On the number of Waring decompositions for a generic polynomial vector

ON THE NUMBER OF WARING DECOMPOSITIONS FOR A GENERIC POLYNOMIAL VECTOR

ELENA ANGELINI, FRANCESCO GALUPPI, MASSIMILIANO MELLA AND GIORGIO OTTAVIANI

Abstract. We prove that a general polynomial vector (f_1, f_2, f_3) in three homogeneous variables of degrees $(3, 3, 4)$ has a unique Waring decomposition of rank 7. This is the first new case we are aware of, and likely the last one, after five examples known since the 19th century and the binary case. We prove that there are no identifiable cases among pairs (f_1, f_2) in three homogeneous variables of degree $(a, a + 1)$, unless $a = 2$, and we give a lower bound on the number of decompositions. The new example was discovered with Numerical Algebraic Geometry, while its proof needs Nonabelian Apolarity.

1. Introduction

Let f_1, f_2 be two general quadratic forms in $n + 1$ variables over $\mathbb{C}$. A well known theorem, which goes back to Jacobi and Weierstrass, says that f_1, f_2 can be simultaneously diagonalized. More precisely there exist linear forms $l_0, \ldots, l_n$ and scalars $\lambda_0, \ldots, \lambda_n$ such that

$$\begin{cases} f_1 &= \sum_{i=0}^{n} l_i^2 \\ f_2 &= \sum_{i=0}^{n} \lambda_i l_i^2 \end{cases} \tag{1.1}$$

An important feature is that the forms l_i are unique (up to order) and their equivalence class, up to multiplication by scalars, depends only on the pencil $\langle f_1, f_2\rangle$, hence also λ_i are uniquely determined after f_1, f_2 have been chosen in this order. The canonical form (1.1) allows us to write easily the basic invariants of the pencil, like the discriminant which takes the form $\prod_{i<j}(\lambda_i - \lambda_j)^2$. We call (1.1) a (simultaneous) Waring decomposition of the pair (f_1, f_2). The pencil (f_1, f_2) has a unique Waring decomposition with $n+1$ summands if and only if its discriminant does not vanish. In the tensor terminology, (f_1, f_2) is *generically identifiable*.

We generalize now the decomposition (1.1) to r general forms, even allowing different degrees. For symmetry reasons, it is convenient not to distinguish f_1 from the other f_j's, so we will allow scalars λ_i^j in the decomposition of each f_j, including f_1. To be precise, let $f = (f_1, \ldots, f_r)$ be a vector of general homogeneous forms of degree $a_1, \ldots, a_r$ in $n + 1$ variables over the complex field $\mathbb{C}$, i.e. $f_i \in Sym^{a_i}\mathbb{C}^{n+1}$ for all $i \in \{1, \ldots, r\}$. Let us assume that $2 \le a_1 \le \ldots \le a_r$.

Definition 1.1. A *Waring decomposition* of $f = (f_1, \ldots, f_r)$ is given by linear forms $\ell_1, \ldots, \ell_k \in \mathbb{P}((\mathbb{C}^{n+1})^\vee)$ and scalars $(\lambda_1^j, \ldots, \lambda_k^j) \in \mathbb{C}^k - \{\underline{0}\}$ with $j \in \{1, \ldots, r\}$ such that

$$f_j = \lambda_1^j \ell_1^{a_j} + \ldots + \lambda_k^j \ell_k^{a_j} \tag{1.2}$$

for all $j \in \{1, \ldots, r\}$ or, in vector notation,

$$f = \sum_{i=1}^{k} \left(\lambda_i^1 \ell_i^{a_1}, \ldots, \lambda_i^r \ell_i^{a_r}\right). \tag{1.3}$$

The geometric argument in §2.2 shows that every f has a Waring decomposition. We consider two Waring decompositions of f as in (1.3) being equal if they differ just by the order of the k summands. The *rank* of f is the minimum number k of summands appearing in (1.3). This definition coincides with the classical one in the case $r = 1$ (the vector f given by a single polynomial).

Due to the presence of the scalars λ_i^j, each form ℓ_i depends essentially only on n conditions. So the decomposition (1.2) may be thought of as a nonlinear system with $\sum_{i=1}^r \binom{a_i+n}{n}$ data (given by f_j) and $k(r+n)$ unknowns (given by kr scalars λ_i^j and k forms ℓ_i). This is a very classical subject, see for example [Re, Lon, Ro, Sco, Te2], although in most of classical papers the degrees a_i were assumed equal, with the notable exception of [Ro].

Definition 1.2. Let $a_1, \ldots, a_r, n$ be as above.
The space $\mathrm{S}ym^{a_1}\mathbb{C}^{n+1} \oplus \ldots \oplus \mathrm{S}ym^{a_r}\mathbb{C}^{n+1}$ is called *perfect* if there exists k such that

$$\sum_{i=1}^{r} \binom{a_i+n}{n} = k(r+n) \tag{1.4}$$

i.e. when (1.2) corresponds to a square polynomial system.

The arithmetic condition (1.4) means that $\sum_{i=1}^r \binom{a_i+n}{n}$ is divisible by $(r+n)$. In particular the number of summands k in the system (1.2) is uniquely determined.

The case with two quadratic forms described in (1.1) corresponds to $r = 2$, $a_1 = a_2 = 2$, $k = n+1$ and it is perfect. The perfect cases are important because, by the above dimensional count, we expect finitely many Waring decompositions for the generic polynomial vector in a perfect space $\mathrm{S}ym^{a_1}\mathbb{C}^{n+1} \oplus \ldots \oplus \mathrm{S}ym^{a_r}\mathbb{C}^{n+1}$.

It may happen that general elements in perfect spaces have no decompositions with the expected number k of summands. The first example, besides the one of a single plane conic, was found by Clebsch in the XIX century and regards ternary quartics, where $r = 1$, $a_1 = 4$ and $n = 2$. Equation (1.4) gives $k = 5$ but in this case the system (1.2) has no solutions and indeed 6 summands are needed to find a Waring decomposition of the general ternary quartic. It is well known that all the perfect cases with $r = 1$ when the system (1.2) has no solutions have been determined by Alexander and Hirschowitz, while more cases for $r \geq 2$ have been found in [CaCh], where a collection of classical and modern interesting examples is listed.

Still, perfectness is a necessary condition to have finitely many Waring decompositions. So two natural questions, of increasing difficulty, arise.

Question 1 Are there other perfect cases for $a_1, \ldots, a_r, n$, beyond (1.1), where a unique Waring decomposition (1.3) exists for generic f, namely where we have generic identifiability?

Question 2 What is the number of Waring decompositions (up to order of summands) for a generic f in any perfect case?

The above two questions are probably quite difficult, but we feel it is worthwhile to state them as guiding problems. Question 2 is open even in the case $r = 1$ of a single polynomial, while Question 1 has been recently solved in [GM] for $r = 1$ improving previous results in [Me1, Me2]. The birational technique used in these papers has been generalized to our setting in §5 of this paper. In the case $r = 1$, some numbers of decompositions for small a_1 and n have been computed (with high probability) in [HOOS] by homotopy continuation techniques, with the numerical software Bertini [Be].

In this paper we contribute to the above two questions. Before stating our conclusions, we need to review other known results on this topic.

In the case $n = 1$ (binary forms) there is a result by Ciliberto and Russo [CR] which completely answers our Question 1.

Theorem 1.3 (Ciliberto-Russo)**.** *Let $n = 1$. In all the perfect cases there is a unique Waring decomposition for generic $f \in Sym^{a_1}\mathbb{C}^2 \oplus \ldots \oplus Sym^{a_r}\mathbb{C}^2$ if and only if $a_1 + 1 \geq \frac{\sum_{i=1}^r (a_i+1)}{r+1}$. (Note the fraction $\frac{\sum_{i=1}^r (a_i+1)}{r+1}$ equals the number k of summands).*

We will provide an alternative proof of Theorem 1.3 by using Apolarity, see Theorem 3.4.

As widely expected, for $n > 1$ generic identifiability is quite a rare phenomenon. It has been extensively investigated in the XIX century and at the beginning of the XX century and the following are the only discovered cases that we are aware of:

$$(1.5) \qquad \begin{cases} (i) & (Sym^2\mathbb{C}^n)^{\oplus 2}, \text{rank } n, \text{Weierstrass [We], as in (1.1)}, \\ (ii) & Sym^5\mathbb{C}^3, \text{rank } 7, \text{Hilbert [Hi], see also [Ri] and [Pa]}, \\ (iii) & Sym^3\mathbb{C}^4, \text{rank } 5, \text{Sylvester Pentahedral Theorem [Sy]}, \\ (iv) & (Sym^2\mathbb{C}^3)^{\oplus 4}, \text{rank } 4, \\ (v) & Sym^2\mathbb{C}^3 \oplus Sym^3\mathbb{C}^3, \text{rank } 4, \text{Roberts [Ro]}. \end{cases}$$

The interest in Waring decompositions was revived by Mukai's work on 3-folds, [Mu1][Mu2]. Since then many authors have devoted their energy to understanding, interpreting and expanding the theory. Cases (ii) and (iii) in (1.5) were explained by Ranestad and Schreyer in [RS] by using syzygies, see also [MM] for an approach via projective geometry and [OO] for a vector bundle approach (called in this paper "Nonabelian Apolarity", see §3). Case (v) was reviewed in [OS] in the setting of Lueroth quartics. (iv) is a classical and "easy" result, there is a unique Waring decomposition of a general 4-tuple of ternary quadrics. There is a very nice geometric interpretation for this latter case. Four points in $\mathbb{P}^5$ define a $\mathbb{P}^3$ that cuts the Veronese surface in 4 points giving the required unique decomposition. See Remark 2.6 for a generalization to arbitrary (d, n).

Our main contribution with respect to unique decompositions is the following new case.

Theorem 1.4. *A general $f \in Sym^3\mathbb{C}^3 \oplus Sym^3\mathbb{C}^3 \oplus Sym^4\mathbb{C}^3$ has a unique Waring decomposition of rank 7, namely it is identifiable.*

The Theorem will be proved in the general setting of Theorem 3.4. Besides the new example found we think it is important to stress the way it arose. We adapted the methods in [HOOS] to our setting, by using the software Bertini [Be] and also the package *Numerical Algebraic Geometry* [KL] in Macaulay2 [M2], with the generous help of Jon Hauenstein and Anton Leykin, who assisted us in writing our first scripts. The computational analysis of perfect cases of forms on $\mathbb{C}^3$ suggested that for $Sym^3\mathbb{C}^3 \oplus Sym^3\mathbb{C}^3 \oplus Sym^4\mathbb{C}^3$ the Waring decomposition is unique. Then we proved it via Nonabelian Apolarity with the choice of a vector bundle. Another novelty of this paper is a unified proof of almost all cases with a unique Waring decomposition via Nonabelian Apolarity with the choice of a vector bundle E, see Theorem 3.4. Finally we borrowed a construction from [MM] to prove, see Theorem 3.7, that whenever we have uniqueness for rank k then the variety parametrizing Waring decompositions of higher rank is unirational.

For $r = 2$ and $n = 2$, the space $\mathrm{Sym}^a\mathbb{C}^3 \oplus \mathrm{Sym}^{a+1}\mathbb{C}^3$ is perfect if and only if $a = 2t$ is even. All the numerical computations we did suggested that identifiability holds only for $a = 2$ (by Robert's Theorem, see (1.5) (v)). Once again this pushed us to prove the non-uniqueness for these pencils of plane curves. Our main contribution to Question 2 regards this case and it is the following.

Theorem 1.5. *A general $f \in \mathrm{Sym}^a\mathbb{C}^3 \oplus \mathrm{Sym}^{a+1}\mathbb{C}^3$ is identifiable if and only if $a = 2$, corresponding to (v) in the list (1.5). Moreover f has finitely many Waring decompositions if and only if $a = 2t$ and in this case the number of decompositions is at least*

$$\frac{(3t-2)(t-1)}{2} + 1.$$

We know by equation (1.5)(v) that the bound is sharp for $t = 1$ and we verified with high probability, using [Be], that it is attained also for $t = 2$. On the other hand we do not expect it to be sharp in general. Theorem 1.5 is proved in section §5. The main idea, borrowed from [Me1], is to bound the number of decompositions with the degree of a tangential projection, see Theorem 5.2. To bound the latter we use a degeneration argument, see Lemma 5.4, that reduces the computation needed to an intersection calculation on the plane.

Acknowledgments. We thank all the participants of the seminar about Numerical Algebraic Geometry held among Bologna, Ferrara, Firenze and Siena in 2014-15, for fruitful and stimulating discussions. We benefited in particular from speaking with A. Bernardi, C. Bocci, A. Calabri, L. Chiantini. We thank J. Hauenstein and A. Leykin for their help with our first numerical computations. We are grateful to the referee for a careful reading. All the authors are members of GNSAGA-INDAM.

2. The Secant construction

2.1. **Secant Varieties.** Let us recall, next, the main definitions and results concerning secant varieties. Let $\mathbb{G}r_k = \mathbb{G}r(k, N)$ be the Grassmannian of k-linear spaces in $\mathbb{P}^N$. Let $X \subset \mathbb{P}^N$ be an irreducible variety of dimension n and let

$$\Gamma_{k+1}(X) \subset X \times \cdots \times X \times \mathbb{G}r_k,$$

be the closure of the graph of

$$\alpha : (X \times \cdots \times X) \setminus \Delta \to \mathbb{G}r_k,$$

taking $(x_0, \ldots, x_k)$ to the $[\langle x_0, \ldots, x_k\rangle]$, for a $(k + 1)$-tuple of distinct points. Observe that $\Gamma_{k+1}(X)$ is irreducible of dimension $(k + 1)n$. Let $\pi_2 : \Gamma_{k+1}(X) \to \mathbb{G}r_k$ be the natural projection. Denote by

$$S_{k+1}(X) := \pi_2(\Gamma_{k+1}(X)) \subset \mathbb{G}r_k.$$

Again $S_{k+1}(X)$ is irreducible of dimension $(k+1)n$. Finally let

$$I_{k+1} = \{(x, [\Lambda]) | x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}r_k,$$

with natural projections π_i onto the factors. Observe that $\pi_2 : I_{k+1} \to \mathbb{G}r_k$ is a $\mathbb{P}^k$-bundle on $\mathbb{G}r_k$.

Definition 2.1. Let $X \subset \mathbb{P}^N$ be an irreducible variety. The *abstract k-Secant variety* is

$$Sec_k(X) := \pi_2^{-1}(S_k(X)) \subset I_k$$

and the *k-Secant variety* is

$$\mathbb{S}\mathrm{ec}_k(X) := \pi_1(Sec_k(X)) \subset \mathbb{P}^N.$$

It is immediate that $Sec_k(X)$ is a $(kn+k-1)$-dimensional variety with a $\mathbb{P}^{k-1}$-bundle structure on $S_k(X)$. One says that X is *k-defective* if

$$\dim \mathbb{S}\mathrm{ec}_k(X) < \min\{\dim Sec_k(X), N\}$$

and calls *k-defect* the number

$$\delta_k = \min\{\dim Sec_k(X), N\} - \dim \mathbb{S}\mathrm{ec}_k(X).$$

Remark 2.2. Let us stress that in our definition $\mathbb{S}\mathrm{ec}_1(X) = X$. A simple but useful feature of the above definition is the following. Let Λ_1 and Λ_2 be two distinct k-secant $(k-1)$-linear spaces to $X \subset \mathbb{P}^N$. Let λ_1 and λ_2 be the corresponding projective $(k-1)$-spaces in $Sec_k(X)$. Then we have $\lambda_1 \cap \lambda_2 = \emptyset$.

Here is the main result we use about secant varieties.

Theorem 2.3 (Terracini Lemma [Te][ChCi]). *Let $X \subset \mathbb{P}^N$ be an irreducible, projective variety. If $p_1, \ldots, p_k \in X$ are general points and $z \in \langle p_1, \ldots, p_k\rangle$ is a general point, then the embedded tangent space at z is*

$$\mathbb{T}_z \mathbb{S}\mathrm{ec}_k(X) = \langle \mathbb{T}_{p_1} X, \ldots, \mathbb{T}_{p_k} X\rangle$$

If X is k-defective, then the general hyperplane H containing $\mathbb{T}_z \mathbb{S}\mathrm{ec}(X)$ is tangent to X along a variety $\Sigma(p_1, \ldots, p_k)$ of pure, positive dimension, containing $p_1, \ldots, p_k$.

2.2. **Secants to a projective bundle.** We show a geometric interpretation of the decomposition (1.2) by considering the k-secant variety to the projective bundle (see [Har, II, §7])

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(a_r)) \subset \mathbb{P}\left(H^0\left(\oplus_i \mathcal{O}_{\mathbb{P}^n}(a_i)\right)\right) = \mathbb{P}^N,$$

where $N = \sum_{i=1}^r \binom{a_i+n}{n} - 1$. We denote by $\pi\colon X \to \mathbb{P}^n$ the bundle projection. Note that $\dim X = r+n-1$ and the immersion in $\mathbb{P}^N$ corresponds to the canonical invertible sheaf $\mathcal{O}_X(1)$ constructed on X ([Har, II, §7]).
Indeed X is parametrized by $\left(\lambda^{(1)}\ell^{a_1}, \ldots, \lambda^{(r)}\ell^{a_r}\right) \in \oplus_{i=1}^r H^0\left(\mathcal{O}_{\mathbb{P}^n}(a_i)\right)$, where $\ell \in \mathbb{C}^{n+1}$ and $\lambda^{(i)}$ are scalars. X coincides with polynomial vectors of rank 1, as defined in the Introduction. It follows that the k-secant variety to X is parametrized by $\displaystyle\sum_{i=1}^k \left(\lambda_i^1 \ell_i^{a_1}, \ldots, \lambda_i^r \ell_i^{a_r}\right)$, where λ_i^j are scalars and $\ell_i \in (\mathbb{C}^{n+1})^\vee$. In the case $a_i = i$ for $i = 1, \ldots, d$, this construction appears already in [CQU]. Since X is not contained in a hyperplane, it follows that any polynomial vector has a Waring decomposition as in (1.3).

Thus, the number of decompositions by means of k linear forms of $f_1, \ldots, f_r$ is equal to the *k-secant degree* of X.
If $a_i = a$ for all $i \in \{1, \ldots, r\}$, then we deal with $\mathbb{P}^{r-1} \times \mathbb{P}^n$ embedded through the Segre-Veronese map with $\mathcal{O}(1, a)$, as we can see in Proposition 1.3 of [DF] or in [BBCC].
Moreover, we remark that perfectness in the sense of Definition 1.2 is equivalent to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^n}(a_r))$ being a *perfect* variety, i.e. $(n+r)|N$.
Theorem 1.3 has the following reformulation (compare with Claim 5.3 and Proposition 1.14 of [CR]):

Corollary 2.4. *If (1.4) and $a_1 + 1 \geq k$ hold, then $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r))$ is k-identifiable, i.e. its k-secant degree is equal to* 1.

Remark 2.5. A formula for the dimension of the k-secant variety of the rational normal scroll X for $n = 1$ has been given in [CaJo, pag. 359] (with a sign mistake, corrected in [CR, Prop. 1.14]).

Remark 2.6. We may consider the Veronese variety $V_{d,n} \subset \mathbb{P}^{\binom{d+n}{n}-1}$. Let $s - 1 = \operatorname{codim} V_{d,n}$. Then s general points determine a unique $\mathbb{P}^{s-1}$ that intersects $V_{d,n}$ in d^n points. The d^n points are linearly independent only if $d^n = s$, that is, either $n = 1$ or $d = n = 2$. This shows that a general vector $f = (f_1, \ldots, f_s)$ of forms of degree d admits $\binom{d^n}{s}$ decompositions, see the table at the end of §4 for some numerical examples. On the other hand, from a different perspective, dropping the requirement that the linear forms giving the decompositions are linearly independent, this shows that there is a unique set of d^n linear forms that decompose the general vector f and span a linear space of dimension $(s-1)$. Note that this time only the forms and not the coefficients are uniquely determined. We will not dwell on this point of view here and leave it for a forthcoming paper.

3. Nonabelian Apolarity and Identifiability

Let V be a complex vector space of dimension $n+1$ and let $f \in Sym^d V$. For any $e \in \mathbb{Z}$, Sylvester constructed the catalecticant map $C_f \colon Sym^e V^* \to Sym^{d-e} V$ which is the contraction by f. Its main property is the inequality $\mathrm{rk}\ C_f \leq \mathrm{rk}\ f$, where the rank on the left-hand side is the rank of a linear map, while the rank on the right-hand side has been defined in the Introduction. In particular the $(k+1)$-minors of C_f vanish on the variety of polynomials with rank bounded by k, which is $\mathbb{S}\mathrm{ec}_k(V_{d,n})$.

The catalecticant map behaves well with polynomial vectors. If $f \in \oplus_{i=1}^r Sym^{a_i} V$, for any $e \in \mathbb{Z}$ we define the catalecticant map $C_f \colon Sym^e V^* \to \oplus_{i=1}^r Sym^{a_i - e} V$ which is again the contraction by f. If f has rank one, this means that there exists $\ell \in V$ and scalars $\lambda^{(i)}$ such that $f = \left(\lambda^{(1)} \ell^{a_1}, \ldots, \lambda^{(r)} \ell^{a_r}\right)$. It follows that $\mathrm{rk}\ C_f \leq 1$, since the image of C_f is generated by $\left(\lambda^{(1)} \ell^{a_1 - e}, \ldots, \lambda^{(r)} \ell^{a_r - e}\right)$, up to factorial factors, which is zero if and only if $a_r < e$. Linearity implies the basic inequality

$$\mathrm{rk}\ C_f \leq \mathrm{rk}\ f.$$

Again the $(k+1)$-minors of C_f vanish on the variety of polynomial vectors with rank bounded by k, which is $\mathbb{S}\mathrm{ec}_k(X)$, where X is the projective bundle defined in §2.2.

A classical example is the following. Assume $V = \mathbb{C}^3$. London showed in [Lon](see also [Sco]) that a pencil of ternary cubics $f = (f_1, f_2) \in Sym^3 V \oplus Sym^3 V$

has *border rank* ≤ 5 (the border rank of f is the smallest number k such that f is in the Zariski closure of the set of polynomial vectors in $Sym^3V \oplus Sym^3V$ of rank k) if and only if $\det C_f = 0$ where $C_f \colon Sym^2V^* \to V \oplus V$ (see [CaCh, Remark 4.2] for a modern reference). Indeed $\det C_f$ is the equation of $\mathbb{S}\mathrm{ec}_5(X)$ where X is the Segre-Veronese variety $\left(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_X(1,3)\right)$. Note that X is 5-defective according to Definition 2.1 and this phenomenon is pretty similar to the case of Clebsch quartics recalled in the introduction.

The following result goes back to Sylvester.

Proposition 3.1 (Classical Apolarity)**.** *Let $f = \sum_{i=1}^k l_i^d \in Sym^dV$, let $Z = \{l_1, \ldots, l_k\} \subset V$. Let $C_f \colon \mathrm{Sym}^eV^* \to \mathrm{Sym}^{d-e}V$ be the contraction by f. Assume the rank of C_f equals k. Then*

$$\mathrm{Bs}\ker\left(C_f\right) \supseteq Z.$$

Proof. The Apolarity Lemma (see [RS]) says that $I_Z \subset f^{\perp}$, which reads in degree e as $H^0(I_Z(e)) \subset \ker C_f$. Look at the subspaces in this inclusion as subspaces of $H^0(\mathbb{P}^n, \mathcal{O}(e))$. The assumption on the rank implies that (compare with the proof of [OO, Prop. 4.3])

$$\mathrm{codim}\, H^0(I_Z(e)) \leq k = \mathrm{r}k\ C_f = \mathrm{codim}\ker C_f,$$

hence we have the equality $H^0(I_Z(e)) = \ker C_f$. It follows that

$$\mathrm{Bs}\ker\left(C_f\right) = \mathrm{Bs}\, H^0(I_Z(e)) \supseteq Z.$$

□

Classical Apolarity is a powerful tool to recover Z from f, hence it is a powerful tool to write down a minimal Waring decomposition of f.

The following Proposition 3.2 is a further generalization and it reduces to classical apolarity when $(X, L) = (\mathbb{P}V, \mathcal{O}(d))$ and $E = \mathcal{O}(e)$ is a line bundle. The vector bundle E may have larger rank which explains the name of Nonabelian Apolarity.

We recall that the natural map $H^0(E) \otimes H^0(E^* \otimes L) \to H^0(L)$ induces the linear map $H^0(E) \otimes H^0(L)^* \to H^0(E^* \otimes L)^*$, then for any $f \in H^0(L)^*$ we have the contraction map $A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$.

Proposition 3.2 (Nonabelian Apolarity)**.** [OO, Prop. 4.3] *Let X be a variety, $L \in Pic(X)$ a very ample line bundle which gives the embedding $X \subset \mathbb{P}\left(H^0(X, L)^*\right) = \mathbb{P}W$. Let E be a vector bundle on X. Let $f = \sum_{i=1}^k w_i \in W$ with $z_i = [w_i] \in X \subset \mathbb{P}W$, let $Z = \{z_1, \ldots, z_k\} \subset \mathbb{P}W$ and let $A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$ be the induced map. Assume that* $\mathrm{rk}A_f = k \cdot \mathrm{rk}E$. *Then* $\mathrm{Bs}(\ker A_f) \supseteq Z$.

In all the cases that we apply the Proposition, we will compute separately $\mathrm{rk}A_f$.

Nonabelian Apolarity enhances the power of Classical Apolarity and may detect a minimal Waring decomposition of a polynomial in some cases when Classical Apolarity fails, see Proposition 3.3. Our main examples start with the quotient bundle Q on $\mathbb{P}^n = \mathbb{P}(V)$. It has rank n and it is defined by the Euler exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \otimes V^* \longrightarrow Q \longrightarrow 0.$$

Let $L = \mathcal{O}(d)$ and $E = Q(e)$. Any $f \in \mathrm{Sym}^dV$ induces the contraction map

$$A_f \colon H^0(Q(e)) \to H^0(Q^*(d-e))^* \simeq H^0(Q(d-e-1))^*. \tag{3.1}$$

The following was the argument used in [OO] to prove cases (*ii*) and (*iii*) of (1.5).

Proposition 3.3. *Let X be a variety, $L \in Pic(X)$ a very ample line bundle and $k = \frac{h^0(X,L)}{\dim X+1}$. Let $[f]$ be a point in $\mathbb{P}(H^0(L)^*)$, E a vector bundle on X with $\mathrm{rk}E = \dim X$, $c_{\mathrm{rkE}}(E) = k$ and $A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$ the contraction map. Assume that for general f, $\mathrm{rk}A_f = k \cdot \mathrm{rk}E$, and there is some f such that the base locus of $\ker A_f$ is given by k points. Then the k-secant map*

$$\pi_k : Sec_k(X) \to \mathbb{P}(H^0(L)^*)$$

is birational. The assumptions are verified in the following cases, corresponding to (ii) and (iii) of (1.5).

(X, L)	$H^0(L)$	*rank*	E
$(\mathbb{P}^2, \mathcal{O}(5))$	$Sym^5\mathbb{C}^3$	7	$Q_{\mathbb{P}^2}(2)$
$(\mathbb{P}^3, \mathcal{O}(3))$	$Sym^3\mathbb{C}^4$	5	$Q^*_{\mathbb{P}^3}(2)$

Specific f's in the statement may be found as random polynomials in [M2]. In order to prove also cases (*iv*) and (*v*) of (1.5) and moreover our Theorem 1.4 we need to extend this result as follows.

Theorem 3.4. *Let $X \xrightarrow{\pi} Y$ be a projective bundle, $L = \mathcal{O}_X(1)$ as in §2.2 which we assume to be very ample and embeds the fibers of π as linear spaces. Let F be a vector bundle on Y and let $E = \pi^*F$. Let $[f]$ be a point in $\mathbb{P}(H^0(L)^*)$, $k = \frac{h^0(X,L)}{\dim X+1}$ and $A_f \colon H^0(E) \to H^0(E^* \otimes L)^*$ the contraction map. Let $a = \frac{\dim Y}{\mathrm{rk}F}$ be an integer and suppose that $(c_{\mathrm{rk}F}F)^a = k$. Assume that X is not k-defective and, for general f, $\mathrm{rk}A_f = k \cdot \mathrm{rk}E$ and there is some f such that the base locus of $\ker A_f$ is given by k fibers of π. Then the k-secant map*

$$\pi_k : Sec_k(X) \to \mathbb{P}(H^0(L)^*)$$

is birational. The assumptions are verified in the following cases.

X	$H^0(L)$	*rank* $= k$	F	a
$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(2)\right)$	$\left(\mathrm{Sym}^2\mathbb{C}^{n+1}\right)^{\oplus 2}$	$n+1$	$Q_{\mathbb{P}^n}(1)$	1
$\left\{ \begin{array}{l} \mathbb{P}\left(\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)\right) \\ \textit{if } k \le a_1+1 \end{array}\right.$	$\oplus_{i=1}^r \mathrm{Sym}^{a_i}\mathbb{C}^2$	$\frac{\sum_{i=1}^r (a_i+1)}{r+1}$	$\mathcal{O}_{\mathbb{P}^1}(k)$	1
$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2}(2)^4\right)$	$(Sym^2\mathbb{C}^3)^{\oplus 4}$	4	$\mathcal{O}_{\mathbb{P}^2}(2)$	2
$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)\right)$	$\mathrm{Sym}^2\mathbb{C}^3 \oplus Sym^3\mathbb{C}^3$	4	$\mathcal{O}_{\mathbb{P}^2}(2)$	2
$\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2}(3)^2 \oplus \mathcal{O}_{\mathbb{P}^2}(4)\right)$	$\left(Sym^3\mathbb{C}^3\right)^{\oplus 2} \oplus \mathrm{Sym}^4\mathbb{C}^3$	7	$Q_{\mathbb{P}^2}(2)$	1

Proof. Let Z be as in Proposition 3.2. We get $Z \subset \mathrm{Bs}(\ker A_f)$, where the base locus can be found by the common zero locus of some sections $s_1, \ldots, s_a$ of E which span $\ker A_f$. Since $E = \pi^*F$ and $H^0(X,E)$ is naturally isomorphic to $H^0(Y,F)$, the zero locus of each section of E corresponds to the pullback through π of the zero locus of the corresponding section of F. By the assumption on the top Chern class of F we expect that the base locus of $\ker A_f$ contains $k = \text{length } (Z)$ fibers of the projective bundle X. The hypothesis guarantees that this expectation is realized for a specific polynomial vector f. By semicontinuity, it is realized for the generic f. This determines the forms l_i in (1.3) for a generic polynomial vector f. It follows that f is in the linear span of the fibers $\pi^{-1}(\pi(l_i))$ where $Z = \{l_1, \ldots, l_k\}$. Fix representatives for the forms l_i for $i = 1, \ldots, k$. Now the scalars λ_i^j in (1.3) are found by solving a linear system. By assumption we have that X is not k-defective (note that this assumption is satisfied in the setting of Proposition 3.3, since otherwise the base locus of $\ker A_f$ should be positive dimensional). In particular the tangent spaces at points in Z, which are general, are independent by the Terracini Lemma. Since each π-fiber is contained in the corresponding tangent space, it follows that the fibers $\pi^{-1}(l_i)$ corresponding to $l_i \in Z$ are independent. It follows that the scalars λ_i^j in (1.3) are uniquely determined and we have generic identifiability. The check that the assumptions are verified in the cases listed has been perfomed with random polynomials with the aid of Macaulay2 package [M2]. □

Remark 3.5. In all the cases listed in Theorem 3.4, by the projection formula we have the natural isomorphism $H^0(X, E^* \otimes L) \simeq H^0(Y, F \otimes \pi_* L)$.

Note that the second case in the list of Theorem 3.4 corresponds to Theorem 1.3 of Ciliberto-Russo. In this case $H^0(E) = \mathrm{Sym}^k\mathbb{C}^2$ has dimension $k+1$, $H^0(E^* \otimes L) = \mathrm{Sym}^{a_1-k}\mathbb{C}^2 \oplus \ldots \oplus \mathrm{Sym}^{a_r-k}\mathbb{C}^2$ has dimension $\sum_{i=1}^r (a_i - k + 1) = k$ (if $k \leq a_1 + 1$) and the contraction map A_f has rank k, with one-dimensional kernel.

The last case in the list of Theorem 3.4 corresponds to Theorem 1.4. A general vector $f \in (\mathrm{Sym}^3\mathbb{C}^3)^{\oplus 2} \oplus \mathrm{Sym}^4\mathbb{C}^3$ induces the contraction $A_f \colon H^0(Q(2)) \to H^0(Q) \oplus H^0(Q) \oplus H^0(Q(1))$ with one-dimensional kernel. Each element in the kernel vanishes on 7 points which give the seven Waring summands of f.

Moreover, observe that $\left(\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2}(2)^4\right), \mathcal{O}_X(1)\right)$ coincides with the Segre-Veronese variety $(\mathbb{P}^3 \times \mathbb{P}^2, \mathcal{O}(1,2))$.

Remark 3.6. The assumption $a_1 + 1 \geq k$ in 1.3 is equivalent to $\frac{1}{r+1}\sum_{i=1}^r (a_i + 1) \leq a_1 + 1$ which means that the a_i are "balanced".

We conclude this section by showing how the existence of a unique decomposition determines the birational geometry of the varieties parametrizing higher rank decompositions. The following is just a slight generalization of [MM, Theorem 4.4].

Theorem 3.7. *Let $X \subset \mathbb{P}^N$ be such that the k-secant map $\pi_k : Sec_k(X) \to \mathbb{P}^N$ is birational. Assume that X is unirational. Then for $p \in \mathbb{P}^N$ general the variety $\pi_h^{-1}(p)$ is unirational for any h such that* $\operatorname{codim} X \geq h \geq k$*, in particular it is irreducible.*

Proof. Let $p \in \mathbb{P}^N$ be a general point. Then for $h > k$ we have $\dim \pi_h^{-1}(p) = h \dim X + h - 1 - N = (h-k)(\dim X + 1)$. Note that, for $q \in \mathbb{P}^N$ general, a general point $x \in \pi_h^{-1}(q)$ is uniquely associated to a set of h points $\{x_1, \ldots, x_h\} \subset X$ and a h-tuple $(\lambda_1, \ldots, \lambda_h) \in \mathbb{C}^h$ with the requirement that

$$q = \sum \lambda_i x_i.$$

Therefore the birationality of π_k allows us to associate, to a general point $q \in \mathbb{P}^N$, its unique decomposition in sum of k factors. That is $\pi_k^{-1}(q) = (q, [\Lambda_k(q)])$ for a general point $q \in \mathbb{P}^N$, where $[\Lambda_k(q)] \subset \mathbb{G}r_{k-1}$ is such that $q \in \Lambda_k(q)$ (see subsection 2.1). Via this identification we may define a map

$$\psi_h : (X \times \mathbb{P}^1)^{h-k} \dashrightarrow \pi_h^{-1}(p)$$

given by

$$(x_1, \lambda_1, \ldots, x_{h-k}, \lambda_{h-k}) \mapsto (p, [\langle x_1, \ldots, x_{h-k}, \Lambda_k(p - \lambda_1 x_1 - \ldots - \lambda_{h-k} x_{h-k})\rangle]).$$

When $\operatorname{codim} X > h$ by the Trisecant Lemma $\Lambda_h(q)$ intersects X in exactly h points. Hence the map ψ_h is generically finite, of degree $\binom{h}{k}$, and dominant. In a similar way, if $\operatorname{codim} X = h$ then ψ_h is generically finite, of degree $\binom{\deg X}{k}$. This is sufficient to show the claim. □

Theorem 3.7 applies to all decompositions that admit a unique form.

Corollary 3.8. *Let $f = (f_1, \ldots, f_r)$ be a vector of general homogeneous forms. Assume f has a unique Waring decomposition of rank k. If*

$$\binom{n+a_1}{n} + \cdots + \binom{n+a_r}{n} - r - n \geq h > k,$$

then the set of rank h decompositions of f is parametrized by a unirational variety.

Remark 3.9. Let's go back to our starting example (1.1) and specialize $f_1 = \sum_{i=0}^n x_i^2$ to the euclidean quadratic form. Then any minimal Waring decomposition of f_1 consists of $n+1$ orthogonal summands, with respect to the euclidean form. It follows that the decomposition (1.1) is equivalent to the diagonalization of f_2 with orthogonal summands. Over the reals, this is possible for any f_2 by the Spectral Theorem.

Also Robert's Theorem, see (v) of (1.5), has a similar interpretation. If $f_1 = x_0^2 + x_1^2 + x_2^2$ and $f_2 \in Sym^3\mathbb{C}^3$ is general, the unique Waring decomposition of the pair (f_1, f_2) consists of four representatives of lines $\{l_1, \ldots, l_4\}$ and scalars $\lambda_1, \ldots, \lambda_4$ such that

$$\begin{cases} f_1 = \sum_{i=1}^4 l_i^2 \\ f_2 = \sum_{i=1}^4 \lambda_i l_i^3 \end{cases} \tag{3.2}$$

Denote by L the 3×4 matrix whose i-th column is given by the coefficients of l_i. Then the first condition in (3.2) is equivalent to the equation

$$LL^t = I. \tag{3.3}$$

This equation generalizes orthonormal bases and the columns of L make a *Parseval* frame, according to [CMS] §2.1. So Robert's Theorem states that the general ternary cubic has a unique decomposition consisting of a Parseval frame.

In general a Parseval frame for a field F is given by $\{l_1, \ldots, l_n\} \subset F^d$ such that the corresponding $d \times n$ matrix L satisfies the condition $LL^t = I$. This is equivalent to the equation $\sum_{i=1}^n (\sum_{j=1}^d l_{ji} x_j)^2 = \sum_{i=1}^d x_i^2$, so again to a Waring decomposition with n summands of the euclidean form in F^d. This makes a connection between our paper and [ORS], which studies frames in the setting of secant varieties and tensor decomposition. For example equation (7) in [ORS] defines a solution to (3.3) with the additional condition that the four columns have unit norm. Note that equation (8) in [ORS] defines a Waring decomposition of the pair (f_1, T). Unfortunately the additional condition about unitary norm does not allow the results of [ORS] to be directly transferred to our setting, but we believe this connection deserves to be pushed further.

It is interesting to notice that the decompositions of moments M_2 and M_3 in [AGHKT, §3] are (simultaneous) Waring decompositions of the quadric M_2 and the cubic M_3.

4. Computational approach

In this section we describe how we can face Question 1 and Question 2, introduced in §1, from the point of view of computational analysis.

With the aid of Bertini [Be], [BHSW] and Macaulay2 [M2] software systems, we can construct algorithms, based on homotopy continuation techniques and monodromy loops, that, in the spirit of [HOOS], yield the number of Waring decompositions of a generic polynomial vector $f = (f_1, \dots, f_r) \in \mathrm{S}ym^{a_1}\mathbb{C}^{n+1} \oplus \dots \oplus \mathrm{S}ym^{a_r}\mathbb{C}^{n+1}$ with high probability. Precisely, given $n, r, a_1, \dots, a_r, k \in \mathbb{N}$ satisfying (1.4) and coordinates $x_0, \dots, x_n$, we focus on the polynomial system

$$\begin{cases} f_1 = \lambda_1^1 \ell_1^{a_1} + \dots + \lambda_k^1 \ell_k^{a_1} \\ \quad \vdots \\ f_r = \lambda_1^r \ell_1^{a_r} + \dots + \lambda_k^r \ell_k^{a_r} \end{cases} \tag{4.1}$$

where $f_j \in \mathrm{S}ym^{a_j}\mathbb{C}^{n+1}$ is a fixed general element, while $\ell_i = x_0 + \sum_{h=1}^n l_h^i x_h \in \mathbb{P}((\mathbb{C}^{n+1})^\vee)$ and $\lambda_i^j \in \mathbb{C}$ are unknown. By expanding the expressions on the right hand side of (4.1) and by applying the identity principle for polynomials, the j-th equation of (4.1) splits into $\binom{a_j+n}{n}$ conditions. Our aim is to compute the number of solutions of $F_{(f_1,\dots,f_r)}([l_1^1, \dots, l_n^1, \lambda_1^1, \dots, \lambda_1^r], \dots\dots, [l_1^k, \dots, l_n^k, \lambda_k^1, \dots, \lambda_k^r])$, the square non linear system of order $k(r+n)$, arising from the equivalent version of (4.1) in which in each equation all the terms are on one side of the equal sign. In practice, to work with general f_j's, we assign random complex values $\overline{l}_h^i$, $\overline{\lambda}_i^j$ to l_h^i, λ_i^j and, by means of $F_{(f_1,\dots,f_r)}$, we compute the corresponding $\overline{f}_1, \dots, \overline{f}_r$, the coefficients of which are so called *start parameters*. In this way, we know a solution $([\overline{l}_1^1, \dots, \overline{l}_n^1, \overline{\lambda}_1^1, \dots, \overline{\lambda}_1^r], \dots\dots, [\overline{\lambda}_1^k, \dots, \overline{l}_n^k, \overline{\lambda}_k^1, \dots, \overline{\lambda}_k^r]) \in \mathbb{C}^{k(r+n)}$ of $F_{(\overline{f}_1,\dots,\overline{f}_r)}$, i.e. a Waring decomposition of $\overline{f} = (\overline{f}_1, \dots, \overline{f}_r)$, which is called a *startpoint*. Then we consider F_1 and F_2, two square polynomial systems of order $k(n+r)$ obtained from $F_{(\overline{f}_1,\dots,\overline{f}_r)}$ by replacing the constant terms with random complex values. We therefore construct 3 segment homotopies

$$H_i : \mathbb{C}^{k(r+n)} \times [0,1] \to \mathbb{C}^{k(r+n)}$$

for $i \in \{0,1,2\}$: H_0 between $F_{(\overline{f}_1,\dots,\overline{f}_r)}$ and F_1, H_1 between F_1 and F_2, H_2 between F_2 and $F_{(\overline{f}_1,\dots,\overline{f}_r)}$. Through H_0, we get a *path* connecting the startpoint to a solution of F_1, called *endpoint*, which therefore becomes a startpoint for the second step given by H_1, and so on. At the end of this loop, we compare the output Waring decomposition with the starting one. If they are equal, this procedure suggests that the case under investigation is identifiable, otherwise we iterate this technique with these two *startingpoints*, and so on. If at a certain point, the number of solutions of $F_{(\overline{f}_1,\dots,\overline{f}_r)}$ stabilizes, then, with high probability, we know the number of Waring decompositions of a generic polynomial vector in $\mathrm{S}ym^{a_1}\mathbb{C}^{n+1} \oplus \dots \oplus \mathrm{S}ym^{a_r}\mathbb{C}^{n+1}$. We have implemented the homotopy continuation technique both in the software Bertini [Be], in conjunction with Matlab, and in the software Macaulay2, with the aid of the package *Numerical Algebraic Geometry* [KL].

Before starting with this computational analysis, we need to check that the variety $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(a_r))$, introduced in §2, is not k-defective, in which case (4.1) has no solutions. In order to do that, by using Macaulay2, we can construct a probabilistic algorithm based on Theorem 2.3, that computes the dimension of the span of the affine tangent spaces to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n}(a_1)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^n}(a_r))$ at k random points and then we can apply semicontinuity properties.

In the following table we summarize the results we are able to obtain by combining numerical and theoretical approaches. Our technique is as follows. We first apply the probabilistic algorithm, checking k-defectivity, described above. If this suggests positive k-defect δ_k, we do not pursue the computational approach. When δ_k is zero, we apply the homotopy continuation technique. If the number of decompositions (up to order of summands) stabilizes to a number, $\#_k$, we indicate it. If homotopy technique does not stabilize to a fixed number, we apply degeneration techniques like in §5 to get a lower bound. If everything fails, we put a question mark. Bold degrees are the ones obtained via theoretical arguments.

r	n	$(a_1,\ldots,a_r)$	k	δ_k	$\#_k$
2	2	$(4,5)$	9	0	3
2	2	$(6,6)$	14	0	≥ 2
2	2	$(6,7)$	16	0	≥ 8
2	3	$(2,4)$	9	2	
3	2	$(2,2,6)$	8	4	
3	2	$(3,3,4)$	7	0	**1**
3	2	$(3,4,4)$	8	0	4
3	2	$(5,5,6)$	14	0	205
3	3	$(3,3,3)$	10	0	56
4	2	$(2,2,4,4)$	7	2	
4	2	$(2,3,3,3)$	6	0	2
4	2	$(4,\ldots,4)$	10	0	?
5	2	$(5,\ldots,5,6)$	16	0	?
6	2	$(2,\ldots,2,3)$	5	3	
6	4	$(2.\ldots,2)$	9	0	45
7	3	$(2,\ldots,2)$	7	0	**8**
8	2	$(3,\ldots,3)$	8	0	**9**
8	2	$(2,\ldots,2,6)$	7	7	
11	4	$(2,\ldots,2)$	11	0	**4368**
13	2	$(4,\ldots,4)$	13	0	**560**
15	2	$(4,\ldots,4,6)$	14	6	
17	3	$(3,\ldots,3)$	17	0	**8436285**
19	2	$(5,\ldots,5)$	19	0	**177100**
26	2	$(6,\ldots,6)$	26	0	**254186856**

5. Identifiability of pairs of ternary forms

In this section we aim to study the identifiability of pairs of ternary forms. In particular we study the special case of two forms of degree a and $a+1$, focusing on

$$X=\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(a)\oplus\mathcal{O}_{\mathbb{P}^2}(a+1)).$$

Note that X can also be seen as a special linear section of $\mathrm{Seg}(\mathbb{P}^2\times\mathbb{P}^2,\mathcal{O}(a,a+1))$. Our main result is the following.

Theorem 5.1. *Let a be an integer. Then a general pair of ternary forms of degree a and $a+1$ is identifiable if and only if $a=2$. Moreover there are finitely many*

decompositions if and only if $a = 2t$ is even, and for such an a the number of decompositions is at least
$$\frac{(3t-2)(t-1)}{2}+1.$$

The Theorem has two directions: on one hand we need to prove that $a = 2$ is identifiable, on the other we need to show that for $a > 2$ a general pair is never identifiable. The former is a classical result we already recalled in (iii) of (1.5) and in Theorem 3.4. For the latter observe that $\dim Sec_k(X) = 4k-1$, therefore if either $4k-1 < N$ or $4k-1 > N$ the general pair is never identifiable. We are left to consider the perfect case $N = 4k-1$. In this case we may assume that X is not k-defective (we will prove that this is always the case in Remark 5.10), otherwise the non identifiability is immediate. Hence the core of the question is to study generically finite maps
$$\pi_k : Sec_k(X) \to \mathbb{P}^N,$$
with $4k = (a+2)^2$. This yields our last numerical constraint, namely, that $a = 2t$ needs to be even.

The first step is borrowed from [Me1][Me2], and it is a slight generalization of [Me1, Theorem 2.1], see also [CR].

Theorem 5.2. *Let $X \subset \mathbb{P}^N$ be an irreducible variety of dimension n. Assume that the natural map $\pi_1 : Sec_k(X) \to \mathbb{P}^N$ is dominant and generically finite of degree d. Let $z \in \mathbb{S}\mathrm{ec}_{k-1}(X)$ be a general point. Consider the projection $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ from the embedded tangent space $\mathbb{T}_z\,\mathbb{S}\mathrm{ec}_{k-1}(X)$. Then $\varphi_{|X} : X \dashrightarrow \mathbb{P}^n$ is dominant and generically finite of degree at most d.*

Proof. Choose a general point z on a general $(k-1)$-secant linear space spanned by $\langle p_1, \ldots, p_{k-1}\rangle$. Let $f : Y \to \mathbb{P}^N$ be the blow up of $\mathbb{S}\mathrm{ec}_{k-1}(X)$ with exceptional divisor E, and fiber $F_z = f^{-1}(z)$. Let $y \in F_z$ be a general point. This point uniquely determines a linear space Π of dimension $(k-1)(n+1)$ that contains $\mathbb{T}_z\,\mathbb{S}\mathrm{ec}_{k-1}(X)$. Then the projection $\varphi_{|X} : X \dashrightarrow \mathbb{P}^n$ is generically finite of degree d if and only if $(\Pi \setminus \mathbb{T}_z\,\mathbb{S}\mathrm{ec}_{k-1}(X)) \cap X$ consists of just d points.

Assume that $\{x_1, \ldots, x_a\} \subset (\Pi \setminus \mathbb{T}_z\,\mathbb{S}\mathrm{ec}_{k-1}(X)) \cap X$. By the Terracini Lemma, Theorem 2.3,
$$\mathbb{T}_z\,\mathbb{S}\mathrm{ec}_{k-1}(X) = \langle \mathbb{T}_{p_1}X, \ldots, \mathbb{T}_{p_{k-1}}X\rangle.$$
Consider the linear spaces $\Lambda_i = \langle x_i, p_1, \ldots, p_{k-1}\rangle$. The Trisecant Lemma, see for instance [ChCi, Proposition 2.6], yields $\Lambda_i \neq \Lambda_j$, for $i \neq j$. Let Λ_i^Y and Π^Y be the strict transforms on Y. Since $z \in \langle p_1, \ldots, p_{k-1}\rangle$ and $y = \Pi^Y \cap F_z$ then Λ_i^Y contains the point $y \in F_z$. In particular we have
$$\Lambda_i^Y \cap \Lambda_j^Y \neq \emptyset.$$

Let $\pi_1 : Sec_k(X) \to \mathbb{P}^N$ be the morphism from the abstract secant variety, and $\mu : U \to Y$ the induced morphism. That is $U = Sec_k(X) \times_{\mathbb{P}^N} Y$. Then there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\mu"] \arrow[d, "p"'] & Y \arrow[d, "f"] \\ Sec_k(X) \arrow[r, "\pi_1"] & \mathbb{P}^N \end{tikzcd}$$

Let λ_i and Λ_i^U be the strict transforms of Λ_i in $Sec_k(X)$ and U respectively. By Remark 2.2 $\lambda_i \cap \lambda_j = \emptyset$, so that

$$\Lambda_i^U \cap \Lambda_j^U = \emptyset.$$

This proves that $\sharp\mu^{-1}(y) \geq a$. But y is a general point of a divisor in the normal variety Y. Therefore $\deg \mu$, and henceforth $\deg \pi_1$, is at least a. □

To apply Theorem 5.2 we need to better understand X and its tangential projections. Recall that a divisor D is a *monoid* if it is irreducible and it is singular in a point with multiplicity $\deg D - 1$. By definition we have

$$X \simeq \mathbb{P}((\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^2}(a+1)),$$

then $X \subset \mathbb{P}^N$ can be seen as the embedding of $\mathbb{P}^3$ blown up in one point q embedded by monoids of degree $a+1$ with vertex q. That is, letting $\mathcal{L} = |\mathcal{I}_{q^a}(a+1)| \subset |\mathcal{O}_{\mathbb{P}^3}(a+1)|$ and $Y = \mathrm{Bl}_q \mathbb{P}^3$, then

$$X = \varphi_{\mathcal{L}}(Y) \subset \mathbb{P}^N.$$

It is now easy, via the Terracini Lemma, to realize that the restriction of the tangential projection $\varphi_{|X} X \dashrightarrow \mathbb{P}^3$ is given by the linear system

$$\mathcal{H} = |\mathcal{I}_{q^a \cup p_1^2 \ldots \cup p_{k-1}^2}(a+1)| \subset |\mathcal{O}_{\mathbb{P}^3}(a+1)|.$$

We already assumed that X is not k-defective, that is, we work under the condition

$$\dim \mathcal{H} = 3. \tag{5.1}$$

Remark 5.3. It is interesting to note that for $a = 2$ the map $\varphi_{|X}$ is the standard Cremona transformation of $\mathbb{P}^3$ given by $(x_0, \ldots, x_3) \mapsto (1/x_0, \ldots, 1/x_3)$.

Let us work out a preliminary Lemma, that we reproduce for lack of an adequate reference.

Lemma 5.4. *Let Δ be a complex disk around the origin, X a variety and $\mathcal{O}_X(1)$ a base point free line bundle. Consider the product $V = X \times \Delta$, with the natural projections, π_1 and π_2. Let $V_t = X \times \{t\}$ and $\mathcal{O}_V(d) = \pi_1^*(\mathcal{O}_X(d))$. Fix a configuration $p_1, \ldots, p_l$ of l points on V_0 and let $\sigma_i : \Delta \to V$ be sections such that $\sigma_i(0) = p_i$ and $\{\sigma_i(t)\}_{i=1,\ldots,l}$ are general points of V_t for $t \neq 0$. Let $P = \cup_{i=1}^l \sigma_i(\Delta)$, and $P_t = P \cap V_t$.*

Consider the linear system $\mathcal{H} = |\mathcal{O}_V(d) \otimes \mathcal{I}_{P^2}|$ on V, with $\mathcal{H}_t := \mathcal{H}_{|V_t}$. Assume that $\dim \mathcal{H}_0 = \dim \mathcal{H}_t = \dim X$, for $t \in \Delta$. Let $d(t)$ be the degree of the map induced by $\mathcal{H}_t$. Then $d(0) \leq d(t)$.

Proof. If, for $t \neq 0$, $\varphi_{\mathcal{H}_t}$ is not dominant the claim is clear. Assume that $\varphi_{\mathcal{H}_t}$ is dominant for $t \neq 0$. Then $\varphi_{\mathcal{H}_t}$ is generically finite and $\deg \varphi_{\mathcal{H}_t}(X) = 1$, for $t \neq 0$. Let $\mu : Z \times \Delta \to V$ be a resolution of the base locus, $V_{Zt} = \mu^* V_t$, and $\mathcal{H}_Z = \mu_*^{-1}\mathcal{H}$ the strict transform linear systems on Z. Then V_{Zt} is a blow up of $V_t = X$, for $t \neq 0$, and $V_{Z0} = \mu_*^{-1} V_0 + R$, for some effective, eventually trivial, residual divisor R. By hypothesis $\mathcal{H}_0$ is the flat limit of $\mathcal{H}_t$, for $t \neq 0$. Hence flatness forces

$$d(t) = \mathcal{H}_Z^{\dim X} \cdot V_{Zt} = \mathcal{H}_Z^{\dim X} \cdot (\mu_*^{-1} V_0 + R) \geq \mathcal{H}_Z^{\dim X} \cdot \mu_*^{-1} V_0 = d(0).$$

□

Lemma 5.4 allows us to work on a degenerate configuration to study the degree of the map induced by $|\mathcal{I}_{q^a \cup p_1^2 \ldots \cup p_{k-1}^2}(a+1)| \subset |\mathcal{O}_{\mathbb{P}^3}(a+1)|$.

Lemma 5.5. *Let $H \subset \mathbb{P}^3 \setminus \{q\}$ be a plane, $B := \{p_1, \ldots, p_b\} \subset H$ a set of $b := (1/2)t(t+3)$ general points, and $C := \{x_1, \ldots, x_c\} \subset \mathbb{P}^3 \setminus \{q \cup H\}$ a set of $c := (1/2)t(t+1)$ general points. Let $a = 2t$ and*

$$\mathcal{H} := |\mathcal{I}_{q^a \cup C^2 \cup B^2}(a+1)| \subset |\mathcal{O}_{\mathbb{P}^3}(a+1)|,$$

be the linear system of monoids with vertex q and double points along $B \cup C$, and $\varphi_{\mathcal{H}}$ the associated map. Then $\dim \mathcal{H} = 3$ and

$$\deg \varphi_{\mathcal{H}} > \frac{(3t-2)(t-1)}{2}.$$

Proof. Note that by construction the lines $\langle q, p_i \rangle$ and $\langle q, x_i \rangle$ are contained in the base locus of $\mathcal{H}$. Let us start computing $\dim \mathcal{H}$. First we prove that there is a unique element in $\mathcal{H}$ containing the plane H.

Claim 5.6. $|\mathcal{H} - H| = 0$.

Proof of the Claim. Let $D \in \mathcal{H}$ be such that $D = H + R$ for a residual divisor in $|\mathcal{O}(a)|$. Then R is a cone with vertex q over a plane curve $\Gamma \subset H$. Moreover R is singular along C and has to contain B. This forces Γ to contain B and to be singular at $\langle q, x_1 \rangle \cap H$. In other words Γ is a plane curve of degree $2t$ with $c = (1/2)t(t+1)$ general double points and passing through $b = (1/2)t(t+3)$ general points. Note that

$$\binom{2t+2}{2} - 3c - b = 1.$$

It is well known, see for instance [AH], that the c points impose independent conditions on plane curves of degree $2t$. Clearly the latter b simple points do the same, therefore there is a unique plane curve Γ satisfying the requirements. This shows that R is unique and in conclusion the claim is proved. □

We are ready to compute the dimension of $\mathcal{H}$.

Claim 5.7. $\dim \mathcal{H} = 3$.

Proof of the Claim. The expected dimension of $\mathcal{H}$ is 3. Then by Claim 5.6 it is enough to show that $\dim \mathcal{H}_{|H} = 2$. To do this observe that $\mathcal{H}_{|H}$ is a linear system of plane curves of degree $2t+1$ with b general double points and c simple general points. As in the proof of Claim 5.6 we compute the expected dimension

$$\binom{2t+3}{2} - 3b - c = 3,$$

and conclude by [AH]. □

Next we want to determine the base locus scheme of $\mathcal{H}_{|H}$. Let $\epsilon : S \to H$ be the blow up of B and $\langle q, x_i \rangle \cap H$, with $\mathcal{H}_S$ the strict transform linear system. We will first prove the following.

Claim 5.8. The scheme base locus of $|\mathcal{I}_{B^2}(2t+1)| \subset |\mathcal{O}_{\mathbb{P}^2}(2t+1)|$ is B^2.

Proof. Let $\mathcal{L}_{ij} := |\mathcal{I}_{B \setminus \{p_i, p_j\}}(t)| \subset |\mathcal{O}_{\mathbb{P}^2}(t)|$, then

$$\dim \mathcal{L}_{ij} = \binom{t+2}{2} - b - 2 - 1 = 2.$$

By the Trisecant Lemma, see for instance [ChCi, Proposition 2.6], we conclude that

$$\operatorname{Bs} \mathcal{L}_{ij} = B \setminus \{p_i, p_j\}.$$

Let $\Gamma_i, \Gamma_j \in \mathcal{L}_{ij}$ be such that $\Gamma_i \ni p_i$ and $\Gamma_j \ni p_j$. Then by construction we have

$$D_{ij} := \Gamma_i + \Gamma_j + \langle p_i, p_j \rangle \in \mathcal{H}.$$

Let D_{ijS}, $\mathcal{L}_{ijS}$ be the strict transforms on S. Note that Γ_h belongs to a pencil of curves in $\mathcal{L}_{hk}$ for any k. These pencils do not have common base locus outside of B since $\mathcal{L}_{ijS}$ is base point free and $\dim \mathcal{L}_{ij} = 2$. Therefore the D_{ijS} have no common base locus. $\square$

Claim 5.9. $\mathcal{H}_S$ is base point free.

Proof. To prove the Claim it is enough to prove that the simple base points associated to C impose independent conditions. Since $C \subset \mathbb{P}^3$ is general this is again implied by the Trisecant Lemma. $\square$

Then we have

$$\deg \varphi_{\mathcal{H}_S} = \mathcal{H}_S^2 = (2t+1)^2 - 4b - c = \frac{(3t-2)(t-1)}{2}.$$

To conclude, observe that, by the same argument as the claims, we can prove that $\varphi_{\mathcal{H}|R}$ is generically finite. Therefore

$$\deg \varphi_{\mathcal{H}} > \deg \varphi_{\mathcal{H}|H} = \deg \varphi_{\mathcal{H}_S} = (2t+1)^2 - 4b - c = \frac{(3t-2)(t-1)}{2}.$$

$\square$

Remark 5.10. Lemma 5.5 proves that $\deg \varphi_{\mathcal{H}}$ is finite. Hence as a byproduct we get that condition (5.1) is always satisfied in our range. That is X is not k-defective for $a = 2t$.

Proof of Theorem 5.1. We already know that the number of decompositions is finite only if $a = 2t$. By Remark 5.10 we conclude that the number is finite when $a = 2t$. Let d be the number of decompositions for a general pair. Then by Theorem 5.2 we know that $d \geq \deg \varphi$ where $\varphi : X \dashrightarrow \mathbb{P}^3$ is the tangential projection. The required bound is obtained combining Lemma 5.4 and Lemma 5.5. $\square$

References

[AH] J. Alexander, A. Hirschowitz, *The blown-up Horace method: application to fourth-order interpolation* Invent. Math. **107** (1992), no. 3, 585–602

[AH2] ——— *Polynomial interpolation in several variables* J. Algebraic Geom. **4** (1995), no. 2, 201–222

[AGHKT] A. Anandkumar, R. Ge, D. Hsu, S. M. Kakade, M. Telgarsky *Tensor Decompositions for Learning Latent Variable Models*, Journal of Machine Learning Research **15** (2014), 2773–2832

[BBCC] E. Ballico, A. Bernardi, M. V. Catalisano, L. Chiantini, *Grassmann secants, identifiability, and linear systems of tensors*, Linear Algebra and its Applications 438 (2013) 121–135

[Be] D. J. Bates, J. D. Hauenstein, A. J. Sommese, C. W. Wampler, *Bertini: Software for numerical algebraic geometry*, available at `bertini.nd.edu`

[BHSW] D. J. Bates, J. D. Hauenstein, A. J. Sommese, C. W. Wampler, *Numerically Solving Polynomial Systems with Bertini*, SIAM, Philadelphia (2013)

[CMS] J. Cahill, D. Mixon, N. Strawn, *Connectivity and irreducibility of algebraic varieties of finite unit norm tight frames*, arXiv:1311.4748

[CaCh] E. Carlini, J. Chipalkatti, *On Warings problem for several algebraic forms*, Comment. Math. Helv., vol. **78**, no. 3, pp. 494–517, 2003

[CaJo] M. Catalano-Johnson, *The possible dimensions of the higher secant varieties*, Amer. Journ. of Math. **118**, no. 2 (1996), 355–361

[ChCi] L. Chiantini, C. Ciliberto *Weakly defective varieties* Trans. Amer. Math. Soc. **354** (2002), no. 1, 151–178

[CR] C. Ciliberto, F. Russo, *Varieties with minimal secant degree and linear systems of maximal dimension on surfaces*, Advances in Mathematics **200** (2006) 1–50

[CQU] P. Comon, Y. Qi, K. Usevich, *A polynomial formulation for joint decompositions of symmetric tensors of different order*, 12th International Conference on Latent Variable Analysis and Signal Separation, Liberec, 2015, Proceedings of LVA/ICA 2015, 22–30

[DF] C. Dionisi, C. Fontanari, *Grassmann defectivity la Terracini*, Le Matematiche **56** (2001) 245–255

[GM] F. Galuppi, M. Mella, *Identifiability of homogeneous polynomials and Cremona Transformations*, arXiv:1606.06895 [math.AG]

[KL] R. Krone, A. Leykin, *package Numerical Algebraic Geometry for Macaulay 2*, available at `www.math.uiuc.edu/Macaulay2`

[Lon] F. London, *Ueber die Polarfiguren der ebenen Kurven dritter Ordnung*, Math. Ann. **36**, (1890), 535–584

[Har] R. Hartshorne, *Algebraic Geometry, GTM 52, Springer, 1977*

[HOOS] J. D. Hauenstein, L. Oeding, G. Ottaviani, A. J. Sommese, *Homotopy techniques for tensor decomposition and perfect identifiability*, arXiv:1501.00090

[Hi] D. Hilbert, *Letter adresseé à M. Hermite*, Gesam. Abh. vol II, 148–153.

[M2] D. Grayson, M. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at `www.math.uiuc.edu/Macaulay2`

[MM] A. Massarenti, M.Mella, *Birational aspects of the geometry of varieties of sums of powers* Adv. Math. 243 (2013), 187–202.

[Me1] M. Mella, *Singularities of linear systems and the Waring problem.* Trans. Amer. Math. Soc. **358** (2006), no. 12, 5523–5538

[Me2] M. Mella, *Base loci of linear systems and the Waring problem.* Proc. Amer. Math. Soc. **137** (2009), no. 1, 91–98.

[Mu1] S. Mukai, *Fano 3-folds* in: Complex Projective Geometry, Trieste, 1989/Bergen, 1989, in: London Math. Soc. Lecture Note Ser., vol. 179, Cambridge Univ. Press, Cambridge, 1992, pp. 255–263.

[Mu2] S. Mukai, *Polarized K3 surfaces of genus 18 and 20*, in: Complex Projective Geometry, in: LMS Lecture Notes Series, Cambridge: University Press, 1992, pp. 264–276.

[OO] L. Oeding and G. Ottaviani. *Eigenvectors of tensors and algorithms for Waring decomposition.* J. Symbolic Comput. **54** (2013), 9–35.

[ORS] L. Oeding, E. Robeva, B. Sturmfels, *Decomposing Tensors into Frames*, Advances in Applied Mathematics **73** (2016) 125–153

[OS] G. Ottaviani, E. Sernesi, *On the hypersurface of Lüroth quartics*, Michigan Math. J. **59**, 365–394 (2010)

[Pa] F. Palatini, *Sulla rappresentazione delle forme ternarie mediante la somma di potenze di forme lineari*, Rom. Acc. L. Rend. **12** (1903) 378–384.

[RS] K. Ranestad and F. Schreyer. *Varieties of sums of powers.* J. Reine Angew. Math. **525** (2000), 147–181.

[Re] T. Reye, *Ueber lineare Systeme und Gewebe von Flächen zweiten grades*, Journal reine angew. Math., 82, 1877, 54–83.

[Ri] H.W. Richmond, *On canonical forms*, Quart. J. Pure Appl. Math. **33** (1904) 967–984.

[Ro] R.A. Roberts, *Note on the plane cubic and a conic*, Proc. London Math. Soc. (1) 21, (1889), 62–69.

[Sco] G. Scorza, *Sopra le figure polari delle curve piane del terzo ordine*, Math. Ann, **51** (1899), 154–157

[Sy] J.J. Sylvester, *Collected Works*, Cambridge University Press, 1904.

[Te] A. Terracini, *Sulle V_k per cui la varietà degli S_h $(h+1)$-seganti ha dimensione minore dell'ordinario*, Rend. Circ. Mat. Palermo, **31** (1911), 392–396

[Te2] A. Terracini, *Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari*, Annali Mat. Pura e Appl., **24**, 1915, 1–10

[We] K. Weierstrass, , *Zur theorie der bilinearen and quadratischen Formen*, Monatsh. Akad. Wisa. Berlin, (1867), pp. 310–338.

(E.Angelini, F. Galuppi, M. Mella) Dipartimento di Matematica e Informatica, Università di Ferrara, Via Machiavelli 35, 44121 Ferrara, Italia
E-mail address: elena.angelini@unife.it, francesco.galuppi@unife.it, mll@unife.it

(G. Ottaviani) Dipartimento di Matematica e Informatica 'Ulisse Dini', Università di Firenze, Viale Morgagni 67/A, Firenze , Italia
E-mail address: ottavian@math.unifi.it