of sets of uncomparable players (AT2) This assumption cannot be dispensed with, in the sense that a positive measure of uncomparable players would allow the existence of Nash equilibria that are not essentially strict. Indeed, if there exists a non-negligible set of players such that every pair cannot be ordered, then the strict single crossing property cannot be employed to rule out that all such players are weakly best responders in equilibrium, and therefore, Nash equilibria need not be essentially strict. The following example illustrates why. Let $||I|| = 1$, and let the set of actions

[22] This last assumption can be easily generalized to any form of trait aggregation, in the same way as it is typically done for aggregative games (see, e.g., Acemoglu and Jensen 2013).

[23] We also note that the existence of a Nash equilibrium is not an issue in this game, e.g., one can invoke Theorem 1, point (i), in Khan et al. (2013a).

A be equal to the real segment [0, 1]. Also, let the set T be such that no $t, t' \in T$ are comparable, so that AT1 is trivially satisfied while AT2 fails. Finally, suppose that $u_i(t, f) = \tau(\{t' : f_{t'} = f_t\})$, meaning that t's payoff only depends on the fraction of players coordinating on her action f_t. It is straightforward to see that any profile where a measure of $\tau(T)/k$ players coordinate on k distinct actions (with k a natural number) is a Nash equilibrium, since each t obtains a payoff of $\tau(T)/k$ which cannot be improved upon by deviating. However, for $k \geq 2$, all $t \in T$ are indifferent between any of the k actions played, and so the Nash equilibrium is not essentially strict.

Separability versus second countability (AA3) A space is called separable if it contains a countable dense subset. Separability is a topological property which is weaker than second countability but plays a similar role: It constrains the topological size of the space.

However, if we assume that the action sets are separable instead of second countable, then our results fail. The following example, which is a modification of a standard argument to illustrate that a separable space need not be second countable, shows that if we replace second countability with separability then there may exist Nash equilibria that are not essentially strict. We consider a unique group of players, and we let the set T be the real line, denoted with $\mathbb{R}$. We let the action set A be the Cartesian product $\mathbb{R} \times \{0, 1\}$. We give A the lexicographic order, i.e., $(r, i) < (s, j)$ if either $r < s$ or else $r = s$ and $i < j$. For every profile of actions f, t's utility function is $u(t, f) = -(t - f_t')^2$, where $f_t' = s$ if $f_t = (s, i)$. In the order topology, A is separable: The set of all points $(q, 0)$ with q rational is a countable dense set. However, f such that $f_t = (t, 0)$ for all $t \in T$ is a Nash equilibrium that is not essentially strict since every agent t is indifferent between $(t, 0)$ and $(t, 1)$.

Axioms of separation (T0, T1) versus order separation (AA4) Intuitively, our assumption on order separation ensures that different weakly best responders can be assigned to actions that are substantially different, in the sense that each action can be associated with a distinct base set. Then, second countability of the action set ensures that this function relating actions to base sets is enumerable.

One might hope to weaken our assumption to something that is more in line with standard separation axioms (like $T0$ or $T1$): For all $a > a'$, there exists an open set $S(a, a')$ such that $a \in S(a, a')$ and $a' \notin S(a, a')$. However, we stress that this attempt would contrast with our technique of proof. Indeed, following the Proof of Lemma 1 (see the "Appendix"), a t that is a weakly best responder might be associated with a set $\widehat{S}$ obtained as $\bigcap_{t' \in R_{i,t}(f), t >_i^T t'} S(g_{i,1}(t), g_{i,1}(t'))$. But then an infinite intersection of open sets need not be open, and this does not allow us to conclude that a base set exists that is included in $\widehat{S}$ and contains the action $g_{i,1}(t)$.

Single crossing versus strict single crossing (AU2) Games of incomplete information are a very important class of games where single crossing properties are usually assumed in order to prove existence of pure Nash equilibria. In these cases, we can apply our Theorem 1 to obtain the existence of an essentially strict Nash equilibrium (see Sect. 4.1). We stress that this result is based on a strict version of the single crossing property, while existence results in games of incomplete information (Athey 2001; McAdams 2003) use weaker assumptions. In particular, they are weaker under two respects. First, they assume single crossing instead of strict single crossing. Second, they require that the property of single crossing holds on a smaller domain: for each player, whenever all other players adopt strategies such that higher types take higher actions. Therefore, one may wonder whether our results still hold if we consider each of the two weakenings of strict single crossing. With respect to the first weakening, the following straightforward counterexample shows that single crossing is not enough. Assume that every agent has a constant utility function, so that everyone is always indifferent between any of her actions. Single crossing property is satisfied, and whatever profile of actions is a weak Nash equilibrium. This trivial example also shows that we cannot recover our main result even if we replace the property of single crossing with the stronger one of increasing differences—i.e., for all $f \in F, i \in I, t' >_i^T t$ and $a' >_i^A a$, we have that $u(t, a', f_{-t}) - u(t, a, f_{-t}) \leq u(t', a', f_{-t'}) - u(t', a, f_{-t'})$.

With respect to the second weakening, we observe that restricting the domain to profiles that are monotone in types and actions for other players is a clever generalization of single crossing when the purpose is to prove the existence of pure Nash equilibria. However, a strict version of this weaker property of single crossing does not work when we want to show that *every* Nash equilibrium is essentially strict. The reason is that it would allow the existence of some weak Nash equilibrium with a profile of actions for which no property of strict single crossing must hold.

Strict increasing difference versus strict single crossing (AU2) One may wonder whether the result in Theorem 1 can be refined to prove strict monotonicity instead of monotonicity. It turns out that this is not the case, even if we adopt the stronger property of strict increasing differences in players (or types) and actions—i.e., for all $f \in F, i \in I, t' >_i^T t$ and $a' >_i^A a$, we have that $u(t, a', f_{-t}) - u(t, a, f_{-t}) < u(t', a', f_{-t'}) - u(t', a, f_{-t'})$—instead of strict single crossing. The following example illustrates why. Let $||I|| = 1$ and let both set T and set A be equal to the real segment [0, 1]. For every profile of actions f, the utility function of t is $u(t, f) = (1 + t) f_t$. It is clear that there exists a unique Nash equilibrium where everybody plays action 1. Hence, monotonicity holds, but strict monotonicity does not.

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Appendix 1: Lemma 1 and its proof

A key result for the Proof of Theorem 1 is that any set of weakly best responders is a countable union of sets having measure zero. Lemma 1 below provides such result.

The logic of the Proof of Lemma 1 goes as follows. The joint use of quasisupermodularity in actions (AU1) and strict single crossing in players and actions (AU2) is similar to that in Theorem 4 of Milgrom and Shannon (1994), and it allows to arrange multiple best replies of different players in a linear order. The crucial economic assumption is the strict single crossing property in players and actions, which implies that the sets of weakly best replies of any two distinct players intersect at most at an extreme point and hence are—roughly speaking—rather separated one from the other. The technical assumptions on countability (AA3) and separation (AA4) complete the job, allowing at most a countable number of such sets (see Sect. 4.3 for a discussion on the importance of the countability and separation properties). Therefore, there can exist only a countable number of comparable players that are weakly best responders; for any such player, there can be many (even uncountable) players that are all uncomparable and weakly best responders, but for the comparability assumption (AT2) their measure is null. This leads to conclude that the set of weakly best responders is formed by countably many sets having measure zero, and hence, its measure is zero as well.

Preliminarily, we define $R_{i,t}(f)$ as the set of best replies to f for $t \in T_i$, namely $R_{i,t}(f) = \{a \in A_i \,:\, u_i(t, a, f_{-t}) \geq u(t, a', f_{-t})$ for all $a' \in A_i\}$.

Lemma 1 *Let Γ be a game that satisfies AT, AA, and AU. Then, for every $i \in I$, $\{t \in T_i \,:\, ||R_{i,t}(f)|| > 1\}$ is a countable union of sets having measure zero.*

Proof This is the outline of the proof. For a generic $i \in I$, first we define a function g_i that maps every $t \in \{t \in T_i \,:\, ||R_{i,t}(f)|| > 1\}$ into a pair (a, a') of her best replies, then we define a function h_i, and we use it to assign (a, a') to a base set. We show that function h_i is injective and that function g_i is such that any set of players assigned to the same pair of actions has measure zero. Finally, we invoke the fact that there exists only a countable number of base sets to obtain the desired result.

For each $i \in I$, we consider the partial orders assumed in AA1 (lattice structure) and AT1 (partial ordering) and we take a function $g_i \,:\, \{t \in T_i \,:\, ||R_{i,t}(f)|| > 1\} \to A_i^2$ such that $g_i(t) = (g_{i,0}(t), g_{i,1}(t))$ with $g_{i,0}(t), g_{i,1}(t) \in R_{i,t}(f)$, $g_{i,0}(t) <_i^A g_{i,1}(t)$, and $g_{i,1}(t) \leq_i^A g_{i,0}(t')$ for $t' >_i^T t$. The following two arguments show that such a function exists for each $i \in I$. First, $a \in R_{i,t}(f)$ and $a' \in R_{i,t}(f)$ imply $a \vee a' \in R_{i,t}(f)$, so that we can set $g_{i,0}(t) = a$ and $g_{i,1}(t) = a \vee a'$, with $a \vee a'$ existing thanks to AA1 (lattice structure). In fact, $u_i(t, a, f_{-t}) \geq u_i(t, a \wedge a', f_{-t})$ since $a \in R_{i,t}(f)$, and hence, $u_i(t, a \vee a', f_{-t}) \geq u_i(t, a', f_{-t})$ by AU1 (quasisupermodularity in actions), which in turn implies that $u_i(t, a \vee a', f_{-t}) = u_i(t, a, f_{-t}) = u_i(t, a', f_{-t})$ since $a \in R_{i,t}(f)$ and $a' \in R_{i,t}(f)$. Second, $a \in R_{i,t}(f)$ and $a' \in R_{i,t'}(f)$ for $t' >_i^T t$ imply $a \leq_i^A a'$. This is true since $u_i(t, a, f_{-t}) \geq u_i(t, a \wedge a', f_{-t})$ due to $a \in R_{i,t}(f)$, and hence, $u_i(t, a \vee a', f_{-t}) \geq u_i(t, a', f_{-t})$ by AU1 (quasisupermodularity in actions), and therefore, $u_i(t', a \vee a', f_{-t}) > u_i(t', a', f_{-t})$ by AU2 (strict single crossing in players and actions), with $a \wedge a'$ existing thanks to AA1 (lattice structure).

For all $i \in I$, by AA2 (topology structure), A_i has a topology and by AA3 (second countability) we can take a countable base $\mathcal{B}_i$ for such a topology. For each $i \in I$, we take a function $h_i \,:\, g_i(\{t \in T_i \,:\, ||R_{i,t}(f)|| > 1\}) \to \mathcal{B}_i$ such that $a_1 \in h_i(a_0, a_1)$ and $a \notin h_i(a_0, a_1)$ for all $a \leq_i^A a_0$. To see that such a function h_i exists, note that by AA4 (order separation) for each $(a_0, a_1) \in g_i(\{t \in T_i \,:\, ||R_{i,t}(f)|| > 1\})$ there exists some open set $S_{a_1} \subset A_i$ such that $a_1 \in S$ and $a \notin S$ for all $a \leq_i^A a_0$; since $\mathcal{B}_i$ is a base, there must exist some $B_{a_1} \in \mathcal{B}_i$ such that $a_1 \in B_{a_1}$ and $B_{a_1} \subseteq S_{a_1}$. We set $h_i(a_0, a_1) = B_{a_1}$.

We check that, for all $i \in I$, g_i is such that, for all $(a, a') \in A_i^2$, $g_i^{-1}(a, a')$ has measure zero. For all $t, t' \in \{t \in T_i : ||R_{i,t}(f)|| > 1\}$, $t <_i^T t'$, we have that $g_{i,0}(t) < g_{i,1}(t) \leq g_{i,0}(t') < g_{i,1}(t')$ from the definition of function g_i. Therefore, $t, t' \in g_i^{-1}(a, a')$ implies $t \not\lesssim_i^T t'$ and $t' \not\lesssim_i^T t$, and AT2 (negligibility of sets of uncomparable players) guarantees that $\tau_i(g_i^{-1}(a, a')) = 0$.

We check that, for all $i \in I$, h_i is injective. For all $(a_0, a_1), (a'_0, a'_1) \in g_i(\{t \in T_i : ||R_{i,t}(f)|| > 1\})$, $(a_0, a_1) \neq (a'_0, a'_1)$, we know that either $a_0 < a_1 \leq a'_0 < a'_1$ or $a'_0 < a'_1 \leq a_0 < a_1$. Suppose, without loss of generality, that $a_0 < a_1 \leq a'_0 < a'_1$. Then, by the definition of function h_i, we know that $a_1 \in h_i(a_0, a_1)$, $a'_1 \in h_i(a'_0, a'_1)$, and $a_1 \notin h_i(a'_0, a'_1)$ since $a_1 \leq a'_0$. Hence, $h_i(a_0, a_1) \neq h_i(a'_0, a'_1)$.

Therefore, $g \circ h$ maps $\{t \in T_i : ||R_{i,t}(f)|| > 1\}$ into $\mathcal{B}_i$ in such a way that for every $B \in \mathcal{B}_i$ such that there exists $t \in T_i$ with $h(g(t)) = B$, we have that $\tau_i(\{t \in T_i : h(g(t)) = B\}) = 0$. Since $\mathcal{B}_i$ is countable, we can conclude that $\{t \in T_i : ||R_{i,t}(f)|| > 1\}$ is the countable union of sets having measure zero. □

Appendix 2: Proof of Proposition 2

We start by checking that Theorem 1 can be applied to Γ^{IS}. Clearly, Γ^{IS} is a special case of Γ^I. First, we note that Γ^I is a specific instance of Γ. To see this, set i's type space $T_i = [0, 1]^h$, with associated probability space $(T_i, \mathcal{T}_i, \tau_i)$ where $\mathcal{T}_i$ is the sigma algebra of all Lebesgue measurable subsets of T_i and measure τ_i is the one induced by ϕ_i, implying that τ_i is atomless since ϕ_i is bounded. Furthermore, set i's action space $A_i = A$. Finally, note that utility u_i^I is a special case of u_i where the utility of type t does not depend on the actions chosen by other types of the same player role.

We next check that all hypotheses of Theorem 1 are satisfied.

AU is satisfied by assumption.

We check AT. Since $[0, 1]^h$ is a partial order, AT1 is satisfied. Take a set $\widehat{T} \subseteq [0, 1]^h$ which is made of types that are all uncomparable. For any $(t_1, t_2, \ldots, t_{h-1}) \in [0, 1]^{h-1}$, there exists at most one $t_h \in [0, 1]$ such that $(t_1, t_2, \ldots, t_{h-1}, t_h) \in \widehat{T}$; otherwise, we would have two elements belonging to $\widehat{T}$ that are comparable. This shows that $\widehat{T}$ is contained in the graph of a function from $[0, 1]^{h-1}$ to $[0, 1]$, which constitutes an hypersurface in $[0, 1]^h$. We know that an hypersurface has Lebesgue measure equal to zero and hence $\widehat{T}$ as well. Therefore, the measure of $\widehat{T}$ according to the marginal density function ϕ_i is null, since the integration of ϕ_i over a zero-measure set is zero. So, AT2 is satisfied.

We check AA. If A is a finite lattice, then AA1–AA4 hold trivially. If $A = [0, 1]^k$, then AA1 and AA2 are satisfied by considering, respectively, the standard order and the Euclidean topology on $[0, 1]^k$. It is well known that the Euclidean space (and any of its subsets) is second countable (it is enough to consider as base the set of all open balls with rational radii and whose centers have rational coordinates). So AA3 is also satisfied. Finally, consider $a, a' \in [0, 1]^k$ such that $a'_i \geq a_i$, $a' \neq a$. Then take an open ball centered at a' with radius lower than the Euclidean distance between a' and a; clearly, a' belongs to the ball, while every $a'' \in [0.1]^k$ such that $a''_i \leq a_i$ does not belong to the ball. This shows that AA4 is satisfied.

So, we can apply Theorem 1 to conclude that every pure-strategy Nash equilibrium must be essentially strict and monotone in types and actions.

Consider now a symmetric pure-strategy Nash equilibrium where every player chooses strategy α. Consider also any strategy α', with $\alpha' \neq \alpha$. We have already shown, by exploiting Theorem 1, that α is essentially strict, and so $u^I(t, \alpha(t), \boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i})) > u^I(t, \alpha'(t), \boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}))$ for almost all $t \in [0, 1]^h$. Therefore,

$$\int_{[0,1]^h} \left(u(t, \alpha(t), \boldsymbol{\alpha}'_{-i}(\mathbf{t}_{-i}))\right) \phi_i(t)\mathrm{d}t > \int_{[0,1]^h} \left(u(t, \alpha'(t), \boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}))\right) \phi_i(t)\mathrm{d}t, \quad (1)$$

which means that $V(\alpha, \alpha) > V(\alpha', \alpha)$. Hence, for ϵ small enough, we can conclude that $(1-\epsilon)V(\alpha, \alpha)+\epsilon V(\alpha, \alpha') > (1-\epsilon)V(\alpha', \alpha)+\epsilon V(\alpha', \alpha')$. We have so established that α is an ESS.

Finally, to show that an ESS exists, we can rely on Theorem 1 in McAdams (2003) that can be applied since AU2 implies the single crossing condition—which is required by the Theorem. Such theorem, if applied to symmetric games, establishes the existence of a symmetric pure-strategy Nash equilibrium.[24] By the previous argument, we conclude that the strategy played in such equilibrium must be an ESS.

References

Acemoglu, D., Jensen, M.K.: Aggregate comparative statics. Games Econ. Behav. **81**, 27–49 (2013)

Alós-Ferrer, C., Ritzberger, K.: Large extensive form games. Econ. Theor. **52**(1), 75–102 (2013)

Athey, S.: Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. Econometrica **69**, 861–889 (2001)

Balbus, Ł., Dziewulski, P., Reffett, K., Woźny, Ł.: Differential information in large games with strategic complementarities. Econ. Theory **59**(1), 201–243 (2013)

Bomze, I., Pötscher, B.: Game Theoretic Foundations of Evolutionary Stability. Springer, Berlin (1989)

Carmona, G.: On the purification of Nash equilibria of large games. Econ. Lett. **85**, 215–219 (2004)

Carmona, G.: Purification of Bayesian–Nash equilibria in large games with compact type and action spaces. J. Math. Econ. **44**, 1302–1311 (2008)

Carmona, G., Podczeck, K.: On the existence of pure-strategy equilibria in large games. J. Econ. Theory **144**, 1300–1319 (2009)

Correa, S., Torres-Martínez, J.P.: Essential equilibria of large generalized games. Econ. Theory **57**(3), 479–513 (2014)

Crawford, V.P.: Nash equilibrium and evolutionary stability in large- and finite-population 'playing the field' models. J. Theor. Biol. **145**, 83–94 (1990)

Dubey, P., Mas-Colell, A., Shubik, M.: Efficiency properties of strategies market games: an axiomatic approach. J. Econ. Theory **22**(2), 339–362 (1980)

Dvoretzky, A., Wald, A., Wolfowitz, J.: Elimination of randomization in certain statistical decision procedures and zero-sum two-person games. Ann. Math. Stat. **22**(1), 1–21 (1951)

Ely, J.C., Sandholm, W.: Evolution in Bayesian games I: theory. Games Econ. Behav. **53**, 83–109 (2005)

Fu, H., Yu, H.: Pareto-undominated and socially-maximal equilibria in non-atomic games. J. Math. Econ. **58**, 7–15 (2015)

Harsanyi, J.C.: Games with randomly disturbed payoffs. Int. J. Game Theory **2**, 1–23 (1973)

Keisler, H.J., Sun, Y.: Why saturated probability spaces are necessary. Adv. Math. **221**(5), 1584–1607 (2009)

[24] Even if we have not found a precise reference, it follows almost directly from the Proof of Theorem 1 in McAdams (2003) that, if we restrict attention to symmetric profiles in a symmetric game, then we are still able to show existence of an isotone pure-strategy equilibrium, which is hence symmetric.

Khan, M.A., Rath, K.P., Sun, Y.: The Dvoretzky–Wald–Wolfowitz theorem and purification in atomless finite-action games. Int. J. Game Theory **34**(1), 91–104 (2006)
Khan, M.A., Rath, K.P., Sun, Y., Yu, H.: Large games with a bio-social typology. *Journal of Economic Theory* (2013a)
Khan, M.A., Rath, K.P., Sun, Y., Yu, H.: Strategic uncertainty and the ex-post nash property in large games. Theoretical Economics **10**(1), 103–129 (2015)
Khan, M.A., Rath, K.P., Yu, H., Zhang, Y.: Large distributional games with traits. Econ. Lett. **118**(3), 502–505 (2013b)
Khan, M.A., Sun, Y.: Pure strategies in games with private information. J. Math. Econ. **24**(7), 633–653 (1995)
Khan, M.A., Sun, Y.: Non-cooperative games with many players. In: Aumann, R.J., Hart, S. (eds.) Handbook of Game Theory with Economic Applications, vol. 3, pp. 1761–1808. Elsevier, Amsterdam (2002). chapter 46
Mas-Colell, A.: On a theorem of Schmeidler. J. Math. Econ. **13**, 206–210 (1984)
McAdams, D.: Isotone equilibrium in games of incomplete information. Econometrica **71**, 1191–1214 (2003)
McAdams, D.: Monotone equilibrium in multi-unit actions. Rev. Econ. Stud. **73**, 1039–1056 (2006)
Milgrom, P., Shannon, J.: Monotone comparative statics. Econometrica **62**, 157–180 (1994)
Milgrom, P.R., Weber, R.J.: Distributional strategies for games with incomplete information. Math. Oper. Res. **10**, 619–632 (1985)
Morris, S.: Purification. In: Durlauf, S., Blume, L. (eds.) The New Palgrave Dictionary of Economics, pp. 779–782. Palgrave Macmillan, New York (2008)
Oechssler, J., Riedel, F.: Evolutionary dynamics on infinite strategy spaces. Econ. Theory **17**, 141–162 (2001)
Oechssler, J., Riedel, F.: On the dynamic foundation of evolutionary stability in continuous models. J. Econ. Theory **107**, 223–252 (2002)
Radner, R., Rosenthal, R.W.: Private information and pure-strategy equilibria. Math. Oper. Res. **7**, 401–409 (1982)
Rashid, S.: Equilibrium points of non-atomic games: asymptotic results. Econ. Lett. **12**, 7–10 (1983)
Reny, P.: On the existence of monotone pure-strategy equilibria in Bayesian games. Econometrica **79**(2), 499–553 (2011)
Reny, P.J., Zamir, S.: On the existence of pure strategy monotone equilibria in asymmetric first-price auctions. Econometrica **72**, 1105–1125 (2004)
Riley, J.G.: Evolutionary equilibrium strategies. J. Theor. Biol. **76**, 109–123 (1979)
Ritzberger, K., Weibull, J.: Evolutionary selection in normal-form games. Econometrica **63**, 1371–1399 (1995)
Sandholm, W.: Evolution in Bayesian games II: stability of purified equilibria. J. Econ. Theory **136**, 641–667 (2007)
Sandholm, W.H.: Population Games and Evolutionary Dynamics. MIT Press, Cambridge (2010)
Sandholm, W.H.: Population games and deterministic evolutionary dynamics. Handb. Game Theory **4**, 703–778 (2015)
Schmeidler, D.: Equilibrium points of non-atomic games. J. Stat. Phys. **7**, 295–300 (1973)
Vickers, G., Cannings, C.: On the definition of an evolutionary stable strategy. J. Theor. Biol. **129**, 349–353 (1987)
Yu, H., Zhang, Z.: Pure strategy equilibria in games with countable actions. J. Math. Econ. **43**(2), 192–200 (2007)