Regular Composition for Slice-Regular Functions of Quaternionic Variable

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Abstract In general (see e.g. Cartan in Elementary Theory of Analytic Functions of One or Several Complex Variables, 1963), given two (formal) power series $g(x) = b_0 + xb_1 + \cdots + x^n b_n + \cdots$ and $f(x) = xa_1 + \cdots + x^n a_n + \cdots$ (with $a_0 = f(0) = 0$) it is well known that the composition of g with f, in symbols $g(f(x))$, is a formal power series when coefficients a_j and b_k are taken in a commutative field. Furthermore, if the constant term a_0 of the power series f is not 0, the existence of the composition $g(f(x))$ has been an open problem for many years and only recently has received some partial answers (see Gan and Knox in Int. J. Math. Math. Sci. 30:761–770, 2002). The notion of slice-regularity, recently introduced by Gentili and Struppa (Adv. Math. 216:279–301, 2007), for functions in the non-commutative division algebra $\mathbb{H}$ of quaternions guarantees their quaternionic analyticity but the non–commutativity of the product in $\mathbb{H}$ requires special attention even to define their multiplication (see also Gentili and Stoppato in Michigan Math. J. 56:655–667, 2008). In this paper we face the problem of defining the (slice-regular) composition $g \odot f$ of two slice-regular functions f, g; this turns out to be defined as an extension of the standard composition $g \circ f$ of functions in a non-commutative setting which takes into account a non-commutative version of Bell polynomials and a generalization of the Faà di Bruno Formula.

1 Introduction to Slice-Regularity in $\mathbb{H}$

We recall that the algebra of quaternions $\mathbb{H}$ consists of numbers $x_0 + ix_1 + jx_2 + kx_3$ where x_l is real ($l = 0, \ldots, 3$), and i, j, k, are imaginary units (i.e. their square equals -1) such that $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. In this way, $\mathbb{H}$ can be considered as a vector space over the real numbers of dimension 4. Given a generic element $q = x_0 + ix_1 + jx_2 + kx_3$ of $\mathbb{H}$ we define in a natural fashion its conjugate $\overline{q} = x_0 - ix_1 - jx_2 - kx_3$, and its square norm $|q|^2 = q\overline{q} = \sum_{k\geq 0}^{3} x_k^2$. The set $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$ will be referred to as the *sphere of imaginary units of* $\mathbb{H}$.

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The following is immediate and yet important

Proposition 1 *For any non-real quaternion* w, *there exist, and are unique,* $x, y \in \mathbb{R}$ *with* $y > 0$, *and an imaginary unit* I_w *such that* $w = x + yI_w$.

Definition 1 Given any imaginary unit I, the set $\mathbb{R} + \mathbb{R}I$ will be denoted by L_I.

Notice that after identifying the imaginary unit I_w in $\mathbb{H}$ with the imaginary unit i of $\mathbb{C}$, the set L_{I_w} may be considered as a complex plane in $\mathbb{H}$ passing through 0, 1 and w. In this way, $\mathbb{H}$ can be obtained as an infinite union of complex planes (which will be also called *slices*).

Definition 2 If Ω is a domain in $\mathbb{H}$, a real differentiable function $f : \Omega \to \mathbb{H}$ is said to be *slice-regular* if, for every $I \in \mathbb{S}_{\mathbb{K}}$, its restriction f_I to the complex line $L_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and I is holomorphic on $\Omega \cap L_I$.

In particular any slice-regular function is C^∞ in $\mathbb{B}(0, R)$.[1] Actually, something more is proven in [5]:

Theorem 1 *A function* $f : B = B(0, R) \to \mathbb{K}$ *is regular if, and only if, it has a series expansion of the form*

$$f(q) = \sum_{n=0}^{+\infty} q^n \frac{1}{n!} \frac{\partial_C^n f}{\partial x^n}(0)$$

converging in B. *In particular if* f *is regular then it is* C^∞ *in* B.

For an introductory survey on slice-regular functions we refer the interested reader to [7].

Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ be a given slice-regular function whose associated quaternionic power series has radius of convergence $R > 0$ and consider the quaternionic power series with real coefficients $\sum_{n=0}^{+\infty} q^n |a_n|$; it also has radius of convergence R, because, according to Hadamard's Formula (see e.g. [1]),

$$R = \begin{cases} 1/\limsup_{n\to\infty} |a_n|^{1/n} & \text{if the lim sup is finite and different from 0} \\ 0 & \text{if } \limsup_{n\to\infty} |a_n|^{1/n} = +\infty \\ +\infty & \text{if } \limsup_{n\to\infty} |a_n|^{1/n} = 0. \end{cases}$$

Therefore the function

$$z \mapsto \sum_{n=0}^{+\infty} q^n |a_n|$$

[1] This smoothness is considered with respect to the so called *Cullen derivative* ∂_C (see also [2, 5]) which is well-defined for slice-regular functions as follows: if $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ then $\partial_C f(q) := \sum_{n=1}^{+\infty} nq^{n-1} a_n$.

(denoted by $f_{abs}(z)$) is a slice-regular function in $B(0, R)$ with the property that for any $I \in \mathbb{S}$, we have $f_{abs}(L_I) \subset L_I$.

2 Non-commutative Bell Polynomials and Slice-Regular Composition

Let $k = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$ be a given multiindex; we set

$$k! := k_1!k_2! \cdots k_n!$$
$$|k| := k_1 + k_2 + \cdots + k_n$$
$$\|k\| := k_1 + 2k_2 + \cdots + nk_n.$$

Then for $n \geq 1$, we define

$$B_n(y_1, \ldots, y_n) := \sum_{\|k\|=n} \frac{n!}{k!} \left(\frac{y_1}{1!}\right)^{k_1} \cdot \left(\frac{y_2}{2!}\right)^{k_2} \cdots \left(\frac{y_n}{n!}\right)^{k_n}$$

where y_j are elements of a commutative algebra; we observe that these polynomials satisfy the recursive equation

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, y_{k+1} \tag{1}$$

with the initial condition $B_0 = 1$. These polynomials are called *Bell polynomials*. If one decomposes B_n into its homogeneous[2] parts $B_{n,d}$ (with $d = 1, 2, \ldots, n$), one can write

$$B_n = \sum_{d=1}^{n} B_{n,d}$$

and obtain

$$B_{n,d}(y_1, \ldots, y_n) := \sum_{\|k\|=n,\ |k|=d} \frac{n!}{k!} \left(\frac{y_1}{1!}\right)^{k_1} \cdot \left(\frac{y_2}{2!}\right)^{k_2} \cdots \left(\frac{y_n}{n!}\right)^{k_n}.$$

The homogeneous polynomials $B_{n,d}$ appear in the expression of the n-th derivative of the chain rule, namely

Proposition 2 (Faà di Bruno Formula) *If $h = g \circ f$, then*

$$h^{(n)}(x) = \sum_{d=1}^{n} B_{n,d}\big(f'(x), f''(x), \ldots, f^{(n)}(x)\big) \cdot g^{(d)}\big(f(x)\big). \tag{2}$$

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[2] A polynomial $B_{n,d}$ is *homogeneous* of degree d if

$$B_{n,d}(\lambda y_1, \lambda y_2, \ldots, \lambda y_n) = \lambda^d B_{n,d}(y_1, y_2, \ldots, y_n)$$

for any λ.

In [8], the authors extend the notion of Bell polynomials to the setting of a non-commutative algebra $\mathscr{A}$ with unit and obtain their explicit expressions, namely

$$\widetilde{B}_{n,d} := \sum_{n_2,\dots,n_d=1}^{n} \binom{n-1}{n_2}\binom{n_2-1}{n_3}\cdots\binom{n_{d-1}-1}{n_d} y_{n_d} y_{n_{d-1}-n_d}\cdots y_{n_2-n_3} y_{n-n_2}$$

for $n \geq d \geq 2$ and $\widetilde{B}_0 = 1$, $\widetilde{B}_1 = y_1$. It is then natural to consider

$$\widetilde{B}_n = \sum_{d=1}^{n} \widetilde{B}_{n,d}.$$

These polynomials also satisfy the analogous of Eq. (1); in particular, because of the non-commutativity of multiplication, if instead of

$$\widetilde{B}_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \widetilde{B}_{n-k} y_{k+1} \tag{3}$$

(with the initial condition $\widetilde{B}_0 = 1$) one considers the condition

$$\widetilde{B}_{n+1} = \sum_{k=0}^{n} \binom{n}{k} y_{k+1} \widetilde{B}_{n-k} \tag{4}$$

then

$$\widetilde{B}_{n,d} := \sum_{n_2,\dots,n_d=1}^{n} \binom{n-1}{n_2}\binom{n_2-1}{n_3}\cdots\binom{n_{d-1}-1}{n_d} y_{n-n_2} y_{n_2-n_3}\cdots y_{n_{d-1}-n_d} y_{n_d}$$

for $n \geq d \geq 2$ and $\widetilde{B}_0 = 1$, $\widetilde{B}_1 = y_1$. Moreover the expressions of $\widetilde{B}_{n,d}$ can be also inductively obtained by means of a derivation[3] D plus the multiplication from the left of an element of $\mathscr{A}$. In fact, if we adopt the notation $D(y) = y'$ and $D(y^{(n-1)}) = y^{(n)}$, the (non-commutative) Bell polynomials which satisfy (3) are uniquely determined by

$$\widetilde{B}_{n+1}\big(y, y', y'', \dots, y^{(n)}\big) = (D+y)\widetilde{B}_n\big(y, y', y'', \dots, y^{(n-1)}\big), \qquad \widetilde{B}_0 = 1,$$

where

$$(D+y)\widetilde{B} := D(\widetilde{B}) + yB;$$

[3] A derivation is an endomorphism

$$D : \mathscr{A} \to \mathscr{A}$$

of an associative (generally non-commutative) algebra $\mathscr{A}$ with unit 1, such that $D(y_1 \cdot y_2) = D(y_1) \cdot y_2 + y_1 \cdot D(y_2)$. In particular $D(1) = 0$.

for (non-commutative) Bell polynomials which satisfy (4) one has to consider the action of $(D+y)$ on the right. Thus one immediately observes that $\widetilde{B}_{n,1}(y_1, y_2, \ldots, y_n) = y_n$ and $\widetilde{B}_{n,n}(y_1, y_2, \ldots, y_n) = y_1^n$; more in general

$$\begin{array}{lllll}
\widetilde{B}_0 & = 1 & & & \\
\widetilde{B}_1(y_1) & = y_1 & & & \\
\widetilde{B}_2(y_1, y_2) & = y_2 + & y_1^2 & & \\
\widetilde{B}_3(y_1, y_2, y_3) & = y_3 + & \underbrace{y_2 y_1 + 2y_1 y_2}_{\widetilde{B}_{3,2}(y_1,y_2,y_3)} & + & y_1^3
\end{array}$$

$$\widetilde{B}_4(y_1, y_2, y_3, y_4) = y_4 + \underbrace{y_3 y_1 + 3y_2^2 + 3y_1 y_3}_{\widetilde{B}_{4,2}(y_1,y_2,y_3)} + \underbrace{y_2 y_1^2 + 2y_1 y_2 y_3 + 3y_1^2 y_2}_{\widetilde{B}_{4,3}(y_1,y_2,y_3,y_4)} + y_1^4$$

$$\vdots \qquad \vdots \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

In any case, it turns out that

$$\begin{aligned}
\left|\widetilde{B}_n(y_1, \ldots, y_n)\right| &\leq \sum_{d=1}^{n} \left|\widetilde{B}_{n,d}(y_1, \ldots, y_n)\right| \\
&\leq \sum_{d=1}^{n} B_{n,d}\big(|y_1|, \ldots, |y_n|\big) = B_n\big(|y_1|, \ldots, |y_n|\big).
\end{aligned} \tag{5}$$

Assume now two slice-regular functions $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ are given. Since $f(0) = a_0$ and $\partial_C^n f(0) = n!a_n$, in analogy to (2), we define $g^{\odot} f(q)$ to be the slice regular function

$$g^{\odot} f(q) := \sum_{n=0}^{+\infty} q^n c_n$$

whose coefficients are given by

$$c_0 = g(a_0), \qquad c_n = \frac{1}{n!} \sum_{d=1}^{n} \widetilde{B}_{n,d}(a_1, 2!a_2, \ldots, n!a_n) \cdot g^{(d)}(a_0), \quad n \geq 1, \tag{6}$$

with $\widetilde{B}_{n,d}$ the corresponding non-commutative Bell polynomial. In particular, if $f(0) = 0 = a_0$, then we have

$$c_n = \frac{1}{n!} \sum_{d=1}^{n} d!\widetilde{B}_{n,d}(a_1, 2!a_2, \ldots, n!a_n) \cdot b_d.$$

Clearly, from (5), for $n \geq 1$, one has

$$\begin{aligned}
|c_n| &\leq \frac{1}{n!} \sum_{d=1}^{n} \left|\widetilde{B}_{n,d}(a_1, 2!a_2, \ldots, n!a_n)\right| \cdot \left|g^{(d)}(a_0)\right| \\
&\leq \frac{1}{n!} \sum_{d=1}^{n} B_{n,d}\big(|a_1|, 2!|a_2|, \ldots, n!|a_n|\big) \cdot \left|g^{(d)}(a_0)\right|
\end{aligned} \tag{7}$$

and the last sum of the right-hand side in the previous inequality corresponds to the coefficients of the power expansion of $g_{abs} \circ f_{abs}$. In other words, we have the following

Proposition 3 *Given two slice-regular functions* $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ *and* $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ *in* $\mathbb{H}$, *if the composition* $g_{abs} \circ f_{abs}$ *of the corresponding associated abs-power series exists and have radius of convergence* R *then it is possible to define the slice-regular function* $g^{\odot} f$ *for* q *such that* $|q| < R$ *in terms of the power series*

$$g^{\odot} f(q) = \sum_{n=0}^{+\infty} q^n c_n$$

where the coefficients are given in (6).

Proof Indeed, the coefficients given in (6) satisfy inequality (7) and this guarantees the convergence of the series $\sum_{n=0}^{+\infty} q^n c_n$ for any q such that $|q| < R$. Finally, because of Theorem 1, the function

$$q \mapsto \sum_{n=0}^{+\infty} q^n c_n$$

is slice-regular. □

Remark 1 The functions f_{abs} and g_{abs} have power series expansions whose coefficients are real numbers, so all the classical results in [1] (and more recent ones in [3]) which provide sufficient conditions for the existence of the composition of formal power series with coefficients in a commutative field apply. In particular the composition $g_{abs} \circ f_{abs}$ exists if $a_0 = 0$.

Remark 2 Since apparently associativity of the product is never applied, a similar result should hold true also for slice-regular functions of octonionic variable (see [6]).

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