The Chow form of the essential variety in computer vision

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Abstract

The Chow form of the essential variety in computer vision is calculated. Our derivation uses secant varieties, Ulrich sheaves and representation theory. Numerical experiments show that our formula can detect noisy point correspondences

Keywords: Chow form, Ulrich sheaf, Pieri resolutions, calibrated cameras, essential variety.

1. Introduction

The *essential variety* $\mathcal{E}$ is the variety of 3×3 real matrices with two equal singular values, and the third one equal to zero ($\sigma_1 = \sigma_2, \sigma_3 = 0$). It was introduced in the setting of computer vision; see (Hartley and Zisserman , 2004, §9.6). Its elements, the so-called *essential matrices*, have the form TR, where T is real skew-symmetric and R is real orthogonal. The essential variety is a cone of codimension 3 and degree 10 in the space of 3×3-matrices, defined by homogeneous cubic equations, that we recall in (2). The complex solutions of these cubic equations define the complexification $\mathcal{E}_{\mathbb{C}}$ of the essential variety. While the real essential variety is smooth, its complexification has a singular locus that we describe precisely in §2.

The *Chow form* of a codimension c projective variety $X \subset \mathbb{P}^n$ is the equation $\text{Ch}(X)$ of the divisor in the Grassmannian $\text{Gr}(\mathbb{P}^{c-1}, \mathbb{P}^n)$ given by those linear subspaces of dimension $c - 1$ which meet X. It is a basic and classical tool that allows one to recover much geometric information about X; for its main properties we refer to (Gelfand et al. , 1994, §4). In (Agarwal et al. , 2015, §4), the problem of computing the Chow form of the essential variety was posed,

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while the analogous problem for the *fundamental variety* was solved, another important variety in computer vision.

The main goal of this paper is to explicitly find the Chow form of the essential variety. This provides an important tool for the problem of detecting if a set of image point correspondences $\{(x^{(i)}, y^{(i)}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid i = 1, \ldots, m\}$ comes from m world points in $\mathbb{R}^3$ and two calibrated cameras. It furnishes an exact solution for $m = 6$ and it behaves well given noisy input, as we will see in §4. Mathematically, we can consider the system of equations:

$$\begin{cases} A\widetilde{X^{(i)}} \equiv \widetilde{x^{(i)}} \\ B\widetilde{X^{(i)}} \equiv \widetilde{y^{(i)}}. \end{cases} \tag{1}$$

Here $\widetilde{x^{(i)}} = (x_1^{(i)} : x_2^{(i)} : 1)^T \in \mathbb{P}^2$ and $\widetilde{y^{(i)}} = (y_1^{(i)} : y_2^{(i)} : 1)^T \in \mathbb{P}^2$ are the given image points. The unknowns are two 3×4 matrices A, B with rotations in their left 3×3 blocks and $m = 6$ points $\widetilde{X^{(i)}} \in \mathbb{P}^3$. These represent calibrated cameras and world points, respectively. A calibrated camera has normalized image coordinates, as explained in (Hartley and Zisserman , 2004, §8.5). Here $\equiv$ denotes equality up to nonzero scale. From our calculation of $\mathrm{Ch}(\mathcal{E}_{\mathbb{C}})$, we deduce:

Theorem 1. *There exists an explicit* 20×20 *skew-symmetric matrix* $\mathcal{M}(x, y)$ *of degree* $\leq (6, 6)$ *polynomials over* $\mathbb{Z}$ *in the coordinates of* $(x^{(i)}, y^{(i)})$ *with the following properties. If* (1) *admits a complex solution then* $\mathcal{M}(x^{(i)}, y^{(i)})$ *is rank-deficient. Conversely, the variety of point correspondences* $(x^{(i)}, y^{(i)})$ *such that* $\mathcal{M}(x^{(i)}, y^{(i)})$ *is rank-deficient contains a dense open subset for which* (1) *admits a complex solution.*

In fact, we will produce two such matrices. Both of them, along with related formulas we derive, are available in ancillary files accompanying the `arXiv` version of this paper, and we have posted them at `http://math.berkeley.edu/~jkileel/ChowFormulas.html`.

Our construction of the Chow form uses the technique of *Ulrich sheaves* introduced in (Eisenbud and Schreyer , 2003). We construct rank 2 Ulrich sheaves on the essential variety $\mathcal{E}_{\mathbb{C}}$. For an analogous construction of the Chow form of $K3$ surfaces, see (Aprodu et al. , 2012).

From the point of view of computer vision, this paper contributes a complete characterization for an 'almost-minimal' problem. Here the motivation is *3D reconstruction*. Given multiple images of a world scene, taken by cameras in an unknown configuration, we want to estimate the camera configuration and a 3D model of the world scene. Algorithms for this are complex, and successful. See (Agarwal et al. , 2009) for a reconstruction from 150,000 images.

By contrast, the system (1) encodes a tiny reconstruction problem. Suppose we are given six point correspondences in two calibrated pictures (the right-hand sides in (1)). We wish to reconstruct both the two cameras and the six world points (the left-hand sides in (1)). If an exact solution exists then it is typically unique, modulo the natural symmetries. However, an exact solution does not always exist. In order for this to happen, a giant polynomial of degree 120 in the 24 variables on the right-hand sides has to vanish. Theorem 1 gives an explicit matrix formula for that polynomial.

The link between minimal or almost-minimal reconstructions and large-scale reconstructions is surprisingly strong. Algorithms for the latter use the former reconstructions repeatedly as core subroutines. In particular, solving the system (1) given $m = 5$ point pairs, instead of $m = 6$, is a subroutine in (Agarwal et al. , 2009). This solver is optimized in (Nistér , 2004). It is used to generate hypotheses inside *random sampling consensus* (RANSAC) (Fischler and Bolles , 1981) schemes for robust reconstruction from pairs of calibrated images. See (Hartley and Zisserman , 2004) for more vision background.

This paper is organized as follows. In §2, we prove that $\mathcal{E}_{\mathbb{C}}$ is a hyperplane section of the variety $PX_{4,2}^s$ of 4×4 symmetric matrices of rank ≤ 2. This implies a determinantal description of $\mathcal{E}_{\mathbb{C}}$; see Proposition 7. A side result of the construction is that $\mathcal{E}_{\mathbb{C}}$ is the secant variety of its singular locus, which corresponds to pairs of isotropic vectors in $\mathbb{C}^3$.

In §3, we construct two Ulrich sheaves on the variety of 4×4 symmetric matrices of rank ≤ 2. One of the constructions we propose is new to our knowledge. Both sheaves are GL(4)-equivariant, and they admit "Pieri resolutions" in the sense of (Sam and Weyman , 2011). We carefully analyze the resolutions using representation theory, and in particular show that their middle differentials may be represented by symmetric matrices; see Propositions 16 and 19.

In §4, we combine the results of the previous sections and we construct the Chow form of the essential variety. The construction from (Eisenbud and Schreyer , 2003) starts with our rank 2 Ulrich sheaves and allows to define two 20×20 matrices in the Plücker coordinates of $\mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^8)$ each of which drops rank exactly when the corresponding subspace $\mathbb{P}^2$ meets the essential variety $\mathcal{E}_{\mathbb{C}}$. It requires some technical effort to put these matrices in skew-symmetric form, and here our analysis from §3 pays off. We conclude the paper with numerical experiments demonstrating the robustness to noise that our matrix formulas in Theorem 1 enjoy.

2. The essential variety is a determinantal variety

2.1. Intrinsic description

Let $\mathcal{E} \subset \mathbb{R}^{3 \times 3}$ be the essential variety defined by the conditions:

$$\mathcal{E} := \{M \in \mathbb{R}^{3 \times 3} \mid \sigma_1(M) = \sigma_2(M), \ \sigma_3(M) = 0\}.$$

The polynomial equations of $\mathcal{E}$ are (see (Faugeras and Maybank , 1990, §4)) as follows:

$$\mathcal{E} = \{M \in \mathbb{R}^{3 \times 3} \mid \det(M) = 0, \ 2(MM^T)M - \mathrm{tr}\left(MM^T\right)M = 0\}. \tag{2}$$

These 10 cubics minimally generate the *real radical ideal* (Bochnak et al. , 1998, p.85) of the essential variety $\mathcal{E}$, and that ideal is prime. Indeed, the real radical property follows from our Proposition 2(i) and (Marshall , 2008, Theorem 12.6.1). We denote by $\mathcal{E}_{\mathbb{C}}$ the projective variety in $\mathbb{P}_{\mathbb{C}}^8$ given by the complex solutions of (2). The essential variety $\mathcal{E}_{\mathbb{C}}$ has codimension 3 and degree 10 (see (Maybank , 1993, Theorem 5.16)). In this section, we will prove that it is isomorphic to a hyperplane section of the variety $PX_{4,2}^s$ of complex symmetric 4×4 matrices of rank ≤ 2. The first step towards this is Proposition 2 below, and that relies on the group symmetries of $\mathcal{E}_{\mathbb{C}}$, which we now explain.

Consider $\mathbb{R}^3$ with the standard inner product Q, and the corresponding action of SO$(3, \mathbb{R})$ on $\mathbb{R}^3$. Complexify $\mathbb{R}^3$ and consider $\mathbb{C}^3$ with the action of SO$(3, \mathbb{C})$, which has universal cover SL$(2, \mathbb{C})$. It is technically simpler to work with the action of SL$(2, \mathbb{C})$. Denoting by U the irreducible 2-dimensional representation of SL$(2, \mathbb{C})$, we have the equivariant isomorphism $\mathbb{C}^3 \cong$

S_2U. Writing Q also for the complexification of the Euclidean product, the projective space $\mathbb{P}(S_2U)$ divides into two $SL(2, \mathbb{C})$-orbits, namely the isotropic quadric with equation $Q(u) = 0$ and its complement. Let V be another complex vector space of dimension 2. The essential variety $\mathcal{E}_\mathbb{C}$ is embedded into the projective space of 3×3-matrices $\mathbb{P}(S_2U \otimes S_2V)$. Since the singular value conditions defining $\mathcal{E}$ are $SO(3, \mathbb{R}) \times SO(3, \mathbb{R})$-invariant, it follows that $\mathcal{E}_\mathbb{C}$ is $SL(U) \times SL(V)$-invariant using (Drusvyatski et al. , 2016, Theorem 2.2).

The following is a new geometric description of the essential variety. From the computer vision application, we start with the set of real points $\mathcal{E}$. However, below we see that the surface $\mathrm{Sing}(\mathcal{E}_\mathbb{C})$ inside $\mathcal{E}_\mathbb{C}$, which has no real points, 'determines' the algebraic geometry. Part (i) of Prop. 2 is proved also in (Maybank , 1993, Prop. 5.9).

Proposition 2. *(i) The singular locus of $\mathcal{E}_\mathbb{C}$ is the projective surface given by:*

$$\mathrm{Sing}(\mathcal{E}_\mathbb{C}) = \left\{ ab^T \in \mathbb{P}(\mathbb{C}^{3\times3}) \,|\, Q(a) = Q(b) = 0 \right\}.$$

(ii) The second secant variety of $\mathrm{Sing}(\mathcal{E}_\mathbb{C})$ equals $\mathcal{E}_\mathbb{C}$.

Proof. Denote by S the variety $\left\{ ab^T \in \mathbb{P}(\mathbb{C}^{3\times3}) \,|\, Q(a) = Q(b) = 0 \right\}$, and let $\widehat{S}$ be the affine cone over it. The line secant variety $\sigma_2(\widehat{S})$ consists of elements of the form $M = a_1 b_1^T + a_2 b_2^T \in \mathbb{C}^{3\times3}$ such that $Q(a_i) = a_i^T a_i = Q(b_i) = b_i^T b_i = 0$ for $i = 1, 2$. We compute that $MM^T = a_1 b_1^T b_2 a_2^T + a_2 b_2^T b_1 a_1^T$ so that $\mathrm{tr}(MM^T) = 2(b_1^T b_2)(a_1^T a_2)$. Moreover $MM^T M = a_1 b_1^T b_2 a_2^T a_1 b_1^T + a_2 b_2^T b_1 a_1^T a_2 b_2^T = (b_1^T b_2)(a_1^T a_2)M$. Hence the equations (2) of $\mathcal{E}_\mathbb{C}$ are satisfied by M. This proves that $\sigma_2(S) \subset \mathcal{E}_\mathbb{C}$. S is a surface and $\sigma_2(S)$ has dimension 5 (see (Chiantini and Ciliberto , 2002, Theorem 1.3)). Since $\sigma_2(S)$ and $\mathcal{E}_\mathbb{C}$ are both of codimension 3 and $\mathcal{E}_\mathbb{C}$ is irreducible, the equality $\sigma_2(S) = \mathcal{E}_\mathbb{C}$ follows. It remains to prove (i). Denote by $[a_i]$ the line generated by a_i. Every element $a_1 b_1^T + a_2 b_2^T$ with $[a_1] \neq [a_2]$, $[b_1] \neq [b_2]$ and $Q(a_i) = Q(b_i) = 0$ for $i = 1, 2$ can be taken by $SL(U) \times SL(V)$ to a scalar multiple of any other element of the same form. This is the open orbit of the action of $SL(U) \times SL(V)$ on $\mathcal{E}_\mathbb{C}$. The remaining orbits are the following:

1. the surface S, with set-theoretic equations $MM^T = M^T M = 0$.
2. $T_1 \setminus S$, where $T_1 = \left\{ ab^T \in \mathbb{P}(\mathbb{C}^{3\times3}) \,|\, Q(a) = 0 \right\}$ is a threefold, with set-theoretic equations $M^T M = 0$.
3. $T_2 \setminus S$, where $T_2 = \left\{ ab^T \in \mathbb{P}(\mathbb{C}^{3\times3}) \,|\, Q(b) = 0 \right\}$ is a threefold, with set-theoretic equations $MM^T = 0$.
4. $\mathrm{Tan}(S) \setminus (T_1 \cup T_2)$, where the *tangential variety* $\mathrm{Tan}(S)$ is the fourfold union of all tangent spaces to S, with set-theoretic equations $\mathrm{tr}(MM^T) = 0, MM^T M = 0$.

It is easy to check they are orbits, in a similar way than in (Gelfand et al. , 1994, Example 14.4.5). One can compute explicitly that the Jacobian matrix of $\mathcal{E}_\mathbb{C}$ at $\begin{pmatrix} 1 & 0 & 0 \\ \sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_1 \setminus S$ has rank 3. The following code in `Macaulay2` (Grayson and Stillman) does that computation:

```
R = QQ[m_(1,1)..m_(3,3)]
M = transpose(genericMatrix(R,3,3))
I = ideal(det(M))+minors(1,2*M*transpose(M)*M - trace(M*transpose(M))*M)
Jac = transpose jacobian I
S = QQ[q]/(1+q^2)
specializedJac = (map(S,R,{1,0,0,q,0,0,0,0,0}))(Jac)
minors(3,specializedJac)
```

4

Hence the points in $T_1 \setminus S$ are smooth points of $\mathcal{E}_\mathbb{C}$. By symmetry, also the points in $T_2 \setminus S$ are smooth. By semicontinuity, the points in $\mathrm{Tan}(S) \setminus (T_1 \cup T_2)$ are smooth. Since points in S are singular for the secant variety $\sigma_2(S)$, this finishes the proof of (i). $\qquad\square$

Remark 3. *From the study of tensor decomposition, the parametric description in Proposition 2 is identifiable, meaning that, from the matrix $a_1 b_1^T + a_2 b_2^T$, all a_i, b_i are determined up to scalar multiple. That shows that real essential matrices have the form $a^T b + \overline{a}^T \overline{b}$ with $a, b \in \mathbb{C}^3$ and $Q(a) = Q(b) = 0$. This may be written in the alternative form $(u^2)^T v^2 + (\overline{u}^2)^T \overline{v}^2 \in S_2(U) \otimes S_2(V)$ with $u \in U$, $v \in V$. This may help in computing real essential matrices. Note that the four non-open orbits listed in the proof of Proposition 2 are contained in the isotropic quadric $\mathrm{tr}(MM^T) = 0$, hence they have no real points.*

Remark 4. *The surface $\mathrm{Sing}(\mathcal{E}_\mathbb{C})$ is more familiar with the embedding by $O(1, 1)$, when it is the smooth quadric surface, doubly ruled by lines. In the embedding by $O(2, 2)$, the two rulings are given by conics. These observations suggests expressing $\mathcal{E}_\mathbb{C}$ as a determinantal variety, as we do next in Proposition 5. Indeed, note that the smooth quadric surface embedded by $O(2, 2)$ is isomorphic to a linear section of the second Veronese embedding of $\mathbb{P}^3$, which is the variety of 4×4 symmetric matrices of rank 1.*

In the following note that $S_2(U \otimes V)$ is 10-dimensional and identifies as the space of symmetric 4×4-matrices.

Proposition 5. *The essential variety $\mathcal{E}_\mathbb{C}$ is isomorphic to a hyperplane section of the variety of rank ≤ 2 elements in $\mathbb{P}(S_2(U \otimes V))$. Concretely, this latter variety identifies as the projective variety of 4×4 symmetric matrices of rank ≤ 2 (see also Subsection 3.2), and the section consists of traceless 4×4 symmetric matrices of rank ≤ 2.*

Proof. The embedding of $\mathbb{P}(U) \times \mathbb{P}(V)$ in $\mathbb{P}(S_2(U) \otimes S_2(V))$ is given by $(u, v) \mapsto u^2 \otimes v^2$. Recall that Cauchy's formula states $S_2(U \otimes V) = (S_2(U) \otimes S_2(V)) \oplus (\wedge^2 U \otimes \wedge^2 V)$, where $\dim(U \otimes V) = 4$. Hence, $\mathbb{P}(S_2(U) \otimes S_2(V))$ is equivariantly embedded as a codimension one subspace in $\mathbb{P}(S_2(U \otimes V))$. The image is the subspace of traceless elements (since that is dimension 8 and invariant), and this map sends $u^2 \otimes v^2 \mapsto (u \otimes v)^2$. By Proposition 2, we have shown that $\mathrm{Sing}(\mathcal{E}_\mathbb{C})$ embeds into a hyperplane section of the variety of rank 1 elements in $\mathbb{P}(S_2(U \otimes V))$. So, $\mathcal{E}_\mathbb{C} = \sigma_2(\mathrm{Sing}(\mathcal{E}_\mathbb{C}))$ embeds into that hyperplane section of the variety of rank ≤ 2 elements. This last variety has degree 10 by Segre formula (Harris and Tu , 1984, Proposition 12 (b)). Comparing dimensions and degrees, the result follows. $\qquad\square$

Remark 6. *In light of the description in Proposition 5, it follows by Example 3.2 and Corollary 6.4 of (Draisma et al. , 2016) that the Euclidean distance degree is $\mathrm{EDdegree}(\mathcal{E}_\mathbb{C}) = 6$. This result has been proved also in (Drusvyatski et al. , 2015), where the computation of $\mathrm{EDdegree}$ was performed in the more general setting of orthogonally invariant varieties. This quantity measures the algebraic complexity of finding the nearest point on $\mathcal{E}$ to a given noisy data point in $\mathbb{R}^{3 \times 3}$.*

2.2. Coordinate description

We now make the determinantal description of $\mathcal{E}_\mathbb{C}$ in Proposition 5 explicit in coordinates. For this, denote $a = (a_1, a_2, a_3)^T \in \mathbb{C}^3$. We have $Q(a) = a_1^2 + a_2^2 + a_3^2$. The $\mathrm{SL}(2, \mathbb{C})$-orbit $Q(a) = 0$

is parametrized by $\left(u_1^2 - u_2^2, 2u_1u_2, \sqrt{-1}(u_1^2 + u_2^2)\right)^T$ where $(u_1, u_2)^T \in \mathbb{C}^2$. Let:

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \in \mathbb{C}^{3\times 3},$$

and define the 4×4 traceless symmetric matrix $s(M)$ (depending linearly on M):

$$s(M) := \frac{1}{2} \begin{pmatrix} m_{11} - m_{22} - m_{33} & m_{13} + m_{31} & m_{12} + m_{21} & m_{23} - m_{32} \\ m_{13} + m_{31} & -m_{11} - m_{22} + m_{33} & m_{23} + m_{32} & m_{12} - m_{21} \\ m_{12} + m_{21} & m_{23} + m_{32} & -m_{11} + m_{22} - m_{33} & -m_{13} + m_{31} \\ m_{23} - m_{32} & m_{12} - m_{21} & -m_{13} + m_{31} & m_{11} + m_{22} + m_{33} \end{pmatrix}. \tag{3}$$

This construction furnishes a new view on the essential variety $\mathcal{E}$, as described in Proposition 7.

Proposition 7. *The linear map s in (3) is a real isometry from the space of 3×3 real matrices to the the space of traceless symmetric 4×4 real matrices. We have that:*

$$M \in \mathcal{E} \iff \operatorname{rk}(s(M)) \le 2.$$

The complexification of s, denoted again by s, satisfies for any $M \in \mathbb{C}^{3\times 3}$:

$$M \in \operatorname{Sing}(\mathcal{E}_{\mathbb{C}}) \iff \operatorname{rk}(s(M)) \le 1,$$
$$M \in \mathcal{E}_{\mathbb{C}} \iff \operatorname{rk}(s(M)) \le 2.$$

Proof. We construct the correspondence over $\mathbb{C}$ at the level of $\operatorname{Sing}(\mathcal{E}_{\mathbb{C}})$ and then we extend it by linearity. Choose coordinates (u_1, u_2) in U and coordinates (v_1, v_2) in V. Consider the following parametrization of matrices $M \in \operatorname{Sing}(\mathcal{E}_{\mathbb{C}})$:

$$M = \begin{pmatrix} u_1^2 - u_2^2 \\ 2u_1u_2 \\ \sqrt{-1}(u_1^2 + u_2^2) \end{pmatrix} \cdot \left(v_1^2 - v_2^2, \ 2v_1v_2, \ \sqrt{-1}(v_1^2 + v_2^2)\right). \tag{4}$$

Consider also the following parametrization of the Euclidean quadric in $U \otimes V$:

$$k = \left(u_2v_2 - u_1v_1, \ -\sqrt{-1}(u_1v_1 + u_2v_2), \ -(u_1v_2 + u_2v_1), \ \sqrt{-1}(u_1v_2 - u_2v_1)\right).$$

The variety of rank 1 traceless 4×4 symmetric matrices is accordingly parametrized by $k^T k$. Substituting (4) into the right-hand side below, a computation verifies that:

$$k^T k = s(M).$$

This proves the second equivalence in the statement above and explains the definition of $s(M)$, namely that it is the equivariant embedding from Proposition 5 in coordinates. The third equivalence follows because $\mathcal{E}_{\mathbb{C}} = \sigma_2(\operatorname{Sing}(\mathcal{E}_{\mathbb{C}}))$, by Proposition 2(ii). For the first equivalence, we note that s is defined over $\mathbb{R}$ and now a direct computation verifies that $\operatorname{tr}\left(s(M)s(M)^T\right) = \operatorname{tr}\left(MM^T\right)$ for $M \in \mathbb{R}^{3\times 3}$. $\square$

Note that the ideal of 3-minors of $s(M)$ is indeed generated by the ten cubics in (2).

Remark 8. *The critical points of the distance function from any data point $M \in \mathbb{R}^{3\times 3}$ to $\mathcal{E}$ can be computed by means of the SVD of $s(M)$, as in* (Draisma et al. , 2016, Example 2.3)*.*

3. Ulrich sheaves on the variety of symmetric 4×4 matrices of rank ≤ 2

Our goal is to construct the Chow form of the essential variety. By the theory of (Eisenbud and Schreyer , 2003), this can be done provided one has an Ulrich sheaf on this variety. The notions of Ulrich sheaf, Chow forms and the construction of (Eisenbud and Schreyer , 2003) will be explained below.

As shown in §2, the essential variety $\mathcal{E}_{\mathbb{C}}$ is a linear section of the projective variety of symmetric 4×4 matrices of rank ≤ 2, which we denote as $PX_{4,2}^s$. If we construct an Ulrich sheaf on $PX_{4,2}^s$, then a quotient of this sheaf by a linear form is an Ulrich sheaf on $\mathcal{E}_{\mathbb{C}}$ provided that linear form is regular for the Ulrich sheaf on $PX_{4,2}^s$. We will achieve this twice, in §3.4 and §3.5.

3.1. Definition of Ulrich modules and sheaves

Definition 9. *A graded module M over a polynomial ring $A = \mathbb{C}[x_0, \ldots, x_n]$ is an* Ulrich *module provided:*

1. *It is generated in degree 0 and has a linear minimal free resolution:*

$$0 \longleftarrow M \leftarrow A^{\beta_0} \longleftarrow A(-1)^{\beta_1} \longleftarrow A(-2)^{\beta_2} \xleftarrow{d_2} \cdots \longleftarrow A(-c)^{\beta_c} \longleftarrow 0. \qquad (5)$$

2. *The length of the resolution c equals the codimension of the support of the module M.*
2.' *The Betti numbers are $\beta_i = \binom{c}{i}\beta_0$ for $i = 0, \ldots, c$.*

One can use either (1) and (2), or equivalently, (1) and (2)' as the definition.

A sheaf $\mathcal{F}$ on a projective space $\mathbb{P}^n$ with support of dimension ≥ 1 is an *Ulrich sheaf* provided it is the sheafification of an Ulrich module. Equivalently, the module of twisted global sections $M = \bigoplus_{d \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(d))$ is an Ulrich module over the polynomial ring A.

Fact 10. *If the support of an Ulrich sheaf $\mathcal{F}$ is a variety X of degree d, then β_0 is a multiple of d, say rd. This corresponds to $\mathcal{F}$ being a sheaf of rank r on X.*

Since there is a one-to-one correspondence between Ulrich modules over A and Ulrich sheaves on $\mathbb{P}^n$, we interchangably speak of both. But in our constructions we focus on Ulrich modules. A prominent conjecture of (Eisenbud and Schreyer , 2003, p.543) states that on any variety X in a projective space, there is an Ulrich sheaf whose support is X.

3.2. The variety of symmetric 4×4 matrices

We fix notation. Let X_4^s be the space of symmetric 4×4 matrices over the field $\mathbb{C}$. This identifies as $\mathbb{C}^{10}$. Let $x_{ij} = x_{ji}$ be the coordinate functions on X_4^s where $1 \leq i \leq j \leq 4$, so the coordinate ring of X_4^s is:

$$A = \mathbb{C}[x_{ij}]_{1 \leq i \leq j \leq 4}.$$

For $0 \leq r \leq 4$, denote by $X_{4,r}^s$ the affine subvariety of X_4^s consisting of matrices of rank $\leq r$. The ideal of $X_{4,r}^s$ is generated by the $(r+1) \times (r+1)$-minors of the generic 4×4 symmetric matrix (x_{ij}). This is in fact a prime ideal, by (Weyman , 2003, Theorem 6.3.1). The rank subvarieties have the following degrees and codimensions by (Harris and Tu , 1984, Proposition 12 (b)):

7

variety	degree	codimension
$X_{4,4}^s$	1	0
$X_{4,3}^s$	4	1
$X_{4,2}^s$	10	3
$X_{4,1}^s$	8	6
$X_{4,0}^s$	1	10

Since the varieties $X_{4,r}^s$ are defined by homogeneous ideals, they give rise to projective varieties $PX_{4,r}^s$ in the projective space $\mathbb{P}^9$. However, in §3.4 and §3.5 it will be convenient to work with affine varieties, and general (instead of special) linear group actions.

The group $GL(4, \mathbb{C})$ acts on X_4^s. If $M \in GL(4, \mathbb{C})$ and $X \in X_4^s$, the action is as follows:

$$M.X = M \cdot X \cdot M^T.$$

Since any complex symmetric matrix can be diagonalized by a coordinate change, there are five orbits of the action of $GL(4, \mathbb{C})$ on X_4^s, one per rank of the symmetric matrix. Let:

$$E = \mathbb{C}^4$$

be a four-dimensional complex vector space. The coordinate ring of X_4^s identifies as $A \cong$ $\mathrm{Sym}(S_2(E))$. The space of symmetric matrices X_4^s may then be identified with the dual space $S_2(E)^*$, so again we see that $GL(E) = GL(4, \mathbb{C})$ acts on $S_2(E)^*$.

3.3. Representations and Pieri's rule

We shall recall some basic representation theory of the general linear group $GL(W)$, where W is a n-dimensional complex vector space. The irreducible representations of $GL(W)$ are given by Schur modules $S_\lambda(W)$ where λ is a generalized partition: a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. When $\lambda = d, 0, \ldots, 0$, then $S_\lambda(W)$ is the dth symmetric power $S_d(W)$. When $\lambda = 1, \ldots, 1, 0, \ldots, 0$, with d 1's, then $S_\lambda(W)$ is the exterior wedge $\wedge^d W$. For all partitions λ there are isomorphisms of $GL(W)$-representations:

$$S_\lambda(W)^* \cong S_{-\lambda_n,\ldots,-\lambda_1}(W) \qquad \text{and} \qquad S_\lambda(W) \otimes (\wedge^n W)^{\otimes r} \cong S_{\lambda + r \cdot \mathbf{1}}(W)$$

where $\mathbf{1} = 1, 1, \ldots, 1$. Here $\wedge^n W$ is the one-dimensional representation $\mathbb{C}$ of $GL(W)$ where a linear map ϕ acts by its determinant.

Denote by $|\lambda| := \lambda_1 + \cdots + \lambda_n$. Assume $\lambda_n, \mu_n \geq 0$. The tensor product of two Schur modules $S_\lambda(W) \otimes S_\mu(W)$ splits into irreducibles as a direct sum of Schur modules:

$$\bigoplus_\nu u(\lambda, \mu; \nu) S_\nu(W)$$

where the sum is over partitions with $|\nu| = |\mu| + |\lambda|$. The multiplicities $u(\lambda, \mu; \nu) \in \mathbb{Z}_{\geq 0}$ are determined by the Littlewood-Richardson rule (Fulton and Harris , 1991, Appendix A). In one case, that will be important to us below, there is a particularly nice form of this rule. Given two partitions λ' and λ, we say that λ'/λ is a *horizontal strip* if $\lambda_i' \geq \lambda_i \geq \lambda_{i+1}'$.

Fact 11 (Pieri's rule). *As $GL(W)$-representations, we have the rule:*

$$S_\lambda(W) \otimes S_d(W) \quad \cong \bigoplus_{\substack{|\lambda'| = |\lambda| + d \\ \lambda'/\lambda \text{ is a horizontal strip}}} S_{\lambda'}(W).$$

8

3.4. The first Ulrich sheaf

We are now ready to describe our first Ulrich sheaf on the projective variety $PX^s_{4,2}$. We construct it as an Ulrich module supported on the variety $X^s_{4,2}$. We use notation from §3.2, so E is 4-dimensional. Consider $S_3(E) \otimes S_2(E)$. By Pieri's rule this decomposes as:

$$S_5(E) \oplus S_{4,1}(E) \oplus S_{3,2}(E).$$

We therefore get a GL(E)-inclusion $S_{3,2}(E) \to S_3(E) \otimes S_2(E)$ unique up to nonzero scale. Since $A_1 = S_2(E)$ from §3.2, this extends uniquely to an A-module map:

$$S_3(E) \otimes A \xleftarrow{\ \alpha\ } S_{3,2}(E) \otimes A(-1).$$

This map can easily be programmed using `Macaulay2` and the package `PieriMaps` (Sam , 2009):

```
R=QQ[a..d]
needsPackage "PieriMaps"
f=pieri({3,2},{2,2},R)
S=QQ[a..d,y_0..y_9]
a2=symmetricPower(2,matrix{{a..d}})
alpha=sum(10,i->contract(a2_(0,i),sub(f,S))*y_i)
```

We can then compute the resolution of the cokernel of α in `Macaulay2`. It has the form:

$$A^{20} \xleftarrow{\ \alpha\ } A(-1)^{60} \longleftarrow A(-2)^{60} \leftarrow A(-3)^{20}.$$

Thus the cokernel of α is an Ulrich module by (1) and (2)' in Definition 9. An important point is that the `res` command in `Macaulay2` computes differential matrices in unenlightening bases. We completely and intrinsically describe the GL(E)-resolution below:

Proposition 12. *The cokernel of α is an Ulrich module M of rank 2 supported on the variety $X^s_{4,2}$. The resolution of M is GL(E)-equivariant and it is:*

$$F_\bullet : S_3(E) \otimes A \xleftarrow{\ \alpha\ } S_{3,2}(E) \otimes A(-1) \xleftarrow{\ \phi\ } S_{3,3,1}(E) \otimes A(-2) \tag{6}$$
$$\xleftarrow{\ \beta\ } S_{3,3,3}(E) \otimes A(-3)$$

with ranks $20, 60, 60, 20$, and where all differential maps are induced by Pieri's rule. The dual complex of this resolution is also a resolution, and these two resolutions are isomorphic up to twist. As in (Sam and Weyman , 2011), we can visualize the resolution by:

$$0 \leftarrow M \longleftarrow \boxed{} \leftarrow \boxed{} \leftarrow \boxed{} \leftarrow \boxed{} \leftarrow 0.$$

Proof. Since M is the cokernel of a GL(E)-map, it is GL(E)-equivariant. So, the support of M is a union of orbits. By Definition 9(2), M is supported in codimension 3. Since the only orbit of codimension 3 is $X^s_{4,2} \backslash X^s_{4,3}$, the support of M is the closure of this orbit, which is $X^s_{4,2}$. It can also easily be checked with `Macaulay2`, by restricting α to diagonal matrices of rank r for $r = 0, \ldots, 4$, that M is supported on the strata $X^s_{4,r}$ where $r \leq 2$. Also, the statement that the rank of M equals 2 is now immediate from Fact 10.

9

Now we prove that the GL(E)-equivariant minimal free resolution of M is $F_\bullet$ as above. By Pieri's rule there is a GL(E)-map unique up to nonzero scalar:

$$S_{3,2}(E) \otimes S_2(E) \longleftarrow S_{3,3,1}(E)$$

and a GL(E)-map unique up to nonzero scalar:

$$S_{3,3,1}(E) \otimes S_2(E) \longleftarrow S_{3,3,3}(E).$$

These are the maps ϕ and β in $F_\bullet$ respectively. The composition $\alpha \circ \phi$ maps $S_{3,3,1}(E)$ to a submodule of $S_3(E) \otimes S_2(S_2(E))$. By (Weyman , 2003, Proposition 2.3.8) the latter double symmetric power equals $S_4(E) \oplus S_{2,2}(E)$, and so this tensor product decomposes as:

$$S_3(E) \otimes S_4(E) \bigoplus S_3(E) \otimes S_{2,2}(E).$$

By Pieri's rule, none of these summands contains $S_{3,3,1}(E)$. Hence $\alpha \circ \phi$ is zero by Schur's lemma. The same type of argument shows that $\phi \circ \beta$ is zero. Thus $F_\bullet$ is a complex.

By our `Macaulay2` computation of Betti numbers before the Proposition, $\ker(\alpha)$ is generated in degree 2 by 60 minimal generators. In $F_\bullet$ these must be the image of $S_{3,3,1}(E)$, since that is 60-dimensional by the hook content formula and it maps injectively to F_1. So $F_\bullet$ is exact at F_1. Now again by the `Macaulay2` computation, it follows that $\ker \phi$ is generated in degree 3 by 20 generators. These must be the image of $S_{3,3,3}(E)$ since that is 20-dimensional and maps injectively to F_2. So $F_\bullet$ is exact at F_2. Finally, the computation implies that β is injective, and $F_\bullet$ is the GL(E)-equivariant minimal free resolution of M.

For the statement about the dual, recall that since $F_\bullet$ is a resolution of a Cohen-Macaulay module, the dual complex, obtained by applying $\text{Hom}_A(-, \omega_A)$ with $\omega_A = A(-10)$, is also a resolution. If we twist this dual resolution with $(\wedge^4 E)^{\otimes 3} \otimes A(7)$, the terms will be as in the original resolution. Since the nonzero GL(E)-map α is uniquely determined up to scale, it follows that $F_\bullet$ and its dual are isomorphic up to twist. □

Remark 13. *The GL(E)-representations in this resolution could also have been computed using the* `Macaulay2` *package* `HighestWeights` *(Galetto , 2015).*

Remark 14. *The dual of this resolution is:*

$$S_{3,3,3}(E^*) \otimes A \leftarrow S_{3,3,1}(E^*) \otimes A(-1) \leftarrow S_{3,2}(E^*) \otimes A(-2) \leftarrow S_3(E^*) \otimes A(-3). \qquad (7)$$

A symmetric form q in $S_2(E^*)$ corresponds to a point in $\text{Spec}(A)$ and a homomorphism $A \to \mathbb{C}$. The fiber of this complex over the point q is then an $\text{SO}(E^*, q)$-complex:

$$S_{3,3,3}(E^*) \leftarrow S_{3,3,1}(E^*) \leftarrow S_{3,2}(E^*) \leftarrow S_3(E^*). \qquad (8)$$

When q is a nondegenerate form, this is the Littlewood complex $L_\bullet^{3,3,3}$ as defined in (Sam et.al. , 2013, §4.2). (The terms of $L_\bullet^{3,3,3}$ can be computed using the plethysm in §4.6 of loc.cit.) This partition $\lambda = (3, 3, 3)$ is not admissible since $3 + 3 > 4$, see (Sam et.al. , 2013, Section 4.1). The cohomology of (8) is then given by (Sam et.al. , 2013, Theorem 4.4) and it vanishes (since here $i_4(\lambda) = \infty$), as it should in agreement with Proposition 12. The dual resolution (7) of the Ulrich sheaf can then be thought of as a "universal" Littlewood complex for the parition $\lambda = (3, 3, 3)$. In other cases when Littlewood complexes are exact, it would be an interesting future research topic to investigate the sheaf that is resolved by the "universal Littlewood complex".

To obtain nicer formulas for the Chow form of the essential variety $\mathcal{E}_\mathbb{C}$ in §4, we now prove that the middle map ϕ in the resolution (6) is symmetric, in the following appropriate sense. In general, suppose that we are given a linear map $W^* \xrightarrow{\mu} W \otimes L^*$ where L is a finite dimensional vector space. Dualizing, we get a map $W \xleftarrow{\mu^T} W^* \otimes L$ which in turn gives a map $W \otimes L^* \xleftarrow{\nu} W^*$. By definition, the map μ is *symmetric* if $\mu = \nu$ and *skew-symmetric* if $\mu = -\nu$. If μ is symmetric and μ is represented as a matrix with entries in L^* with respect to dual bases of W and W^*, then that matrix is symmetric, and analogously when μ is skew-symmetric. Note that the map μ also induces a map $L \xrightarrow{\eta} W \otimes W$.

Fact 15. *The map μ is symmetric if the image of η is in the subspace $S_2(W) \subseteq W \otimes W$ and it is skew-symmetric if the image is in the subspace $\wedge^2 W \subseteq W \otimes W$.*

Proposition 16. *The middle map ϕ in the resolution* (6) *is symmetric.*

Proof. Consider the map ϕ in degree 3. It is:

$$S_{3,2}(E) \otimes S_2(E) \longleftarrow S_{3,3,1}(E) \cong S_{3,2}(E)^* \otimes (\wedge^4 E)^{\otimes 3}$$

and it induces the map:

$$S_{3,2}(E) \otimes S_{3,2}(E) \longleftarrow S_2(E)^* \otimes (\wedge^4 E)^{\otimes 3} \cong S_{3,3,3,1}(E).$$

By the Littlewood-Richardson rule, the right representation above occurs with multiplicity 1 in the left side. Now one can check that $S_{3,3,3,1}(E)$ occurs in $S_2(S_{3,2}(E))$. This follows by Corollary 5.5 in (Carré and Leclerc , 1995) or one can use the package `SchurRings` (Stillman et.al.) in `Macaulay2`:

```
needsPackage "SchurRings"
S = schurRing(s,4,GroupActing=>"GL")
plethysm(s_2,s_{3,2})
```

Due to Fact 15, we can conclude that the map ϕ is symmetric. □

3.5. The second Ulrich sheaf

We construct another Ulrich sheaf on $PX_{4,2}^s$ and analyze it similarly to as above. This will lead to a second formula for $\mathrm{Ch}(\mathcal{E}_\mathbb{C})$ in §4. Consider $S_{2,2,1}(E) \otimes S_2(E)$. By Pieri's rule:

$$S_{2,2,1}(E) \otimes S_2(E) \cong S_{4,2,1}(E) \oplus S_{3,2,2}(E) \oplus S_{3,2,1,1}(E) \oplus S_{2,2,2,1}(E).$$

Thus there is a GL(E)-map, with nonzero degree 1 components unique up to scale:

$$S_{2,2,1}(E) \otimes A \xleftarrow{\alpha} (S_{3,2,2}(E) \oplus S_{3,2,1,1}(E) \oplus S_{2,2,2,1}(E)) \otimes A(-1).$$

This map can be programmed in `Macaulay2` using `PieriMaps` as follows:

```
R=QQ[a..d]
needsPackage "PieriMaps"
f1= transpose pieri({3,2,2,0},{1,3},R)
f2=transpose pieri({3,2,1,1},{1,4},R)
```

11

```
f3=transpose pieri({2,2,2,1},{3,4},R)
f = transpose (f1||f2||f3)
S=QQ[a..d,y_0..y_9]
a2=symmetricPower(2,matrix{{a..d}})
alpha=sum(10,i->contract(a2_(0,i),sub(f,S))*y_i)
```

We can then compute the resolution of $\mathrm{coker}(\alpha)$ in `Macaulay2`. It has the form:

$$A^{20} \xleftarrow{\alpha} A(-1)^{60} \longleftarrow A(-2)^{60} \longleftarrow A(-3)^{20}.$$

Thus the cokernel of α is an Ulrich module, and moreover we have:

Proposition 17. *The cokernel of α is an Ulrich module M of rank 2 supported on the variety $X_{4,2}^s$. The resolution of M is $\mathrm{GL}(E)$-equivariant and it is:*

$$
\begin{aligned}
F_\bullet : S_{2,2,1}(E) \otimes A \quad &\xleftarrow{\alpha} (S_{3,2,2}(E) \oplus S_{3,2,1,1}(E) \oplus S_{2,2,2,1}(E)) \otimes A(-1) \\
&\xleftarrow{\phi} (S_{4,2,2,1}(E) \oplus S_{3,3,2,1}(E) \oplus S_{3,2,2,2}(E)) \otimes A(-2) \qquad (9) \\
&\xleftarrow{\beta} S_{4,3,2,2}(E) \otimes A(-3)
\end{aligned}
$$

with ranks $20, 60, 60, 20$. The dual complex of this resolution is also a resolution and these two resolutions are isomorphic up to twist. We can visualize the resolution by:

$$0 \leftarrow M \longleftarrow \boxplus \leftarrow \boxplus \oplus \boxplus \oplus \boxplus \leftarrow \boxplus \oplus \boxplus \oplus \boxplus \leftarrow \boxplus \leftarrow 0.$$

Proof. The argument concerning the support of M is exactly as in Proposition 12.

Now we prove that the minimal free resolution of M is of the form above, differently than in 12. To start, note that the module $S_{4,2,2,1}(E)$ occurs by Pieri once in each of:

$$S_{3,2,2}(E) \otimes S_2(E), \quad S_{3,2,1,1}(E) \otimes S_2(E), \quad S_{2,2,2,1}(E) \otimes S_2(E).$$

On the other hand, it occurs in:

$$S_{2,2,1}(E) \otimes S_2(S_2(E)) \cong S_{2,2,1}(E) \otimes S_4(E) \oplus S_{2,2,1}(E) \otimes S_{2,2}(E)$$

only twice, as seen using Pieri's rule and the Littlewood-Richardson rule. Thus $S_{4,2,2,1}(E)$ occurs at least once in the degree 2 part of $\ker(\alpha)$. Similarly we see that each of $S_{3,3,2,1}(E)$ and $S_{3,2,2,2}(E)$ occurs at least once in $\ker(\alpha)$ in degree 2. But by the `Macaulay2` computation before this Proposition, we know that $\ker(\alpha)$ is a module with 60 generators in degree 2. And the sum of the dimensions of these three representations is 60. Hence each of them occurs exactly once in $\ker(\alpha)$ in degree 2, and they generate $\ker(\alpha)$.

Now let C be the 20-dimensional vector space generating $\ker(\phi)$. Since the resolution of M has length equal to $\mathrm{codim}(M)$, the module M is Cohen-Macaulay and the dual of its resolution, obtained by applying $\mathrm{Hom}_A(-, \omega_A)$ where $\omega_A \cong A(-4)$, is again a resolution of $\mathrm{Ext}_A^3(M, \omega_A)$. Thus the map from $C \otimes A(-3)$ to each of:

$$S_{4,2,2,1}(E) \otimes A(-2), \quad S_{3,3,2,1}(E) \otimes A(-2), \quad S_{3,2,2,2}(E) \otimes A(-2)$$

is nonzero. In particular C maps nontrivially to:

$$S_{3,2,2,2}(E) \otimes S_2(E) \cong S_{5,2,2,2}(E) \oplus S_{4,3,2,2}(E).$$

Each of the right-hand side representations have dimension 20, so one of them equals C. However only the last one occurs in $S_{3,3,2,1}(E) \otimes S_2(E)$, and so $C \cong S_{4,3,2,2}(E)$. We have proven that the $GL(E)$-equivariant minimal free resolution of M indeed has the form $F_\bullet$.

For the statement about the dual, recall that each of the three components of α in degree 1 are nonzero. Also, as the dual complex is a resolution, here obtained by applying $\mathrm{Hom}_A(-, \omega_A)$ with $\omega_A = A(-10)$, all three degree 1 components of β are nonzero. If we twist this dual resolution with $(\wedge^4 E)^{\otimes 4} \otimes A(7)$, the terms will be as in the original resolution. Because each of the three nonzero components of the map α are uniquely determined up to scale, the resolution $F_\bullet$ and its dual are isomorphic up to twist. $\qquad\square$

Remark 18. *Again the $GL(E)$-representations in this resolution could have been computed using the* `Macaulay2` *package* `HighestWeights`.

Proposition 19. *The middle map ϕ in the resolution* (9) *is symmetric.*

Proof. We first show that the three 'diagonal' components of ϕ in (9) are symmetric:

$$S_{3,2,2}(E) \otimes S_2(E) \xleftarrow{\phi_1} S_{4,2,2,1}(E)$$

$$S_{3,2,1,1}(E) \otimes S_2(E) \xleftarrow{\phi_2} S_{3,3,2,1}(E)$$

$$S_{2,2,2,1}(E) \otimes S_2(E) \xleftarrow{\phi_3} S_{3,2,2,2}(E).$$

Twisting the third component ϕ_3 with $(\wedge^4 E^*)^{\otimes 2}$, it identifies as:

$$E^* \otimes S_2(E) \longleftarrow E$$

and so ϕ_3 is obviously symmetric. Twisting the second map ϕ_2 with $\wedge^4 E^*$ it identifies as:

$$S_{2,1}(E) \otimes S_2(E) \longleftarrow S_{2,2,1}(E) = (S_{2,1}(E)^*) \otimes (\wedge^4 E)^{\otimes 2},$$

which induces the map:

$$S_{2,1}(E) \otimes S_{2,1}(E) \longleftarrow S_2(E)^* \otimes (\wedge^4 E)^{\otimes 2} = S_{2,2,2}(E).$$

By the Littlewood-Richardson rule, the left tensor product contains $S_{2,2,2}(E)$ with multiplicity 1. By Corollary 5.5 in (Carré and Leclerc , 1995) or `SchurRings` in `Macaulay2`, this is in $S_2(S_{2,1}(E))$:

```
needsPackage "SchurRings"
S = schurRing(s,4,GroupActing=>"GL")
plethysm(s_2,s_{2,1})
```

So by Fact 15, the component ϕ_2 is symmetric. The first map ϕ_1 may be identified as:

$$S_{3,2,2}(E) \otimes S_2(E) \longleftarrow (S_{3,2,2}(E))^* \otimes (\wedge^4 E)^{\otimes 4},$$

which induces the map:

$$S_{3,2,2}(E) \otimes S_{3,2,2}(E) \longleftarrow S_2(E)^* \otimes (\wedge^4 E)^{\otimes 4} = S_{4,4,4,2}(E).$$

Again by Littlewood-Richardson, $S_{4,4,4,2}(E)$ is contained with multiplicity 1 in the left side. By Corollary 5.5 in (Carré and Leclerc , 1995) or the package `SchurRings` in `Macaulay2`, this is in $S_2(S_{3,2,2}(E))$:

```
needsPackage "SchurRings"
S = schurRing(s,4,GroupActing=>"GL")
plethysm(s_2,s_{3,2,2})
```

It is now convenient to tensor the resolution (9) by $(\wedge^4 E^*)^{\otimes 2}$, and to let:

$$T_1 = S_{1,0,0,-2}(E), \quad T_2 = S_{1,0,-1,-1}(E), \quad T_3 = S_{0,0,0,-1}(E).$$

We can then write the middle map as:

$$T_1 \otimes A(-1) \oplus T_2 \otimes A(-1) \oplus T_3 \otimes A(-1) \xleftarrow{\begin{pmatrix} \phi_1 & \mu_2 & \nu_2 \\ \mu_1 & \phi_2 & 0 \\ \nu_1 & 0 & \phi_3 \end{pmatrix}} T_1^* \otimes A(-2) \oplus T_2^* \otimes A(-2) \oplus T_3^* \otimes A(-2) \quad (10)$$

Note indeed that the component:

$$S_{1,0,-1,-1}(E) \otimes S_2(E) = T_2 \otimes S_2(E) \longleftarrow T_3^* \cong S_1(E)$$

must be zero, since the left tensor product does not contain $S_1(E)$ by Pieri's rule. Similarly the map $T_3 \otimes S_2(E) \longleftarrow T_2^*$ is zero.

We know the maps ϕ_1, ϕ_2 and ϕ_3 are symmetric. Consider:

$$T_2 \otimes A(-1) \xleftarrow{\mu_1} T_1^* \otimes A(-2), \quad T_1 \otimes A(-1) \xleftarrow{\mu_2} T_2^* \otimes A(-2).$$

Since the resolution (9) is isomorphic to its dual, either both μ_1 and μ_2 are nonzero, or they are both zero. Suppose both are nonzero. The dual of μ_2 is (up to twist) $T_2 \otimes A(-1) \xleftarrow{\mu_2^T} T_1^* \otimes A(-2)$. But such a GL$(E)$-map is unique up to scalar, as is easily seen by Pieri's rule. Thus whatever the case we can say that $\mu_1 = c_\mu \mu_2^T$ for some nonzero scalar c_μ. Similarly we get $\nu_1 = c_\nu \nu_2^T$. Composing the map (10) with the automorphism on its right given by the block matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\mu & 0 \\ 0 & 0 & c_\nu \end{pmatrix},$$

we get a middle map:

$$T_1 \otimes A(-1) \oplus T_2 \otimes A(-1) \oplus T_3 \otimes A(-1) \xleftarrow{\begin{pmatrix} \phi_1 & \mu_2' & \nu_2' \\ \mu_1 & \phi_2' & 0 \\ \nu_1 & 0 & \phi_3' \end{pmatrix}} T_1^* \otimes A(-2) \oplus T_2^* \otimes A(-2) \oplus T_3^* \otimes A(-2)$$

where the diagonal maps are still symmetric, and $\mu_1 = (\mu_2')^T$ and $\nu_1 = (\nu_2')^T$. So we get a symmetric map, and the result about ϕ follows. $\qquad\square$

14

This second Ulrich module constructed above in Proposition 17 is a particular instance of a general construction of Ulrich modules on the variety of symmetric $n \times n$ matrices of rank $\leq r$; see (Weyman , 2003), §6.3 and Exercise 34 in §6. We briefly recall the general construction. Let $W = \mathbb{C}^n$ and G be the Grassmannian $\mathrm{Gr}(n - r, W)$ of $(n - r)$-dimensional subspaces of W. There is a tautological exact sequence of algebraic vector bundles on G:

$$0 \to \mathcal{K} \to W \otimes \mathcal{O}_G \to \mathcal{Q} \to 0,$$

where r is the rank of $\mathcal{Q}$. Let $X = X_n^s$ be the affine space of symmetric $n \times n$ matrices, and define Z to be the incidence subvariety of $X \times G$ given by:

$$Z = \{((W \xrightarrow{\phi} W), (\mathbb{C}^{n-r} \xhookrightarrow{i} W)) \in X \times G \,|\, \phi \circ i = 0\}.$$

The variety Z is the affine geometric bundle $\mathbb{V}_G(S_2(\mathcal{Q}))$ of the locally free sheaf $S_2(\mathcal{Q})$ on the Grassmannian G. There is a commutative diagram:

$$
\begin{array}{ccc}
Z & \longrightarrow & X \times G \\
\downarrow & & \downarrow \\
X_{n,r}^s & \longrightarrow & X
\end{array}
$$

in which Z is a desingularization of $X_{n,r}^s$. For any locally free sheaf $\mathcal{E}$, the Schur functor S_λ applies to give a new locally free sheaf $S_\lambda(\mathcal{E})$. Consider then the locally free sheaf:

$$\mathcal{E}(n, r) = S_{(n-r)^r}(\mathcal{Q}) \otimes S_{n-r-1, n-r-2, \cdots, 1, 0}(\mathcal{K})$$

on the Grassmannian $\mathrm{Gr}(n - r, W)$. Note that $S_{(n-r)^r}(\mathcal{Q}) = (\det(\mathcal{Q}))^{n-r}$ is a line bundle and $\mathcal{E}(n, r)$ is a locally free sheaf of rank $2^{\binom{n-r}{2}}$. Let $Z \xrightarrow{p} G$ be the projection map. By pullback we get the locally free sheaf $p^*(\mathcal{E}(n, r))$ on Z. The pushforward of this locally free sheaf down to $X_{n,r}^s$ is an Ulrich sheaf on this variety. Since $X_{n,r}^s$ is affine this corresponds to the module of global sections $H^0(Z, p^*\mathcal{E})$. The Ulrich module in Proposition 17 is that module when $n = 4$ and $r = 2$. For our computational purposes realized in §4, we worked out the equivariant minimal free resolution as above. Interestingly, we do not know yet whether the 'simpler' Ulrich sheaf presented in §3.4, which is new to our knowledge, generalizes to a construction for other varieties.

4. The Chow form of the essential variety

4.1. Grassmannians and Chow divisors

The Grassmannian variety $\mathrm{Gr}(c, n + 1) = \mathrm{Gr}(\mathbb{P}^{c-1}, \mathbb{P}^n)$ parametrizes the linear subspaces of dimension $c - 1$ in $\mathbb{P}^n$, i.e the $\mathbb{P}^{c-1}$'s in $\mathbb{P}^n$. Such a linear subspace may be given as the rowspace of a $c \times (n + 1)$ matrix. The tuple of maximal minors of this matrix is uniquely determined by the linear subspace up to scale. The number of such minors is $\binom{n+1}{c}$. Hence we get a well-defined point in the projective space $\mathbb{P}^{\binom{n+1}{c}-1}$. This defines an embedding of the Grassmannian $\mathrm{Gr}(c, n+1)$ into that projective space, called the Plücker embedding. Somewhat more algebraically, let W be a vector space of dimension $n + 1$ and let $\mathbb{P}(W)$ be the space of lines in W through the origin. Then a linear subspace V of dimension c in W defines a line $\wedge^c V$ in $\wedge^c W$, and so it defines a point in $\mathbb{P}(\wedge^c W) = \mathbb{P}^{\binom{n+1}{c}-1}$. Thus the Grassmannian $\mathrm{Gr}(c, W)$ embeds into $\mathbb{P}(\wedge^c W)$.

15

If X is a variety of codimension c in a projective space $\mathbb{P}^n$, then a linear subspace of dimension $c-1$ will typically not intersect X. The set of points in the Grassmannian $\mathrm{Gr}(c, n+1)$ that do have nonempty intersection with X forms a divisor in $\mathrm{Gr}(c, n+1)$, called the *Chow divisor*. This is seen by counting dimensions in the incidence diagram:

$$X \longleftarrow \mathcal{X} = \{(x, L) \in X \times \mathrm{Gr}(\mathbb{P}^{c-1}, \mathbb{P}^n) \,\big|\, x \in L\} \longrightarrow \mathrm{Gr}(\mathbb{P}^{c-1}, \mathbb{P}^n).$$

In detail, the fibers of the left projection are isomorphic to $\mathrm{Gr}(\mathbb{P}^{c-2}, \mathbb{P}^{n-1})$, so they have dimension $(c-1)(n-c+1)$. So

$$\dim(\mathcal{X}) = (c-1)(n-c+1) + (n-c) = c(n+1-c) - 1.$$

Since the right arrow is degree 1 onto its image, that image has dimension $\dim(\mathcal{X})$, which is 1 less than $\dim(\mathrm{Gr}(c, n+1))$. Next recall that the divisor class group of $\mathrm{Gr}(c, n+1)$ is isomorphic to $\mathbb{Z}$. Considering the Plücker embedding $\mathrm{Gr}(c, n+1) \subseteq \mathbb{P}^{\binom{n+1}{c}-1}$, any hyperplane in the latter projective space intersects the Grassmannian in a divisor which generates the divisor class group of $\mathrm{Gr}(c, n+1)$. This follows from an application of (Hartshorne , 1977, Chapter II, Proposition 6.5(c)). The homogeneous coordinate ring of this projective space $\mathbb{P}^{\binom{n+1}{c}-1} = \mathbb{P}(\wedge^c W)$ is $\mathrm{Sym}(\wedge^c W^*)$. Note that here $\wedge^c W^*$ are the linear forms, i.e. the elements of degree 1. If X has degree d, then its Chow divisor is cut out by a single form $\mathrm{Ch}(X)$ of degree d unique up to nonzero scale, called the *Chow form,* in the coordinate ring of the Grassmannian $\mathrm{Sym}(\wedge^c W^*)/I_{\mathrm{Gr}(c,n+1)}$. As the parameters n, c, d increase, Chow forms become unwieldy to even store on a computer file. Arguably, the most efficient (and useful) representations of Chow forms are as determinants or Pfaffians of a matrix with entries in $\wedge^c W^*$. As we explain next, Ulrich sheaves can give such formulas.

4.2. Construction of Chow forms

We now explain how to obtain the Chow form $\mathrm{Ch}(X)$ of a variety X from an Ulrich sheaf $\mathcal{F}$ whose support is X. The reference for this is (Eisenbud and Schreyer , 2003, p.552-553). Let $M = \oplus_{d \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(d))$ be the graded module of twisted global sections over the polynomial ring $A = \mathbb{C}[x_0, \ldots, x_n]$. We write W^* for the vector space generated by the variables $x_0, \ldots, x_n$. Consider the minimal free resolution (5) of M. The map d_i may be represented by a matrix D_i of size $\beta_i \times \beta_{i+1}$, with entries in the linear space W^*. Since (5) is a complex the product of two successive matrices $D_{i-1}D_i$ is the zero matrix. Note that when we multiply the entries of these matrices, we are multiplying elements in the ring $A = \mathrm{Sym}(W^*) = \mathbb{C}[x_0, \ldots, x_n]$.

Now comes the shift of view: Let $B = \oplus_{i=0}^n \wedge^i W^*$ be the exterior algebra on the vector space W^*. We now consider the entries in the D_i (which are all degree one forms in $A_1 = W^* = B_1$) to be in the ring B instead. We then multiply together all the matrices D_i corresponding to the maps d_i. The multiplications of the entries are performed in the skew-commutative ring B. We then get a product:

$$D = D_0 \cdot D_1 \cdots D_{c-1},$$

where c is the codimension of the variety X which supports $\mathcal{F}$. If $\mathcal{F}$ has rank r and the degree of X is d, the matrix D is a nonzero $rd \times rd$ matrix. The entries in the product D now lie in $\wedge^c W^*$. Now comes the second shift of view: We consider the entries of D to be linear forms in the polynomial ring $\mathrm{Sym}(\wedge^c W^*)$. Then we take the determinant of D, computed in this polynomial ring, and get a form of degree rd in $\mathrm{Sym}(\wedge^c W^*)$. When considered in the coordinate ring of the Grassmannian $\mathrm{Sym}(\wedge^c W^*)/I_G$, then $\det(D)$ equals the rth power of the Chow form of X. For more information on the fascinating links between the symmetric and exterior algebras, the reader can start with the Bernstein-Gel'fand-Gel'fand correspondence as treated in (Eisenbud et al. , 2003).

4.3. Skew-symmetry of the matrices computing the Chow form of $PX_{4,2}^s$

In §3 we constructed two different Ulrich modules of rank 2 on the variety $PX_{4,2}^s$ of symmetric 4×4 matrices of rank ≤ 2. That variety has degree 10. The matrix D thus in both cases is 20×20, and its determinant is a square in $\mathrm{Sym}(\wedge^c W^*)$ as we now show. In fact, and here our analysis of the equivariant resolutions pays off, the matrix D in both cases is skew-symmetric when we use the bases distinguished by representation theory for the differential matrices:

Lemma 20. *Let A, B, C be matrices of linear forms in the exterior algebra. Their products behave as follows under transposition:*

1. $(A \cdot B)^T = -B^T \cdot A^T$
2. $(A \cdot B \cdot C)^T = -C^T \cdot B^T \cdot A^T.$

Proof. Part (1) is because $uv = -vu$ when u and v are linear forms in the exterior algebra. Part (2) is because $uvw = -wvu$ for linear forms in the exterior algebra. $\qquad\square$

The resolutions (6) and (9) of our two Ulrich sheaves, have the form:

$$F \xleftarrow{\alpha} G \xleftarrow{\phi} G^* \xleftarrow{\beta} F^*. \tag{11}$$

Dualizing and twisting we get the resolution:

$$F \xleftarrow{\beta^T} G \xleftarrow{\phi^T} G^* \xleftarrow{\alpha^T} F^*.$$

Since $\phi = \phi^T$, both β and α^T map isomorphically onto the same image. We can therefore replace the map β in (11) with α^T, and get the $\mathrm{GL}(E)$-equivariant resolution:

$$F \xleftarrow{\alpha} G \xleftarrow{\phi} G^* \xleftarrow{\alpha^T} F^*.$$

Let $\underline{\alpha}, \underline{\phi}$ and $\underline{\alpha}^T$ be the maps in the resolution above, but now considered to live over the exterior algebra. The Chow form associated to the two Ulrich sheaves is then the Pfaffian of the matrix:

$$\underline{\alpha}\,\underline{\phi}\,\underline{\alpha}^T.$$

Proposition 21. *The Chow form $\mathrm{Ch}(PX_{4,2}^s)$ constructed from the Ulrich sheaf is, in each case, the Pfaffian of a 20×20 skew-symmetric matrix.*

Proof. The Chow form squared is the determinant of $\underline{\alpha}\,\underline{\phi}\,\underline{\alpha}^T$ and we have:

$$(\underline{\alpha}\,\underline{\phi}\,\underline{\alpha}^T)^T = -(\underline{\alpha}^T)^T\,\underline{\phi}^T\,\underline{\alpha}^T = -\underline{\alpha}\,\underline{\phi}\,\underline{\alpha}^T. \qquad\square$$

4.4. Explicit matrices computing the Chow form of $PX_{4,2}^s$

Even though our primary aim is to compute the Chow form of the essential variety, we get explicit matrix formulas for the Chow form of $PX_{4,2}^s$ as a by-product of our method. We carried out the computation in Proposition 21 in `Macaulay2` for both Ulrich modules on $PX_{4,2}^s$. We used the package `PieriMaps` to make matrices D_1 and D_2 representing α and ϕ with respect to the built-in choice of bases parametrized by semistandard tableaux. We had to multiply D_2 on the right by a change of basis matrix to get a matrix representative with respect to dual bases, i.e. symmetric. For example in the case of the first Ulrich module (6) this change of basis matrix

computes the perfect pairing $S_{3,2}(E) \otimes S_{3,3,1}(E) \to (\wedge^4 E)^{\otimes 3}$. Let us describe the transposed inverse matrix that represents the dual pairing. Columns are labeled by the semistandard Young tableaux S of shape $(3, 2)$, and rows are labeled by the semistandard Young tableaux T of shape $(3, 3, 1)$. The (S, T)-entry in the matrix is obtained by fitting together the tableau S and the tableau T rotated by $180°$ into a tableau of shape $(3, 3, 3, 3)$, straightening, and then taking the coefficient of $\begin{smallmatrix}\boxed{0}\boxed{0}\boxed{0}\\\boxed{1}\boxed{1}\boxed{1}\\\boxed{2}\boxed{2}\boxed{2}\\\boxed{3}\boxed{3}\boxed{3}\end{smallmatrix}$. To finish for each Ulrich module, we took the product $D_1 D_2 D_1^T$ over the exterior algebra.

The two resulting explicit 20×20 skew-symmetric matrices are available as `arXiv` ancillary files or at this paper's webpage[1]. Their Pfaffians equal the Chow form of $PX_{4,2}^s$, which is an element in the homogeneous coordinate of the $\mathrm{Gr}(3, 10) = \mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^9)$. To get a feel for the 'size' of this Chow form, note that this ring is a quotient of the polynomial ring $\mathrm{Sym}(\wedge^3 \mathrm{Sym}_2(E))$ in 120 Plücker variables, denoted $\mathbb{Q}[p_{\{11,12,13\}}, \ldots, p_{\{33,34,44\}}]$ on our website, by the ideal minimally generated by 2310 Plücker quadrics. We can compute that the degree 10 piece where $\mathrm{Ch}(PX_{4,2}^s)$ lives is a 108,284,013,552-dimensional vector space.

Both 20×20 matrices afford extremely compact formulas for this special element. Their entries are linear forms in $p_{\{11,12,13\}}, \ldots, p_{\{33,34,44\}}$ with one- and two-digit relatively prime integer coefficients. No more than 5 of the p-variables appear in any entry. In the first matrix, 96 off-diagonal entries equal 0. The matrices give new expressions for one of the two irreducible factors of a discriminant studied since 1879 by (Salmon , 1960) and as recently as 2011 (Plauman , 2011), as we see next in Remark 22.

Remark 22. *From the subject of plane curves, it is classical that every ternary quartic form $f \in \mathbb{C}[x, y, z]_4$ can be written as $f = \det(xA + yB + zC)$ for some 4×4 symmetric matrices A, B, C. Geometrically, this expresses $\mathrm{V}(f)$ inside the net of plane quadrics $\langle A, B, C \rangle$ as the locus of singular quadrics. By Theorem 7.5 of (Plauman , 2011), that plane quartic curve $\mathrm{V}(f)$ is singular if and only if the* Vinnikov discriminant*:*

$$\Delta(A, B, C) = \mathbf{M}(A, B, C)\mathbf{P}(A, B, C)^2$$

evaluates to 0. Here $\mathbf{M}$ is a degree $(16, 16, 16)$ polynomial known as the tact invariant *and $\mathbf{P}$ is a degree $(10, 10, 10)$ polynomial. The factor $\mathbf{P}$ equals the Chow form $\mathrm{Ch}(PX_{4,2}^s)$ after substituting Plücker coordinates for Stiefel coordinates:*

$$p_{\{i_1 j_1, i_2 j_2, i_3 j_3\}} = \det \begin{pmatrix} a_{i_1 j_1} & a_{i_2 j_2} & a_{i_3 j_3} \\ b_{i_1 j_1} & b_{i_2 j_2} & b_{i_3 j_3} \\ c_{i_1 j_1} & c_{i_2 j_2} & c_{i_3 j_3} \end{pmatrix}.$$

4.5. Explicit matrices computing the Chow form of $\mathcal{E}_{\mathbb{C}}$

We now can put everything together and solve the problem raised by Agarwal, Lee, Sturmfels and Thomas in (Agarwal et al. , 2015) of computing the Chow form of the essential variety. In Proposition 7, we constructed a linear embedding $s: \mathbb{P}^8 \hookrightarrow \mathbb{P}^9$ that restricts to an embedding $\mathcal{E}_{\mathbb{C}} \hookrightarrow PX_{4,2}^s$. Both of our Ulrich sheaves supported on $PX_{4,2}^s$ pull back to Ulrich sheaves supported on $\mathcal{E}_{\mathbb{C}}$, and their minimal free resolutions pull back to minimal free resolutions:

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[1] http://math.berkeley.edu/~jkileel/ChowFormulas.html

$$s^*F \xleftarrow{\ s^*\alpha\ } s^*G \xleftarrow{\ s^*\phi\ } s^*G^* \xleftarrow{\ s^*\alpha^t\ } s^*F^*.$$

Here we verified in `Macaulay2` that s^* quotients by a linear form that is a nonzero divisor for the two Ulrich modules. So, to get the Chow form $\mathrm{Ch}(\mathcal{E}_{\mathbb{C}})$ from Propositions 12 and 17, we took matrices D_1 and D_2 symmetrized from above, and applied s^*. That amounts to substituting $x_{ij} = s(M)_{ij}$, where $s(M)$ is from §2.2. We then multiplied $D_1 D_2 D_1^T$, which is a product of a 20×60, a 60×60 and a 60×20 matrix, over the exterior algebra.

The two resulting explicit 20×20 skew-symmetric matrices are available at the paper's webpage. Their Pfaffians equal the Chow form of $\mathcal{E}_{\mathbb{C}}$, which is an element in the homogeneous coordinate of $\mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^8)$. We denote that ring as the polynomial ring in 84 (dual) Plücker variables $\mathbb{Q}[q_{\{11,12,13\}}, \ldots, q_{\{31,32,33\}}]$ modulo 1050 Plücker quadrics. Here $\mathrm{Ch}(\mathcal{E}_{\mathbb{C}})$ lives in the 9,386,849,472-dimensional subspace of degree 10 elements.

Both matrices are excellent representations of $\mathrm{Ch}(\mathcal{E}_{\mathbb{C}})$. Their entries are linear forms in $q_{\{11,12,13\}}, \ldots, q_{\{31,32,33\}}$ with relatively prime integer coefficients less than 216 in absolute value. In the first matrix, 96 off-diagonal entries vanish, and no entries have full support.

Bringing this back to computer vision, we can now prove our main result stated in §1:

Proof of Theorem 1. We first construct $\mathcal{M}(x^{(i)}, y^{(i)})$, and then we prove that it has the desired properties. For the construction, let Z denote the 6×9 matrix:

$$
\begin{pmatrix}
y_1^{(1)}x_1^{(1)} & y_1^{(1)}x_2^{(1)} & y_1^{(1)} & y_2^{(1)}x_1^{(1)} & y_2^{(1)}x_2^{(1)} & y_2^{(1)} & x_1^{(1)} & x_2^{(1)} & 1 \\
y_1^{(2)}x_1^{(2)} & y_1^{(2)}x_2^{(2)} & y_1^{(2)} & y_2^{(2)}x_1^{(2)} & y_2^{(2)}x_2^{(2)} & y_2^{(2)} & x_1^{(2)} & x_2^{(2)} & 1 \\
y_1^{(3)}x_1^{(3)} & y_1^{(3)}x_2^{(3)} & y_1^{(3)} & y_2^{(3)}x_1^{(3)} & y_2^{(3)}x_2^{(3)} & y_2^{(3)} & x_1^{(3)} & x_2^{(3)} & 1 \\
y_1^{(4)}x_1^{(4)} & y_1^{(4)}x_2^{(4)} & y_1^{(4)} & y_2^{(4)}x_1^{(4)} & y_2^{(4)}x_2^{(4)} & y_2^{(4)} & x_1^{(4)} & x_2^{(4)} & 1 \\
y_1^{(5)}x_1^{(5)} & y_1^{(5)}x_2^{(5)} & y_1^{(5)} & y_2^{(5)}x_1^{(5)} & y_2^{(5)}x_2^{(5)} & y_2^{(5)} & x_1^{(5)} & x_2^{(5)} & 1 \\
y_1^{(6)}x_1^{(6)} & y_1^{(6)}x_2^{(6)} & y_1^{(6)} & y_2^{(6)}x_1^{(6)} & y_2^{(6)}x_2^{(6)} & y_2^{(6)} & x_1^{(6)} & x_2^{(6)} & 1
\end{pmatrix}
$$

where the columns of Z are labeled by $11, 12, \ldots, 33$ respectively. Now set q_{ijk} to be the determinant of Z with columns i, j, k removed, and substitute into either of 20×20 skew-symmetric matrices above that compute the Chow form $\mathrm{Ch}(\mathcal{E}_{\mathbb{C}})$. This constructs $\mathcal{M}(x^{(i)}, y^{(i)})$.

Observe that $\mathcal{M}(x^{(i)}, y^{(i)})$ drops rank if and only if the subspace $\ker(Z) \subseteq \mathbb{P}^8$ intersects $\mathcal{E}_{\mathbb{C}}$. This follows by definition of the Chow form, since the Plücker coordinates of $\ker(Z)$ equal the maximal minors of Z when Z is full-rank (Gelfand et al. , 1994, p.94). Said differently:

$$\mathcal{M}(x^{(i)}, y^{(i)}) \text{ drops rank} \iff \exists M \in \mathcal{E}_{\mathbb{C}} \text{ such that } \forall i = 1, \ldots 6$$

$$\begin{pmatrix} y_1^{(i)} & y_2^{(i)} & 1 \end{pmatrix} M \begin{pmatrix} x_1^{(i)} \\ x_2^{(i)} \\ 1 \end{pmatrix} = 0. \tag{12}$$

Indeed, (12) is a linear system for $M \in \mathcal{E}_{\mathbb{C}}$, while Z is the coefficient matrix of that system.

In the rest of the proof of Theorem 1, we relate solutions of (1) to solutions of (12). As goes computer vision parlance, we will move between cameras and world points to relative

19

poses, and then back. In the first direction, given $\{(x^{(i)}, y^{(i)})\}$, suppose that we have a solution $A, B, \widetilde{X^{(1)}}, \ldots, \widetilde{X^{(6)}}$ to (1). Note that the group:

$$G := \{g \in \mathrm{GL}(4, \mathbb{C}) \,|\, (g_{ij})_{1 \le i, j \le 3} \in \mathrm{SO}(3, \mathbb{C}) \text{ and } g_{41} = g_{42} = g_{43} = 0\}$$

equals the stabilizer of the set of calibrated camera matrices inside $\mathbb{C}^{3 \times 4}$, with respect to right multiplication. We now make two simplifying assumptions about our solution to (1).

- Without loss of generality, $A = [\,\mathrm{id}_{3\times3}\,|\,0\,]$. For otherwise, select $g \in G$ so that $Ag = [\,\mathrm{id}_{3\times3}\,|\,0\,]$, and then $Ag, Bg, g^{-1}\widetilde{X^{(1)}}, \ldots, g^{-1}\widetilde{X^{(6)}}$ is also a solution to (1).

- Denoting $B = [\,R\,|\,t\,]$ for $R \in \mathrm{SO}(3, \mathbb{C})$ and $t \in \mathbb{C}^3$, then without loss of generality, $t \ne 0$. For otherwise, we may zero out the last coordinate of each $\widetilde{X^{(i)}}$ and replace B by $[\,R\,|\,t'\,]$ for any $t' \in \mathbb{C}^3$, and then we still have a solution to the system (1).

Denote $[\,t\,]_\times := \begin{pmatrix} 0 & t_3 & -t_2 \\ -t_3 & 0 & t_1 \\ t_2 & -t_1 & 0 \end{pmatrix}$. Set $M = [\,t\,]_\times R$. Then $M \in \mathcal{E}_\mathbb{C}$ and M is a solution to (12), as for each $i = 1, \ldots, 6$ we have:

$$\begin{pmatrix} y_1^{(i)} & y_2^{(i)} & 1 \end{pmatrix} M \begin{pmatrix} x_1^{(i)} \\ x_2^{(i)} \\ 1 \end{pmatrix} \equiv (B\widetilde{X^{(i)}})^T M (A\widetilde{X^{(i)}})$$

$$= \widetilde{X^{(i)}}^T \left([\,R\,|\,t\,]^T [\,t\,]_\times R \,[\,\mathrm{id}_{3\times3}\,|\,0\,] \right) \widetilde{X^{(i)}}$$

$$= \widetilde{X^{(i)}}^T \left([\,R\,|\,0\,]^T [\,t\,]_\times [R\,|\,0\,] \right) \widetilde{X^{(i)}}$$

$$= 0.$$

Here the second-to-last equality is because $t^T [\,t\,]_\times = 0$, and the last equality is because the matrix in parentheses is skew-symmetric. We have shown the second sentence in Theorem 1.

Conversely, given $\{(x^{(i)}, y^{(i)})\}$, let us start with a solution $M \in \mathcal{E}_\mathbb{C}$ to system (12). From this, we will produce a solution to (1) *provided that M is sufficiently nice*. More precisely, assume:

1. M may be factored as a skew-symmetric matrix times a rotation matrix, i.e. $M = [\,t\,]_\times R$ where $t \in \mathbb{C}^3$ and $R \in \mathrm{SO}(3, \mathbb{C})$.
2. For each $i = 1, \ldots, 6$, we have that $\begin{pmatrix} y_1^{(i)} & y_2^{(i)} & 1 \end{pmatrix} M \ne 0$ and $M \begin{pmatrix} x_1^{(i)} & x_2^{(i)} & 1 \end{pmatrix}^T \ne 0$.

For readers of (Hartley and Zisserman , 2004, §9.2.4), condition 2 means that $\{(x^{(i)}, y^{(i)})\}$ avoids the *epipoles* of M. Supposing conditions 1 and 2 hold, we set $A = [\,\mathrm{id}_{3\times3}\,|\,0\,]$ and $B = [\,R\,|\,t\,]$. Then there exists $\widetilde{X^{(i)}} \in \mathbb{P}^3$ such that $A\widetilde{X^{(i)}} \equiv x^{(i)}$ and $B\widetilde{X^{(i)}} \equiv y^{(i)}$. Indeed (dropping i for convenience), we take $\widetilde{X} = \begin{pmatrix} x_1 & x_2 & 1 & \lambda \end{pmatrix}^T$ where $\lambda \in \mathbb{C}$ satisfies $R\widetilde{x} + \lambda t \equiv \widetilde{y}$. To find such λ, we solve $\begin{pmatrix} R\widetilde{x} & t & \widetilde{y} \end{pmatrix}_{3\times3} \begin{pmatrix} 1 \\ \lambda \\ -\mu \end{pmatrix} = 0$ for $\lambda, \mu \in \mathbb{C}$ with $\mu \ne 0$. These equations are soluble since:

- $0 = \widetilde{y}^T M \widetilde{x} = \widetilde{y}^T [\,t\,]_\times R\widetilde{x} = \det \begin{pmatrix} R\widetilde{x} & t & \widetilde{y} \end{pmatrix}$ by Laplace expansion along the third column.

- $0 \neq M\widetilde{x} = t_\times R\widetilde{x} \Rightarrow$ columns t and $R\widetilde{x}$ are linearly independent. Likewise, $0 \neq \widetilde{y}^T M = \widetilde{y}^T[t]_\times R \Rightarrow$ columns $\widetilde{y}$ and t are linearly independent.

This produces cameras and world points $A, B, \widetilde{X}^{(1)}, \ldots, \widetilde{X}^{(6)}$ satisfying (1), given an essential matrix M satisfying the epipolar constraints (12) as well as the two regularity conditions above. To complete the proof of Theorem 1, it is enough to show that in the variety of point correspondences $\{(x^{(i)}, y^{(i)})\}$ where (12) is soluble, there is a dense, open subset of point correspondences where all solutions $M \in \mathcal{E}_\mathbb{C}$ satisfy conditions 1 and 2 above.

To this end, consider the diagram below, with projections π_1 and π_2 respectively:

$$(\mathbb{C}^2 \times \mathbb{C}^2)^6 \longleftarrow (\mathbb{C}^2 \times \mathbb{C}^2)^6 \times \mathcal{E}_\mathbb{C} \longrightarrow \mathcal{E}_\mathbb{C}.$$

Inside $(\mathbb{C}^2 \times \mathbb{C}^2)^6 \times \mathcal{E}_\mathbb{C}$ with coordinates (x, y, M), we consider three incidence varieties:

$$\mathcal{I}_0 := \left\{ (x, y, M) \ \Big| \ \widetilde{y^{(i)}}^T M \widetilde{x^{(i)}} = 0 \ \text{for each} \ i = 1, \ldots, 6 \right\}$$

$$\mathcal{I}_1 := \left\{ (x, y, M) \in \mathcal{I}_0 \ \Big| \ M \ \text{does \underline{not} factor as} \ M = [t]_\times R \ \text{for any} \ t \in \mathbb{C}^3, R \in \mathrm{SO}(3, \mathbb{C}) \right\}$$

$$\mathcal{I}_2 := \left\{ (x, y, M) \in \mathcal{I}_0 \ \Big| \ M\widetilde{x^{(i)}} = 0 \ \text{or} \ \widetilde{y^{(i)}}^T M = 0 \ \text{for some} \ i = 1, \ldots, 6 \right\}.$$

So, $\pi_1(\mathcal{I}_0)$ is the variety of point correspondences where (12) is soluble, i.e. the hypersurface $V(\det(\mathcal{M}(x^{(i)}, y^{(i)}))) \subset (\mathbb{C}^2 \times \mathbb{C}^2)^6$, while $\pi_1(\mathcal{I}_0) \setminus (\pi_1(\mathcal{I}_1) \cup \pi_1(\mathcal{I}_2))$ consists of those point correspondences where (12) is soluble and all solutions to (12) satisfy conditions 1 and 2 above. We will show that $\pi_1(\mathcal{I}_1)$ and $\pi_1(\mathcal{I}_2)$ are closed subvarieties with dimension < 23.

For $\mathcal{I}_1$, note that $\pi_2(\mathcal{I}_1) \subset \mathcal{E}_\mathbb{C}$ is a closed subvariety, the complement of the open orbit from the proof of Proposition 2, i.e. the 4-dimensional $\mathrm{Tan}(S)$. Also each fiber of $\pi_2|_{\mathcal{I}_1}$ is 18-dimensional. It follows that $\mathcal{I}_1$ is closed and 22-dimensional. So $\pi_1(\mathcal{I}_1)$ is closed and has dimension ≤ 22.

Next for $\mathcal{I}_2$, note that $\pi_2|_{\mathcal{I}_2}$ surjects onto $\mathcal{E}_\mathbb{C}$ and has general fibers that are 17-dimensional. So $\mathcal{I}_2$ is closed and 22-dimensional, implying that $\pi_1(\mathcal{I}_1)$ is closed and has dimension ≤ 22.

At this point, we have shown the converse in Theorem 1, and this completes the proof. $\square$

We illustrate the main theorem with two examples. Note that since the first example is a 'positive', it is a strong (and reassuring) check of correctness for our formulas.

Example 23. *Consider the image data of 6 point correspondences* $\{(\widetilde{x^{(i)}}, \widetilde{y^{(i)}}) \in \mathbb{P}^2 \times \mathbb{P}^2 \,|\, i = 1, \ldots, m\}$ *given by the corresponding rows of the two matrices:*

$$[\widetilde{x^{(i)}}] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & -\frac{1}{2} & 1 \\ -3 & 0 & 1 \\ \frac{3}{2} & -\frac{5}{2} & 1 \\ 1 & \frac{1}{7} & 1 \end{pmatrix} \qquad [\widetilde{y^{(i)}}] = \begin{pmatrix} \frac{8}{11} & \frac{16}{11} & 1 \\ \frac{7}{22} & \frac{5}{22} & 1 \\ \frac{8}{29} & \frac{34}{29} & 1 \\ \frac{17}{20} & -1 & 1 \\ \frac{1}{7} & \frac{1}{7} & 1 \\ \frac{9}{4} & \frac{3}{4} & 1 \end{pmatrix}.$$

In this example, they do come from world points $\widetilde{X^{(i)}} \in \mathbb{P}^3$ *and calibrated cameras* A, B:

$$
[\widetilde{X^{(i)}}] = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & -2 & 4 & 1 \\ 3 & 0 & -1 & 1 \\ 3 & -5 & 2 & 1 \\ 7 & 1 & 7 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} & 0 \\ \frac{4}{9} & -\frac{8}{9} & \frac{1}{9} & 1 \end{pmatrix}.
$$

To detect this, we form the 6×9 *matrix* Z *from the proof of Theorem* 1:

$$
Z = \begin{pmatrix} 0 & 0 & \frac{8}{11} & 0 & 0 & \frac{16}{11} & 0 & 0 & 1 \\ \frac{7}{22} & -\frac{7}{22} & \frac{7}{22} & \frac{5}{22} & -\frac{5}{22} & \frac{5}{22} & 1 & -1 & 1 \\ 0 & -\frac{4}{29} & \frac{8}{29} & 0 & -\frac{17}{29} & \frac{34}{29} & 0 & -\frac{1}{2} & 1 \\ -\frac{51}{20} & 0 & \frac{17}{20} & 3 & 0 & -1 & -3 & 0 & 1 \\ \frac{3}{14} & -\frac{5}{14} & \frac{1}{7} & \frac{3}{14} & -\frac{5}{14} & \frac{1}{7} & \frac{3}{2} & -\frac{5}{2} & 1 \\ \frac{9}{4} & \frac{9}{28} & \frac{9}{4} & \frac{3}{4} & \frac{3}{28} & \frac{3}{4} & 1 & \frac{1}{7} & 1 \end{pmatrix}.
$$

We substitute the maximal minors of Z *into the matrices computing* $\mathrm{Ch}(\mathcal{E}_{\mathbb{C}})$ *in* `Macaulay2`. *The determinant command then outputs 0. This computation recovers the fact that the point correspondences are images of 6 world points under a pair of calibrated cameras.*

Example 24. *Random data* $\{(x^{(i)}, y^{(i)}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid i = 1, \ldots, 6\}$ *is expected to land outside the Chow divisor of* $\mathcal{E}_{\mathbb{C}}$. *We made an instance using the* `random(QQ)` *command in* `Macaulay2` *for each coordinate of image point. The coordinates ranged from* $\frac{1}{8}$ *to 5 in absolute value. We carried out the substitution from Example* 23, *and got two full-rank skew-symmetric matrices with Pfaffians* $\approx 5.5 \times 10^{25}$ *and* $\approx 1.3 \times 10^{22}$, *respectively. These matrices certified that the system* (1) *admits no solutions for that random input.*

The following proposition is based on general properties of Chow forms, collectively known as the U-resultant method to solve zero-dimensional polynomial systems. In our situation, it gives a connection with the 'five-point algorithm' for computing essential matrices. The proposition is computationally inefficient as-is for that purpose, but see (Murao et.al. , 1993) for a more efficient algorithm that would exploit our matrix formulas for $\mathrm{Ch}(\mathcal{E}_{\mathbb{C}})$. Implementing the algorithms in (Murao et.al. , 1993) for our matrices is one avenue for future work.

Proposition 25. *Given a generic 5-tuple* $\{(x^{(i)}, y^{(i)}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid i = 1, \ldots, 5\}$, *if we make the substitution from the proof of Theorem* 1, *then the Chow form* $\mathrm{Ch}(\mathcal{E}_{\mathbb{C}})$ *specializes to a polynomial in* $\mathbb{R}[x_1^{(6)}, x_2^{(6)}, y_1^{(6)}, y_2^{(6)}]$. *Over* $\mathbb{C}$, *this specialization completely splits as:*

$$
\prod_{i=1}^{10} \begin{pmatrix} y_1^{(6)} & y_2^{(6)} & 1 \end{pmatrix} M^{(i)} \begin{pmatrix} x_1^{(6)} \\ x_2^{(6)} \\ 1 \end{pmatrix}.
$$

Here $M^{(1)}, \ldots, M^{(10)} \in \mathcal{E}_{\mathbb{C}}$ *are the essential matrices determined by the given five-tuple.*

Proof. By the proof of Theorem 1, any zero of the above product is a zero of the specialization of $\mathrm{Ch}(\mathcal{E}_{\mathbb{C}})$. By Hilbert's Nullstellensatz, this implies that the product divides the specialization. But both polynomials are inhomogeneous of degree 20, so they are $\equiv$. $\square$

4.6. Numerical experiments with noisy point correspondences

In this final subsection, we discuss how our Theorem 1 is actually resistant to a common complication in concrete applications of algebra: noisy data. Indeed, on real image data, correctly matched point pairs will only come to the computer vision practitioner with finite accuracy. In other words, they differ from exact correspondences by some noise.

Question 26. *While in Theorem 1 the matrix $\mathcal{M}(x, y)$ drops rank when there is an exact solution to* (1)*, how can we tell if there is an approximate solution?*

The answer is to calculate the Singular Value Decomposition of the matrices $\mathcal{M}(x, y)$ from Theorem 1, when a noisy six-tuple of image point correspondences is plugged in. Since Singular Value Decomposition is numerically stable (Demmel , 1997, §5.2), we expect approximately rank-deficient SVD's when there exists an approximate solution to (1). To summarize, since we have **matrix** formulas, we can look at **spectral gaps** in the presence of noise.

We offer experimental evidence that this works. For our experiments, we assumed uniform noise from unif $[-10^{-r}, 10^{-r}]$; this arises in image processing from pixelation (Bovik , 2005, §4.5). For each $r = 1, 1.5, 2, \ldots, 15$, we executed five hundred of the following trials:

- *Pseudo-randomly generate an exact six-tuple of image point correspondences*

$$\{(x^{(i)}, y^{(i)}) \in \mathbb{Q}^2 \times \mathbb{Q}^2 \,|\, i = 1, \ldots, 6\}$$

  with coordinates of size $O(1)$.

- *Corrupt each image coordinate in the six-tuple by adding an independent and identically distributed sample from* unif $[-10^{-r}, 10^{-r}]$.

- *Compute the SVD's of both 20×20 matrices $\mathcal{M}(x, y)$, derived from the first and second Ulrich sheaf respectively, with the above noisy image coordinates plugged in.*

These experiments were performed in `Macaulay2` using double precision for all floating-point arithmetic. Since it is a little subtle, we elaborate on our algorithm to pseudo-randomly generate exact correspondences in the first bullet. It breaks into three steps:

1. Generate calibrated cameras $A, B \in \mathbb{Q}^{3 \times 4}$. To do this, we sample twice from the Haar measure on $SO(3, \mathbb{R})$ and sample twice from the uniform measure on the radius 2 ball centered at the origin in $\mathbb{R}^3$. Then we concatenate nearby points in $SO(3, \mathbb{Q})$ and $\mathbb{Q}^3$ to obtain A and B. To find the nearby rotations, we pullback under $\mathbb{R}^3 \longrightarrow S^3 \backslash \{N\} \longrightarrow SO(3, \mathbb{R})$, we take nearby points in $\mathbb{Q}^3$, and then we pushforward.
2. Generate world points $X^{(i)} \in \mathbb{Q}^3$ ($i = 1, \ldots, 6$). To do this, we sample six times from the uniform measure on the radius 6 ball centered at the origin in $\mathbb{R}^3$ (a choice fitting with some real-world data) and then we replace those by nearby points in $\mathbb{Q}^3$.
3. Set $\widetilde{x^{(i)}} \equiv A\widetilde{X^{(i)}}$ and $\widetilde{y^{(i)}} \equiv B\widetilde{X^{(i)}}$.

The most striking takeaway of our experiments is stated in the following result concerning the bottom spectral gaps we observed. Bear in mind that since $\mathcal{M}(x, y)$ is skew-symmetric, its singular values occur with multiplicity two, so $\sigma_{19}(\mathcal{M}(x, y)) = \sigma_{20}(\mathcal{M}(x, y))$.
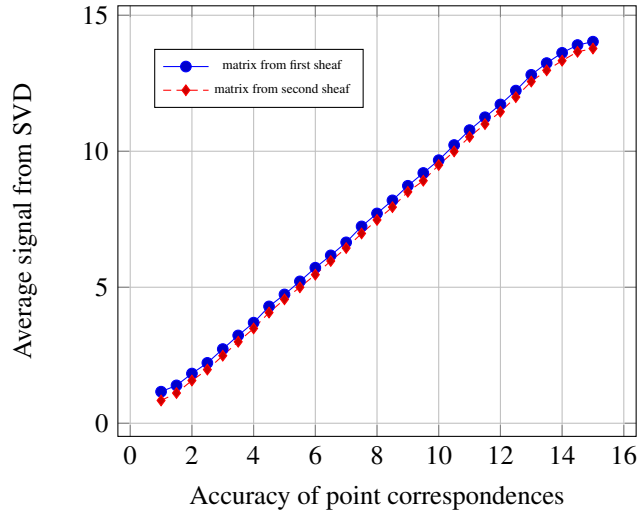
FIGURE. Both matrices satisfying Theorem 1 detect approximately consistent point pairs.

Empirical 27. *In the experiments described above, we observed for both matrices:*

$$\frac{\sigma_{18}(\mathcal{M}(x,y))}{\sigma_{20}(\mathcal{M}(x,y))} = O(10^r).$$

Here $\mathcal{M}(x,y)$ has r-noisy image coordinates, and σ_i denotes the ith largest singular value.

The figure above plots $\log_{10}\left(\frac{\sigma_{18}(\mathcal{M}(x,y))}{\sigma_{20}(\mathcal{M}(x,y))}\right)$ averaged over the five hundred trials against r.

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