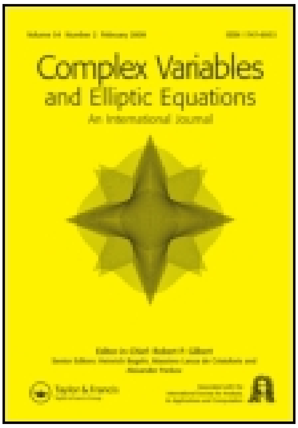

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A Julia's Lemma for the symmetrized bidisc $\#_2$

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A Julia's Lemma for the symmetrized bidisc $\mathbb{G}_2$

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In this article we provide a result that may be considered as an extension of the Julia's Lemma to the case of holomorphic self-maps in the intriguing domain known as the symmetrized bidisc. Julia's Lemma is a classical result for holomorphic self-maps in the (poly)disc, and it turns out to be one of the starting points for the study of iterates of holomorphic self-maps. In the setting of the symmetrized bidisc, this kind of study towards a description of behaviour of iterates of holomorphic self-maps is of great interest and is partially still under investigation. The techniques involved in this article seem to be very well suited for the case of symmetric bidisc and resemble most of the analogous properties in the case of the polydisc.

Keywords: Symmetrized bidisc; Julia's Lemma; Busemann sublevel sets; complex geodesics

AMS Subject Classifications: 32A40; 32A07

1. Introduction

Let $\mathbb{G}_2$ be the symmetrized bidisc,[1] that is the image of the bidisc $\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, \ |z_2| < 1\}$ under $\pi = (\pi_1, \pi_2)$, with $\pi_j : \mathbb{D}^2 \to \mathbb{D}$, $j = 1, 2$, and $\pi_1(z_1, z_2) = z_1 + z_2 := s$, $\pi_2(z_1, z_2) = z_1 \cdot z_2 := p$. The symmetrized bidisc can be considered as a special case of the symmetrized n-disc ($n \geq 2$) which is defined to be the image of the n polydisc $\mathbb{D}^n$ under $\pi = (\pi_1, \dots, \pi_n)$, where each π_j is the symmetric polynomial of degree j in n variables, namely

$$\pi(z_1, \dots, z_n) = \left(\sum_{1 \leq j_1 < \cdots < j_k \leq n} z_{j_1} \cdots z_{j_k} \right)_{k=1,\dots,n}.$$

It can be proven (see, e.g. [1]) that, equivalently,

$$\mathbb{G}_2 = \{(s, p) \in \mathbb{C}^2 : |s - \bar{s}p| + |p|^2 < 1\} = \left\{ (s, p) \in \mathbb{C}^2 : \sup_{a \in \partial\mathbb{D}} \left| \frac{2ap - s}{2 - as} \right| < 1 \right\}. \tag{1.1}$$

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The symbols adopted as coordinates in $\mathbb{G}_2$ remind us that $\pi_1(z_1, z_2)$ gives the sum of z_1 and z_2, whereas $\pi_2(z_1, z_2)$ is the product of z_1 and z_2.

After putting $f_a(s,p) = \frac{2ap-s}{2-as}$, it is clear that f_a is holomorphic in $(\mathbb{C}\backslash\{2/a\}) \times \mathbb{C}$ and that actually for any $a \in \mathbb{D}$ the function f_a maps $\mathbb{G}_2$ in $\mathbb{D}$. Moreover, the following important relation (see, e.g. [2]) can be found:

$$k^*_{\mathbb{G}_2}((s_1,p_1),(s_2,p_2)) = \max_{|a|=1}\{m_{\mathbb{D}}(f_a(s_1,p_1),f_a(s_2,p_2))\}, \tag{1.2}$$

where $k^*_{\mathbb{G}_2}$ is the Lempert function in $\mathbb{G}_2$ and $m_{\mathbb{D}}$ is the pseudo-metric in $\mathbb{D}$. In particular (see again [2]), the Kobayashi pseudo-distance[2] $k_{\mathbb{G}_2}$ can be obtained as follows: $k_{\mathbb{G}_2} = \tanh^{-1}(k^*_{\mathbb{G}_2})$. Notice that, similarly, $\tanh^{-1}(m_{\mathbb{D}})$ provides the so-called Poincaré distance in $\mathbb{D}$ which will be denoted in the sequel by ω.

Finally, the Šilov boundary of $\mathbb{G}_2$ is defined as $\pi(\partial\mathbb{D} \times \partial\mathbb{D}) := \check{\partial}\mathbb{G}_2$. We recall that $\partial\mathbb{D} \times \partial\mathbb{D}$ is precisely the Šilov boundary of $\mathbb{D}^2$. Clearly $\check{\partial}\mathbb{G}_2 \subset \overline{\mathbb{G}_2}$ (the closure of $\mathbb{G}_2$ in $\mathbb{C}^2$), but notice that the condition $|s-\bar{s}p| + |p|^2 \le 1$ does not imply that $(s,p) \in \overline{\mathbb{G}_2}$, since $(5/2, 1) \notin \overline{\mathbb{G}_2}$. The previous counterexample is made more precise in the following Proposition [1].

Proposition 1

(i) $(s,p) \in \overline{\mathbb{G}_2} \Leftrightarrow |s-\bar{s}p| + |p|^2 \le 1$ *and* $|s| \le 2$

(ii) $(s,p) \in \check{\partial}\mathbb{G}_2 \Leftrightarrow |f_a(s,p)| = 1\ \forall a \in \partial\mathbb{D}$.

Proof

(i) Clearly, if $(s,p) \in \overline{\mathbb{G}_2}$ then $|s-\bar{s}p| + |p|^2 \le 1$ and $|s| \le 2$.
Viceversa, let (s,p) be such that $|s-\bar{s}p| + |p|^2 \le 1$ and $|s| \le 2$. Thus

(a) if $s \ne p\bar{s}$, then $(ts,p) \in \mathbb{G}_2$ for $0<t<1$;
(b) if $s = p\bar{s}$, then $(ts, t^2p) \in \mathbb{G}_2$ for $0<t<1$.

(ii) We first observe that
$(s,p) \in \check{\partial}\mathbb{G}_2 \Leftrightarrow s = \bar{s}p,\ |p| = 1$ and $|s| \le 2$; hence, from

$$f_a(s,p) = \frac{2ap-s}{2-as} = \frac{ap(2-\overline{as})}{2-as}$$

we make the desired conclusion. ∎

We can therefore start with the following proposition [1].

Proposition 2 *Let h be a holomorphic automorphism of $\mathbb{D}$, then the relation*

$$H_h(\pi(z_1,z_2)) = \pi(h(z_1), h(z_2))$$

defines a holomorphic mapping $H_h : \mathbb{G}_2 \to \mathbb{G}_2$ with the following properties:

(a) $H_h \in \mathrm{Aut}(\mathbb{G}_2)$;
(b) *if* $h(z) = \lambda z$, (*with* $|\lambda| = 1$), *then* $H_h(s,p) = (\lambda s,\ \lambda^2 p)$;
(c) *for any* $a \in \mathbb{D}$, *if* $h_a(z) = (z-a)/(1-\bar{a}z)$, *then* $H_{h_a}(\pi(a,a)) = (0,0)$.

Proof Clearly $H_h \in \mathrm{Aut}(\mathbb{G}_2)$, since $H_h^{-1} = H_{h^{-1}}$. Points (b) and (c) are then straightforward to prove. ∎

To be more precise on point (a) of the previous proposition, one can actually prove (see, e.g. [1]) the following proposition.

PROPOSITION 3 $\mathrm{Aut}(\mathbb{G}_2)=\{H_h : h\in\mathrm{Aut}(\mathbb{D})\}$.

This result has been recently extended to the case of the symmetrized n-disc of $\mathbb{C}^n$. Notice, furthermore, that if $f:\mathbb{D}^2\to\mathbb{C}$ is a *symmetric* holomorphic map, then a holomorphic map $H_f:\mathbb{G}_2\to\mathbb{C}$ is defined by $H_f(\pi(z_1,z_2))=f(z_1,z_2)$. We define $\{(2z,z^2)=\pi(z,z):z\in\mathbb{D}\}:=\mathcal{S}$ to be the *royal variety of* $\mathbb{G}_2$.

Remark 4 It has to be observed that $\mathrm{Aut}(\mathbb{G}_2)$ does not act transitively on $\mathbb{G}_2$. However $\mathrm{Aut}(\mathbb{G}_2)$ acts transitively on the royal variety $\mathcal{S}$ of $\mathbb{G}_2$ since, from (c) of Proposition 2, it is easily seen that, for any point $(2a,a^2)\in\mathcal{S}$, we have $H_{h_a}(2a,a^2)=(0,0)$. In the same way, one can deduce that $\mathrm{Aut}(\mathbb{G}_2)$ acts transitively on $\overline{\mathcal{S}}\cap\check{\partial}\mathbb{G}_2$. Furthermore, since $\mathrm{Aut}(\mathbb{D})$ acts doubly transitively on $\partial\mathbb{D}$, given two points $x,y\in\check{\partial}\mathbb{G}_2\setminus\partial\mathcal{S}$, we can always find an automorphism $H_h\in\mathrm{Aut}(\mathbb{G}_2)$ such that $H_h(x)=y$. This is so since if $x=\pi(e^{i\alpha},e^{i\beta})$, with $\alpha\neq\beta$, and $y=\pi(e^{i\gamma},e^{i\delta})$, with $\gamma\neq\delta$, then we can find $h\in\mathrm{Aut}(\mathbb{D})$ such that $h(e^{i\alpha})=e^{i\gamma}$ and $h(e^{i\beta})=e^{i\delta}$, which defines H_h as required.

We recall now the following general definition.

Definition 5 If X is a complex manifold, any $\varphi\in\mathrm{Hol}(\mathbb{D},X)$ which is an isometry for Poincaré and Kobayashi distance will be called a *complex geodesic*.

We remind here that, from the definition, in $\mathbb{D}$ any complex geodesic is a holomorphic automorphism of $\mathbb{D}$ and that in the polydisc $\mathbb{D}^n$ a complex geodesic is of the form $\mathbb{D}\ni z\mapsto(\varphi_1(z),\dots,\varphi_n(z))$ where at least one of the φ_j is an automorphism of $\mathbb{D}$ (see, e.g. [3,4]); in particular, this is the case if, for instance, $\mathbb{D}\ni z\mapsto(z,\varphi_2(z),\dots,\varphi_n(z))$. A first link between complex geodesics in $\mathbb{G}_2$ and holomorphic automorphisms of $\mathbb{G}_2$ is given in the following lemma (see [1]).

LEMMA 6 *Let $\varphi:\mathbb{D}\to\mathbb{G}_2$ be a mapping of the form*

$$\varphi=\left(\frac{P}{R},\frac{Q}{R}\right),$$

with P, Q, R polynomials of degree ≤ 2 and such that $R^{-1}(0)\cap\overline{\mathbb{D}}=\emptyset$. If $\varphi(\partial\mathbb{D})\subset\check{\partial}\mathbb{G}_2$ and, for some ξ, $\eta\in\partial\mathbb{D}$, $\varphi(\xi)=(2\eta,\eta^2)$, then

$$\mathrm{h}:=\mathrm{f}_{\overline{\eta}}\circ\varphi=\frac{2\overline{\eta}\mathrm{Q}-\mathrm{P}}{2\mathrm{R}-\overline{\eta}\mathrm{P}}$$

is an automorphism of $\mathbb{D}$ and φ is a complex geodesic. Furthermore, if $g\in\mathrm{Aut}(\mathbb{D})$, then $\psi:=H_g\circ\varphi$ satisfies the same assumptions of φ.

We finally recall that for any two points $(s_1,p_1),(s_2,p_2)\in\mathbb{G}_2$ there exists a complex geodesic φ such that (s_1,p_1), $(s_2,p_2)\in\varphi(\mathbb{D})$ (see [1, p. 37]) and that a complete description of complex geodesics in $\mathbb{G}_2$ is given in [1]; it essentially depends on the possible intersection of the complex geodesic with the royal variety $\mathcal{S}$, namely

PROPOSITION 7 *Let $\varphi:\mathbb{D}\to\mathbb{G}_2$ be holomorphic. Then*

(a) *if $\#(\varphi(\mathbb{D})\cap\mathcal{S})\geq 2$, then φ is a complex geodesic if and only if $\varphi(z)=(-2z,z^2)$, modulo* $\mathrm{Aut}(\mathbb{D})$;
(b) *if $\#(\varphi(\mathbb{D})\cap\mathcal{S})=1$, then φ is a complex geodesic if and only if $\varphi(z)=\pi(B(\sqrt{z}),B(-\sqrt{z}))$, modulo* $\mathrm{Aut}(\mathbb{D})$, *with B a Blaschke product of order ≤ 2 such that $B(0)=0$;*

(c) *if* $\varphi(\mathbb{D}) \cap \mathcal{S} = \emptyset$, *then* φ *is a complex geodesic if and only if* $\varphi = \pi(h_1, h_2)$, *where* $h_1, h_2 \in \mathrm{Aut}(\mathbb{D})$ *are such that* $h_1 - h_2$ *has no zero in* $\mathbb{D}$.

Notice that if φ is a complex geodesic in $\mathbb{G}_2$ such that $\#(\varphi(\mathbb{D}) \cap \mathcal{S}) \geq 2$, then $\varphi(\mathbb{D}) = \mathcal{S}$. In what follows we will be particularly interested in finding relationships between complex geodesics in $\mathbb{G}_2$ and in $\mathbb{D}^2$, which are also classified (see, e.g. [3]). Thus, let $\varphi : \mathbb{D} \to \mathbb{G}_2$ be a complex geodesic:

(a) if $\#(\varphi(\mathbb{D}) \cap \mathcal{S}) \geq 2$, then, modulo $\mathrm{Aut}(\mathbb{D})$, $\varphi = \pi \circ \widetilde{\varphi}$, with $\widetilde{\varphi}(z) = (z, z)$, modulo $\mathrm{Aut}(\mathbb{D})$, and $\widetilde{\varphi} : \mathbb{D} \to \mathbb{D}^2$ is clearly a complex geodesic;
(b) if $\#(\varphi(\mathbb{D}) \cap \mathcal{S}) = 1$, then, modulo $\mathrm{Aut}(\mathbb{D})$, $\varphi(z) = \pi(B(\sqrt{z}), B(-\sqrt{z}))$, with B a Blaschke product of order ≤ 2 such that $B(0) = 0$. Therefore, we can assume $\varphi(\mathbb{D}) \cap \mathcal{S} = (0, 0)$ and so

$$\varphi(z) = \begin{cases} (0, -c_0^2 z) & \text{if } B(z) = c_0 z \\ \left(2c_1\sqrt{z} \cdot \dfrac{\sqrt{z} - z_1}{1 - \overline{z_1}\sqrt{z}},\ c_1^2 z\left(\dfrac{\sqrt{z^2} - z_1^2}{1 - \overline{z_1^2}\sqrt{z^2}}\right)\right) & \text{if } B(z) = c_1 z \cdot \dfrac{z - z_1}{1 - \overline{z_1} z} \end{cases}$$

with $c_0, c_1 \in \mathbb{C}$ different from zero. In both cases, there is no complex geodesic $\widetilde{\varphi}$ in $\mathbb{D}^2$ such that $\varphi = \pi \circ \widetilde{\varphi}$, since for a complex geodesic in $\mathbb{D}^2$ at least one of its components belongs to $\mathrm{Aut}(\mathbb{D})$. Notice furthermore that there cannot be any complex geodesic $\varphi(z) = \pi(B(\sqrt{z}), B(-\sqrt{z}))$ passing through any point of $\mathcal{S}$ and $(2, 1) \in \check{\partial}\mathbb{G}_2$;
(c) if $\varphi(\mathbb{D}) \cap \mathcal{S} = \emptyset$, then $\varphi = \pi(h_1, h_2)$, where $h_1, h_2 \in \mathrm{Aut}(\mathbb{D})$ are such that $h_1 - h_2$ has no zero in $\mathbb{D}$. Finally, observe that for any $a \in \mathbb{R}$, $-1 < a < 1$, the map $h_a(z) := (z - a)/(1 - az)$ belongs to $\mathrm{Aut}(\mathbb{D})$ and is such that $h_a(1) = 1$. Hence, for any $a, b \in \mathbb{R}$ with a, b such that $-1 < a < 1$, $-1 < b < 1$, $a \neq b$, the map $\varphi(z) = \pi(h_a(z), h_b(z))$ is a complex geodesic such that $\varphi(1) = (2, 1)$; more in general, since any automorphism of $\mathbb{D}$ maps the boundary of $\mathbb{D}$ into itself, then if $\#(\varphi(\mathbb{D}) \cap \mathcal{S}) \neq 1$, necessarily $\lim_{\mathbb{R} \ni t \to 1^-} \varphi(t) \in \check{\partial}\mathbb{G}_2$ and, in particular, if $\varphi(t) = \pi(t, t)$, then $\lim_{\mathbb{R} \ni t \to 1^-} \varphi(t) \in \check{\partial}\mathbb{G}_2 \cap \overline{\mathcal{S}}$.

Finally, since the above description of complex geodesics in $\mathbb{G}_2$ is provided as a projection of a pair of Blaschke products or automorphisms of $\mathbb{D}$ *modulo* $\mathrm{Aut}(\mathbb{D})$, and since (as observed in Remark 4) $\mathrm{Aut}(\mathbb{D})$ acts transitively on $\partial\mathbb{D}$ and $\mathrm{Aut}(\mathbb{G}_2)$ acts transitively on $\overline{\mathcal{S}} \cap \partial\mathbb{G}_2$, for the sake of simplicity, if $\varphi(z) = \pi(h_1(z), h_2(z))$ is a complex geodesic in $\mathbb{G}_2$ with $h_1, h_2 \in \mathrm{Aut}(\mathbb{D})$ and such that $\varphi(\mathbb{D}) \cap \partial\overline{\mathcal{S}} \neq \emptyset$, we will often assume that $h_1(z) = z$ and write $\varphi(z) = \pi(z, \vartheta(z))$, with $\vartheta = h_1^{-1} \circ h_2 \in \mathrm{Aut}(\mathbb{D})$. Notice that in this way $\varphi(1) = (2, 1)$.

It is worth noticing here that a geodesic in $\mathbb{G}_2$ is not necessarily a projection of a complex geodesic in the bidisc $\mathbb{D}^2$.

2. Horocycles and Julia's Lemma in $\mathbb{G}_2$

According to the analogous definition given in $\mathbb{D}^2$ (see [5]), which generalizes the notion of horocycles[3] in the unit disc $\mathbb{D}$, we introduce the following definition.

Definition 8 If $k_{\mathbb{G}_2}$ is the Kobayashi distance in $\mathbb{G}_2$, let φ be a complex geodesic in $\mathbb{G}_2$ such that[4]

$$\lim_{\mathbb{R}\ni t\to 1^-}\varphi(t)=x\in\check{\partial}\mathbb{G}_2.$$

We say that $z\in\mathbb{G}_2$ belongs to the Busemann[5] sublevel set of centre $x\in\check{\partial}\mathbb{G}_2$ and radius $R>0$ if and only if

$$\lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(z,\varphi(t))-k_{\mathbb{G}_2}(\varphi(0),\varphi(t))]<\frac{1}{2}\log R$$

and summarize it – in symbols – by putting $z\in\mathbb{B}^{\varphi}_{\mathbb{G}_2}(x,R)$.

In what follows, we will see that the functions $f_a:\mathbb{G}_2\to\mathbb{D}$ (defined in Section 1) may be considered as the analogue in $\mathbb{G}_2$ of the projections in $\mathbb{D}^2$ given by $(z,w)\mapsto z$ or $(z,w)\mapsto w$.

Proposition 9 *Consider $x\in\partial\mathbb{G}_2$ and let φ be a complex geodesic in $\mathbb{G}_2$ such that $\lim_{\mathbb{R}\ni t\to 1^-}\varphi(t)=x$. Then $z\in\mathbb{B}^{\varphi}_{\mathbb{G}_2}(x,R)$ if and only if $f_a(z)\in E\big(f_a(x),\beta_{f_a\circ\varphi}(1)\cdot R\big)$ for any $a\in\overline{\mathbb{D}}$, where $\beta_{f_a\circ\varphi}(1)$ is the boundary dilation coefficient at 1 of the holomorphic self-map $f_a\circ\varphi:\mathbb{D}\to\mathbb{D}$.*

Proof We recall that $z\in\mathbb{B}^{\varphi}_{\mathbb{G}_2}(x,R)$ if and only if

$$1/2\log R>\lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(z,\varphi(t))-k_{\mathbb{G}_2}(\varphi(0),\varphi(t))];$$

from (1.2), we deduce that

$$\begin{aligned}&\lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(z,\varphi(t))-k_{\mathbb{G}_2}(\varphi(0),\varphi(t))]\\&\quad=\lim_{\mathbb{R}\ni t\to 1^-}\left\{\max_{|a|=1}[\omega(f_a(z),f_a(\varphi(t))]-\omega(0,t)\right\}\\&\quad=\lim_{\mathbb{R}\ni t\to 1^-}\left\{\max_{|a|=1}[\omega(f_a(z),f_a(\varphi(t))]-\omega(0,f_{a_0}(\varphi(t)))+\omega(0,f_{a_0}(\varphi(t)))-\omega(0,t)\right\}.\end{aligned}$$

Hence, given $a_0\in\partial\mathbb{D}$, we obtain that

$$\begin{aligned}1/2\log R>&\lim_{\mathbb{R}\ni t\to 1^-}[\omega(f_{a_0}(z),f_{a_0}(\varphi(t))-\omega(0,f_{a_0}(\varphi(t)))]\\&+\lim_{\mathbb{R}\ni t\to 1^-}[\omega(0,f_{a_0}(\varphi(t)))-\omega(0,t)].\end{aligned}\tag{2.2}$$

The second part of the limit in (2.2) is equivalent to

$$\lim_{\mathbb{R}\ni t\to 1^-}\frac{1}{2}\log\left(\frac{1+|f_{a_0}(\varphi(t))|}{1+|t|}\cdot\frac{1-|t|}{1-|f_{a_0}(\varphi(t))|}\right).\tag{2.3}$$

Since $f_{a_0}\circ\varphi$ is a holomorphic self-map of $\mathbb{D}$ such that

$$\lim_{\mathbb{R}\ni t\to 1^-}f_{a_0}(\varphi(t)):=f_{a_0}(x)\in\partial\mathbb{D},$$

from the characterization of $\beta_{f_{a_0}\circ\varphi}(1)$, the boundary dilation coefficient at 1 of the holomorphic self-map $f_a \circ \varphi$ (see [3]), we can write

$$\beta_{f_{a_0}\circ\varphi}(1) = \lim_{\mathbb{R}\ni t\to 1^-} \frac{1-|f_{a_0}(\varphi(t))|}{1-|t|} = \lim_{\mathbb{R}\ni t\to 1^-} \frac{1-|f_{a_0}(\varphi(t))|}{1-t};$$

thus the proof is completed. ■

The reason why we adopted this general approach in defining subsets of $\mathbb{G}_2$ which resemble the definition of horocycles is related to the peculiar geometric properties of the symmetrized bidisc $\mathbb{G}_2$ which has no analogue in the case of the bidisc $\mathbb{D}^2$. Indeed, the following example shows that a Busemann sublevel set cannot be always obtained as the projection of a horocycle in the bidisc $\mathbb{D}^2$.

Example 10 Since for any $z \in \mathbb{D}$ we have $0 \le |z|^2 = |z^2| < |z|$, consider $r \in \mathbb{R}$ such that $|z|^2 < r < |z|$ and define $R = \frac{r}{1-r}$. Then there exists $z \in \mathbb{D}$ such that $z \in E(1,R)$ and $\lambda z^2 \in E(1,R)$ for any $\lambda \in \overline{\mathbb{D}}$, but $-z \notin E(1,R)$. Clearly, $\pi(z,-z) = (0,-z^2)$ and $f_a(0,-z^2) = -az^2$; since $a \in \overline{\mathbb{D}}$, we conclude that $f_a(0,-z^2) \in E(1,R)$ and so $(0,-z^2) \in \mathbb{B}^{\varphi}_{\mathbb{G}_2}((2,1),R)$ (with $\varphi = \pi(t,t)$) even though $(z,-z) \notin E(1,R) \times E(1,R)$.

The boundary dilation coefficient of a holomorphic self-map in $\mathbb{D}$ at a point of $\partial\mathbb{D}$ is not necessarily finite and, actually, it is in general infinite (see [3]); this is not the case if one considers $x = (e^{i\alpha}, e^{i\beta}) \in \check{\partial}\mathbb{G}_2$ and $\varphi(z) = \pi(h_1(z), h_2(z))$ with $h_1, h_2 \in \mathrm{Aut}(\mathbb{D})$ such that $\lim_{\mathbb{R}\ni t\to 1^-} h_1(t) = e^{i\alpha}$, $\lim_{\mathbb{R}\ni t\to 1^-} h_2(t) = e^{i\beta}$ and $h_1 - h_2 \ne 0$ in $\mathbb{D}$, that is when φ is a complex geodesic in $\mathbb{G}_2$ passing through x and such that $\#(\varphi(\mathbb{D}) \cap \mathcal{S}) \ne 1$. Given $a \in \mathbb{D}$, we define

$$f_a(e^{i\alpha}, e^{i\beta}) = \frac{2ae^{i(\alpha+\beta)} - (e^{i\alpha} + e^{i\beta})}{2 - a(e^{i\alpha} + e^{i\beta})} := \tau_a.$$

Since

$$\lim_{\mathbb{R}\ni t\to 1^-} \frac{\tau_a - f_a(\varphi(t))}{1-t} = \frac{4a(e^{i\alpha}h_2'(1) + e^{i\beta}h_1'(1)) - 2a^2(e^{i\alpha}h_2'(1) + e^{i\beta}h_1'(1)) - 2(h_1'(1) + h_2'(1))}{[2 - a(e^{i\alpha} + e^{i\beta})]^2}$$

is finite for any $a \in \mathbb{D}$, from Julia–Wolff–Carathéodory Theorem (see, e.g. [3]), so is the boundary dilation coefficient of $f_a \circ \varphi$ at 1. In particular, consider a complex geodesic $\varphi : \mathbb{D} \to \mathbb{G}_2$ such that $\lim_{\mathbb{R}\ni t\to 1^-} \varphi(t) = (2,1) \in \check{\partial}\mathbb{G}_2$ and $\#(\varphi(\mathbb{D}) \cap \mathcal{S}) \ne 1$; we can consider φ parameterized as $\varphi(t) = (t, \vartheta(t))$, with $\vartheta \in \mathrm{Aut}(\mathbb{D})$ such that $\vartheta(1) = 1$. Hence, after some calculations, we deduce that

$$\begin{aligned}\lim_{\mathbb{R}\ni t\to 1^-} \frac{-1 - f_a(\varphi(t))}{1-t} &= \lim_{\mathbb{R}\ni t\to 1^-} \frac{\frac{-2 + at + a\vartheta(t) - 2at\vartheta(t) + t + \vartheta(t)}{2 - at - a\vartheta(t)}}{1-t} \\ &= \lim_{\mathbb{R}\ni t\to 1^-} \frac{at(1-\vartheta(t)) + (a\vartheta(t) - 1)\cdot(1-t) - (1-\vartheta(t))}{(1-t)\cdot(2 - at - a\vartheta(t))} \\ &= \begin{cases} -\dfrac{1+\vartheta'(1)}{2} & \text{if } a \ne 1 \\ -\dfrac{2\vartheta'(1)}{1+\vartheta'(1)} & \text{if } a = 1. \end{cases}\end{aligned}$$

Notice that the previous limit does not depend[6] on a and actually implies (from the Julia–Wolff–Carathéodory Theorem in $\mathbb{D}$ (see [3])) that

$$\beta_{f_a\circ\varphi}(1) = \begin{cases} \dfrac{1+\vartheta'(1)}{2} & \text{if } a \neq 1 \\ \dfrac{2\vartheta'(1)}{1+\vartheta'(1)} & \text{if } a = 1. \end{cases}$$

Furthermore, from the above calculations, we can show that, if $\varphi(t) = \pi(t, \vartheta(t))$ is a complex geodesic in $\mathbb{G}_2$ passing through $(2, 1)$, then, given any $h \in \mathrm{Aut}(\mathbb{D})$ such that $h(1) = 1$, we have

$$\lim_{t\to 1}[k_{\mathbb{G}_2}(0, \varphi(t)) - \omega(0, t)] = \lim_{t\to 1}[k_{\mathbb{G}_2}(0, \varphi(h(t))) - \omega(0, h(t))].$$

Indeed,

$$\begin{aligned} &\lim_{t\to 1}[k_{\mathbb{G}_2}(0, \varphi(h(t))) - \omega(0, h(t))] \\ &\quad = \lim_{t\to 1}[k_{\mathbb{G}_2}(0, \varphi(h(t))) - \omega(0, t) + \omega(0, t) - \omega(0, h(t))] \\ &\quad = \lim_{t\to 1}[\max_{a\in\partial\mathbb{D}} \omega(0, f_a(\varphi(h(t)))) - \omega(0, t)] + \lim_{t\to 1}[\omega(0, t) - \omega(0, h(t))]. \end{aligned}$$

Thus

$$\begin{aligned} &\lim_{t\to 1}[\omega(0, f_a(\varphi(h(t)))) - \omega(0, t)] + \lim_{t\to 1}[\omega(0, t) - \omega(0, h(t))] \\ &\quad = \begin{cases} -\dfrac{1}{2}\log\dfrac{h'(1)(\vartheta'(1)+1)}{2} + \dfrac{1}{2}\log h'(1) = \dfrac{1}{2}\log\dfrac{2}{1+\vartheta'(1)} & \text{if } a \neq 1 \\ -\dfrac{1}{2}\log\dfrac{h'(1)(\vartheta'(1)+1)}{2\vartheta'(1)} + \dfrac{1}{2}\log h'(1) = \dfrac{1}{2}\log\dfrac{1+\vartheta'(1)}{2\vartheta'(1)} & \text{if } a = 1. \end{cases} \end{aligned}$$

Similarly,

$$\lim_{t\to 1}[k_{\mathbb{G}_2}(0, \varphi(t)) - \omega(0, t)] = \lim_{t\to 1}[\max_{a\in\partial\mathbb{D}} \omega(0, f_a(\varphi(t))) - \omega(0, t)]$$

and

$$\lim_{t\to 1}[\omega(0, f_a(\varphi(t))) - \omega(0, t)] = \begin{cases} \dfrac{1}{2}\log\dfrac{2}{\vartheta'(1)+1} & \text{if } a \neq 1 \\ \dfrac{1}{2}\log\dfrac{1+\vartheta'(1)}{2\vartheta'(1)} & \text{if } a = 1. \end{cases}$$

Therefore we can always assume that a complex geodesic φ in $\mathbb{G}_2$ passing through $(2, 1)$ is parameterized as $t \mapsto \pi(t, \vartheta(t))$, with $\vartheta \in \mathrm{Aut}(\Delta)$ such that $\vartheta(1) = 1$ and $\vartheta'(1) \leq 1$; indeed, if $\vartheta'(1) > 1$, then we may simply consider $\varphi_1 = \varphi \circ \vartheta^{-1}$ instead of φ. Notice that

$$\frac{1+\vartheta'(1)}{2\vartheta'(1)} \geq \frac{2}{1+\vartheta'(1)}$$

and equality holds if and only if $\vartheta'(1) = 1$. If F is a holomorphic self-map of $\mathbb{G}_2$ and $\varphi: \mathbb{D} \to \mathbb{G}_2$ is a complex geodesic such that $\lim_{\mathbb{R}\ni t\to 1^-} \varphi(t) = x \in \check{\partial}\mathbb{G}_2$, then we give the following definition.

Definition 11

$$\frac{1}{2}\log\alpha^F_{a,\varphi}(x) = \lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(0,\varphi(t)) - \omega(0, f_a(F(\varphi(t))))].$$

Remark 12 The definition of $\alpha^F_{a,\varphi}(x)$ is independent from the parameterization of the geodesic φ. Before showing this we observe that, from the definition of boundary dilation coefficient at 1,

$$\begin{aligned}
\frac{1}{2}\log\alpha^F_{a,\varphi}(x) &= \lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(0,\varphi(t)) - \omega(0, f_a(F(\varphi(t))))] \\
&= \lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(0,\varphi(t)) - \omega(0,t) + \omega(0,t) - \omega(0, f_a(F(\varphi(t))))] \\
&= \lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(0,\varphi(t)) - \omega(0,t)] + \lim_{\mathbb{R}\ni t\to 1^-}[\omega(0,t) - \omega(0, f_a(F(\varphi(t))))] \\
&= \frac{1}{2}\log\frac{\beta_{f_a\circ F\circ\varphi}(1)}{\beta_{f_a\circ\varphi}(1)}.
\end{aligned}$$

Now consider $\vartheta \in \mathrm{Aut}(\mathbb{D})$ such that $\vartheta(1)=1$ and $\psi = \varphi\circ\vartheta$, then

$$\begin{aligned}
&\lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(0,\psi(t)) - \omega(0, f_a(F(\psi(t))))] \\
&\quad = \lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(0,\varphi(\vartheta(t))) - \omega(0, f_a(F(\varphi(\vartheta(t)))))] \\
&\quad = [k_{\mathbb{G}_2}(\varphi(0),\varphi(\vartheta(t))) - \omega(0, f_a(F(\varphi(\vartheta(t))))) \\
&\qquad + k_{\mathbb{G}_2}(0,\varphi(\vartheta(t))) - k_{\mathbb{G}_2}(\varphi(0),\varphi(\vartheta(t)))] \\
&\quad = \lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(\varphi(0),\psi(t)) - \omega(0, f_a(F(\varphi(\vartheta(t))))) \\
&\qquad + k_{\mathbb{G}_2}(0,\varphi(\vartheta(t))) - k_{\mathbb{G}_2}(\varphi(0),\psi(t))].
\end{aligned}\tag{2.4}$$

Since $k_{\mathbb{G}_2}(\varphi(0),\varphi(\vartheta(t))) = \omega(0,\vartheta(t))$ and $\psi = \varphi\circ\vartheta$, we can actually write

$$\lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(\varphi(0),\psi(t)) - \omega(0, f_a(F(\varphi(\vartheta(t))))) + k_{\mathbb{G}_2}(0,\varphi(\vartheta(t))) - k_{\mathbb{G}_2}(\varphi(0),\psi(t))] =$$
$$\lim_{\mathbb{R}\ni t\to 1^-}[\omega(0,\vartheta(t)) - \omega(0,t) + \omega(0,t) - \omega(0, f_a(F(\varphi(\vartheta(t))))) + k_{\mathbb{G}_2}(0,\varphi(\vartheta(t))) - \omega(0,\vartheta(t))].$$

We remind that

$$\lim_{\mathbb{R}\ni t\to 1^-}[\omega(0,\vartheta(t)) - \omega(0,t)] = -\frac{1}{2}\log\beta_\vartheta(1)$$
$$\lim_{\mathbb{R}\ni t\to 1^-}[\omega(0,t) - \omega(0, f_a(F(\varphi(\vartheta(t)))))] = \frac{1}{2}\log\beta_{f_a\circ F\circ\varphi\circ\vartheta}(1)$$

For the remaining part in the limit 2.4, recall that $k_{\mathbb{G}_2}(0,\varphi(\vartheta(t))) = \max_{a\in\partial\mathbb{D}}\omega(0, f_a(\varphi(t))$ and remember that, for any $a\in\overline{\mathbb{D}}$,

$$\lim_{\mathbb{R}\ni t\to 1^-}[\omega(0, f_a(\varphi(\vartheta(t)))) - \omega(0,\vartheta(t))] = -\frac{1}{2}\log\beta_{f_a\circ\varphi}(1);$$

so we can conclude that

$$\lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(0,\psi(t)) - \omega(0, f_a(F(\psi(t))))] = \lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(0,\varphi(t)) - \omega(0, f_a(F(\varphi(t))))].$$

We now want to show how to obtain explicit examples of holomorphic self-maps in $\mathbb{G}_2$ whose boundary dilation coefficients are finite and not zero.

Example 13 Consider a pair of holomorphic functions $f, g \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ such that

$$\text{K-}\lim_{z\to 1} f(z) = 1, \quad \text{K-}\lim_{z\to 1} g(z) = 1$$

and with finite boundary dilation coefficients $\beta_f(1)$, $\beta_g(1)$ at 1. For $(s,p) \in \mathbb{G}_2$, define $u := \frac{s+\sqrt{s^2-4p}}{2}$ and $v := \frac{s-\sqrt{s^2-4p}}{2}$ and consider the holomorphic map F in $\mathbb{G}_2$ as follows:

$$F(s,p) = (f(u) + g(v), f(u) \cdot g(v)).$$

Clearly, F maps holomorphically $\mathbb{G}_2$ into itself; furthermore, if φ is a complex geodesic in $\mathbb{G}_2$ passing through $x = (2,1)$ and parameterized as $t \mapsto \pi(t, \vartheta(t))$, with $\vartheta \in \mathrm{Aut}(\mathbb{D})$ such that $\vartheta(1) = 1$ and $\vartheta'(1) \leq 1$, then

$$\lim_{\mathbb{R}\ni t\to 1^-} F(\varphi(t)) = (2,1) =: x.$$

Given

$$f_a(s,p) = \frac{2ap - s}{2 - as},$$

the holomorphic map $f_a \circ F \circ \varphi$ belongs to $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ for any $a \in \overline{\mathbb{D}}$ and

$$f_a(F(\varphi(t))) = \frac{2af(t)\, g(\vartheta(t)) - f(t) - g(\vartheta(t))}{2 - af(t) - ag(\vartheta(t))}.$$

In order to evaluate the boundary dilation coefficient of $f_a \circ F \circ \varphi$ at 1, one considers

$$\begin{aligned}\lim_{\mathbb{R}\ni t\to 1^-} \frac{-1 - f_a(F(\varphi(t)))}{1-t} &= \lim_{\mathbb{R}\ni t\to 1^-} \frac{-1 - \frac{2af(t)g(\vartheta(t)) - f(t) - g(\vartheta(t))}{2 - af(t) - ag(\vartheta(t))}}{1-t} \\ &= \lim_{\mathbb{R}\ni t\to 1^-} \frac{-2 + af(t) + ag(\vartheta(t)) - 2af(t)g(\vartheta(t)) + f(t) + g(\vartheta(t))}{(1-t)(2 - af(t) - ag(\vartheta(t)))}.\end{aligned}$$

The above limit can also be written as follows:

$$\begin{aligned}\lim_{\mathbb{R}\ni t\to 1^-} \Bigg[& af(t) \frac{1 - g(\vartheta(t))}{(1-t)(2 - af(t) - ag(\vartheta(t)))} + ag(\vartheta(t)) \frac{1 - f(t)}{(1-t)(2 - af(t) - ag(\vartheta(t)))} \\ & - \frac{1 - f(t)}{(1-t)(2 - af(t) - ag(\vartheta(t)))} - \frac{1 - g(\vartheta(t))}{(1-t)(2 - af(t) - ag(\vartheta(t)))} \Bigg],\end{aligned}$$

which implies that $\beta_{f_a \circ F \circ \varphi}(1) = \frac{\beta_g(1)\vartheta'(1) + \beta_f(1)}{2}$ if $a \neq 1$. If $a = 1$,

$$\lim_{\mathbb{R}\ni t\to 1^-} \frac{-1 - f_1(F(\varphi(t)))}{1-t} = -2 \frac{(1 - f(t))(1 - g(\vartheta(t)))}{(1-t)(2 - f(t) - g(\vartheta(t)))}$$

– or, equivalently, –

$$= \lim_{\mathbb{R}\ni t\to 1^-} -2 \frac{1 - f(t)}{1-t} \cdot \frac{1 - g(\vartheta(t))}{1 - \vartheta(t)} \cdot \frac{1 - \vartheta(t)}{1-t} \cdot \frac{1}{\frac{1-f(t)}{1-t} + \frac{1-g(\vartheta(t))}{1-\vartheta(t)} \frac{1-\vartheta(t)}{1-t}},$$

which leads to the conclusion that

$$\lim_{\mathbb{R}\ni t\to 1^-} \frac{-1 - f_1(F(\varphi(t)))}{1-t} = \frac{-2\beta_f(1)\cdot\beta_g(1)\cdot\vartheta'(1)}{\beta_f + \beta_g\cdot\vartheta'(1)}.$$

Since – see Remark 12 and the calculations developed after Example 10 –

$$\alpha^F_{a,\varphi}(x) = \frac{\beta_{f_a\circ F\circ\varphi}(1)}{\beta_{f_a\circ\varphi}(1)},$$

we can conclude that

$$\alpha^F_{a,\varphi}(x) = \begin{cases} \dfrac{\beta_g(1)\vartheta'(1) + \beta_f(1)}{1+\vartheta'(1)} & \text{if } a \neq 1 \\ \dfrac{\beta_g(1)\cdot\beta_f(1)(1+\vartheta'(1))}{\beta_f(1)+\beta_g(1)\vartheta'(1)} & \text{if } a = 1; \end{cases}$$

hence $\alpha^F_{a,\varphi}(x)$ is finite and not zero for any $a\in\partial\mathbb{D}$.

Remark 14 If, for a given a, it turns out that $\alpha^F_{a,\varphi}(x) < +\infty$, then necessarily $\lim_{t\to 1^-} f_a(F(\varphi(t)))\in\partial\mathbb{D}$. Therefore, if $\alpha^F_{a,\varphi}(x) < +\infty$ for any $a\in\partial\mathbb{D}$, then $\lim_{t\to 1^-} |f_a(F(\varphi(t)))| = 1$ for any $a\in\partial\mathbb{D}$; but this, from Proposition 1, is equivalent to saying that $\lim_{\mathbb{R}\ni t\to 1^-} F(\varphi(t))\in\check{\partial}\mathbb{G}_2$.

If $x\in\overline{\mathcal{S}}\cap\check{\partial}\mathbb{G}_2$, and the coefficient $\alpha^F_{a,\varphi}(x)$ is finite for a complex geodesic φ, we can prove that $\alpha^F_{a,\varphi}(x)$ is also finite for any other complex geodesic ψ in $\mathbb{G}_2$ passing through x, namely the following lemma.

LEMMA 15 *Let F be a holomorphic self-map of $\mathbb{G}_2$ and $\varphi:\mathbb{D}\to\mathbb{G}_2$ a complex geodesic such that $\lim_{\mathbb{R}\ni t\to 1^-}\varphi(t) = x\in\overline{\mathcal{S}}\cap\check{\partial}\mathbb{G}_2$. If, for $a\in\partial\mathbb{D}$, it turns out that $\alpha^F_{a,\varphi}(x)$ is finite, then $\alpha^F_{a,\psi}(x)$ is also finite, where $\psi:\mathbb{D}\to\mathbb{G}_2$ is any other complex geodesic in $\mathbb{G}_2$ such that $\lim_{\mathbb{R}\ni t\to 1^-}\psi(t) = x\in\overline{\mathcal{S}}\cap\check{\partial}\mathbb{G}_2$.*

Proof Assume that $x=(2,1)$ and, as usual, $\varphi(z)=\pi(z,\vartheta(z))$, $\psi(z)=\pi(z,\eta(z))$, with $\vartheta,\eta\in\mathrm{Aut}(\mathbb{D})$ such that $\vartheta(1)=\eta(1)=1$, then

$$\begin{aligned} k_{\mathbb{G}_2}(0,\varphi(t)) - \omega(0,f_a(F(\varphi(t))) &= k_{\mathbb{G}_2}(0,\varphi(t)) - k_{\mathbb{G}_2}(0,\psi(t)) \\ &\quad + \omega(0,f_a(F(\psi(t))) - \omega(0,f_a(F(\varphi(t))) \\ &\quad + k_{\mathbb{G}_2}(0,\psi(t)) - \omega(0,f_a(F(\psi(t))). \end{aligned} \tag{2.5}$$

From the triangle inequality, we have

$$k_{\mathbb{G}_2}(0,\varphi(t)) - k_{\mathbb{G}_2}(0,\psi(t)) \le k_{\mathbb{G}_2}(\psi(t),\varphi(t))$$

and

$$\omega(0,f_a(F(\psi(t))) - \omega(0,f_a(F(\varphi(t))) \le \omega(f_a(F(\psi(t)),f_a(F(\varphi(t)));$$

hence

$$k_{\mathbb{G}_2}(0,\varphi(t)) - k_{\mathbb{G}_2}(0,\psi(t)) \le \omega(\eta(t),\vartheta(t))$$

and

$$\omega(0,f_a(F(\psi(t))) - \omega(0,f_a(F(\varphi(t))) \le \omega(\eta(t),\vartheta(t)).$$

Since

$$\omega(\eta(t),\vartheta(t)) = \frac{1}{2}\log\frac{1+\left|\frac{\eta(t)-\vartheta(t)}{1-\overline{\eta(t)}\vartheta(t)}\right|}{1-\left|\frac{\eta(t)-\vartheta(t)}{1-\overline{\eta(t)}\vartheta(t)}\right|}$$

and

$$\left|\frac{\eta(t)-\vartheta(t)}{1-\overline{\eta(t)}\vartheta(t)}\right| = \left|\frac{\eta(t)-1+1-\vartheta(t)}{1-\vartheta(t)+\vartheta(t)-\overline{\eta(t)}\vartheta(t)}\right| = \left|\frac{\frac{1-\vartheta(t)}{1-t}-\frac{1-\eta(t)}{1-t}}{\frac{1-\vartheta(t)}{1-t}+\vartheta(t)\cdot\frac{1-\overline{\eta(t)}}{1-\eta(t)}\cdot\frac{1-\eta(t)}{1-t}}\right|,$$

after some calculations, we easily deduce that

$$\lim_{\mathbb{R}\ni t\to 1^-}\omega(\eta(t),\vartheta(t)) = \begin{cases} \frac{1}{2}\log\frac{\vartheta'(1)}{\eta'(1)} & \text{if } \vartheta'(1)\geq \eta'(1) \\ \frac{1}{2}\log\frac{\eta'(1)}{\vartheta'(1)} & \text{if } \vartheta'(1) < \eta'(1). \end{cases} \tag{2.6}$$

In any case $\lim_{\mathbb{R}\ni t\to 1^-}\omega(\eta(t),\vartheta(t))$ is finite so that from (2.5) we conclude that

$$\lim_{\mathbb{R}\ni t\to 1^-}[k_{\mathbb{G}_2}(0,\psi(t)) - \omega(0,f_a(F(\psi(t)))] = \alpha^F_{a,\varphi} + 2\lim_{\mathbb{R}\ni t\to 1^-}\omega(\eta(t),\vartheta(t)) < +\infty.$$

■

We are now ready to prove the following version of the Julia's Lemma in the symmetrized bidisc $\mathbb{G}_2$.

Proposition 16 *Let F be a holomorphic self-map of $\mathbb{G}_2$ and consider a complex geodesic φ in $\mathbb{G}_2$ passing through $x\in\check{\partial}\mathbb{G}_2$ such that $\alpha^F_{a,\varphi}(x)$ is finite for any $a\in\partial\mathbb{D}$. Then there exist $y\in\check{\partial}\mathbb{G}_2$ and a complex geodesic ψ in $\mathbb{G}_2$ passing through y, such that, for any $R>0$ (and with the notations so-far introduced)*

$$F(\mathbb{B}^{\varphi}_{\mathbb{G}_2}(x,R)) \subseteq \mathbb{B}^{\psi}_{\mathbb{G}_2}(y,\alpha^F(x)\cdot R),$$

where $\alpha^F(x) = \max_{a\in\partial\mathbb{D}}\alpha^F_{a,\varphi}(x)$.

Proof Take $z\in\mathbb{B}^{\varphi}_{\mathbb{G}_2}(x,R)$ and consider $F(z)$. From the inequality

$$\frac{1}{2}\log\alpha^F_{a,\varphi}(x) \geq \frac{1}{2}\log\frac{\beta_{f_a\circ F\circ\varphi}(1)}{\beta_{f_a\circ\varphi}(1)}$$

proven in Remark 12, we deduce that, since $\alpha^F_{a,\varphi}(x)$ is finite for any $a\in\partial\mathbb{D}$, necessarily $\beta_{f_a\circ F\circ\varphi}(1)$ is finite for any $a\in\partial\mathbb{D}$. Thus, from Julia–Wolff–Carathéodory Theorem, we conclude that, for any $a\in\partial\mathbb{D}$, the following limit exists:

$$\lim_{\mathbb{R}\ni t\to 1^-} f_a(F(\varphi(t))) := \tau_{a,\varphi}.$$

Then

$$\lim_{w\to\tau_{a,\varphi}}[\omega(f_a(F(z),w) - \omega(0,w)] = \lim_{t\to 1}[\omega(f_a(F(\varphi(t))),f_a(F(z)) - \omega(0,f_a(F(\varphi(t))))]$$

since from (2.1), the definition of a horocycle in $\mathbb{D}$ of centre $f_{a_0}(y)\in\partial\mathbb{D}$ is independent of the way of approaching the centre of the horocycle. Hence, since the holomorphic map $f_a \circ F$ is a contraction,

$$\begin{aligned}
&\lim_{w\to\tau_{a,\varphi}}[\omega(f_a(F(z)),w)-\omega(0,w)] \leq \lim_{t\to 1}[k_{\mathbb{G}_2}(z,\varphi(t))-\omega(0,f_a(F(\varphi(t))))]\\
&\quad=\lim_{t\to 1}[k_{\mathbb{G}_2}(z,\varphi(t))-\omega(0,t)]+\lim_{t\to 1}[\omega(0,t)-k_{\mathbb{G}_2}(0,\varphi(t))]\\
&\quad\quad+\lim_{t\to 1}[k_{\mathbb{G}_2}(0,\varphi(t))-\omega(0,f_a(F(\varphi(t))))]\leq\frac{1}{2}\log\alpha^F_{a,\varphi}(x)\cdot R\\
&\quad\quad+\lim_{t\to 1}[\omega(0,t)-k_{\mathbb{G}_2}(0,\varphi(t))]
\end{aligned}$$

because $z\in\mathbb{B}^{\varphi}_{\mathbb{G}_2}(x,R)$. Finally, from

$$\lim_{t\to 1}[\omega(0,t)-k_{\mathbb{G}_2}(0,\varphi(t))]\leq\lim_{t\to 1}[\omega(0,t)-\omega(0,f_a(\varphi(t)))]\quad\forall a\in\partial\mathbb{D}$$

we conclude that

$$\lim_{w\to\tau_{a,\varphi}}[\omega(w,f_a(F(z))-\omega(0,w)]\leq\frac{1}{2}\log\alpha^F_{a,\varphi}(x)\cdot R\cdot\beta_{f_a\circ\varphi}(1)$$

or equivalently that

$$f_a(F(z))\in E(\tau_{a,\varphi},\alpha^F_{a,\varphi}(x)\cdot R\cdot\beta_{f_a\circ\varphi}(1))\quad\forall a\in\partial\mathbb{D}$$

and so, using Proposition 9, we conclude that $F(z)$ belongs to a Busemann sublevel set $B^{\psi}_{\mathbb{G}_2}(y,c(\alpha^F_{a,\varphi}(x),\beta_{f_a\circ\varphi}(1))\cdot R))$, where $f_a(y)=\tau_{a,\varphi}$. We recall that, from Remark 14, one has $y\in\check{\partial}\mathbb{G}_2$ *independent* from the choice of φ. Indeed, if φ_1 is another complex geodesic in $\mathbb{G}_2$ passing through $x\in\check{\partial}\mathbb{G}_2$ such that $\alpha^F_{a,\varphi_1}(x)$ is finite for any $a\in\partial\mathbb{D}$, then assume by contradiction that for a given a_0 $\tau_{a_0,\varphi}\neq\tau_{a_0,\varphi_1}$. Since, for any $R>0$,

$$f_{a_0}(B^{\varphi}_{\mathbb{G}_2}(x,R))\subseteq E(\tau_{a,\varphi},c\cdot R)$$

and

$$f_{a_0}(B^{\varphi_1}_{\mathbb{G}_2}(x,R))\subseteq E(\tau_{a,\varphi_1},c_1\cdot R),$$

we can always find a suitable $R_0>0$ such that $B^{\varphi}_{\mathbb{G}_2}(x,R_0)\cap B^{\varphi_1}_{\mathbb{G}_2}(x,R_0)\neq\emptyset$ but $E(\tau_{a,\varphi},c\cdot R_0)\cap E(\tau_{a,\varphi_1},c_1\cdot R_0)=\emptyset$. It is now left to determine a complex geodesic ψ in $\mathbb{G}$ (passing through y) for the definition of the Busemann sublevel set $B^{\psi}_{\mathbb{G}_2}(y,c(\alpha^F_{a,\varphi}(x),\beta_{f_a\circ\varphi}(1))\cdot R)$. We will distinguish the possible cases:

(a) if $x,y\in\partial\mathcal{S}\cap\check{\partial}\mathbb{G}_2$, then from Remark 4 and Lemma 6 we can assume that $x=y$ so that $\psi=\varphi$;
(b) if $x,y\in\check{\partial}\mathbb{G}_2\setminus\partial\mathcal{S}$, then, again, from Remark 4 and Lemma 6 we can assume that $x=y$ so that $\psi=\varphi$;
(c) if $x\in\partial\mathcal{S}\cap\check{\partial}\mathbb{G}_2$ and $y\in\check{\partial}\mathbb{G}_2\setminus\partial\mathcal{S}$, from Remark 4 we can assume $x=(2,1)$ and $y=(0,-1)$. We have seen that, $\forall a\in\partial\mathbb{D}$,

$$f_a(F(z))\in E(\tau_{a,\varphi},\alpha^F_{a,\varphi}(x)\cdot\beta_{f_a\circ\varphi}(1)\cdot R);$$

if $a \neq 1$, $\beta_{f_a \circ \varphi}(1) = \frac{1+\vartheta'(1)}{2}$ so that we can also say (see the text after Example 10) that

$$
\begin{aligned}
f_a(F(z)) &\in E\left(\tau_{a,\varphi}, \alpha^F_{a,\varphi}(x) \cdot \frac{1+\vartheta'(1)}{2} \cdot R\right) \\
&\subseteq E\left(\tau_{a,\varphi}, \alpha^F_{a,\varphi}(x) \cdot \left[\frac{1+\vartheta'(1)}{2} \cdot (1-\Re a) + (1+\Re a)\right] \cdot R\right).
\end{aligned}
$$

Hence we can take a complex geodesic ψ passing through $(0,-1)$ of the form $\psi(z) = \pi(z, \eta(z))$ with $\eta \in \mathrm{Aut}(\mathbb{D})$ such that $\eta(1) = -1$ and $\eta'(1) = \frac{1+\vartheta'(1)}{2}$ since $\beta_{f_a \circ \psi}(1) = \eta'(1) \cdot (1-\Re a) + (1+\Re a)$. Finally, if $a = 1$, then $\beta_{f_a \circ \varphi}(1) = \frac{2\vartheta'(1)}{1+\vartheta'(1)}$; but since $\frac{2\vartheta'(1)}{1+\vartheta'(1)} \leq \frac{1+\vartheta'(1)}{2}$ for $\vartheta'(1) \leq 1$, we can repeat the above argument.

(d) if $x \in \check{\partial}\mathbb{G}_2 \setminus \partial\mathcal{S}$ and $y \in \partial\mathcal{S} \cap \check{\partial}\mathbb{G}_2$, from Remark 4 we can assume that $x = (0,-1)$ and $y = (2,1)$. We have seen that (see endnote 6), $\forall a \in \partial\mathbb{D}$,

$$f_a(F(z)) \in E(\tau_{a,\varphi}, 2\alpha^F_{a,\varphi}(x) \cdot R)$$

and we can take a complex geodesic ψ passing through $(2,1)$ of the form $\psi(z) = \pi(z, \eta(z))$ with $\eta \in \mathrm{Aut}(\mathbb{D})$ such that $\eta(1) = 1$ and $\eta'(1) = 2$. ■

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Notes

1. This domain (and its further generalizations) has been attracting the interest of many mathematicians for several reasons. Among them, we want to recall here that in Engineering Mathematics these domains have been investigated for robust stabilization which is related to the so-called spectral Nevanlinna–Pick interpolation problem. More recently, the domain $\mathbb{G}_2$ (which is not convex, since $(2,1)$ and $(2i,-1) \in \overline{\mathbb{G}_2}$ but $(1+i, 0) \notin \overline{\mathbb{G}_2}$) turned out to be not biholomorphically equivalent to a convex domain. This is the first example known of a domain which actually cannot be exhausted by domains biholomorphic to convex domains whose (Möbius) distance equals the Lempert functions in the sense of (1.1).
2. For a definition of the Kobayashi (pseudo) metric and distance (see, e.g. [6,7]), we only remind here that the Kobayashi distance in the unit disc $\mathbb{D}$, $k_{\mathbb{D}}$ coincides with the Poincaré distance.
3. We recall that a horocycle in $\mathbb{D}$ of centre $x \in \partial\mathbb{D}$ and radius $R > 0$ is (also) defined (see [3]) as follows:
$$E(x,R) = \left\{ z \in \mathbb{D} \;:\; \lim_{w \to x} [\omega(z,w) - \omega(0,w)] < \frac{1}{2}\log R \right\}. \tag{2.1}$$
4. In short, we will often say that the complex geodesic φ passes through or tends to the point x of the Šilov boundary of $\mathbb{G}_2$.
5. Sometimes the function $B^{\varphi}(z) := \lim_{\mathbb{R} \ni t \to 1^-} [k_{\mathbb{G}_2}(z, \varphi(t)) - k_{\mathbb{G}_2}(\varphi(0), \varphi(t))]$ is called the *Busemann function*.
6. This is not always the case. Consider, for instance, a complex geodesic ψ passing through $(0,-1) = \pi(1,-1) = \pi(-1,1)$ of the form $\psi(z) = \pi(z, \eta(z))$ with $\eta \in \mathrm{Aut}(\mathbb{D})$ such that

$\eta(1) = -1$. Then, $f_a(0, -1) = -a$ and so

$$\lim_{\mathbb{R}\ni t\to 1^-} \frac{-a - f_a(\psi(t))}{1-t} = \lim_{\mathbb{R}\ni t\to 1^-} -2a\frac{1 + t\eta(t)}{(2 - at - a\eta(t))(1-t)} = \eta'(1)\frac{(a-1)^2}{2} - \frac{(a+1)^2}{2},$$

hence $\beta_{f_a\circ\psi}(1) = -\eta'(1)\frac{(a-1)^2}{2a} + \frac{(a+1)^2}{2a}$. Notice that (as expected) $\beta_{f_a\circ\psi}(1)$ is a positive real number since

$$-\eta'(1)\frac{(a-1)^2}{2a} + \frac{(a+1)^2}{2a} = \eta'(1)(1 - \Re a) + (1 + \Re a)$$

and $-1 \leq \Re a \leq 1$. Finally we observe that, for any $a \in \partial\mathbb{D}$,

$$2\eta'(1) \geq \eta'(1)(1 - \Re a) + (1 + \Re a)$$

and

$$\eta'(1)(1 - \Re a) + (1 + \Re a) > \eta'(1)$$

if $\eta'(1) \leq 1$.

References

[1] M. Jarnizki and P. Pflug, *Invariant distances and metrics in complex analysis – revisited*, Dissertationes Mathematicae, 2005.
[2] M. Jarnizki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, Berlin, New York, 1993.
[3] M. Abate, *Iteration Theory of Holomorphic Mappings on Taut Manifolds*, Mediterranean Press, Rende, 1989.
[4] E. Vesentini, *Capitoli Scelti Della Teoria Delle Funzioni Olomorfe*, Unione Matematica Italiana, Paris, 1984.
[5] C. Frosini, *Busemann Functions and Julia–Wolff–Carathéodory Theorem for Polydiscs*, Adv. Geom. 10 (2010), pp. 435–463.
[6] S. Kobayashi, *Intrinsic distances, measures and geometric function theory*, Bull. Am. Math. Soc. 82 (1976), pp. 357–416.
[7] L. Lempert, *La metrique Kobayashi et les representation des domains sur la boule*, Bull. Soc. Math. France 109 (1981), pp. 427–474.