Indag. Mathem., N.S., 13 (4), 537–549 December 16, 2002

A new rigidity result for holomorphic maps

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Communicated by Prof. M.S. Keane at the meeting of September 30, 2002

ABSTRACT

In this paper we determine which vanishing order of a holomorphic map f at a point of the (non necessarily regular) boundary of a very generic domain of $\mathbb{C}$ is required for f to be constant. In particular this vanishing order is 1 if the boundary is Dini-smooth whereas it is at least β/α if f locally maps a Dini-smooth corner of opening $\pi\alpha$ into a Dini-smooth corner of opening $\pi\beta$. Finally an analogous result is stated for the case of a holomorphic map f which maps a cusp into a cusp.

1. INTRODUCTION

There are several results about the rigidity of holomorphic self maps in the unit disc of $\mathbb{C}$ (see [7], [8] and [12]); these results state that, under suitable hypotheses, when the boundary expansion of a self map f agrees up to the certain order to the identity or to a specific rational map, then f is actually the identity or a completely determined rational map. We are interested to determine boundary conditions for a holomorphic map f from a certain domain D of $\mathbb{C}$ into $\mathbb{C}$ which forces f to be a constant map. A number of similar results, with various smoothness assumptions on the image of the boundary of D (see [3], [4], [5], [6], [9]) have been proved by assuming *infinite vanishing order of f at a boundary point $p \in \partial D$* (we recall that f has vanishing order $k \geq 0$ at p if, as z tends to p, the ratio $f(z)/|z-p|^N$ tends to 0 for all $N \leq k$; if as z tends to p, the

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ratio $f(z)/|z-p|^k$ tends to 0 for all k, then the vanishing order of f at p is infinite). In this paper, we shall study the rigidity of some holomorphic functions which map a point $p \in \partial D$ into the boundary of the image of f. This requirement is somehow deeply related to the geometry of the domain D and of the image $f(D)$ and it is assumed to avoid the spiraling case (see [3]) which occurs around the origin for a map such as

$$f(z) = \exp\left(\frac{z+1}{z-1}\right),$$

that vanishes to infinite order at the boundary point 1 of the unit disc but it is *not* constant (see [10]). We first consider the case of a domain D with smooth boundary at the point $p \in \partial D$ and we prove that if $f(D)$ is contained in the inner domain of a Dini-smooth curve and the vanishing order of the function f is at least 1, then f is constant. Afterwards, we consider holomorphic maps between domains whose boundaries have Dini-smooth corners; in this case, we prove that, to have f constant, the minimal vanishing order of f is strictly related to the openings of the angles at the corners; the special (and remaining) case concerning the rigidity of holomorphic maps between domains with outward cusps at the boundary is treated in the last section. The authors would like to thank Graziano Gentili for many helpful conversations and precious suggestions for this work.

2. PRELIMINARY RESULTS

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc of $\mathbb{C}$ and $H = \{w \in \mathbb{C} : Re w > 0\}$ be the right-half plane of $\mathbb{C}$; for $\tau \in \partial\Delta$, the map $\varphi_\tau(z) = \frac{\tau+z}{\tau-z}$ is a biholomorphism of Δ onto H with inverse $\varphi_\tau^{-1}(w) = \tau\frac{w-1}{w+1}$. If f is a holomorphic map from Δ to itself and if $\sigma, \tau \in \partial\Delta$, then the positive real number defined by

$$\beta_f(\sigma,\tau) = \sup_{z\in\Delta}\left\{\frac{|\tau - f(z)|^2}{1-|f(z)|^2}\Big/\frac{|\sigma - z|^2}{1-|z|^2}\right\},$$

is called the *boundary dilatation coefficient* of f with respect to σ and τ (see e.g. [1]). It turns out that, given $\sigma \in \partial\Delta$, then there is at most one point $\tau \in \partial\Delta$ such that $\beta_f(\sigma,\tau)$ is finite. Take $\tau \in \partial\Delta$ and $M > 0$. The *Stolz region* $K(\tau, M)$ of vertex τ and amplitude M is

$$K(\tau, M) = \left\{z \in \Delta : \frac{|\tau - z|}{1-|z|} < M\right\}.$$

Geometrically, $K(\tau, M)$ is a kind of angular sector of vertex τ and opening less than π. A function $f : \Delta \to \mathbb{C}$ has *non tangential limit* l at $\sigma \in \partial\Delta$, and we shall write

$$\operatorname*{K-lim}_{z\to\sigma} f(z) = l,$$

if $f(z) \to l$ as z tends to σ within $K(\sigma, M)$ for all $M > 1$. We recall the *Julia-Wolff-Carathéodory* Theorem (see [1]):

Theorem 2.1. *Let f be a holomorphic map from the disc Δ to itself and $\sigma, \tau \in \partial\Delta$. Then*

$$\operatorname*{K-lim}_{z \to \sigma} \frac{\tau - f(z)}{\sigma - z} = \tau\bar{\sigma}\beta_f(\sigma, \tau).$$

If $\beta_f(\sigma, \tau)$ is finite, then

$$\operatorname*{K-lim}_{z \to \sigma} f(z) = \tau \quad \textit{and} \quad \operatorname*{K-lim}_{z \to \sigma} f'(z) = \tau\bar{\sigma}\beta_f(\sigma, \tau).$$

Let C be a Jordan curve of $\mathbb{C}$, i.e. a plane, continuous, closed, simple curve, and let $\Omega \subset \mathbb{C}$ be the bounded component of $\mathbb{C} \setminus C$; such a domain will be called the *inner* domain of C. Let $\varphi : \Delta \to \Omega$ map Δ conformally onto the inner domain of C. The *Carathéodory Theorem* (see [11]) states that φ has a continuous and injective extension to $\overline{\Delta}$. The smoothness of C does not imply that φ' extends to $\overline{\Delta}$ too (see [11]); to guarantee the extension of φ' we need the notion of *Dini-smooth* curve.

Definition 2.2. *Let the function ϕ be uniformly continuous on the connected set $A \subset \mathbb{C}$. Its* modulus of continuity *is defined by*

$$\omega(\delta) \equiv \omega(\delta, \phi, A) = \sup_{\delta \geq 0}\{|\phi(z_1) - \phi(z_2)| : z_1, z_2 \in A, |z_1 - z_2| \leq \delta\}.$$

The function ϕ is called Dini-continuous *if, for any positive constant a,*

$$\int_0^a t^{-1}\omega(t)dt < +\infty.$$

If, for example, the function ϕ is *Hölder-continuous* with exponent λ $(0 < \lambda \leq 1)$ on A, i.e. if there exists $M > 0$ such that

$$|\phi(z_1) - \phi(z_2)| \leq M|z_1 - z_2|^\lambda$$

for all $z_1, z_2 \in A$, then ϕ is Dini-continuous (see [11] p. 46). We say that a curve is *Dini-smooth* if it has a parametrization $\gamma : [0, b] \to \mathbb{C}$ such that γ' is Dini-continuous and non zero on $[0, b]$. The following results can be found in [11].

Theorem 2.3. *Let φ map Δ conformally onto the inner domain of a Dini-smooth Jordan curve. Then φ' has a continuous extension to $\overline{\Delta}$ and*

$$\frac{\varphi(\zeta) - \varphi(z)}{\zeta - z} \to \varphi'(z) \neq 0 \quad \textit{for} \quad \zeta \to z, \quad \zeta, z \in \overline{\Delta}.$$

If φ maps Δ conformally onto the inner domain Ω of a Jordan curve, let $\zeta = e^{i\vartheta} \in \partial\Delta$. We say that Ω has a *corner* of opening $\pi\alpha$ $(0 \leq \alpha \leq 2)$ at $\varphi(\zeta) \in \partial\Omega$ if

$$\arg[\varphi(e^{it}) - \varphi(e^{i\vartheta})] \to \begin{cases} \beta & \text{as } t \to \vartheta^+, \\ \beta + \pi\alpha & \text{as } t \to \vartheta^-, \end{cases}$$

for some $\beta \in \mathbb{R}$. If $\alpha = 1$ we have a *tangent* of direction angle β; if $\alpha = 0$ we have an *outward pointing cusp*; if $\alpha = 2$ we have an *inward pointing cusp*.

Definition 2.4. *A (bounded) domain $D \subset \mathbb{C}$ has a* Dini-smooth corner *at $\sigma \in \partial D$ of opening $\pi\alpha$ $(0 \leq \alpha \leq 2)$ if*

(i) *D is simply connected with locally connected boundary;*

(ii) *there exists a biholomorphism $\varphi : \Delta \to D$, with $\varphi(\zeta) = \sigma$ for some $\zeta \in \partial\Delta$;*

(iii) *there exist closed arcs $A^{\pm} \subset \partial\Delta$ ending at $\zeta \in \partial\Delta$ and lying on opposite sides of ζ which are mapped by (the continuous extension of) φ onto Dini-smooth Jordan arcs $C^{\pm} \subset \partial D$ and form the angle $\pi\alpha$ at $\varphi(\zeta) = \sigma$.*

We say that a domain $D \subset \mathbb{C}$ has a *local* Dini-smooth corner at $\sigma \in \partial D$ if there exists a neighbourhood V of σ in $\mathbb{C}$ such that $V \cap D$ satisfies the previous conditions.

Remark 2.5. Definition 2.4 and the following results clearly hold if we assume the domain D unbounded with $\sigma \neq \infty$; in fact everything carries over to the case of domains in the Riemann sphere $\widehat{\mathbb{C}}$ if the spherical metric is used instead of the Euclidean metric (see [11]).

For a domain Ω, which is biholomorphic to Δ through a map φ and has a corner of opening $\pi\alpha$, the following theorem gives an estimate of the behaviour of $\varphi : \Delta \to \Omega$ in a neighbourhood of the vertex of such a corner. This estimate will play a key role in the sequel.

Theorem 2.6. *Let Ω have a Dini-smooth corner of opening $\pi\alpha$ $(0 < \alpha \leq 2)$ at $\varphi(\zeta) \in \partial\Omega$, where $\varphi : \Delta \to \Omega$ is the biholomorphism defining the corner and $\zeta \in \partial\Delta$. Then the function*

$$z \mapsto \frac{\varphi(\zeta) - \varphi(z)}{(\zeta - z)^{\alpha}}$$

is continuous, non zero and different from ∞ on $\overline{\Delta} \cap \{z \in \mathbb{C} : |z - \zeta| < \rho\}$ for some $\rho > 0$.

3. RIGIDITY IN SMOOTH DOMAINS

We shall start with this simple but crucial

Lemma 3.1. *Let $f : \Delta \to H$ be a holomorphic function and let $\sigma \in \partial\Delta$ be such that*

$$\text{(3.1)} \qquad \operatorname*{K\text{-}lim}_{z \to \sigma} \frac{f(z)}{z - \sigma} = 0.$$

Then $f \equiv 0$.

Proof. Let $\tau \in \partial\Delta$: we define the holomorphic map $h = -\varphi_{\tau}^{-1} \circ f$ which maps Δ into Δ and is such that, for any $z \in \Delta$,

$$\frac{\tau - h(z)}{\sigma - z} = 2\tau \frac{f(z)}{(\sigma - z)(f(z) + 1)}.$$

Passing to the non tangential limits as $z \to \sigma$ in both sides of the above equation, by (3.1) and by applying Theorem 2.1 to the map h, we obtain that

$$\beta_h(\sigma, \tau) = 0.$$

Hence the function h is identically equal to a constant and since, from (3.1) it follows that f has non tangential limit 0 as $z \to \sigma$, this constant is $-\varphi_\tau^{-1}(0) = \tau$; therefore $f \equiv 0$. $\square$

We state the same result for holomorphic self maps of the half plane H.

Corollary 3.2. *Let $f : H \to H$ be a holomorphic function and let $\sigma \in \partial H$ be such that*

$$\lim_{w \to \sigma} \frac{f(w)}{w - \sigma} = 0. \tag{3.2}$$

Then $f \equiv 0$.

Proof. Assume $\sigma = 0 \in \partial H$. Let $\tau \in \partial \Delta$ and $\varphi_\tau : \Delta \to H$ be a biholomorphism such that $\varphi_\tau(1) = 0$. We consider the holomorphic function $F = f \circ \varphi_\tau$ which maps Δ into H; we have

$$\text{K-}\lim_{z \to 1} \frac{F(z)}{z - 1} = \text{K-}\lim_{z \to 1} \frac{f(\varphi_\tau(z))}{z - 1}.$$

By (3.2) and since $\varphi_\tau'(1)$ is non zero and different from ∞, we obtain

$$\text{K-}\lim_{z \to 1} \frac{f(\varphi_\tau(z))}{\varphi_\tau(z)} \cdot \frac{\varphi_\tau(z)}{z - 1} = 0.$$

Therefore, by the previous lemma, we can conclude that $f \equiv 0$. $\square$

Remark 3.3. The above Lemma 3.1 can be stated in the following form: let $f, g : \Delta \to \mathbb{C}$ be holomorphic functions such that:

$$\operatorname{Re} f(z) \geq \operatorname{Re} g(z) \quad \forall z \in \Delta \tag{3.3}$$

$$\text{K-}\lim_{z \to \sigma} \frac{f(z) - g(z)}{\sigma - z} = 0 \tag{3.4}$$

for some $\sigma \in \partial \Delta$; then $f \equiv g$ in Δ. This version recall Theorem 2.4 in [12]; notice that condition (3.4) seems weaker than those required in [12], even though the idea of the proof of Lemma 3.1 is very similar to the one in [12] and essentially relies upon a consequence of Julia-Wolff-Carathéodory Theorem, namely that any holomorphic self map of the unit disc with zero non tangential derivative at its (boundary) Wolff point is constant. On the other hand, condition (3.1) can be regarded as a condition on the first term of the boundary expansion of the map f; sometimes it is also considered as an estimate of the vanishing order of a map (see [3], [4], [5], [6], [9]). From this point of view, the most general approach is in [3], where for a map f - to have finite vanishing order at a boundary point - it is only required that it has an image of the boundary on which the logarithm is not multiply defined (this geometric property is recalled in [3] as *non spiraling at the origin*). In our case, since $f(\Delta) \subset H$, and there exists $\sigma \in \partial\Delta$ such that

$$\text{K-}\lim_{z \to \sigma} f(z) = 0 \in \partial H,$$

then $\partial f(\Delta)$ is definitively non spiraling at the origin as defined in [3].

The hypotheses of Lemma 3.1 require $f(\Delta)$ to have a *support straight line* in $0 \in \partial f(\Delta)$, i.e. to lie entirely on one side of the closed half-plane determined by a straight line at 0 (not necessarily the tangent line). The next result shows that it is enough to require that $f(\Delta)$ has a "good" support curve passing through $0 \in \partial f(\Delta)$ to prove that (3.1) implies $f \equiv 0$.

Theorem 3.4. *Let $f : \Delta \to \mathbb{C}$ be a (bounded) holomorphic function. Suppose that there is a Dini-smooth Jordan curve C passing through $0 \in \partial f(\Delta)$ such that $f(\Delta)$ is contained in the inner domain of C. If $\sigma \in \partial\Delta$ is such that*

$$\text{K-}\lim_{z \to \sigma} \frac{f(z)}{z - \sigma} = 0, \tag{3.5}$$

then $f \equiv 0$.

Proof. Let Ω be the inner domain of C and assume that $f(\Delta) \subset \Omega$. By the Carathéodory Theorem any conformal map φ of Δ onto Ω is continuous on $\overline{\Delta}$: we may assume $\varphi(1) = 0$. We also know, by Theorem 2.3, that φ' extends continuously and is non zero on $\overline{\Delta}$. Let $\psi = \varphi^{-1}$; the map ψ' is bounded on $\overline{\Omega}$. We define the holomorphic map $F = \varphi_T \circ \psi \circ f$, where $\varphi_T : \Delta \to H$ is such that $\varphi_T(1) = 0$; then F maps Δ into H and by (3.5)

$$\text{K-}\lim_{z \to \sigma} \frac{F(z)}{z - \sigma} = \text{K-}\lim_{z \to \sigma} \frac{\varphi_T(\psi(f(z)))}{\psi(f(z)) - 1} \cdot \frac{\psi(f(z)) - 1}{f(z)} \cdot \frac{f(z)}{z - \sigma} = 0.$$

Therefore, by Lemma 3.1, we can conclude that $F \equiv 0$ and then $f \equiv 0$. $\square$

The previous result can be easily extended for any holomorphic map $f : D \to \mathbb{C}$, where D is a domain of $\mathbb{C}$ whose boundary admits at $\sigma \in \partial D$ a small disc Δ^* internally tangential to D and such that $f(\partial\Delta^*)$ is contained in the inner domain of a Dini-smooth Jordan curve. More generally we can prove the following

Proposition 3.5. *Let $D \subset \mathbb{C}$ be a domain and $\sigma \in \partial D$ have a neighbourhood U in $\mathbb{C}$ such that $\partial D \cap U$ is a Dini-smooth Jordan arc. Let $f : D \to H$ be a holomorphic map such that*

$$\lim_{w \to \sigma} \frac{f(w)}{w - \sigma} = 0. \tag{3.6}$$

Then $f \equiv 0$.

Proof. Let V be a simply connected neighbourhood of σ such that $V \cap \overline{D} = \widetilde{V}$ is the inner part of a Dini-smooth Jordan curve. Let $\varphi : \Delta \to \widetilde{V}$ be a Riemann map with $\varphi(1) = \sigma$ and consider $F = f \circ \varphi : \Delta \to H$. Since, by Theorem 2.3, the Riemann map φ can be continuously extended on $\overline{\Delta}$ in such a way that $\varphi'(1) \neq \infty$, then, by (3.7), we have

$$\operatorname*{K-lim}_{z \to 1} \frac{F(z)}{z-1} = \operatorname*{K-lim}_{z \to 1} \frac{f(\varphi(z))}{\varphi(z) - \varphi(1)} \cdot \frac{\varphi(z) - \varphi(1)}{z-1} = 0.$$

Therefore, by Lemma 3.1, we can conclude that $F \equiv 0$ and then $f \equiv 0$. $\square$

Notice that the vanishing order one is not always enough to have f constant if the boundary of the domain is not a smooth curve; the domain $D = \{z \in \Delta : \mathrm{Re}z - 1 \leq \mathrm{Im}z \leq -\mathrm{Re}z + 1\}$ has a corner of opening $\pi/2$ in $1 \in \partial D$ and the map $f(z) = (z-1)^2$ defined in D is such that

(i) $f(D) \subset H$;

(ii) $\displaystyle\lim_{z \to 1} \frac{f(z)}{z-1} = 0$;

(iii) $\displaystyle\lim_{z \to 1} \frac{f(z)}{(z-1)^2} \neq 0$.

4. RIGIDITY IN DOMAINS WITH CORNERS

The next result provides suitable hypotheses for an extension of Proposition 3.5 in the case of a holomorphic map f defined on a domain with a local Dini-smooth corner.

Theorem 4.1. *Let $D \subset \mathbb{C}$ be a domain with a local Dini-smooth corner at $\sigma \in \partial D$ of opening $\pi\alpha$ $(0 < \alpha \leq 2)$ and let $f : D \to H$ be a holomorphic function. If there exists $N \geq 1/\alpha$ such that*

$$\lim_{w \to \sigma} \frac{f(w)}{(w-\sigma)^N} = 0, \tag{4.1}$$

then $f \equiv 0$.

Proof. Let φ be the conformal map of Δ onto a suitable neighbourhood $V \cup D$ of σ as defined in Section 2; since φ is continuous on $\overline{\Delta}$, we can suppose $\varphi(1) = \sigma$. We consider the holomorphic map from Δ into H defined by $F = f \circ \varphi$ and we claim that

$$\operatorname*{K-lim}_{z \to 1} \frac{F(z)}{z-1} = 0.$$

Indeed, by (4.1) and by Theorem 2.6, we have

$$\operatorname*{K-lim}_{z\to 1}\left|\frac{F(z)}{z-1}\right| = \operatorname*{K-lim}_{z\to 1}\frac{|f(\varphi(z))|}{|\varphi(z)-\sigma|^N}\cdot\frac{|\varphi(z)-\sigma|^N}{|z-1|^{\alpha N}}\cdot|z-1|^{\alpha N-1} =$$
$$= \operatorname*{K-lim}_{z\to 1}\frac{|f(\varphi(z))|}{|\varphi(z)-\sigma|^N}\cdot\left[\frac{|\varphi(z)-\sigma|}{|z-1|^{\alpha}}\right]^N\cdot|z-1|^{\alpha N-1} = 0.$$

Again, by Lemma 3.1, we can conclude that $f\equiv 0$. □

Remark 4.2. When D is an angular region $A(\alpha)=\{w\in H : |\arg w| < \frac{\alpha}{2}\pi\}$, $(0<\alpha\le 2)$, then the conformal mapping φ of H onto $A(\alpha)$ is of the form

$$\varphi(z)=\frac{z^\alpha}{\alpha},$$

with $z\in H$ (see [2]). For such a domain the proof of Theorem 4.1 can be done by explicit and direct calculations; indeed, if $F=f\circ\varphi$ maps H into H, we find that, since $N\ge 1/\alpha$,

$$\lim_{z\to 0}\frac{F(z)}{z} = \lim_{z\to 0}\frac{f(\varphi(z))}{\varphi(z)^N}\cdot\frac{\varphi(z)^N}{z} =$$
$$= \lim_{z\to 0}\frac{f(\varphi(z))}{\varphi(z)^N}\cdot\frac{z^{\alpha N-1}}{\alpha^N} = 0.$$

Therefore, by Corollary 3.2, we have $f\equiv 0$. Condition (4.1) is sharp: the function $f(z)=z^2$ maps $A(1/2)$ onto H and its vanishing order is less than 2.

We are now interested in the case of holomorphic maps from Δ into the inner domain Ω of a Jordan curve which has a Dini-smooth corner at $0\in\partial\Omega$ of opening $\pi\alpha$ $(0<\alpha\le 2)$.

Theorem 4.3. *Let $f:\Delta\to\mathbb{C}$ be a (bounded) holomorphic function. Suppose that $f(\Delta)$ is contained in the inner domain Ω of a Jordan curve which has a Dini-smooth corner at $0\in\partial\Omega$ of opening $\pi\alpha$ $(0<\alpha\le 2)$. If there exist $N\ge\alpha$ and $\sigma\in\partial\Delta$ such that*

$$\operatorname*{K-lim}_{z\to\sigma}\frac{f(z)}{(z-\sigma)^N}=0, \tag{4.2}$$

then $f\equiv 0$.

Proof. We may assume that $\sigma=1\in\partial\Delta$. Let φ be the biholomorphism of Δ onto Ω defined in (2.4) of Section 2; the function φ has a continuous and injective extension to $\overline{\Delta}$ and, up to rotation, we can suppose $\varphi(1)=0$. The function $\psi=\varphi^{-1}$ is also continuous and injective on $\overline{\Omega}$. By Theorem 2.6, after the change of parameter $w=\varphi(z)$ and $z=\psi(w)$, we obtain that the function

$$w\mapsto\frac{w}{(\psi(w)-1)^\alpha} \tag{4.3}$$

is continuous and different from 0 and ∞ in a neighbourhood of $0\in\partial\Omega$. We define the holomorphic map $F = \varphi_T \circ \psi \circ f$, where $\varphi_T : \Delta \to H$ is such that $\varphi_T(1) = 0$. Then F maps Δ into H and, from (4.14) and (4.13), we have

$$\begin{aligned}\operatorname*{K-lim}_{z\to 1}\left|\frac{F(z)}{z-1}\right|^\alpha &= \operatorname*{K-lim}_{z\to 1}\left|\frac{1-\psi(f(z))}{1+\psi(f(z))}\right|^\alpha \cdot \frac{1}{|z-1|^\alpha} = \\ &= \operatorname*{K-lim}_{z\to 1}\frac{|1-\psi(f(z))|^\alpha}{|f(z)|}\cdot\frac{|f(z)|}{|z-1|^N}\cdot\frac{|z-1|^{N-\alpha}}{|1+\psi(f(z))|^\alpha} = 0.\end{aligned}$$

Hence, Lemma 3.1 implies $F \equiv 0$ and therefore $f \equiv 0$. $\square$

Remark 4.4. When Ω is an angular region $A(\alpha)$, as in remark (4.2), we can directly and explicitly prove Theorem 4.3; the function ψ, which maps $A(\alpha)$ onto H, is of the form

$$\psi(w) = cw^{\frac{1}{\alpha}},$$

with $c = \alpha^{\frac{1}{\alpha}}$ and $w \in A(\alpha)$. So $F = \psi \circ f$ maps Δ into H and, by (4.13), we have

$$\begin{aligned}\operatorname*{K-lim}_{z\to 1}\left|\frac{F(z)}{z-1}\right| &= \operatorname*{K-lim}_{z\to 1}\left|\frac{\psi(f(z))}{z-1}\right| = \operatorname*{K-lim}_{z\to 1}\left|\frac{cf(z)^{\frac{1}{\alpha}}}{z-1}\right| = \\ &= \operatorname*{K-lim}_{z\to 1}\frac{c|f(z)|^{\frac{1}{\alpha}}}{|z-1|^{\frac{N}{\alpha}}}\cdot|z-1|^{\frac{N}{\alpha}-1} = \\ &= \operatorname*{K-lim}_{z\to 1} c\left|\frac{f(z)}{(z-1)^N}\right|^{\frac{1}{\alpha}}\cdot|z-1|^{\frac{N}{\alpha}-1} = 0.\end{aligned}$$

Finally, the previous results immediately yield

Corollary 4.5. *Let $D \subset \mathbb{C}$ be a domain with a local Dini-smooth corner at $\sigma \in \partial D$ of opening $\pi\alpha$ $(0 < \alpha \leq 2)$ and let $f : D \to \mathbb{C}$ be a (bounded) holomorphic function. Suppose that $f(D)$ is contained in the inner domain Ω of a Jordan curve which has a Dini-smooth corner at $0 \in \partial\Omega$ of opening $\pi\beta$ $(0 < \beta \leq 2)$. If there exists $N \geq \beta/\alpha$ such that*

$$\lim_{w\to\sigma}\frac{f(w)}{(w-\sigma)^N} = 0, \tag{4.4}$$

then $f \equiv 0$.

5. RIGIDITY IN DOMAINS WITH CUSPS

Theorems 4.1 and 4.3 also hold if $\alpha = 2$, that is when there is an inward pointing cusp at the boundary of the domain: in this case the vanishing order required for f to be constant is at least $k = 1/2$ in Theorem 4.1 and $k = 2$ in Theorem 4.3. But in the case of an outward pointing cusp (that is when $\alpha = 0$) the analogous results do not hold and the next two examples will illustrate this fact. Before stating the first example, we introduce the following result (see [11], p. 266).

Proposition 5.1. *Let $w \in \mathbb{C}$, $c > 0$ and let*

$$G_a = \{w + \varrho e^{i\vartheta} \; : \; 0 < \varrho < c, \; \vartheta_-(\varrho) < \vartheta < \vartheta_+(\varrho)\}$$

where $\vartheta_\pm$ are locally absolutely continuous functions such that

$$(5.1) \qquad \vartheta_+(\varrho) - \vartheta_-(\varrho) \sim a\varrho \text{ as } \varrho \to 0, \quad \int_0^c \vartheta'_\pm(\varrho)\, d\varrho < \infty$$

with $a > 0$. Then any biholomorphism $\varphi : \Delta \to G_a$ with (finite) non tangential limit $\varphi(\sigma)$ at $\sigma \in \partial\Delta$ is such that

$$(5.2) \qquad |\varphi(z) - \varphi(\sigma)| = \frac{\pi/a + o(1)}{-\log|z - \sigma|},$$

as $z \to \sigma$ in Δ.

The first of conditions (5.1) geometrically means that the domain G_a has an outward pointing cusp whose "opening" is somehow measured by the parameter a.

Example 1. Suppose that Theorem 4.1 still holds for $\alpha = 0$ in the following sense: if $D \subset \mathbb{C}$ is a domain with a local Dini-smooth corner at $\sigma \in \partial D$ of opening 0 and $f : D \to H$ is a holomorphic function such that

$$\lim_{w \to \sigma} \frac{f(w)}{(w - \sigma)^N} = 0 \quad \text{for any} \quad N > 0,$$

then $f \equiv 0$ in D. Take $\sigma = 0$ and $D = G_a$ defined in Proposition 5.1 and let $\varphi : \Delta \to G_a$ be the Riemann mapping with $\varphi(1) = 0$. By the result stated above, we know that

$$(5.3) \qquad |\varphi(z)| = \frac{c + o(1)}{-\log|z - 1|},$$

as $z \to 1$ in Δ, with $c > 0$. Let $\psi = \varphi^{-1}$. After changing parameters in (5.3) we obtain

$$|\psi(w) - 1| = \exp\left\{-\frac{c + o(1)}{|w|}\right\},$$

as $w \to 0$ in G_a. We define the biholomorphic function $f = \varphi_T \circ \psi$, where $\varphi_T : \Delta \to H$ is such that $\varphi_T(1) = 0$: then f maps G_a into H and vanishes to infinite order at 0, since

$$\lim_{w \to 0}\left|\frac{f(w)}{w^N}\right| = \lim_{w \to 0} \frac{|1 - \psi(w)|}{|1 + \psi(w)||w|^N} = \lim_{w \to 0} \frac{1}{\exp\left\{\frac{c + o(1)}{|w|}\right\}|w|^N} \cdot \frac{1}{|1 + \psi(w)|} = 0,$$

for all $N > 0$. But f does not constantly vanish in Δ.

Example 2. Let D be the inner domain of a Jordan curve which has a Dini-smooth corner at $0 \in \partial D$ of opening 0. Since D is bounded and simply connected there exists a biholomorphism f of Δ onto D. Now, by *Carathéodory Theorem*, since ∂D is a Jordan curve, the function f is a homeomorphism of $\partial\Delta$ on ∂D; thus there exists $\sigma \in \partial\Delta$ such that $f(\sigma) = 0 \in \partial D$. Notice that f does not vanish on Δ, even though

$$\operatorname*{K-lim}_{z\to\sigma} f(z) = 0.$$

(This would be the case if Theorem 4.3 were true for $\alpha = 0$).

We are now ready for the following

Theorem 5.2. *Let $D \subset \mathbb{C}$ be a domain with a local Dini-smooth corner at $0 \in \partial D$ of opening $\pi\alpha$ $(0 < \alpha \leq 2)$ and let $G_b \subset \mathbb{C}$ be (following the notation of Proposition 5.1) a domain with an outward pointing cusp at 0. Let $f : D \to G_b$ be a holomorphic function. If there exists $N \geq 1/\alpha$ such that*

$$\lim_{w\to 0}\left|\frac{f(w)}{w^N}\right| = L < +\infty, \tag{5.4}$$

then $f \equiv 0$.

Proof. We can assume that there exist simply connected neighbourhoods (respectively in D and G_b) of the corner and of the cusp; call them V_α and W_b. Then, if $\varphi : \Delta \to V_\alpha$ and $\psi_b : W_b \to \Delta$ are two Riemann mappings such that $\varphi(1) = 0$ and $\psi_b(0) = 1$, then the map $F = \varphi_\tau \circ \psi_b \circ f \circ \varphi$ (where φ_τ is the Cayley transform from Δ onto H such that $\varphi_\tau(1) = 0$) is a holomorphic map from Δ into H such that

$$\left|\frac{F(z)}{z-1}\right| = \frac{|1-\psi_b(f(\varphi(z)))|}{|1+\psi_b(f(\varphi(z)))|}\cdot\frac{1}{|z-1|} =$$

$$= \exp\left\{\frac{-\pi/b + o(1)}{|f(\varphi(z))|}\right\}\cdot\frac{1}{|f(\varphi(z))|}\cdot\frac{|f(\varphi(z))|}{|z-1|}\cdot\frac{1}{|1+\psi_b(f(\varphi(z)))|} =$$

$$= \exp\left\{\frac{-\pi/b + o(1)}{|f(\varphi(z))|}\right\}\cdot\frac{1}{|f(\varphi(z))|}\cdot\frac{|f(\varphi(z))|}{|\varphi(z)|^N}\cdot\frac{|\varphi(z)|^N}{|z-1|^{N\alpha}}\cdot\frac{|z-1|^{N\alpha-1}}{|1+\psi_b(f(\varphi(z)))|}.$$

Now, as z tends to 1 non tangentially in Δ, the first two factors tend to 0, whereas the third and fourth factors are finite (because of condition (5.4) and Theorem 2.6) and the last factors converge if $N \geq 1/\alpha$; therefore, by Lemma 3.1, we are done. $\square$

Remark 5.3. Notice that, with the same notation of the previous theorem, there exists a non constant holomorphic function $f : G_b \to D$ such that

$$\lim_{w \to 0} \frac{f(w)}{w^N} = 0 \quad \text{for any} \quad N \geq 0;$$

in fact it suffices to consider the map $f = \varphi \circ \psi_b$ and observe that

$$\left|\frac{f(w)}{w^N}\right| = \frac{|\varphi(\psi_b(w))|}{|\psi_b(w) - 1|^\alpha} \cdot \exp\left\{\frac{-\pi/b + o(1)}{|w|}\right\}^\alpha \cdot \frac{1}{w^N}$$

is infinitesimal as $w \to 0$ in G_b for any $N \geq 0$.

Finally, the remaining case is examined in

Theorem 5.4. *Let G_a and G_b be (following the notation of Proposition 5.1) two domains of $\mathbb{C}$ with outward pointing cusps at 0. If $f : G_a \to G_b$ is a holomorphic function such that*

$$\lim_{w \to 0} \left|\frac{f(w)}{w}\right| = L < a/b, \tag{5.5}$$

then $f \equiv 0$.

Proof. We can assume that there exist simply connected neighbourhoods (respectively in G_a and G_b) of the cusps; call them W_a and W_b. Then, if $\varphi_a : \Delta \to W_a$ and $\psi_b : W_b \to \Delta$ are two Riemann mappings such that $\varphi_a(1) = 0$ and $\psi_b(0) = 1$, then the map $F = \varphi_\tau \circ \psi_b \circ f \circ \varphi_a$ (where φ_τ is the Cayley transform from Δ onto H such that $\varphi_\tau(1) = 0$) is a holomorphic map from Δ into H such that

$$\left|\frac{F(z)}{z-1}\right| = \frac{|1 - \psi_b(f(\varphi(z)))|}{|1 + \psi_b(f(\varphi(z)))|} \cdot \frac{1}{|z-1|} =$$

$$= \exp\left\{\frac{-\pi/b + o(1)}{|f(\varphi_a(z))|}\right\} \cdot \exp\left\{\log|z-1|^{-1}\right\} \cdot \frac{1}{|1 + \psi_b(f(\varphi_a(z)))|} =$$

$$= \exp\left\{\frac{-\pi/b + o(1)}{|f(\varphi_a(z))|} - \log|z-1|\right\} \cdot \frac{1}{|1 + \psi_b(f(\varphi_a(z)))|}.$$

Now, since, by (5.2),

$$-\log|z-1| = \frac{\pi/a + o(1)}{|\varphi_a(z)|},$$

we know that

$$\left|\frac{F(z)}{z-1}\right| = \exp\left\{\frac{-\pi/b + o(1)}{|f(\varphi_a(z))|} + \frac{\pi/a + o(1)}{|\varphi_a(z)|}\right\} \cdot \frac{1}{|1 + \psi_b(f(\varphi_a(z)))|} =$$

$$= \exp\left\{\frac{1}{|\varphi_a(z)|}\left[\frac{-\pi/b + o(1)}{\frac{|f(\varphi_a(z))|}{|\varphi_a(z)|}} + \pi/a + o(1)\right]\right\} \cdot \frac{1}{|1 + \psi_b(f(\varphi_a(z)))|};$$

therefore,

$$\operatorname*{K\text{-}lim}_{z \to 1} \left| \frac{F(z)}{z-1} \right| = 0$$

and so, by Lemma 3.1, we are done. $\square$

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(Received May 2002)