Some minimum bias window functions.

Firenze, September 29, 1976

Some efficient window functions for minimum bias spectral estimation are defined, starting from a general quantum-mechanical analogy. Following this approach, some new interesting functions are in particular obtained. Efficiency comparisons regarding these and other known functions are reported.

1. - INTRODUCTION.

It is of increasing interest to define efficient window functions for optimal spectral estimation and digital filtering.

Many window functions are up to now available; recently interesting window functions were proposed [1] [2]. *In particular, through a quantum-mechanical analogy, efficient window functions were defined and proposed for minimum bias spectral estimation* [5].

In this paper efficiency comparisons are presented among different window functions and in particular the last ones above recalled, here presented in more general form.

2. - DERIVATION OF PAPOULIS-TYPE MINIMUM BIAS WINDOW FUNCTIONS.

It has been shown by Papoulis [1] *that the optimum minimum-bias window function (for spectral estimates)*

(1)

$$w_0(t) = \frac{1}{\pi}\left|\sin\pi\frac{t}{\tau}\right| + \left(1-\frac{|t|}{\tau}\right)\cos\pi\frac{t}{\tau} \qquad |t| \leqslant \tau$$

with Fourier transform $W_0(f)$ *(see Table I) can be determined by minimizing the bias integral*

$$\int_{-\infty}^{+\infty} f^2\, W(f)\, df \tag{2}$$

with the condition $W(f) \geqslant 0$ *for every* f *and*

$$w(0) = \int_{-\infty}^{+\infty} W(f)\, df = 1 \tag{3}$$

It should be remarked the close analogy which exists between this problem and the quantum-mechanical problem of the ground energy of a particle in a one-dimensional potential box [3]. *In this case the total energy of the particle in the quantum state described by the wave function* $\psi(x)$ *is proportional to*

$$\int_{-\infty}^{+\infty} p^2\, |X(p)|^2\, dp \tag{4}$$

where $X(p)$ *is the Fourier transform of* $\psi(x)$.

On this line by putting, according to relation (2) *and* (4),

$$|X(f)|^2 = W(f) \tag{5}$$

the probability density of the particle in the one-dimensional box in the momentum representation (we use the variables t *and* f *instead of* x *and* p *respectively) gives the window function* $W(f)$ *for spectral estimates. For a box extending from* $t = -\tau/2$ *to* $t = \tau/2$ *the wave function of the particle* $\psi(t)$ *vanishes for* $|t| \geqslant \tau/2$. *From relation* (5) *and the convolution theorem it follows that*

$$w(t) = \int_{-\frac{\tau}{2}+t}^{\tau/2} \psi(\theta)\,\psi(\theta - t)\, d\theta, \quad w(0) = 1 \tag{6}$$

where, according to the generalized Fejer-Riesz theorem [4], $w(t)$ *is different from zero for* $|t| < \tau$.

The minimum-bias Papoulis function $w_0(t)$ *is obtained from eq. (6) if the wave function* $\psi(t)$ *is the ground state eigenfunction*

$$\psi_0(t) = \sqrt{\frac{2}{\tau}}\cos\pi\frac{t}{\tau} \qquad |t| \leqslant \tau/2 \tag{7}$$

of the corresponding one-dimensional box Schroedinger problem.

This conclusion can be generalized by considering a particle in a potential $V_\lambda(t)$ *which reduces to the one-dimensional potential box as the parameter* λ

Estratto da « ALTA FREQUENZA », N. 8 vol. XLVI - 1977 - da pag. 364 — 188 E a pag. 193 E — 369

TABLE I. - *Window functions and their Fourier transforms (see fig. 3).*

Window functions *	Fourier transforms
$w_0(t) = \frac{1}{\pi}\left\vert\sin\frac{\pi t}{\tau}\right\vert + \left(1-\frac{\vert t\vert}{\tau}\right)\cos\frac{\pi t}{\tau}$	$W_0(f) = \frac{4\tau}{\pi^2}\;\frac{1+\cos 2\pi f\tau}{(4f^2\tau^2-1)^2}$
$w_{1/2}(t) = \frac{1}{3}\left[\left(1-\frac{\vert t\vert}{\tau}\right)\left(1+2\cos^2\frac{\pi t}{\tau}\right) + \frac{3}{\pi}\cos\frac{\pi t}{\tau}\left\vert\sin\frac{\pi t}{\tau}\right\vert\right]$	$W_{1/2}(f) = \frac{2\tau}{3\pi^2(f^2\tau^2-1)^2}\left(\frac{\sin\pi f\tau}{f\tau}\right)^2$
$w_1(t) = \frac{1}{5}\left[\left(1-\frac{\vert t\vert}{\tau}\right)\left(3+2\cos^2\frac{\pi t}{\tau}\right)\cos\frac{\pi t}{\tau} + \frac{1}{3\pi}\left(4+11\cos^2\frac{\pi t}{\tau}\right)\left\vert\sin\frac{\pi t}{\tau}\right\vert\right]$	$W_1(f) = \frac{1152\tau}{5\pi^2}\;\frac{1+\cos 2\pi f\tau}{(4f^2\tau^2-9)^2(4f^2\tau^2-1)^2}$
$w_A(t) = 1-6t^2/\tau^2+6\vert t\vert^3/\tau^3$ for $\vert t\vert \leqslant \tau/2$; $= 2(1-\vert t\vert/\tau)^3$ for $\vert t\vert \geqslant \tau/2$	$W_A(f) = \frac{12\tau}{\pi^4}\left(\frac{\sin\pi f\tau/2}{f\tau}\right)^4$
$w_B(t) = 1-5\left(\frac{t}{\tau}\right)^2 + 5\left\vert\frac{t}{\tau}\right\vert^3 - \left\vert\frac{t}{\tau}\right\vert^5$	$W_B(f) = \frac{15}{2}\;\frac{\tau}{\pi^6}\;\frac{1}{(f\tau)^6}\;(\sin\pi f\tau - \pi f\tau\cos\pi f\tau)^2$
$u_0(t) = \cos\left(\frac{\pi}{2}\;\frac{t}{\tau}\right)$	$U_0(f) = \frac{4\tau}{\pi}\;\frac{\cos 2\pi f\tau}{1-16f^2\tau^2}$
$u_{1/2}(t) = \frac{1}{2}\left[1+\cos\frac{\pi t}{\tau}\right]$	$U_{1/2}(f) = \frac{\tau}{2\pi}\;\frac{\sin 2\pi f\tau}{f\tau}\;\frac{1}{1-4f^2\tau^2}$
$u_1(t) = \cos^3\left(\frac{\pi}{2}\;\frac{t}{\tau}\right)$	$U_1(f) = \frac{24\tau}{\pi}\;\frac{\cos 2\pi f\tau}{(16f^2\tau^2-9)(16f^2\tau^2-1)}$
$u_A(t) = 1-\vert t\vert/\tau$	$U_A(f) = \frac{\tau}{\pi^2}\left(\frac{\sin\pi f\tau}{f\tau}\right)^2$
$u_B(t) = 1-t^2/\tau^2$	$U_B(f) = \frac{1}{\pi^2\tau}\;\frac{1}{f^2}\left(\frac{\sin 2\pi f\tau}{2\pi f\tau} - \cos 2\pi f\tau\right)$

* All $w(t)$ and $u(t)$ functions are identically zero for $|t| > \tau$.

goes to a suitable value λ_0. *In this case the ground state eigenfunction (which is a function of* λ*) of the corresponding Schroedinger problem goes to the eigenfunction* $\psi_0(t)$. *The particular class of ground state eigenfunctions*

$$(8)\qquad \psi_\lambda(t) = A\left(\cos\pi\frac{t}{\tau}\right)^{2\lambda+1} \qquad |t| \leqslant \tau/2$$

has been proposed in a previous paper [5]. *In eq.* (8) $A = \left[\frac{\pi\,\Gamma(2\lambda+1)(2\lambda+1)}{\tau\,\Gamma(1/2)\,\Gamma(2\lambda+3/2)}\right]^{1/2}$ *is the normalization constant where* $\Gamma(x)$ *is the Eulero* Γ*-function.*

The function $\psi_\lambda(t)$ *corresponds to the potential*

$$(9)\qquad V_\lambda(t) = \frac{h^2\pi^2}{8m\tau^2}\left[(4\lambda+1)^2-1\right]\operatorname{tg}^2\left(\pi\frac{t}{\tau}\right) \quad |t| < \tau/2$$

where h is the Planck constant and m is the mass of the particle and $\psi_\lambda(t)$ *reduces to* $\psi_0(t)$ *as* λ *goes to zero. The general form of the window pairs* $w_\lambda(t) \leftrightarrow W_\lambda(f)$, *as obtained from eqs.* (8), (6) *and* (5) *is given in reference* [5]. *For* $\lambda = 0, 1/2, 1$ *the window functions* $w_0(t)$, $w_{1/2}(t)$, $w_1(t)$ *and their transforms* $W_0(f)$, $W_{1/2}(f)$, $W_1(f)$ *are reported in Table I. The Papoulis window function* ($\lambda = 0$) *corresponds to the more extended function in the time domain and, consequently, to the more concentrated function in the frequency domain.*

Returning to the ground state eigenfunction $\psi_0(t)$, *we observe that the non-stationary quantum mechanical states*

$$(10)\qquad \psi_A(t) = \sqrt{\frac{3}{\tau}}\left(\frac{1}{2}-\frac{|t|}{\tau}\right), \qquad \psi_B(t) = \sqrt{\frac{30}{\tau}}\left(\frac{1}{4}-\frac{t^2}{\tau^2}\right), \quad |t| < \tau/2$$

correspond to approximate $\psi_0(t)$ *by a triangular function and a semicircular function respectively. From eqs.* (6) *and* (5) *we obtain the window functions*

$w_A(t)$, $w_B(t)$ and their transforms $W_A(f)$, $W_B(f)$, which approximate, at some extent, the Papoulis pair $w_0(t) \leftrightarrow W_0(f)$. These functions are reported in Table I. Here we note only that the pair $w_A(t) \leftrightarrow W_A(f)$ is known as the Parzen window pair [6] and that the pair $w_B(t) \leftrightarrow W_B(f)$ is a very good approximation to the Papoulis pair as consequence of the fact that the state $\psi_B(t)$ closely approximates the ground state $\psi_0(t)$ [5].

As a further step we consider now the functions $\psi_\lambda(t)$, $\psi_A(t)$ and $\psi_B(t)$. With the substitution $t \to t/2$ we obtain respectively the new functions

$$(11)\qquad u_\lambda(t) = \left[\cos\left(\frac{\pi t}{2\tau}\right)\right]^{2\lambda+1} \qquad |t| < \tau$$

$$(12)\qquad u_A(t) = \left(1 - \frac{|t|}{\tau}\right);\ u_B(t) = 1 - \frac{t^2}{\tau^2}\ |t| < \tau$$

where we have normalized the u-functions so that $u(0) = 1$.

The so constructed u-functions can be considered as window functions. The functions u_0, $u_{1/2}$, u_1, u_A and u_B are reported in table I together with their Fourier transforms. It can be observed that $u_A(t)$ is the Fejer-Bartlett triangular window [7] and $u_{1/2}(t)$ is the Hanning-Tukey window [7]

$$(13)\qquad u_\beta(t) = \beta + (1-\beta)\cos\pi\frac{t}{\tau}$$

in the particular case $\beta = 1/2$.

We remark that the procedure of this paper, based on a quantum-mechanical analogy, while simple and intuitive, permits through easy rules to obtain the more important known window functions. Further, on this line, new window functions can be defined. In order to show the characteristics of these new classes w_λ and u_λ of window functions, we will go in the following in more details on the bias criteria for the selection of spectral windows.

3. - Efficiency comparison of the proposed spectral windows.

As is known [7] the efficiency of a window $w(t)$ can be estimated at a given frequency f by the bias term B_f defined as

$$(14)\qquad B_f = \int_{-\infty}^{\infty} S(f-f')\,W(f')\,df' - S(f)$$

where $S(f)$ is the true spectrum and $W(f)$ the Fourier transform of $w(t)$.

Palmer [8] has found an upper bound for the normalized bias $B_f/S(f)$ based on suitable models of the true spectrum $S(f)$ in regions of width Δf. The bound is given by $K A_t(\Delta f)$ in the transition region between peaks and valleys of the true spectrum and by $A_p(\Delta f) + K A_t(\Delta f)$ in the region of a prominent peak or valley, where K is a suitable finite real number and

$$(15)\qquad A_t(\Delta f) = 2\int_{\Delta f/2}^{\infty} |W(f)|\,df$$

$$(16)\qquad A_p(\Delta f) = \frac{4}{\Delta f^2}\int_0^{\Delta f/2} f^2\,W(f)\,df$$

Palmer has also given plots of $A_t(\Delta f)$ (dB) and $A_p(\Delta f)$ (dB) for some typical window functions, from which the well known 'Hanning' window (called cos 1 in Palmer's paper [8]) appears to be the best one on the basis of bias criteria. In fig. 1 and fig. 2 plots of $A_t(\Delta f)$ (dB) and $A_p(\Delta f)$ (dB) are shown for the most significant window functions listed in table I compared with the behaviour of Hanning window (cos 1).

The transition region parameter $A_t(\Delta f)$, as Palmer observes, has greater importance because it appears in the upper bound for both regions of interest of the true spectrum and because the differences among the peak/valley parameters $A_p(\Delta f)$ of different windows are considerably smaller than for $A_t(\Delta f)$. Therefore, according to Palmer's criterion, some of the functions listed in Table I may show better behaviour than cos 1, depending on the length Δf of the region of validity of the true spectrum model proposed by Palmer. At very small Δf one can see that $u_A(t)$ is the best one, while at large

Table II. - *Window functions and their properties.*

	Energy (1)	Bias coefficient (2)	Asymptotic attenuation (3)
$w_0(t)$	0.535 τ	$\frac{\pi^2}{2\tau^2}$	$\frac{1}{2\pi^2\tau^3 f^4}$
$w_{1/2}(t)$	0.418 τ	0.667 π^2/τ^2	$\frac{2}{3\pi^2\tau^5 f^6}$
$w_1(t)$	0.415 τ	0.900 π^2/τ^2	$\frac{9}{5\pi^2\tau^7 f^8}$
$w_A(t)$	0.539 τ	$\frac{0.6\,\pi^2}{\tau^2}$	$\frac{12}{\pi^4\tau^3 f^4}$
$w_B(t)$	0.602 τ	0.500 π^2/τ^2	$\frac{15}{2\pi^4\tau^3 f^4}$
$u_0(t)$	1.00 τ	undetermined	$\frac{1}{4\pi\tau f^2}$
$u_{1/2}(t)$ (cos 1) (Hanning)	0.750 τ	$\frac{\pi^2}{4\tau^2}$	$\frac{1}{8\pi\tau^2 f^3}$
$u_1(t)$	0.625 τ	0.375 π^2/τ^2	$\frac{3}{32\pi\tau^3 f^4}$
$u_A(t)$	0.667 τ	∞	$\frac{1}{\pi^2\tau f^2}$
$u_B(t)$	1.065 τ	undetermined	$\frac{1}{\pi^2\tau f^2}$

(1) Energy $E = \int_{-\infty}^{+\infty} W^2(f)\,df.$

(2) Bias coefficient $D = 2\pi^2 \int_{-\infty}^{+\infty} f^2\,W(f)\,df.$

(3) Asymptotic attenuation $A = W(f)$ for large f.

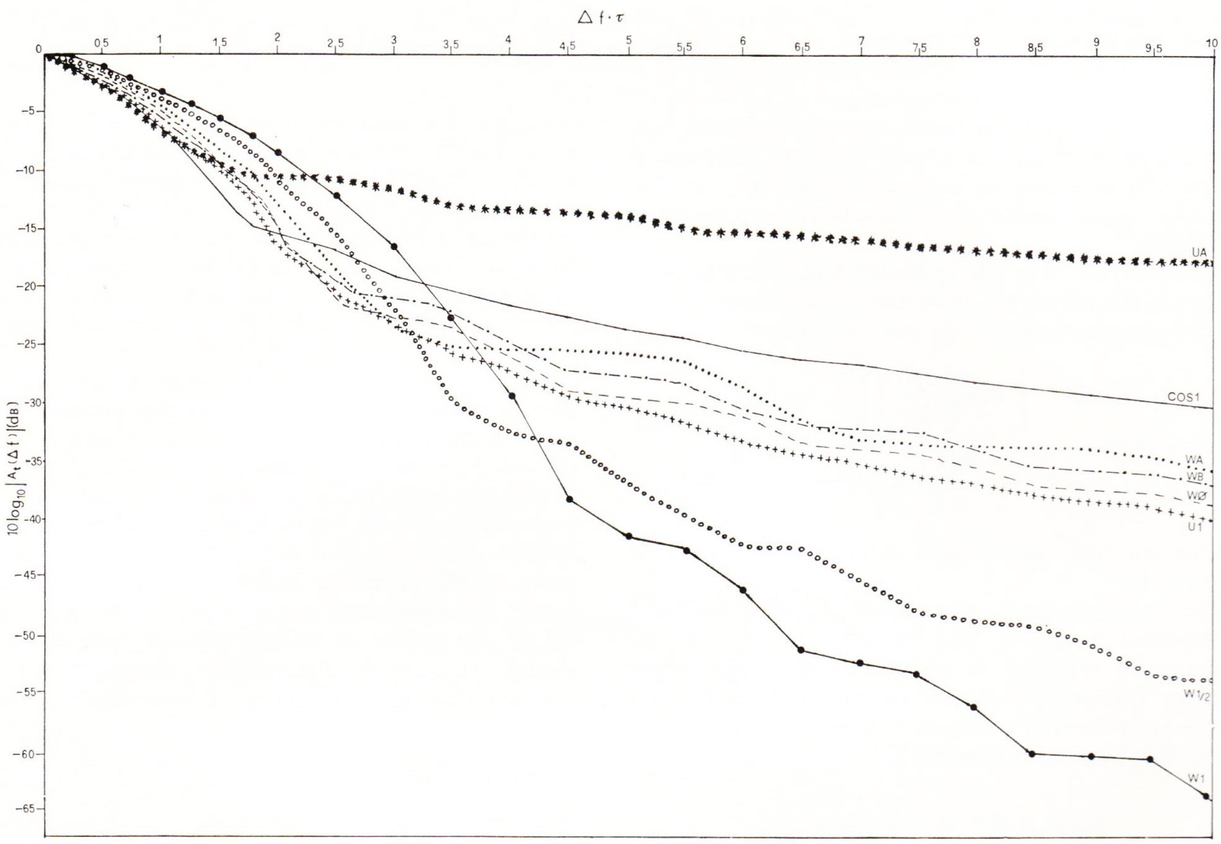

Fig. 1. - Transition region bias parameter $A_t\ (\Delta\ f)$ verus $\Delta\ f$ for some spectral window function of interest.

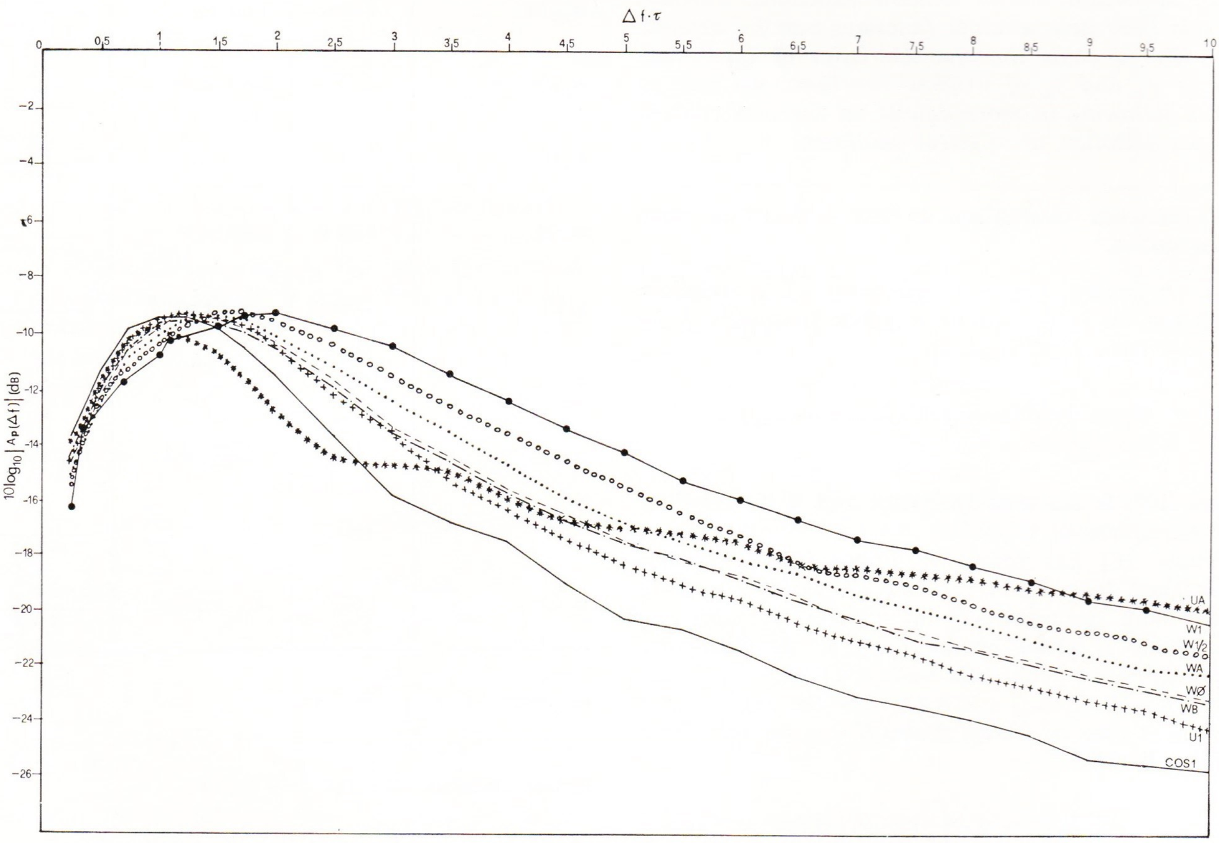

Fig. 2. - Peak/valley region bias parameter $A_p\ (\Delta\ f)$ versus $\Delta\ f$ for some spectral window functions of interest.

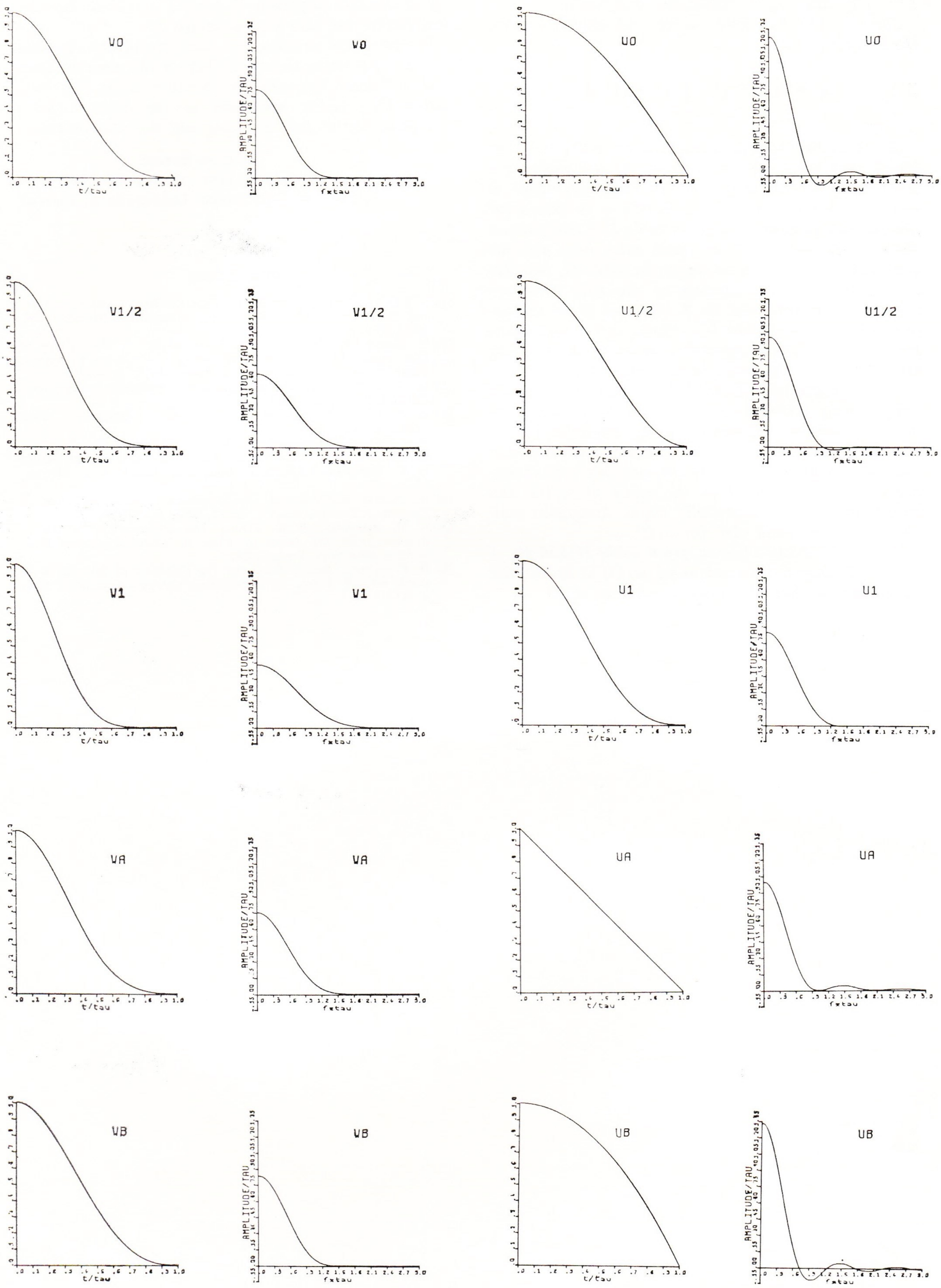

Fig. 3. - Plots of the window functions and their Fourier transforms of table I.

Δf $w_1(t)$ *and* $w_{1/2}(t)$ *result the best ones.*

Papoulis [1] *has given a general expression for the bias term* (14)

$$B_f \simeq 2\pi^2 S''(f) \int_{-\infty}^{\infty} f'^2 W(f')\, df' \tag{17}$$

($S''(f)$ *is the second derivative of* $S(f)$*) and has found that the window function* (1) *minimizes the integral in* (17)*.In table II the bias coefficient D — i.e. the coefficient of* $S''(f)$ *in* (17) *— is shown for the window functions listed in table I. A comparison among different windows must take into account also their energy E, also shown in table II, because the variance of the estimate of the true spectrum is directly proportional to E* [1], *and their asymptotic attenuation (table II), which is an important parameter for the suppression of interference from distant large components in the true spectrum.*

From the analysis of table II, we can observe that the Hanning window has the minimum bias coefficient; however, as remarked by Papoulis [1], *a reasonable comparison must take into account also the value of the energy E. If the energy of the Hanning window is reduced to the value of* $w_0(t)$, *the bias term becomes essentially equal. Analogous considerations are valid also for* $u_1(t)$.

Moreover we can observe, from table II and fig. 1 *and fig.* 2, *that the behaviour of* $w_B(t)$ *is essentially equivalent to that of Papoulis' window* $w_0(t)$.

The window functions of table I and their Fourier transforms are also plotted in fig. 3.

We conclude pointing out that the utility of these window functions is not limited to the area of spectral estimates but extends to other areas (filtering, antennas) and in particular to the finite impulse response digital filtering (one and two dimensional).

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