HAUSDORFF MEASURE ON O-MINIMAL STRUCTURES

A. FORNASIERO AND E. VASQUEZ RIFO

Abstract. We introduce the Hausdorff measure for definable sets in an o-minimal structure, and prove the Cauchy–Crofton and co-area formulae for the o-minimal Hausdorff measure. We also prove that every definable set can be partitioned into "basic rectifiable sets", and that the Whitney arc property holds for basic rectifiable sets.

§1. Introduction. Let K be an o-minimal structure expanding a field. We introduce, for every $e \in \mathbb{N}$, the e-dimensional Hausdorff measure $\mathscr{H}^e$ for definable sets, which is the generalization of the usual Hausdorff measure for real sets [Morgan98]. The definition of $\mathscr{H}^e$ uses the measure defined by Berarducci-Otero in [BO04] together with nontrivial results on partitions of definable sets into manageable pieces (§3). Such partitions were first obtained for the real numbers by Kurdyka [K92], then for arbitrary o-minimal expansions of fields by Pawłucki [P08] and independently by the second author [VR06].

With this definition of Hausdorff measure we generalize some well known results from geometric measure theory, such as the co-area formula (a version of Fubini's theorem), to the o-minimal context. We also prove the following version of the Cauchy–Crofton formula (which computes the Hausdorff measure of a set as the average number of points of intersection with hyperplanes of complementary dimension):

THEOREM 1.1 (Cauchy–Crofton Formula). *Let $A \subseteq \mathring{K}^n$ be definable of dimension e where $\mathring{K}$ is the finite part of K. Then,*

$$\mathscr{H}^e(A) = \frac{1}{\beta} \int_{\mathscr{AG}_{n-e}(K^n)} \#(E \cap A)\, \mathrm{d}E.$$

where the integral is over the set of all affine hyperplanes of codimension e, the measure is induced from Haar measure on the group of rigid motions, and

$$\beta := \Gamma\left(\tfrac{e+1}{2}\right)\Gamma\left(\tfrac{n-e+1}{2}\right)\Gamma\left(\tfrac{n+1}{2}\right)^{-1}\pi^{-1/2}.$$

When $K = \mathbb{R}$ our results follow from $\mathscr{C}^1$ cell decomposition and standard geometric measure theory. However, there are significant technical obstacles that must be overcome when working over an arbitrary field K. For instance, the naive definition of Hausdorff measure given by

$$\mathscr{H}^e(X) := \mathscr{H}^e_{\mathbb{R}}(\mathrm{st}(X)), \tag{1}$$

where st denotes the standard part, does not work because the resulting "measure" is not additive: see Example 5.8. The correct definition for the e-dimensional Hausdorff measure is defining it first for "basic e-rectifiable sets" (see Definition 3.9) via (1), and then extending it to definable sets by using a partition into basic e-rectifiable pieces. Such a partition is obtained by using partitions into M_n-cells (as remarked above, such partitions are obtained in [K92], [P08], and [VR06]), another consequence of which is the Whitney arc property for basic e-rectifiable sets (§4).

1.1. The standard part map. Let K be a $(2^{\aleph_0})^+$-saturated o-minimal structure, expanding a field. Let $\mathring{K} := \{x \in K : \exists n \in \mathbb{N}\ |x| < n\}$ be the set of finite elements of K. Notice that $\mathring{K}$ is a valuation ring; by our saturation assumption, the corresponding residue field is (canonically) isomorphic to $\mathbb{R}$. Let st : $\mathring{K} \to \mathbb{R}$ be the "standard part" map: the quotient map from $\mathring{K}$ to its residue field. We also denote by st : $\mathring{K}^n \to \mathbb{R}^n$ the function mapping $\bar{x}$ to the n-tuple of standard parts of the components of $\bar{x}$. Given a subset $A \subset K^n$, we put $\mathrm{st}(A) = \mathrm{st}(A \cap \mathring{K}^n) \subseteq \mathbb{R}^n$.

Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be the extended real line. Occasionally in dealing with K-valued functions, we extend st to a map st : $K \to \bar{\mathbb{R}}$.

Let $\mathbb{R}_K$ be the structure on $\mathbb{R}$ generated by the sets of the form $\mathrm{st}(U)$, where U varies among the definable subsets of K^n. By [BP98], $\mathbb{R}_K$ is o-minimal. [M09] gives a more precise result, which helps clarify the situation:

FACT 1.2. [M09] *The subsets of $\mathbb{R}^n$ definable in $\mathbb{R}_K$ are exactly the finite unions of differences $\mathrm{st}(X) \setminus \mathrm{st}(Y)$ with definable $X, Y \subseteq K^n$. The closed subsets of $\mathbb{R}^n$ definable in $\mathbb{R}_K$ are exactly the sets $\mathrm{st}(X)$ with definable $X \subseteq K^n$.*

§**2. Lebesgue measure on o-minimal structures.** The definitions of measure theory are taken from [Halmos50].

For every $n \in \mathbb{N}$, let $\mathscr{L}^n_{\mathbb{R}}$ be the n-dimensional Lebesgue measure (on $\mathbb{R}^n$). If n is clear from context we drop the superscript. Let $\mathscr{L}^n_1$ be the o-minimal measure on $\mathring{K}^n$ defined in [BO04]. More precisely, $\mathscr{L}^n_1$ is a measure on the σ-ring R_n generated by the definable subsets of $\mathring{K}^n$; thus, $(\mathring{K}^n, R_n, \mathscr{L}^n_1)$ is a measure space. Moreover, since $\mathring{K}^n \in R_n$, R_n is actually a σ-algebra.

Notice that $\mathscr{L}^n_1$ can be extended in a natural way to a measure $\mathscr{L}^n_2$ on the σ-ring $\mathscr{B}_n$ generated by the definable subsets of K^n of finite diameter. Finally, we denote by $\mathscr{L}^n$ the completion of $\mathscr{L}^n_2$, and if n is clear from context we drop the superscript. Notice that the σ-ring $\mathscr{B}_n$ is not a σ-algebra.

FACT 2.1. [BO04, Thm. 4.3] *If $C \subset \mathring{K}^n$ is definable, then $\mathscr{L}^n(C)$ is the Lebesgue measure of $\mathrm{st}(C)$.*

DEFINITION 2.2. For $A \subseteq K^n$ and $f : K^n \to K^m$ we define $\mathrm{st}(f) : A \to \bar{\mathbb{R}}^m$ by $\mathrm{st}(f)(x) = \mathrm{st}(f(x))$.

REMARK 2.3. If $A \subseteq \mathring{K}^n$ and $f : A \to K$ are definable, then $\mathrm{st}(f)$ is an $\mathscr{L}^n$-measurable function.

DEFINITION 2.4. Let $A \subseteq \mathring{K}^n$ and $f : A \to K$ be definable. If $\mathrm{st}(f)$ is $\mathscr{L}^n$-integrable we will denote its integral by

$$\int\limits_A f \, \mathrm{d}\mathscr{L}^n; \quad \int\limits_A f(x) \, \mathrm{d}x; \quad \int\limits_A f(x) \, \mathrm{d}\mathscr{L}^n(x) \quad \text{or} \int\limits_A f.$$

REMARK 2.5. If $A \subseteq \mathring{K}^n$ and $f : A \to \mathring{K}$ are definable, then $\mathrm{st}(f)$ is $\mathscr{L}$-integrable.

LEMMA 2.6. *Let $B := [0,1]^n \subset K^n$. Let $f : B \to K^m$ be a definable $\mathscr{C}^1$ function, such that all the partial derivatives of f are bounded by $C \in \mathbb{R}_{>0}$. Then, $\mathrm{st}(\Gamma(f))$ is the graph of a function $g : [0,1]^n \to \mathbb{R}^m$.*

PROOF. Let $b \in B$. Assume, for a contradiction, that there exist $a_1 \neq a_2$ in $\mathbb{R}^m$ such that $\langle b, a_1 \rangle$ and $\langle b, a_2 \rangle$ are both in $\mathrm{st}(\Gamma(f))$. Let $x_1, x_2 \in B$ and $y_1, y_2 \in K^m$ such that $y_i = f(x_i)$ and $\mathrm{st}(\langle x_i, y_i \rangle) = \langle b, a_i \rangle$, $i = 1, 2$. By the o-minimal mean value theorem (see [Dries98, Theorem 7.2.3]), $|y_2 - y_1| \leq mC|x_2 - x_1|$. Thus, $|a_2 - a_1| \leq 0$, so $a_1 = a_2$. ⊣

LEMMA 2.7. *Let $U \subseteq K^n$ be a definable set. Then, $\dim(\mathrm{st}(U)) \leq \dim(U)$.*

PROOF. First assume $U \subseteq \mathring{K}^n$. Let $d := \dim(U)$. W.l.o.g., U is the graph of a $\mathscr{C}^1$ definable function $f : V \to \mathring{K}^{n-d}$, where $V \subset \mathring{K}^d$ is a definable, bounded set. By the proof of [BO04, Lemma 2.6], after shrinking V if necessary, w.l.o.g. we can assume that $V = [0,1]^d$ and that all the partial derivatives of f are bounded on V. The conclusion then follows immediately from Lemma 2.6.

For general $U \subseteq K^n$, for each $m \in \mathbb{Z}_{>0}$ observe that by the preceding case the dimension of $\mathrm{st}(U \cap [-m,m]^n)$ is at most $\dim(U)$. Since $\mathrm{st}(U) = \bigcup_{m=1}^\infty \mathrm{st}(U \cap [-m,m]^n)$ and dimension may be computed locally we are done. ⊣

DEFINITION 2.8. A function f is Lipschitz if there is $C \in \mathring{K}$ such that, for all $x, y \in \mathrm{dom}(f)$, we have $|f(x) - f(y)| < C|x - y|$ (notice the condition on C being finite). An invertible function f is bi-Lipschitz if both f and f^{-1} are Lipschitz.

REMARK 2.9. Let $U \subset \mathring{K}^n$ and $f : U \to \mathring{K}$ be definable, with $f \geq 0$. Then,

$$\int\limits_U f \, \mathrm{d}\mathscr{L}^n = \mathscr{L}^{n+1}\big(\{\langle \bar{x}, y \rangle \in U \times K : 0 \leq y \leq f(\bar{x})\}\big).$$

LEMMA 2.10 (Change of variables). *Let $U, V \subseteq \mathring{K}^n$ be open and definable, and let $A \subseteq U$ be definable. Let $f : U \to V$ be definable and bi-Lipschitz and $g : V \to \mathring{K}$ be definable, then*

$$\int\limits_{f(A)} g = \int\limits_A |\det \mathrm{D}f| \, g \circ f.$$

Before proving the above lemma we need some preliminary definitions and results.

Let $U \subseteq \mathring{K}^n$ be open and let $f : U \to K$ be definable. The set of points where f takes infinite values is

$$I_f = \mathrm{st}(\{x \in U \mid f(x) \notin \mathring{K}\}).$$

By [BP98] or [M09], I_f is definable.

LEMMA 2.11. *Let $U \subset \mathring{K}^n$ be open and let $f : U \to K$ be definable. Assume that the set I_f has dimension strictly less than n. Then there is a $\mathbb{R}_K$-definable function $\overline{f} : C \to \mathbb{R}$, where $C \subset \mathrm{st}(U)$ is an open set, such that*

i) $E := \left(\mathrm{st}(U) \setminus C \right)$ *is $\mathscr{L}_{\mathbb{R}}^n$-negligible (therefore $\mathrm{st}^{-1}(E)$ is $\mathscr{L}^n$-negligible).*

ii) f *and $\overline{f}$ are $\mathscr{C}^1$ on $U \setminus \mathrm{st}^{-1}(E)$ and C, respectively.*

iii) *For every $x \in U \cap \mathrm{st}^{-1}(C)$ we have $\mathrm{st}(f(x)) = \overline{f}(\mathrm{st}(x))$.*

iv)

$$\int\limits_U f = \int\limits_C \overline{f}.$$

v) $\mathrm{D}f$ *is finite on $U \cap \mathrm{st}^{-1}(C)$ and $\mathrm{D}(\overline{f})(\mathrm{st}\, x) = \mathrm{st}(\mathrm{D}f(x))$.*

PROOF. By cell decomposition, we may assume that f is a function of class $\mathscr{C}^2$, and that U is an open cell. If the fiber $\mathrm{st}(\Gamma(f))_x$ for $x \in \mathrm{st}(U)$ is empty then $x \in I_f$. Let E be the union of the following sets:

1) the closure (in $\mathrm{st}(U)$) of I_f;
2) the set of points $x \in \mathrm{st}(U)$ such that the fiber $\mathrm{st}(\Gamma(f))_x$ has dimension 1;
3) the set $\mathrm{st}(U) \cap \mathrm{st}(K^n \setminus U)$.

Put $C = \mathrm{st}(U) \setminus E$. Since $\dim(\Gamma(f)) = n$ we have $\dim(\mathrm{st}(\Gamma(f)) \leq n$ by Lemma 2.7, and therefore the set in 2) is negligible. Moreover, the set in 3) is equal to $\mathrm{st}(\mathrm{bd}(U))$, and therefore it is also negligible. It follows that E is negligible. By cell decomposition, there are $\mathbb{R}_K$-definable functions g_k of class $\mathscr{C}^1$ for $k = 1, \ldots, r$ such that $\mathrm{st}(\Gamma(f)) \cap (C \times \mathbb{R})$ is the union of the graphs of the functions g_i. We claim that $r = 1$:

In fact, if g_1, g_2 are two different such functions, and say $g_1 < g_2$, then there is an $x_0 \in \mathrm{st}(U)$ such that $\langle x_0, g_1(x_0) \rangle, \langle x_0, g_2(x_0) \rangle \in \mathrm{st}(\Gamma(f))$. Let $c \in \mathbb{R}$ be such that $g_1(x_0) < c < g_2(x_0)$ and let $y_i \in U$ such that $\mathrm{st}(\langle y_i, f(y_i) \rangle) = \langle x_0, g_i(x_0) \rangle$, $i = 1, 2$. Choose $d \in K$ such that $st(d) = c$. Notice that, by adding the set in 3) to E, we ensured that $\mathrm{st}^{-1}(x_0) \subset U$. Since f is continuous, the o-minimal intermediate value theorem implies that there exists z lying on the segment $[y_0, y_1]$, such that $f(z) = d$ (notice that $[y_0, y_1] \subset \mathrm{st}^{-1}(x_0) \subset U$). Thus, $\langle x_0, c \rangle = \mathrm{st}(\langle z, d \rangle) \in \mathrm{st}(\Gamma(f))$, and we have shown that $\{ \langle x_0, y \rangle : y \in (g_1(x_0), g_2(x_0)) \} \subset \mathrm{st}(\Gamma(f))$. On the other hand, $\{ \langle x_0, y \rangle : \langle x_0, y \rangle \in \mathrm{st}(\Gamma(f)) \}$ is a finite set $\{ \langle x_0, g_1(x_0) \rangle, \ldots, \langle x_0, g_r(x_0) \rangle \}$, absurd.

Let $\overline{f} := g_1$. ii) holds, and for every $x \in U$ with $\mathrm{st}(x) \in C$ we have $\mathrm{st}(f(x)) = \overline{f}(\mathrm{st}(x))$. The equality of the integrals in iv) follows from Remark 2.9.

To obtain v) we will enlarge E by a negligible set. For $i = 1, \ldots, n$ let

$$E_i := \mathrm{st}\left(\left\{ x \in U : \frac{\partial f}{\partial x_i}(x) \notin \mathring{K} \right\} \right).$$

By [BP98], E_i is $\mathbb{R}_K$-definable. If $\dim(E_i) = n$, then E_i contains an open ball B. By dimension considerations, there is a line segment $L \subseteq B$ parallel to the x_i-axis such that $L \cap I_f$ is a finite set of points. This contradicts Lemma 2.5 of [BO04] by which every definable, one variable function into $\mathring{K}$ has finite derivative except on $\mathrm{st}^{-1}(A)$, for a finite set A. It follows that each set E_i is negligible and therefore, after enlarging E, we may assume that $\mathrm{D}(f)$ is finite on $U \setminus \mathrm{st}^{-1}(E)$. We may now apply parts i), ii), and iii) to each of the functions $\frac{\partial f}{\partial x_i}$ to define $\overline{\mathrm{D}f}$.

As before, we will enlarge E (and shrink C) by a closed negligible set. Using what we have already proved, we may assume that all second derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are finite on $U \setminus \mathrm{st}^{-1}(E)$. Let $V := \{x \in C : \mathrm{D}(\overline{f})(x) \neq \overline{\mathrm{D}f}(x)\}$. The set V is $\mathbb{R}_K$-definable and open. If V is nonempty, then it contains an open ball and therefore w.l.o.g. we may assume that V is an open ball centered at 0. We may also assume $f(0) = 0$. After subtracting from f a linear function, we can assume that $\frac{\partial f}{\partial x_i}(0) = 0$ and $\frac{\partial \overline{f}}{\partial x_i}(0) = 3\varepsilon > 0$ for some index $i = 1, \ldots, n$. Let g be the restriction of f to the x_i-axis in K^n so that $g' = \frac{\partial f}{\partial x_i}$ and g'' is finite.

Since g'' is finite there is $\delta \in \mathbb{R}_{>0}$ such that $g'(x) < \varepsilon$ for all $x \in K$ with $|x| < \delta$, and $\frac{\partial \overline{f}}{\partial x_i}(x) > 2\varepsilon$ for all $x \in \mathbb{R}^n$ with $|x| < \delta$. Thus there exists $x \in C$ with $|f(x)| < |x|\varepsilon$ and $\overline{f}(x) \geq 2|x|\varepsilon$ contradicting iii). We conclude that V is empty. $\dashv$

REMARK 2.12. Let $f : U \to K$ and $g : V \to U$ be continuous definable functions on open subsets $U \subseteq \mathring{K}^n$ and $V \subseteq \mathring{K}^m$. If $g^{-1}(A)$ is negligible whenever A is, then by part iii) of Lemma 2.11, outside a negligible closed set, $\overline{(f \circ g)} = \overline{f} \circ \overline{g}$, where $\overline{g}$ is defined coordinate-wise.

PROOF OF LEMMA 2.10. The fact that f is bi-Lipschitz implies that $\overline{f}$ is injective (since it is also bi-Lipschitz).

Claim 1. Let $C \subset \mathrm{st}(V)$ be Lebesgue measurable. Then,

$$\mathscr{L}^n(C) = \int_{(\mathrm{st}\,f)^{-1}(C)} \mathrm{st}(|\det \mathrm{D}f|).$$

In fact, by the change of variables formula (on the reals) and Lemma 2.11,

$$\mathscr{L}^n(C) = \int_{\overline{f}^{-1}(C)} |\det \mathrm{D}\overline{f}|) = \int_{(\mathrm{st}\,f)^{-1}(C)} \mathrm{st}(|\det \mathrm{D}f|).$$

Claim 2. Let $h : V \to \bar{\mathbb{R}}$ be an integrable function. Then,

$$\int_V h = \int_U \mathrm{st}(|\det \mathrm{D}f|)\, h \circ f.$$

Claim 1 implies that the statement is true if h is a simple function. By continuity, the statement is true for any integrable function h.

In particular, we can apply Claim 2 to the function

$$h : x \mapsto \begin{cases} \mathrm{st}(g(x)) & \text{if } x \in f(A), \\ 0 & \text{otherwise,} \end{cases}$$

and obtain the conclusion. $\dashv$

LEMMA 2.13 (Fubini's theorem). $\mathscr{L}^{n+m}$ *is the completion of the product measure* $\mathscr{L}^n \times \mathscr{L}^m$. *Therefore, if D is the interval $[0,1] \subset K$ and given $f : D^{n+m} \to D$ definable,*

$$\int_{D^{n+m}} f(x,y)\, \mathrm{d}\mathscr{L}^{n+m}(x,y) = \iint_{D^m D^n} f(x,y)\, \mathrm{d}\mathscr{L}^m(x)\, \mathrm{d}\mathscr{L}^n(y).$$

PROOF. Follows from the definition of $\mathscr{L}^n$ in [BO04]. $\dashv$

2.1. Measure on semialgebraic sets.

DEFINITION 2.14. We say that $E \subseteq K^n$ is $\emptyset$-semialgebraic if E is definable without parameters in the language of pure fields. If $E \subseteq K^n$ is $\emptyset$-semialgebraic we denote the subset of $\mathbb{R}^n$ defined by the same formula that defines E by $E_{\mathbb{R}}$.

REMARK 2.15. Let $E \subseteq \mathring{K}^n$ be $\emptyset$-semialgebraic. Then, $\mathrm{st}(E) = \overline{E_{\mathbb{R}}}$.

Let $E \subseteq K^n$ be closed and $\emptyset$-semialgebraic submanifold. Working in local charts, from [BO04] one can easily define a measure $\mathscr{L}^E$ on the σ-ring generated by the definable subsets of E of bounded diameter. We will denote in the same way the completion of $\mathscr{L}^E$. Notice that $\mathscr{L}^{K^n} = \mathscr{L}^n$.

REMARK 2.16. Let E be a closed, $\emptyset$-semialgebraic submanifold of K^n of dimension e, $F := \mathrm{st}(E)$, and $C \subseteq E$ be definable and bounded. Then, $\mathscr{L}^E(C) = \mathscr{L}^F_{\mathbb{R}}(\mathrm{st}(C))$, where $\mathscr{L}^F_{\mathbb{R}}$ is the e-dimensional Hausdorff measure on F.

One could also take the above remark as the definition of $\mathscr{L}^E$ on $E \cap \mathring{K}^n$.

§3. **Rectifiable partitions.** Theorem 3.8 shows that every definable set $A \subset \mathring{K}^n$ has a partition into definable sets which are M_n-cells after an orthonormal change of coordinates (where $M_n \in \mathbb{Q}$ depends only on n). In [P08], Pawłucki shows that a permutation of the coordinates suffices. The proof of 3.8 follows closely that of [K92]. The partition in 3.8 is then used in Corollary 3.11 to show that definable sets have a rectifiable partition.

DEFINITION 3.1. Let $L : V \to W$ be a linear map between normed K-vector spaces. The norm of L is given by

$$\|L\| := \sup_{|v|=1} |L(v)|.$$

For V, W in the Grassmannian of e-dimensional linear subspaces of K^n, namely $\mathscr{G}_e(K^n)$, let π_V and $\pi_W \in \mathrm{End}_K(K^n)$ be the orthogonal projections onto V and W respectively. In this way we have a canonical embedding $\mathscr{G}_e(K^n) \subset \mathrm{End}_K(K^n)$. The *distance function* on the Grassmannian is given by the inclusion above:

$$\delta(V, W) := \|\pi_V - \pi_W\|.$$

For P in $\mathscr{G}_1(K^n)$ and $X \in \mathscr{G}_k(K^n)$, define

$$\delta(P, X) := |v - \pi_X(v)|,$$

where π_X is the orthogonal projection onto X, and v is a generator of P of norm 1. Note that $\delta(P, X) = 0$ if and only if $P \subset X$, $0 \le \delta(P, X) \le 1$ and $\delta(P, X) = 1$ if and only if $P \perp X$. Note also that $\delta(P, X)$ is the definable analog of the sine of the angle between P and X.

LEMMA 3.2. *Let $n \in \mathbb{N}_{>0}$. Then there exists an $\varepsilon_n \in \mathbb{Q}_{>0}$, $\varepsilon_n < 1$, such that for any $X_1, \dots, X_{2n} \in \mathscr{G}_{n-1}(K^n)$, there is a line $P \in \mathscr{G}_1(K^n)$ such that whenever $Y_1, \dots, Y_{2n} \in \mathscr{G}_{n-1}(K^n)$ and*

$$\delta(X_i, Y_i) < \varepsilon_n, \quad i = 1, \dots, 2n, \quad then$$
$$\delta(P, Y_i) > \varepsilon_n, \quad i = 1, \dots, 2n.$$

PROOF. For $\varepsilon > 0$ define $S_i(\varepsilon) = \{v \in S^{n-1} : |v - \pi_{X_i}(v)| \leq 2\varepsilon\}$. If $K = \mathbb{R}$, let $\varepsilon_n \in \mathbb{Q}_{>0}$ be small enough so that $2n \operatorname{Vol}(S_1(\varepsilon_n)) < \operatorname{Vol}(S^{n-1})$, where Vol is the measure $\mathscr{L}^{S^{n-1}}$ defined in §2.1. Then

$$\operatorname{Vol}(\textstyle\bigcup_{i=1}^{2n} S_i(\varepsilon_n)) \leq 2n \operatorname{Vol}(S_1(\varepsilon_n)) < \operatorname{Vol}(S^{n-1})$$

and therefore

$$\bigcup_{i=1}^{2n} S_i(\varepsilon_n) \neq S^{n-1}. \tag{2}$$

The same ε_n will necessarily satisfy (2) for any field K containing $\mathbb{R}$.

Now, we choose

$$v \in S^{n-1} - \bigcup_{i=1}^{2n} S_i(\varepsilon_n)$$

and let $P := \langle v \rangle$. Then

$$\delta(P, Y_i) = |v - \pi_{Y_i} v| \geq |v - \pi_{X_i} v| - |\pi_{X_i} v - \pi_{Y_i} v| > \varepsilon_n. \qquad \dashv$$

DEFINITION 3.3. Let $\varepsilon > 0$. A definable embedded submanifold M of K^n is ε-*flat* if for each $x, y \in M$ we have $\delta(TM_x, TM_y) < \varepsilon$, where TM_x denotes the tangent space to M at x.

LEMMA 3.4. *Let $A \subset K^n$ be a definable submanifold of dimension e and $\varepsilon \in \mathbb{R}_{>0}$. Then there is a cell decomposition $A = \bigcup_{i=0}^{k} A_i$ of A such that for every i we have either $\dim(A_i) < \dim(A)$ or A_i is an ε-flat submanifold of K^n.*

PROOF. Cover $\mathscr{G}_e(K^n)$ by a finite number of balls B_i of radius $\varepsilon/2$; and consider the Gauss map $G : A \to \mathscr{G}_e(K^n)$ taking an element a of A to TA_a. Take a cell decomposition of K^e compatible with A and partitioning each $G^{-1}(B_i)$. Then the e-dimensional cells contained in A are ε-flat. $\dashv$

LEMMA 3.5. *Let $\varepsilon \in \mathbb{Q}_{>0}$, and let $A \subset \mathring{K}^n$ be an open definable set. Then there are open, pairwise disjoint cells $A_1, \ldots, A_p \subset A$ such that*

(i) $\dim(A - \bigcup A_i) < n$.

(ii) *For each i, there are definable, pairwise disjoint sets $B_1, \ldots, B_k$ (with k depending on i) such that*

   (a) $k \leq 2n$;

   (b) *each B_j is a definable subset of ∂A_i and an ε-flat, $(n-1)$-dimensional, $\mathscr{C}^1$-submanifold of K^n;*

   (c) $\dim(\partial A_i - \bigcup_{j=1}^{k} B_j) < n - 1$.

PROOF. By induction on n. The lemma is clear for $n = 1$. Assume that $n > 1$ and the lemma holds for smaller values of n.

Take a cell decomposition of $\overline{A}$ compatible with A into $\mathscr{C}^1$-cells. Let C be an open cell in this decomposition; it suffices to prove the lemma for C. Note that $C = (f, g)_X$, where X is an open cell in K^{n-1} and f, g are definable $\mathscr{C}^1$-functions on X. Take finite covers of $\Gamma(f)$ and $\Gamma(g)$ by open, definable sets U_i and V_j, respectively, such that each $U_i \cap \Gamma(f)$ and each $V_j \cap \Gamma(g)$ is ε-flat (to do this, take a finite cover of the Grassmannian by ε-balls and pull it back via the Gauss maps for $\Gamma(f)$ and $\Gamma(g)$). The collection of all sets $\pi(U_i) \cap \pi(V_j)$ is an open cover $\mathscr{O}$ of X. By the cell decomposition theorem, there is a $\mathscr{C}^1$-cell decomposition of X

partitioning each set in $\mathscr{O}$. Let S be an open cell in this decomposition, and let $C_0 := (f, g)_S$. It suffices to prove the lemma for C_0. By the inductive hypothesis, we can find $A_1', \ldots, A_p' \subset S$ and $B_1', \ldots, B_k' \subset \partial A_i'$ satisfying the conditions (i) and (ii) above (with n replaced by $n - 1$). Define

$$A_i := (f, g)_{A_i'}, \qquad i = 1, \ldots, p.$$

Then $\dim(C_0 - \bigcup_{i=1}^p A_i) < n$. For $j = 1, \ldots, k$, the set $(B_j' \times K) \cap \partial A_i$ is definable. Take a $\mathscr{C}^1$-cell decomposition of this set, and let B_j be the union of the $(n-1)$-dimensional cells in this decomposition (note that B_j may be empty). Then B_j is an ε-flat $\mathscr{C}^1$-submanifold of K^n and

$$\dim\big(((B_j' \times K) \cap \partial A_i) - B_j\big) < n - 1.$$

Define $B_{k+1} := \Gamma(f|A_i')$ and $B_{k+2} := \Gamma(g|A_i')$; by construction these are ε-flat. It is routine to see that $\partial A_i \subset B_{k+1} \cup B_{k+2} \cup (\partial A_i' \times K)$. Thus

$$\begin{aligned}
\partial A_i - \bigcup_{j=1}^{k+2} B_j &\subset ((\partial A_i' \times K) \cap \partial A_i) - \bigcup_{j=1}^k B_j \\
&= (\bigcup_{j=1}^k ((B_j' \times K) \cap \partial A_i) \cup E) - \bigcup_{j=1}^k B_j \\
&\subset \bigcup_{j=1}^k (((B_j' \times K) \cap \partial A_i) - B_j) \cup E,
\end{aligned}$$

where E is a definable set with $\dim(E) < n - 1$. Therefore $\dim(\partial A_i - \bigcup_{j=1}^{k+2} B_j) < n - 1$. Since $k \le 2(n-1)$, we get $k + 2 \le 2n$ and the lemma is proved. $\dashv$

DEFINITION 3.6. Let $U \subseteq K^n$ be open and let $f : U \to K^m$ be definable. Given $0 < M \in K$, we say that f is an M-function if $|\mathrm{D}f| \le M$. We say that f has finite derivative if $|\mathrm{D}f|$ is finite.

Notice that, by ω-saturation of K, if f is definable and has finite derivative, then it is an M-function for some finite M.

Let $M \in K_{>0}$. An M-cell is a $\mathscr{C}^1$-cell where the $\mathscr{C}^1$ functions that define the cell are M-functions. More precisely:

DEFINITION 3.7. Let $(i_1, \ldots, i_m)$ be a sequence of zeros and ones, and $M \in K_{>0}$. An $(i_1, \ldots, i_m)$-M-cell is a subset of K^m defined inductively as follows:

(i) A (0)-M-cell is a point $\{r\} \subset K$, a (1)-M-cell is an interval $(a, b) \subset K$, where $a, b \in K$.

(ii) An $(i_1, \ldots, i_m, 0)$-M-cell is the graph $\Gamma(f)$ of a definable M-function $f : X \to K$ of class $\mathscr{C}^1$, where X is an $(i_1, \ldots, i_m)$-M-cell; an $(i_1, \ldots, i_m, 1)$-M-cell is a set

$$(f, g)_X := \{(x, r) \in X \times K : f(x) < r < g(x)\},$$

where X is an $(i_1, \ldots, i_m)$-M-cell and $f, g : X \to K$ are definable M-functions of class $\mathscr{C}^1$ on X such that for all $x \in X$, $f(x) < g(x)$.

THEOREM 3.8. *Let $A \subset \mathring{K}^n$ be definable. Then there are definable, pairwise disjoint sets A_i, $i = 1, \ldots, s$, such that $A = \bigcup_i A_i$ and for each A_i, there is a change of coordinates $\sigma_i \in O_n(K)$ such that $\sigma_i(A_i)$ is an M_n-cell, where $M_n \in \mathbb{Q}_{>0}$ is a constant depending only on n.*

Proof. We will make use of the following fact:

Let $\varepsilon \in [0,1]$, $P \in \mathscr{G}_1(K^n)$, $X \in \mathscr{G}_k(K^n)$ and and $w \in X$ be a unit vector. Suppose $\delta(P,X) > \varepsilon$. If $\pi_P(w) \geq 1/2$, where π_P is the orthogonal projection onto P, then

$$|\pi_P(w) - w| \geq |\pi_P(w) - \pi_X(\pi_P(w))| > |\pi_P(w)|\varepsilon \geq 1/2\varepsilon.$$

If $\pi_P(w) < 1/2$, then $|w| \leq |\pi_P(w)| + |\pi_p(w) - w| \leq 1/2 + |\pi_p(w) - w|$. In either case, we have

$$|\pi_P(w) - w| \geq \frac{1}{2}\varepsilon. \tag{3}$$

We prove the theorem by induction on n; for $n = 1$ the theorem is clear. We assume that $n > 1$ and that the theorem holds for smaller values of n. We also proceed by induction on $d := \dim(A)$. It's clear for $d = 0$; so we assume that $d > 0$ and the theorem holds for definable bounded subsets B of K^n with $\dim(B) < d$.

Case I: $\dim(A) = n$. In this case A is an open, bounded, definable subset of K^n, so by using the inductive hypothesis and Lemma 3.5, we can reduce to the case where there are pairwise disjoint, definable $B_1, \ldots, B_k \subset \partial A$ such that $k \leq 2n$, $\dim(\partial A - \bigcup_{j=1}^k B_j) < n - 1$ and each B_j is an ε_n-flat submanifold, where ε_n is as in Lemma 3.2. By Lemma 3.2, there is a hyperplane L such that for each B_j and all $x \in B_j$, we have $\delta(L^\perp, T_x B_j) > \varepsilon_n$. Take a cell decomposition $\mathscr{B}$ of K^n, with respect to orthonormal coordinates in the L, $L^\perp$ axis, partitioning each B_j. Let

$$\mathscr{S} := \{C \in \mathscr{B} : \dim(C) = n - 1, C \subset \bigcup_{j=1}^k B_j\}$$

and note that $\dim(\partial A \setminus \bigcup_{C \in \mathscr{S}} C) < n - 1$. Furthermore,

$$\mathrm{BAD} := \{x \in A : \pi_L^{-1}(\pi_L(x)) \cap \partial A \not\subset \bigcup_{c \in \mathscr{S}} C\}$$

has dimension smaller than n. Let $U_1, \ldots, U_l$ be the elements of $\{\pi_L(C) : C \in \mathscr{S}\}$. Then the set

$$\{x \in A : x \notin \pi_L^{-1}(\bigcup_{i=1}^l U_i)\}$$

is contained in BAD, and therefore has dimension smaller than n.

By using the inductive hypothesis, we only need to find the required partition for each of the sets $A \cap \pi_L^{-1}(U_i)$, $i = 1, \ldots, l$. Fix $i \in \{1, \ldots, l\}$ and let $U := U_i$, $A' := A \cap \pi_L^{-1}(U)$. Take $C \in \mathscr{S}$ with $\pi_L(C) = U$. Then $C = \Gamma(\phi)$ for a definable $\mathscr{C}^1$-map $\phi : U \to L^\perp$ and for all $x \in C$,

$$T_x C = \{(v, \mathrm{D}\phi(v)) : v \in T_{\pi_L(x)} U\}.$$

Let $v \in T_{\pi_L(x)} U$ be a unit vector; since we have $\delta(L^\perp, T_x C) > \varepsilon_n$ and $|(v, \mathrm{D}\phi(v))| = \sqrt{1 + |\mathrm{D}\phi(v)|^2}$, it follows from equation (3) that

$$\tfrac{1}{2}\varepsilon_n \leq \frac{1}{\sqrt{1 + |\mathrm{D}\phi(v)|^2}} |\pi_{L^\perp}((v, \mathrm{D}\phi(v))) - (v, \mathrm{D}\phi(v))| = \frac{1}{\sqrt{1 + |\mathrm{D}\phi(v)|^2}} |v|.$$

Therefore,

$$|\mathrm{D}\phi(v)| \leq \sqrt{\frac{4}{\varepsilon_n^2} - 1}.$$

Let $M_n \in \mathbb{Q}$ be bigger than $\max \left\{ M_{n-1}, \sqrt{\frac{4}{\varepsilon_n^2} - 1} \right\}$.

We have proved that for each $C_j \in \mathcal{S}$ with $\pi_L(C_j) = U$ there is a definable $\mathscr{C}^1$-map $\phi_j : U \to K$, such that $|\mathrm{D}\phi_j| < M_n$ and $C_j = \Gamma(\phi_j)$.

By the inductive hypothesis, there is a partition $\mathscr{P}$ of U such that each piece $P \in \mathscr{P}$ is a M_{n-1}-cell after a change of coordinates of L. We have

$$A' = \coprod_{\substack{P \in \mathscr{P} \\ (\phi_r, \phi_s)_P \subset A'}} (\phi_r, \phi_s)_P,$$

and $(\phi_r, \phi_s)_P$ is a M_n-cell after a coordinate change.

Case II: $\dim(A) < n$. In this case, by Lemma 3.4, we can partition A into cells which are ε_n-flat. Therefore we may assume that A is an ε_n-flat submanifold, where ε_n is as in Lemma 3.2. As in case I, there is a hyperplane L such that A is the graph of a function $f : U \to K$, $U \subset L$ and $|\mathrm{D}f| < M_n$. By the inductive hypothesis, we can partition U into M_{n-1}-cells. The graphs of f over the cells in this partition give the required partition of A.   ⊣

DEFINITION 3.9. Let $A \subseteq K^n$ and $e \leq n$. A is basic e-rectifiable with bound M if, after a permutation of coordinates, A is the graph of an M-function $f : U \to K^{n-e}$, where $U \subset K^e$ is an open M-cell for some finite M.

LEMMA 3.10. Let $A \subset \mathring{K}^n$ be an M-cell of dimension e. If $M \in \mathring{K}$ then A is a basic e-rectifiable set, and the bound of A can be chosen depending only on M and n.

PROOF. We proceed by induction on n. If $n = 0$ or $n = 1$ the result is trivial, so assume $n \geq 2$. By definition, there exists an M-cell $B \subset \mathring{K}^{n-1}$ such that

(1) either $A = \Gamma(g)$ for some M-function $g : B \to \mathring{K}$, or
(2) $A = (g, h)_B$ for some M-functions $g, h : B \to \mathring{K}$, with $g < h$.

By inductive hypothesis, there is a permutation of the first $n - 1$ coordinates so that $B = \Gamma(f)$ where $f : C \to K^{n-1-d}$ is an L-function for some L depending only on M and n and $C \subseteq K^d$ is an open L-cell.

In case (1) $d = e$. Define $l : C \to K^{n-e}$ by $l(x) = \langle f(x), g(x, f(x)) \rangle$. It is easy to see that l is an L'-function for some L' depending only on M and n, and that $A = \Gamma(l)$.

In case (2), $d = e - 1$. For $x \in C$ define $\tilde{g}(x) := g(x, f(x)), \tilde{h}(x) := h(x, f(x))$, and $\tilde{B} := (\tilde{g}, \tilde{h})_C$. Given $\bar{x} \in C$ and $\tilde{g}(x) < y < \tilde{h}(x)$ with $\langle \bar{x}, y \rangle \in \tilde{B}$, define $l(\bar{x}, y) := f(\bar{x})$. We have that $\tilde{B}$ is an open e-dimensional L-cell, $l : \tilde{B} \to K^{n-e}$ is an L'-function for some L' depending only on M and n, and after a permutation of coordinates $A = \Gamma(l)$.   ⊣

COROLLARY 3.11. Let $A \subseteq \mathring{K}^n$ be a definable set of dimension at most e. Then there is a partition $A = \bigcup_{i=0}^{k} A_i$ with $\dim(A_0) < e$ and such that for each $i > 0$ there is $\sigma_i \in O_n(K)$ with $\sigma_i(A_i)$ basic e-rectifiable. Moreover, the bounds of each $\sigma_i(A_i)$ can be chosen to depend only on n.

PROOF. Apply Theorem 3.8 and 3.10.   ⊣

A partition $(A_0, \ldots, A_k)$ of A as in Corollary 3.11 is a basic e-rectifiable partition of A.

Notice that a similar result has also been proved in [PW06, Theorem 2.3] (where they also take arbitrarily small bounds): however, in [PW06] they don't require that

the functions parametrizing the set A are injective (which is essential for our later uses).

§**4. Whitney decomposition.** The fact that the functions that define an M-cell are actually Lipschitz function follows from the following property of M-cells:

Every pair of points x, y in an M-cell $C \subset K^n$ can be connected by a definable $\mathscr{C}^1$ curve $\gamma : [0, 1] \to C$ with $|\gamma'(t)| < N|x - y|$, where N is a constant depending only on M and n which is finite if M is (Lemma 4.3 or [VR06] 3.10 & 3.11).

The same property implies that an N-function f on an M-cell is Lipschitz where the Lipschitz constant is finite if M and N are (Corollary 4.5). This last property will be needed for defining Hausdorff measure.

REMARK 4.1. Let $U \subset \mathring{K}^n$ be open and definable, and $f : U \to \mathring{K}$ be an M-function (for some finite M). It is not true in general that f is L-Lipschitz for some finite L: this is the reason why we needed to prove Theorem 3.8.

DEFINITION 4.2. Let $A \subset K^n$, $B \subset K^m$ be definable sets. Let $\lambda \subset A \times ([0, 1] \times B) \subset K^n \times K^{1+m}$ be a definable set such that for every $x \in A$, the fiber over x

$$\lambda_x := \{y \in [0, 1] \times B : \langle x, y \rangle \in \lambda\}$$

is a curve $\lambda_x : [0, 1] \to B$. We view λ as describing the family of curves $\{\lambda_x\}_{x \in A}$. Such a family is a definable family of curves (in B, parametrized by A).

An L-cell is an L-Lipschitz cell if the functions that define the L-cell are L-Lipschitz.

LEMMA 4.3. *Fix $L \in K_{>0}$ and $n \in \mathbb{N}_{>0}$. Then, there is a constant $K(n, L) \in K_{>0}$ depending only on n and L, that is finite if L is, such that for every L-Lipschitz cell $C \subset K^n$ there is a definable family of curves $\gamma \subset C^2 \times ([0, 1] \times C)$ such that: For all $x, y \in C$, $\gamma_{x,y} : [0, 1] \to C$ is a $\mathscr{C}^1$-curve with*

(i) $\gamma_{xy}(0) = x, \gamma_{xy}(1) = y$;
(ii) $|\gamma'_{xy}(t)| \leq K(n, L)|x - y|$, *for all $t \in [0, 1]$.*

PROOF. By induction on n. For $n = 1$ the lemma is clear. Take $n \geq 1$, and assume that the lemma holds for n. Let $C \subset K^{n+1}$ be an L-Lipschitz cell. Then $C = \Gamma(f)$ or $C = (g, h)_X$ for some L-Lipschitz cell $X \subset K^{n-1}$ and definable, $\mathscr{C}^1$, L-Lipschitz functions f, g, h with $g < h$, and $|Df|, |Dg|, |Dh| \leq L$. By induction, there are a constant $k := K(n - 1, L)$ and a definable family of $\mathscr{C}^1$-curves β in X with the required properties. Let $\pi_n : K^{n+1} \to K^n$ be the projection onto the first n coordinates.

If $C = \Gamma(f)$, we lift β to C via f: fix $x, y \in C$ and let $\gamma_{x,y}(t) := (\alpha(t), f(\alpha(t)))$, where for all $t \in [0, 1]$ $\alpha(t) := \beta_{\pi_n(x), \pi_n(y)}(t)$. Then we have $|\gamma'_{xy}(t)| \leq (1+L)k|x-y|$.

If $C = (g, h)_X$, we lift β as follows: Fix $x, y \in C$ and let $\alpha := \beta_{\pi_n(x), \pi_n(y)}$. Let $\pi : K^{n+1} \to K$ be the projection onto the last coordinate and take $u, v \in (0, 1)$ with

$$\pi(x) = uh(\alpha(0)) + (1 - u)g(\alpha(0)),$$
$$\pi(y) = vh(\alpha(1)) + (1 - v)g(\alpha(1)).$$

Let $l(t) := tv + (1 - t)u$, for $t \in [0, 1]$. We define $\gamma_{x,y}(t) := (\alpha(t), l(t)h(\alpha(t)) + (1 - l(t))g(\alpha(t)))$, and note that

$$|\gamma'_{xy}(t)| \leq k|x - y| + |(v - u)(h(\alpha(t)) - g(\alpha(t)))| + 2Lk|x - y|,$$

since $l(t), 1 - l(t)$ are between 0 and 1 and $|\mathrm{D}h(\alpha'(t))|, |\mathrm{D}g(\alpha'(t))| \leq L|\alpha'(t)|$. Let $f := h - g$. We want to bound $|(v - u)f(\alpha(t))|$, which equals

$$\big|\pi y - \pi x - v(f(\alpha(1)) - f(\alpha(t))) +$$
$$+ u(f(\alpha(0)) - f(\alpha(t))) + g(\alpha(0)) - g(\alpha(1))\big|.$$

But

$$|f(\alpha(1)) - f(\alpha(t))| \leq L|\alpha(1) - \alpha(t)| = L|1 - t|\left|\frac{\alpha(1) - \alpha(t)}{1 - t}\right| \leq L|\alpha'(t_0)|$$

for some t_0 between t and 1. Similarly, $|f(\alpha(0)) - f(\alpha(t))| \leq L|\alpha'(t_1)|$, for some t_1 between t and 1. Since $u, v \in [0, 1]$, we get

$$|(v - u)f(\alpha(t))| \leq |\pi y - \pi x| + 2Lk|x - y| + L|x - y|;$$

thus $|\gamma'_{xy}(t)| \leq K(n, L)|x - y|$ for some constant $K(n, L)$ depending only on n and L which is finite if L is. The collection of the curves γ_{xy} for $x, y \in C$ constitutes the required family of curves. $\dashv$

THEOREM 4.4. *Let $L > 0$, and let $C \subset K^n$ be an L-cell. Then C is a $k(n, L)$-Lipschitz cell, where $k(n, L)$ depends only on n and L, and is finite if L is.*

PROOF. By induction on n; the theorem is clear for $n = 1$. Assume that $n > 1$ and that the theorem holds for $n - 1$. Then $C = \Gamma(f)$ or $C = (g, h)_X$, where $X \subset K^{n-1}$ is a $k(n - 1, L)$-Lipschitz cell and f, g, h are $\mathscr{C}^1$-functions on X such that $|\mathrm{D}f|, |\mathrm{D}g|, |\mathrm{D}h| \leq L$. We need to show that f, g, h are Lipschitz.

Since X is a k-Lipschitz cell, $k := k(n - 1, L)$, it follows from Lemma 4.3 that there is a constant $K(n - 1, k)$ such that whenever $x, y \in X$, there is a definable, $\mathscr{C}^1$-curve γ joining x and y with $|\gamma'(t)| \leq K(n - 1, k)|x - y|$ for all $t \in [0, 1]$. Let $g := f \circ \gamma$, and let $t_0 \in (0, 1)$ be such that

$$|f(x) - f(y)| = |g'(t_0)| = |\mathrm{D}f(\gamma'(t_0))| \leq L|\gamma'(t_0)| \leq LK(n - 1, k)|x - y|.$$

Thus f is $LK(n - 1, k)$-Lipschitz. We set $k(n, L) := LK(n - 1, k)$. $\dashv$

COROLLARY 4.5. *Let C be an M-cell and f be a definable M-function. Then f is Lipschitz, and with finite Lipschitz constant if M is finite.*

PROOF. By Theorem 4.4, C has a definable family of curves as in Lemma 4.3. The result therefore follows from the mean value theorem. $\dashv$

DEFINITION 4.6. A definable set $A \subset K^n$ satisfies the Whitney arc property if there is a constant $K \in \mathring{K}_{>0}$ such that for all $x, y \in A$ there is a definable curve $\gamma : [0, 1] \to A$ with $\gamma(0) = x$, $\gamma(1) = y$ and $\mathrm{length}(\gamma) := \int_0^1 |\gamma'| \leq K|x - y|$.

LEMMA 4.7. *Let $C \subset \mathring{K}^n$ be an M-cell, $M \in \mathring{K}$. Then, C satisfies the Whitney arc property.*

PROOF. It follows from Theorem 4.4 and Lemma 4.3. The finiteness of C and M guarantee that $\mathrm{length}(\gamma)$ can be defined. $\dashv$

THEOREM 4.8. *Let $A \subset \mathring{K}^n$ be definable. Then, A can be partitioned into finitely many definable sets, each of them satisfying the Whitney arc property.*

PROOF. This follows from Lemma 4.7, Theorem 3.8 and the fact that the Whitney arc property is invariant under an orthonormal change of coordinates.   ⊣

§**5. Hausdorff measure.** For an introduction to geometric measure theory, and to the Hausdorff measure, see [Morgan98].

DEFINITION 5.1. Let $U \subseteq K^n$ be open and let $f : U \to \mathring{K}^m$ be a definable function. If $a \in U$, $e \leq n$ and M is the set of the $e \times e$ minors of $\mathrm{D}f(a)$ we define

$$J_e f(a) = \begin{cases} +\infty & \text{if } f \text{ is not differentiable at } a \\ & \qquad \text{or } \mathrm{rank}(\mathrm{D}f(a)) > e, \\ \sqrt{\sum_{m \in M} m^2} & \text{otherwise;} \end{cases}$$

(cf. [Morgan98, §3.6]).

Notice that if $e = n = m$, then $J_n f = |\det(\mathrm{D}f)|$.

DEFINITION 5.2. Let $U \subseteq \mathring{K}^e$ be an open M-cell for some $M \in \mathbb{N}$, and let $f : U \to \mathring{K}^m$ be a definable function with finite derivative. Let $F : U \to \mathring{K}^{m+e}$ be $F(x) := \langle x, f(x) \rangle$ and $C := \Gamma(f) = F(U)$ (notice that C has bounded diameter). We define

$$\mathscr{H}^e(C) := \int_U J_e F \, \mathrm{d}\mathscr{L}^e.$$

LEMMA 5.3. *If $C \subseteq \mathring{K}^n$ is a basic e-rectifiable set, then $\mathscr{H}^e(C) = \mathscr{H}^e_{\mathbb{R}}(\mathrm{st}(C))$, where $\mathscr{H}^e_{\mathbb{R}}$ is the e-dimensional Hausdorff measure on $\mathbb{R}^n$.*

PROOF. Let $A \subset \mathring{K}^e$ and $f : A \to \mathring{K}^{n-e}$ be as in Definition 3.9, and $F : A \to \mathring{K}^n$ as in Definition 5.2. Let $B := \mathrm{st}(A)$. Then, using the real Area formula [Morgan98],

$$\int_A J_e F \, \mathrm{d}\mathscr{L}^e = \int_B J_e(\overline{F}) \, \mathrm{d}\mathscr{L}^e_{\mathbb{R}} = \mathscr{H}^e_{\mathbb{R}}(\overline{F}(B)) = \mathscr{H}^e_{\mathbb{R}}(\mathrm{st}(C)).$$
                                                                                    ⊣

DEFINITION 5.4. Let $A \subseteq \mathring{K}^n$ be definable of dimension at most e, and $(A_0, \ldots, A_k)$ be a basic e-rectifiable partition of A. For $i > 0$, define $\mathscr{H}^e(A_i) := \mathscr{H}^e(\sigma_i(A_i))$, where $\sigma_i \in O_n(K)$ is as in Corollary 3.11; and

$$\mathscr{H}^e(A) := \sum_{i=1}^k \mathscr{H}^e(\sigma(A_i)).$$

LEMMA 5.5. *If A is as in the above definition, then $\mathscr{H}^e(A)$ does not depend on the choice of the basic e-rectifiable partition $(A_0, \ldots, A_k)$.*

PROOF. It suffices to prove the following: if C is a basic e-rectifiable set and $(A_0, \ldots, A_k)$ is a basic e-rectifiable partition of C, then $\mathscr{H}^e(C) = \sum_{i=1}^k \mathscr{H}^e(A_i)$, where $\mathscr{H}^e(C)$ and $\mathscr{H}^e(A_i)$ are defined using 5.2. For every $i = 1, \ldots, n$ let U and V_i be M-cells, $f : U \to K^{n-e}$ and $g_i : V_i \to K^{n-e}$ be definable functions with finite derivative, $\sigma_i \in O_n(K)$, $F : K^e \to K^n$ defined by $F(x) := (x, f(x))$, and $G_i : K^e \to K^n$ defined by $G(x) = \sigma_i(x, g_i(x))$ such that $C = F(U)$ and

$A_i = G_i(V_i)$. Define $U_i := F^{-1}(A_i) \subseteq U$, and $H_i := G_i^{-1} \circ F : U_i \to V_i$. Notice that each H_i is a bi-Lipschitz bijection, that U is the disjoint union of the U_i, and that $\dim(U_0) < e$. Hence,

$$\mathcal{H}^e(C) = \int_U J_e F \, d\mathcal{L}^e = \sum_{i=1}^n \int_{U_i} J_e F \, d\mathcal{L}^e = \sum_{i=1}^n \int_{U_i} J_e(G_i \circ H_i) \, d\mathcal{L}^e =$$

$$\sum_{i=1}^n \int_{U_i} (J_e(G_i) \circ H_i) \cdot |\det(\mathrm{D}H_i)| \, d\mathcal{L}^e = \sum_{i=1}^n \int_{V_i} J_e G_i \, d\mathcal{L}^e = \sum_{i=1}^n \mathcal{H}^e(A_i)$$

where we used Lemma 2.10, the fact that each σ_i is a linear function with determinant ± 1, and that $J_e(G \circ H) = (J_e(G) \circ H) \cdot |\det(\mathrm{D}H)|$.   ⊣

LEMMA 5.6. $\mathcal{H}^e$ *does not depend on* n. *That is, let* $m \geq n$, *and* $A \subset \mathring{K}^n$ *definable, and* $\psi : K^n \to K^m$ *be the embedding* $x \mapsto (x, 0)$. *Then,* $\mathcal{H}^e(A) = \mathcal{H}^e(\psi(A))$.

PROOF. Obvious from the definition and Lemma 5.5.   ⊣

Notice that $\mathcal{H}^0(C)$ is the cardinality of C.

It is clear that $\mathcal{H}^e$ can be extended to the σ-ring generated by the definable subsets of K^n of finite diameter and dimension at most e; we will also denote the completion of this extension by $\mathcal{H}^e$.

LEMMA 5.7. $\mathcal{H}^e$ *is a measure on the* σ-*ring generated by the definable subsets of* K^n *of bounded diameter and dimension at most* e.

PROOF. Since K is $\aleph_1$-saturated, it suffices to show that, for every A and B disjoint definable subsets of K^n of finite diameter and dimension at most e, $\mathcal{H}^e(A \cup B) = \mathcal{H}^e(A) + \mathcal{H}^e(B)$. But this follows immediately from Lemma 5.5.   ⊣

EXAMPLE 5.8. In Lemma 5.3, the assumption that C is basic e-rectifiable is necessary. For instance, take $\varepsilon > 0$ infinitesimal, and X be the following subset of K^2

$$X := ([0, 1] \times \{0\}) \cup \{\langle x, y \rangle : 0 \leq x \leq 1 \ \& \ y = \varepsilon x\}.$$

Then, $\mathrm{st}(X) = [0, 1] \times \{0\}$, and thus $\mathcal{H}^1(X) = 2$, while $\mathcal{H}^1_{\mathbb{R}}(\mathrm{st}(X)) = 1$. This is the source of complication in the theory, and one of the reasons why we had to wait until this section to introduce $\mathcal{H}^e$.

§6. Cauchy–Crofton formula. Give $e \leq n$, define

$$\beta := \Gamma\left(\tfrac{e+1}{2}\right)\Gamma\left(\tfrac{n-e+1}{2}\right)\Gamma\left(\tfrac{n+1}{2}\right)^{-1}\pi^{-1/2}.$$

DEFINITION 6.1. Let $\mathscr{AG}_e(K^n)$ be the Grassmannian of affine e-dimensional subspaces of K^n and let $\mathscr{AG}_e(\mathbb{R}^n)$ be the Grassmannian of affine e-dimensional subspaces of $\mathbb{R}^n$. Fix an embedding of $\mathscr{AG}_e(\mathbb{R}^n)$ into some $\mathbb{R}^m$, such that $\mathscr{AG}_e(\mathbb{R}^n)$ is a $\emptyset$-semialgebraic closed submanifold of $\mathbb{R}^m$, and the restriction to $\mathscr{AG}_e(\mathbb{R}^n)$ of the $\dim(\mathscr{AG}_e(\mathbb{R}^n))$-dimensional Hausdorff measure coincides with the Haar measure on $\mathscr{AG}_e(\mathbb{R}^n)$.

DEFINITION 6.2. Given $A \subseteq K^n$ and $E \in \mathscr{AG}_{n-e}(K^n)$, we define $f_A(E) := \#(A \cap E)$.

THEOREM 6.3 (Cauchy–Crofton Formula). *Let $A \subseteq \mathring{K}^n$ be definable of dimension e. Then*,

$$\mathscr{H}^e(A) = \frac{1}{\beta} \int\limits_{\mathscr{AG}_{n-e}(K^n)} f_A \, \mathrm{d}\mathscr{L}^{\mathscr{AG}_{n-e}(K^n)}.$$

We prove the theorem by reducing it to the known case of $K = \mathbb{R}$. This is done by showing that $\#(A \cap E)$ equals $\#(\mathrm{st}\, A \cap \mathrm{st}\, E)$ almost everywhere.

DEFINITION 6.4. Let $f : U \to \mathring{K}^m$ be definable, with $U \subset \mathring{K}^n$ open. Let $E \subset \mathbb{R}^n$ and $\overline{f}$ be as in Lemma 2.11. We say that $b \in \mathbb{R}^n$ is an S-regular point of $\overline{f}$ if

i) $b \in \mathrm{st}(U) \setminus \overline{E}$;
ii) b is a regular point of $\overline{f}$.

Otherwise, we say that b is an S-singular point and $\overline{f}(b)$ is an S-singular value of $\overline{f}$. If $c \in \mathbb{R}^m$ is not an S-singular value, we say that c is an S-regular value of $\overline{f}$.

REMARK 6.5. Let S be the set of S-regular points of $\overline{f}$. Then, S is open and definable in $\mathbb{R}_K$.

LEMMA 6.6 (Morse-Sard). *Assume that $m \geq n$. Then, the set of S-singular values of $\overline{f}$ is $\mathscr{L}^m_{\mathbb{R}}$-negligible*,

PROOF. By Lemma 2.11, E is negligible; since E is also $\mathbb{R}_K$-definable, it has empty interior and therefore $\dim(E) < n$. Since $m \geq n$, it follows that $\overline{f}(E)$ is negligible. The set of S-singular values of $\overline{f}$ is the union of $\overline{f}(E)$ and the set of singular values of $\overline{f}$; it is therefore negligible. ⊣

LEMMA 6.7 (Fixed point theorem). *Let X be a definable nonempty closed subset of K^n. Let $f : X \to X$ be a definable map. Assume that*:

i) *Either X is bounded, and, for every distinct x and y in X, $|f(x) - f(y)| < |x - y|$*;
ii) *Or there exists C in K such that $0 \leq C < 1$ and, for every x and y in X, $|f(x) - f(y)| \leq C|x - y|$.*

Then, f has a unique fixed point.

PROOF. Uniqueness of the fixed point of f is clear; thus, we have only to show existence.

If i), define $g : X \to K$, $g := |x - f(x)|$. Let $d := \min(g(X))$. If f has no fixed point, then $d > 0$. Let $x_1 \in X$ such that $g(x_1) = d$; define $x_2 := f(x_1)$. Then, $g(x_2) = |f(x_2) - x_2| < |x_2 - x_1| = d$, absurd.

If ii), pick any $x_1 \in X$; let $x_2 := f(x_1)$, $r_0 := |x_2 - x_1|$, $r := r_0/(1 - C)$, and $Y := \{x \in X : |x - x_0| \leq r\}$. Then, Y is definable, closed, and bounded, and $f(Y) \subseteq Y$. Hence, by i), f has a fixed point in Y. ⊣

Lemma 6.7 (with the same proof) holds in any K which is a definably complete expansion of a field [Miller01].

LEMMA 6.8 (Implicit Function). *Assume that $m = n$. Let $b \in \mathbb{R}^n$. If b is an S-regular point of $\overline{f}$ then, for every $y \in \mathrm{st}^{-1}(\overline{f}(b))$ there exists a unique $x \in \mathrm{st}^{-1}(b)$ such that $f(x) = y$.*

PROOF. Choose $x_0 \in \mathrm{st}^{-1}(b)$. Let $A := (\mathrm{D}f(x_0))^{-1}$. Since b is a regular point of $\overline{f}$, $\|A\|$ is finite. Thus we can choose $r, \rho \in \mathbb{Q}_{>0}$ such that $B := \overline{B(b; \rho)}$ is contained

in the set of S-regular points of $\overline{f}$, and

$$\|\mathrm{D}\overline{f}(b') - \mathrm{D}\overline{f}(b)\| < \frac{1}{2n\|A\|}, \text{for every } b' \in B$$
$$r \leq \frac{\rho}{2\|A\|}.$$

Moreover, we can pick ρ such that $B' := \overline{B(x_0; \rho)} \subset U$. Given $y \in K^n$ such that $|y - f(x_0)| < r$, consider the mapping

$$T_y : B' \to K^n$$
$$T_y(x) := x + A \cdot (y - f(x)).$$

T_y is definable and Lipschitz, with Lipschitz constant $1/2$. Furthermore, $T_y(B') \subset B'$. Therefore, for every $y \in B(f(x_0); r)$ there exists a unique $x \in B'$ such that $T_y(x) = x$. Thus, there is a unique $x \in B$ with $f(x) = y$. It remains to show that, given $y \in \mathrm{st}^{-1}(\overline{f}(b))$ and $x \in B'$ such that $f(x) = y$, we have $x \in \mathrm{st}^{-1}(b)$. We can verify that

$$\overline{T}_y : B \to B$$
$$\overline{T}_y(b') = b' + (\mathrm{D}\overline{f}(b))^{-1} \cdot (\overline{f}(b) - \overline{f}(b'))$$

is also a contraction, and therefore it has a unique fixed point, namely b. Since $\overline{T}_y(\mathrm{st}(x)) = \mathrm{st}(x)$, we must have $\mathrm{st}(x) = b$.   ⊣

REMARK 6.9. Let $U \subset \mathring{K}^m$. If $f : U \to \mathring{K}^n$ is definable and M-Lipschitz (for some finite M), $n \geq m$ and E is $\mathscr{L}_{\mathbb{R}}^m$-negligible, then the set $f(\mathrm{st}^{-1}(E))$ is $\mathscr{L}^n$-negligible.

PROOF. We can cover E with a polyrectangle Y whose measure is an arbitrarily small rational number λ and such that Y covers $\mathrm{st}^{-1}(E)$. Since $f(Y)$ has measure at most $CM^n\lambda$ (C depends only on m and n) the result follows.   ⊣

LEMMA 6.10. Let $A \subseteq \mathring{K}^n$ be a basic e-rectifiable set of dimension e. Consider $V := K^e$ as embedded in K^n via the map $x \mapsto \langle x, 0 \rangle$. Identify each $p \in V$ with the $(n-e)$-dimensional affine space which is orthogonal to V and intersects V in p. Then, for almost every $p \in V$, we have $\#(p \cap A) = \#(\mathrm{st}(p) \cap \mathrm{st}(A))$.

PROOF. Let $\pi : K^n \to V$ be the orthogonal projection. Let $U \subset \mathring{K}^e$ be an open M-cell and $f : U \to K^{n-e}$ be a definable M-function (M finite) such that $A = \Gamma(f)$. Let $F(x) := \langle x, f(x) \rangle$. Let $h := \pi \circ F : U \to V$, and consider $\overline{h} : C \to \mathrm{st}(V)$, $C \subset \mathrm{st}(U)$ as in Lemma 2.11. For almost every $p \in V$, $\#(p \cap A) = \#(h^{-1}(p))$, and $\#(\mathrm{st}\,p \cap \mathrm{st}A) = \#(\overline{h}^{-1}(\mathrm{st}\,p))$ because $F : U \to A$ and $\overline{F} : C \to \mathrm{Im}(\overline{F})$ are bijections. Thus, it suffices to prove that, for almost every $p \in V$, $\#(h^{-1}(p)) = \#(\overline{h}^{-1}(\mathrm{st}\,p))$. Let E be as in Lemma 2.11. By Remark 6.9, $h(\mathrm{st}^{-1}(E))$ is $\mathscr{L}^e$-negligible. Let S be the set of S-singular values of $\overline{h}$, by Lemma 6.6, S is negligible.

Let $p \in V \setminus (\mathrm{st}^{-1}(S) \cup h(\mathrm{st}^{-1}(E)))$. Then for every x in $h^{-1}(p)$, $\mathrm{st}(x)$ is an S-regular point of $\overline{h}$, and therefore Lemma 6.8 implies that $\#(h^{-1}(p)) = \#(\overline{h}^{-1}(\mathrm{st}\,p))$.   ⊣

Notice that the above lemma does not hold if A is only definable, instead of basic e-rectifiable.

PROOF OF THEOREM 6.3. By Corollary 3.11, w.l.o.g. A is basic e-rectifiable. Let $B := \mathrm{st}(A)$, and $f_B(F) := \#(B \cap F)$, for every $F \in \mathscr{AG}_e(\mathbb{R}^n)$. By Lemma 6.10,

$$\int\limits_{\mathscr{AG}_{n-e}(K^n)} f_A \, \mathrm{d}\mathscr{L}^{\mathscr{AG}_{n-e}(K^n)} = \int\limits_{\mathscr{AG}_{n-e}(\mathbb{R}^n)} f_B \, \mathrm{d}\mathscr{L}^{\mathscr{AG}_{n-e}(\mathbb{R}^n)}.$$

By the usual Cauchy–Crofton formula [Morgan98, 3.16], the right-hand side in the above identity is equal to $\mathscr{H}^e_{\mathbb{R}}(B) = \mathscr{H}^e(A)$, where we applied Lemma 5.3. ⊣

§7. Further properties of Hausdorff measure and the Co-area formula.

THEOREM 7.1. *Let $e \leq n$ and $C \subseteq K^n$ be a bounded and definable set of dimension at most e.*

1. $\mathscr{H}^e$ *is invariant under isometries.*
2. *For every $r \in \mathring{K}$, $\mathscr{H}^e(rC) = \mathrm{st}(r)^e \mathscr{H}^e(C)$.*
3. *If C is $\emptyset$-semialgebraic, then $\mathscr{H}^e(C) = \mathscr{H}^e(C_{\mathbb{R}}) = \mathscr{H}^e(\mathrm{st}(C))$.*
4. *if $\dim(C) < e$, then $\mathscr{H}^e(C) = 0$; the converse is not true.*
5. $\mathscr{H}^e(C) < +\infty$.
6. *If $\big(C(r)\big)_{r \in K^d}$ is a definable family of bounded subsets of K^n, then there exists a natural number M such that $\mathscr{H}^n(C(r)) < M$ for every $r \in K^d$.*
7. *If K' is either an elementary extension or an o-minimal expansion of K, then $\mathscr{H}^e(C_{K'}) = \mathscr{H}^e(C)$.*
8. *If $n = e$, then $\mathscr{H}^e(C) = \mathscr{L}^n(C)$.*
9. *If C is a subset of an e-dimensional affine space E, then $\mathscr{H}^e(C) = \mathscr{L}^E(C)$.*

PROOF. (1) Use the Cauchy–Crofton formula.
(2), (4) and (7) Apply the definition of $\mathscr{H}^e$ and Lemma 5.5.
(3) Apply Corollary 3.11 to $C_{\mathbb{R}}$ and use Lemma 5.3.
(5) and (6) Apply the Cauchy–Crofton formula 6.3 and then use the same argument as in [Dries03, Proposition 4.1].
(8) Apply Lemma 5.3.
(9) Since $\mathscr{H}^e$ is invariant under isometries, w.l.o.g. E is the coordinate space K^e. By Lemma 5.6, the measure $\mathscr{H}^e$ inside K^n is equal to the measure $\mathscr{H}^e$ inside K^e, and the latter is equal to $\mathscr{L}^e$. The conclusion follows from Fact 2.1. ⊣

The following theorem is the adaption to o-minimal structures of the Co-area formula, a well-known generalization of Fubini's theorem. Let $D := [0, 1] \subset K$.

THEOREM 7.2 (Co-area Formula). *Let $A \subset D^m$ be definable, and $f : D^m \to D^n$ be a definable Lipschitz function, with $m \geq n$. Then, $J_n f$ is $\mathscr{L}^m_K$-integrable, and*

$$\int\limits_A J_n f \, \mathrm{d}\mathscr{L}^m = \int\limits_{D^n} \mathscr{H}^{m-n}(A \cap f^{-1}(y)) \, \mathrm{d}\mathscr{L}^n(y).$$

SKETCH OF PROOF. W.l.o.g., A is an open subset of D^m. By Lemma 6.6, w.l.o.g. all points of A are S-regular for $\overline{f}$. Apply the real co-area formula [Morgan98] to $g := \overline{f}$ and $B := \mathrm{st}(A)$, and obtain

$$\int\limits_A J_n f \, \mathrm{d}\mathscr{L}^m = \int\limits_B J_n g \, \mathrm{d}\mathscr{L}^m_{\mathbb{R}} = \int\limits_{D^n_{\mathbb{R}}} \mathscr{H}^{m-n}_{\mathbb{R}}(B \cap g^{-1}(z)) \, \mathrm{d}\mathscr{L}^n_{\mathbb{R}}(z).$$

By the Implicit Function Theorem and Lemma 5.3, for almost every $y \in D_{\mathbb{R}}^n$, we have

$$\mathscr{H}^{m-n}(A \cap f^{-1}(y)) = \mathscr{H}_{\mathbb{R}}^{m-n}(B \cap g^{-1}(\mathrm{st}\, y)). \qquad \dashv$$

REFERENCES

[BO04] A. Berarducci and M. Otero, *An additive measure in o-minimal expansions of fields*, **The Quarterly Journal of Mathematics**, vol. 55 (2004), no. 4, pp. 411–419.

[BP98] Y. Baisalov and B. Poizat, *Paires de structures o-minimales*, this Journal, vol. 63 (1998), no. 2, pp. 570–578.

[Dries98] L. van den Dries, **Tame topology and o-minimal structures**, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998.

[Dries03] ———, *Limit sets in o-minimal structures*, **O-minimal Structures, Proceedings of the RAAG Summer School Lisbon 2003** (M. Edmundo, D. Richardson, and A. Wilkie, editors), Lecture Notes in Real Algebraic and Analytic Geometry, Cuvillier Verlag, 2005.

[Halmos50] P. R. Halmos, **Measure Theory**, D. Van Nostrand Company, Inc., New York, 1950.

[K92] K. Kurdyka, *On a subanalytic stratification satisfying a Whitney property with exponent 1*, **Real algebraic geometry proceedings (Rennes, 1991)**, Lecture Notes in Mathematics, vol. 1524, Springer, Berlin, 1992, pp. 316–322.

[M09] J. Maříková, *The structure on the real field generated by the standard part map on an o-minimal expansion of a real closed field*, **Israel Journal of Mathematics**, vol. 171 (2009), pp. 175–195.

[Miller01] C. Miller, *Expansions of dense linear orders with the intermediate value property*, this Journal, vol. 66 (2001), no. 4, pp. 1783–1790.

[Morgan88] F. Morgan, **Geometric measure theory**, Academic Press Inc., 1988, An introduction to Federer's book by the same title.

[P08] W. Pawłucki, *Lipschitz cell decomposition in o-minimal structures. I*, **Illinois Journal of Mathematics**, vol. 52 (2008), no. 3, pp. 1045–1063.

[PW06] J. Pila and A. J. Wilkie, *The rational points of a definable set*, **Duke Mathematical Journal**, vol. 133 (2006), no. 3, pp. 591–616.

[VR06] E. Vasquez Rifo, **Geometric partitions of definable sets**, Ph.D. thesis, The University of Wisconsin-Madison, August 2006.

[W00] F. Warner, **Foundations of differentiable manifolds and lie groups**, Springer-Verlag, 2000.

INSTITUT FÜR MATHEMATISCHE LOGIK
WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER
EINSTEINSTR. 62, 48149 MÜNSTER, GERMANY
E-mail: antongiulio.fornasiero@googlemail.com

INSTITUTO DE MATEMÁTICA Y FÍSICA
UNIVERSIDAD DE TALCA
TALCA, CHILE
E-mail: evasquez@inst-mat.utalca.cl