A GENERALISATION OF A THEOREM OF WIELANDT

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Abstract. In 1974, Helmut Wielandt proved that in a finite group G, a subgroup A is subnormal if and only if it is subnormal in every $\langle A, g\rangle$ for all $g \in G$. In this paper, we prove that the subnormality of an odd order nilpotent subgroup A of G is already guaranteed by a seemingly weaker condition: A is subnormal in G if for every conjugacy class C of G there exists $c \in C$ for which A is subnormal in $\langle A, c\rangle$. We also prove the following property of finite non-abelian simple groups: if A is a subgroup of odd prime order p in a finite almost simple group G, then there exists a cyclic p'-subgroup of $F^*(G)$ which does not normalise any non-trivial p-subgroup of G that is generated by conjugates of A.

1. Introduction

The main result of our paper is the following criterion for the existence of a non-trivial normal p-subgroup in a finite group:

Theorem A. *Let G be a finite group and p be an odd prime. Let A be a p-subgroup of G such that*

$(*)$ *for every conjugacy class C of G there exists $g \in C$ with A subnormal in $\langle A, g\rangle$.*

Then $A \leq O_p(G)$.

As an immediate consequence we have that

Corollary B. *If A is an odd order nilpotent subgroup of a finite group G satisfying condition $(*)$, then A is subnormal in G.*

This can be considered a generalisation of the following result due to H. Wielandt (see [14, 7.3.3]):

Theorem (Wielandt). *Let A be a subgroup of a finite group G. Then the following conditions are equivalent.*

(i) *A is subnormal in G;*
(ii) *A is subnormal in $\langle A, g\rangle$ for all $g \in G$;*
(iii) *A is subnormal in $\langle A, A^g\rangle$ for all $g \in G$;*
(iv) *A is subnormal in $\left\langle A, A^{a^g}\right\rangle$ for all $a \in A$, $g \in G$.*

Our proof of Theorem A makes use of a reduction argument to arrive at a question about finite almost simple groups and then prove a property of these groups which may be of independent interest:

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Theorem C. *Let G be a finite almost simple group with simple socle S and $p > 2$ be a prime dividing $|G|$. Let $A \leq G$ be cyclic of order p. Then there exists a cyclic p'-subgroup $X \leq S$ such that*

$$\text{И}_G^A(X, p) = \varnothing.$$

Here, $\text{И}_G^A(X, p)$ denotes the set of non-trivial p-subgroups of G generated by conjugates of A and normalised by X.

Our proof is therefore related to (and relies on) the classification of finite simple groups. It should be noted that for $p = 2$ the conclusions of Theorem A and Theorem C are no longer true. In particular condition $(*)$ does not imply that $A \leq O_2(G)$. An easy example is reported at the end of Section 3.

In Section 2 we give, after some preparations, the proof of Theorem C, and then in Section 3 show the reduction of Theorem A to the case of almost simple group.
We end with Section 4, where we analyse similar variations related to the other criteria for subnormality given by the original Theorem of Wielandt, namely conditions (iii), better known as the Baer-Suzuki Theorem. We show that in general these generalisations fail to guarantee the subnormality of odd p-subgroups. For other variations on the Baer–Suzuki Theorem the interested reader may consult [17], [10], [6], [7], [8] and [9].

2. Almost simple groups

2.1. **Notation and preliminary results.** In this section we let S be a non-abelian finite simple group and G any group such that $S \leq G \leq \mathrm{Aut}(S)$. For p a prime divisor of $|G|$ denote by $\mathcal{S}_p(G)$ the set of all (possibly trivial) p-subgroups of G. For a p'-subgroup X of S we denote by $\text{И}_G(X, p)$ the set of p-subgroups of G normalised by X, namely

$$\text{И}_G(X, p) = \{Y \in \mathcal{S}_p(G) \mid X \leq N_G(Y)\}.$$

Also for $A \in \mathcal{S}_p(G)$ set

$$\text{И}_G^A(X, p) := \{Y \in \text{И}_G(X, p) \mid Y \text{ is generated by } G\text{-conjugates of } A\}.$$

Note that if $A \leq S$, then $\text{И}_G^A(X, p) \subseteq \text{И}_S(X, p)$, otherwise if $A \not\leq S$ then no $E \in \text{И}_G^A(X, p)$ lies in S.

We aim to prove Theorem C, which we restate:

Theorem 2.1. *Let G be a finite almost simple group with simple socle S. Then with the same notation as above, for every odd prime p dividing $|G|$ and every $A \leq G$ of order p, there exists a cyclic p'-subgroup $X \leq S$ such that $\text{И}_G^A(X, p) = \varnothing$.*

In [2], a similar condition is considered. The authors investigate finite groups G and primes p that have the following property:

(R2) all nilpotent hyperelementary p'-subgroups X of $F^*(G)$ satisfy $\text{И}_G(X, p) \neq 1$

where a *hyperelementary group* X is one for which $O^q(X)$ is cyclic, for some prime q; basically a nilpotent hyperelementary p'-group X is a direct product of a Sylow q-subgroup for some prime $q \neq p$, and a cyclic p'-group. They show ([2, Thm. 2]) that the only almost simple group G satisfying (R2) at a prime p is for $S = \mathrm{L}_3(4)$, $p = 2$, and 4 dividing $|G : S|$.

Note that assumption (R2) implies our assumption. See also [3] for an analogous condition and its related subnormality criteria.

Lemma 2.2. *In the situation of Theorem 2.1 assume that $A \not\leq S$. Let X be a non-trivial p'-subgroup of S and $E \in И^A_G(X,p)$. Then X commutes with some non-trivial p-element in $G \setminus S$.*

Proof. By the coprime action of X on E, we have that $E = [E,X]C_E(X)$. As E is generated by conjugates of A and $A \not\leq S$, we necessarily have that $C_E(X) \not\leq S$. □

Lemma 2.3. *In the situation of Theorem 2.1 if $И^A_G(X,p) \neq \varnothing$ then X normalises a non-trivial elementary abelian p-subgroup of S or it centralises a non-trivial p-element of G.*

Proof. Let E be a non-trivial p-subgroup of G normalised by X. If X does not centralise any non-trivial p-element, then $A \leq S$ by Lemma 2.2 and hence $E \leq S$. Now X also normalises $Z(E)$, and then also $\Omega_1(Z(E))$, the largest elementary abelian subgroup of $Z(E)$. □

The following is a well-known consequence of the classification of finite simple groups and can be found for example in [5, 2.5.12].

Proposition 2.4. *Let S be non-abelian simple, $S < G \leq \mathrm{Aut}(S)$ and assume that $x \in G \setminus S$ has odd prime order p. Then S is of Lie type and one of the following occurs:*

(1) *x is a field automorphism of S;*
(2) *x is a diagonal automorphism of S and one of*
  (2.1) *$S = \mathrm{L}_n(q)$ with $p|(n, q-1)$,*
  (2.2) *$S = \mathrm{U}_n(q)$ with $p|(n, q+1)$,*
  (2.3) *$p = 3$, $S = E_6(q)$ with $3|(q-1)$,*
  (2.4) *$p = 3$, $S = {}^2E_6(q)$ with $3|(q+1)$; or*
(3) *$p = 3$ and x is a graph or graph-field automorphism of $S = \mathrm{O}^+_8(q)$.*

We prove Theorem 2.1 by treating separately the cases: S is alternating, sporadic or a simple group of Lie type.

2.2. **The case of alternating groups.** Throughout the rest of this subsection we assume $S = \mathfrak{A}_n$, with $n \geq 5$ and G such that $S \leq G \leq \mathrm{Aut}(S)$. For X a cyclic p'-subgroup of S we denote by E any element of $И_G(X,p)$. Also, we tacitly assume that any such E is elementary abelian (see Lemma 2.3).

The following elementary result [2, Lemma 3] will be used several times.

Lemma 2.5. *Let $X \leq G \leq \mathfrak{S}_n$ and $E \in И_G(X,p)$. If E acts non-trivially on some X-orbit $\mathcal{O}$, then p divides $|\mathcal{O}|$.*

Proof. As X acts on $\mathrm{fix}_{\mathcal{O}}(E)$, $\mathrm{fix}_{\mathcal{O}}(E) = \varnothing$ and $|\mathcal{O}| \equiv |\mathrm{fix}_{\mathcal{O}}(E)| \equiv 0 \pmod p$. □

Proposition 2.6. *If $И_G(X,p) \neq 1$ for every cyclic p'-subgroup X of S, then $S = \mathfrak{A}_6$ and $(G,p) \in \{(\mathrm{PGL}_2(9), 2), (\mathrm{Aut}(\mathfrak{A}_6), 2)\}$. In particular for every odd prime p and every subgroup A of order p there exists a cyclic p'-subgroup X for which $И^A_G(X,p) = \varnothing$.*

Proof. The cases $n = 5$ and $n = 6$, with $G \leq \mathfrak{S}_6$, follow quite immediately by taking, as subgroup X a Sylow 5-subgroup, if $p = 2$ or $p = 3$, and a cyclic subgroup of order 3 if $p = 5$. Assume that $n = 6$ and $G \not\leq \mathfrak{S}_6$. As $|G : \mathfrak{A}_6|$ is either 2 or 4, if p is odd then $И_G(X,p) = И_{\mathfrak{A}_6}(X,p)$, for every $X \leq \mathfrak{A}_6$. Therefore, by what we have just proved,

there is some cyclic p'-subgroup X of $\mathfrak{A}_6$ for which $\text{И}_G(X,p)=1$. Let $p=2$. The group $G=M_{10}$ contains no elements of order 10 (see [4]). We take X a cyclic subgroup of order 5 of G. Note that if E is any 2-subgroup of G normalised by X, then $E=[E,X]$ and so E lies in $\mathfrak{A}_6$, but then $\text{И}_G(X,2)=\text{И}_{\mathfrak{A}_6}(X,2)=1$. Let now $G=\mathrm{PGL}_2(9)$ or $G=\mathrm{Aut}(\mathfrak{A}_6)$. Then every element of odd order normalises a non-trivial 2-subgroup of G, basically the elements of order 3 normalise a copy of $\mathfrak{A}_4$ lying in $\mathfrak{A}_6$, while the elements of order 5 centralise always an outer involution (see [4]). Therefore we have that $\text{И}_G(X,2)\neq 1$ for every cyclic odd order subgroup X of G.
Assume for the rest of the proof that $n\geq 7$ and argue by contradiction. We treat separately the two cases: 1) n is even and 2) n is odd.

Case 1. n is even.
Assume first that $p|(n-1)$. In particular p is odd. We take as cyclic p'-subgroup X of $\mathfrak{A}_n$ the one generated by $x=(12)(3\dots n)$. Let E be a non-trivial element of $\text{И}_G(X,p)$. Now the set $\{1,2\}$ cannot lie in $\mathrm{fix}(E)$, otherwise E acts non-trivially on $\{3,\dots,n\}$, and by Lemma 2.5 we would have that p divides $n-2$, which is not the case being a divisor of $n-1$. Also from the fact that $E=E^x$, it follows that none of 1 and 2 are fixed by E. But then, as $p>2$, EX is transitive on $\{1,2,\dots,n\}$, and since p does not divide n we have a contradiction.

Assume now that $p\nmid(n-1)$.
We choose first $X=\langle x\rangle$ with $x=(2\dots n)$ and let $1\neq E\in \text{И}_G(X,p)$. By Lemma 2.5 we have that E does not fix 1. Then EX is transitive on $\{1,2,\dots,n\}$. In particular we have that n is a power of p and since it is even $p=2$. Say $n=2^r\geq 8$.
We can now change our testing subgroup $X=\langle x\rangle$ and choose now $x=(123)(4\dots n)$. This is of course a cyclic p'-subgroup of $\mathfrak{A}_n$. Let E be a non-trivial elementary abelian 2-subgroup lying in $\text{И}_G(X,2)$. Note that E acts fixed-point-freely on $\{1,2,\dots,n\}$. Indeed if E fixes a point, then as X normalises E, we have that either $\{1,2,3\}$ or $\{4,\dots,n\}$ lie in $\mathrm{fix}(E)$. In any case we reach a contradiction with Lemma 2.5. Now we claim that E is transitive. Let $\mathcal{O}$ be the E-orbit containing 1. Then $\mathcal{O}$ cannot be contained in $\{1,2,3\}$, otherwise E being a 2-group, there will be a fixed point of E in $\{1,2,3\}$, which is not the case. Let therefore $e\in E$ be such that $1e=i\in\{4,\dots,n\}$. The subgroup $\langle x^3\rangle$ is transitive on $\{4,\dots,n\}$, as 3 is coprime to $n-3=2^r-3$, thus for every $j\in\{4,\dots,n\}$ we may take some $y\in\langle x^3\rangle$ such that $iy=j$. But then

$$1(e^y)=1(y^{-1}ey)=1(ey)=i(y)=j$$

and since $e^y\in E$ the element j lies in $\mathcal{O}$. In particular we have proved that $\{4,\dots,n\}\subseteq\mathcal{O}$ and, since $n-2=2^r-2$ is not a power of 2 as $n\geq 8$, we have that $\mathcal{O}=\{1,2,\dots,n\}$ and E acts regularly on it. Now let e be the unique element of E that maps 1 to 2, then e^x maps 2 to 3. Now as $[e,e^x]=1$ we have that

$$1(ee^x)=2(e^x)=3=1(e^xe)$$

which means that $1(e^x)=3(e^{-1})$, and so $1e^x\notin\{1,2,3\}$. If we set $1e^x=j$, for some $j\in\{4,\dots,n\}$, we reach a contradiction, since

$$e=(12)(3,j)\dots \quad\text{and}\quad e^x=(23)(1,j)\dots$$

but also as $n\geq 8$, $jx\neq j$ and so $e^x=(1x,2x)(3x,jx)\dots=(23)(1,jx)\dots$.

Case 2. n is odd.
In this situation we have that $p|n$. Indeed if this is not the case, we take X the subgroup generated by an n-cycle of $\mathfrak{A}_n$. Now any $1 \neq E \in И_G(X,p)$ acts non-trivially on $\{1,2,\ldots,n\}$, and therefore we reach a contradiction to Lemma 2.5. Thus $p|n$; in particular p is odd.
We take now x the $(n-2)$-cycle $(3,4,\ldots,n)$ and $X = \langle x \rangle$. Then any $1 \neq E \in И_G(X,p)$ does not fix both 1 and 2, otherwise by Lemma 2.5, $p|(n-2)$ which is not the case as $p|n$ and p is odd. Assume that 1 is not fixed by E (otherwise argue considering 2 in place of 1) and let $\mathcal{O}_1$ be the E-orbit containing 1. Since p is odd there is some $e \in E$ such that $1e = i$ for some $i \in \{3,\ldots,n\}$. Now as X is transitive on $\{3,\ldots,n\}$, for every $j \in \{3,\ldots,n\}$ there is some power m of x such $ix^m = j$. But then

$$1(e^{x^m}) = 1(x^{-m}ex^m) = 1(ex^m) = i(x^m) = j,$$

and as $e^{x^m} \in E$ we have proved that $\{3,\ldots,n\} \subseteq \mathcal{O}_1$. Since $p \nmid n-1$ we conclude that E is transitive on $\{1,\ldots,n\}$. We show now that E is regular. The stabiliser $\mathrm{Stab}_E(1)$ is normalised by X, and therefore if this is non-trivial, then by Lemma 2.5 we reach the contradiction $p|(n-2)$. It follows that E is regular on $\{1,2,\ldots,n\}$ and so $n = |E| = p^r$, and any non-trivial element σ of E is a product of exactly p^{r-1} cycles of length p. In particular there exists a unique $\sigma \in E$ which maps 1 to 2. We write

$$\sigma = \sigma_1\sigma_2\cdots\sigma_{p^{r-1}}$$

with $\sigma_1 = (12u_3\ldots u_p)$, a p-cycle. Since $\sigma^x \in E$ maps 1 to 2, we necessarily have that $\sigma = \sigma^x$, but this is not the case as $\sigma_1^x = (12u_3x\ldots u_px) \neq (12u_3\ldots u_p) = \sigma_1$. □

2.3. **The case of sporadic groups.** We assume now that S is one of the 27 sporadic simple groups (including the Tits simple group ${}^2F_4(2)'$), and $S \leq G \leq \mathrm{Aut}(S)$. As before X will denote a cyclic p'-subgroup of S and E a non-trivial elementary abelian p-subgroup of G normalised by X. Our basic reference for properties of sporadic groups is [4].

Proposition 2.7. *Let S be a simple sporadic group, $S \leq G \leq \mathrm{Aut}(S)$ and p a prime. Then there exists a cyclic p'-subgroup X of S such that $И_G(X,p) = 1$. In particular, for every odd prime p and every subgroup A of G of order p, there exists a cyclic p'-subgroup X such that $И_G^A(X,p) = \varnothing$.*

Proof. We extend a little our notation. Given a prime p and a positive integer q coprime to p, we write $И_G(q,p)$ for the set of p-subgroups of G that are normalised by some cyclic q-subgroup of S.

Table 1 summarises the situation for the sporadic groups and their automorphism groups. For every group S, we list a pair $(q;r)$ of primes such that $И_G(q,p) = 1$ for all $p \neq q$, and $И_G(r,q) = 1$. For four groups, $И_G(q,p) \neq 1$ for another prime $p \neq q$, in which case either $И_G(r,p) = 1$, or we give a further integer s such that $И_G(s,p) = 1$. Our choice of $(q;r)$, respectively $(q;r;s)$ works for both S and $\mathrm{Aut}(S)$.

We prove the validity of Table 1 by considering the individual groups in turn.

The groups $S = M_{11}$, J_1, J_2, M_{23}, M_{24}, Co_3, Co_2, Ru, Ly, J_4, Fi_{23} have trivial outer automorphism group. The validity of our claim is immediate from the known lists of maximal subgroups [4].

Table 1. The case of sporadic groups.

M_{11}	$(11;3)$	Co_3	$(23;7)$	B	$(47;31)$
M_{12}	$(11;3)$	Co_2	$(23;5)$	M	$(59;71)$
M_{22}	$(11;3)$	Co_1	$(23\,(p\neq 2);33\,(p=2);13)$	J_1	$(19;11)$
M_{23}	$(23;5)$	He	$(17;7)$	$O'N$	$(31;19)$
M_{24}	$(23;5)$	Fi_{22}	$(13\,(p\neq 3);11)$	J_3	$(19;17)$
J_2	$(7;5)$	Fi_{23}	$(23\,(p\neq 2);17)$	Ly	$(67;37)$
Suz	$(13;11)$	Fi_{24}'	$(29;23)$	Ru	$(29;13)$
HS	$(11;3)$	HN	$(19;11)$	J_4	$(43;37)$
M^cL	$(11\,(p\neq 2);7)$	Th	$(31\,(p\neq 2);19)$	${}^2F_4(2)'$	$(13;5)$

For $S = M_{12}$, M_{22}, HS, He, J_3, $O'N$, HN, Th, ${}^2F_4(2)'$ we have $|\mathrm{Out}(S)| = 2$. For $G = S$ we can argue as before, while for $G = \mathrm{Aut}(S)$ we invoke Lemma 2.2 for a suitable subgroup $X \le S$ of prime order as listed in Table 1. We deal with the remaining groups in some more detail.

$S = Suz$. Here $|\mathrm{Out}(S)| = 2$. The maximal subgroups of S of order divisible by 13 are isomorphic to $G_2(4)$, $\mathrm{L}_3(3){:}2$ or $\mathrm{L}_2(25)$. As these have no elements of order 11, we immediately obtain $И_{Suz}(5,11) = 1$. Moreover, for any of these groups, a Sylow 13-subgroup does not normalise any other p-subgroup, for $p \neq 13$. Thus $И_{Suz}(13,p) = 1$ for every prime $p \neq 13$. Finally the outer involutions do not centralise any element of order 13, forcing the same conclusions for $\mathrm{Aut}(Suz)$.

$S = M^cL$. Here $|\mathrm{Out}(S)| = 2$. The maximal subgroups of S of order divisible by 11 are isomorphic to M_{11} and M_{12}. Therefore $И_{M^cL}(7,11) = И_{M^cL}(11,p) = 1$ for every $p \neq 11$. Now consider $G = \mathrm{Aut}(M^cL)$. Again, X of order 11 shows that there are no examples except possibly when $p = 2$. In the latter case for X we take a cyclic subgroup of order 7. The Atlas [4] shows that $C_G(X) \le S$, and so there can be no example for G by Lemma 2.2.

$S = Co_1$. Here $\mathrm{Out}(S) = 1$. The maximal subgroups of S of order divisible by 23 are isomorphic to Co_2, $2^{11}{:}M_{24}$ or Co_3. Since these groups have no elements of order 13, we obtain that $И_{Co_1}(13,23) = 1$. Moreover if Y is any of these maximal subgroups $И_Y(23,p) = 1$, for every p different from 23 and 2. Finally Co_1 has elements of order 33 which are auto-centralising. As Co_1 and Co_2 do not contain elements of order 33, the unique maximal subgroups of S that have such an element are: $\mathrm{U}_6(2){:}\mathfrak{S}_3$, $3^6{:}2M_{12}$ and $3^{\cdot}Suz{:}2$. Now, a cyclic subgroup of order 33 in $\mathrm{U}_6(2){:}\mathfrak{S}_3$ does not lie completely in $\mathrm{U}_6(2)$; therefore, if such a subgroup normalises a non-trivial 2-subgroup, then, since S has no elements of order 66, we should have that $И_{\mathrm{U}_6(2)}(11,2) \neq 1$. This is not the case as in $\mathrm{U}_6(2)$ the maximal subgroups of order divisible by 11 are M_{22} and $\mathrm{U}_5(2)$. Consider now a cyclic subgroup Y of order 33 inside $3^6{:}2M_{12}$. This is the direct product of a subgroup of order 3 in 3^6 by a Sylow 11-subgroup of $2M_{12}$. Assume that X is a non-trivial 2-subgroup of $3^6{:}2M_{12}$ normalised by Y. Then $X \in И_{2M_{12}}(11,2)$, and since $И_{M_{12}}(11,2) = 1$ we deduce that X is centralised by a Sylow 11-subgroup of M_{12}, and thus by the whole Y, which is a contradiction since in S there are no elements of order 66. Finally, a similar argument shows that if X is a non-trivial element of $И_{3^{\cdot}Suz:2}(33,2)$, then $X \cap Suz$ is a non-trivial element of $\text{И}_{Suz}(11, 2)$. This is impossible since the maximal subgroups of Suz of order divisible by 11 are: $\mathrm{U}_5(2)$, $3^5{:}M_{11}$ and $M_{12}{:}2$, forcing $\text{И}_{Suz}(11, 2) = 1$.

$S = Fi_{22}$. Here $|\mathrm{Out}(S)| = 2$. The maximal subgroups of S of order divisible by 13 are isomorphic to ${}^2F_4(2)$ or $\mathrm{O}_7(3)$. Since both these groups have orders not divisible by 11, we have that $\text{И}_S(11, 13) = 1$. Now, $\text{И}_{{}^2F_4(2)}(13, p) = 1$ for every $p \neq 13$, since the maximal subgroups of ${}^2F_4(2)$ containing a Sylow 13-subgroup are $\mathrm{L}_2(25)$ and $\mathrm{L}_3(3){:}2$ and $C_S(13) = 13$. In $\mathrm{O}_7(3)$ there are three isomorphism classes of maximal subgroups of order divisible by 13, namely $G_2(3), \mathrm{L}_4(3){:}2$ and $3^{3+3}{:}\mathrm{L}_3(3)$. We have that $\text{И}_{\mathrm{O}_7(3)}(13, p) = 1$ if $p \neq 3$ (and $p \neq 13$), forcing $\text{И}_S(13, p) = 1$ for every prime p different from 3 and 13. To deal with the case $p = 3$, we look at the maximal subgroups of S of order divisible by 11. These are isomorphic to one of the following: M_{12}, $2^{10}{:}M_{22}$ and $2\,{}^{\cdot}\mathrm{U}_6(2)$. For any of these groups Y we have $\text{И}_Y(11, 3) = 1$, thus the same happens in S. Since $|\mathrm{Out}(S)| = 2$, we only need to show that $\text{И}_{\mathrm{Aut}(S)}(q, 2) = 1$ for some odd integer q. This is guaranteed by the fact that $\text{И}_S(13, 2) = 1$ and $C_{\mathrm{Aut}(S)}(13) = 13$.

For the last three groups, the Atlas does not contain complete lists of maximal subgroups, so we need to give a different argument.

$S = Fi_{24}'$. Here $|S| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$, $|\mathrm{Out}(S)| = 2$. Here, a subgroup of order 29 cannot act faithfully on an elementary abelian p-subgroup for $p \neq 29$, by the order formula. On the other hand, subgroups of order 29 are not normalised by elements of order 23.

$S = B$. Here $|S| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$, $\mathrm{Out}(S) = 1$. >From the order formula it is clear that a subgroup of order 47 cannot act non-trivially on an elementary abelian p-subgroup of S, except possibly for $p = 2$. Since elements of order 47 are self-centralising, and not normalised by an element of order 31, we must have $p = 2$. But the 2-rank of S is 14 by [13], too small for an action of C_{47}.

$S = M$. Here $|S| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$, $\mathrm{Out}(S) = 1$. A subgroup of order 59 cannot act faithfully on an elementary abelian p-subgroup for $p \neq 59$, by the order formula. On the other hand, subgroups of order 59 are not normalised by elements of order 71. $\square$

2.4. **Classical groups of Lie type.** We consider the following setup. Let S be a finite simple group of Lie type. There exists a simple linear algebraic group $\mathbf{H}$ of adjoint type defined over the algebraic closure of a finite field and a Steinberg endomorphism $F : \mathbf{H} \to \mathbf{H}$ such that the finite group of fixed points $H = \mathbf{H}^F$ satisfies $S = [H, H]$.

We now make use of the fact that groups of Lie type possess elements of orders which cannot occur in their Weyl group, and with small centraliser. These can be found, for example, in the Coxeter tori. For this we need the existence of *Zsigmondy primitive prime divisors* (see [11, Thm. 3.9]):

Lemma 2.8. *Let q be a power of a prime and $e > 2$ an integer. Then unless $(q, e) = (2, 6)$ there exists a prime ℓ dividing $q^e - 1$, but not dividing $q^f - 1$ for any $f < e$, and $\ell \geq e + 1$.*

In Table 2 we have collected for each type of classical group two maximal tori T_1, T_2 of H (indicated by their orders). Then the order of T_i is divisible by a Zsigmondy prime divisor ℓ_i of $q^{e_i} - 1$, with e_i given in the table (unless $e_i = 2$ or $(e_i, q) = (6, 2)$).

Table 2. Two tori for classical groups.

H		$\lvert T_1\rvert$	$\lvert T_2\rvert$	e_1	e_2
A_{n-1}	$(n \geq 2)$	$(q^n-1)/(q-1)$	$q^{n-1}-1$	n	$n-1$
${}^2A_{n-1}$	$(n \geq 3$ odd$)$	$(q^n+1)/(q+1)$	$q^{n-1}-1$	$2n$	$n-1$
	$(n \geq 4$ even$)$	$q^{n-1}+1$	$(q^n-1)/(q+1)$	$2n-2$	n
B_n, C_n	$(n \geq 2$ even$)$	q^n+1	$(q^{n-1}+1)(q+1)$	$2n$	$2n-2$
	$(n \geq 3$ odd$)$	q^n+1	q^n-1	$2n$	n
D_n	$(n \geq 4$ even$)$	$(q^{n-1}+1)(q+1)$	$(q^{n-1}-1)(q-1)$	$2n-2$	$n-1$
	$(n \geq 5$ odd$)$	$(q^{n-1}+1)(q+1)$	q^n-1	$2n-2$	n
${}^2D_n$	$(n \geq 4)$	q^n+1	$(q^{n-1}+1)(q-1)$	$2n$	$2n-2$

Proposition 2.9. *Assume that S is of classical Lie type not in characteristic p. Then Theorem 2.1 holds for all $S \leq G \leq \mathrm{Aut}(S)$.*

Proof. Let $\mathbf{H}, H$ be as above so that $S = [H,H]$. We distinguish three cases.

Case 1: $A \leq S$.

The cases when $e_i \leq 2$, that is, H is of type A_1, A_2, ${}^2A_2$ or B_2, will be considered in Proposition 2.10. For all other types, for X we choose a maximal cyclic subgroup of $T_i \cap S$ for $i = 1,2$, with T_i from Table 2. Note that the orders of $T_1 \cap S, T_2 \cap S$ are coprime, and T_i is the centraliser in H of any $s_i \in T_i$ of order ℓ_i. Assume that $И_S^A(X,p) \neq \varnothing$. By Lemma 2.3 and the fact that $A \leq S$, X normalises a non-trivial elementary abelian p-subgroup E of S. Let $\pi : \tilde{\mathbf{H}} \to \mathbf{H}$ be a simply-connected covering of $\mathbf{H}$, and hence $\ker(\pi) = Z(\tilde{\mathbf{H}})$. We let $\tilde{E}$ be a (normal) Sylow p-subgroup of the full preimage of E in $\tilde{\mathbf{H}}$. Then $\tilde{E}$ is normalised by the full preimage of X. First assume that $|Z(\tilde{\mathbf{H}})|$ is prime to p. Then $\tilde{E} \cong E$ is abelian. As $\tilde{\mathbf{H}}$ is simply-connected, $p > 2$ is not a torsion prime of $\tilde{\mathbf{H}}$ (see [15, Tab. 14.1]), so an inductive application of [15, Thm. 14.16] to a sequence of generators of the abelian group $\tilde{E}$ shows that $\mathbf{C} := C_{\tilde{\mathbf{H}}}(\tilde{E})$ contains a maximal torus of $\tilde{\mathbf{H}}$ and is connected reductive, hence a subsystem subgroup of $\tilde{\mathbf{H}}$ of maximal rank. Then $N_{\tilde{\mathbf{H}}}(\mathbf{C}) = \mathbf{C}N_{\tilde{\mathbf{H}}}(\mathbf{T})$ for any maximal torus $\mathbf{T}$ of $\tilde{\mathbf{H}}$, so $N_{\tilde{\mathbf{H}}}(\mathbf{C})/\mathbf{C}$ is isomorphic to a section of the Weyl group W of $\tilde{\mathbf{H}}$. As $N_{\tilde{\mathbf{H}}}(\tilde{E}) \leq N_{\tilde{\mathbf{H}}}(C_{\tilde{\mathbf{H}}}(\tilde{E})) = N_{\tilde{\mathbf{H}}}(\mathbf{C})$ we see that $N_{\mathbf{H}}(E)/C_{\mathbf{H}}(E)$ is a section of W.

Now note that the order of the Weyl group of $\mathbf{H}$ is not divisible by any prime larger than e_i, except for H of type D_n, with $n \geq 4$ even and $e_2 = n-1$. Here, $\ell_2 \geq e_2 + 1 = n$, but n is even so that in fact $\ell_2 > n$ does not divide the order of the Weyl group either. This shows that elements $s_i \in X$ of order ℓ_i must centralise E, for $i = 1,2$. So p divides the order of $C_S(s_i) = T_i \cap S$ for $i = 1,2$, a contradiction as these orders are coprime.

The cases when $(q, e_i) = (2,6)$, that is, $S = \mathrm{L}_6(2), \mathrm{L}_7(2), \mathrm{U}_6(2), \mathrm{O}_7(2), \mathrm{O}_8^{\pm}(2), \mathrm{O}_9(2)$ will be handled in Proposition 2.11, while $S = \mathrm{U}_4(2) \cong \mathrm{S}_4(3)$ will be treated in Proposition 2.10.

Now assume that $\tilde{E}$ is non-abelian. Then p divides $|Z(\tilde{\mathbf{H}})|$ and thus $S = \mathrm{L}_n(q)$ or $S = \mathrm{U}_n(q)$. Let E_1 be a minimal non-cyclic characteristic subgroup of $\tilde{E}$. Then E_1 is of symplectic type, hence extra-special (see [1, (23.9)]) and normalised by X. Write $|E_1| = p^{2a+1}$, then $p^a \leq n$ as $E_1 \leq \mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$. Now the outer automorphism group of E_1 is $\mathrm{Sp}_{2a}(p)$, and all prime divisors of its order are at most $(p^a+1)/2 < n$. But our Zsigmondy prime divisors ℓ_i of $|X|$ satisfy $\ell_i \geq n$, so again we conclude that X must centralise an element of order p. We conclude as before.

Case 2: $A \not\leq S$ contains diagonal automorphisms.
In this case by Proposition 2.4 we have $S = \mathrm{L}_n(q)$ or $S = \mathrm{U}_n(q)$. Here let X be generated by a regular unipotent element. By Lemma 2.2, if X normalises a non-trivial p-subgroup generated by conjugates of A, X must centralise some non-trivial element of order p. But the centraliser of a regular unipotent element in the group $\mathrm{PGL}_n(q)$ resp. $\mathrm{PGU}_n(q)$ of inner-diagonal automorphisms is obviously unipotent, hence this case does not occur, as by assumption p is not the defining characteristic.

Case 3: $A \not\leq S$ does not contain diagonal automorphisms.
By Proposition 2.4, A contains field, graph or graph-field automorphisms. Now in all cases, a maximal cyclic subgroup X of $T_1 \cap S$ can be identified to a subgroup of the multiplicative group of $\mathbb{F}_{q^{e_1}}$ by viewing some isogeny version of H as a classical matrix group. The normaliser in S of X then acts by field automorphisms of $\mathbb{F}_{q^{e_1}}/\mathbb{F}_q$. Using the embedding into a matrix group one sees that the field automorphisms of S act on X as the field automorphisms of $\mathbb{F}_q/\mathbb{F}_r$, where r is the characteristic of $\mathbf{H}$. In particular they induce automorphisms of X different from those induced by $N_S(X)$. So with this choice of X field automorphisms cannot lead to examples by Lemma 2.2. Finally, if $S = \mathrm{O}_8^+(q)$ and A contains graph or graph-field automorphisms of order 3 then we choose X to be generated by an element x of order $(q^2+1)/d$ in a maximal torus $T \leq S$ of order $(q^2+1)^2/d^2$, where $d = \gcd(q-1, 2)$. The normaliser $N_S(T)$ acts by the complex reflection group $G(4,2,2)$ of 2-power order, while in the extension by a graph or graph-field automorphism it acts by the primitive reflection group G_5. These automorphisms hence induce further non-trivial elements normalising X, and not centralising x. □

We now complete the proof for the small rank cases.

Proposition 2.10. *Assume that* $S = \mathrm{L}_2(q)$ *(*$q \geq 8$*),* $\mathrm{L}_3(q)$*,* $\mathrm{U}_3(q)$ *(*$q > 2$*), or* $\mathrm{S}_4(q)$ *(*$q > 2$*), and* $p \nmid q$*. Then Theorem 2.1 holds for all* $S \leq G \leq \mathrm{Aut}(S)$*.*

Proof. We just need to deal with the case that $G = S$, since the other possibilities were already discussed in the proof of Proposition 2.9. First assume that $S = \mathrm{L}_2(q)$. If $q \geq 8$ is even, then elements of order $q+1$ do not normalise any non-trivial p-subgroup with p dividing $q-1$, while elements of order $q-1$ do not normalise any with $p|(q+1)$. If $q = r^f \geq 9$ is odd, elements of order r do not normalise non-trivial p-subgroups for $2 < p|(q^2-1)$.

Next let $S = \mathrm{L}_3(q)$. Elements of order $(q^2+q+1)/\gcd(3, q-1)$ do not normalise non-trivial p-subgroups for p dividing q^2-1, while elements of order 2 do not normalise non-trivial p-subgroups for p dividing $(q^2+q+1)/\gcd(3, q-1)$. Similarly for $S = \mathrm{U}_3(q)$, $q > 2$, we can argue using elements of order $(q^2-q+1)/\gcd(3, q+1)$, respectively of order 2.

Finally assume that $S = \mathrm{S}_4(q)$. Using X of order q^2+1 we see that we must have $p|(q^2+1)$. In this case, take X of order 3. □

Proposition 2.11. *Assume that* S *is one of* $\mathrm{L}_6(2), \mathrm{L}_7(2), \mathrm{U}_6(2), \mathrm{O}_7(2), \mathrm{O}_8^{\pm}(2)$ *or* $\mathrm{O}_9(2)$*. Then Theorem 2.1 holds for all* $S \leq G \leq \mathrm{Aut}(S)$*.*

Table 3. Some groups over $\mathbb{F}_2$.

$\mathrm{L}_6(2)$	$(2;7)$	$\mathrm{U}_6(2)$	$(11;3)$	$\mathrm{O}_8^+(2)$	$(7;5)$
$\mathrm{L}_7(2)$	$(127;31)$	$\mathrm{O}_7(2)$	$(7;5)$	$\mathrm{O}_8^-(2)$	$(17;3)$
$\mathrm{O}_9(2)$	$(17;5)$				

Proof. In all of these groups, just one of the two Zsigmondy primes ℓ_i exists. By the argument given in the proof of Proposition 2.9, we still obtain that either elements of order ℓ_i centralise a p-element or that $S \neq G$. We may then conclude as in the proof of Proposition 2.7, using an additional prime as in Table 3, except in the two cases when $|\mathrm{Out}(S)| > 2$:

Let $S = \mathrm{U}_6(2)$. Here $\ell_1 = 11$ shows that $p = 11$ if $G = S$, and since a subgroup of order 7 does not normalise one of order 11, we reach a contradiction in this case. We assume therefore that $p = 3$ and $A \not\leq S$. Let first X be a subgroup of S of order 11 and Y a maximal subgroup of S containing X. Then $Y \simeq \mathrm{U}_5(2)$ or M_{22}, and therefore $И_Y(X,3) = И_S(X,3) = 1$. Now if $E \in И_G^A(X,3)$ we have that $E \cap S \in И_S(X,3) = 1$ and so $E = C_E(X)$ has order 3, and, E being generated by conjugates of A, we have that E is a conjugate of A. Now the group S has four classes of outer elements of order three, denoted $3D, 3E, 3F$ and $3G$ in [4]. Amongst these just $3D$ has centraliser in S divisible by 11, namely $C_S(3D) \simeq \mathrm{U}_5(2)$. We have therefore that $A = \langle x \rangle$ for some x in $3D$. Now take X a subgroup of S of order 7. We may argue as before. Since a maximal subgroup Y of S containing X is isomorphic to one of

$$M_{22},\ 2^9.\mathrm{L}_3(4),\ \mathrm{U}_4(3){:}2,\ \mathrm{S}_6(2),\ \mathrm{L}_3(4){:}2,$$

we have that $И_Y(X,3) = И_S(X,3) = 1$, and therefore any element $E \in И_G^A(X,3)$ is a cyclic subgroup conjugate to A and centralised by X. This is a contradiction since 7 does not divide $|\mathrm{U}_5(2)|$.

Let $S = \mathrm{O}_8^+(2)$. The prime $\ell_2 = 7$ shows that $p = 7$ if $A \leq S$. Since a subgroup of order 5 does not normalise any non-trivial 7-subgroup, we reach a contradiction if $A \leq S$. Let $p = 3$ and $A \not\leq S$. In G there are three classes of outer 3-elements, two of order 3 and one of order 9. In all cases 5 does not divide the order of their centralisers in S. Thus if X is a cyclic subgroup of order 5 we reach a contradiction with Lemma 2.2. □

2.5. **Groups of exceptional type.** In this section we prove Theorem 2.1 when S is one of the exceptional groups of Lie type. We keep the setting from the beginning of the previous subsection. Note that we need not treat ${}^2B_2(2)$ (which is solvable), $G_2(2) \simeq \mathrm{U}_3(3).2$, ${}^2G_2(3) \simeq \mathrm{L}_2(8).3$ and ${}^2F_4(2)'$ (see Section 2.3).

As in the case of classical groups we provide in Table 4 for each type of group two maximal tori of H, indicated by their orders. Here, we denote by Φ_n the n-th rational cyclotomic polynomial evaluated at q, and moreover we let $\Phi_8' = q^2 + \sqrt{2}q + 1$, $\Phi_8'' = q^2 - \sqrt{2}q + 1$, $\Phi_{24}' = q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1$, $\Phi_{24}'' = q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1$ for $q^2 = 2^{2f+1}$, and $\Phi_{12}' = q^2 + \sqrt{3}q + 1$, $\Phi_{12}'' = q^2 - \sqrt{3}q + 1$ for $q^2 = 3^{2f+1}$. We then have $\gcd(|T_1|, |T_2|) = d$, where $d = (3, q-1)$, $(3, q+1)$ and $d = (2, q-1)$ for $S = E_6(q), {}^2E_6(q), E_7(q)$ respectively, and $d = 1$ otherwise. Furthermore, $|H : S| = |T_i : T_i \cap S| = d$ in all cases.

Table 4. Two tori for exceptional groups.

H		$\lvert T_1\rvert$	$\lvert T_2\rvert$	H	$\lvert T_1\rvert$	$\lvert T_2\rvert$	d
${}^2B_2(q^2)$	$(q^2 \geq 8)$	Φ_8'	Φ_8''	$F_4(q)$	Φ_8	Φ_{12}	1
${}^2G_2(q^2)$	$(q^2 \geq 27)$	Φ_{12}'	Φ_{12}''	$E_6(q)$	$\Phi_3\Phi_{12}$	Φ_9	$(3, q-1)$
$G_2(q)$	$(q \geq 3)$	Φ_3	Φ_6	${}^2E_6(q)$	$\Phi_6\Phi_{12}$	Φ_{18}	$(3, q+1)$
${}^3D_4(q)$		Φ_3^2	Φ_{12}	$E_7(q)$	$\Phi_2\Phi_{14}$	$\Phi_1\Phi_7$	$(2, q-1)$
${}^2F_4(q^2)$	$(q^2 \geq 8)$	Φ_{24}'	Φ_{24}''	$E_8(q)$	Φ_{15}	Φ_{30}	1

Proposition 2.12. *Assume that S is of exceptional Lie type not in characteristic p. Then Theorem 2.1 holds for all $S \leq G \leq \mathrm{Aut}(S)$.*

Proof. First assume that $A \leq S$. Using for X maximal cyclic subgroups of $S \cap T_i$, for T_i as listed in Table 4, we conclude by the same arguments as in the proof of Proposition 2.9 that in a possible counterexample to Theorem 2.1 the prime p would divide the orders of both $S \cap T_i$, $i = 1, 2$, which is a contradiction as their orders are coprime, unless possibly if p is a torsion prime for H. The torsion primes for groups of exceptional Lie type are just the bad primes (see [15, Tab. 14.1]); in particular $p \leq 5$, and even $p = 3$ unless $S = E_8(q)$. The maximal rank of an elementary abelian p-subgroup of H, for p odd, is at most the rank m of H, see e.g. [5, Thm. 4.10.3]. It is easy to check that for all bad primes $p \nmid q$ and $s \leq m$, $p^s - 1$ is not divisible by ℓ_i for $i \in \{1, 2\}$, so again T_i must centralise a non-trivial p-element, which contradicts the fact that $\gcd(|T_1|, |T_2|) = 1$, except for the groups $F_4(2), E_6(2)$ and ${}^2E_6(2)$ (each with $p = 3$). In all of the latter cases, at least one of the ℓ_i does not divide $3^s - 1$ for $s \leq 4$, and does not divide the centraliser order of an element of order 3 either, so again we are done.

Now assume that $A \not\leq S$. Then either A contains field automorphisms, in which case taking for X a maximal cyclic subgroup of T_1 shows that no example arises by Lemma 2.2 as field automorphisms induce proper non-inner automorphisms on this torus. Or, we have that $S = E_6(q)$ or ${}^2E_6(q)$, $p = 3$, and A contains diagonal automorphisms. In this case we take for X the subgroup generated by a regular unipotent element; this has a unipotent centraliser in the group of inner-diagonal automorphisms and thus we are done. $\square$

2.6. **Groups of Lie type in defining characteristic.** If p is the defining prime for S, we can again make use of the two tori T_1, T_2 introduced before.

Proposition 2.13. *Assume that S is of Lie type in characteristic p. Then Theorem 2.1 holds for all $S \leq G \leq \mathrm{Aut}(S)$.*

Proof. Let $A \leq H$ be cyclic of order p and $1 \neq P \leq H$ be a p-subgroup generated by conjugates of A. First assume that $A \leq S$. Then P is a non-trivial unipotent subgroup of S, hence its normaliser $N_S(P)$ is contained in some proper parabolic subgroup of S (see [15, Thm. 26.5]). Let s be a regular semisimple element of S in the torus T_1 as given in Table 2 when S is classical, or in Table 4 in case S is exceptional. Then the centraliser $C_S(s)$ is contained in T_1, in particular s does not centralise any non-trivial split torus of $\mathbf{H}$ and so is not contained in a proper parabolic subgroup of S. Thus $И_S^A(X, p) = \varnothing$.

If $A \not\leq S$, then by Proposition 2.4 either A contains a field automorphism of S, or $p = 3$ and A contains a graph or graph-field automorphism. According to Lemma 2.2, X is centralised by an outer p-element. Now as pointed out in the proof of Propositions 2.9 and 2.12, field automorphisms do not enlarge the centraliser of X as defined above, so we may assume that $S = \mathrm{O}_8^+(q)$, $p = 3$ and H involves a graph or graph-field automorphism. In this case take X generated by a semisimple element of order $(q^2+1)/\gcd(2, q-1)$ and conclude as in the proof of Proposition 2.9. $\square$

3. Proof of Theorem A

In this section we complete the proof of Theorem A. We need the following result, whose proof can be found in [12, Thm. 4.2].

Lemma 3.1. *Let G be a finite group and V a faithful irreducible G-module. Assume that p is an odd prime number different from the characteristic of V and that A is a subgroup of G of order p that lies in $O_p(G)$. Then there exists an element $v \in V$ such that*

$$A \not\subseteq \bigcup_{g \in G} C_G(v)^g.$$

Given a finite group G and a subgroup $A \leq G$ we say that the pair (G, A) satisfies $(*)$ if for every conjugacy class C of G there exists $g \in C$ such that A is subnormal in $\langle A, g\rangle$.

Proof of Theorem A. We argue by contradiction: assume that G is a finite group, A an odd p-subgroup of G, $A \not\leq O_p(G)$, and the pair (G, A) satisfies the condition $(*)$. Moreover we assume that that $|G| + |A|$ is minimal with respect to these conditions. We proceed by steps.

Step 1. We have $O_p(G) = 1$.

Indeed, note that $(G/O_p(G), AO_p(G)/O_p(G))$ satisfies $(*)$, therefore if $O_p(G) \neq 1$ by our minimal assumption, we would have that $AO_p(G)/O_p(G) \leq O_p(G/O_p(G)) = 1$, which is a contradiction.

Step 2. We have $|A| = p$.

Let B be a proper subgroup of A and note that (G, B) satisfies $(*)$. By the minimal choice, every proper non-trivial subgroup of A lies in $O_p(G)$. Since $O_p(G) = 1$, we conclude that $B = 1$, i.e., A has order p.

$>$From now on we set $A = \langle a\rangle$.

Step 3. G has a unique minimal normal subgroup M.

Assume that M and N are two distinct minimal normal subgroups of G and assume also that $A \not\leq N$. Then $(G/N, AN/N)$ satisfies $(*)$ and so, by our minimal choice we have that $AN/N \leq O_p(G/N)$. In particular, $AN \lhd\!\lhd G$. Then also $AN \cap M \lhd\!\lhd G$. If $A \leq M$, then $A = A(N \cap M) = AN \cap M$, and we have that $A \lhd\!\lhd G$. Since A is a p-subgroup, then $A \leq O_p(G)$, which is not the case. Therefore $A \not\leq M$ and by the same arguments as for N above, we conclude that AM, and thus also $AM \cap AN$, is subnormal in G. Finally note that $M \cap AN = 1$. Indeed, otherwise we have that $A \leq MN$, as $|A| = p$ and $M \cap N = 1$. Now MN/N is a minimal normal subgroup of G/N and, being isomorphic to M, it is not a p-subgroup by Step 1. Then $MN/N \cap O_p(G/N) = 1$, forcing $AN/N = 1$ a contradiction. Thus $M \cap AN = 1$ and $A = A(M \cap AN) = AM \cap AN$, therefore is subnormal in G, which again contradicts Step 1.

Step 4. M is non-abelian.

Assume that M is an elementary abelian q-group, with q a prime different from p, by Step 1. Let $Y/M = O_p(G/M)$, then by our minimal assumption $A \leq Y$. We take $P \in \mathrm{Syl}_p(Y)$ such that $A \leq P$. By the Frattini argument $G = YN = MN$, with $N = N_G(P)$. Now $[N_M(P), P] \leq M \cap P = 1$, thus $N_M(P) = C_M(P)$. Also, M being normal and abelian, $C_M(P)$ is normalised by both M and $N_G(P)$, thus $C_M(P) \trianglelefteq G$. As M is the unique minimal normal subgroup of G, we have that either $C_M(P) = 1$ or $C_M(P) = M$. Note that in the latter case P is normal in G, which contradicts Step 1. Therefore we have that G is a split extension $G = M \rtimes N$. Moreover, since M is the unique minimal normal subgroup of G, $C_N(M) = 1$, i.e., N acts faithfully on M. Let m be an arbitrary non-trivial element of M. By condition $(*)$ there exists some $n \in N$ such that $A \lhd\lhd \langle A, m^n \rangle$. In particular the subgroup $V := \left\langle a, a^{m^n} \right\rangle$ is a p-group. As $m^n \in M$, $MV = MA$, and therefore

$$V = MA \cap V = (M \cap V)A = A,$$

as M and V have coprime orders. Therefore $\langle a \rangle = \left\langle a^{m^n} \right\rangle$, i.e., m^n normalises A. In particular, as M is a normal q-subgroup, we have that

$$[a, m^n] \in A \cap M = 1,$$

which means that $A \subseteq C_N(m)^n$. By the arbitrary choice of m in M we have reached a contradiction with Lemma 3.1.

Step 5.

Let $M = S_1 \times S_2 \times \ldots \times S_n$ be the unique minimal normal subgroup of G, with all the S_i's isomorphic to a finite non-abelian simple group S. Denote by π_i the projection map of M onto S_i, for every $i = 1, 2, \ldots n$. Let also $1 = x_1, x_2, \ldots, x_n$ be elements of G such that $S_1^{x_i} = S_i$, for $i = 1, 2, \ldots, n$. Let K be the kernel of the permutation action of G on the set $\mathcal{S} := \{S_1, S_2, \ldots, S_n\}$, i.e.,

$$K := \bigcap_{i=1}^{n} N_G(S_i).$$

We treat separately the two cases: $A \nleq K$ and $A \leq K$.

Case 1. $A \nleq K$.

Set $\overline{G} := G/K$ and use the "bar" notation to denote subgroups and elements of $\overline{G}$. By induction, we have that $\overline{A} \leq O_p(\overline{G})$. Since p is odd, by Gluck's Theorem ([16, Cor. 5.7]) there exists a proper non-empty subset $\mathcal{R} \subset \mathcal{S}$ such that

$$\overline{G}_{\mathcal{R}} \cap O_p(\overline{G}) = \overline{1},$$

where $G_{\mathcal{R}}$ denotes the stabiliser in G of the set $\mathcal{R}$. Without loss of generality, we may assume $\mathcal{R} = \{S_1, \ldots, S_r\}$ for some $r < n$. Let q be any prime different from p dividing $|S_1|$, and let $s_1 \in S_1$ be any non-trivial q-element. Set

$$s_{\mathcal{R}} := s_1 s_1^{x_2} \ldots s_1^{x_r} \in M.$$

By assumption, there exists a G-conjugate of $s_{\mathcal{R}}$, say $y := s_{\mathcal{R}}^g$, such that A is subnormal in $\langle a, y \rangle$, in particular $\langle a, a^y \rangle$ is a p-subgroup. Thus $[a, y] = a^{-1}a^y$ is a p-element. Also, $[a, y]$ is G-conjugate to $[a^{g^{-1}}, s_{\mathcal{R}}]$. Since $\overline{a^{g^{-1}}}$ is a non-trivial element of $O_p(\overline{G})$, $a^{g^{-1}}$ does not stabilise $\mathcal{R}$, therefore there exists some $i \in \mathcal{R}$ such that $(S_i)^{a^{g^{-1}}} = S_j$ for some $j \notin \mathcal{R}$, this forces that $\pi_j([a^{g^{-1}}, s_{\mathcal{R}}])$ is a non-trivial q-element of S_j, and since $p \neq q$, $[a^{g^{-1}}, s_{\mathcal{R}}]$ cannot be a non-trivial p-element of M. So we have $[a, y] = 1$, but then $\langle a \rangle$ stabilises $\mathcal{R}$, which is in contradiction with $\overline{G}_{\mathcal{R}} \cap O_p(\overline{G}) = \overline{1}$.

Case 2. $A \leq K$.
We consider first the case in which $A \leq C_G(S_i)$, for every $i = 1, \ldots, n$. Then

$$A \leq \bigcap_{i=1}^{n} C_G(S_i) = (C_G(S_1))_G,$$

the normal core of S_1 in G. Since M is the unique minimal normal subgroup of G and $M \not\leq (C_G(S_1))_G$, we necessarily have that $(C_G(S_1))_G = 1$, and so $A = 1$, which is a contradiction.
Assume now that A does not centralise some S_i, say S_1. Let $1 \neq s_1 \in S_1$ and let $m = s_1 s_1^{x_2} \ldots s_1^{x_n} \in M$. Let $g \in G$ be such that $A \lhd\!\lhd \langle A, m^g \rangle$. Writing $m^g = h_1 k$, with $h_1 = s_1^{x_i g} \in S_1$ for some $i = 1, \ldots, n$, and $k = \prod_{j \neq i} s_1^{x_j g} \in S_2 \times \ldots \times S_n$ we have that for every $u, v \in \mathbb{N}$

$$[a^u, (m^g)^v] = [a^u, h_1^v k^v] = [a^u, k^v][a^u, h_1^v]^{k^v} = [a^u, h_1^v][a^u, k^v],$$

since A normalises each S_i and S_1 is centralised by S_j, for every $j \neq 1$. In particular we have that

$$[A, \langle m^g \rangle] = [A, \langle h_1 \rangle] \times [A, \langle k \rangle].$$

Therefore $\pi_1([A, \langle m^g \rangle]) = [A, \langle h_1 \rangle]$ is a p-subgroup of S_1. Finally note that $h_1 = s_1^{x_i g}$, and so for the arbitrary element $s_1 \in S_1$ there exists $x_i g \in N_G(S_1)$ such that $A \lhd\!\lhd \langle A, s_1^{x_i g} \rangle$. In particular if s_1 is chosen to be a p'-element of S we have that s_1 normalises a non-trivial p-subgroup of G which is generated by G-conjugates of A. Therefore, as $A \not\leq C_G(S_1)$, we have proved that the almost simple group $\tilde{G} := N_G(S_1)/C_G(S_1)$ contains a non-trivial subgroup of order p, namely $\tilde{A} := AC_G(S_1)/C_G(S_1)$, such that for every cyclic p'-subgroup $\widetilde{X}$ of $F^*(\tilde{G})$ we have that $\text{И}_{\tilde{G}}^{\tilde{A}}(\tilde{X}, p) \neq \varnothing$. This is in contradiction to Theorem 2.1. $\square$

We end this section by showing with an easy example that for $p = 2$ Theorem A is no more true.

Example 3.2. Let H be a Sylow 2-subgroup of $\mathrm{GL}_2(3)$, namely H is a semidihedral group of order 16, acting, in the natural way, on the natural module $M \simeq C_3 \times C_3$. Let G be the semidirect product $M \rtimes H$, and a a non-central involution of H. Since $O_2(G) = 1$, $\langle a \rangle$ is not subnormal in G. We show that $\langle a \rangle$ satisfies $(*)$. A non-trivial element of G has order either a 2-power, or 3, or 6. In the first case, it is conjugate to an element g of H and so $\langle a \rangle$ is subnormal in the 2-group $\langle a, g \rangle$. In the second case, the element lies in M, but note that every element of M centralises a conjugate in H of a, i.e., $M = \bigcup_{x \in H} C_M(a^x)$, thus there exists an $x \in H$ such that $\langle a \rangle$ is subnormal in $\langle a, g^x \rangle \simeq C_6$. In the latter case, up to conjugation, we have $\langle a, g \rangle = \langle g \rangle$.

4. Other conditions for subnormality

As stated in the Introduction, in this Section we briefly analyse similar variations related to the other criteria for subnormality given by the original Theorem of Wielandt (namely conditions (iii) and (iv)). We see that in general these generalisations fail to guarantee the subnormality of odd p- subgroups.

Given a finite group G and an odd p-subgroup A of G, we consider the following condition:

$(**)$ *for every conjugacy class C of G there exists $g \in C$ such that $A \lhd\lhd \langle A, A^g \rangle$.*

It is trivial that condition $(*)$ implies $(**)$.
The next result shows that $(**)$ is enough to guarantee the subnormality of A in the class of finite solvable groups.

Theorem 4.1. *Let G be a finite solvable group and p a prime. If there exists a p-subgroup A of G satisfying $(**)$ then $A \leq O_p(G)$.*

Proof. As A is nilpotent, every subgroup of A also satisfies $(**)$. Therefore we can assume that A is cyclic, say $A = \langle a \rangle$.
We argue by induction on the order of G.
Note that if M is a minimal normal subgroup of G the assumption holds for the group G/M. Thus in particular, we may assume that G admits a unique minimal normal subgroup, say M, and that $aM \in O_p(G/M) := Y/M$. Now if M is a p-group we are done. Let M be an elementary abelian q-group, with $q \neq p$. Take P a Sylow p-subgroup of Y containing a, so that by the Frattini argument $G = YN = MN$, with $N = N_G(P)$. Being M minimal normal in G, we have that $C_M(P) = M \cap N_G(P) = 1$ (otherwise $Y = M \times P$ and $a \in O_p(G)$). Thus $G = M \rtimes N$. Since also M is the unique minimal normal subgroup of G we have $C_N(M) = 1$. Let m be a non-trivial element of M. By assumption there exists $n \in N$ such that the subgroup $V := \left\langle a, a^{m^n} \right\rangle$ is nilpotent. In particular, as $m^n \in M$, $MV = M\langle a \rangle$, and therefore

$$V = M\langle a \rangle \cap V = (M \cap V) \times \langle a \rangle ,$$

forcing $\langle a \rangle = \left\langle a^{m^n} \right\rangle$. So

$$[a, m^n] \in A \cap M = 1,$$

which means that $A \subseteq C_N(m)^n$. By the arbitrary choice of m in M we have reached a contradiction with Lemma 3.1. $\square$

For non-solvable groups the situation is completely different and the following example shows that there are almost simple groups with non-trivial p-subgroups satisfying $(**)$.

Example 4.2. Every subgroup of $\mathfrak{S}_8$ generated by a 3-cycle satisfies $(**)$. Indeed, let $A = \langle (123) \rangle$ and C a conjugacy class of $\mathfrak{S}_8$. If every element of C is the product of at least three disjoint cycles of length > 1, then C contains $g = (14...)(25...)(36...)...$ and so A is subnormal in the abelian subgroup $\langle A, A^g \rangle$. If C contains a k-cycle, then $k \geq 6$ otherwise there exists g in C fixing pointwise $\{1, 2, 3\}$ and so $\langle A, A^g \rangle = A$. But then take $g = (142536...) \in C$ and argue as before. The remaining case is when the elements of C are products of two disjoint cycles and fix at most two points. If one of these cycle is a 3-cycle, then $g = (123)... \in C$, forcing again $A^g = A$. We can then assume that one cycle is at least a 2-cycle and the other a 4-cycle, but then take $g = (14...)(2536...) \in C$ and conclude as before.

A similar behaviour can be noticed for every prime p, if n is big enough.

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