Opinion Dynamics in an Open Community

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Abstract. We here discuss the process of opinion formation in an open community where agents are made to interact and consequently update their beliefs. New actors (birth) are assumed to replace individuals that abandon the community (deaths). This dynamics is simulated in the framework of a simplified model that accounts for mutual affinity between agents. A rich phenomenology is presented and discussed with reference to the original (closed group) setting. Numerical findings are supported by analytical calculations.

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1 Introduction

Opinion dynamics modeling represents a challenging field where ideas from statistical physics and non-linear science can be possibly applied to understand the emergence of collective behaviors, like consensus or polarization in social groups. Several toy models have been proposed in the past to reproduce the key elements that supposedly drive the process of opinion making [1] . This large production has not been always accompanied by an adequate benchmarking effort to the relevant psychological literature, and doubts are consequently being cast on the interpretative ability of the proposed mathematical formulations. Despite these intrinsic limitations, several models

of interacting agents display a rich and intriguing dynamics which deserves to be fully unraveled.

Opinion dynamics models can be classified in two large groups. On the one hand, opinions are represented as discrete (spin–like) variables where the system behaves similarly to spin glasses models [2]. On the other, each individual bears a continuous opinion which span a pre–assigned range [3]. In both approaches, a closed system is generally assumed, meaning that the same pool of actors is made to interact during the evolution. This can be interpreted by assuming that the inherent dynamical timescales (e.g. opinion convergence time) are much faster than those associated to the processes (e.g. migration, birth/death) responsible for a modification of the group composition, these latter effects having being therefore so far neglected. Such an implicit assumption is certainly correct when the debate is bound to a small community of individuals, thus making it possible to eventually achieve a rapid convergence towards the final configuration. Conversely, it might prove inaccurate when applied to a large ensemble of interacting agents, as the process becomes considerably slower and external perturbations need to be accounted for. Given the above, it is therefore of interest to elucidate the open system setting, where the population is periodically renewed.

To this end, we refer to the model presented in [4], where the role of affinity among individuals is introduced as an additional ingredient. This novel quantity measures the degree of inter–personal intimacy and sharing, an effect of paramount importance in real social system [5] .

Indeed, the outcome of an hypothetic binary interaction relies on the difference of opinions, previously postulated, but also on the quality of the mutual relationships. The affinity is dynamically coupled to the opinion, and, in this respect, it introduces a memory bias into the system: affinity between agents increases when their opinion tends to converge.

The aforementioned model is here modified to accommodate for a death/born like process. In this formulation, M agents are randomly eliminated from the system, every T time steps. When an agent exits from the community (virtually, dies), he is immediately replaced by a new element, whose opinion and affinity with respect to the group are randomly assigned. As we shall see, the perturbation here prescribed alters dramatically the behavior of the system, with reference to the ideal close–system configuration. To understand such modifications via combined numerical and analytical tools, constitutes the object of the investigations here reported.

The paper is structured as follows. We first introduce the model, then we present the obtained analytical and numerical results and, finally, we sum up and draw our conclusions.

2 The model

In the following, we will shortly review the model previously introduced in [4] and present the additional features that are here under inspection. The interested reader can thus refer to the original paper [4] for a more detailed account on the model characteristics.

Consider a population of N agents and assume that at time t they bear a scalar opinion $O_i^t \in [0, 1]$. We also introduce the $N \times N$ time dependent matrix α^t, whose elements α_{ij}^t belong to the interval $[0, 1]$. The quantities α_{ij}^t specify the affinity of individual i vs. j, at time t: Larger values of α_{ij}^t are associated to more trustable relationships.

Both the affinity matrix and the agents opinions are randomly initialized time $t = 0$. At each time step t, two agents, say i and j, are selected according to the following extraction rule: first the agent i is randomly selected, with a uniform probability. Then, the agent j which is closer to i in term of the social metric D_{ij}^η is selected for interaction. The quantity D_{ij}^η results from the linear superposition of the so–called social distance, d_{ij}, and a stochastic contribution η_j, namely:

$$D_{ij}^\eta = d_{ij}^t + \eta_j(0, \sigma). \qquad (1)$$

Here $\eta_j(0, \sigma)$ represents a normally distributed noise, with mean zero and variance σ, the latter being named social temperature. The social distance is instead defined as:

$$d_{ij}^t = \Delta O_{ij}^t (1 - \alpha_{ij}^t) \qquad j = 1, ..., N \qquad j \neq i, \quad (2)$$

with $\Delta O_{ij}^t = |O_i^t - O_j^t|$.

The smaller the value of d_{ij}^t the closer the agent j to i, both in term of affinity and opinion. The additive noise $\eta_j(0, \sigma)$ acts therefore on a fictitious 1D manifold, which is introduced to define the pseudo–particle (agent) interaction on the basis of a nearest neighbors selection mechanism and, in this respect, set the degree of mixing in the community.

When the two agents i and j are selected on the basis of the recipe prescribed above, they interact and update their characteristics according to the following scheme [1]:

$$\begin{cases} O_i^{t+1} & = O_i^t - \frac{1}{2}\Delta O_{ij}^t \Gamma_1\left(\alpha_{ij}^t\right) \\ \alpha_{ij}^{t+1} & = \alpha_{ij}^t + \alpha_{ij}^t[1 - \alpha_{ij}^t]\Gamma_2\left(\Delta O_{ij}\right), \end{cases} \qquad (3)$$

where the functions Γ_1 and Γ_2 respectively read:

$$\Gamma_1\left(\alpha_{ij}^t\right) = \frac{1}{2}\left[\Theta(\alpha_{ij}^t - \alpha_c) + 1\right] \qquad (4)$$

and

$$\Gamma_2\left(\Delta O_{ij}\right) = -\Theta(|\Delta O_{ij}^t| - \Delta O_c) \qquad (5)$$

and the symbol $\Theta(\cdot)$ stands for the Heaviside step–function [2]. More specifically, Γ_1 is 0 or 1 while Γ_2 is -1 or 1, depending on the value of their respective arguments. In the above expressions α_c and ΔO_c are constant parameters. Notice that, for $\alpha_c \to 0$, the opinion and affinity are formally decoupled in (3), and the former evolves following the Deffuant et al. scheme [3] with $\mu = 0.5$.

In [4], a complete analysis of the qualitative behavior of the model as a function of the involved parameters is reported. Asymptotic clusters of opinion are formed, each agglomeration being different in size and centered around distinct opinion values. Individuals sharing the same be-

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[1] The evolution of the quantities $O_j(t)$ and $\alpha_{ij}(t)$ is straightforwardly obtained by switching the labels i and j in the equations.

[2] In [4], the switchers Γ_1 and Γ_2 are smooth functions constructed from the hyperbolic tangent. We shall here limit the discussion to considering the Heaviside approximation, which is recovered by formally sending $\beta_{1,2}$ to infinity in Eqs. (3) and (4) of [4].
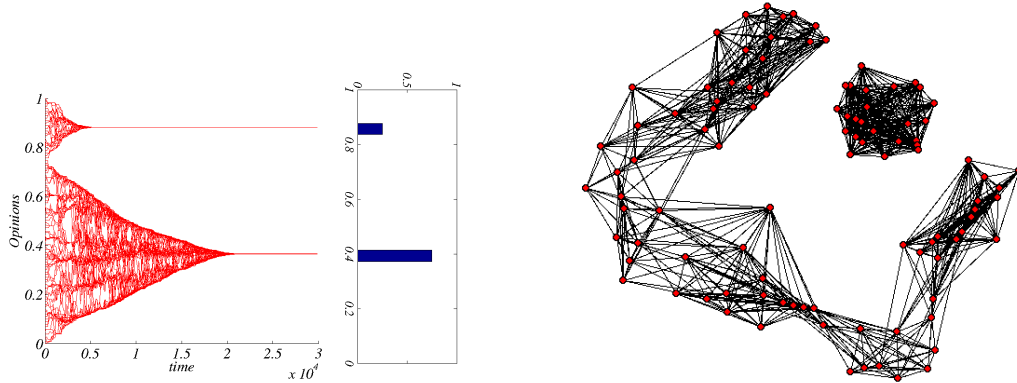
Fig. 1. Left panel: Typical evolution of the opinion versus time, i.e. number of iterations and the asymptotic distribution of agents' opinion in a closed community. Right panel: Final affinity matrix, represented by the underlying network: node i is linked to node j if $\alpha_{ij} > \bar{\alpha} = 0.8$. Here $\sigma = 4 \cdot 10^{-4}$, $\Delta O_c = 0.5$, $\alpha_c = 0.5$ and $\rho = 0$, i.e. no agent can leave the group. Initial opinion are (random) uniformly distributed within the interval $[0, 1]$, while α_{ij}^0 is initialized with uniform (random) values between 0 an 0.5.

lieves are also characterized by a large affinity scores, as it is exemplified in Figure 1.

More quantitatively, the system is shown to undergo a continuous phase transition: above a critical value of the control parameter $(\sigma \alpha_c)^{-1/2}$ the system fragments into several opinion clusters, otherwise convergence to a single group is numerically shown to occur. The interested reader can refer to [4] for a speculative interpretation of this finding in term of its psychological implications. We shall here simply notice that a significant degree of mixing (large social temperature σ) brings the system towards the single–cluster final configuration.

Starting from this setting, we introduce the birth/death process, which in turn amounts to place the system in contact with an external reservoir. The perturbation here hypothesized is periodic and leaves the total number of agent unchanged. Every T time steps (i.e. encounter events) M agents, randomly selected, are forced to abandon the system (death). Every removed individual is instantaneously replaced by a new element, whose initial opinion and affinity are randomly fished, with uniform probability, from the respective intervals $[0, 1]$ and $[0, \alpha_{max}]$. Further, we introduce $\rho = \frac{M}{T}$ to characterize the departure density, a crucial quantity that will play the role of the control parameter in our subsequent developments. As a final remark, it should be emphasized that no aging mechanisms are introduced: agents are mature enough to experience peer to peer encounters from the time they enter the system.

3 Results

Numerical simulations are performed and the evolution of the system monitored. Qualitatively, the system shows the

typical critical behavior and asymptotic clusters of opinion, as observed in its original formulation [4]. However, peculiar distinctions are found, some of those being addressed in the following discussion.

First, an apparently smooth transition is also observed within this novel formulation, which divides the mono– and multi–clustered phases. Interestingly, the transition point is now sensitive to the departure density ρ. To further elucidate this point, we draw in Figure 2 the average number of observed clusters versus a rescaled temperature. A clear transition towards an ordered (single-clustered) phase is observed, as the temperature increases. The parameter σ_c in Figure 2 plays the role of an effective temperature, and it is numerically adjusted to make distinct curves collapse onto the same profile, which hence holds irrespectively of the value of ρ. The inset of Figure 2 shows that there is a linear correlation between σ_c and ρ. The larger the departure density ρ, the larger the effective temperature σ_c. In other words, when ρ is made to increase (i.e. the system is experiencing the effect of a more pronounced external perturbation), one needs to augment the degree of mixing, here controlled by the social temperature σ, if a convergence to the final mono–cluster is sought. The death/birth process here modeled is in fact acting against the thermal contribution, which brings into contact otherwise socially distant individuals. While this latter effect enhances the chances of convergence, the former favors the opposite tendency to spread.

To further elucidate the role of the external perturbation, we shall refer to the dynamical regime where the
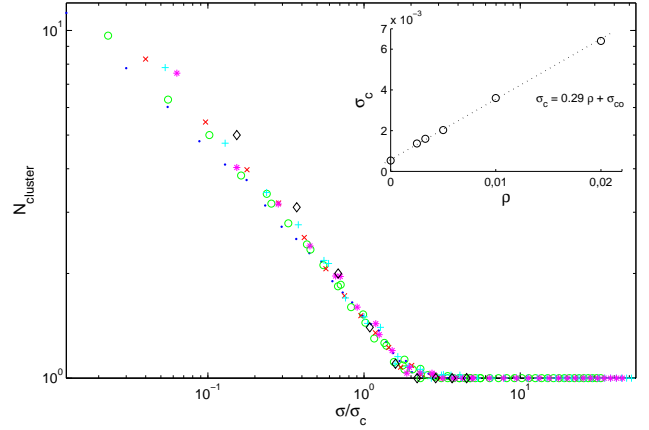

Fig. 2. Main plot: Average number of clusters as function of the rescaled quantity $\frac{\sigma}{\sigma_c}$ for different values of the density ρ. Diamonds, stars, crosses, x-marks, circles and solid diamonds represent, respectively, simulations run at $\rho = 0$, $\rho = 0.0025$, $\rho = 0.0033$, $\rho = 0.005$, $\rho = 0.01$ and $\rho = 0.05$. In all simulations, here and after reported, unless otherwise specified, $\alpha_c = \Delta O_c = 0.5$; O_i^0 and α_{ij}^0 are random variables uniformly distributed in the intervals $[0, 1]$ and $[0, \alpha_{max}]$ – being $\alpha_{max} = 0.5$ – respectively. Inset : σ_c as a function of ρ. The open circles represent the values calculated from the transitions shown in the main plot. The dotted lines represent the best linear fit : $\sigma_c = 0.3\rho + \sigma_{co}$, with $\sigma_{co} = 5.5 \cdot 10^{-4}$. Notice that $\sigma_c = \sigma_{co}$ is eventually recovered in the closed-system setting, which in turn corresponds to $\rho = 0$.

agents converge to a single cluster. When ρ is set to zero, the final shared opinion is 0.5 to which all agents eventually agree, see Figure 3a. In other words, the final distribution is a Dirac delta, with the peak positioned in $O = 0.5$. Conversely, for $\rho > 0$, the final distribution of opinions presents a clear spreading around the most probable value, still found to be 0.5. This scenario is clearly depicted in Figure 3b.
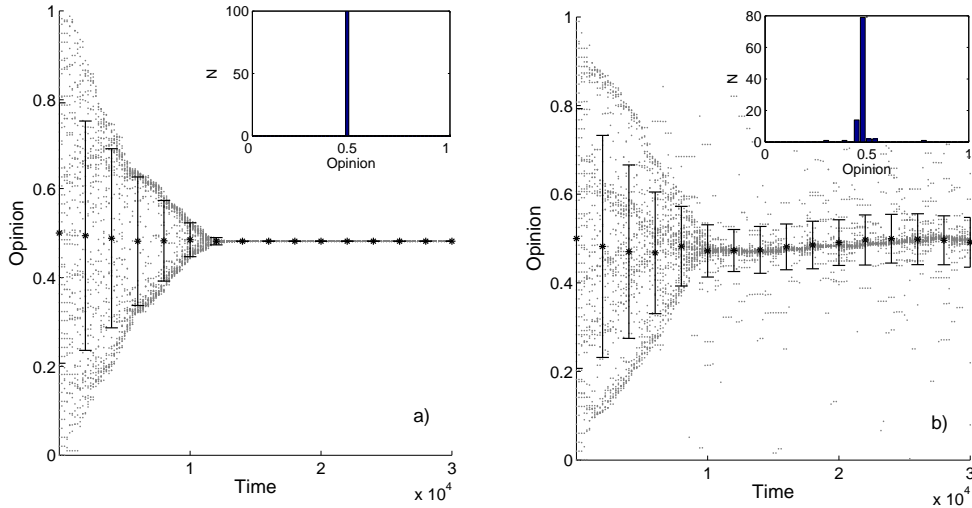
Fig. 3. Opinion evolution versus time, the latter being quantified through the number of iterations. The black stars represent the mean opinion and the error bar the standard deviation of the opinion distribution. In the insets, the histogram of asymptotic distribution of agents' opinion. In panel a) the system is closed (i.e. $\rho = 0$), in b) $\rho = 5 \cdot 10^{-3}$.

The associated standard deviation v is deduced, from a series of simulations, and shown to depend on the selected value of ρ. The result of the analysis is reported in Figure 4, where the calculated value of v (symbols) is plotted versus the departure density amount ρ.

For small values of the control parameter ρ, the standard deviation v of the cluster scales proportionally to $\sqrt{\rho}$, Figure 3. Numerics indicates that the proportionality coefficient gets smaller, as α_c grows. In the opposite limit, namely for large values of the density ρ, the standard deviation v rapidly saturates to a asymptotic value, v_c. The latter is universal, meaning that it neither scales with α_c, nor it does with the social temperature σ. Our best numerical estimates returns $v_c = 0.28$.

The solid lines in Figure 5 represent the function :

which straightforwardly follows from a simple analytical argument, developed hereafter. In the above expression T_c, stands for the time of convergence of the opinion cluster. In [6], working within the Deffuant's scheme [3], i.e. in the closed community case without affinity, authors showed that the time needed to form a coherent assembly from a sequence of binary encounters scales as $N \log N$, where N refers to the population size. In [4], it was also proven that the affinity slows down the convergence rate and a correction for T_c that accounts for the role of α_c was worked out. More precisely, it was found that :

$$v^2 = \frac{M}{12N[1 - \frac{N-M}{N}(\frac{T_c - T}{T_c})^2]}, \qquad (6)$$

$$T_c(\alpha_c) \approx \frac{N \log(N)}{2 \log(2)(\alpha_* - \alpha_c)}. \qquad (7)$$

approximated form:

$$v = \sqrt{\frac{T_c(\alpha_c)}{24N}}\sqrt{\frac{M}{T}} = \gamma_t\sqrt{\rho}, \qquad (8)$$

which presents the same dependence of v versus $\sqrt{\rho}$, as observed in the numerical experiments. Moreover, the coefficient γ_t is predicted to decay when increasing the cutoff in affinity α_c, just as expected from the numerics. For $\rho \gg 1$, Eq. (6) implies:

$$v = \sqrt{\frac{1}{12}} \approx 0.28, \qquad (9)$$

in excellent agreement with our numerical findings.
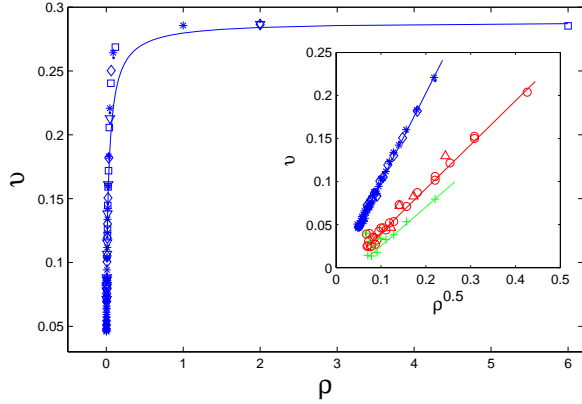
Fig. 4. Main: The standard deviation of the final mono–cluster v as a function of the departure density ρ. Each point results from averaging out 100 independent runs. Symbols refer to numerical simulation performed with different values of α_c, M and σ (stars: $\alpha_c = 0.5$, $M = 1$, $\sigma = 0.07$, diamonds: $\alpha_c = 0.5$, $M = 2$, $\sigma = 0.07$, squares: $\alpha_c = 0.5$, $M = 6$, $\sigma = 0.07$, dots: $\alpha_c = 0.5$, $M = 2$, $\sigma = 0.25$, down – triangles: $\alpha_c = 0.5$, $M = 2$, $\sigma = 1$, open circles: $\alpha_c = 0.3$, $M = 2$, $\sigma = 0.28$, up – triangles: $\alpha_c = 0.3$, $M = 6$, $\sigma = 0.28$ and crosses: $\alpha_c = 0$, $M = 2$, $\sigma = 0.28$. The solids line refers to the theoretical prediction (6). Inset: v versus $\sqrt{\rho}$, for $\rho \in [0, 0.5]$. This zoomed view confirms the correctness of the scaling dependence derived in (8).

where $\alpha_* \simeq 0.64$ [3]. Before turning to discuss the analytical derivation of Eq. (6), we wish to test its predictive adequacy with reference to the two limiting cases outlined above. Indeed, for $\rho \ll 1$, Eq. (6) can be cast in the

To derive Eq. (6) let us suppose that at time t the death/birth process takes place and the system experience an injection of new individuals. Label with v_t the standard deviation of the agents opinion distribution $f^t(O)$, at time t. It is reasonable to assume that $f^t(O)$ is centered around $1/2$. After T interactions between agents, when the next perturbation will occur (M agents are randomly removed from the community and replaced by M new actors with random opinion and affinity scores) the distribution has been already modified, because of the underlying dynamical mechanism specified through Eqs. (3). More concretely, the opinions slightly converge around the peak value $1/2$, an effect that certainly translates into a reduction of the associated standard deviation. To provide a quantitative estimate of such phenomenon, we recall that in the relevant (O, t) plan, the convergence process fills an ideal *triangular* pattern, whose hight measures T_c. This topological observation enables us to put forward the following linear ansatz:

$$v_{t+T}^{conv} = \frac{T_c - T}{T}v_t, \qquad (10)$$

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[3] α_* is an effective value deduced via numerical fit, as discussed in [4]. A fully analytical derivation of T_c will be presented in [7].

where v_{t+T}^{conv} labels the standard deviation just before the insertion of the next pool of incoming agents. Recalling that the newly inserted elements are uniformly distributed, and labeling with $G(\cdot, \cdot)$ the opinion distribution (the two entries referring respectively to mean and the standard deviation), the updated variance is:

$$
\begin{aligned}
v_{t+T}^2 &= \frac{M}{N} \int_0^1 \left(O - \frac{1}{2}\right)^2 dO \\
&+ \frac{N-M}{N} \int_0^1 G\left(\frac{1}{2}, v_{t+T}^{conv}\right) \left(O - \frac{1}{2}\right)^2 dO \\
&= \sqrt{\frac{M}{12N} + \frac{N-M}{N} \left(\frac{T_c - T}{T}\right)^2 v_t^2}.
\end{aligned}
$$

The asymptotic stationary solution correspond to $v_{t+T} = v_t$, a condition that immediately leads to Eq. (6) when plugged into (11). The above analysis also suggests that the final fate of the system is not affected by the time when the perturbation is first applied, t_{in}. This conclusion is also confirmed by direct numerical inspection: The asymptotic value of the standard deviation v does not depend on t_{in}, but solely on ρ. Even in the extreme condition, when the death/birth perturbation is switched on after the agents have already collapsed to the mean opinion 0.5, one observes that, after a transient, the cluster spreads and the measured value of v agrees with the theoretical prediction (6).

Aiming at further characterizing the system dynamics, we also studied the case where, initially, agents share the same belief O_0. The initial distribution of opinions is therefore a Dirac delta $f^0(O) = \delta(O - O_o)$. Such condition is a stationary solution, for any given O_o when the death/birth process is inactivated. Conversely, when the death/birth applies, the system evolves toward a final state, being characterized by a single cluster, centered in $O = 0.5$ and with standard deviation given by Eq. 6. It is also observed that the time needed by the system to complete the transition T_{conv} depends on the value of ρ and the critical affinity α_c, see Figure 5. A simple theoretical argument enables us to quantitatively explain these findings. The initial distribution of opinions is modified after the first death/birth event as:

$$
f^1(O) = \frac{M}{N} + \frac{N-M}{N} \delta(O - O_o). \tag{11}
$$

The first term refers to the freshly injected actors, while the second stands for the remaining Delta-distributed individuals. Hence, the mean opinion value reads:

$$
\bar{O}_1 = \int_0^1 f^1(O) O \, dO = \frac{M}{2N} + \frac{N-M}{N} O_o. \tag{12}
$$

We can suppose that between the occurrence of two consecutive perturbations (separated by T iterations), the group average opinion does not significantly change. Notice that the probability of interaction of a newborn agent with another belonging to the main group is in fact proportional to M/N. Moreover several consecutive encounters of this type are necessary to induce a macroscopic change of the averaged opinion. Under this hypothesis the next death/birth event makes the average opinion change as:

$$
\bar{O}_2 = \frac{M}{2N} + \frac{N-M}{N} \bar{O}_1. \tag{13}
$$

After n death/birth iterations, the opinion mean value reads :

$$
\bar{O}_n = \frac{M}{2N} + \frac{N-M}{N} \bar{O}_{n-1}. \tag{14}
$$

From Eq. (14) one easily gets that the asymptotic equilibrium is reached for $\bar{O}_\infty = 0.5$, as found in our numerical

experiments; in fact the following relation is straightfor-wardly obtained:

$$\bar{O}_n = \left(\frac{N-M}{N}\right)^n O_o + \sum_{l=0}^{n-1} \frac{M}{2N}\left(\frac{N-M}{N}\right)^l, \quad (15)$$

being O_o the initial common believe. By setting $\alpha = \frac{N-M}{N}$ and $\beta = \frac{M}{2N}$, the solution of Eq. (15) reads:

$$\bar{O}_n = \beta\frac{1-\alpha^n}{1-\alpha} + \alpha^n O_o, \quad (16)$$

whose asymptotic solution is given by $\bar{O}_n \to \bar{O}_\infty = 0.5$.

Expression (16) reproduces quite well the dynamics of the cluster mean, as seen in the simulations. The adequacy of (16) is in fact clearly demonstrated in Figure 5a. Let us define the convergence time T_{conv} as the number of iterations needed to bring the average opinion ϵ close to its asymptotic value $1/2$. Solving Eq. (16) for n and recalling that $T_{conv} = nT$ yield:

$$T_{conv} = T\log_\alpha\left[\frac{1}{|O_o - \frac{1}{2}|}(\epsilon - \frac{\beta}{1-\alpha})\right], \quad (17)$$

The above estimate is in excellent agreement with the numerical results reported in Figure 5b.

4 Conclusions

In this paper we have discussed the process of opinion making in an open group of interacting subjects. The model postulates the coupled dynamical evolution of both individuals' opinion and mutual affinity, according to the rules formulated in [4]. At variance with respect to the toy—model [4], the system is now open to contact with an external reservoir of potentially interacting candidates. Every T iterations the M agents are instantaneously replaced by newborn actors, whose opinion and affinity scores
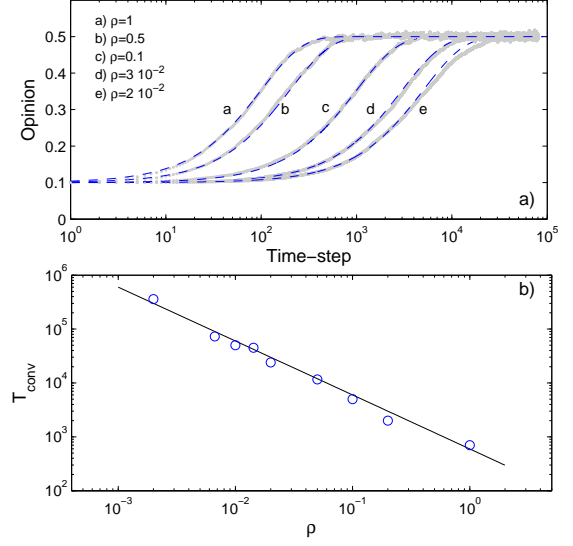

Fig. 5. a) The opinion as a function of time for the case where all the agents are initialized with the same opinion $O_o = 0.1$. The solid (red) line represent the numerical data while the segmented (blue) line represents the theoretical prediction from Eq. (16). b) The transition time T_{conv} as a function of the death density ρ. Symbols stand for the results of the simulations and the solid line represents the prediction (17). Here $N = 100$, $\alpha_c = \Delta O_c = 0.5$, $\epsilon = 0.001$ and $\sigma = 0.07$.

are randomly generated according to a pre–assigned (here uniform) probability distribution. The ratio $\rho = M/T$, here termed departure density plays the role of a control parameter. The occurrence of a phase transition is found which separates between two macroscopically different regimes: For large values of the so–called social temperature the system collapses to a single cluster in opinion, while in the opposite regime a fragmented phase is detected. The role of ρ is elucidated and shown to enter in the critical threshold as a linear contribution. Two phenomena are then addressed, with reference to the sin-

gle clustered phase. On the one side, the external perturbation, here being hypothesized to mimic a death/birth process, induces a spreading of the final cluster. The associated variance is numerically shown to depend on the density amount ρ, the functional dependence being also analytically explained. On the other, we also show that the birth/death events imposed at a constant pace can produce the progressive migration of a cluster, initially localized around a given opinion value. A theoretical argument is also developed to clarify this finding. As a general comment, we should emphasize that the effect of opening up the system to external influences changes dramatically its intrinsic dynamics revealing peculiar, potentially interesting, features which deserves to be further explored.

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