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Representations admitting a toric reduction

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Dottorando:
Francesco Panelli

Tutori:
Prof. Alexander Lytchak
Prof. Fabio Podestà

Coordinatore
Prof. Graziano Gentili

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Introduction

For an orthogonal representation (G, V) of a compact Lie group G on a finite dimensional euclidean vector space V, the orbit space V/G is one of the most important invariants: indeed, its metric structure encodes the information about the horizontal geometry of V with respect to the orbits. It seems then an interesting task to try to understand which algebraic properties of the representation can be recovered by the metric space V/G.

This new point of view in Representation Theory has been recently introduced by Claudio Gorodski and Alexander Lytchak (cf. [19]). In order to address the issue, they call two representations (G, V), (G', V') *quotient-equivalent* if their orbit spaces are isometric; in this way one is interested in looking for properties which depend only on quotient-equivalent classes. For instance, in [19] it is proved that, for a representation (G, V), all G-invariant subspaces of V can be recognized metrically; in particular irreducibility of (G, V) is a property that does not change in its quotient-equivalence class. Also the cohomogeneity of (G, V) depends only on the quotient space V/G, since it equals $\dim V/G$.

In Mathematics one often tries to reduce a given problem to a lower dimensional one having the same feautures one is interested in. In our context we are interested in lowering the dimension of the group; more precisely, given a representation (G, V), we need to consider representations (G', V') which are quotient equivalent to (G, V) and satisfy $\dim G' < \dim G$. Such representations are called *reductions* of (G, V). If in addition $\dim G'$ is minimal, then (G', V') is said to be *reduced*, and the number $\dim G'$ is called the *abstract copolarity* of (G, V) (cf. [19]). Since the horizontal information is unchanged, one then hopes to recover propeties of (G, V) by studying (G', V').

An extremal situation occurs when (G, V) is polar (cf. [4, 12, 35, 36]). In this case, there is a subspace Σ of V, called a *section*, which intersects all orbits perpendicularly. The subgroup N of G consisting of the elements which preserves Σ acts on it with a kernel that we quotient out; the effective action (Γ, Σ) obtained with this procedure is quotient-equivalent to (G, V) and $\dim \Gamma = 0$. For a representation, the existence of sections is equivalent to the integrability of the horizontal distribution; in particular, polarity is a property that depends only on quotient-equivalent classes. In fact, it is possible to prove that a representation is polar if and only if it has abstract copolarity 0 (cf. [19]).

In [22] the notion of polarity is generalized. Given a representation (G, V), a subspace Σ of V is called a *generalized section*, or, more precisely, a k-section, if it intersects all orbits, and contains the normal space to principal orbits it meets at each intersection point with codimension k. The minimum of all such k is called the *copolarity* of ρ. Denoting by N (resp. Z) the subgroup of G consisting of the elements that preserve Σ (resp. that fix Σ pointwisely), the group $\bar{N} := N/Z$ acts on Σ and the representations (G, V), $(\bar{N}, \Sigma)$ are quotient-equivalent. Since the copolarity of (G, V) equals $\dim \bar{N}$, we see that it bounds above the abstract copolarity of (G, V). If (G, V) has non-trivial principal isotropy group H, the fixed point space V^H of H on V provides a proper generalized section for (G, V).

At this point it seems interesting to look for relationships between cohomogeneity, abstract copolarity and copolarity of a given representation (G, V) (cf. [19, Question 1.15]). Note that the first two invariants are constant within the quotient-equivalent class of (G, V), but it is not known whether also the third one is.

The problem above has been investigated only in some particulare cases (cf. [22, 19, 37]). For instance, we have already mentioned the fact that a representation is polar (i.e. is of copolarity 0) if and only if it has abstract copolarity 0. One of the goals of this thesis is to show that a representation has copolarity 1 if and only if it has abstract copolarity 1 (cf. Corollary 3.2.4, or [37]).

Concerning relationships between copolarity and cohomogeneity, polar representations seem to behave very wildly: indeed, for any $n \in \mathbb{N}$ there exists a polar representation with cohomogeneity n. If we allow the copolarity to be greater than zero, the situation becomes much more rigid. For instance, in [19, 22] it has been proved that if an irreducible representation has copolarity k, $1 \leq k \leq 6$, then its cohomogeneity is $k + 2$. More generally, an irreducible representation which admits a minimal reduction (G, V) where the identity component G° of G is a k-dimensional torus must be of cohomogeneity $k + 2$ (cf. [19]). However the picture is far from being fully-understood, and in fact in [19] the authors provide an example of an irreducible representation of cohomogeneity 5 and copolarity 7.

The first goal of this thesis is to generalize the above statements concerning the relationships between copolarity and abstract copolarity to the reducible case for low values of the copolarity. Since cohomogeneity is an invariant which depends only on the quotient-equivalent class of a representation, it is more convenient to deal with abstract copolarity instead of copolarity; however we shall see that the theorems obtained in this case imply analogous ones for copolarity (cf. Remark 3.2.13).

Namely, we shall analyze whether a reducible representation of abstract copolarity $k \in \{1,2\}$ has cohomogeneity $k+2$. Actually, this statement turns out to be false, and trivial counterexamples are constructed as follows. Consider any representation (H,W) and, for any $n \in \mathbb{N}$, a polar representation (H',W') of cohomogeneity n. Then the product $(H \times H', W \oplus W')$, where the group $H \times H'$ acts componentwisely, has the same copolarity of (H,W) but cohomogeneity $\geq n$. Thus we have a family of representations with the same copolarity but arbitrarily large cohomogeneity. This means that the problem we are studying is interesting only for a smaller class of representations, namely those which are *indecomposable*. A representation is called *decomposable* if it has the same orbits as the direct product of two subrepresentations. In the case of abstract copolarity one, we shall show that the counterexamples described above are the only ones that can occur (cf. [37]):

Theorem 3.2.1. *Let $\rho = (H,W)$ be a non-reduced, indecomposable representation of a connected, compact Lie group H of abstract copolarity* 1*. Then ρ has cohomogeneity* 3*.*

The case of abstract copolarity 2 is slightly more complicated, and contains some more counterexamples (cf. [37]):

Theorem 3.2.5. *Let $\rho = (H,W)$ be a non-reduced, indecomposable representation of a compact, connected Lie group H of abstract copolarity* 2*. Then either:*

1. *ρ has cohomogeneity* 4*, or*
2. *$\rho = \rho_1 \oplus \rho_2$, where*
   - *(a) $\rho_1 = (H,W_1)$ is orbit-equivalent to the isotropy representation of a rank* 2 *real grassmannian,*
   - *(b) $\rho_2 = (H,W_1^{\perp})$ is orbit-equivalent to a non-polar* $\mathbf{U}(1)$*-representation without non-trivial fixed points.*

Conversely, let ρ_1 be the isotropy representation of a rank 2 *real grassmannian, ρ_2 be a non-polar* $\mathbf{U}(1)$*-representation without non-trivial fixed points and set $\rho := \rho_1 \oplus \rho_2$. Then ρ is indecomposable, has cohomogeneity $\neq 4$, and both its copolarity and abstract copolarity are equal to* 2*.*

We feel the need to remark here that the proof of these statements heavily relies on the fact that, if the abstract copolarity of (H,W) is either 1 or 2, then the identity component of the group G of any minimal reduction (G,V) of (H,W) must be a torus. As a Corollary, we shall see that representations of copolarity 1 are exactly those of abstract copolarity 1.

The second goal of this thesis is to study more closely the class of representations that admit a toric reduction. This class generalizes that of polar representations; and indeed, representations that admit a toric reduction have many features in common with the polar ones. For instance, it is well-known that any slice representation and any invariant subspace of a polar representation is polar as well. Similarly, we shall see that any slice representation and any invariant subspace of a representation admitting a toric reduction also admits a toric reduction.

Irreducible representations (H, W) with a toric reduction have been classified in [20]. In this thesis we shall classify the reducible ones when the group H is simple. Namely, we shall prove the following:

Theorem 3.3.1. *Assume that H is a compact, connected, simple Lie group, and let (H, W) be an effective, non-reduced, reducible, indecomposable representation of H that admits a toric reduction. If (H, W) is non-polar, then it is one of the following representations:*

H	W
$\mathbf{SO}(n)$	$\mathbb{R}^n \oplus \mathbb{R}^n$
$\mathbf{G}_2$	$\mathbb{R}^7 \oplus \mathbb{R}^7$
$\mathbf{Spin}(7)$	$\mathbb{R}^8 \oplus \mathbb{R}^8$
$\mathbf{Sp}(2)$	$\mathbb{R}^5 \oplus \mathbb{C}^4$
$\mathbf{Spin}(9)$	$\mathbb{R}^9 \oplus \mathbb{R}^{16}$
$\mathbf{Spin}(8)$	$\mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-$

Here $\mathbb{R}^8_0$ denotes the standard representation of $\mathbf{Spin}(8)$, while $\mathbb{R}^8_+$, $\mathbb{R}^8_-$ denote its half spin representations.

The work is organized as follows. Chapter 1 is dedicated to the Theory of Actions of Lie groups. After having recalled the most basic results in this field (Sections 1.1-1.4), we pass to define, following [22], the notion of copolarity for an isometric action (Section 1.5). Then, in Section 1.6, we formally define the metric distance on the orbit space of an action, and recall its main properties (cf. [7, 9]). Finally, in Section 1.7, we intoduce and discuss the notion of reduction for an action, showing in particular that any proper generlized section gives rise to a reduction (cf. [19]).

Chapter 2 is dedicated to Representation Theory. First, in Section 2.1, we racall the main basic tools and definitions from the theory (mainly following [1, 8]). In Sections 2.2, 2.3 we apply the theory of reductions to the case of representations, stating and proving the most simple results from [19] that will be used later. In particular, Section 2.2 is mainly dedicated to the proof of the fact that the invariant subspaces of a representation (G, V) can be recognized metrically in the orbit space V/G. Here we point out a simple consequence of this fact, well-known to the experts, for which, as far as we know, there is still no proof in literature: namely, we show that also the isotypical components of a representation (G, V) can be recognized metrically in V/G (cf. Proposition 2.2.13). In Section 2.4 we introduce one of the most important technical notions of this thesis, namely decomposability of representations. In particular, following [37], we explain the definition, and go through its main properties. Finally, in Section 2.5, we study, following [37], representations of tori, proving two criteria for their indecomposability and for the existence in them of proper generalized sections (cf. Propositions 2.5.8, 2.5.12).

Chapter 3 is the heart of the thesis. In Section 3.1 we formally define representations admitting a toric reduction, and prove some of their basic properties (some of which can be also found in [37]). Section 3.2 is dedicated to the proof of Theorems 3.2.1, 3.2.5, which represent the main results of [37]. Finally in Section 3.3 we prove Theorem 3.3.1.

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Chapter 1

Lie group actions

1.1 Elementary definitions

Let M be a smooth manifold and G a Lie group.

Definition 1.1.1. A (smooth) *action* of G on M is a smooth map

$$\begin{aligned} \Theta : G \times M &\to M \\ (g,p) &\mapsto g \cdot p \end{aligned}$$

so that:

1. $g \cdot (h \cdot p) = (gh) \cdot p$ for all $g, h \in G$, $p \in M$;

2. $e \cdot p = p$ for all $p \in M$, if e denotes the identity element of G.

If Θ, Θ' are actions of G on the manifolds M, M' respectively, we say that a smooth map $f : M \to M'$ is *G-equivariant* if

$$f(g \cdot p) = g \cdot f(p), \qquad \forall\, g \in G, \;\; \forall\, p \in M.$$

Θ and Θ' are said to be *equivalent* if there exists a G-equivariant diffeomorphism $f : M \to M'$.

Note that an action Θ of the group G on a manifold M induces an homomorphism $\rho_\Theta : G \to \mathrm{Diff}(M)$ of G into the group $\mathrm{Diff}(M)$ of all diffeomorphism of M. Clearly $\ker \rho_\Theta$ is a closed normal subgroup of G; we say that Θ is:

1. *effective*, or *faithful*, if $\ker \rho_\Theta$ is trivial;

2. *quasi-effective* if $\ker \rho_\Theta$ is finite.

Given $p \in M$, the *isotropy group at* p is the closed subgroup G_p of G given by

$$G_p := \{g \in G \mid g \cdot p = p\}.$$

An action is called *free* if G_p is trivial for all $p \in M$.

If $p \in M$, the G*-orbit through* p is the set

$$G \cdot p := \{g \cdot p \mid g \in G\}.$$

Clearly the orbits provide a partition of the manifold M. The space M/G of all orbits, endowed with the quotient topology, is called *orbit space.* We consider now the map

$$\begin{aligned} \iota : G/G_p &\to M \\ gG_p &\mapsto g \cdot p, \end{aligned}$$

whose range coincides with the orbit $G \cdot p$. It is not hard to show that ι is smooth, injective and that its differential has maximum rank at each point; in particular $G \cdot p$ is a submanifold of M. Under some additional hypothesis on the action (i.e. properness), it is possible to prove also that ι is a homeomorphism onto its image if and only if the set $G \cdot p$ is relatively closed in M. However we remark here that in the most important case to us, namely when G is compact, all orbits are obviously regularly embedded in M.

1.2 The Slice Theorem

We shall always assume henceforth that the Lie group G is compact; however much of what we are going to say can be proved in the more general setting of proper actions (cf. [15] for more details).

The main tool in the study of Lie group actions is the concept of slice, which we are going to define:

Definition 1.2.1. Let M be a smooth manifold on which a compact Lie group G is acting, and fix $p \in M$. A submanifold S through p is called a *slice* if:

1. the set $G \cdot S := \{g \cdot s \mid g \in G,\ s \in S\}$ is open in M;
2. if $g \in G$, $s \in S$, then $g \cdot s \in S$ if and only if $g \in G_p$.

Note that by definition G_p acts on S and we can define a G-equivariant map $f : G \cdot S \to G/G_p$ by $f(g \cdot s) = gG_p$. The next result provides a slice through any point $p \in M$.

Theorem 1.2.2 (Slice Theorem). *Let G a compact Lie group acting on a manifold M. If $p \in M$, there exists a slice S through p.*

Proof. Clearly G_p is compact, so there exists a G_p-invariant Riemannian metric $\mathbf{g}$ on M. For $r > 0$ we define $D(r) := \{v \in T_p(G \cdot p)^\perp \mid \mathbf{g}(v, v) < r\}$; we shall prove that $S_r := \exp_p(D(r))$ is a slice if r is chosen small enough.

Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G and G_p respectively, and let $\mathfrak{m}$ be a complement of $\mathfrak{k}$ in $\mathfrak{g}$. We choose a neighborhood V of the origin in $\mathfrak{m}$ so that the exponential map $\exp|_V$ of the group G is a diffeomorphism onto its image, and set $P := \{\exp(v) \mid v \in V\}$. Next we define the map

$$\alpha : S_r \times P \to M$$
$$(s, g) \mapsto g \cdot s$$

it is easily verified that $\mathrm{d}\alpha_{(p,e)}$ has maximal rank, so α provides a diffeomorphism of an open neighborhood $U \times W$ of (p, e) onto an open neighborhood U_p of p in M. We may assume $U = S_r$, $W = P$, up to choose a smaller r and a smaller V; so $A := \alpha(S_r \times P)$ is open in M. Then if $g_0 \in G$, $s_0 \in S$, we have that $g_0 \cdot A$ is an open neighborhood of $g_0 \cdot s_0$ contained in $G \cdot S_r$, showing that the latter is an open set.

Suppose now that, for any $r > 0$, S_r does not verify property (2) in Definition 1.2.1. Then we may find two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ converging to p so that $x_n, y_n \in S_{\frac{1}{n}}$, $g_n \cdot x_n = y_n$, for suitable $g_n \in G$ and for all $n \in \mathbb{N}$. Since G is compact we may assume $\{g_n\}_{n\in\mathbb{N}} \to g$, for some $g \in G$. Clearly $g \in G_p$ and, up to change x_n with $g \cdot x_n$ and g_n with $g_n g^{-1}$ we may suppose also $g = e$.

Now, the map $\mathfrak{m} \times \mathfrak{k} \ni (\xi, \eta) \mapsto \exp(\xi)\exp(\eta)$ is a diffeomorphism around the origin onto a neighborhood of $e \in G$, so, if n is big enough, we can write $g_n = a_n b_n$ for some $a_n \in P$, $b_n \in G_p$, $a_n \neq e$. For such values of n we then have

$$\alpha(b_n \cdot x_n, a_n) = (a_n b_n) \cdot x_n = g_n \cdot x_n = y_n = \alpha(y_n, e).$$

Since $b_n \cdot x_n \in S_r$ and α is injective, we deduce $b_n \cdot x_n = y_n$ and $a_n = e$, contradiction. □

Using the Slice Theorem we can investigate some basic topological properties of the orbit space. First we denote by $\pi : M \to M/G$ the canonical projection, and recall that by definition M/G is endowed with the quotient topology induced by π. Then notice that π is open, so M/G is second countable if M is. The next result shows that, moreover, M/G is locally compact and Hausdorff.

Lemma 1.2.3. *If G is compact, the quotient space M/G is locally compact and Hausdorff.*

Proof. Fix $\pi(x) \in M/G$, and let S be a slice through x. Clearly $\pi(S) = \pi(G \cdot S)$ is open in M/G; moreover, since $S = \exp_x(D(r))$ for some $r > 0$, where $D(r)$ is a ball in an euclidean space, we have $\bar{S} = \exp_x(\overline{D(r)})$, so it is compact. Hence $\pi(S)$ is a relatively compact neighborhood of $\pi(x)$ in M/G.

In order to show that M/G is Hausdorff, we need only to prove that the set $R := \{(x, g \cdot x) \in M \times M \mid x \in M, g \in G\}$ is closed in $M \times M$. Indeed, if $\{(x_n, g_n \cdot x_n)\}_{n\in\mathbb{N}} \subseteq R$ converges to (x, y), we may assume $\{g_n\}_{n\in\mathbb{N}} \to g \in G$, so $y = \lim g_n \cdot x_n = g \cdot x$ and $(x, y) \in R$. □

1.3 Isometric actions

In the following pages we shall be mainly interested in actions of Lie groups on Riemannian manifolds. The notion of compatibility between the action and the Riemannian structure is given in the following:

Definition 1.3.1. Let G be a compact Lie group acting on a Riemannian manifold $(M, \mathbf{g})$. We say that the action is *isometric* if for any $g \in G$ the diffeomorphism $M \to M$ given by $p \mapsto g \cdot p$ is an isometry for the Riemannian structure $\mathbf{g}$. In this case we say that $\mathbf{g}$ is a *G-invariant* metric on M.

In other words an action is called isometric if the induced homomorphism $\rho : G \to \mathrm{Diff}(M)$ takes its values in $\mathrm{Iso}(M, \mathbf{g})$ (the group of all isometries of $(M, \mathbf{g})$). Note that conversely, $\mathrm{Iso}(M, \mathbf{g})$ being a Lie group (cf. [27]), any Lie group homomorphism $\rho : G \to \mathrm{Iso}(M, \mathbf{g})$ gives rise to an isometric action of G on $(M, \mathbf{g})$. We shall often denote such an by $\rho = (G, M, \mathbf{g})$, or simply by $\rho = (G, M)$ if the metric is clear from the context.

Remark 1.3.2. If G is compact and M is second countable, any action of G on M is isometric with respect to a suitable Riemannian structure on M. It is possible to prove that such a metric can be chosen to be complete (cf. [33]).

Some simple, nevertheless important, properties of isometric actions are given in the following Lemmas:

Lemma 1.3.3. *Let (G, M) be an isometric action, and $\gamma : [a, b] \to M$ a geodesic. If γ is perpendicular to the orbit $G \cdot \gamma(t_0)$ for a suitable $t_0 \in [a, b]$, then it is perpendicular to the orbit $G \cdot \gamma(t)$ for every $t \in [a, b]$.*

Proof. Denote by $\mathfrak{g}$ the Lie algebra of G, and, for any $X \in \mathfrak{g}$, define a vector field X^* on M by:

$$X^*_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX) \cdot p, \qquad \forall\, p \in M. \tag{1.1}$$

Since G acts by isometries, X^* is Killing, so, denoting by $\langle\cdot,\cdot\rangle$ the G-invariant Riemannian metric on M, we have that the quantity $\langle X^* \circ \gamma, \gamma' \rangle$ is constant. Now, it is easily seen that the set $\{X^*_p \mid X \in \mathfrak{g}\}$ coincides with $T_p(G \cdot p)$, therefore the assumption implies

$$\langle X^* \circ \gamma, \gamma' \rangle|_{t_0} = 0, \qquad \forall\, X \in \mathfrak{g},$$

and the result follows. □

Lemma 1.3.4. *Let (G, M) be an isometric action of the compact Lie group G on the connected, complete Riemannian manifold M. If $p \in M$ and $\mathcal{O}$ is a G-orbit, there exist a minimizing geodesic $\gamma : [a, b] \to G$ so that $\gamma(a) = p$, $\gamma(b) \in \mathcal{O}$. Morover such geodesic is perpendicular to every orbit it meets.*

Proof. Denote by $\langle\cdot,\cdot\rangle$ the complete G-invariant Riemannian metric on M, and by d the distance induced by $\langle\cdot,\cdot\rangle$ on M. Since $\mathcal{O}$ is compact there exists $q \in \mathcal{O}$ so that $\mathrm{d}(p, q) = \mathrm{d}(p, \mathcal{O})$, and since $\langle\cdot,\cdot\rangle$ is complete there exists a minimizing geodesic $\gamma : [a, b] \to M$ so that $\gamma(a) = p$, $\gamma(b) = q \in \mathcal{O}$. Now, by the first variation formula, γ is perpendicular to $\mathcal{O}$, so the result follows from Lemma 1.3.3. □

We now wish to describe locally an isometric action of a compact Lie group G on a complete Riemannian manifold $(M, \mathbf{g})$; observe that, by Remark 1.3.2, the assumptions of isometricity of the action and completeness of the metric are not restrictive.

If $\mathcal{O}$ is an orbit of (G, M), we can consider the *normal bundle to* $\mathcal{O}$, i.e. the bundle on $\mathcal{O}$ defined by

$$\nu\mathcal{O} := \bigcup_{p \in \mathcal{O}} \nu_p\mathcal{O},$$

where

$$\nu_p\mathcal{O} := \{v \in T_pM \mid \mathbf{g}(v, T_p\mathcal{O}) = 0\} = (T_p\mathcal{O})^\perp.$$

We shall now see how $\nu\mathcal{O}$ can be used to describe the action of G on M in a suitable neighborhood of $\mathcal{O}$. First we define an action of G on $\nu\mathcal{O}$:

$$g \cdot v := \mathrm{d}g_{\sigma(v)}(v), \qquad \forall\, v \in \nu\mathcal{O},\ \forall g \in G, \tag{1.2}$$

where $\sigma : \nu\mathcal{O} \to \mathcal{O}$ is the canonical projection. Since $(M, \mathbf{g})$ is complete we have a well-defined map

$$\begin{aligned} \mathrm{Exp} : \nu\mathcal{O} &\to M \\ v &\mapsto \exp_{\sigma(v)}(v), \end{aligned}$$

satisfying:

$$g \cdot \mathrm{Exp}(v) = \mathrm{Exp}(g \cdot v), \qquad \forall\, v \in \nu\mathcal{O},\ \forall\, g \in G.$$

Now fix $p \in \mathcal{O}$, end consider $\mathrm{dExp}_p|_{T_p\mathcal{O}}$, $\mathrm{dExp}_p|_{\nu_p\mathcal{O}}$ (here we are identifying $\mathcal{O}$ with the zero section in $\nu\mathcal{O}$). Since they both are the identity, there exists a neighborhood U_p of p in $\nu\mathcal{O}$ so that Exp is a diffeomorphism of U_p onto an open set of M containing p.

Choose $\mathbb{R} \ni r > 0$ so that $B_r := \{v \in \nu_p\mathcal{O} \mid \mathbf{g}_p(v,v) < r^2\} \subseteq U_p$, and define

$$A_r := G \cdot B_r = \{v \in \nu\mathcal{O} \mid \mathbf{g}_{\sigma(v)}(v,v) < r^2\}.$$

It is clear that A_r is G-invariant and that $\mathrm{Exp}|_{A_r}$ is locally a diffeomorphism. We shall show that $\mathrm{Exp}|_{A_r}$ is in fact a diffeomorphism onto its image if we choose $r > 0$ small enough.

Assume by contradiction that this is not true; then there exist two sequences $\{v_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}}$ so that

$$v_n, w_n \in A_{1/n}, \qquad v_n \neq w_n, \qquad \mathrm{Exp}(v_n) = \mathrm{Exp}(w_n), \qquad \forall\, n \in \mathbb{N}.$$

Using that G is compact, we may assume that $\sigma(v_n) \to p$, so

$$\mathrm{Exp}(w_n) = \mathrm{Exp}(v_n) \to p.$$

Let d be the distance on M induced by $\mathbf{g}$, and observe that

$$\mathrm{d}(\sigma(w_n), \mathrm{Exp}(w_n)) = \mathrm{d}(\sigma(w_n), \exp_{\sigma(w_n)}(w_n)) \leq \frac{1}{n}.$$

We can now choose $\delta > 0$ so that

$$C := \{v \in \nu\mathcal{O} \mid \mathrm{d}(p, \sigma(v)) < \delta,\ \mathbf{g}(v,v) < \delta^2\} \subseteq U_p.$$

From

$$\mathrm{d}(p, \sigma(w_n)) \leq \mathrm{d}(p, \mathrm{Exp}(w_n)) + \mathrm{d}(\sigma(w_n), \mathrm{Exp}(w_n))$$

we get $\mathrm{d}(p, \sigma(w_n)) \to 0$, hence $\{w_n\}_{n\in\mathbb{N}} \subseteq C$ for any n large enough. On the other hand $\{v_n\}_{n\in\mathbb{N}} \subseteq U_p$ for any n large enough, so, by the injectivity of Exp on U_p we deduce $v_n = w_n$ for such values of n, contradiction.

So we have proved that $\mathrm{Exp}|_{A_r}$ is a diffeomorphism for a suitable value of $r > 0$. The image $\mathfrak{T}(\mathcal{O}) := \mathrm{Exp}(A_r)$ is a G-invariant open neighborhood of $\mathcal{O}$ which is called *tubular neighborhood*, or simply *tube*, around $\mathcal{O}$.

Notice that the action of G on $\nu\mathcal{O}$ is equivalent to that on $\mathfrak{T}(\mathcal{O})$. Indeed, a G-equivariant diffeomorphism between $\nu\mathcal{O}$ and $\mathfrak{T}(\mathcal{O})$ is given by the map $\Phi_{\mathcal{O}} := \mathrm{Exp} \circ \psi : \nu\mathcal{O} \to \mathfrak{T}(\mathcal{O})$, where

$$\begin{aligned} \psi : \nu\mathcal{O} &\to A_r \\ v &\mapsto \frac{rv}{1 + \mathbf{g}(v,v)^{1/2}}. \end{aligned}$$

Hence the study of the action of G on the neighborhood $\mathfrak{T}(\mathcal{O})$ of $\mathcal{O}$ reduces to the study of the action of G on $\nu\mathcal{O}$.

Finally we notice that, if $p \in \mathcal{O}$, formula (1.2) defines an action of the compact Lie group G_p on the vector space $\nu_p\mathcal{O}$; it is called the *slice representation* at p. It is easily seen that the map

$$\begin{aligned} G \times_{G_p} \nu_p\mathcal{O} &\to \nu\mathcal{O} \\ [(g,v)] &\mapsto g \cdot v \end{aligned}$$

is a G-equivariant diffeomorphism; here $G \times_{G_p} \nu_p\mathcal{O}$ denotes the quotient space $\frac{G\times\nu_p\mathcal{O}}{G_p}$, where G_p acts on $G \times \nu_p\mathcal{O}$ by the rule

$$h \cdot (g,v) \mapsto (gh^{-1}, h \cdot v), \qquad \forall\, g,h \in G,\ \forall\, v \in \nu_p\mathcal{O}.$$

The G_p-orbit of an element $(g,v) \in G \times \nu_p\mathcal{O}$ is denoted by $[(g,v)]$. Note that G acts on $G \times_{G_p} \nu_p\mathcal{O}$ as follows:

$$a \cdot [(g,v)] := [(ag,v)], \qquad \forall\, a,g \in G,\ \forall\, v \in \nu_p\mathcal{O}.$$

We summarize what we have obtained so far in the following:

Theorem 1.3.5 (Tube Theorem)**.** *Let G be a compact Lie group acting on a second countable manifold M. Then for any orbit $\mathcal{O} = G \cdot p$ of the action, there exist a G-invariant open neighborhood $\mathfrak{T}(\mathcal{O})$ of $\mathcal{O}$ and a G-equivariant diffeomorphism*

$$\Psi : G \times_{G_p} \nu_p\mathcal{O} \to \mathfrak{T}(\mathcal{O}),$$

where $\nu_p\mathcal{O}$ denotes the normal space to $\mathcal{O}$ at p with respect to any G-invariant complete Riemannian metric on M.

The identification Ψ above requires a rescaling to pass from the whole vector space $\nu_p\mathcal{O}$ to the ball $B_r \subseteq \nu_p\mathcal{O}$ centred at the origin and of radius r. We can avoid the scaling process simply considering the space $G \times_{G_p} B_r$ instead of $G \times_{G_p} \nu_p\mathcal{O}$, and observing that the map $\Phi : G \times_{G_p} B_r \to \mathfrak{T}$ defined by

$$\Phi : [(g,v)] \mapsto \operatorname{Exp}(g \cdot v) = \exp_p(g \cdot v), \qquad \forall\, g \in G,\ \forall\, v \in B_r$$

is a G-equivariant diffeomorphism. Note that, if r is chosen sufficiently small, the set consisting of the points $[(e,v)]$, $v \in B_r$, identifies with $\exp_p(B_r)$, which is a slice through p.

1.4 Orbit types

Let G be a compact Lie group acting on a second countable manifold M; in this section we shall describe, following [6, Chapter 9], how to hierarchize the orbit space M/G. Thanks to Remark 1.3.2 we may, and always will, assume that M is endowed with a complete G-invariant Riemannian metric $\mathbf{g}$. We begin with some definitions:

Definition 1.4.1. We shall denote by $\mathcal{T}$ the set of conjugacy classes of compact subgroups of G. For every $p \in M$, we call *orbit type* of p, or simply *type* of p, the class $[G_p]$ of G_p in $\mathcal{T}$.

Clearly two points of the same orbit are of the same type; and two orbits are of the same type if and only if they are G-equivalent.

Definition 1.4.2. For every $t \in \mathcal{T}$ denote by $M_{(t)}$ the set of points of M of type t, i.e. the union of orbits of type t; this is a G-invariant subset of M. If $H \in t$, we also write $M_{(H)}$ for $M_{(t)}$.

For example, $M_{(G)}$ is the closed subspace of M consisting of the points fixed by G. It will be often denoted by M^G.

We can define an order $\preceq$ on $\mathcal{T}$ as follows: given $t, t' \in \mathcal{T}$, we say that $t \preceq t'$ if there exixst $H \in t$, $H' \in t'$ so that $H' \subseteq H$. This relation is actually well-defined and it is obviously reflexive. In order to prove anti-simmetry, we observe that if $t \preceq t'$ and $t' \preceq t$, then there exist $H, K \in t$, $H', K' \in t'$ so that $H \supseteq H'$, $K' \supseteq K$. On the other hand $H' = gK'g^{-1}$, $K = hHh^{-1}$ for suitable $g, h \in G$, so $H \supseteq H' \supseteq gKg^{-1} = ghH(gh)^{-1}$, which imples $H = ghH(gh)^{-1}$ (since they are two Lie subgroups of G with the same identity component and the same finite number of connected components). So $[H] = [H']$, i.e. $t = t'$. Transitivity of $\preceq$ is proved similarly.

Remark 1.4.3. The same kind of argument used above to prove the shew-simmetry of $\preceq$ allows us to show that for any chain in $(\mathcal{T}, \preceq)$ of the form

$$\cdots \preceq t_i \preceq t_{i+1} \preceq t_{i+2} \preceq \cdots$$

there exists $N \in \mathbb{N}$ such that $t_n = t_{n+1}$ for every $n \geq N$. Indeed, we can associate to any such chain a decreasing sequence $\{H_n\}_{n \in \mathbb{N}}$ of compact subgroups of G; so for n large enough H_n and H_{n+1} share the same identity component and the same finite number of connected components and are, therefore, equal to each other.

If $\mathcal{O}$, $\mathcal{O}'$ are orbits of the action (G, M) of type t, t' respectively, then $t \preceq t'$ if and only if there exists a G-equivariant map (necessarily a surjective submersion) $\mathcal{O}' \to \mathcal{O}$. If $p, p' \in M$ are of type t, t' respectively, then $t \leq t'$ if and only if there exists $g \in G$ so that $gG_{p'}g^{-1} \subseteq G_p$.

Theorem 1.4.4. *Let G be a compact Lie group acting on a second countable manifold M.*

1. *Let $p \in M$ and $t \in \mathcal{T}$ its orbit type; there exists a G-invariant open neighborhood $\mathfrak{T}$ of p so that, for any point $q \in \mathfrak{T}$, the orbit type of q is $\succeq t$.*

2. *For all $t \in \mathcal{T}$, $M_{(t)}$ is an embedded G-invariant submanifold of M, and the canonical projection $M_{(t)} \to M_{(t)}/G$ is a bundle.*

3. *If the orbit space M/G is connected, the set $\mathcal{T}$ has a (unique) maximum τ. Moreover $M_{(\tau)}$ is a dense open subset of M and $M_{(\tau)}/G$ is connected.*

Proof. Fix $p \in M$. In order to prove (1) and (2) we may restrict to a tubular neighborhood $\mathfrak{T}$ of the orbit $\mathcal{O} := G \cdot p$, hence we may suppose that M is of the form $G \times_H \nu_p\mathcal{O}$, where $H := G_p$. In this identification p corresponds to the point $[(e, O)] \in G \times_H \nu_p\mathcal{O}$.

If $q := [(g, v)] \in G \times_H \nu_p\mathcal{O}$, $a \in G$ fixes q if and only if $a \in gH_vg^{-1}$. So G_q is conjugate to the subgroup H_v of $H = G_p$, proving that q is of type $\succeq t$, hence (1). Moreover, q is of type t if and only if G_q is conjugate to H in G, or equivalently if H_v is conjugate to H in G. Since $H_v \subseteq H$ this implies $H_v = H$, so v is fixed by H. If W denotes the subspace of $\nu_p\mathcal{O}$ consisting of the elements fixed by H, it follows that $M_{(t)}$ can be identified with $G \times_H W = \frac{G}{H} \times W$, hence (2).

We now prove (3) by induction on $\dim M$, the assertion being clear for $\dim M = 0$. Assume now the result true in any dimension $< n = \dim M$, and let τ be a maximal element in $(\mathcal{T}, \preceq)$ (which exists by Remark 1.4.3).

We claim that if $A \subseteq M_{(\tau)}$ is a non-empty G-invariant set which is open and closed in $M_{(\tau)}$, then A is dense in M.

It is enough to show that, for any $x \in A$, $\mathfrak{T}_x \cap A$ is dense in $\mathfrak{T}_x$, $\mathfrak{T}_x$ being a tubular neighborhood around the orbit $G \cdot x$. So we may assume that M is of the form $G \times_H V$, where $V := \nu_x(G \cdot x)$ and $H := G_x$; in this identification x corresponds to the point $[(e, O)]$. In order to simplify the notation we set $\mathfrak{T} := G \times_H V$.

Suppose first that H acts trivially on V. Then $\mathfrak{T}/G = V$, which is connected, and A/G is open and closed in $\mathfrak{T}/G$, so $A/G = \mathfrak{T}/G$ and $A = \mathfrak{T}$.

Now assume that H acts non-trivially on V, and let S be the unit sphere in V. We note that S/H is connected: indeed this is obvious if $\dim S \geq 1$; if instead $\dim S = 0$ and S consists of two points, we have that S/H is the one-point space because H acts non-trivially on V and hence on S.

Set $Y := G \times_H S$, which is a closed, G-invariant submanifold of $\mathfrak{T}$ of codimension 1, and notice that $Y/G = S/H$ is connected. By the induction hypothesis, there exists a maximal orbit type θ for Y, the set $Y_{(\theta)}$ is open and dense in Y, and $Y_{(\theta)}/G$ is connected. Consider now the action of $\mathbb{R}^+$ on $\mathfrak{T}$ given by

$$\lambda \cdot [(g, v)] := [(g, \lambda v)], \qquad \forall\, \lambda \in \mathbb{R}^+,\ \forall\, [(g, v)] \in \mathfrak{T}.$$

Clearly $\mathfrak{T}_{(\theta)}$ contains $\mathbb{R}^+ \cdot Y_{(\theta)}$, which is open and dense in $\mathfrak{T}$. Since $\mathfrak{T}_{(\tau)}$ is open in $\mathfrak{T}$ by (1), we deduce $\mathfrak{T}_{(\tau)} \cap \mathfrak{T}_{(\theta)} \neq \varnothing$, so $\theta = \tau$ and $\mathfrak{T}_{(\tau)}$ is dense in $\mathfrak{T}$. Now, $\mathfrak{T}_{(\tau)}/G$ contains $(\mathbb{R}^+ \cdot Y_{(\theta)})/G$ as an open and dense subset, and the latter is connected since it is homeomorphic to $\mathbb{R}^+ \times \frac{Y_{(\theta)}}{G}$. Thus $\mathfrak{T}_{(\tau)}/G$ is itself connected, and must coincide with A/G, the latter being open and closed in the former. This implies $A = \mathfrak{T}_{(\tau)}$, which is dense in $\mathfrak{T}$, as claimed.

We may now finish the proof of (3). First $M_{(\tau)}$ is open in M by (1) and dense in M by the claim. Let $B \subseteq M_{(\tau)}/G$ be a non-empty open and closed subset, and set $A := \pi^{-1}(B)$, $\pi : M_{(\tau)} \to M_{(\tau)}/G$ being the canonical projection. By the claim $\bar{A} = M$, hence $A = M_{(\tau)}$ and $B = M_{(\tau)}/G$. This proves that $M_{(\tau)}/G$ is connected.

Finally, let $t \in \mathcal{T}$ be any orbit type, and let U be an open neighborhood of a point $p \in M$ of type t satisfying the condition in (1). Since $M_{(\tau)}$ is dense, it intersects U, so $\tau \succeq t$ and τ is the unique maximum of $(\mathcal{T}, \preceq)$. □

Remark 1.4.5. We notice here that, if $t \in \mathcal{T}$, the submanifold $M_{(t)}$ of M might be disconnected, and its connected components might have different dimensions. Indeed, if p, q are of the same type t, the fixed point space W_p of G_p in $\nu_p(G \cdot p)$ and W_q of G_q in $\nu_q(G \cdot q)$ need not have the same dimension.

Remark 1.4.6. With the notation of Theorem 1.4.4(3), we have that a point $p \in M$ is of type τ if and only if the slice representation at p is trivial. This easily follows from the proof of Theorem 1.4.4(1).

We can now give the following definitions:

Definition 1.4.7. Let G be a compact Lie group acting on a second countable manifold M, and denote by τ the maximum of the partially ordered set $(\mathcal{T}, \preceq)$. A point $p \in M$ is called:

1. *principal* if the slice representation at p is trivial (i.e. if it belongs to $M_{(\tau)}$);
2. *exceptional* if the slice representation at p is non-trivial and has discrete orbits;
3. *singular* if it is neither principal nor exceptional.

An orbit $\mathcal{O}$ of the action (G, M) is called *principal* (resp. *exceptional*, *singular*) if it contains principal (resp. exceptional, singular) points.

With the notation of Theorem 1.4.4(3), we shall often denote the open dense set $M_{(\tau)}$ by M_{pr}.

We shall now give an alternative description of principal (and exceptional) orbits. Fix $p \in M$ so that the orbit $G \cdot p$ has maximal dimension, and identify, as usual, a tubular neighborhood $\mathfrak{T}$ of $\mathcal{O}$ with $G \times_H V$, where $H := G_p$ and $V := \nu_p \mathcal{O}$.

Note that the set consisting of the points of the form $[(e,v)]$, $v \in V$, identifies with a slice through p. By Theorem 1.4.4(3) there exists a principal point $q \in \mathfrak{T}$ of the form $[(e,v)]$; notice that $G_q \subseteq G_p$. On the other hand, q being principal, $G \cdot q$ has maximal dimension, so $\dim G_q = \dim G_p$ and G_q, G_p have the same identity component. Hence p is principal if and only if G_q and G_p have the same (finite) number of connected components. In any case, for all $w \in V$, the point $q' := [(e,w)]$ satisfies $G_{q'} = (G_p)_{q'} \subseteq G_p$, hence $\dim G_{q'} = \dim G_p$ is minimal and the orbit $G_p \cdot q'$ has dimension $\dim G_p - \dim(G_p)_{q'} = 0$, so is discrete. Summarizing, we have proved the following:

Proposition 1.4.8. *Let $p \in M$ a point so that the orbit $G \cdot p$ has maximal dimension. Then p is either principal or exceptional, and p is principal if and only if G_p has a minimal number of connected components among the isotropy subgroups of the action (G, M) with the same dimension as G_p.*

The next result shows that the orbits of an action are determined by the principal ones:

Proposition 1.4.9. *Let G_1, G_2 be two compact Lie groups acting isometrically on a complete Riemannian manifold M. If there exists $p \in M$ principal for both actions (G_1, M), (G_2, M) such that $G_1 \cdot p = G_2 \cdot p$, then $G_1 \cdot q = G_2 \cdot q$ for all $q \in M$.*

Proof. Fix $q \in M$, set $\mathcal{O} := G_1 \cdot p = G_2 \cdot p$ and let $\gamma : [0, \ell] \to M$ be a minimizing geodesic joining $\mathcal{O}$ and q; without loss of generality we may assume $\gamma(0) = p$, $\gamma(\ell) = q$; furthermore, by Lemma 1.3.4, there exists $v \in \nu_p(G \cdot p)$ such that $\gamma(t) = \exp_p(tv)$, for all $t \in [0, \ell]$. Clearly, if $g_i \in G_i$, $i = 1, 2$, we have

$$\begin{aligned} g_1 \cdot q = g_1 \cdot \exp_p(\ell v) = \exp_{g_1 \cdot p}(\ell g_1 \cdot v), \\ g_2 \cdot q = g_2 \cdot \exp_p(\ell v) = \exp_{g_2 \cdot p}(\ell g_2 \cdot v), \end{aligned} \tag{1.3}$$

where we interpret g_1 and g_2 as isometries $M \to M$ in the ususal way. Since $G_1 \cdot p = G_2 \cdot p$, for any fixed $g_1 \in G_1$ there exists $g_2 \in G_2$ such that $g_1 \cdot p = g_2 \cdot p$. Now, $g_2^{-1} g_1$ is an isometry $M \to M$ which fixes p and whose differential at p fixes pointwisely the entire $\nu_p(G \cdot p)$ (indeed, p is principal; cf. Remark 1.4.6). Therefore g_1 and g_2 act as the same isometry on $\nu_p(G \cdot p)$ and by (1.3) we deduce

$$g_1 \cdot q = g_2 \cdot q \subseteq G_2 \cdot q.$$

Since $g_1 \in G_1$ was arbitrarily fixed, we get $G_1 \cdot q \subseteq G_2 \cdot q$. In the same way we prove the other inclusion. □

We can now introduce one of the most important (at list from our viewpoint) invariants that are associated to an action:

Definition 1.4.10. If $\mathcal{O}$ is a principal orbit of the action $\rho = (G, M)$, the number $\dim M - \dim \mathcal{O}$ is called *cohomogeneity* of ρ and is denoted by $\mathrm{chm}(\rho)$ or by $\mathrm{chm}(G, M)$.

Clearly, if M is connected, $\mathrm{chm}(G, M) = 0$ if and only if G acts transitively on M.

In order to compute the cohomogeneity of an action, the following result is the most important tool:

Proposition 1.4.11. *The cohomogeneity of an action $\rho = (G, M)$ coincides with the cohomogeneity of the slice representation at any point $p \in M$.*

Proof. Fix $p \in M$, and identify a tubular neighborhood around the orbit $\mathcal{O} := G \cdot p$ with $G \times_H V$ where, as usual, $H := G_p$ and $V := \nu_p \mathcal{O}$. Since a slice through p identifies with the set consisting of the points in $G \times_H V$ of the form $[(e, v)]$, $v \in V$, the slice representation identifies with the action defined by

$$h \cdot [(e, v)] := [(e, h \cdot v)], \qquad \forall\, h \in H,\ \forall\, v \in V.$$

We claim that if $q \in V$ is principal for the slice representation, then it is principal for the action (G, M) as well. Indeed, if $q' \in V$ we have $(G_p)_{q'} = G_{q'}$, so G_q is conjugate to a subgroup of $G_{q'}$ for any $q' \in V$. If we choose such $q' \in V$ principal for (G, M) the claim follows. We have then

$$\begin{aligned}
\mathrm{chm}(G_p, V) &= \dim V - \dim G_p \cdot q \\
&= (\dim M - \dim G + \dim G_p) - (\dim G_p - \dim G_q) \\
&= \dim M - \dim G + \dim G_q \\
&= \dim M - \dim G \cdot q = \mathrm{chm}(G, M).
\end{aligned}$$

□

Remark 1.4.12. If G is a disconnected group acting on a manifold M, it is clear that the orbits of the induced action (G°, M), G° being the identity component of G, are the connected components of the orbits of (G, M). In particular

$$\mathrm{chm}(G, M) = \mathrm{chm}(G^\circ, M).$$

In the next Proposition we compute the cardinality of the set of all orbit types of an action:

Proposition 1.4.13. *If the compact Lie group G acts on a second countable connected manifold M, the set of all orbit types of (G, M) is at most countable. Moreover, $\mathcal{S}$ is finite in the following cases:*

1. *if M/G is compact;*

2. *if $M = V$ is a finite-dimensional vector space on which G acts linearly.*

Proof. We use induction on $\dim M$, the case $\dim M = 0$ being obvious. Since the result is clear when G acts transitively, we may also assume $\mathrm{chm}(G, M) \geq 1$.

Now suppose the result true for all manifolds with dimension $< \dim M$; by second countability of M, it is enough to prove that, if $\mathfrak{T}$ is a tubular neighborhood of M, then the orbit types occurring in $\mathfrak{T}$ are finite.

Assume $\mathfrak{T}$ is a tubular neighborhood around the orbit $G \cdot p$. Note first that the orbit types in $\mathfrak{T}$ are in 1-1 correspondance with the orbit types of the slice representation $(G_p, \nu_p(G \cdot p))$; moreover, since a point in $\nu_p(G \cdot p)$ different from the origin has the same orbit type of its projection on the unit sphere, the orbit types of the slice representation $(G_p, \nu_p(G \cdot p))$ have finite cardinality if and only if the orbit types of the induced action of G_p on the unit sphere in $\nu_p(G \cdot p)$ have. We conclude then using the induction hypothesis (case (1)).

If M/G is compact, we can corver M with a finite number of tubular neighborhood, hence the same argument as above shows that the set of all orbit types of M is finite in this case.

If M is a finite-dimensional vector space V, then any point different from the origin has the same orbit type of its projection on the unit sphere, and we conclude using again the induction hypothesis (case (1)). □

We conclude this Section studying more closely the structure on M induced by the orbit types.

Definition 1.4.14. Let the compact Lie group G act on M, and fix $p \in M$. Set $x := G \cdot p \in M/G$. The *stratum* through p, $\mathrm{Str}(p)$, is the connected component containing p of the set $M_{(G_p)}$. The canonical projection of $\mathrm{Str}(p)$ in M/G is called *stratum* through x and is denoted $\mathrm{Str}(x)$.

Now let $p \in M$, and set $H := G_p$, $\mathcal{O} := G \cdot p$. Let $\mathfrak{T}$ be a tubular neighborhood around $\mathcal{O}$, which identifies with $G \times_H \nu_p\mathcal{O}$. From the proof of Theorem 1.4.4 we have then that $\mathfrak{T} \cap M_{(H)}$ identifies with $\frac{G}{H} \times (\nu_p\mathcal{O})^H$; in particular

$$\dim \mathrm{Str}(p) = \dim G - \dim H + \dim(\nu_p\mathcal{O})^H,$$

and the tangent space to $\mathrm{Str}(p)$ at p identifies with $T_p\mathcal{O} \oplus (\nu_p\mathcal{O})^H$. In order to obtain another useful formula for $\dim \mathrm{Str}(p)$, denote by M_0^H the connected component through p of the set $M^H := \{q \in M \mid h \cdot q = q, \forall\, h \in H\}$. We consider the surjective map

$$\begin{aligned} \chi : (G/H) \times (M_0^H \cap \mathfrak{T}) &\to G \cdot M_0^H \cap \mathfrak{T} = \mathrm{Str}(p) \cap \mathfrak{T} \\ (gH, q) &\mapsto g \cdot q \end{aligned}$$

Observing that the normalizer $N_G(H)$ of H in G acts on $M_0^H \cap \mathfrak{T}$, we easily deduce that χ factors through a diffeomorphism

$$(G/H) \times_{N_G(H)/H} (M_0^H \cap \mathfrak{T}) \to G \cdot M_0^H \cap \mathfrak{T} = \mathrm{Str}(p) \cap \mathfrak{T},$$

where here $(G/H) \times_{N_G(H)/H} (M_0^H \cap \mathfrak{T})$ denotes the orbit space of $(G/H) \times (M_0^H \cap \mathfrak{T})$ by the action of $N_G(H)/H$ defined as follows:

$$nH \cdot (gH, q) := (gn^{-1}H, n \cdot q).$$

Hence

$$\dim \mathrm{Str}(p) = \dim G - \dim N_G(H) + \dim M_0^H, \tag{1.4}$$

which is the desired formula. Turning to the dimension of strata in M/G, we note that, by Theorem 1.4.4, the canonical projection $\mathrm{Str}(p) \to \mathrm{Str}(x)$ is a bundle (with fibre G/H), where x is the orbit $\mathcal{O} = G \cdot p$ thought as a point of M/G. Hence the tangent space of $\mathrm{Str}(x)$ at x identifies with $(\nu_p\mathcal{O})^H$, and

$$\dim \mathrm{Str}(x) = \dim(\nu_p\mathcal{O})^H.$$

Denote now the orthogonal complement of $(\nu_p\mathcal{O})^H$ in $\nu_p\mathcal{O}$ by $(\nu_p\mathcal{O})^\dagger$. We are led to give the following:

Definition 1.4.15. Using the notation introduced above, the *quotient codimension of* $\mathrm{Str}(x)$ is the number

$$\mathrm{qcodim}(\mathrm{Str}(x)) := \mathrm{chm}(H, (\nu_p\mathcal{O})^\dagger).$$

The *boundary* (in the sense of Alexandrov) of M/G, $\partial(M/G)$, is the closure in M/G of the union of all strata of quotient codimension 1.

Points in M projecting to strata of quotient codimension 1 will play a fundamental role in the sequel; for this reason, they will be called *important*, or, more precisely, G-*important*.

Remark 1.4.16. The definition of quotient codimension is motiveted by the fact that the cohomogeneity of an action coincides with the Hausdorff dimension of the orbit space (with respect to a canonical metric structure which will be introduced in Section 1.6).

1.5 Copolarity of isometric actions

In this Section we are going to associate another very important invariant to an action (G, M) of a compact Lie group G on a second countable manifold M. This invariant, called *copolarity*, was firstly introduced by Claudio Gorodski, Carlos Olmos and Ruy Tojeiro in [22]; it generalizes the concept of *polar action*, which goes back to Jiri Dadok, Richard S. Palais and Chuu-Lian Terng ([12, 35]). Throughout the following, $\mathbf{g}$ will denote a fixed complete, G-invariant Riemannian metric on M (cf. Remark 1.3.2).

Definition 1.5.1 (cf. [22])**.** Let (G, M) be an isometric action of a compact Lie group G on a complete Riemannian manifold M. Let $k \geq 0$ be an integer. A k-*section* for (G, M) is a complete, connected, embedded submanifold Σ of M such that:

1. Σ is totally geodesic in M;

2. Σ intersects every G-orbit;

3. for any $p \in M_{\mathrm{pr}} \cap \Sigma$, $T_p\Sigma$ contains $\nu_p(G \cdot p)$ as a subspace of codimension k;

4. for any $p \in M_{\mathrm{pr}} \cap \Sigma$ and $g \in G$, $g \cdot p \in \Sigma \Rightarrow g\Sigma = \Sigma$.

Note that if there exists a k-section Σ through a point $p \in M$, then $g \cdot \Sigma$ is a k-section through the point $g \cdot p$. In particular, if a k-section exists, then there exists a k-section through any point of M. A k-section will be called a *generalized section* if we are not interested in the integer k.

Moreover, since a k-section is a connected and totally geodesic submanifold, we see, using Lemma 1.3.4, that if Σ_1 is a k_1-section through a principal point $p \in M$, and Σ_2 is a k_2-section through the same point p, then then the connected component containing p of the intersection $\Sigma_1 \cap \Sigma_2$ is a k-section with $k \leq \min\{k_1, k_2\}$. Since the whole manifold M is trivially an n-section containing p, where n is the dimension of a principal orbit, we see that there exists exactly one k_0-section passing through p, with k_0 minimal. The number k_0 clearly does not depend on the principal point p, but only on the action.

Definition 1.5.2 (cf. [22])**.** The *copolarity* of the isometric action (G, M), G a compact Lie group, M a complete Riemannian manifold, is the number k_0 defined above. It will be denoted by $\mathrm{c}(G, M)$.

Clearly $\mathrm{c}(G, M) \in \{0, 1, \cdots, n\}$, where n is the dimension of a principal orbit. If $\mathrm{c}(G, M) = n$, i.e. if M contains no proper generalized section, we say, following [22], that (G, M) has *trivial copolarity.*

Definition 1.5.3 (cf. [4, 12, 35, 36])**.** An isometric action (G, M) is called *polar* if $\mathrm{c}(G, M) = 0$. In this case, a 0-section is simply called a *section.*

Remark 1.5.4. If we drop the condition (4) in Definition 1.5.1, the corresponding submanifold Σ is called a *pre-section*. Note that for a pre-section Σ and $g \in G$, the set $g \cdot \Sigma$ is again a pre-section, and also the intersection of two pre-sections is again a pre-section. Now it is clear that a minimal pre-section fulfills (4), so it is actually a generalized section. In particular, in order to show that an isometric action has copolarity $\leq k$, it is enough to construct a pre-section which contains the normal space to all principal orbits it meets with codimension k.

Remark 1.5.5. Since generalized sections are complete and totally geodesic submanifolds, Lemma 1.3.4 implies that, in Definition 1.5.1, condition (2) follows from condition (3), provided that the latter is non-empty, i.e. if we assume that Σ contains a principal point.

We shall now prove some useful properties about (generalized) sections and polar representations that will be used later on. We begin with a Lemma:

Lemma 1.5.6 (cf. [22]). *Let (G, M) be an isometric action, $p \in M$ and $v \in V := \nu_p(G \cdot p)$. The following statements are equivalent:*

1. *v is principal for the slice representation at p;*
2. *there exists $\varepsilon > 0$ so that the points $\exp_p(tv)$, $0 < t < \varepsilon$, are principal for the action (G, M).*
3. *there exists $t_0 > 0$ such that $\exp_p(t_0 v)$ is principal for the action (G, M).*

Proof. (1) $\Leftrightarrow$ (2). By the Tube Theorem, we may identify a tubular neighborhood $\mathfrak{T}$ around $G \cdot p$ to $G \times_{G_p} B_r$, where B_r is the ball in $\nu_p(G \cdot p)$ centred at the origin and of radius r, if we choose $r > 0$ small enough. Then, if we set $\varepsilon := r$, the points $\exp_p(tv)$, $0 < t < \varepsilon$, are contained in $\mathfrak{T}$ and correspond to $[(e, tv)]$ under the above identification. The same argument as in the proof of Proposition 1.4.11 shows then that v is principal for the slice representation if and only if $[(e, tv)] \equiv \exp_p(tv)$, $0 < t < \varepsilon$, is principal for (G, M), as claimed.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (2). Set $q := \exp(t_0 v)$, choose $0 < \varepsilon < \min\{r, t_0\}$ where r is as above, and notice that

$$G_{\exp_p(tv)} = (G_p)_v \subseteq G_q \subseteq G_{\exp_p(tv)}, \qquad 0 < t < \varepsilon;$$

here the last inclusion follows from the fact that the slice representation at q is trivial (since q is principal for (G, M)), hence an element g which fixes q must fix any geodesic through q perpendicular to the orbit $G \cdot q$. So, if $0 < t < \varepsilon$, $G_{\exp_p(tv)} = G_q$ and $\exp_p(tv)$ is principal. $\square$

As a consequence we deduce:

Proposition 1.5.7. *If Σ is a generalized section for the isometric action (G, M), the set $\Sigma_{pr} := \Sigma \cap M_{pr}$ is open and dense in Σ.*

Proof. Clearly $\Sigma_{\rm pr}$ is open in Σ. Fix now $p \in \Sigma$, $q \in \Sigma_{\rm pr}$, and let $\gamma : [0, a] \to M$ a minimizing geodesic from p to q. Since $q \in \Sigma$ is principal and Σ is totally geodesic, $\gamma([0, a]) \subseteq \Sigma$. Moreover Lemma 1.5.6 implies that $\gamma(t) \in M_{\rm pr}$ for every $t > 0$ sufficiently small, and the result follows. □

The next two Lemmas are trivial but often useful:

Lemma 1.5.8. *Let (G, M) be an isometric action, and denote by G° the identity component of G. If Σ is a generalized section of (G°, M), then it is also a generalized section of (G, M).*

In the special case of 0-section we can be more precise:

Lemma 1.5.9. *Σ is a section for an isometric action (G, M) if and only if it is a section for the action (G°, M). In particular (G, M) is polar if and only if (G°, M) is polar.*

Generalized sections in M provide generalized sections of the slice representation:

Proposition 1.5.10. *Let (G, M) be an isometric action and Σ a k-section. For each $p \in \Sigma$, the intersection $T_p\Sigma \cap \nu_p(G \cdot p)$ is a k_1-section of the slice representation at p with $k_1 \leq k$.*

Proof. Set $V_p := T_p\Sigma \cap \nu_p(G \cdot p)$; clearly it is a connected, complete, embedded submanifold of $\nu_p(G \cdot p)$. We shall now prove conditions (1)-(4) in Definition 1.5.1. Condition (1) is obvious. Condition (3) is equivalent to show that $V_p^\perp \subseteq T_v(G_p \cdot v)$ for any $v \in V_p$ principal for the slice representation (here $V_p^\perp$ denotes the orthogonal complement of V_p in $\nu_p(G \cdot p)$). Since the slice representation is linear, and using the Tube Theorem, up to a rescaling of v we may assume that $q := \exp_p(v)$ belongs to a slice S_p through p. Then q is principal for (G, M). Choose $w \in V_p^\perp$, and let J be the Jacobi field along $\exp(tv)$ such that $J(0) = O$, $J'(0) = w$. Clearly J is everywhere tangent to S_p; in particular $\mathrm{d}(\exp_p)_v(w) = J(1) \in \nu_q\Sigma \subseteq T_q(G \cdot q)$, since Σ fulfills (3). The intersection of a G-orbit with the slice S_p is parametrized by G_p, hence $J(1) \in T_q(G_p \cdot q)$, which is the image of $T_v(G_p \cdot v)$ under the differential of the normal exponential map, therefore $w \in T_v(G_p \cdot v)$, as claimed.

Condition (2) will follows if can prove that V_p contains principal points for the slice representation (cf. Remark 1.5.5). Let $\mathcal{O}$ be a principal G-orbit, and choose a connected component $\mathcal{B}$ of $\Sigma \cap \mathcal{O}$. Let $\gamma = \exp_p(tv)$, $0 \leq t \leq a$, be a minimizing geodesic in Σ from p to $\mathcal{B}$. Then $\gamma'(a) \in T_{\gamma(a)}\Sigma = T_{\gamma(a)}\mathcal{B} \oplus \nu_{\gamma(a)}(G \cdot \gamma(a))$. On the other hand, since γ is minimizing, $\gamma'(a)$ must be perpendicular to $\mathcal{B}$; therefore

$\gamma'(a) \in \nu_{\gamma(a)}(G \cdot \gamma(a))$. By Lemma 1.3.3, $v = \gamma'(0) \in T_p\Sigma \cap \nu_p(G \cdot p) = V_p$, and we deduce that v is regular for the slice representation at p thanks to Lemma 1.5.6.

Finally we prove Condition (4). Let $v \in V_p$ be a principal point for the slice representation, and let $h \in G_p$ such that $h \cdot v \in V_p$. Recalling the definition this means $\mathrm{d}h_p(v) \in V_p$, where h is thought here as the isometry $M \to M$ given by $q \mapsto h \cdot q$. Note that v, $\mathrm{d}h_p(v)$ are mapped by the exponential map to two points of the form p', $h \cdot p'$. After possibly a rescaling of v we may assume that p', $h \cdot p'$ belongs to a slice S_p through p. Since Σ is totally geodesic and satisfies (4) we deduce $p', h \cdot p' \in \Sigma$ and $h \cdot \Sigma = \Sigma$. Differentiation implies now $\mathrm{d}h_p(T_p\Sigma) = T_p\Sigma$, and, since h is an isometry, $\mathrm{d}h_p(V_p) = V_p$. This means $h \cdot V_p = V_p$ with respect to the slice representation, and the proof is finished. □

Corollary 1.5.11. *If an isometric action (G, M) has copolarity k, then any slice representation of (G, M) has copolarity $\leq k$.*

Corollary 1.5.12. *Every slice representation of a polar action (G, M) is polar. Moreover, if Σ is a section through a point p, then $T_p\Sigma$ is a section for the slice representation at p.*

We conclude this Section proving some result about polar actions. The first one shows that sections always meet the orbit perpendicularly.

Proposition 1.5.13. *Let (G, M) be a polar isometric action, and let Σ be a section. Then Σ intersects every orbit perpendicularly.*

Proof. If $p \in \Sigma$ is a principal point, by definition we have $T_p\Sigma = \nu_p(G \cdot p)$, so Σ intersects perpendicularly every principal orbit. The general case follows from Proposition 1.5.7. □

We observe now that, if (G, M) is an isometric action with canonical projection $\pi : M \to M/G$, the Tube Theorem easily implies that the restriction of π to the principal stratum M_{pr}

$$\pi : M_{\mathrm{pr}} \to M_{\mathrm{pr}}/G$$

is a Riemannian submersion; the vertical distribution $\mathcal{V}$ is clearly tangent to the orbits. For a polar action the horizontal distribution $\mathcal{H} := \mathcal{V}^{\perp}$ (defined only over M_{pr}) is integrable, the sections being its integral submanifolds. Conversely, if $\mathcal{H}$ is integrable, then the action admits sections (cf. [2, 25]); however, such sections may not be regularly embedded in M (as required by Definition 1.5.1). An example of this situation is provided in [35].

There is an important case in which the integrability of the horizontal distributions implies the existence of regularly embedded sections, namely the case of simply connected space forms. This case will be of particular interest to us, since from the next chapter on M will always be a vector space endowed with a G-invariant inner product.

Theorem 1.5.14. *If M is a simply connected space form, an isometric action (G, M) is polar if and only if the horizontal distribution over the principal stratum M_{pr} is integrable.*

Proof. If $\tilde{\Sigma}$ is a leaf of $\mathcal{H}$, then $\tilde{\Sigma}$ is totally geodesic (cf. [42]). Since M is a simply connected space form, say of costant sectional curvature κ, there exists a complete, connected, totally geodesic submanifold Σ containing $\tilde{\Sigma}$, with $\dim \Sigma = \dim \tilde{\Sigma}$; furthermore, Σ is isometric to a simply connected space form $M'(\kappa)$, which is canonically embedded in M (cf. [4]). Clearly Σ contains principal points, since it contains $\tilde{\Sigma}$, which is defined only on the principal stratum M_{pr}; moreover, if $p \in \Sigma$ is principal, then $T_p\Sigma = \nu_p(G \cdot p)$ and we are done. □

1.6 Metric structure of the orbit space

We begin this section introducing a natural metric structure on the orbit space of an isometric action, and studying its main properties.

Assume than that G is a compact Lie group acting isometrically on the complete Riemannian manifold $(M, \mathbf{g})$. We can define a distance d in the orbit space M/G in the following way: if $x = G \cdot p$, $y = G \cdot q$ are points in M/G, we set

$$\mathrm{d}(x, y) := \min\{\mathrm{d}_M(g \cdot p, h \cdot q) \mid g, h \in G\} = \min\{\mathrm{d}_M(p, g \cdot q) \mid g \in G\},$$

where d_M denotes the Riemannian distance in $(M, \mathbf{g})$. In other words, the distance between the points $G \cdot p$ and $G \cdot q$ in G/M is, by definition, the distance between the sets $G \cdot p$ and $G \cdot q$ in M.

Note that d is well-defined because the orbits of (G, M) are compact submanifolds.

Remark 1.6.1. The distance between two orbits in M is realized by the length of a minimizing geodesic, orthogonal to all orbit it meets, by Lemma 1.3.4. Hence the projection of a metric ball in M is a metric ball of the same radius in M/G.

From Remark 1.6.1 we easily get:

Lemma 1.6.2. *The topology on M/G induced by the distance* d *coincides with the quotient topology.*

In order to state some more interesting properties of the metric space $(M/G, \mathrm{d})$ we briefly recall here some definitions from Metric Geometry; we refer to [7, 9] for more details.

Definition 1.6.3. Let (X, d) be a metric space. The length of a continuous path $\gamma : [a, b] \to X$ is defined by

$$L(\gamma) := \sup \left\{ \sum_{i=0}^{n} d(\gamma(t_i), \gamma(t_{i+1})) \;\middle|\; a = t_0 < t_1 < \cdots < t_n = b \right\}.$$

The path γ is called *rectifiable* if $L(\gamma)$ is finite.

To any metric space (X, d) we can then associate a new distance, $\bar{d}$, defined, for every $x, y \in X$, by

$$\bar{d}(x, y) := \inf\{L(\gamma) \mid \gamma : [a, b] \to X \text{ is a continuous path, } \gamma(a) = x,\ \gamma(b) = y\}.$$

Clearly $\bar{d}(x, y) = \infty$ if and only if there are no rectifiable paths joining x and y.

Remark 1.6.4. The following statments are easily proved using the definition:

1. $\bar{d}$ is a well-defined distance on X;
2. $\bar{d}(x, y) \geq d(x, y)$ for all $x, y \in X$;
3. if $\gamma : [a, b] \to X$ is continuous with respect to the topology induced by $\bar{d}$, then it is continuous with respect to the topology induced by d;
4. if $\gamma : [a, b] \to X$ is a continuous and rectifiable path in (X, d), then it is a continuous and rectifiable path in $(X, \bar{d})$;
5. the length of a path in $(X, \bar{d})$ is the same as its length in (X, d);
6. $\bar{\bar{d}} = \bar{d}$.

Definition 1.6.5. A metric space (X, d) is called a *length space* if $\bar{d} = d$. In this case we say that d is a *length distance.*

By Remark 1.6.4(6), we can associate a length space to any matric space (X, d), namely $(X, \bar{d})$.

One of the reasons which make length spaces so interesting is that for them an analogous of the usual Hopf-Rinow Theorem for Riemannian manifolds holds. In order to state precisely such result we need to introduce the concept of geodesic in any metric space.

Definition 1.6.6. Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$, or, more briefly, a *geodesic* from x to y, is a continuous path γ from a closed interval $[0, \ell] \subseteq \mathbb{R}$ to X such that $\gamma(0) = x$, $\gamma(\ell) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, \ell]$. If $\gamma(0) = x$, then γ is said to *issue* from x. The image of γ is called a *geodesic segment* with endpoints x and y.

A *geodesic ray* in X is a continuous map $\gamma : [0, \infty) \to X$ so that $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \geq 0$. Similarly, a *geodesic line* is a continuous map $\gamma : \mathbb{R} \to X$ so that $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in \mathbb{R}$

A *local geodesic* in X is a continuous map from an interval $I \subseteq \mathbb{R}$ to X with the property that for every $t \in I$, there exists $\varepsilon > 0$ such that $d(\gamma(t'), \gamma(t'')) = |t' - t''|$ for all $t', t'' \in I$ with $|t - t'| + |t - t''| < \varepsilon$.

(X, d) is said to be a *geodesic metric space* or, more briefly, a *geodesic space*, if every two points in X are joined by a geodesic. Similarly, given $r > 0$, (X, d) is said to be *r-geodesic* if for every pair of points $x, y \in X$ with $d(x, y) < r$ there is a geodesic joining x to y.

A subset C of a metric space (X, d) is said to be *convex* if every pair of points $x, y \in C$ can be joined by a geodesic in X and the image of every such geodesic is contained in C. If this condition holds for all points $x, y \in C$ with $d(x, y) < r$, then C is said to be *r-convex*.

We can now state the version of the Hopf-Rinow Theorem for length spaces (cf. [9] for a proof):

Theorem 1.6.7 (Hopf, Rinow, Cohn-Vossen)**.** *For a locally compact length space (X, d) the following assertions are equivalent:*

1. *d is complete;*
2. *every closed metric ball in (X, d) is compact;*
3. *every local geodesic $\gamma : [0, \ell) \to X$ can be extended to a continuous path $\bar{\gamma} : [0, \ell] \to X$.*

Moreover, these conditions imply that (X, d) is a geodesic space.

We are now ready to prove the following:

Theorem 1.6.8. *Let G be a compact Lie group acting isometrically on a complete Riemannian manifold $(M, \mathbf{g})$. Then the metric space $(M/G, \mathrm{d})$ is a complete, locally compact length space.*

Proof. By Lemmas 1.2.3 and 1.6.1 we have that $(M/G, \mathrm{d})$ is a locally compact metric space. Since moreover the distance between two orbits in M is realized by a minimizing geodesic, we easily deduce that

$$\mathrm{d}_M(p, q) \geq \mathrm{d}(G \cdot p, G \cdot q), \qquad \forall\, p, q \in M,$$

therefore the canonical projection $\pi : M \to M/G$ does not increase the distances. Hence the projection of a rectifiable curve in M is a rectifiable curve in M/G, and since (by definition) the Riemannian distance d_M in M is a length distance, the same must be true for d, proving that $(M/G, \mathrm{d})$ is a length space.

Finally, M being complete, every closed metric ball in M is compact, therefore, by Remark 1.6.1, the same is true for $(M/G, \mathrm{d})$. The Hopf-Rinow Theorem for length spaces then implies that d is complete. □

Remark 1.6.9. At this point it is clear that any two points in M/G can be joined by a geodesic. On the other hand observe that the projections of horizontal geodesics in M (i.e. geodesics perpendicular to all orbits they meet) are suitable concatenations of geodesics in M/G. Indeed, it follows from the Tube Theorem that such a projection γ has the property that for each t in its domain, there exists $\varepsilon > 0$ such that $\gamma|_{[t-\varepsilon,t]}$, $\gamma|_{[t,t+\varepsilon]}$ are shortest paths.

We now proceed with some other definitions for general metric spaces. In what follows we shall denote by $\mathbb{E}^2$ the euclidean space $\mathbb{R}^2$ endowed with the standard metric.

Definition 1.6.10. Let (X, d) be a metric space. A *comparison triangle* in $\mathbb{E}^2$ for a triple of points (p, q, r) in X is a triangle in the euclidean plane with vertices $\bar{p}$, $\bar{q}$, $\bar{r}$ such that $d(p, q) = d(\bar{p}, \bar{q})$, $d(q, r) = d(\bar{q}, \bar{r})$ and $d(p, r) = d(\bar{p}, \bar{r})$. Such triangle is unique up to isometry, and shall be denoted $\bar{\Delta}(p, q, r)$. The interior angle of $\bar{\Delta}(p, q, r)$ at $\bar{p}$ is called the *comparison angle* between q and r at p and is denoted by $\bar{\angle}_p(q, r)$.

Note that the comparison angle is well-defined provided that q and r are both distinct from p.

Definition 1.6.11. Let (X, d) be a metric space, and $\gamma : [0, a] \to X$, $\tilde{\gamma} : [0, \tilde{a}] \to X$ be two geodesic paths with $\gamma(0) = \tilde{\gamma}(0)$. If $t \in (0, a]$, $\tilde{t} \in (0, \tilde{a}]$, we consider the comparison triangle $\bar{\Delta}(\gamma(0), \gamma(t), \tilde{\gamma}(\tilde{t}))$ and the comparison angle $\bar{\angle}_{\gamma(0)}(\gamma(t), \tilde{\gamma}(\tilde{t}))$. The *angle* between the geodesic paths γ, $\tilde{\gamma}$ is the number $\angle_{\gamma,\tilde{\gamma}} \in [0, \pi]$ defined by:

$$\angle_{\gamma,\tilde{\gamma}} := \lim_{\varepsilon \to 0} \sup_{0<t,\tilde{t}<\varepsilon} \bar{\angle}_{\gamma(0)}(\gamma(t), \tilde{\gamma}(\tilde{t})).$$

The angle $\angle_{\gamma,\tilde{\gamma}}$ can be expressed purely in term of the distance by noting that

$$\bar{\angle}_{\gamma(0)}(\gamma(t), \tilde{\gamma}(\tilde{t})) = \arccos\left(\frac{t^2 + \tilde{t}^2 - d(\gamma(t), \tilde{\gamma}(\tilde{t}))^2}{2t\tilde{t}}\right)$$

Remark 1.6.12. The angle between γ and $\tilde{\gamma}$ depends only on the germs of these paths at 0: if $\hat{\gamma} : [0, b] \to X$ is any geodesic path for which there exists $\varepsilon > 0$ such that $\hat{\gamma}|_{[0,\varepsilon]} = \tilde{\gamma}|_{[0,\varepsilon]}$, then the angle between γ and $\hat{\gamma}$ is the same as that between γ and $\tilde{\gamma}$.

It is now just a matter of patience (otherwise cf. [7, p. 10]) to use the definition to prove the following:

Lemma 1.6.13. *Let (X, d) be a metric space, and let γ, $\tilde{\gamma}$, $\hat{\gamma}$ be three geodesics paths issuing from the same point p. Then*

$$\angle_{\gamma,\hat{\gamma}} \leq \angle_{\gamma,\tilde{\gamma}} + \angle_{\tilde{\gamma},\hat{\gamma}}. \tag{1.5}$$

Definition 1.6.14. Let (X, d) be a metric space. Two non-trivial geodesics γ, $\tilde{\gamma}$ issuing from a point $p \in X$ are said to define the *same direction* at p if the angle between them is zero. In this case we write $\gamma \sim \tilde{\gamma}$. The triangle inequality (1.5) implies that $\sim$ is an equivalence relation on the set of non-trivial geodesics issuing from p, and $\angle$ induces a distance on the set of equivalence classes. The resulting metric space is called the *space of directions* at p and is denoted by $S_p(X)$. The euclidean cone over S_pX is called the *tangent cone* at p and is denoted by $C_p(X)$.

Note that two geodesics issuing from p with the same direction might have images intersecting only at p.

Remark 1.6.15. If X is a Riemannian manifold, then $S_p(X)$ is isometric to the unit sphere in the tangent space T_pX, while $C_p(X)$ is isometric to T_pX itself.

The following proposition is a simple consequence of the Tube Theorem:

Proposition 1.6.16. *Let G be a compact Lie group acting isometrically on a complete Riemannian manifold $(M, \mathbf{g})$. Fix $p \in M$ and set $x := G \cdot p \in M/G$. Then $C_x(M/G)$ is isometric to the orbit space of the slice representation at p.*

Remark 1.6.17. From Proposition 1.6.16 it follows that strata in M/G can be characterized as the connected components of the sets of points in M/G with isometric tangent cones.

Now we briefly recall the definition of an Alexandrov space (we refer to [7] for more details).

Denote by $M^n(\kappa)$ the simply connected space form of dimension $n \in \mathbb{N}$ with costant sectional curvature $\kappa \in \mathbb{R}$, and set $D_\kappa := \infty$ if $\kappa \leq 0$, $D_\kappa := \pi/\sqrt{\kappa}$ if $\kappa > 0$; D_κ is called the *diameter* of $M^n(\kappa)$. Note that, by classical results in Riemannian Geometry, two points $x, y \in M^n(\kappa)$ are joined by a unique minimizing geodesic if and only if $d(x, y) < D_\kappa$.

Definition 1.6.18. A *geodesic triangle* Δ in a metric space X consists of three points p, q, $r \in X$, called *vertices*, and a choice of three geodesic segments $[p, q]$, $[q, r]$, $[r, p]$ joining them, called *sides*. Such a geodesic triangle will be denoted by $\Delta([p, q], [q, r], [r, p])$ or, less accurately if in X there are pairs of points joined by more than one geodesic segment, by $\Delta(p, q, r)$.

The main tool in the definition of Alexandrov spaces is the following lemma (cf. [7, p. 24] for a proof):

Lemma 1.6.19 (Alexandrov)**.** *Let κ be a real number, and let p, q, r be three points in a metric space (X, d); if $\kappa > 0$ assume that $d(p, q) + d(q, r) + d(r, p) < 2D_k$. Then there exist points $\bar{p}, \bar{q}, \bar{r} \in M^2(\kappa)$ such that $d(p, q) = d(\bar{p}, \bar{q})$, $d(p, r) = d(\bar{p}, \bar{r})$, $d(q, r) = d(\bar{q}, \bar{r})$. The geodesic triangle $\bar{\Delta}(\bar{p}, \bar{q}, \bar{r})$ in $M^2(\kappa)$ is called a* comparison triangle *for the triple (p, q, r); it is unique up to an isometry of $M^2(\kappa)$.*

Let $\Delta(p,q,r)$ a geodesic triangle in the metric space (X,d), and let $\bar{\Delta}(\bar{p},\bar{q},\bar{r})$ be a comparison triangle for $\Delta(p,q,r)$ (i.e. a comparison triangle for the triple (p,q,r)) in $M^2(\kappa)$, $\kappa \in \mathbb{R}$. A point $\bar{x} \in [q,r]$ is called a *comparison point* for $x \in [q,r]$ if $d(q,x) = d(\bar{q},\bar{x})$. Comparison points on $[\bar{p},\bar{q}]$ and $[\bar{p},\bar{r}]$ are defined similarly.

Definition 1.6.20. Let (X,d) be a metric space, and let κ be a real number. Let Δ be a geodesic triangle in X; if $\kappa > 0$ assume that the perimeter of Δ is less than $2D_\kappa$. Let $\bar{\Delta} \subseteq M^2_\kappa$ be a comparison triangle for Δ (which exists by Alexandrov's Lemma). Then Δ is said to satisfy the CAT(κ) *inequality* if, for all $x, y \in \Delta$, and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$ we have

$$d(x,y) \geq d(\bar{x},\bar{y}).$$

If $\kappa \leq 0$, then X is called a CAT(κ)-*space* if X is a geodesic space all of whose geodesic triangles satisfy the CAT(κ)-inquality.

If $\kappa > 0$, then X is called a CAT(κ)-*space* if X is a D_κ-geodesic space all of whose geodesic triangles with perimeter less than $2D_\kappa$ satisfy the CAT(κ)-inquality.

Definition 1.6.21. A metric space (X,d) is said to be of *curvature* $\geq \kappa$ *in the sense of Alexandrov* if it is locally a CAT(κ)-space.

An *Alexandrov space* is a metric space (X,d) with curvature bounded below in the sense of Alexandrov.

The following result holds (see [9, p. 356] for a proof):

Theorem 1.6.22. *Let G be a compact Lie group acting isometrically on a complete Riemannian manifold $(M,\mathbf{g})$, and assume that the sectional curvatures of M are bounded below by $\kappa \in \mathbb{R}$. Then the orbit space M/G is of curvature $\geq \kappa$ in the sense of Alexandrov.*

1.7 Basic theory of reductions

We have seen in the previous section that the orbit space is a very important metric object that can be associated to an isometric action (G,M) of a compact Lie group G on a complete Riemannian manifold M. An interesting question which has been recently rised by Claudio Gorodski and Alexander Lytchak asks how much information about the action can be recovered from its orbit space (cf. [19]); in other words, we are looking for algebraic properties of (G,M) which can be read of the quotient M/G. In order to make these statements more precise, we give, following [19], the following definition:

Definition 1.7.1 ([19])**.** Two actions (G_i, M_i), $i = 1, 2$, of the compact Lie groups G_1, G_2 on the complete Riemannian manifolds M_1, M_2 respectively are said to be *quotient-equivalent* if their orbit spaces M_1/G_1, M_2/G_2 are isometric.

This means that we are interested in those properties of an action which depend only on the quotient-equivalence class of the action.

Example 1.7.2. If two actions ρ_1, ρ_2 are quotient-equivalent, $\mathrm{chm}(\rho_1) = \mathrm{chm}(\rho_2)$. Indeed, the cohomogeneity of an action coincides with the Hausdorff dimension of the orbit space.

The problem of understending whether a given property of an action (G, M) is constant within quotient-equivalent classes seems rather difficult, and in fact there are simple algebraic invariant of (G, M) (such as $\dim M$, cf. Corollary 2.3.18 or [19, Prop. 1.1], and the examples of Chapter 3) which are not invariant of the quotient. Some partial results in this direction have been given by Claudio Gorodski and Alexander Lytchak when M is a finite dimensional vector space V on which G acts linearly, i.e. the case of representations (cf. [19, 20]); we shall describe them in Chapter 2, which will be dedicated to Representation Theory. The goal of this thesis is to extend some of such results. We remark here that, however, the kind of problems we have just introduced has originated some new interesting classes of representations, which generalize the polar ones, that will be widely discussed in the following chapters.

Before going on, we point out a special case of quotient-equivalence:

Definition 1.7.3. Two actions (G_i, M_i), $i = 1, 2$, of the compact Lie groups G_1, G_2 on the complete Riemannian manifolds M_1, M_2 respectively are said to be *orbit-equivalent* if there is an isometry $f : M_1 \to M_2$ which maps orbits to orbits.

Since we work exclusively with actions of *compact* Lie groups on *complete Riemannian* manifolds, for the sake of brevity we shall always omit to mention these assumptions.

In order to study a quotient-equivalence class of actions, it is useful to hierarchize its elements as follows:

Definition 1.7.4 ([19])**.** Given two actions $\rho_i = (G_i, M_i)$, $i = 1, 2$, we say that ρ_1 is a *reduction* of ρ_2 if ρ_1 and ρ_2 are quotient-equivalent and $\dim G_1 < \dim G_2$.

Definition 1.7.5 ([19])**.** Let $\mathcal{X}$ be a quotient-equivalence class of actions. We say that $(G, M) \in \mathcal{X}$ is *reduced* if $\dim G$ is minimal in $\mathcal{X}$. In this case the number $\dim G$ is called the *abstract copolarity* of any $\rho \in \mathcal{X}$, and is denoted by $\mathrm{ac}(\rho)$.

Definition 1.7.6 ([19])**.** Given two actions ρ_i, $i = 1, 2$, we say that ρ_1 is a *minimal reduction* of ρ_2 if ρ_1 is a reduction of ρ_2 and is reduced.

Remark 1.7.7. It is not known whether copolarity is an invariant of the quotient (cf. [19, p. 69]); for this reason, it is often convenient to work with abstract copolarity when dealing with problems that involve quotient-equivalent classes. In any case we shall prove later in this section that, for any action ρ, $\mathrm{ac}(\rho) \leq \mathrm{c}(\rho)$.

It seems that most of actions do not admit a reduction; indeed, the existence of a reduction entails the presence of interesting geometric properties and bounds the complexity of the action and its orbit space. As we have already mentioned, most of the results in this field have been obtained for representations, so they will be described later. The rest of this section is dedicated to show that proper generalized sections give rise to reductions in a standard way (cf. [32, 22]).

Let Σ be a generalized section of an action (G, M), and set

$$\begin{aligned} N(\Sigma) &:= \{g \in G \mid g \cdot \Sigma = \Sigma\}, \\ Z(\Sigma) &:= \{g \in G \mid g \cdot p = p, \, \forall \, p \in \Sigma\}, \\ W(\Sigma) &:= N(\Sigma)/Z(\Sigma). \end{aligned}$$

The groups $N(\Sigma)$, $Z(\Sigma)$, $W(\Sigma)$ are respectively called the *normalizer* of Σ, the *centralizer* of Σ and the *Weyl group* of Σ. Clearly $N(\Sigma)$ acts on Σ with kernel $Z(\Sigma)$. We shall prove the following result:

Theorem 1.7.8 ([32, 22, 35])**.** *Let Σ be a generalized section of (G, M), and $W := W(\Sigma)$ the corresponding Weyl group. Then the action (W, Σ) is effective and quotient-equivalent to (G, M).*

Note that if Σ is properly contained in M, then $\dim W < \dim G$ and (W, Σ) is a reduction of (G, M).

We need the following lemmas:

Lemma 1.7.9. *Let Σ be a k-section of (G, M). Given $p \in \Sigma$, denote by $\mathcal{K}_p := \{g \cdot \Sigma \mid g \in G, \, p \in g \cdot \Sigma\}$ the set of k-sections through p which are G-translates of Σ. Then the isotropy group G_p acts transitively on $\mathcal{K}_p$.*

Proof. If p is a principal point, $|\mathcal{K}_p| = 1$ by condition (4) in Definition 1.5.1 and there is nothing to prove. So let $p \in \Sigma$ a non-principal point, and let $\Sigma_1, \Sigma_2 \in \mathcal{K}_p$. Denote by S_p a slice at p. By Proposition 1.5.10, $\Sigma_i \cap S_p$ intersects all G_p-orbit in S_p, $i = 1, 2$. Let $q \in S_p$ be G-principal; we can find $h_i \in G_p$ such that $h_i \cdot q \in \Sigma_i$; therefore $q \in h_1^{-1} \cdot \Sigma_1 \cap h_2^{-1} \cdot \Sigma_2$ and it follows $h_1^{-1} \cdot \Sigma_1 = h_2^{-1} \cdot \Sigma_2$. Hence $h_2 h_1^{-1} \in G_p$ is so that $(h_2 h_1^{-1}) \cdot \Sigma_1 = \Sigma_2$, and we are done. □

Lemma 1.7.10. *Let Σ be a k-section of (G, M), and set $N := N(\Sigma)$. Then*

$$N \cdot p = \Sigma \cap (G \cdot p), \qquad \forall \, p \in \Sigma.$$

Proof. Since N acts on Σ we have clearly $N\cdot p \subseteq \Sigma\cap(G\cdot p)$. Conversely, let $g\in G$, $p\in\Sigma$ so that $g\cdot p\in\Sigma$. Then Σ and $g\cdot\Sigma$ are both k-sections through $g\cdot p$ and by Lemma 1.7.9 there exists $h\in G_{g\cdot p}$ such that $(hg)\cdot\Sigma=\Sigma$, so $hg\in N$. On the other hand $g\cdot p = h\cdot(g\cdot p)=(hg)\cdot p\in N\cdot p$, proving that $N\cdot p\supseteq\Sigma\cap(G\cdot p)$. □

Proof of Theorem 1.7.8. We shall show that the inclusion $\iota:\Sigma\to M$ induces an isometry $I:\Sigma/N\to M/G$, where $N:=N(\Sigma)$.

First observe that I is well-defined (because $N\subseteq G$), continuous (since the orbit spaces are endowed with the quotient topology) and satisfies

$$\mathrm{d}(I(x),I(y))\leq \mathrm{d}_\Sigma(x,y),\qquad \forall\, x,y\in\Sigma/N$$

since Σ is totally geodesic; here d (resp. d_Σ) denotes the usual distance in M/G (resp. Σ/N). Moreover I is injective by Lemma 1.7.10.

Let now $p,p'\in\Sigma$ be G-principal points. The distance between the points $G\cdot p$, $G\cdot p'\in M/G$ is the distance between the orbits $G\cdot p, G\cdot p'$ in M, and is realized, by Lemma 1.3.4, by a geodesic γ which is perpendicular to every orbit it meets. We may suppose that γ starts from p, and call $r\in G\cdot p'$ the ending point of γ. Since Σ is totally geodesic and complete, the image of γ is contained in Σ; moreover from $G\cdot p'=G\cdot r$ and the injectivity of I we deduce $N\cdot p'=N\cdot r$. Thus the length of γ equals the distance in Σ/N between $N\cdot p$, $N\cdot p'$, and

$$\mathrm{d}_\Sigma(N\cdot p,N\cdot p')\leq \mathrm{d}(G\cdot p,G\cdot p'),$$

showing that I is an isometry between Σ_{pr}/N and M_{pr}/G. We finish the proof invoking Proposition 1.5.7, since I is continuous and Σ/N, M/G are complete metric spaces. □

As a consequence we obtain:

Corollary 1.7.11. *Let Σ be a generalized section of (G,M), and $W=W(\Sigma)$ the corresponding Weyl group. Then $\mathrm{chm}(G,M)=\mathrm{chm}(N,\Sigma)$ and $\Sigma_{\mathrm{pr}}:=\Sigma\cap M_{\mathrm{pr}}$ is exactly the principal stratum of the action (W,Σ).*

Proof. Denote by $\tilde\Sigma_{\mathrm{pr}}$ the set of principal points of (W,Σ), and let $I:\Sigma/N\to M/G$ be the isometry introduced in the proof of Theorem 1.7.8. Then I maps $\tilde\Sigma_{\mathrm{pr}}/N$ onto M_{pr}/G; indeed, by Remark 1.6.17, they are exactly the set of points in Σ/N, M/G respctively admitting a neighborhood isometric to a Riemannian manifold. Therefore, $\mathrm{chm}(N,\Sigma)=\dim\tilde\Sigma_{\mathrm{pr}}/N=\dim M_{\mathrm{pr}}/G=\mathrm{chm}(G,M)$ and $p\in\tilde\Sigma_{\mathrm{pr}}$ if and only if $p\in M_{\mathrm{pr}}\cap\Sigma=\Sigma_{\mathrm{pr}}$. □

We now describe, following [22], a very useful way to construct generalized sections of a given action (G,M). First we fix a principal isotropy subgroup H, and consider the fixed point space M^H of H. Denote by $M^H_{\text{pr}} := M^H \cap M_{\text{pr}}$ the subset of principal points of M^H. The closure in M of M^H_{pr} is called the *core* of M, and is denoted by M_{c}. It can be shown that M_{c} consists of those connected components of M^H which contain principal points of M (cf. [24, 39]). Now we prove that any connected component Σ of M_{c} is a k-section for (G, M), where $k := \dim \Sigma - \text{chm}(G, M)$. Condition (1) in Definition 1.5.1 is clearly satisfied since obviously Σ is a totally geodesic submanifold of M. Condition (3) follows from the definition of Σ and the fact that the slice representation at a principal point is trivial. Moreover, if a principal point $p \in \Sigma$ and $g \in G$ are so that $g \cdot p \in \Sigma$, then the isotropy subgroup at both p, $g \cdot p$ is H, therefore g normalizes H and hence fixes Σ. This is Condition (4). Finally Condition (2) is satisfied by Remark 1.5.5.

Notice that, if (G, M) is effective, then $M^H = M$ if and only if H is trivial. Thus:

Proposition 1.7.12. *If an effective action has non-trivial principal isotropy, then it has non-trivial copolarity and admits reductions.*

Corollary 1.7.13. *Let (G, M), (G', M') be quotient-equivalent actions, and assume that (G, M) is a minimal reduction of (G', M'). Then $\dim M < \dim M'$.*

Proof. By Proposition 1.7.12, (G, M) is faithful and has trivial principal isotropy groups. Again by Proposition 1.7.12 we may assume without loss of generality that also (G', M') is faithful with trivial principal isotropy groups. Now

$$\dim M - \dim G = \text{chm}(G, M) = \text{chm}(G', M') = \dim M' - \dim G',$$

so the assertion follows from the fact that, by definition, $\dim G < \dim G'$. □

We can now prove the the abstract copolarity of a representation is bounded above by the copolarity:

Proposition 1.7.14. *Let Σ be a minimal generalized section of (G, M), and $W = W(\Sigma)$ the corresponding Weyl group. Then $\dim W = \text{c}(G, M)$. In particular*

1. $\text{ac}(G, M) \leq \text{c}(G, M)$*;*
2. *if (G, M) is polar, then W is finite.*

Proof. First we observe that (W, Σ) has trivial principal isotropy groups. Indeed otherwise, by the discussion above, we could find a proper generalized section $\hat{\Sigma}$ for the action (W, Σ). This would be also a generalized section for (G, M), contradicting the minimality of Σ.

Using Corollary 1.7.11 we then have

$$\begin{aligned}\text{c}(G, M) &= \dim \Sigma - \text{chm}(G, M) \\ &= \dim \Sigma - \text{chm}(W, \Sigma) = \dim \Sigma - (\dim \Sigma - \dim W) = \dim W,\end{aligned}$$

as claimed. Now, (1) follows from the definition of abstract copolarity and Theorem 1.7.8, while (2) is obvious (cf. Definition 1.5.3). □

If M is a finite dimensional vector space V on which G acts linearly, the core of M coincides with the fixed point sets V^H of a principal isotropy subgroup H. Since $N(V^H) = N_G(H)$, $Z(V^H) = H$, we have the following:

Proposition 1.7.15 (Reduction principle for representations)**.** *Let* (G, V) *be a representations and let* H *be a principal isotropy subgroup. If* H *is not trivial, then* $(N_G(H)/H, V^H)$ *is a reduction of* (G, V)*.*

The reduction of a representation described in Proposition 1.7.15 is called the *Luna-Richardson-Straume reduction*, *LRS-reduction* in the sequel (cf. [23, 28, 40]).

Chapter 2

Lie group representations

2.1 Basic representation theory

In this Section we shall recall some basic definitions and facts about Representation Theory of compact Lie groups. We refer to [1, 8] for a more detailed exposition.

Throughout the following pages, we shall denote by $\mathbb{K}$ one of the three classical fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

Definition 2.1.1. Let G be a compact Lie group and V a finite-dimensional vector space over $\mathbb{K}$. A representation of G on V is a map

$$\begin{aligned} \Theta : G \times V &\to V \\ (g, v) &\mapsto g \cdot v \end{aligned}$$

satisfying the following properties:

1. $g \cdot (h \cdot v) = (gh) \cdot v$ for all $g, h \in G$, $v \in V$;
2. $e \cdot v = v$ for all $v \in V$ if e denotes the identity element of G;
3. for all $g \in G$, the map $V \to V$ given by $v \mapsto g \cdot v$ is $\mathbb{K}$-linear.

In this case we shall say that V is a $\mathbb{K}G$-space. Moreover, if $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we shall say accordingly that the representation is *real*, *complex*, *quaternionic*.

Notice that, equivalently, a representation of G over V is a Lie group homomorphism

$$\theta : G \to \mathrm{Aut}(V),$$

where $\mathrm{Aut}(V)$ denotes the Lie group of all $\mathbb{K}$-linear automorphisms of V.

With the terminology of Chapter 1, a representation of a group G on a $\mathbb{K}$-vector space V is an action of G on V satisfying condition (3) in Definition 2.1.1.

Definition 2.1.2. A $\mathbb{K}G$-map between two $\mathbb{K}G$-spaces V, W is a $\mathbb{K}$-linear map $f : V \to W$ which commutes with the action:

$$f(g \cdot v) = g \cdot f(v), \qquad \forall\, g \in G,\ \forall\, v \in V.$$

V, W are said to be *equivalent* if there exists a bijective $\mathbb{K}G$-map $f : V \to W$. In this case we write $V \simeq W$.

Definition 2.1.3. Let V be a $\mathbb{K}G$-space. A $\mathbb{K}$-subspace U of V is said to be *invariant* if $g \cdot u \in U$ for all $u \in U$, $g \in G$. We say moreover that V, or the representation of G on V, is *irreducible over* $\mathbb{K}$, or simply *irreducible* when $\mathbb{K}$ is understood, if V has no invariant $\mathbb{K}$-subspaces except V and $\{O\}$.

We shall see that real and quaternionic representations can be described by means of complex representations. In order to do this we need the following:

Definition 2.1.4. Let V be $\mathbb{C}G$-space. A *structure map* on V is a $\mathbb{C}$-antilinear map $J : V \to V$ which commutes with the action of G and stidfies $J^2 = \pm \mathrm{id}_V$.

Definition 2.1.5. An irreducible $\mathbb{C}G$-space is said to be:

1. *of real type* if it admits a structure map J with $J^2 = \mathrm{id}_V$;
2. *of quaternionic type* if it admits a structure map J with $J^2 = -\mathrm{id}_V$;
3. *of complex type* if it admits no structure maps.

This Definition is motivated by the following remark:

Remark 2.1.6. Assume that V is a $\mathbb{C}G$-space admitting a structure map J so that $J^2 = -\mathrm{id}_V$. Then V becames an $\mathbb{H}G$-space by defining $jv := J(v)$, for all $v \in V$. Conversely, if V is an $\mathbb{H}G$-space, we can obtainin a $\mathbb{C}G$-space V' by ignoring the quaternionic strucutre.Clearly V' admits a structure map J with $J^2 = -\mathrm{id}_V$, namely the map $v \mapsto jv$.

Similarly, let V be an $\mathbb{R}G$-space admitting a structure map J so that $J^2 = \mathrm{id}_V$, and denote by V' the eigenspace of J corresponding to the eigenvalue $+1$. Then V' is a real, G-invariant subspace of V, i.e. an $\mathbb{R}G$-space. Conversely, if V is an $\mathbb{R}G$-space, the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ of V is a $\mathbb{C}G$-space in a natural manner:

$$g \cdot (v \otimes z) := (g \cdot v) \otimes z, \qquad \forall\, g \in G,\ \forall\, v \in V,\ \forall\, z \in \mathbb{C}.$$

It admits a structure map J satisfying $J^2 = \mathrm{id}_V$, namely complex conjugation:

$$v \otimes z \mapsto v \otimes \bar{z}, \qquad \forall\, v \in V,\ \forall\, z \in \mathbb{C}.$$

If V, W are $\mathbb{K}G$-spaces, the direct sum $V \oplus W$ is a $\mathbb{K}G$-space by making G act on it by:

$$g \cdot (v, w) = (g \cdot v, g \cdot w), \qquad \forall\, g \in G,\ \forall\, v \in V\ \forall\, w \in W.$$

Note that if $\mathbb{K} = \mathbb{C}$ and J_V, J_W are, respectively, structure maps on V, W such that $J_V^2 = J_W^2$, then $J_V \oplus J_W$ is a structure map on $V \oplus W$.

If V, W are $\mathbb{C}G$-spaces, the tensor product $V \otimes_{\mathbb{C}} W$ becomes a $\mathbb{C}G$-space by defining

$$g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w), \qquad \forall\, g \in G,\ \forall\, v \in V\ \forall\, w \in W.$$

If J_V, J_W are structure maps on V, W respectively, then $J_V \otimes J_W$ is a structure map on $V \otimes_{\mathbb{C}} W$. Assume also $J_V^2 = \varepsilon_V \mathrm{id}_V$, $J_W^2 = \varepsilon_W \mathrm{id}_V$, where $\varepsilon_V, \varepsilon_W \in \{\pm 1\}$. Then $(J_V \otimes J_W)^2 = \varepsilon_V \varepsilon_W \mathrm{id}_V$ and we have three cases:

1. If $\varepsilon_V = \varepsilon_W = 1$, then $\varepsilon_V \varepsilon_W = 1$. In this case V, W, $V \otimes_{\mathbb{C}} W$ all admit a G-invariant real subspace as in Remark 2.1.6, say V', W', $(V \otimes_{\mathbb{C}} W)'$. Then $(V \otimes_{\mathbb{C}} W)'$ is equivalent to $V' \otimes_{\mathbb{R}} W'$, seen as a $\mathbb{R}G$-space in the natural manner.

2. If $\varepsilon_V = 1$, $\varepsilon_W = -1$, then $\varepsilon_V \varepsilon_W = -1$. In this case V admits a real G-invariant subspace V', while W, $V \otimes_{\mathbb{C}} W$ become $\mathbb{H}G$-spaces W', $(V \otimes_{\mathbb{C}} W)'$ as in Remark 2.1.6. Then $(V \otimes_{\mathbb{C}} W)'$ is G-equivalent to the $\mathbb{H}G$-space obtained by the real tensor product $V' \otimes_{\mathbb{R}} W$ making $\mathbb{H}$ act by

$$q(v' \otimes w) := v' \otimes (qw), \qquad \forall\, q \in \mathbb{H},\ \forall\, v' \in V',\ \forall\, w \in W.$$

3. If $\varepsilon_V = \varepsilon_W = -1$, then $\varepsilon_V \varepsilon_W = 1$. The real G-invariant subspace of $V \otimes_{\mathbb{C}} W$ defined as in Remark 2.1.6 is denoted in this case by $V \otimes_{\mathbb{H}} W$.

If V, W are $\mathbb{K}G$-space, we denote by $\mathrm{Hom}_{\mathbb{K}}(V, W)$ the set of all $\mathbb{K}$-linear maps from V to W. It is a vector space over $\mathbb{R}$ if $\mathbb{K} = \mathbb{R}, \mathbb{H}$ and over $\mathbb{C}$ if $\mathbb{K} = \mathbb{C}$. We can make G act on $\mathrm{Hom}_{\mathbb{K}}(V, W)$ by

$$(g \cdot f)(v) := g \cdot f(g^{-1} \cdot v), \qquad \forall\, g \in G,\ \forall\, v \in V,\ \forall f \in \mathrm{Hom}_{\mathbb{K}}(V, W).$$

Note that the subspace $\mathrm{Hom}_{\mathbb{K}G}(V, W)$ of elements which are invariant by G is exactly the space of $\mathbb{K}G$-maps from V to W.

Definition 2.1.7. Let V be a $\mathbb{C}G$-space.

1. The *dual* V^* of V is defined as the $\mathbb{C}G$-space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, where we make G act trivially on the target space $\mathbb{C}$.

2. The *conjugate* $\mathrm{t}V$ of V is the $\mathbb{C}G$-space defined as follows: as a vector space, $\mathrm{t}V$ is the conjugate of V, i.e. the multiplication by complex scalar is given by
$$z \cdot v := \bar{z}v, \qquad \forall\, z \in \mathbb{C},\ \forall\, v \in V;$$
the action of G on $\mathrm{t}V$ is the same as on V.

We have the following:

Proposition 2.1.8. *Let V be a $\mathbb{C}G$-space, then the $\mathbb{C}G$-spaces V^* and $\mathrm{t}V$ are equivalent.*

Proof. Since G is compact, V carries a positive definite G-invariant hermitian form H (say, $\mathbb{C}$-antilinear in the first argument and $\mathbb{C}$-linear in the second one). Then the map $\alpha : \mathrm{t}V \to V^*$ given by
$$\alpha(v)(w) := H(v, w), \qquad \forall\, v, w \in V$$
is a bijective $\mathbb{C}G$-map. □

Definition 2.1.9. Let V be a $\mathbb{C}G$-space. We say that V is *self dual*, or *self conjugate*, if it is equivalent to V^*.

Remark 2.1.10. If a $\mathbb{C}G$-space V admits a structure map J, then it is self-dual. Indeed, J defines a bijective $\mathbb{C}G$-map between $\mathrm{t}V$ and V.

Next we analyze the structure of a $\mathbb{K}G$-space in terms of its invariant subspaces.

Proposition 2.1.11. *Every $\mathbb{K}G$-space is the direct sum of irreducible G-invariant $\mathbb{K}$-subspaces.*

Proof. We proceed by induction on $\dim_{\mathbb{K}}V$. If V is irreducible (in particular, if $\dim_{\mathbb{K}}V = 1$) there is nothing to prove. So it is enough to show that, if V is reducible, V is the direct sum of two G-invariant $\mathbb{K}$-subspaces of dimension $< \dim_{\mathbb{K}} V$.

Since the group G is assumed to be compact, V carries a positive definite G-invariant inner product (hermitian if $\mathbb{K} = \mathbb{C}, \mathbb{H}$). Let U be a G-invariant $\mathbb{K}$-subspace of V with $0 < \dim_{\mathbb{K}} U < \dim_{\mathbb{K}} V$. Then $V = U \oplus U^{\perp}$, where $U^{\perp}$ is the orthogonal complement of U in V with respect to the chosen inner product, and $0 < \dim_{\mathbb{K}} U^{\perp} < \dim_{\mathbb{K}} V$. This finishes the proof. □

Unfortunately, the decomposition of a $\mathbb{K}G$-space into a sum of irreducible summands may not be unique. For instance, if G acts trivially on a vector space of dimension ≥ 2 there are infinitely many of such decompositions. In order to obtain a new decomposition of the space wich is unique, we consider a representative set

$\{U_i\}_{i\in I}$ for the equivalence classes of irreducible $\mathbb{K}G$-spaces. If V is any $\mathbb{K}G$-space, for any $i \in I$ we denote by V_i the sum of all irreducible G-invariant $\mathbb{K}$-subspaces of V which are equivalent to U_i. Clearly there exists a finite subset $J \subseteq I$ such that $V_i \neq \{O\}$ if and only if $i \in J$, and

$$V = \bigoplus_{i\in J} V_i \tag{2.1}$$

Such V_i, $i \in J$ are called the *isotypical components* of V.

Proposition 2.1.12. *For any $\mathbb{K}G$-space V, the decomposition (2.1) into a direct sum of isotypical components is unique.*

Proof. We adopt the following notation. If $0 \neq n \in \mathbb{N}$ and W is a $\mathbb{K}G$-space, then nW denotes the direct sum of n copies of W. We set moreover $0W := \{O\}$. All we need to show is that if

$$\bigoplus_{i\in I} n_i V_i \simeq \bigoplus_{i\in I} m_i V_i, \tag{2.2}$$

then $n_i = m_i$ for all $i \in I$; here the n_i's, m_i's are non-negative integers, all of which but a finite number are zero. So assume that (2.2) holds; we have

$$\mathrm{Hom}_{\mathbb{K}G}\left(V_j, \bigoplus_{i\in I} n_i V_i\right) \simeq \mathrm{Hom}_{\mathbb{K}G}\left(V_j, \bigoplus_{i\in I} m_i V_i\right), \qquad \forall\, j \in I,$$

and also

$$\bigoplus_{i\in I} n_i \mathrm{Hom}_{\mathbb{K}G}(V_j, V_i) \simeq \bigoplus_{i\in I} m_i \mathrm{Hom}_{\mathbb{K}G}(V_j, V_i), \qquad \forall\, j \in I.$$

Now, $\mathrm{Hom}_{\mathbb{K}G}(V_j, V_i) = \{O\}$ if $i \neq j$, so we deduce

$$n_j \mathrm{Hom}_{\mathbb{K}G}(V_j, V_j) \simeq m_j \mathrm{Hom}_{\mathbb{K}G}(V_j, V_j), \qquad \forall\, j \in I.$$

Taking the dimension of both sides, and noting that $\mathrm{Hom}_{\mathbb{K}G}(V_j, V_j) \neq \{O\}$, we deduce $n_j = m_j$ for all $j \in I$, as claimed. □

We now characterize the existence of a structure map on a $\mathbb{C}G$-space in term of the existence of suitable invariant bilinear forms. We begin with some preliminary Lemmas:

Lemma 2.1.13. *Let V be a $\mathbb{C}G$-space admitting a structure map J. Then V carries a positive definite G-invariant hermitian form H satisfying*

$$H(J(v), J(w)) = \overline{H(v, w)}, \qquad \forall\, v, w \in V. \tag{2.3}$$

Proof. If K is a positive definite G-invariant hermitian form on V, we can choose

$$H(v,w) := \frac{1}{2}\{K(v,w) + \overline{K(J(v),J(w))}\}.$$

□

Lemma 2.1.14 (Schur)**.** *Let V, W be irreducible $\mathbb{K}G$-spaces.*

1. *If $f : V \to W$ is a $\mathbb{K}G$-map then either f is identically zero, or it is an isomorphism.*
2. *If $\mathbb{K} = \mathbb{C}$ and $f : V \to V$ is a $\mathbb{C}G$-map, then there exists $\lambda \in \mathbb{C}$ such that $f = \lambda \mathrm{id}_V$.*

Proof. (1) Clearly $\ker(f)$ is a G-invariant $\mathbb{K}$-subspace of V, so f is either identically zero or injective. In the latter case f must be an isomorphism because $\mathrm{im}(f)$ is a non-zero G-invariant $\mathbb{K}$-subspace of W.

(2) Consider the map $f - \lambda \mathrm{id}_V$, $\lambda \in \mathbb{C}$. For some λ such a map is not an isomorphism, hence it must be identically zero by (1). □

The desired result is the following:

Proposition 2.1.15. *A $\mathbb{C}G$-space V admits a structure map J satisfying $J^2 = \mathrm{id}_V$ if and only if there exists a non-singular, symmetric, G-invariant, $\mathbb{C}$-bilinear form $B : V \otimes_{\mathbb{C}} V \to \mathbb{C}$.*

Similarly, a $\mathbb{C}G$-space V admits a structure map J satisfying $J^2 = -\mathrm{id}_V$ if and only if there exists a non-singular, skew-symmetric, G-invariant, $\mathbb{C}$-bilinear form $B : V \otimes_{\mathbb{C}} V \to \mathbb{C}$.

Proof. First suppose that V carries a structure map J such that $J^2 = \varepsilon \mathrm{id}_V$, $\varepsilon \in \{\pm 1\}$. By Lemma 2.1.13 we can impose on V a positive definite hermitian form H which is G-invariant and satisfies (2.3). Define

$$B(v,w) := H(J(v),w), \qquad \forall\, v,w \in V.$$

Then B is clearly $\mathbb{C}$-bilinear, non-singular and G-invariant. Moreover

$$B(w,v) = H(J(w),v) = \overline{H(v,J(w))} = H(J(v),\varepsilon w) = \varepsilon B(v,w),$$

so B is symmetric or skew-symmetric according to the sign of ε.

Conversely, suppose given on V a non-singular, G-invariant, $\mathbb{C}$-bilinear form $B : V \otimes_{\mathbb{C}} V \to \mathbb{C}$ satisfying

$$B(w,v) = \varepsilon B(v,w), \qquad \forall\, v,w \in V,$$

where $\varepsilon \in \{\pm 1\}$. By Lemma 2.1.13 we can also suppose that V carries a positive definite, G-invariant hermitian form H. Thus we can define a $\mathbb{C}$-antilinear, G-invariant, bijective map $f : V \to V$ by

$$H(v, w) = B(f(v), w), \qquad \forall\, v, w \in V.$$

Now, f^2 is a $\mathbb{C}G$-map, so, by Proposition 2.1.11 and Lemma 2.1.14, V splits as a sum of eigenspaces of f^2:

$$V = V_1 \oplus \cdots \oplus V_k.$$

Denote by λ_i the eigenvalue associated to V_i, $i = 1, \dots, k$. First note that each V_i is preserved by f. Moreover, if $O \neq v \in V_i$, we have

$$\begin{aligned} H(f(v_i), f(v_i)) &= B(f^2(v_i), f(v_i)) \\ &= \lambda_i B(v_i, f(v_i)) = \lambda_i \varepsilon B(f(v_i), v_i) = \lambda_i \varepsilon H(v_i, v_i), \end{aligned}$$

therefore $\lambda_i \varepsilon$ is real and > 0. Defining $J : V \to V$ by

$$J|_{V_i} := (\lambda_i \varepsilon)^{-1/2} f|_{V_i}, \qquad \forall\, i = 1, \dots, k$$

we get a $\mathbb{C}$-antilinear, G-invariant, bijective map satisfying $J^2 = \varepsilon \mathrm{id}_V$. □

Thanks to Proposition 2.1.15 we may prove the following characterization of the types of an irreducible $\mathbb{C}G$-space:

Proposition 2.1.16. *Let V be an irreducible $\mathbb{C}G$-space.*

1. *If V is self dual, then it is either of real type or of quaternionic type, but not both.*

2. *If V is not self dual, then it is of complex type.*

Proof. If V is not self dual, then it is of complex type by Remark 2.1.10.

Assume then that V is self dual. The $\mathbb{C}G$-space $V^* \otimes_{\mathbb{C}} V^*$ is the space of $\mathbb{C}$-bilinear maps $V \otimes_{\mathbb{C}} V \to \mathbb{C}$, and decomposes as

$$V^* \otimes_{\mathbb{C}} V^* = S^2(V^*) \oplus \Lambda^2(V^*),$$

where $S^2(V^*)$ denotes is the subspace of symmetric $\mathbb{C}$-bilinear forms, while $\Lambda^2(V^*)$ is the subspace of skew-symmetric $\mathbb{C}$-bilinear forms. On the other hand

$$V^* \otimes_{\mathbb{C}} V^* \simeq \mathrm{Hom}_{\mathbb{C}}(V, V^*).$$

Since V is self dual, $V \simeq V^*$ are both irreducible, so $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(V, V^*) = 1$ by Lemma 2.1.14. Hence, denoting $S^2_G(V^*)$ (resp. $\Lambda^2_G(V^*)$) the space of G-invariant $\mathbb{C}$-bilinear forms $V \otimes_{\mathbb{C}} V \to \mathbb{C}$, we have

$$\dim_{\mathbb{C}} S^2_G(V^*) + \dim_{\mathbb{C}} \Lambda^2_G(V^*) = 1.$$

Finally observe that a non-zero G-invariant $\mathbb{C}$-bilinear form B corresponds to a non-zero $\mathbb{C}G$-map $V \to V^*$, which is an isomorphism again by Lemma 2.1.14, so B is non-singular. We have now two cases:

1. If $\dim_{\mathbb{C}} S^2_G(V^*) = 1$, $\dim_{\mathbb{C}} \Lambda^2_G(V^*) = 0$, V carries a symmetric non-singular G-invariant $\mathbb{C}$-bilinear form, but no skew-symmetric non-singular G-invariant $\mathbb{C}$-bilinear forms, so it is of real type but not of quaternionic type by Proposition 2.1.15.

2. If $\dim_{\mathbb{C}} S^2_G(V^*) = 0$, $\dim_{\mathbb{C}} \Lambda^2_G(V^*) = 1$, V carries a skew-symmetric non-singular G-invariant $\mathbb{C}$-bilinear form, but no symmetric non-singular G-invariant $\mathbb{C}$-bilinear forms, so it is of quaternionic type but not of real type again by Proposition 2.1.15.

This concludes the proof. □

Before stating and proving the final result of this Section, we introduce the following notation:

1. If V is an $\mathbb{R}G$-space, we denote by $\mathrm{c}V$ the complexification $\mathbb{C} \otimes_{\mathbb{R}} V$, which becomes a $\mathbb{C}G$-space making G act trivially on $\mathbb{C}$.

2. If V is an $\mathbb{H}G$-space, we denote by $\mathrm{c}'V$ the $\mathbb{C}G$-space obtained from V forgetting the quaternionic structure.

3. If V is a $\mathbb{C}G$-space, we denote by:
   (a) $\mathrm{r}V$ the $\mathbb{R}G$-space obtained from V forgetting the complex structure;
   (b) $\mathrm{q}V$ the tensor product $\mathbb{H} \otimes_{\mathbb{C}} V$, regarded as an $\mathbb{H}G$-space in the obvious way (i.e. making G act trivially on $\mathbb{H}$), and as a left $\mathbb{H}$-module.

The next Lemma is easily proved:

Lemma 2.1.17. *The following relations hold:*

$$\begin{array}{lll} \mathrm{rc} = 2, & \mathrm{cr} = 1 + \mathrm{t}, & \mathrm{qc}' = 2, \\ \mathrm{c}'\mathrm{q} = 1 + \mathrm{t}, & \mathrm{tc} = \mathrm{c}, & \mathrm{rt} = \mathrm{r}, \\ \mathrm{tc}' = \mathrm{c}', & \mathrm{qt} = \mathrm{q}, & \mathrm{t}^2 = 1. \end{array}$$

These equations are to be interpretated as saying that $\mathrm{rc}V \simeq V \oplus V$ for every $\mathbb{R}G$-space V, $\mathrm{cr}V \simeq V \oplus \mathrm{t}V$ for every $\mathbb{C}G$-space V, and so on.

Theorem 2.1.18. *Let G be a fixed compact Lie group. It is possible to choose a family of $\mathbb{R}$-irreducible $\mathbb{R}G$-spaces $\{U_m\}$, a family of $\mathbb{C}$-irreducible $\mathbb{C}G$-spaces $\{V_n\}$ and a family of $\mathbb{H}$-irreducible $\mathbb{H}G$-spaces $\{W_p\}$, with m, n, p varying in suitable sets of indices, which satisfy the following conditions:*

1. *the non-equivalent irreducible real representations are precisely the* U_m*,* $\mathrm{r}V_n$ *and* $\mathrm{rc}'W_p$*;*

2. *the non-equivalent irreducible complex representations are precisely the* $\mathrm{c}U_m$*,* V_n*,* $\mathrm{t}V_n$ *and* $\mathrm{c}'W_p$*;*

3. *the non-equivalent irreducible quaternionic representations are precisely the* $\mathrm{qc}U_m$*,* $\mathrm{q}V_n$*, and* W_p*.*

Proof. Let V be an irreducible complex representation. If V is not self dual, $V \not\simeq \mathrm{t}V$, so for each pair $(V, \mathrm{t}V)$ we choose exactly one representation V_n. If V is self dual, by Proposition 2.1.16 it is either of real type or of quaternionic type. So we can choose a real representation U_m or a quaternionic representation W_p so that $\mathrm{c}U_m$ and $\mathrm{c}'W_p$ give such V. At this point Condition (2) is satisfied.

We shall prove now that U_m, $\mathrm{r}V_n$, $\mathrm{rc}'W_p$ are irreducible over $\mathbb{R}$. In fact, U_m is irreducible over $\mathbb{R}$ because cU_m is irreducible over $\mathbb{C}$. Moreover, by Lemma 2.1.17, we have

$$\mathrm{cr}V_n \simeq V_n \oplus \mathrm{t}V_n, \qquad \mathrm{crc}'W_p \simeq 2\mathrm{c}'W_p,$$

so irreducibility of $\mathrm{r}V_n$ follows from the fact that $V_n \not\simeq \mathrm{t}V_n$, while irreducibility of $\mathrm{rc}'W_p$ follows from the fact that $\mathrm{c}'W_p$ is not of real type (since it is of quaternionic type). Similarly we prove that $\mathrm{qc}U_m$, $\mathrm{q}V_n$, W_p are irreducible over $\mathbb{H}$.

It remains to prove that there are no further real or quaternionic representations. We shall consider only the real case, the quaternionic on being similar. First we notice that any irreducible complex representation occurs as an irreducible summand in one of

$$\mathrm{c}U_m, \qquad \mathrm{cr}V_n = V_n \oplus \mathrm{t}V_n, \qquad \mathrm{crc}'W_p \simeq 2\mathrm{c}'W_p.$$

The assertion then follows from the fact that, if U, U' are irreducible inequivalent real representations, then $\mathrm{c}U$, $\mathrm{c}U'$ do not share common $\mathbb{C}$-irreducible summands. Indeed, by Lemma 2.1.14,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(\mathrm{c}U, \mathrm{c}U') = \dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{R}G}(U, U') = 0,$$

and the proof is concluded. □

Remark 2.1.19. The irreducible complex representations occurring in

$$\{\mathrm{c}U_m\}, \qquad \{V_n, \mathrm{t}V_n\}, \qquad \{\mathrm{c}'W_p\}$$

are exactly the representations of real type, complex type, quaternionic type respectively.

This fact suggests the following:

Definition 2.1.20. An irreducible real representation (resp. an irreducible quaternionic representation) is said to be:

1. of *real type* if it occurs in $\{U_m\}$ (resp. in $\{\mathrm{qc}U_m\}$);
2. of *complex type* if it occurs in $\{\mathrm{r}V_n\}$ (resp. in $\{\mathrm{q}V_n\}$);
3. of *quaternionic type* if it occurs in $\{\mathrm{rc}'W_p\}$ (resp. in $\{W_p\}$).

2.2 Reductions of representations

In Section 1.7 we introduced the problem of understanding which (algebraic) properties of an isometric action can be recovered from the metric structure of the orbit space, or, more precisely, which properties of the action depend only on its quotient-equivalent class. Such a problem, that was rised by Claudio Gorodski and Alexander Lytchak in [19], has been studied by the authors in the same paper, where some results have been obtained in the case of representations.

In this and the next Sections we shall describe such results, including the proof of the most simple ones for the sake of completeness. Therefore, from now on, we shall consider only real representations of compact Lie groups. In particular we shall write $\rho = (G, V)$ to denote a $\mathbb{R}G$-space V, with G compact. In this case we shall suppose also that V is endowed with a fixed G-invrariant inner product; so ρ can be also thought as a Lie group homorphism $G \to \mathbf{O}(V)$. It is important to remark now that, following [19], from now on we shall work in the category of representations, this meaning that all Definitions given in Section 1.7 will always be considered as reformulated in such a category.

We begin with an observation which will be used several times.

Proposition 2.2.1 ([2, 25, 19])**.** *For a representation $\rho = (G, V)$, the following statement are equivalent:*

1. *ρ is polar;*
2. *ρ has abstract copolarity 0;*
3. *the quotient V_{pr}/G is flat.*

Proof. (1) $\Rightarrow$ (2). It follows from Proposition 1.7.14.

(2) $\Rightarrow$ (3). If ρ has abstract copolarity 0, then ρ has a reduction to a representation (Γ, Σ) of a finite group Γ. Let I denote an isometry $\Sigma/\Gamma \to V/G$. The canonical projection $\Sigma_{\mathrm{pr}} \to \Sigma_{\mathrm{pr}}/\Gamma$ is a local isometry (because Γ is finite), thus $\Sigma_{\mathrm{pr}}/\Gamma$ is flat because Σ (which is a vector space) is. The assertion follows by noting that I maps $\Sigma_{\mathrm{pr}}/\Gamma$ onto V_{pr}/G by Remark 1.6.17.

(3) ⇒ (1). The canonical projection $\pi : V_{\mathrm{pr}} \to V_{\mathrm{pr}}/G$ is a Riemannian submersion, so, denoting by $\tilde{\sec}$, sec, respectively, the sectional curvatures of V_{pr}, V_{pr}/G, we can apply the O'Neill formula (cf. [34])

$$\sec(X,Y) = \tilde{\sec}(\tilde{X},\tilde{Y}) + \frac{3}{4}\|\mathcal{V}[\tilde{X},\tilde{Y}]\|^2,$$

where $\|\cdot\|$ denotes the norm associated to the inner product in V, $\mathcal{V}$ is the vertical projection in the tangent bundle of V_{pr}, $\{X,Y\}$ is any orthonormal basis of a 2-plane in the tangent bundle of V_{pr}/G, and $\{\tilde{X},\tilde{Y}\}$ is its horizontal lift. Since both V_{pr} and V_{pr}/G are flat, such a formula implies that $\mathcal{V}[\tilde{X},\tilde{Y}] = O$ for all tangent vectors of V_{pr}, hence the horizontal distribution of π over the principal stratum V_{pr} is integrable. Then ρ is polar by Theorem 1.5.14. □

Remark 2.2.2. Proposition 2.2.1 implies that polarity of a representation ρ is a property that depends only on the quotient-equivalence class of ρ.

A very important class of polar representations is described in the following:

Example 2.2.3. If $M = G/K$ is symmetric space, then the isotropy representation of K on the tangent space $T_{eK}M$ is polar (cf. [26]). Indeed, it turns out that a section is given by $T_{eK}S$, where S is any flat totally geodesic submanifold of M through eK with maximal dimension. In particular, $\mathrm{chm}(K, T_{eK}M) = \mathrm{rk}(M)$ is the rank of the symmetric space M.

Actually, isotropy representations of symmetric spaces constitute a large part of the class of polar representations. Indeed, Jiri Dadok has proved in [12] that any irreducible polar representation (G,V), where G is compact and connected, is orbit-equivalent to the isotropy representation of a symmetric space. More precisely, he has shown that every irreducible polar representation (G,V) is the isotropy representation of a symmetric space with the following exceptions:

G	V	Cond.	G	V	Cond.
$\mathbf{G}_2$	$\mathbb{R}^7$	–	$\mathbf{SU}(p)\times\mathbf{SU}(q)$	$\mathbb{C}^p\otimes_{\mathbb{C}}\mathbb{C}^q$	$p\neq q$
$\mathbf{Spin}(7)$	$\mathbb{R}^8$	–	$\mathbf{SU}(n)$	$\mathbb{C}^n$	$n\geq 2$
$\mathbf{SO}(3)\times\mathbf{Spin}(7)$	$\mathbb{R}^3\otimes\mathbb{R}^8$	–	$\mathbf{SU}(n)$	$\Lambda^2\mathbb{C}^n$	n odd
$\mathbf{U}(1)\times\mathbf{Sp}(n)$	$\mathbb{C}\otimes_{\mathbb{C}}\mathbb{H}^n$	–	$\mathbf{Sp}(n)\times\mathbf{Sp}(1)$	$\mathbb{H}^n\otimes_{\mathbb{H}}\mathbb{H}$	$n\geq 1$
$\mathbf{SO}(2)\times\mathbf{G}_2$	$\mathbb{R}^2\otimes\mathbb{R}^7$	–	$\mathbf{Spin}(10)$	$\mathbb{C}^{32}$	–
$\mathbf{SO}(2)\times\mathbf{Spin}(7)$	$\mathbb{R}^2\otimes\mathbb{R}^8$	–	–	–	–

From the family of symmetric spaces we get also an important class of representations with non trivial copolarity:

Example 2.2.4 (cf. [22, 19]). Let (H, V) be the isotropy representation of an irreducible hermitian symmetric space of rank $r \geq 2$, where H is connected. Then $H = S^1 \cdot G$, where G is the semisimple part of H. Assume that the restriction (G, V) of (H, V) is irreducible and not orbit-equivalent to (H, V) (this second assumption is precisely associated to non-compact hermitian symmetric spaces of *tube type*, cf. [11, Section 9]). Then (G, V) has non-trivial copolarity equal to $r - 1$. Indeed, a minimal generalized section is given by the complexification of a section of (H, V) (see [22]). There are four families of representations of this kind; namely (cf. [16, 5])

G	V	Symmetric Space	Conditions
$\mathbf{SU}(n)$	$S^2\mathbb{C}^n$	$\mathbf{Sp}(n)/\mathbf{U}(n)$	$n \geq 3$
$\mathbf{SU}(n)$	$\Lambda^2\mathbb{C}^n$	$\mathbf{SO}(2n)/\mathbf{U}(n)$	$n = 2p \geq 6$
$\mathbf{SU}(n) \times \mathbf{SU}(n)$	$\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^n$	$\mathbf{SU}(2n)/\mathrm{S}(\mathbf{U}(n) \times \mathbf{U}(n))$	$n \geq 3$
$\mathbf{E}_6$	$\mathbb{C}^{27}$	$\mathbf{E}_7/S^1 \cdot \mathbf{E}_6$	–

We now proceed to analyze some properties of a representation which are invariant of the quotient, recalling the most simple results from [19]. The first one that we consider is the existence of non-trivial fixed points:

Remark 2.2.5. Suppose that (G, V), (G', V') are quotient-equivalent representation, and recall that V^G denotes the set of fixed points of G on V. Clearly the quotient V/G splits as $V^G \times (V^G)^\perp/G$, and V^G is an euclidean factor. Let p, $\bar{p}$ two points in the unit sphere $S(V)$ of V. The distance in V between p and the orbit $G \cdot \bar{p}$ is ≤ 2, and equality holds if $G \cdot \bar{p}$ is the antipodal point $-p$; in this case $\bar{p} \in V^G$. Therefore, if $\bar{p} \notin V^G$, a geodesic beween $G \cdot p$ and $G \cdot \bar{p}$ does not cross the origin O_V in V, and its projection to V/G does not cross the vertex $O_{V/G}$ (i.e. the orbit $\{O_V\}$ of O_V); therefore $G \cdot p$ and $G \cdot \bar{p}$ are not contained in an euclidean factor. This proves that V^G is the maximal euclidean factor in V/G; so any isometry $I : V/G \to V'/G'$ splits into a product $I = I_1 \times I_2$ of isometries

$$I_1 : V^G \to V'^{G'}, \qquad I_2 : \frac{(V^G)^\perp}{G} \to \frac{(V'^{G'})^\perp}{G'},$$

since it must map euclidean factors to euclidean factors.

The following proposition now clearly holds:

Proposition 2.2.6 ([19]). *Let (G, V), (G', V') be quotient-equivalent representations. Then $\dim V^G = \dim V'^{G'}$. In particular, (G, V) has non-trivial set of fixed points if and only if (G', V') has non-trivial set of fixed points.*

Note that, by changing an isometry along the maximal euclidean factor, we always find an isometry $I : V/G \to V'/G'$ which map the vertex $O_{V/G}$ to the vertex $O_{V'/G'}$.

We will show that in fact all invariant subspaces can be recognized metrically in the quotient, i.e. they are invariant under quotient-equivalence. The main tool in the proof of such a result is the following:

Proposition 2.2.7. *A representation* (G, V) *has non-trivial fixed points if and only if* $\mathrm{diam}(S(V)/G) > \pi/2$. *In this case,* $\mathrm{diam}(S(V)/G) = \pi$.

Before proving Proposition 2.2.7 we need to recall some facts from spherical geometry, and prove some preliminary results.

Definition 2.2.8. A subset $B \subseteq S(V)$ is said to be *bounded* if it is contained in an open geodesic ball.

The *radius* of a bounded subset $B \subseteq S(V)$ is the number:

$$r := \inf\{r' > 0 \mid B \subseteq B_{r'}(p) \text{ for some } p \in S(V)\};$$

here $B_{r'}(p)$ denotes the geodesic ball centred at p with radius r'.

A point $p \in S(V)$ is said to be a *centre* of the bounded set B if $B \subseteq \overline{B_r(p)}$, r being the radius of B.

If $(G, S(V))$ is an isometric action of the compact Lie group G on the unit sphere $S(V)$ of an euclidean vector space V, we shall denote the metric distance on $S(V)$ by d_s, and the induced distance on the orbit space $S(V)/G$ by d_q. Observe that any representation (G, V) induces an isometric action $(G, S(V))$ by restriction.

Lemma 2.2.9. *Let* $(G, S(V))$ *an isometric action, and let* $p \in S(V)$. *If* $r > \pi/2$ *is fixed, the set* $\mathcal{C} := \{q \in S(V) \mid \mathrm{d}_s(q, G \cdot p) \geq r\}$ *is convex.*

Proof. Let $q, q' \in \mathcal{C}$, and denote by $\gamma : [0, \ell] \to S(V)$ a minimal geodesic joining $q = \gamma(0)$ to $q' = \gamma(\ell)$. For any fixed $t \in [0, \ell]$, there exists $g_t \in G$ such that the point $g_t \cdot p$ realizes the distance between the point $\gamma(t)$ and the orbit $G \cdot p$. Now, q, q' are contained in the complement $S(V) \backslash B_r(g_t \cdot p)$, which is a closed geodesic ball $\bar{B}$ of radius $\pi - r < \frac{\pi}{2}$. Therefore γ is unique and contained in $\bar{B}$. In particular, $\mathrm{d}_s(\gamma(t), G \cdot p) = \mathrm{d}_s(\gamma(t), g_t \cdot p) \geq r$ for all $t \in [0, \ell]$. □

Lemma 2.2.10. *A bounded set* $B \subseteq S(V)$ *with radius* $r < \pi/2$ *has a unique centre.*

Proof. Existence. Choose a sequence $\{p_n\}_{n \in \mathbb{N}} \subseteq S(V)$, and corresponding $r_n > r$, such that $B \subseteq B_{r_n}(p_n)$ for all $n \in \mathbb{N}$, and $r_n \to r$. We shall prove that $\{p_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $S(V)$. Since $S(V)$ is a complete metric space, such a sequence will have a (unique) limit point $\bar{p}$, which clearly must satisfy $B \subseteq \overline{B_r(\bar{p})}$. Thus $\bar{p}$ will be a centre of B.

Fix a point $x \in S(V)$. Since the isometry group of $S(V)$ is transitive on $S(V)$, for every $q \in B$ we can find an isometry $I_q : S(V) \to S(V)$ mapping q to x. Note moreover that the image $I_q(p_n)$ is contained in a ball of radius r_n around x, for all $n \in \mathbb{N}$. We shall estimate the distance between p_n, p_m by estimating the distance between their images $I_q(p_n)$, $I_q(p_m)$.

Fix $\varepsilon > 0$, and consider the annulus $\mathcal{S}_{R,R'} := B_R(x) \backslash B_{R'}(x)$ for R, R' such that $R' < r < R < \frac{\pi}{2}$. We claim that we may choose R, R' so that each geodesic segment in $\mathcal{S}_{R,R'}$ has length $< \varepsilon$. Indeed, assume that for some $\varepsilon > 0$, and for all pair (R, R') as above, there exists a geodesic segment $\gamma_{R,R'}$ in $\mathcal{S}_{R,R'}$ with $L(\gamma_{R,R'}) \geq \varepsilon$. For $R, R' \to r$, $\gamma_{R,R'}$ converges to a geodesic segment γ contained in the distance sphere $S_r(x)$ and with length $L(\gamma) \geq \varepsilon > 0$. This is absurd, since the asumption $r < \frac{\pi}{2}$ implies that $S_r(x)$ does not contain geodesic segments of positive length.

Now, for n, m large enough, r_n, $r_m < R$. Let p denote the midpoint of the geodesic segment joining p_n, p_m. Since $R < \frac{\pi}{2}$, the ball $B_R(x)$ is convex and, for all $q \in B$, the image $I_q(p)$ is contained in it. If $I_q(p) \in B_{R'}(x)$ for all $q \in B$, then $B \subseteq B_{R'}(p)$, contradicting $R' < r$. Thus there exists a point $q \in B$ such that $I_q(p) \subseteq \mathcal{S}_{R,R'}$. Therefore at least one of the two geodesic segments joining $I_q(p_n)$, $I_q(p)$ and $I_q(p)$, $I_q(p_m)$ is contained in $\mathcal{S}_{R,R'}$ and has length $< \varepsilon$ by the claim above. Hence we deduce $\mathrm{d}_s(p_n, q_n) = \mathrm{d}_s(I_q(p_n), I_q(p_m)) < 2\varepsilon$.

Uniqueness. If $\hat{p}$ is another point such that $B \subseteq \overline{B_r(\hat{p})}$, then $\hat{p}, \bar{p} \in \overline{B_r(q)}$, for all $q \in \bar{B}$. The midpoint p' of the geodesic segment joining $\hat{p}$ and $\bar{p}$ belongs to $B_r(q)$ for all $q \in \bar{B}$, thus $\bar{B} \subseteq B_r(p')$, contradicting the choice of r. □

Proof of Proposition 2.2.7. Clearly, if (G, V) has non-trivial fixed points, then $\mathrm{diam}(S(V)/G) = \pi$.

On the other hand, assume that $\mathrm{diam}(S(V)/G) > \frac{\pi}{2}$; it is enough to prove that in this case (G, V) has non-trivial fixed-points. Indeed, our assumption implies that, for some $\varepsilon > 0$ and $p \in S(V)$, the set

$$\mathcal{B} := \left\{ q \in S(V) \mid \mathrm{d}_s(q, G \cdot p) \geq \frac{\pi}{2} + \varepsilon \right\}$$

is non-empty; moreover it is clearly compact and G-invariant. Finally, from the proof of Lemma 2.2.9, we see that $\mathcal{B}$ is convex and, more precisely, that for each pair of points $q, q' \in \mathcal{B}$, the minimizing geodesic between q and q' in $S(V)$ is unique and contained in $\mathcal{B}$. Therefore $\mathcal{B}$ does not contain a great sphere. Now, a convex and compact subset of $S(V)$ is either a great k-sphere, a k-hemisphere, or a proper subset of a k-hemisphere ($k \geq 1$). Since $\mathcal{B}$ does not contain great spheres, we deduce then $\mathcal{B} \subseteq H_k \backslash \partial H_k$, where H_k is a suitable k-hemisphere. In particular, $\mathcal{B}$ is bounded and of radius $r < \frac{\pi}{2}$. By Lemma 2.2.10, $\mathcal{B}$ has a unique centre $\bar{p}$. Now, for $g \in G$ and $q \in \mathcal{B}$, we have $\mathrm{d}_s(q, g \cdot \bar{p}) = \mathrm{d}_s(g^{-1} \cdot q, \bar{p}) \leq r$, since

$\mathcal{B}$ is G-invariant. Then uniqueness of the centre implies $g \cdot \bar{p} = \bar{p}$, for all $g \in G$, so $\bar{p}$ is a non-trivial fixed point of (G, V) . □

We can finally prove:

Proposition 2.2.11 ([19]). *Let (G,V) and (G',V') be quotient-equivalent representations without fixed points. If $W \subseteq V$ is a G-invariant subspace, then there exists a G'-invariant subspace $W' \subseteq V'$ such that $W/G \simeq W'/G'$. If $I : V/G \to V'/G'$ is an isometry, and $\pi : V \to V/G$, $\pi' : V' \to V'/G'$ are the canonical projections, then we have $W' = (\pi')^{-1}(I(\pi(W)))$.*

Proof. By Remark 2.2.5 we may suppose that (G,V), (G',V') have no non-trivial fixed points.

Consider now the restricted action of G on the unit sphere $S(V)$. We claim that a closed subset $Z \subseteq S(V)/G$ has the form $Z = S(W)/G$ for a suitable G-invariant subspace W of V if and only if there exists a subset $Z' \subseteq S(V)/G$ such that

$$Z = \{z \in S(V)/G \mid \mathrm{d}_q(z, z') = \pi/2,\ \forall\, z' \in Z'\}.$$

Since the quotient spaces V/G, V'/G' are isometric, the assertion of the Proposition follows immediately from the claim.

If $Z = S(W)/G$ for a suitable G-invariant subspace $W \subseteq V$, $Z' := S(W^{\perp})/G$ satisfies the requirement.

Conversely, assume that Z is given in terms of Z' as above. First we show that $\pi^{-1}(Z)$ is convex. Fix $p' \in \pi^{-1}(Z')$, and set

$$\mathcal{Z}_{p'} := \{p \in S(V) \mid \mathrm{d}_s(p, G \cdot p') = \pi/2\}.$$

Since, by hypothesis, $\pi^{-1}(Z)$ is the intersection of the sets $\mathcal{Z}_{p'}$, $p' \in \pi^{-1}(Z')$, it is enough to prove that the latter are convex for all $p' \in \pi^{-1}(Z')$.

Let $p, q \in \mathcal{Z}_{p'}$, and consider a minimizing geodesic $\gamma : [0, \ell] \to S(V)$ between p and q.

If $q \neq -p$, then γ is unique. For any fixed $t \in [0, \ell]$, we find a point $g_t \cdot p' \in G \cdot p'$ ($g_t \in G$) such that $\mathrm{d}_s(\gamma(t), g_t \cdot p') = \mathrm{d}_s(\gamma(t), G \cdot p')$; note that this distance is $\leq \frac{\pi}{2}$ since $\mathrm{diam}(S(V)/G) = \frac{\pi}{2}$ (and this is true because we are assuming that (G,V) has no non-trivial fixed points). Now, p, q lie on the sphere with distance $\frac{\pi}{2}$ from the point $g_t \cdot p'$, hence γ is also contained in this sphere. Therefore $\mathrm{d}_s(\gamma(t), g_t \cdot p) = \frac{\pi}{2}$, and $\gamma(t) \in \mathcal{Z}_{p'}$. Since t was fixed arbitrarily in $[0, \ell]$ we conclude that γ has image contained in $\mathcal{Z}_{p'}$.

If $q = -p$, let $n := \dim S(V)$. Since $\mathrm{d}_s(p, G \cdot p') = \mathrm{d}_s(-p, G \cdot p') = \frac{\pi}{2}$, we have that $G \cdot p'$ is contained in the sphere $S^{n-1}(p)$ of distance $\frac{\pi}{2}$ from p. On the other hand $G \cdot p'$ is G-invariant, so

$$G \cdot p' \subseteq \bigcap_{g \in G} S^{n-1}(g \cdot p) \subseteq S^{n-2}(p),$$

because $G \cdot p \neq \{p, -p\}$. Hence p, $-p$ belong to the complementary sphere S^2, and there exists a minimizing geodesic γ joining p to $-p$ with image contained in S^2, i.e. with $\mathrm{d}_s(\gamma(t), G \cdot p') = \frac{\pi}{2}$, for all t in the domain of γ; in other words, the image of γ is contained in $\mathcal{Z}_{p'}$.

Summarizing, we have shown that $\mathcal{Z}_{p'}$ is convex for all $p' \in \pi^{-1}(Z')$, and therefore $\pi^{-1}(Z)$ is convex as well. Clearly $\pi^{-1}(Z)$ is also compact and G-invariant. Now, a convex and compact subset of $S(V)$ is either a great k-sphere, or is contained in a k-hemisphere ($k \geq 1$). Suppose that $\pi^{-1}(Z) \subseteq H_k$, where H_k is a k-hemisphere. Note that H_k has the form $\overline{(S^k \backslash S^{k-1})_0}$, where $(S^k \backslash S^{k-1})_0$ denotes one of the two connected components of $S^k \backslash S^{k-1}$. Now, if G leaves H_k invariant, then it preserves also the vector space V^{k+1} spanned by S^k and the great sphere S^{k-1} (i.e. it preserves ∂H_k). Therefore G fixes pointwisely the complementary 0-sphere of S^{k-1} in S^k. This contradicts the fact that (G, V) has no non-trivial fixed points. Hence $\pi^{-1}(Z)$ is a great sphere in $S(V)$, and has the form $S(W)$ for a suitable G-invariant subspace W of V. In other words $Z = S(W)/G$ and the claim is proved. □

Corollary 2.2.12. *Let (G, V), (G', V') be quotient-equivalent representations. Then (G, V) is irreducible if and only if (G', V') is irreducible.*

Proposition 2.2.11 tells us that invariant subspaces of a representation can be metrically recognised. The same is true for isotypical components; this simple fact is well-known to the experts, however, as far as we know, it is not present in literature yet.

Proposition 2.2.13. *Let (G, V), (G', V') be quotient-equivalent representations. If $W \subseteq V$ is an isotypical component of (G, V), then there exists an isotypical component $W' \subseteq V'$ of (G', V') such that $W/G \simeq W'/G'$. If $I : V/G \to V'/G'$ is an isometry, and $\pi : V \to V/G$, $\pi' : V' \to V'/G'$ are the canonical projections, then we have $W' = (\pi')^{-1}(I(\pi(W)))$.*

Proof. Since W is a G-invariant subspace of V, by Proposition 2.2.11 there exists a G'-invariant subspace $W' \subseteq V$ such that $W/G \simeq W'/G'$.

Assume by contradiction that W' has two G'-invariant irreducible subspaces, W_1', W_2' which are not G'-equivalent. Using again Proposition 2.2.11 (and Corollary 2.2.12), there exist two G-invariant irreducible subspaces W_1, W_2 such that $W_i/G \simeq W_i'/G'$, $i = 1, 2$. Since the representations $(G, W_1 \oplus W_2)$, $(G', W_1' \oplus W_2')$ are quotient-equivalent, we obtain a contradiction with Proposition 2.2.11: indeed, W_1, W_2 being G-equivalent, $W_1 \oplus W_2$ has infinitely many G-invariant subspaces, while $W_1' \oplus W_2'$ has only two G'-invariant subspaces. □

2.3 Orbifold points and boundary points

Given a representation (G, V), consider the identity component G° of the compact Lie group G. Since G° is a normal subgroup of G, the finite group G/G° acts by isometries on the metric space V/G° with quotient V/G. If (G, V) is a minimal reduction of a representation (H, W) with H connected, it turns out (cf. [19]) that the action of G/G° on V/G° is one of the main tools in the analysis of the propertis of (the quotient-equivalence class of) (H, W); in particular, the results of this thesis (cf. also [37]) rely on the fact that the group G/G° is generated by special elements, called *nice involutions*, which act on V/G° in a suitable manner (cf. Theorem 2.3.13 below, or [19]).

In order to study how G/G° acts on V/G°, we need to consider a subset of V/G which is bigger than V_{pr}/G, namely the *orbifold part*, $(V/G)_{\mathrm{orb}}$, of V/G. Since the definition of this set requires some background knowledge about Riemannian orbifolds, we dedicate the first part of this Section to briefly recall the main basic material on this topic; we refer to [7, 14, 43, 13] for a more detailed exposition.

Definition 2.3.1. A (differentiable) orbifold structure $\mathcal{Q}$ of dimension n on a Hausdorff, paracompact topological space Q is given by the following data:

1. An open cover $\{V_i\}_{i\in I}$ of Q.
2. For each $i \in I$, a finite subgroup Γ_i of the group of diffeomorphisms of a simply connected n-dimensional manifold X_i and a continuous map
$$q_i : X_i \to V_i,$$
called *chart*, such that q_i induces a homeomorphism from X_i/Γ_i onto V_i.
3. For all $x_i \in X_i$, $x_j \in X_j$ such that $q_i(x_i) = q_j(x_j)$, there is a diffeomorphism h from an open connected neighborhood W of x_i to a neighborhood of x_j such that $q_j \circ h = q_i|_W$. Such a map h is called a *change of chart*; it is well defined up to composition with an element of Γ_j (cf. [14]). In particular if $i = j$, then h is the restriction of an element of Γ_i.

The family $\{(X_i, q_i, \Gamma_i)\}_{i\in I}$ is called an *atlas* for the orbifold structure $\mathcal{Q}$. We say that two such atlases $\{(X_i, q_i, \Gamma_i)\}_{i\in I_1}$, $\{(X_i, q_i, \Gamma_i)\}_{i\in I_2}$ *define the same orbifold structure* on Q if $\{(X_i, q_i, \Gamma_i)\}_{i\in I_1\cup I_2}$ satisfies the compatibily condition (3).

The pair $(Q, \mathcal{Q})$ is called a (differentiable) *orbifold*; if the topological space Q is understood, we shall often say, for the sake of brevity, that $\mathcal{Q}$ *is an orbifold.*

Note that if all groups Γ_i are trivial, then $\mathcal{Q}$ is simply a differentiable manifold.

A *Riemannian metric* on an orbifold $\mathcal{Q}$ with atlas $\{(X_i, q_i, \Gamma_i)\}_{i\in I}$ is a family $\{\mathbf{g}_i\}_{i\in I}$, where, for all $i \in I$, $\mathbf{g}_i$ is a Γ_i-invariant Riemannian metric on X_i, with respect to which each change of charts is an isometry. Any differentiable orbifold (over a paracompact topological space) admits a Riemannian metric (cf. [14]). A *Riemannian orbifold* is an orbifold endowed with a Riemannian metric.

Recall that a *pseudogroup of local diffeomorphisms* of a differentiable manifold X is a collection $\mathcal{H}$ of diffeomorphisms $h : V \to W$ of open sets of X such that:

1. $\mathcal{H}$ contains the identity map $\mathrm{id}_X : X \to X$;
2. the restriction of an element of $\mathcal{H}$ to any open subset of X belongs to $\mathcal{H}$;
3. $\mathcal{H}$ is closed under taking inverses, compositions (whenever possible) and unions of its elements

Given a family H of local diffeomorphisms of a manifold X containing id_X, we can form the *pseudogroup generated by* H, which is obtained by taking restrictions of the elements in H to open subsets of X, together with their inverses, compositions and unions.

Two points $x, y \in X$ are said to *belong to the same orbit* of $\mathcal{H}$ if there exists an element $h \in \mathcal{H}$ such that $h(x) = y$. This defines an equivalence relation on X whose classes are called the *orbits* of $\mathcal{H}$. The quotient of X by this equivalence relation, endowed with the quotient topology, is denoted by $X/\mathcal{H}$.

If $\mathcal{Q}$ is an orbifold with atlas $\{(X_i, q_i, \Gamma_i)\}_{i\in I}$, let X be the disjoint union of the X_i's, and call q the union of the maps q_i's. The changes of charts generate a pseudogroup $\mathcal{H}_{\mathcal{Q}}$ of local diffeomorphisms of X, called the *pseudogroup of changes of charts of the orbifold* $\mathcal{Q}$ (with respect to the chosen atlas). It contains in particular all the elements of the groups Γ_i's. Denoting by Q the underlying topological space of $\mathcal{Q}$, note that the map $q : X \to Q$ induces a homeomorphism $X/\mathcal{H}_{\mathcal{Q}} \to Q$.

If $\mathcal{Q}$ is a Riemannian orbifold, and $\{\mathbf{g}_i\}_{i\in I}$ is a Riemannian metric on $\mathcal{Q}$, then $\mathcal{H}_{\mathcal{Q}}$ consists of local isometries of X, with respect to the Riemannian metric obtained by taking the union of the $\mathbf{g}_i$'s. In this case the homeomorphism $X/\mathcal{H}_{\mathcal{Q}} \to Q$ induces a distance on Q, which coincides with the local distances defined on each V_i by the homeomorphism $X_i/\Gamma_i \to V_i$.

Let p be a point of an orbifold $\mathcal{Q}$, and let (X_i, q_i, Γ_i) be a chart of $\mathcal{Q}$ at p. For a fixed $x_i \in X_i$ such that $q_i(x_i) = p$, we denote by $\Gamma^i_{x_i}$ the isotropy group of the action (Γ_i, X_i) at x_i. Clearly the isomorphism class of $\Gamma^i_{x_i}$ is independent of the lift x_i of p. Moreover, if (X_j, q_j, Γ_j) is another chart at p, if $x_j \in X_j$ satisfies $q_j(x_j) = p$, and if h is the corresponding change of charts defined in a neighborhood W of x, then $\Gamma^j_{x_j} \simeq h \circ \Gamma^i_{x_i} \circ h^{-1}$. Thus, for a given $p \in \mathcal{Q}$, the isomorphism class of the isotropy groups $\Gamma^i_{x_i}$ is independent of both the chart at p and the lift of p within the chart. We shall denote this isomorphism class by Γ_x, and we shall refer to it as the *local group at* x.

If Λ is a finite group, a *stratum of type* Λ in an orbifold $\mathcal{Q}$, $\mathcal{Q}_{(\Lambda)}$, is a connected component of the set of points $p \in \mathcal{Q}$ whose local group is isomorphic to Λ. Note that strata are manifolds: indeed, $\mathcal{Q}$ is locally homeomorphic to X/Γ for a suitable manifold X and a suitable finite group Γ of $\mathrm{Diff}(X)$; therefore, if Λ is a subgroup of Γ, the stratum $\mathcal{Q}_{(\Lambda)}$ is locally homeomorphic to $X_{(\Lambda)}/\Gamma$, which is a manifold (cf. Theorem 1.4.4).

If Γ is the trivial group consisting of the sole identity, the stratum $\mathcal{Q}_{(\Gamma)}$ is called the *principal stratum*, and is denoted $\mathcal{Q}_{\mathrm{pr}}$; it is open and dense in Q, the underlying topological space of $\mathcal{Q}$, and it consists of all points of Q admitting a neighborhood homeomorphic to a manifold.

In the following pages, an important role will be played by strata of type $\mathbb{Z}/2\mathbb{Z}$; they are called *strata of codimension* 1 in $\mathcal{Q}$. The closure of the union of all strata of codimension 1 is denoted by $\partial\mathcal{Q}$ and is called the *boundary* of $\mathcal{Q}$.

Definition 2.3.2. Assume that $\mathcal{Q}$, $\mathcal{R}$ are two orbifolds with atlases $\{(X_i, q_i, \Gamma_i)\}_{i\in I}$, $\{(Y_j, r_j, \Gamma_j)\}_{j\in J}$ respectively. An *orbifold-map* between $\mathcal{Q}$, $\mathcal{R}$ is a continuous map $f : Q \to R$ between their underlying topological spaces such that, for each $x \in X$, there exist charts (X_i, q_i) at x, (Y_j, r_j) at $f(x)$ with the following properties: the map $f : q_i(X_i) \to r_j(Y_j)$ can be lifted to a smooth map $\tilde{f} : X_i \to Y_j$ satisfying $r_j \circ \tilde{f} = f \circ q_i$. If in addition $f : R \to Q$ is a homeomorphism, we shall say that the corresponding orbifold-map is an *orbifold-diffeomorphism.*

We shall use the same letter to denote both the orbifold-map and the corresponding continuous map between the underlying topological spaces.

Definition 2.3.3. If $\mathcal{Q}$, $\mathcal{R}$ are Riemannian orbifolds, an orbifold-diffeomorphism $f : \mathcal{Q} \to \mathcal{R}$ is called and *orbifold-isometry* if each local lift of the corresponding map $f : Q \to R$ between the underlying topological spaces is an isometry.

Given two (Riemannian) orbifolds $\mathcal{Q}$, $\mathcal{Q}'$ with underlying topological spaces Q, Q' and atlases $\{(X_i, q_i, \Gamma_i)\}_{i\in I}$, $\{(X_i', q_i', \Gamma_i')\}_{i\in I'}$ respectively, an *orbifold-covering* is a surjective orbifold-map $p : \mathcal{Q}' \to \mathcal{Q}$ such that, for each $x \in X$, there exists a chart (X_i, q_i, Γ_i) at x with the following property: for any connected component U_i of $p^{-1}(q_i(X_i))$ in Q', there is a chart (X_j', q_j', Γ_j') with $\Gamma_j' \leq \Gamma_i$ satisfying $U_i \simeq X_j'/\Gamma_j'$. Note that the underlying map $p : Q' \to Q$ is not in general a covering. An orbifold map $f : \mathcal{Q}' \to \mathcal{Q}'$ such that $p \circ f = p$ is called a *deck transformation.*

The *orbifold universal covering* $p : \tilde{\mathcal{Q}} \to \mathcal{Q}$ of a connected (Riemannian) orbifold $\mathcal{Q}$ is an orbifold-covering with the following property: for any orbifold-covering $p' : \mathcal{Q}' \to \mathcal{Q}$, there exists an orbifold-covering $\bar{p} : \tilde{\mathcal{Q}} \to \mathcal{Q}'$ such that $p = p' \circ \bar{p}$.

Thurston proved in [43] that any connected orbifold $\mathcal{Q}$ admits a (unique) orbifold universal covering $\tilde{\mathcal{Q}} \to \mathcal{Q}$ (cf. [10] for a detailed proof). Its deck transformation group is called *orbifold fundamental group* and is denoted by $\pi_1^{\mathrm{orb}}(\mathcal{Q})$. If

$\pi_1^{\text{orb}}(\mathcal{Q})$ is trivial, $\mathcal{Q}$ is called *simply connected*. Note that if an orbifold is simply connected, then its underlying topological space is simply connected as well; however, there exist connected orbifolds $\mathcal{Q}$ with $\pi_1^{\text{orb}}(\mathcal{Q}) \neq \{e\}$ whose underlying topological space is simply connected.

Definition 2.3.4. Let $\mathcal{Q}$ be a Riemannian orbifold. An orbifold-isometry $f : \mathcal{Q} \to \mathcal{Q}$ is called a *reflection* if its restriction to the principal stratum $\mathcal{Q}_{\text{pr}}$ fixes a submanifold of dimension 1. If Γ is a discrete subgroup of orbifold-isometries of $\mathcal{Q}$, we shall denote by Γ_{refl} the subgroup of Γ generated by the reflections of $\mathcal{Q}$ which belong to Γ. Since the conjugate of a reflection is a reflection, Γ_{refl} is a normal subgroup of Γ.

Lemma 2.3.5. *Let $\mathcal{Q}$ be a Riemannian orbifold, and Γ a discrete subgroup of orbifold-isometries of $\mathcal{Q}$. Then $\mathcal{Q}/\Gamma$ is a Riemannian orbifold, and the natural projection $\mathcal{Q} \to \mathcal{Q}/\Gamma$ is an orbifold-covering.*

Proof. Since the elements of Γ are orbifold-isometries, the action of Γ on $\mathcal{Q}$ is proper; thus, for any fixed $x \in \mathcal{Q}$, there exists a neighborhood U of x invariant under the action of the isotropy Γ_x at x so that Γ_x parametrizes the orbits of Γ on U (cf. [14]). In particular, U/Γ_x identifies with an open subset of Q/Γ (Q being the underlying topological space of $\mathcal{Q}$) by means of the map $\Gamma_x \cdot y \mapsto \Gamma \cdot y$, $y \in U$. Without loss of generality, we assume this neighborhood U to be $X_i/\Gamma_i = q_i(X_i)$ for a suitable chart (X_i, q_i, Γ_i) of $\mathcal{Q}$. Now, any element of Γ_x is an isometry of $U \simeq X_i/\Gamma_i$, so (cf. [7, p. 133]) it lifts to an isometry of X_i. Denote by $\tilde{\Gamma}_i$ the set of all this lifts; the finiteness of both Γ_i and Γ_x implies the finitenes of $\tilde{\Gamma}_i$, so, denoting by $\pi_i : X_i \to X_i/\tilde{\Gamma}_i$ the projection, we see that $(X_i, \pi_i, \tilde{\Gamma}_i)$ is a chart for Q/Γ. The set of all these charts forms an atlas for an orbifold structure $\mathcal{Q}/\Gamma$ on Q/Γ, with respect to which it is a Riemannian orbifold. Since Γ_i identifies with a subgroup of $\tilde{\Gamma}_i$ the last assertion in the statement follows. □

In particular, if M is a (Riemannian) manifold, considered as an orbifold with the obvious orbifold structure, and if Γ is a discrete group of isometries of M, then the quotient M/Γ is a (Riemannian) orbifold.

Definition 2.3.6. A *good*, or *developable*, orbifold, is an orbifold of the form M/Γ, where M is a Riemannian manifold, and Γ is a discrete group of isometries of M. An orbifold is called *bad* if it is not good.

For instance, if (G, M) is a polar isometric action of the compact group G on the complete Riemannian manifold M, the orbit space M/G is a good orbifold.

Given a (Riemannian) orbifold $\mathcal{Q}$ with underlying topological space Q, we denote by $\mathcal{Q}_*$ the topological space $Q \backslash \partial\mathcal{Q}$ (which is open in Q) with the obvious Riemannian orbifold structure.

Lemma 2.3.7 ([19])**.** *Let $\mathcal{Q}$ be a connected Riemannian orbifold, and let Γ a discrete group of isometries of $\mathcal{Q}$. Set $\mathcal{Q}' := \mathcal{Q}/\Gamma$. If $\pi_1^{\mathrm{orb}}(\mathcal{Q}'_*)$ is trivial, then Γ is generated by reflections.*

Proof. The quotient group $\bar{\Gamma} := \Gamma/\Gamma_{\mathrm{refl}}$ acts by isometries on $\bar{\mathcal{Q}} := \mathcal{Q}/\Gamma_{\mathrm{refl}}$ with quotient $\mathcal{Q}'$. We want to show that the projection $\bar{\mathcal{Q}} \to \mathcal{Q}'$ induces an orbifold covering $\bar{\mathcal{Q}}_* \to \mathcal{Q}'_*$. For this, assume first that an element $\omega \in \bar{\Gamma}$ acts as a reflection on $\bar{\mathcal{Q}}$. Then there exists a point $\bar{p} \in \bar{\mathcal{Q}}_{\mathrm{pr}}$ whose local group consists only of ω and the identity, and therefore projects to a point $p' \in \mathcal{Q}'$ lying in a codimension 1 stratum. Clearly, a lift $p \in \mathcal{Q}$ of p' is contained in the principal stratum $\mathcal{Q}_{\mathrm{pr}}$, and so it is fixed by a reflection in Γ not contained in Γ_{refl}, contradiction. Hence, there are no elements in $\bar{\gamma}$ that act as a reflection on $\bar{\mathcal{Q}}$, and the projection $\bar{\mathcal{Q}} \to \mathcal{Q}'$ has the property that the preimage of any boundary point in $\mathcal{Q}'$ is a boundary point in $\bar{\mathcal{Q}}$. Therefore the projection $\bar{\mathcal{Q}} \to \mathcal{Q}'$ restricts to an orbifold covering $\bar{\mathcal{Q}}_* \to \mathcal{Q}'_*$ and $\mathcal{Q}'_* = \bar{\mathcal{Q}}_*/\bar{\Gamma}$. On the other hand, $\mathcal{Q}'_*$ is simply connected by assumption, so $\bar{\Gamma}$ has to act trivially on the dense subset $\bar{\mathcal{Q}}_*$ of $\bar{\mathcal{Q}}$. Thus $\bar{\Gamma}$ acts trivially on all of $\bar{\mathcal{Q}}$, i.e. $\Gamma = \Gamma_{\mathrm{refl}}$ as claimed. □

We now turn back to representations.

Definition 2.3.8. Given a representation (G, V) of a compact Lie group G with orbit space $X := V/G$, we say that a point in X is said to be *orbifold point* if it has a neighborhood isometric to a Riemannian orbifold. The subset of X consisting of the orbifold points is denoted by X_{orb}.

Note that $X_{\mathrm{orb}} \supseteq X_{\mathrm{pr}} := V_{\mathrm{pr}}/G$ and is open and dense in X. Moreover, it has a natural orbifold structure, denoted by $\mathcal{X}_{\mathrm{orb}}$.

Assume that $x \in X$ and let $v \in V$ be a lift of x. If the slice representation at v is polar, then x is an orbifold point by Lemma 2.3.5. The fact that the converse also holds is due to Alexander Lytchak and Gudlaugur Thorbergsson (cf. [31]). In [31] it is also proved that representations (G, V) with the properties that the slice representation at any $O \neq v \in V$ is polar are exactly those for which the quotient $S(V)/G$ is a Riemannian orbifold. By the way, such representations are called *infinitesimally polar*, and have been classified in [21].

Definition 2.3.9. If (G, V) is a representation, an isometry f of the quotient $X := V/G$ is called a *reflection* if its restriction to V_{pr}/G fixes a submanifold of dimension 1.

The following Lemma is well-known to the experts, and its proof can be found in [29]:

Lemma 2.3.10. *Consider a representation (H, W) with H connected, and set $X := W/H$. Then the set $(\mathcal{X}_{orb})_* := \mathcal{X}_{orb} \backslash \partial \mathcal{X}_{orb}$ coincides with the set of non-singular G-orbits, and has trivial orbifold fundamental group.*

The following, is the first fundamental result of this Section:

Theorem 2.3.11 ([19])**.** *Let (H,W), (G,V) be quotient-equivalent representations with H connected. Then the group G/G° of connected components of G is generated by reflections on V/G°.*

Proof. Set $X := V/G^\circ$, $X' := V/G \simeq W/H$, $\Gamma := G/G^\circ$. Since H is connected, $(\mathcal{X}'_{\mathrm{orb}})_* = \mathcal{X}'_{\mathrm{orb}} \backslash \partial \mathcal{X}'_{\mathrm{orb}}$ has trivial orbifold fundamental group by Lemma 2.3.10. Now Γ is a finite group of isometries of $\mathcal{X}_{\mathrm{orb}}$, and the natural projection $\mathcal{X}_{\mathrm{orb}} \to \mathcal{X}'_{\mathrm{orb}} = \mathcal{X}_{\mathrm{orb}}/\Gamma$ is an orbifold covering, so the assertion follows by Lemma 2.3.7. □

We wish now to define a special set of generators for G/G°. Consider a representation (G,V) of G (which is faithful and) with trivial principal isotropy groups (in our applications, (G,V) will be a minimal reduction of another representation (H,W)). Then all slice representations have trivial principal isotropy groups as well. If $v \in V$ is a G-important point (i.e. it projects to a stratum in V/G of quotient codimension 1), then G_v acts transitively on the unit sphere S^a in $(\nu_v \mathcal{O})^\dagger$, and $G_v = S^a$. Here we have used the notation introduced in Section 1.4; more precisely, $\mathcal{O} := G \cdot v$, and $(\nu_v \mathcal{O})^\dagger$ denotes the orthogonal complement in $\nu_v \mathcal{O}$ of the fixed point space $(\nu_v \mathcal{O})^{G_v}$ of G_v. By a well-known result in Lie group Theory this can happen only for $a \in \{0,1,3\}$. Clearly in this case

$$\dim \mathrm{Str}(v) = \dim V - a - 1 \tag{2.4}$$

Note that if $a = 1,3$, then G_v is contained in G°, hence v is also a G°-important point. Conversely, if v is a G°-important point, it cannot belong to an exceptional orbit by Lemma 2.3.10; then the slice representation at v cannot have discrete orbits, and $a \neq 0$.

Finally observe that if v is G-important but not G°-important, i.e. if $a = 0$, then G_v containes exactly one element which is not the identity; such an element belongs to $G \backslash G^\circ$, normalizes G° and acts as a reflection on V/G°.

Suppose now in addition that G/G° is generated by reflections on V/G° (this is the case if (G,V) is a minimal reduction by Theorem 2.3.11). Given $\eta \in G/G^\circ$ one of these reflections, we take a G°-principal point $x \in V/G^\circ$ which is fixed by η. Let v be a preimage of x in V; clearly v is G-important, thus, by the discussion above, G_v has exactly one element different from the identity, ω, which has the form $g_0\eta$ for some $g_0 \in G^\circ$. Clearly ω is an involution in G (i.e. $\omega^2 = e$). Comparing formulas (1.4) and (2.4) for $\dim \mathrm{Str}(v)$ we deduce moreover that ω satisfies:

$$\dim V - \dim V^\omega = \dim G - \dim Z_G(\omega) + 1, \tag{2.5}$$

where V^ω is the subspace of points in V which are fixed by ω, while $Z_G(\omega)$ is the subgroup of G consisting of the elements commuting with ω. Noting that η and

ω are equivalent modulo G°, we deduce that G/G° is generated by projections of involutions satisfying (2.5).

The costruction given above is due to Claudio Gorodski and Alexander Lytchak (cf. [19]); following their terminology, we give the following:

Definition 2.3.12. Under the above notation, an involution $\omega \in G$ satisfying (2.5) is called *nice*.

Summarizing, we have proved the second fundamental result of this Section:

Theorem 2.3.13 ([19]). *Let (G, V) be a faithful representation of a compact Lie group G with trivial principal isotropy groups. Assume moreover that G/G° is generated by reflections on V/G°. Then G/G° admits a set of generators whose elements are projections of nice involutions of G.*

Nice involutions have a very useful property:

Lemma 2.3.14 ([19]). *Under the same assumptions of Theorem 2.3.13, let $U_\pm$ be G°-invariant subspaces of V with $U_+ \cap U_- = \{O\}$. If $\omega \in G$ is a nice involution such that $\omega(U_-) = U_+$, then the action of G° on $U_\pm$ is of cohomogeneity* 1.

Proof. Let V^ω be the fixed point space of ω; by the definition of ω, if $p \in V^\omega$, the stratum through p has codimension 1 in V (cf. (2.4)). Since $G \cdot V^\omega$ and $\mathrm{Str}(p)$ locally coincide, we deduce that $G \cdot V^\omega$ has codimension 1 in V. Similarly, $G \cdot V^\omega$ and $G^\circ \cdot V^\omega$ locally coincide, therefore $G^\circ \cdot V$ has codimension 1 in V as well. Now, let $U_\perp$ denote the orthogonal complement of $U_+ \oplus U_-$ in V; clearly $(U_+ \oplus U_-) \oplus U_\perp$ is a decomposition invariant under G° and ω. Therefore, if we set $V_0^\omega := V^\omega \cap (U_+ \oplus U_-)$, we see that $V^\omega = V_0^\omega \oplus (V^\omega \cap U_\perp)$ and $G^\circ \cdot V_0^\omega$ has codimension ≤ 1 in $U_+ \oplus U_-$. Let Δ be the subset of $U_+ \oplus U_-$ consisting of the elements $u_+ + u_-$ such that $|u_+| = |u_-|$. Since V_0^ω is the set $\{u + \omega(u) \mid u \in U_+\}$, we see that $G^\circ \cdot V^\omega$ is contained in Δ and therefore it must have codimension 1 in V. At this point it is clear that also Δ has codimension 1 in V.

Now, up to the origin, both Δ and $G^\circ \cdot V_0^\omega$ are locally manifolds of the same dimension; hence $G^\circ \cdot V_0^\omega$ contains an open subset of Δ. Thus, we can find u in the unit sphere of U_+, and an open set V of the unit sphere in U_- such that $u + V \subseteq G^\circ \cdot V_0^\omega$. This means that for all $v \in V$, there exist $h \in G^\circ$, $u' \in U_+$ so that $h \cdot u = u'$, $h \cdot v = \omega(u')$. In particular $h^{-1}\omega h \cdot u = v$, therefore $(\omega h^{-1}\omega)h \cdot u = \omega \cdot v$. Hence the orbit $G^\circ \cdot u$ contains the open subset $\omega(V)$ of the unit sphere in U_+. If $\dim U_+ \geq 2$, then G° acts transitively on the unit sphere in U_+, as claimed. If $\dim U_+ = 1$, the statement is clear anyway. ☐

We conclude this Section proving one last result from [19]. First we need the following:

Definition 2.3.15. For a good Riemannian orbifold M/Γ, where M is a Riemannian manifold, and Γ is a discrete group of isometries of M, an *orbifold geodesic* $\bar{\gamma}$ is the projection of a geodesic γ of M. In this case, the *index* of $\bar{\gamma}$ is, by definition, the index of γ.

Proposition 2.3.16. *Let $\rho_i = (G_i, V_i)$, $i = 1, 2$, be quotient-equivalent representations. If $\partial(V_1/G_1^\circ) = \varnothing$, then $\dim V_1 \leq \dim V_2$. In particular, if $V_1/G_1 \simeq V_2/G_2$ has empty boundary, then $\dim V_1 = \dim V_2$.*

Proof. If $\partial(V_1/G_1^\circ) = \varnothing$, we can find a horizontal geodesic $\bar{\gamma}$ in $S(V_1)$ of length π entirely contained in the principal stratum. Thus the projection of $\bar{\gamma}$ is a geodesic γ in $S(V_1)/G_1^\circ$ entirely contained in $S(V_1)_{\mathrm{pr}}/G_1^\circ$. Consider now the projection λ of γ to $S(V_1)/G_1 \simeq S(V_2)/G_2$. Clearly λ is contained in the quotient of the manifold $S(V_1)_{\mathrm{pr}}/G_1^\circ$ by the finite group of isometries G/G°; so λ is an orbifold geodesic in $(S(V_1)/G_1)_{\mathrm{orb}}$. Consider its lift $\bar{\lambda}$ to a G_2-horizontal geodesic in $S(V_2)$. We may assume that $\bar{\gamma}$ was chosen so that all of the above curves start at a principal point.

Let m be the index of γ, i.e. the number of conjugate points along γ, counted with their multiplicities, and denote by $\mathcal{O}_1^\circ$ the G_1°-orbit through the initial point of $\bar{\gamma}$. Then $\bar{\gamma}$ is a $\mathcal{O}_1^\circ$-geodesic, and m is also the $\mathcal{O}_1^\circ$-index of $\bar{\gamma}$, i.e. the number of $\mathcal{O}_1^\circ$-focal points along $\bar{\gamma}$, counted with their multiplicities. On the other hand, in the unit sphere $S(V_1)$, the $\mathcal{O}_1^\circ$-index of an $\mathcal{O}_1^\circ$-geodesic of length π is $\dim \mathcal{O}_1^\circ$. Thus $m = \dim \mathcal{O}_1^\circ$.

Now let $\mathcal{O}_2$ be the G_2-orbit through the initial point of $\bar{\lambda}$. It has been proved in [31] that the $\mathcal{O}_2$-index of the $\mathcal{O}_2$-geodesic $\bar{\lambda}$ is the sum of the index of the orbifold geodesic η, i.e. m, with a non-negative integer that counts the number of intersections of $\bar{\lambda}$ with G_2-singular orbits. In particular, the $\mathcal{O}_2$-index of $\bar{\lambda}$ is $\geq m$. On the other hand the $\mathcal{O}_2$-index of $\bar{\lambda}$ equals $\dim \mathcal{O}_2$, so we deduce $\dim \mathcal{O}_2 \geq \dim \mathcal{O}_1^\circ$, and therefore $\dim V_2 \geq \dim V_1$.

Since any boundary point of V_1/G_1° projects to a boundary point of V_1/G_1, the last assertion follows. □

Corollary 2.3.17. *Let (G, V) be a faithful representation such that $\partial(V/G^\circ) = \varnothing$. Then (G, V) is reduced.*

Proof. Suppose (G, V) is not reduced; then it admits a minimal reduction (G', V'). By Propositon 2.3.16 and Corollary 1.7.13 we obtain $\dim V \leq \dim V' < \dim V$, a contradiction. □

Corollary 2.3.18. *Let $\mathcal{C}$ be a quotient-equivalent class of representations, and assume that the orbit space of its elements has empty boundary. Then:*

1. *the dimension of the representation space is constant within $\mathcal{C}$;*

2. *if $(G, V) \in \mathcal{C}$, the principal isotropy groups of (G, V) coincide with $\ker(G, V)$;*

3. if $(G, V) \in \mathcal{C}$ is faithful, then it is reduced.

Proof. (1) is contained in Proposition 2.3.16. Let H be a principal isotropy group of (G, V). By (1) and Proposition 1.7.12 $V^H = V$, so $H \subseteq \ker(G, V)$ and (2) follows. Now it is clear that, if (G, V) is faithful, then it has trivial principal isotropy groups. So, for any other $(G', V') \in \mathcal{C}$, the relation $\text{chm}(G, V) = \text{chm}(G', V')$, together with (1), gives

$$\dim G = \dim G' - \dim H' \leq \dim G',$$

where H' denotes a principal isotropy group of (G', V'), and (3) follows. □

2.4 Decomposability of representations

Consider two representations $\rho_i = (G_i, V_i)$, $i = 1, 2$. The *product representation* of ρ_1 and ρ_2 is the representation $\rho_1 \times \rho_2 = (G_1 \times G_2, V_1 \oplus V_2)$ where we let $G_1 \times G_2$ act on $V_1 \oplus V_2$ componentwisely; namely

$$(g_1, g_2) \cdot (v_1, v_2) := (g_1 \cdot v_1, g_2 \cdot v_2), \qquad \forall\, (g_1, g_2) \in G_1 \times G_2,\ \ \forall\, (v_1, v_2) \in V_1 \oplus V_2.$$

The goal of this Section is to study more closely quotient-equivalence classes containig products of representations. We begin remarking that, with the notation introduced above, the orbit space of $\rho_1 \times \rho_2$ is isometric to the product of the orbit spaces of ρ_1 and ρ_2:

$$\frac{V_1 \oplus V_2}{G_1 \times G_2} \simeq \frac{V_1}{G_1} \times \frac{V_2}{G_2}.$$

This fact suggests to give the following:

Definition 2.4.1 ([37])**.** Let $\rho = (G, V)$ be a representation of the compact Lie group G. We say that ρ is *decomposable* if the orbit space V/G splits as a product

$$V/G \simeq X_1 \times X_2$$

for suitable metric spaces X_1, X_2 neither of which is isometric to a single point.

We remark that the notion of decomposability has been introduced in the more general setting of singular Riemannian foliations by Alexander Lytchak in [30] and by Marco Radeschi in [38].

By definition, if a representation ρ is decomposable, then any representation quotient-equivalent to ρ is decomposable as well. Our first goal is to show that quotient-equivalence classes of decomposable representations contain always a product representation; so they can be characterized by this fact.

Proposition 2.4.2 ([37]). *Let (G, V) be a decomposable representation of the compact Lie group G, and set $V/G = X_1 \times X_2$. Then there exists a G-invariant splitting $V = V_1 \oplus V_2$ of V such that X_i is isometric to the orbit space of the induced representation $\rho_i := (G, V_i)$, $i = 1, 2$. In particular, (G, V) is orbit-equivalent to the product representation $(G_1 \times G_2, V_1 \oplus V_2)$, where $G_i := \rho_i(G)$.*

Note however that Proposition 2.4.2 gives us much more information: indeed, it implies that the decomposability of a representation can be analyzed within the representation itself.

Before proving Proposition 2.4.2 we need to recall some basic constructions from Metric Geometry (cf. [7]).

Given a metric space (Y, d), the (*flat*) *cone* $C(Y)$ over Y is the metric space defined as follows. As a set, $C(Y)$ is the quotient of $[0, \infty) \times Y$ by the equivalence relation

$$(t, y) \sim (t', y') \quad \Leftrightarrow \quad t = t' = 0 \quad \text{or} \quad t = t' > 0,\ y = y'$$

The quotient equivalence classe of $(t, y) \in [0, \infty) \times Y$ is denoted by ty. The class of $(0, y)$, $y \in Y$, is denoted by O and is called the *vertex* of the cone, or the *cone point*. Let d_π be the distance on Y defined by

$$d_\pi(y, y') := \min(\pi, d(y, y')), \qquad \forall\, y, y' \in Y.$$

The distance between $x = ty$, $x' = t'y' \in C(Y)$ is then defined by

$$d(x, x')^2 := t^2 + (t')^2 - 2tt' \cos(d_\pi(y, y')).$$

Let now (Y_i, d^i), $i = 1, 2$, be two metric spaces. The *spherical join* $Y_1 * Y_2$ of Y_1 and Y_2 is the metric space defined as follows. As a set, $Y_1 * Y_2$ is the quotient of $[0, \pi/2] \times Y_1 \times Y_2$ modulo the equivalence relation which identifies (θ, y_1, y_2), (θ', y_1', y_2') whenever:

1. $\theta = \theta' = 0$ and $y_1 = y_1'$, or
2. $\theta = \theta' = \pi/2$ and $y_2 = y_2'$, or
3. $\theta = \theta' \in (0, \pi/2)$, $y_1 = y_1'$, $y_2 = y_2'$.

The equivalence class of (θ, y_1, y_2) will normally be denoted $y_1 \cos\theta + y_2 \sin\theta$. Sometimes we shall denote the class of $(0, y_1, y_2)$ (resp. $(\pi/2, y_1, y_2)$) simply by y_1 (resp. y_2), thus implicitly identifying Y_1 and Y_2 to subsets of $Y_1 * Y_2$.

We define a distance d on $Y_1 * Y_2$ by requiring that the distance between the points $y = y_1 \cos\theta + y_2 \sin\theta$, $y' = y_1' \cos\theta' + y_2 \sin\theta'$ be at most π, and that d satisfy the formula

$$\cos(d(y, y')) = \cos\theta \cos\theta' \cos(d^1_\pi(y_1, y_1')) + \sin\theta \sin\theta' \cos(d^2_\pi(y_2, y_2')). \tag{2.6}$$

Remark 2.4.3. If one equips Y_1 and Y_2 with the truncated distances d^1_π, d^2_π (or if d_1 and d_2 do not assume values greater than π), then the natural inclusion of each space into $Y_1 * Y_2$ is an isometry onto its image.

Remark 2.4.4. Y_1 (resp. Y_2) is exactly the subset of $Y_1 * Y_2$ consisting of the points with distance $\pi/2$ from Y_2 (resp. Y_1).

The next result explains the connection between cones and spherical joins:

Lemma 2.4.5. *For any metric spaces Y_1, Y_2, there is a natural isometry*

$$\Phi : C(Y_1 * Y_2) \to C(Y_1) \times C(Y_2).$$

Proof. Define

$$\Phi(t(y_1 \cos\theta + y_2 \sin\theta)) := (t\cos\theta y_1, t\sin\theta y_2)$$

for all $t \in [0, \infty)$, and all $y_1 \cos\theta + y_2 \sin\theta \in Y_2 * Y_2$. Clearly Φ is surjective; we claim that it is an isometry. Indeed, given two points $x = t(y_1 \cos\theta + y_2 \sin\theta)$ and $x' = t'(y'_1 \cos\theta' + y'_2 \sin\theta')$ in $C(Y_1 * Y_2)$ we have

$$d(x, x')^2 = t^2 + (t')^2 - 2tt'[\cos\theta\cos\theta'\cos(d^1_\pi(y_1, y'_1)) + \sin\theta\sin\theta'\cos(d^2_\pi(y_2, y'_2))].$$

On the other hand,

$$\begin{aligned} d(\Phi(x), \Phi(x'))^2 = t^2\cos^2\theta + (t')^2\cos^2\theta' - 2tt'\cos\theta\cos\theta'\cos(d^1_\pi(y_1, y'_1)) \\ + t^2\sin^2\theta + (t')^2\sin^2\theta' - 2tt'\sin\theta\sin\theta'\cos(d^2_\pi(y_2, y'_2)). \end{aligned}$$

These expressions are obviously equal. □

Remark 2.4.6. The fact that formula (2.6) does indeed define a distance on $Y_1 * Y_2$ is contained in the proof of Lemma 2.4.5.

We can now give the proof of Proposition 2.4.2:

Proof of Proposition 2.4.2. Let $S(V)$ denote, as usual, the unit sphere in V, and set $S := S(V)/G$, $X := V/G$. Notice that, since $\operatorname{diam}(S) \leq \pi$, X coincides with the flat cone $C(S)$ over the metric space S; moreover, denoting by O the vertex of X, we see that S is the unit sphere in X centred at O. So the assumption of decomposability of (G, V) implies that we have a splitting of the cone $C(S)$ as a product

$$C(S) = X_1 \times X_2.$$

Identify X_i with the X_i-fibre through O, and set $Y_i := S \cap X_i$, $i = 1, 2$. We claim that $Y_i \neq \varnothing$ and that X_i is isometric to $C(Y_i)$, $i = 1, 2$. We shall prove the claim for $i = 1$, the other case being similar.

Denote by O_1 the projection of O in X_1. Since by definition X_1 is not a single point, it contains an element $x_1 \neq O_1$. Now, X_1 is a complete, locally compact length space (since X is), therefore Theorem 1.6.7 implies that there is a minimizing geodesic between O_1 and x_1. Clearly this curve is also a minimizing geodesic between O and (x_1, O_2) in X, O_2 being the projection of O in X_2, and so is the restriction to the interval $[0, 1]$ of the geodesic $\gamma(t) = tx_1$, $t \in [0, \infty)$. By construction $\gamma(t) \in X_1$ for $0 \leq t \leq 1$; so the projection of γ in X_2, which is a geodesic in X_2 (not necessarily parametrized by the arc length, cf. [7, p. 56]), is constantly equal to O_2 for $0 \leq t \leq 1$, and hence for all $t \in [0, \infty)$. This proves that γ is entirely contained in X_1. Since there exists $t_0 > 0$ such that $\gamma(t_0) \in S$, we get $\gamma(t_0) \in Y_1$.

Now define $\Phi_1 : C(Y_1) \to X_1$ by setting $\Phi_1(ty_1) = ty_1$, where the first occurrence of ty_1 denotes the equivalence class of $(t, y_1) \in [0, \infty) \times Y_1$ in $C(Y_1)$, while the second one denotes the equivalence class of $(t, y_1) \in [0, \infty) \times S$ in $C(S)$. By the same argument as above, the entire geodesic ty_1 of $C(S)$ is contained in X_1 (since $y_1 \in X_1$), therefore Φ_1 is well-defined and surjective. Moreover Φ_1 is clearly an isometry, and the claim is proved.

Summarizing, there are the following relations of isometricity:

$$C(S) \simeq X_1 \times X_2 \simeq C(Y_1) \times C(Y_2) \simeq C(Y_1 * Y_2),$$

where we have used Lemma 2.4.5. Since $\operatorname{diam}(S) \leq \pi$ we easily get

$$S \simeq Y_1 * Y_2.$$

Invoking Remark 2.4.4 and the claim used in the proof of Proposition 2.2.11 we deduce that there exist two G-invariant complementary subspaces V_1, V_2 of V such that $Y_i = S(V_i)/G$, i.e. $X_i = V_i/G$, $i = 1, 2$.

Now we show that the product representation $(G_1 \times G_2, V_1 \oplus V_2)$ and (G, V) have the same orbits. Indeed, if $v \in V$, it is clear that $G \cdot v \subseteq (G_1 \times G_2) \cdot v$. On the other hand, the representations above are quotient-equivalent, hence there exists $v' \in V$ such that $G \cdot v = (G_1 \times G_2) \cdot v'$. Thus $(G_1 \times G_2) \cdot v' = G \cdot v \subseteq (G_1 \times G_2) \cdot v$, and we get $G \cdot v = (G_1 \times G_2) \cdot v$ exploiting the fact that any two orbits with non-empty intersection must coincide. □

A first useful consequence of Proposition 2.4.2 is the following:

Corollary 2.4.7 ([37]). *Let ρ_1, ρ_2 be representations, $\rho_1 \times \rho_2$ their product. Then*

$$\operatorname{ac}(\rho_1 \times \rho_2) = \operatorname{ac}(\rho_1) + \operatorname{ac}(\rho_2).$$

Proof. Assume $\rho_i = (H_1, W_i)$ for suitable compact Lie groups H_i and suitable euclidean spaces W_i, $i = 1, 2$. Then by definition $\rho_1 \times \rho_2 = (H_1 \times H_2, W_1 \oplus W_2)$, where $H_1 \times H_2$ acts componentwisely. Let (G, V) be a minimal reduction of $\rho_1 \times \rho_2$, and let (G_i, V_i) be a minimal reduction of ρ_i, $i = 1, 2$. By Proposition 2.4.2, (G, V) is orbit-equivalent to a product representation $(G'_1 \times G'_2, V'_1 \oplus V'_2)$, where $V = V'_1 \oplus V'_2$ and $V'_i/G'_i \simeq W_i/H_i$, $i = 1, 2$. Now

$$\frac{V_1}{G_1} \times \frac{V_2}{G_2} \simeq \frac{W_1}{H_1} \times \frac{W_2}{H_2} \simeq \frac{W_1 \oplus W_2}{H_1 \times H_2} \simeq \frac{V}{G},$$

thus $(G_1 \times G_2, V_1 \oplus V_2)$ is quotient-equivalent to (G, V) and we get

$$\mathrm{ac}(\rho_1 \times \rho_2) = \dim G \leq \dim G_1 + \dim G_2 = \mathrm{ac}(\rho_1) + \mathrm{ac}(\rho_2). \tag{2.7}$$

Notice also that

$$\frac{V_i}{G_i} \simeq \frac{W_i}{H_i} \simeq \frac{V'_i}{G'_i} \qquad i = 1, 2,$$

therefore $\dim G_1 + \dim G_2 \leq \dim G'_1 + \dim G'_2$. If the equality in (2.7) didn't hold, we would get

$$\dim G < \dim G'_1 + \dim G'_2,$$

i.e. (G, V) would be a minimal reduction of $(G'_1 \times G'_2, V'_1 \oplus V'_2)$. Thus $\dim V < \dim V'_1 + \dim V'_2$ by Corollary 1.7.13, contradicting $V = V'_1 \oplus V'_2$. □

Consider now a representation (G, V) of a compact Lie group G. We wish to split (G, V) into a product of indecomposable subrepresentations. In order to do this we need a version of the de Rham decomposition theorem for metric spaces due to Thomas Foertsch and Alexander Lytchak (cf. [17]).

Definition 2.4.8. A metric space X is called *irreducible* if, for any decomposition $X = Y \times Z$ into a product of metric spaces, one of the factors Y or Z must be a point.

Theorem 2.4.9 ([17])**.** *Let X be a geodesic metric space of finite Hausdorff dimension. Then X admits a decomposition as a direct product*

$$X = Y_0 \times Y_1 \times \cdots \times Y_n,$$

where Y_0 is a euclidean space (possibly a point), and where the Y_i, $i = 1, \ldots, n$, are irreducible metric spaces not isometric to the real line, nor to a point. If there is another direct product decomposition $X = Z_0 \times Z_1 \times \cdots \times Z_m$ of the same kind, then we have $m = n$ and there exists a permutation σ of $\{0, 1, \ldots, n\}$ such that for each point $x \in X$, the Y_i-fibre through x coincides with the $Z_{\sigma(i)}$-fibre through x, for all $i = 0, 1, \ldots, n$.

Since the orbit space V/G of (G, V) is geodesic (cf. Theorems 1.6.7, 1.6.8) and of finite Hausdorff dimension (cf. Example 1.7.2), by Theorem 2.4.9 it splits as a product of metric spaces

$$V/G = Y_0 \times Y_1 \times \cdots \times Y_n, \tag{2.8}$$

where Y_0 is euclidean (possibly trivial), and where Y_i, $i = 1, \ldots, n$, is a non-trivial, non-euclidean irreducible metric space. Then Proposition 2.4.2 immediately yields:

Theorem 2.4.10 ([37]). *Let (G, V) be a representation of a compact Lie group G. Then there exists a G-invariant splitting $V = V_0 \oplus V_1 \oplus \cdots \oplus V_n$ of V such that:*

1. *the induced representation $\rho_0 := (G, V_0)$ is trivial and $\dim V_0 \geq 0$;*
2. *the induced representation $\rho_i := (G, V_i)$ is indecomposable, non-trivial and $\dim V_i \geq 1$, for $i = 1, \ldots, n$;*
3. *(G, V) is orbit-equivalent to the product representation*

$$(G_0 \times G_1 \times \cdots \times G_n, V_0 \oplus V_1 \oplus \cdots \oplus V_n),$$

where $H_0 := \{e\}$, $H_i := \rho_i(H_i)$, $i = 1, \ldots, n$.

We observed above that the (in)decomposability of a representation $\rho = (G, V)$ is a property which depends only on the quotient-equivalent class $\mathcal{C}$ of ρ. We conclude this section proving that also (in)decomposability of the induced representation (G°, V) is a property invariant of the quotient, at least if $\mathcal{C}$ is non-polar and contains a representation (H, W) with H connected. More precisely, we have the following:

Proposition 2.4.11 ([37]). *Let (H, W) be a non-polar indecomposable representation of a connected, compact Lie group H. If (G, V) is quotient-equivalent to (H, W), then the induced representation (G°, V) is also indecomposable.*

Proof. Our assumptions imply that the representation (G, V) is indecomposable and non-polar (cf. Remark 2.2.2); moreover the finite group G/G° is generated by reflection on V/G° (cf. Theorem 2.3.11). Now decompose the orbit space V/G° of the induced representation (G°, V) as in (2.8):

$$V/G^\circ = Y_0 \times Y_1 \times \cdots \times Y_n. \tag{2.9}$$

By Proposition 2.4.2, each Y_i is the orbit space of a suitable subrepresentation (G°, V_i) of (G°, V); we shall say that Y_i is flat if its principal stratum $(V_i)_{\text{pr}}/G^\circ$ is. Clearly Y_0 is flat, since it is the quotient of a trivial representation. Up to a reordering of the other factors in (2.9) we may assume that $Y_1, \ldots, Y_m$ are non-flat, while $Y_{m+1}, \dots, Y_n$ are flat. We have then $m \geq 1$, otherwise V/G° would be flat and, by Proposition 2.2.1, (G°, V) would be polar: this is absurd because (G, V) is non-polar (cf. Lemma 1.5.9).

Now consider a reflection $\gamma \in G/G^\circ$. Since it is an isometry of the cone V/G°, it must preserve its vertex, hence the uniqueness part in Theorem 2.4.9 implies that γ permutes the Y_i's. On the other hand γ fixes a 1-dimensional submanifold of the principal stratum V_{pr}/G° and satisfies $\gamma^2 = \mathrm{id}$, therefore either γ preserves all Y_i's, acting as a reflection on one of them and fixing pointwisely the others, or it interchanges Y_j, Y_k, for some $j, k \in \{0, \dots, n\}$, $j \neq k$, and fixes pointwisely each Y_i with $i \neq j, k$. Note that in the latter case Y_j, Y_k must have dimension 1: this means that the corresponding subrepresentations (G°, V_j), (G°, V_k) have cohomogeneity 1, hence they are polar and Y_j, Y_k are flat. From these observations we easily get

$$\frac{V}{G} = \frac{Y_{\mathrm{flat}}}{G} \times \frac{Y_1}{G} \times \cdots \times \frac{Y_m}{G},$$

where $Y_{\mathrm{flat}} := Y_0 \times Y_{m+1} \times \cdots \times Y_n$ is the product of all flat factors in (2.9). Using indecomposability of (G, V) we finally deduce that Y_{flat} is trivial, and that $m = 1$; therefore (G°, V) is indecomposable. □

Remark 2.4.12. The analogous of Proposition 2.4.11 for irreducibility does not hold: indeed, there exist quotient-equivalent representations (H, W), (H', W') such that the induced representation (H°, W) is irreducible, while the induced representation $((H')^\circ, W')$ is not. In [19, Theorem 1.7], it is shown that a quotient equivalent class with this property has a unique representation (G, V) with $\dim G$ minimal, and that G° is a torus. Representations admitting a reduction (G, V) where G° a torus will be studied more closely in Chapter 3.

The following Lemma gives a criterion to decide whether a reducible representations is indecomposable.

Lemma 2.4.13. *Let G be a compact Lie group, and (G, V_1), (G, V_2) two irreducible representations of G. Set $c_i := \mathrm{chm}(G, V_i)$, $i = 1, 2$, and $c := \mathrm{chm}(G, V_1 \oplus V_2)$. Then $(G, V_1 \oplus V_2)$ is decomposable if and only if $c = c_1 + c_2$.*

Proof. Assume that $(G, V_1 \oplus V_2)$ is decomposable; then it is orbit-equivalent to a product representation of the form $(G_1 \times G_2, V_1' \oplus V_2')$, where $V_1 \oplus V_2 = V_1' \oplus V_2'$, and the V_i' are G-invariant (cf. Proposition 2.4.2). Clearly we may suppose $V_1 = V_1'$, $V_2 = V_2'$. Let $\rho_i := (G, V_i)$. Since $G_i = \rho_i(G)$, we see then that $(G, V_1 \oplus V_2)$ and the product $(G \times G, V_1 \oplus V_2)$ have the same orbits. Thus $c = c_1 + c_2$.

Conversely, assume $c = c_1 + c_2$; we shall prove that $(G, V_1 \oplus V_2)$ has the same orbits as the product representation $(G \times G, V_1 \oplus V_2)$. First we note that we can choose $v = v_1 + v_2 \in V_1 \oplus V_2$ so that v is principal for $(G, V_1 \oplus V_2)$, while v_i,

$i = 1, 2$, is principal for (G, V_i). Clearly $G \cdot v \subseteq (G \times G) \cdot (v_1 + v_2)$. Fix now $g_1 \cdot v_1 + g_2 \cdot v_2 \in (G \times G) \cdot (v_1 + v_2)$. From $c = c_1 + c_2$ we easily deduce that the isotropy group G_{v_1} acts on V_2 with the same orbits as G; thus there exists $\bar{g}_1 \in G_{v_1}$ such that $\bar{g}_1 \cdot v_2 = g_2 \cdot v_2$. Similarly, there exists $\bar{g}_2 \in G_{g_2 \cdot v_2}$ such that $\bar{g}_2 \cdot v_1 = g_1 \cdot v_1$. Hence

$$\bar{g}_2\bar{g}_1 \cdot v = \bar{g}_2\bar{g}_1 \cdot v_1 + \bar{g}_2\bar{g}_1 \cdot v_2 = \bar{g}_2 \cdot v_1 + \bar{g}_2 g_2 \cdot v_2 = g_1 \cdot v_1 + g_2 \cdot v_2,$$

and $G \cdot v = (G \times G) \cdot (v_1 + v_2)$. The claim now follows by Proposition 1.4.9. □

2.5 Representations of tori

The goal of this Section is to prove two useful criteria to understand whether a real representation $(\mathbb{T}, V)$ of a torus $\mathbb{T}$ is indecomposable and whether it has non-trivial copolarity (cf. Propositions 2.5.8, 2.5.12 below). We begin recalling some standard facts about representations of tori; sometimes, on the following pages, we shall need to consider also complex representations.

Lemma 2.5.1. *Let $(\mathbb{T}, U)$ be a complex representation of the torus $\mathbb{T}$ which is irreducible over $\mathbb{C}$. Then $\dim_{\mathbb{C}} U = 1$.*

Proof. Every $t \in \mathbb{T}$ can be considered as a bijective $\mathbb{C}$-linear map $t : U \to U$, namely the map given by $u \mapsto t \cdot u$, for all $u \in U$. Since $\mathbb{T}$ is abelian, we have

$$t' \cdot (t \cdot u) = t \cdot (t' \cdot u), \qquad \forall\, t' \in \mathbb{T},$$

thus $t : U \to U$ is a $\mathbb{C}\mathbb{T}$-map. Irreducublity of U implies (by Lemma 2.1.14) that there exists $\phi(t) \in \mathbb{C}$ with $t \cdot u = \phi(t)u$, for all $u \in U$. Hence all complex subspaces of U are $\mathbb{T}$-invariant and $\dim_{\mathbb{C}} U = 1$. □

If $(\mathbb{T}, U)$ is a complex representation of $\mathbb{T}$ which is irreducible over $\mathbb{C}$, we see from the proof of Lemma 2.5.1 that there is a map $\phi : \mathbb{T} \to \mathbb{C}$ so that $t \cdot u = \phi(t)u$, for all $t \in \mathbb{T}$, $u \in U$. Obviously this map takes its values in $\mathbb{C} \backslash \{0\}$ and is a Lie group homomorphism. More precisely, let $(\cdot, \cdot)$ be a $\mathbb{T}$-invariant hermitian metric on U. If $u \in U$ we have then

$$(u, u) = (t \cdot u, t \cdot u) = |\phi(t)|^2 (u, u), \qquad \forall\, t \in \mathbb{T},$$

therefore ϕ is a Lie group homomorphism $\mathbb{T} \to S^1$. We have then the following:

Proposition 2.5.2. *There is a bijective correspondance between irreducible complex representations of a torus $\mathbb{T}$ and Lie group homomorphisms $\mathbb{T} \to S^1$.*

Proof. We have seen above that we can associate a Lie group homorphism $\mathbb{T} \to S^1$ to any irreducible complex representation $(\mathbb{T}, U)$. Conversely, given a Lie group homomorphism $\phi : \mathbb{T} \to S^1$, we can define an irreducible complex representation of $\mathbb{T}$ on $U := \mathbb{C}$ by setting

$$t \cdot u := \phi(t)u, \qquad \forall\, t \in \mathbb{T},\ \forall\, u \in U.$$

Clearly these constructions are inverse of each other. □

Definition 2.5.3. Given an irreducible complex representation $(\mathbb{T}, U)$ of the torus $\mathbb{T}$, the Lie group homomprhism $\phi : \mathbb{T} \to S^1$ introduced in the proof of Lemma 2.5.1 is called the *character* of $(\mathbb{T}, U)$.

Remark 2.5.4. If $(\mathbb{T}, U)$ is an irreducible complex representation of the torus $\mathbb{T}$ with character $\phi : \mathbb{T} \to S^1$, then the character of the dual representation $(\mathbb{T}, \mathrm{t}U)$ is the conjugate of ϕ, i.e. the map $\bar{\phi} : \mathbb{T} \to S^1$ given by $t \mapsto \overline{\phi(t)}$. In particular, $(\mathbb{T}, U)$ is of complex type if and only if it is not the trivial representation (cf. Proposition 2.1.16). Clearly the trivial representation of $\mathbb{T}$ is of real type.

Remark 2.5.5. Any complex representation of a torus $\mathbb{T}$ (not necessarily irreducible) is completely determined by the characters and the complex dimensions of its isotypical components.

For the sake of completeness we give a complete description of Lie group homomorphism $\mathbb{T} \to S^1$.

Proposition 2.5.6. *Let $\mathbb{T}$ be a k-dimensional torus, and $\phi : \mathbb{T} \to S^1$ a Lie group homomorphism. Identify $\mathbb{T}$ with the product of k copies of S^1. Then there exist $n_1, \ldots, n_k \in \mathbb{Z}$ so that*

$$\phi(z_1, \ldots, z_k) = z_1^{n_1} \cdots z_k^{n_k}, \qquad \forall\, (z_1, \ldots, z_k) \in \mathbb{T}.$$

Proof. Since $\mathbb{T} \simeq S^1 \times \cdots \times S^1$ (k times), and the restriction of ϕ to any S^1-factor is a Lie group homomorphism $S^1 \to S^1$, it is enough to prove the case $k = 1$.

We consider the covering $\mathbb{R} \to S^1$ given by $x \mapsto \mathrm{e}^{2\pi i x}$ for all $x \in \mathbb{R}$. Any Lie group homomorphism $\phi : S^1 \to S^1$ can be lifted to a Lie group homomorphism $\tilde{\phi} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{\phi}(1) \in \mathbb{Z}$. Set $n := \tilde{\phi}(1)$. Then $\tilde{\phi}(a) = na$, and $\tilde{\phi}(a/b) = na/b$, for all $a, b \in \mathbb{Z}$, $b \neq 0$. By continuity $\tilde{\phi}(x) = nx$ for all $x \in \mathbb{R}$, hence $\phi(z) = z^n$ for all $z \in S^1$. □

We have already mentioned that we are mainly interested in real representations. So consider a real representation $(\mathbb{T}, V)$ of the torus $\mathbb{T}$ which is irreducible over $\mathbb{R}$. If $(\mathbb{T}, V)$ is not trivial, then, using the notation introduced in Section 2.1 just before Lemma 2.1.17, we have $V = \mathrm{r}U$ for some irreducible complex representation $(\mathbb{T}, U)$ of $\mathbb{T}$ (cf. Theorem 2.1.18). Thus Lie group homomorphism $\mathbb{T} \to S^1$

determine also all real representations of $\mathbb{T}$. Notice however that $\mathrm{r}U = \mathrm{rt}U$ for any complex representation $(\mathbb{T}, U)$ of $\mathbb{T}$, therefore any (non-trivial) homomorphism $\mathbb{T} \to S^1$ determines the same real representation as its conjugate.

Given any real (or complex) representation $(\mathbb{T}, V)$ of the torus $\mathbb{T}$ and a Lie group homorphism $\phi : \mathbb{T} \to S^1$, we shall denote by V_ϕ the sum of all irreducible summands of V with character ϕ. If V_ϕ is not zero, then it is an isotypical component of V; in this case ϕ will be called a *character* of $(V, \mathbb{T})$. If $(V, \mathbb{T})$ is a real representation, then the discussion above implies $V_\phi = V_{\bar{\phi}}$. If in addition ϕ is not trivial we have also

$$V_\phi = V \cap ((\mathrm{c}V)_\phi \oplus (\mathrm{c}V)_{\bar{\phi}}).$$

Identifying the Lie algebra of S^1 with the real line, and denoting by $\mathfrak{t}$ the Lie algebra of $\mathbb{T}$, we may associate to any character $\phi : \mathbb{T} \to S^1$ of $(\mathbb{T}, V)$ an element of the dual $\mathfrak{t}^*$, namely $\mathrm{d}\phi_e : \mathfrak{t} \to \mathbb{R}$.

Definition 2.5.7. Given a real (or complex) representation $(\mathbb{T}, V)$ of the torus $\mathbb{T}$, the elements of $\mathfrak{t}^*$ associated to the characters of $(\mathbb{T}, V)$ as described above are called the *weights* of $(\mathbb{T}, V)$.

We can now state and prove a useful criterion for the indecomposability of a representation of a torus. From this moment on, all representations considered will be tacitly assumed to be real.

Proposition 2.5.8 ([37]). *Let $\rho = (\mathbb{T}, V)$ be a representation of the torus $\mathbb{T}$, and let $\Theta \subseteq \mathfrak{t}^*$ denote the set of all weights of ρ. Then ρ is decomposable if and only if there exist two non-empty subsets $\Theta_1, \Theta_2 \subseteq \Theta$ such that $\Theta_1 \cup \Theta_2 = \Theta$ and $\langle \Theta_1 \rangle \cap \langle \Theta_2 \rangle = \{O\}$.*

Proof. Assume first that ρ is decomposable. Then, by Proposition 2.4.2, there exists a $\mathbb{T}$-invariant splitting $V = V_1 \oplus V_2$ so that ρ has the same orbits as the product representation $(\mathbb{T}_1 \times \mathbb{T}_2, V_1 \oplus V_2)$, where, if we denote by ρ_i the induced representation $(\mathbb{T}, V_i)$, $\mathbb{T}_i = \rho_i(\mathbb{T})$, $i = 1, 2$. Note in particular that $\mathbb{T}_i$ is a torus. Now we may assume that ρ has trivial kernel; so, $\mathbb{T}$ being abelian, ρ has trivial principal isotropy group. Note that also $(\mathbb{T}_i, V_i)$ has trivial kernel and hence trivial principal isotropy groups. Since a principal orbit of ρ is isometric to a principal orbit of $(\mathbb{T}_1 \times \mathbb{T}_2, V_1 \oplus V_2)$, we deduce $\dim \mathbb{T}_1 + \dim \mathbb{T}_2 = \dim \mathbb{T}$. Now let $\tilde{\mathbb{T}}_i$ be the subtorus of $\mathbb{T}$ with the same Lie algebra $\mathfrak{k}_j$ of $\ker(\rho_j)$, $i, j \in \{1, 2\}$, $i \neq j$. Notice that $\dim \tilde{\mathbb{T}}_i = \dim \mathbb{T}_i$, $i = 1, 2$, so $\dim \tilde{\mathbb{T}}_1 + \dim \tilde{\mathbb{T}}_2 = \dim \mathbb{T}$. Since $(\mathbb{T}, V)$ has no kernel, we deduce also $\tilde{\mathbb{T}}_1 \cap \tilde{\mathbb{T}}_2 = \{e\}$ so $\mathbb{T} = \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$. Denote by Θ_i the set of all weights of the subrepresentation $(\tilde{\mathbb{T}}_i, V_i)$. Since, for $i \neq j$, the action $(\tilde{\mathbb{T}}_i, V_j)$ is trivial, we see that $\Theta_1 \cup \Theta_2 = \Theta$. Moreover $\langle \Theta_1 \rangle \subseteq \mathfrak{k}_2^*$ and, similarly, $\langle \Theta_2 \rangle \subseteq \mathfrak{k}_1^*$, so $\langle \Theta_1 \rangle \cap \langle \Theta_2 \rangle \subseteq \mathfrak{k}_1^* \cap \mathfrak{k}_2^* = \{O\}$.

Conversely, let Θ_1, Θ_2 be as in the statement, and denote by V_i the sum of all isotypical components of ρ corresponding to weights in Θ_i, $i = 1, 2$. Denote moreover by $\mathfrak{a}_i \subseteq \mathfrak{t}$ the annihilator of $\langle\Theta_i\rangle$, $i = 1, 2$. From $\langle\Theta_1\rangle \cap \langle\Theta_2\rangle = \{O\}$ we get $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{t}$ (direct sum if and only if ρ has discrete kernel), while from $\Theta_1 \cup \Theta_2 = \Theta$ we get $V_1 \oplus V_2 = V$. Let T_i be the subtorus of $\mathbb{T}^k$ with Lie algebra $\mathfrak{a}_i$. Any element $t \in \mathbb{T}$ can be written as $t = t_1 t_2$ for suitable $t_i \in T_i$, $i = 1, 2$. Since the induced representation (T_i, V_i) is trivial, we deduce that ρ and the product representation $(T_2 \times T_1, V_1 \oplus V_2)$ have the same orbits, so ρ is decomposable. □

Remark 2.5.9. Let $\rho = (\mathbb{T}, V)$ be a representation of a torus $\mathbb{T}$, and denote by $\Theta \subseteq \mathfrak{t}^*$ the set of all weights of ρ. Then $\langle\Theta\rangle = \mathfrak{t}^*$ if and only if ρ has discrete kernel. Indeed, an element $x \in \mathfrak{t}$ belongs to the Lie algebra of $\ker(\rho)$ if and only if it is annihilated by all elements of Θ.

In order to state the main Corollary of Proposition 2.5.8, we need to give the following:

Definition 2.5.10. If $\rho = (\mathbb{T}, V)$ is a representation of a torus $\mathbb{T}$, we say that a line $\mathfrak{s} \subseteq \mathfrak{t}^*$ is *induced* by ρ if ρ has a weight θ such that $\langle\theta\rangle = \mathfrak{s}$.

Corollary 2.5.11 ([37]). *Let $\rho = (\mathbb{T}, V)$ be a faithful representation of a k-dimensional torus $\mathbb{T}$, $k \geq 2$. If ρ is indecomposable, then it induces at least $k+1$ lines in $\mathfrak{t}^*$.*

Proof. By contradiction assume that ρ induces only $\ell \leq k$ lines $\mathfrak{s}_1, \ldots, \mathfrak{s}_\ell$ in $\mathfrak{t}^*$, and let Θ be the set of all weights of ρ. Clearly $\langle\mathfrak{s}_1, \ldots, \mathfrak{s}_\ell\rangle = \langle\Theta\rangle = \mathfrak{t}^*$, where the last equality holds because ρ is faithful (cf. Remark 2.5.9). In particular $\ell = k$, and the lines $\mathfrak{s}_i$'s are linearly independent. Now define Θ_1 as the subset of Θ consisting of the weights θ such that $\langle\theta\rangle = \mathfrak{s}_1$, and, analogously, define Θ_2 as the subset of Θ consisting of the weights θ such that $\langle\theta\rangle = \mathfrak{s}_j$ for some $j \geq 2$. Notice that Θ_2 is non-empty because $k \geq 2$. Then Θ_1 and Θ_2 clearly satisfy the assumptions of Proposition 2.5.8, therefore ρ is decomposable. □

Especially for toric representations, the absence of proper generalized sections can be read of the quotient:

Proposition 2.5.12. *Let $\rho = (\mathbb{T}, V)$ be a representation of a torus $\mathbb{T}$. Then ρ has trivial copolarity if and only if $\partial(V/\mathbb{T}) = \varnothing$.*

Proof. We may assume without loss of generality that ρ is faithful and has no non-trival fixed points. Since $\mathbb{T}$ is abelian, this in particular implies that ρ has trivial principal isotropy groups.

If $\partial(V/\mathbb{T}) = \varnothing$, ρ is reduced by Corollary 2.3.18.

Conversely, assume $\partial(V/\mathbb{T}) \neq \varnothing$, and let $p \in V$ be a $\mathbb{T}$-important point, i.e. a point projecting onto a stratum of quotient codimension 1 in $V/\mathbb{T}$. The non-trivial part of the slice representation at p has trivial principal isotropy groups and cohomogeneity 1, therefore $\mathbb{T}_p$ is a sphere S^a (cf. the discussion preceding Theorem 2.3.13 in Section 2.3). Since $\mathbb{T}$ is abelian, we get $a \in \{0, 1\}$, and, by (2.4), $\dim V^{\mathbb{T}_p} = \dim \mathrm{Str}(p) = \dim V - a - 1$, where, as usual, $V^{\mathbb{T}_p}$ denotes the fixed point space of $\mathbb{T}_p$.

Assume first $a = 0$, and let ω be the generator of $\mathbb{T}_p \simeq \mathbb{Z}/2\mathbb{Z} \simeq S^0$. Then V^ω has codimension 1 and ω is a reflection in V. On the other hand, $\omega \in S^1 \subseteq \mathbf{SO}(V)$, a contradiction.

Then $a = 1$, and we have a $\mathbb{T}$-invariant decomposition $V = V^{\mathbb{T}_p} \oplus \bar{V}$, where $\dim \bar{V} = 2$; we shall write accordingly $\rho = \tilde{\rho} \oplus \bar{\rho}$. Notice that $\bar{\rho}$ is irreducible, since ρ has no non-trivial fixed points. Let $\tilde{\Theta}$ be the set of weights corresponding to isotypical components in $V^{\mathbb{T}_p}$, and denote by $\bar{\theta}$ the weight of $\bar{\rho}$. Clearly any weight of ρ either belongs to $\tilde{\Theta}$, or it is equal to $\bar{\theta}$. Moreover all weights in $\tilde{\Theta}$ vanish on the 1-dimensional Lie algebra of $\mathbb{T}_p$, $\mathfrak{t}_p$: indeed, fix a non-zero vector $v \in V^{\mathbb{T}_p}$, and notice that, for all $t \in \mathbb{T}_p$, we have $v = t \cdot v = \phi(t)v$, i.e. $\phi(t) = 1$, if ϕ is any character of ρ corresponding to an isotypical component in $V^{\mathbb{T}_p}$. Now, faithfulness of ρ implies that $\bar{\theta}$ cannot vanish on $\mathfrak{t}_p$, so $\langle\tilde{\Theta}\rangle \cap \langle\bar{\theta}\rangle = \{O\}$. By (the proof of) Proposition 2.4.2 ρ is orbit-equivalent to the product representation $(\tilde{\rho}(\mathbb{T}^k) \times S^1, V^{\mathbb{T}_p} \oplus \bar{V})$; but $(S^1, \bar{V})$ is polar, so ρ has non-trivial copolarity. □

As a corollary, we find a family of representations for which copolarity and abstract copolarity do coincide:

Corollary 2.5.13. *Let G be a compact Lie group such that G° is a torus, and consider a representation (G, V) of G.*

1. *If (G, V) is faithful and has trivial copolarity, then it is reduced.*
2. *We have* $\mathrm{c}(G, V) = \mathrm{ac}(G, V)$.

Proof. (1) The induced representation (G°, V) has trivial copolarity (cf. Lemma 1.5.8), so $\partial(V/G^\circ) = \varnothing$ by Proposition 2.5.12. The assertion follows from Corollary 2.3.17.

(2) Let Σ be a generalized section of (G, V), and denote by N, Z respectively the normalizer and the centralizer of Σ. By definition, $\Gamma := N/Z$ is the Weyl group of Σ. Since N° is a compact and connected subgroup of G°, we see that it is a torus. Thus also $\Gamma^\circ \simeq \frac{N^\circ}{Z \cap N^\circ}$ is a torus. Now, (Γ, Σ) is reduced by (1), and is quotient-equivalent to (G, V). Therefore $\dim \Gamma$ coincides with both $\mathrm{c}(G, V)$ (by Proposition 1.7.14) and $\mathrm{ac}(G, V)$. □

Chapter 3

Toric reductions

Given a representation $\rho = (G, V)$ of a compact Lie group G, we have associated to ρ three important invariants: cohomogeneity, copolarity and abstract copolarity. A natural problem that arises at this point is then to look for general relationships between them (cf. [19, Question 1.5]).

We have seen that $\mathrm{ac}(\rho) \leq \mathrm{c}(\rho)$ (cf. Proposition 1.7.14) and, moreover, that $\mathrm{ac}(\rho) = 0$ if and only if $\mathrm{c}(\rho) = 0$, i.e. if and only if ρ is polar (cf Proposition 2.2.1). Polar representations are extremely wild in terms of cohomogeneity: indeed, for any $n \in \mathbb{N}$, $n \geq 0$, there exists a polar representation of cohomogeneity n (cf. Example 2.2.3). If we allow the copolarity to be greater than zero, the situation seems to become much more rigid: for instance, C. Gorodski and A. Lytchak have proved in [19] the following result:

Theorem 3.0.14. *Let (H, W) be an irreducible non-reduced representation of a compact, connected Lie group with copolarity k, $1 \leq k \leq 6$. Then (H, W) is of cohomogeneity $k + 2$.*

The case $k = 1$ was already known, and is due to Claudio Gorodski, Carlos Olmos and Ruy Tojeiro (cf. [22]).

However we are still far from having a general picture of the behaviour of copolarity and cohomogeneity even for an irreducible representation. Indeed, in [19] it is also shown that $(\mathbf{U}(3) \times \mathbf{Sp}(2), \mathbb{C}^3 \otimes_{\mathbb{C}} \mathbb{C}^4)$, which is irreducible, has copolarity 7 and cohomogeneity 5.

Actually, it seems that the reason why Theorem 3.0.14 holds might be linked to the fact that the group G of a minimal reduction (G, V) of a representation (H, W) satisfying the assumptions of Theorem 3.0.14 is a finite extension of a torus, i.e. G° is a torus. In fact, it is shown in [19] that $\mathrm{c}(H, W) = \mathrm{ac}(H, W) = \mathrm{chm}(H, W) - 2$ for any irreducible representation (H, W), with H connected, admitting a minimal reduction (G, V) where G° is a torus.

Representations satisfying the latter property are said to *admit a toric reduction* and they are the main object of our study in this thesis.

This Chapter is organized as follows. In Section 3.1 we discuss some general features of representations admitting a toric reduction; in Section 3.2 we shall see how to generalize Theorem 3.0.14 to the reducible case when the copolarity is either 1 or 2 (cf. [37]). Namely, we shall consider a reducible, non-reduced indecomposable representation (H, W) with H connected, and we shall show that if it has abstract copolarity 1, then its cohomogeneity is 3. If instead $\mathrm{ac}(H, W) = 2$, we shall see that $\mathrm{chm}(H, W)$ is not necessarily 4; moreover all counterexamples will be completely described. We feel the need to remark here that the proof of these statements heavily relies on the fact that, if $\mathrm{ac}(H, W) = 1, 2$, the identity component of the group G of any minimal reduction (G, V) of (H, W) must be a torus. As a Corollary, we shall see that representations of copolarity 1 are exactly those of abstract copolarity 1.

Irreducible non-polar representations (H, W), with H connected, admitting a toric reduction have been completely described by Claudio Gorodski and Alexander Lytchak in [20]; in Section 3.3, we shall extend such classification to the case of a reducible (H, W) with H simple.

3.1 Representations admitting a toric reduction

As mentioned above, the study object of this chapter consists of representations admitting a toric reduction. We begin with their formal definition:

Definition 3.1.1. A representation (H, W) of a compact Lie group H is said to *admit a toric reduction* if there exists a representation (H', W') quotient-equivalent to (H, W) such that the identity component $(H')^\circ$ of H' is a torus (possibly of dimension 0). In this case (H', W') is called a *toric reduction* of (H, W).

Remark 3.1.2. By Corollary 2.5.13, if (H, W) admits a toric reduction, then it admits a *minimal* reduction (G, V) such that G° is a torus.

Representations admitting toric reductions are obviously a generalization of polar representations. Now, subrepresentations and slice representations of a polar representation are polar (cf. [12]); the following Lemma shows that in fact analogous results are true for representations admitting a toric reduction.

Lemma 3.1.3. *Assume that (H, W) admits a toric reduction.*

1. *If $W' \subseteq W$ is an H-invariant subspace, then the induced representation (H, W') admits a toric reduction;*

2. *If $w \in W$, then the slice representation at w admits a toric reduction.*

Proof. Let (G, V) be a (minimal) reduction of (H, W) so that G° is a torus.

(1) By Proposition 2.2.11, there exists a G-invariant subspace V' of V such that V'/G is isometric to W'/H. Thus (G, V') is toric reduction of (H, W').

(2) Set $X := W/H$, $Y := V/G$, and let $I : X \to Y$ be an isometry. Let p be the projection of w onto X, and $v \in V$ a point projecting onto $I(p)$. The orbit spaces of the slice representations at w and v are respectively isometric to the tangent cones $C_p(X)$, $C_{I(p)}(Y)$. Since I induces an isometry between these cones, we deduce that such slice representations are quotient-equivalent. The assertion follows observing that the isotropy at v is a finite extension of a torus. □

We now fix an indecomposable, non-reduced representation (H, W) of a connected group H, and consider a *minimal* reducton (G, V) of (H, W) such that G° is a torus $\mathbb{T}$. We set $k := \dim \mathbb{T}$, and suppose $k \geq 1$ (i.e. that neither (G, V) nor (H, W) is polar). Moreover, we shall denote by $\mathfrak{t}$ the Lie algebra of $\mathbb{T}$. The remaining part of this section is dedicated to the study of the induced representation $(\mathbb{T}, V)$, which will play a central role later on. Note first that (G, V) is faithful and has trivial copolarity; in particular, it also has trivial principal isotropy groups. Next Lemma contains some simple remarks:

Lemma 3.1.4. *Under the above assumptions, the group G is disconnected, and contains a set of nice involutions whose projection in $G/\mathbb{T}$ is a set of generators. Moreover the induced representation $(\mathbb{T}, V)$ of the identity component of G is indecomposable and reduced, and its orbit space $V/\mathbb{T}$ has empty boundary.*

Proof. First we observe that G cannot be connected by Proposition 2.5.12 and Corollary 2.3.18. Since G has trivial principal isotropy groups and $G/\mathbb{T}$ acts as a reflection group on $V/\mathbb{T}$ (cf. Theorem 2.3.11), $G/\mathbb{T}$ is generated by a set Ω all of whose elements can be lifted to a nice involution in G (cf. Theorem 2.3.13). Since (G, V) is non-polar and indecomposable, Proposition 2.4.11 implies moreover that the induced representation $(\mathbb{T}, V)$ is indecomposable as well. We now claim that $V/\mathbb{T}$ has no boundary, so that $(\mathbb{T}, V)$ is reduced (cf. Corollary 2.3.17). Indeed, if $\partial(V/\mathbb{T}) \neq \varnothing$, then Proposition 2.5.12 would yield a proper generalized section for $(\mathbb{T}, V)$, and hence a proper generalized section for (G, V) (cf. Lemma 1.5.8), contradiction. □

We now decompose the space V into its isotypical components with respect to the induced representation $(\mathbb{T}, V)$:

$$V = V_1 \oplus \cdots \oplus V_m. \tag{3.1}$$

Our goal is to understand how nice involutions act on these components. First we need a Lemma with some elementary, nevertheless important, facts:

Lemma 3.1.5. *With the notation introduced above, the following holds:*

1. *the induced action of* $\mathbb{T}$ *on each* V_i *in (3.1) is not trivial;*

2. $\dim V$ *is even and* $\geq 2k+2$.

Proof. (1) Indeed, otherwise $(\mathbb{T}, V)$ would be decomposable, and this is impossible by Lemma 3.1.4.

(2) We have that $\dim V$ is even by (1). If $k = 1$, $\dim V$ cannot be equal to 2, otherwise $(\mathbb{T}, V)$, and hence also (G, V), would be polar, a contradiction; thus $\dim V \geq 4$. If $k \geq 2$, let ℓ be the number of lines induced by $(\mathbb{T}, V)$. Clearly $m \geq \ell$, and $\ell \geq k+1$ by Lemma 2.5.11 (since $(\mathbb{T}, V)$ is indecomposable). Thus $m \geq k+1$, and $\dim V \geq 2m \geq 2k+2$. □

Consider now a nice involution $\omega \in G$, and denote by $Z_G(\omega)$, V^ω respectively the subgroup of G consisting of the elements commuting with ω and the fixed point space of ω in V. Since we have $0 \leq \dim Z_G(\omega) \leq k$, formula (2.5) yields $1 \leq \operatorname{codim}_V V^\omega \leq k+1$.

Lemma 3.1.6. *If* $\omega \in G$ *is a nice involution, then* $\operatorname{codim}_V V^\omega \neq 1$ *or, equivalently,* $\dim Z_G(\omega) \neq k$.

Proof. Assume that $\omega \in G$ is a nice involution such that $\dim Z_G(\omega) = k$; then the identity component of G, $\mathbb{T}$, is contained in $Z_G(\omega)$, and ω preserves all $\mathbb{T}$-invariant subspaces of V. Since ω satisfies also $\operatorname{codim}_V V^\omega = 1$, it is a reflection in V, therefore we may conclude that there exists a $\mathbb{T}$-irreducible subspace U of V such that $\omega(U) = U$, and $\omega|_U$ is a reflection. If $\dim U = 2$, then $(\mathbb{T}, U)$ is equivalent to an irreducible S^1-representation, which is given by rotations in $U \simeq \mathbb{R}^2$; this is impossible since $Z_G(\omega) \supseteq \mathbb{T}$. Hence $\dim U = 1$, and $\mathbb{T}$ fixes U pointwisely. This contradicts Lemma 3.1.5(1). □

We immediately deduce the following:

Corollary 3.1.7. *Let* $\omega \in G$ *be a nice involution. Then* $0 \leq \dim Z_G(\omega) \leq k-1$ *or, equivalently,* $2 \leq \operatorname{codim}_V V^\omega \leq k+1$.

In order to study the action of a nice involution ω on the decomposition (3.1), we distinguish the cases $\operatorname{codim}_V V^\omega = k+1$, $\operatorname{codim}_V V^\omega \in \{2, \ldots, k\}$ or, equivalently, $\dim Z_G(\omega) = 0$, $\dim Z_G(\omega) \in \{1, \ldots, k-1\}$.

The case $\operatorname{codim}_V V^\omega = k+1$

Note that the Lie algebra of $Z_G(\omega)$ coincides with the fixed point space of $\operatorname{Ad}(\omega)$ on $\mathfrak{t}$; so in this case $\operatorname{Ad}(\omega) : \mathfrak{t} \to \mathfrak{t}$ is an involution with no non-trivial fixed points, and conjugation c_ω in $\mathbb{T}$ with respect to ω coincides with the inversion map $t \mapsto t^{-1}$. If $\phi : \mathbb{T} \to S^1$ is a homomorphism, we have then $\phi \circ c_\omega = \bar{\phi}$, therefore ω preserves all $\mathbb{T}$-isotypical components of V (i.e. ω preserves decomposition (3.1)).

Lemma 3.1.8. *If $\omega \in G$ is a nice involution such that* $\mathrm{codim}_V V^\omega = k+1$, *and* $U \subseteq V$ *is a non-trivial* $\mathbb{T}$*-invariant subspace of* V *which is preserved by* ω, *then the action of* ω *on* U *is not trivial.*

Proof. If ω acts trivially on U we have

$$\bar{t} \cdot u = c_\omega(t) \cdot u = t \cdot u, \qquad \text{i.e.} \qquad t^2 \cdot u = u, \qquad \forall\, t \in \mathbb{T},\ \forall\, u \in U.$$

Since the map $\mathbb{T} \to \mathbb{T}$ given by $t \mapsto t^2$ is surjective, we deduce that $\mathbb{T}$ acts trivially on U, therefore $U = \{O\}$ by Lemma 3.1.5. □

The desired result is the following:

Proposition 3.1.9. *Suppose that G contains a nice involution ω which satisfies* $\mathrm{codim}_V V^\omega = k+1$.

1. *If $k = 1$, then* $\dim V = 4$.
2. *If $k \geq 2$, then* $\dim V = 2k+2$ *and all the* $\mathbb{T}$*-isotypical components of* V *are irreducible; moreover,* ω *acts as a reflection on each of them.*

In any case, $\mathrm{chm}(G,V) = k+2$.

Proof. As in (3.1), let V_i, $i = 1, \dots, m$, be the $\mathbb{T}$-isotypical components of V, and let ω_i be the restriction of ω to V_i. Since

$$\sum \mathrm{codim}_{V_i} V_i^{\omega_i} = \mathrm{codim}_V V^\omega = k+1,$$

Lemma 3.1.8 implies $m \leq k+1$.

If $k \geq 2$, then Lemma 2.5.11 implies $m = k+1$. Thus, using again Lemma 3.1.8, each V_i has dimension 2 and $\dim V = 2k+2$. Note that in this case we have $\mathrm{codim}_{V_i} V_i^{\omega_1} = 1$ for all $i = 1, \dots, k+1$, thus ω acts as a reflection on each V_i.

If $k = 1$, then either $m = 2$ and the same argument as above yields $\dim V = 4$, or $m = 1$. In the latter case, Lemma 3.1.8 implies that V has at most two $\mathbb{T}$-irreducible components. Since V cannot be $\mathbb{T}$-irreducible, otherwise $(\mathbb{T}, V)$ would be polar, this means $\dim V = 4$, as claimed. □

The case $\mathrm{codim}_V V^\omega \in \{2, \dots, k\}$

In this case $\mathrm{Ad}(\omega) : \mathfrak{t} \to \mathfrak{t}$ is an involution, so it is diagonalizable with eigenvalues ± 1. Denote by $U_\pm \subseteq \mathfrak{t}$ the eigenspace corresponding to the eigenvalue ± 1. Clearly U_+ is equal to the Lie algebra of $Z_G(\omega)$, so $\dim U_+ = \dim Z_G(\omega) \in \{1, \dots, k-1\}$; in particular $\mathrm{Ad}(\omega) \neq \pm \mathrm{id}_{\mathfrak{t}}$. Call now $V_\pm$ the sum of all isotypical components of $(\mathbb{T}, V)$ whose weight θ belongs to $\mathrm{ann}(U_\pm)$, i.e. satifies $\theta(x) = O$, for all $x \in U_\pm$. Then

$$V = V_+ \oplus V_- \oplus \bar{V},$$

where $\bar{V} := (V_+ \oplus V_-)^\perp$.

Lemma 3.1.10. *We have $\bar{V} \neq \{O\}$.*

Proof. Assume by contradiction $\bar{V} = \{O\}$, and let $\Theta_\pm$ the set of all weights induced by $\mathbb{T}$-isotypical components contained in $V_\pm$. Then

$$\langle \Theta_+ \rangle \cap \langle \Theta_- \rangle \subseteq \operatorname{ann}(U_+) \cap \operatorname{ann}(U_-) = \{O\},$$

and of course $\Theta_+ \cup \Theta_-$ is the set of all weights induced by $(\mathbb{T}, V)$. We show now that $\Theta_\pm \neq \varnothing$. Assume for instance $\Theta_- = \varnothing$, i.e. $\Theta = \Theta_+$. Then U_+ is annihilated by all weights of $(\mathbb{T}, V)$, i.e. U_+ is contained in the Lie algebra of $\ker(\mathbb{T}^k, V)$. But the latter is $\{O\}$, so also $U_+ = \{O\}$, contradiction. This means that $\Theta_- \neq \varnothing$, and similarly one proves that $\Theta_+ \neq \varnothing$. Thus, by Proposition 2.5.8, $(\mathbb{T}, V)$ is decomposable, and this is impossible by Lemma 3.1.4. □

Thanks to Lemma 3.1.10 we may consider a $\mathbb{T}$-isotypical component $V_1 \subseteq \bar{V}$. Then also $V_2 := \omega(V_1)$ is a $\mathbb{T}$-isotypical component contained in $\bar{V}$. Denote by $\mathfrak{s}_i$, $i = 1, 2$, the line in $\mathfrak{t}^*$ induced by V_i. Clearly $\operatorname{Ad}(\omega)$, via its natural action on $\mathfrak{t}^*$, cannot fix $\mathfrak{s}_1$, so $V_1 \neq V_2$ and $V_1 \cap V_2 = \{O\}$.

Applying Lemma 2.3.14 we deduce then that $(\mathbb{T}, V_i)$, $i = 1, 2$, has cohomogeneity 1. In particular, since $(\mathbb{T}, V_i)$ can be effectivized to a representation (S^1, V_i), we get $\dim V_i = 2$, so V_1 and V_2 are $\mathbb{T}$-irreducible.

We next show that $\bar{V} = V_1 \oplus V_2$. If this were not the case, we could find two irreducible $\mathbb{T}$-isotypical components V_3, $V_4 \subseteq \bar{V}$ which are interchanged by ω and so that $V_3 \oplus V_4$ has trivial intersection with $V_1 \oplus V_2$. Exploiting again Lemma 2.3.14, we deduce that $(\mathbb{T}, V_1 \oplus V_3)$, $(\mathbb{T}, V_2 \oplus V_4)$ are of cohomogeneity 1, and this is impossible. Such a contradiction proves that $\bar{V} = V_1 \oplus V_2$. In particular, ω preserves all $\mathbb{T}$-isotypical components of V, except V_1 and V_2 which are interchanged.

We now set $j := \operatorname{codim}_V V^\omega \in \{2, \ldots, k\}$, and denote by $\omega_\pm$, $\bar{\omega}$ the restriction of ω to $V_\pm$, $\bar{V}$ respectively. Since $\operatorname{codim}_{\bar{V}} \bar{V}^{\bar{\omega}} = 2$, we deduce

$$\operatorname{codim}_{V_+} V_+^{\omega_+} + \operatorname{codim}_{V_-} V_-^{\omega_-} = j - 2. \tag{3.2}$$

We need to study the action of ω on $V_\pm$.

Lemma 3.1.11. *If $v \in V_\pm$, then for any $t \in \mathbb{T}$ we have*

$$c_\omega(t) \cdot v = t^{\mp 1} \cdot v$$

Proof. We may suppose that v belongs to a $\mathbb{T}$-isotypical component with character ϕ and weight $\theta = \mathrm{d}\phi_e$. Then, if $x \in \mathfrak{t}$ satisfies $t = \exp(x)$, and if $x = x_+ + x_-$, where $x_\pm \in U_\pm$,

$$c_\omega(t) \cdot v = \exp(\theta(\operatorname{Ad}(\omega)x))v = \exp(\mp\theta(x_\mp))v = \phi(\exp(\mp x_\mp))v.$$

Similarly

$$t^{\mp 1} \cdot v = \exp(\mp\theta(x))v = \phi(\exp(\mp x_{\mp}))v.$$

□

Fix now a $\mathbb{T}$-irreducible subspace $W \subseteq V_+$. By Lemma 3.1.11 $\omega(W)$ is a $\mathbb{T}$-irreducible subspace of V_+. Set $V_+^1 := W$, $V_+^2 := \omega(W)$, and assume $V_+^1 \neq V_+^2$. By Lemma 2.3.14, $\mathbb{T}$ acts on $V_i \oplus V_i^+$, $i = 1, 2$, with cohomogeneity 1, and this is absurd. Then $\omega(W) = W$ for any $\mathbb{T}$-irreducible subspace $W \subseteq V_+$. Consider a decomposition of V_+ into $\mathbb{T}$-irreducible summands:

$$V_+ = V_+^1 \oplus \cdots \oplus V_+^\ell. \tag{3.3}$$

Then any V_+^i has dimension 2 and is preserved by ω. Suppose that ω acts on V_+^i as $\pm\mathrm{id}_{V_+^i}$. Then, using Lemma 3.1.11, for $t \in \mathbb{T}$, $v \in V_+^i$ we have:

$$\pm t \cdot v = \omega \cdot (t \cdot v) = (\omega t \omega) \cdot (\omega \cdot v) = \pm(\omega t \omega) \cdot v = \pm t^{-1} \cdot v,$$

and this yields $t^2 \cdot v = v$ for all $t \in \mathbb{T}$, $v \in V_+^i$, i.e. $\mathbb{T}$ acts trivially on V_i^+, contradiction (cf. Lemma 3.1.5). Since $\dim V_+^i = 2$ this implies that ω acts as a reflection on it.

Let ω_+^i, $i = 1, \ldots, \ell$, be the restriction of ω to V_+^i. By formula (3.2) we obtain:

$$\ell = \sum_{i=1}^{\ell} \mathrm{codim}_{V_+^i} (V_+^i)^{\omega_+^i} = \mathrm{codim}_{V_+} V_+^{\omega_+} = j - 2 - \mathrm{codim}_{V_-} V_-^{\omega_-} \leq j - 2. \tag{3.4}$$

If α is the number of lines induced in $\mathfrak{t}^*$ by V_+, we get then

$$\alpha \leq \ell \leq j - 2.$$

Suppose by contradiction that $\alpha \leq j - 3$, and introduce the following notation:

- $\mathfrak{s}_1$, $\mathfrak{s}_2$ are the lines in $\mathfrak{t}^*$ induced by V_1, V_2 respectively;
- $\mathfrak{r}_1^+, \ldots, \mathfrak{r}_\alpha^+$ are the lines in $\mathfrak{t}^*$ induced by V_+;
- $\mathfrak{r}_1^-, \ldots, \mathfrak{r}_\beta^-$ are the lines in $\mathfrak{t}^*$ induced by V_-.

Define Θ_+ to be the set of all weights θ induced by $(\mathbb{T}, V)$ such that $\langle\theta\rangle \in \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{r}_1^+, \ldots, \mathfrak{r}_\alpha^+\}$, and define Θ_- to be the set of all weights induced by $(\mathbb{T}, V)$ such that $\langle\theta\rangle \in \{\mathfrak{r}_1^-, \ldots, \mathfrak{r}_\beta^-\}$. Clearly $\Theta := \Theta_+ \cup \Theta_-$ is the set of all weights induced by $(\mathbb{T}, V)$, and $\Theta_+ \neq \varnothing$. Moreover $\Theta_- \neq \varnothing$ otherwise $V = V_+ \oplus V_1 \oplus V_2$ and, by (3.4),

$$\dim V = 2\ell + 4 \leq 2j \leq 2k,$$

contradicting Lemma 3.1.5. Since $(\mathbb{T}, V)$ is faithful, $\langle\Theta_+\rangle + \langle\Theta_-\rangle = \langle\Theta\rangle = \mathfrak{t}^*$. Moreover $|\Theta_+| = \alpha+2 \leq j-1$, so $\dim\langle\Theta_+\rangle \leq j-1$, and $\dim\langle\Theta_-\rangle \leq \dim \operatorname{ann}(U_-) = \dim U_+ = \dim Z_G(\omega) = k-j+1$ (cf. (2.5)). Hence

$$\begin{aligned}\dim(\langle\Theta_+\rangle \cap \langle\Theta_-\rangle) &= \dim\langle\Theta_+\rangle + \dim\langle\Theta_-\rangle - \dim(\langle\Theta_+\rangle + \langle\Theta_-\rangle) \\ &\leq (j-1) + (k-j+1) - k = 0,\end{aligned}$$

i.e. $\langle\Theta_+\rangle \cap \langle\Theta_-\rangle = \{O\}$. Lemma 2.5.8 implies then that $(\mathbb{T}, V)$ is decomposable, in contradiction with Lemma 3.1.4. Therefore $\alpha = \ell = \operatorname{codim}_{V_+} V_+^{\omega_+} = j-2$, and

$$\operatorname{codim}_{V_-} V_-^{\omega_-} = j - 2 - \operatorname{codim}_{V_+} V_+^{\omega_+} = 0.$$

In particular ω acts as the identity on V_-, and (3.3) is the decomposition of V_+ into $\mathbb{T}$-isotypical components (which in particular are irreducible). Summarizing, we have proved the following:

Proposition 3.1.12. *Let $\omega \in G$ a nice involution such that* $\operatorname{codim}_V V^\omega = j \in \{2, 3, \ldots, k\}$. *Then V decomposes as*

$$V = V_- \oplus V_+^1 \oplus \cdots \oplus V_+^{j-2} \oplus V_1 \oplus V_2$$

where:

1. *the V_+^i's, V_1 and V_2 are irreducible isotypical components of* $(\mathbb{T}, V)$*;*
2. *ω preserves V_- and acts trivially on it;*
3. *ω preserves V_+^i and is a reflection on it, for all $i = 1, \ldots, j-2$;*
4. *ω interchanges V_1, V_2.*

We finish this section proving a consequence of Propositions 3.1.9, 3.1.12.

Proposition 3.1.13. *Let (H, W) be a non-reduced indecomposable representation of a compact, connected Lie group H admitting a toric reduction, and let (G, V) be a minimal reduction of (H, W) such that $G^\circ = \mathbb{T}$ is a torus of dimension $k \geq 2$. Assume that there is an isotypical component $U \subseteq V$ of the induced representation $(\mathbb{T}, V)$ which is not $\mathbb{T}$-irreducible. Then there exists an isotypical component U' of (H, W) such that $U'/H \simeq U/\mathbb{T}$ and the induced representation (H, U') is equivalent to an S^1-representation.*

In particular, all the isotypical components of $(\mathbb{T}, V)$ are irreducible in the following cases:

1. *if the semisimple part of H has no non-trival fixed points on W;*

2. if H is semisimple.

Proof. Under our assumptions, all nice involutions $\omega \in G$ must satisfy the condition $2 \leq \mathrm{codim}_V V^\omega \leq k$ by Proposition 3.1.9. For any such ω, we consider the corresponding decomposition

$$V = V_-^\omega \oplus V_+^{1,\omega} \oplus \cdots \oplus V_+^{j-2,\omega} \oplus V_1^\omega \oplus V_2^\omega$$

as in Proposition 3.1.12, and set $V_\cap := \bigcap_\omega V_-^\omega$.

Assume by contradiction $U \not\subseteq V_\cap$. Then $U \not\subseteq V_-^\sigma$ for some nice involution $\sigma \in G$. Exploiting Proposition 3.1.12 we deduce then $U \subseteq (V_-^\sigma)^\perp$ and $\dim U = 2$, contradicting the hypothesis.

Then $U \subseteq V_\cap$ and all nice involutions act trivially on U, so U is G-invariant, and also an isotypical component of (G, V). Note then that the induced representation (G, U) is equivalent to (S^1, U). Let U' be an H-invariant subspace of W so that U'/H is isometric to U/G (cf. Proposition 2.2.11). Clearly U'/H is isometric also to $U/\mathbb{T} = U/S^1$. Since $\dim U > 2$, (S^1, U) is not polar, therefore $\partial(U'/H) = \partial(U/S^1) = \varnothing$ (cf. Proposition 2.5.12). Now, set $K := \ker(H, U')$. The faithful representation $(H/K, U')$ is equivalent to (H, U') and we have $\frac{H}{K} \simeq S^1$ by Corollary 2.3.18. □

3.2 Copolarity and cohomogeneity

The goal of ths Section is to generalize Theorem 3.0.14 to the case of reducible representations of copolarity 1 and 2. Since cohomogeneity is a property that depends only on the quotient-equivalence class of a representation, it is more convenient to deal with abstract copolarity instead of copolarity; however we shall see that the theorems obtained in this case imply analogous ones for copolarity (cf. Remark 3.2.13). All results in this Section have been obtained in collaboration with Carolin Pomrehn, and can be found in [37].

More precisely, we are interested in understanding whether a reducible representation (H, W) of a connected, compact Lie group H such that $\mathrm{ac}(H, W) = k$, $k = 1, 2$, is of cohomogeneity $k + 2$. We notice immediately that there are trivial counterexamples to this statement. Indeed, for any $n \in \mathbb{N}$, there exists a polar representation (H', W') of a connected, compact Lie group H' with cohomogeneity n; so the product

$$(H \times H', W \oplus W')$$

is a representation of a connected, compact Lie group with the same abstract copolarity as (H, W) and cohomogeneity $\geq n$. In this way, we have a family of representations of connected, compact Lie groups with constant abstract copolarity, but arbitrarily large cohomogeneity. This means that the problem we are studying is interesting only for representations which do not split as a product of subrepresentations, i.e. the indecomposable ones. In the case of abstract copolarity one we see that the counterexamples described above are the only ones that can occur:

Theorem 3.2.1 ([37]). *Let $\rho = (H, W)$ be a non-reduced, indecomposable representation of a connected, compact Lie group H of abstract copolarity 1. Then ρ has cohomogeneity 3.*

Proof. Let (G, V) be a minimal reduction of ρ; then G is a finite extension of the 1-dimensional torus $\mathbb{T}^1$, and by Lemma 3.1.4 it contains a nice involution ω. Applying Lemma 3.1.7 we deduce $\operatorname{codim}_V V^\omega = 2$, so $\operatorname{chm}(\rho) = \operatorname{chm}(G, V) = 3$ by Proposition 3.1.9. □

We easily deduce the following:

Corollary 3.2.2. *Let $\rho = (H, W)$ be a non-reduced, indecomposable representation of a connected, compact Lie group H. The following statements are equivalent:*

1. *ρ has copolarity 1;*
2. *ρ has abstract copolarity 1;*
3. *ρ is non-polar and has cohomogeneity 3;*
4. *ρ is orbit-equivalent to one of the following representations (H', W'):*

H'	W'	*Conditions*
$\mathbf{SO}(n)$	$\mathbb{R}^n \oplus \mathbb{R}^n$	$n \geq 3$
$\mathbf{U}(2)$	$\mathbb{R}^3 \oplus \mathbb{C}^2$	–
$\mathbf{Sp}(1) \times \mathbf{Sp}(2)$	$\mathbb{R}^5 \oplus \mathbb{C}^2 \otimes_{\mathbb{H}} \mathbb{C}^4$	–
$\mathbf{Spin}(9)$	$\mathbb{R}^{16} \oplus \mathbb{R}^9$	–
$\mathbf{U}(1) \times \mathbf{SU}(n) \times \mathbf{U}(1)$	$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}^n \oplus \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}$	$n \geq 2$
$\mathbf{Sp}(1) \times \mathbf{Sp}(n) \times \mathbf{Sp}(1)$	$\mathbb{C}^2 \otimes_{\mathbb{H}} \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \otimes_{\mathbb{H}} \mathbb{C}^2$	$n \geq 2$
$\mathbf{SO}(2) \times \mathbf{Spin}(9)$	$\mathbb{R}^2 \otimes \mathbb{R}^{16}$	–
$\mathbf{U}(2) \times \mathbf{Sp}(n)$	$\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^n$	$n \geq 2$
$\mathbf{Sp}(1) \times \mathbf{Sp}(n)$	$S^3\mathbb{C}^2 \otimes_{\mathbb{H}} \mathbb{C}^{2n}$	$n \geq 2$

Proof. (1) ⇒ (2). Since a representation has abstract copolarity 0 if and only if it is polar (cf. Proposition 2.2.1), it follows from Proposition 1.7.14 that if ρ has copolarity 1, then it has abstract copolarity 1.

(2) ⇒ (3). It follows from Theorem 3.2.1.

(3) ⇒ (4) ⇒ (1). They follow from Straume's classification of indecomposable representation of cohomogeneity 3 (cf. [40]). □

Remark 3.2.3. The list up to *equivalence* of non-reduced, indecomposable representations (H, W), with H compact and connected, satisfying the conditions of Corollary 3.2.2 can be found in [41].

We can be a little more precise:

Corollary 3.2.4. *A representation $\rho = (H, W)$ of a connected, compact Lie group H has abstract copolarity 1 if and only if it has copolarity 1.*

Proof. The same argument as in the proof of Corollary 3.2.2 shows that, if ρ has copolarity 1, then it has abstract copolarity 1.

Conversely, assume that ρ has abstract copolarity 1. If it is indecomposable, then either it is reduced and there is nothing to prove, or it is not reduced and the assertion follows by Corollary 3.2.2.

Assume next that ρ is decomposable. Exploiting Corollary 2.4.7 we may suppose that ρ is orbit-equivalent to a product representation $(H_1 \times H_2, W_1 \oplus W_2)$, where (H_1, W_1) is polar and (H_2, W_2) is indecomposable with abstract copolarity 1. By the above discussion (H_2, W_2) has copolarity 1, so the same is true for ρ since clearly the copolarity of a product representation is the sum of the copolarities of the factors. ☐

The case of a representation of abstract copolarity 2 is slightly more complicated. Our goal is to prove the following:

Theorem 3.2.5 ([37])**.** *Let $\rho = (H, W)$ be a non-reduced, indecomposable representation of a compact, connected Lie group H of abstract copolarity 2. Then either:*

1. *ρ has cohomogeneity 4, or*

2. *$\rho = \rho_1 \oplus \rho_2$, where*

   (a) *$\rho_1 = (H, W_1)$ is orbit-equivalent to the isotropy representation of a rank 2 real grassmannian,*

   (b) *$\rho_2 = (H, W_1^{\perp})$ is orbit-equivalent to a non-polar $\mathbf{U}(1)$-representation without non-trivial fixed points.*

Conversely, let ρ_1 be the isotropy representation of a rank 2 real grassmannian, ρ_2 be a non-polar $\mathbf{U}(1)$-representation without non-trivial fixed points and set $\rho := \rho_1 \oplus \rho_2$. Then ρ is indecomposable, has cohomogeneity $\neq 4$, and both its copolarity and abstract copolarity are equal to 2.

We now begin the work that will lead us to the proof of Theorem 3.2.5; in particular, from now on $\rho = (H, W)$ will denote a non-reduced, indecomposable representation of a connected, compact Lie group H of abstract copolarity 2, and (G, V) a minimal reduction of ρ. Clearly G is a finite extension of the 2-dimensional torus $\mathbb{T}^2$, and contains a nice involution ω thanks to Lemma 3.1.4. By Corollary 3.1.7 we know moreover that $\mathrm{codim}_V V^\omega \in \{2, 3\}$. If $\mathrm{codim}_V V^\omega = 3$, Proposition 3.1.9 implies $\mathrm{chm}(\rho) = \mathrm{chm}(G, V) = 4$ and we are done; so we may assume that all nice involutions $\omega \in G$ satisfy $\mathrm{codim}_V V^\omega = 2$.

Let $\omega_1 \in G$ be a fixed nice involution. By Proposition 3.1.12 we can decompose V as

$$V = V_- \oplus V_1 \oplus V_2.$$

Here V_1, V_2 are irreducible $\mathbb{T}^2$-isotypical components of V inducing two distinct lines $\mathfrak{s}_1$, $\mathfrak{s}_2$ in $(\mathfrak{t}^2)^*$, where $\mathfrak{t}^2$ denotes the Lie algebra of $\mathbb{T}^2$. Moreover, ω_1 acts as the identity on V_-, and interchanges V_1, V_2. Clearly in this case the eigenspaces $U_\pm$ of $\mathrm{Ad}(\omega) : \mathfrak{t}^2 \to \mathfrak{t}^2$ corresponding to the eigenvalues ± 1 are both 1-dimensional. So V_- induces exactly one line $\mathfrak{s}^-$ in $(\mathfrak{t}^2)^*$.

Now, any other nice involution $\omega_2 \in G$ permutes the $\mathbb{T}^2$-isotypical components of V, so, via its natural action on $(\mathfrak{t}^2)^*$, it permutes the lines $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}^-$. If ω_2 does not fix $\mathfrak{s}^-$, then $\dim V_- = 2$, $\dim V = 6$ and $\mathrm{chm}(\rho) = \mathrm{chm}(G, V) = 4$, so Theorem 3.2.5 holds. Note that in this case all $\mathbb{T}^2$-isotypical components of V are irreducible and G acts as the full permutation group on them, therefore (G, V), as well as the original representation (H, W), is irreducible.

In particular, we may suppose that all nice involutions in G fix $\mathfrak{s}^-$ and interchange $\mathfrak{s}_1$, $\mathfrak{s}_2$. It is not hard to prove that in this case all nice involutions project onto the same element in G/G°, thus $G/G^\circ = \mathbb{Z}/2\mathbb{Z}$.

Lemma 3.1.5 implies now that $\dim V_- \geq 2$; if $\dim V_- = 2$, then $\dim V = 6$ and $\mathrm{chm}(\rho) = \mathrm{chm}(G, V) = 4$, so we shall assume $\dim V_- > 2$, i.e. $\dim V_- \geq 4$, and show that we are in case (2) of Theorem 3.2.5.

Notice that $V_1 \oplus V_2$ and V_- are G-invariant subspaces of V; by Proposition 2.2.11 we can then find two H-invariant orthogonal complementary subspaces W_1, $W_2 \subseteq W$ such that

$$W_1/H \simeq (V_1 \oplus V_2)/G, \qquad W_2/H \simeq V_-/G.$$

Let ρ_i be the induced representation (H, W_i), $i = 1, 2$; then $\rho = \rho_1 \oplus \rho_2$ by construction.

First observe that the same argument used in the proof of Proposition 3.1.13 implies that (H, W_2) is equivalent to an S^1-representation. Moreover (H, W_2) is not polar because $(G, V_-) \simeq (\mathbb{T}, V_-)$ isn't (indeed, $\dim V_- \geq 4$), and has no non-trivial fixed points (cf. Lemma 3.1.5 and Proposition 2.2.6).

We now study $(G, V_1 \oplus V_2)$. First notice that, by Corollary 2.5.11,

$$\frac{V_1 \oplus V_2}{\mathbb{T}^2} \simeq \frac{V_1}{S^1} \times \frac{V_2}{S^1},$$

so $(\mathbb{T}^2, V_1 \oplus V_2)$ is polar and of cohomogeneity 2. The same is true for $(G, V_1 \oplus V_2)$ thanks to Lemma 1.5.9, Remark 1.4.12, and obviously also for ρ_1. In particular the latter is orbit-equivalent to the isotropy representation of a rank 2 symmetric space (cf. the discussion following Remark 2.2.3). The proof that this is in fact a real grassmannian will require a few remarks and lemmas.

We begin with the observation that $\rho_2(H) = S^1$, hence H cannot be semisimple; we shall write then $H = Z^\circ \cdot H_{\mathrm{s}}$, where Z°, the identity component of the centre $Z(H)$ of H, is a torus $\mathbb{T}^a$, $a \geq 1$, and H_{s} is the semisimple part of H. The same argument implies moreover that the induced representation (H_{s}, W_2) is trivial.

Lemma 3.2.6. *We have* $Z^\circ \not\subseteq \ker \rho_1$

Proof. Suppose the action of Z° on W_1 is trivial; we shall show that (H, W) is orbit-equivalent to the product representation $(H_{\mathrm{s}} \times Z^\circ, W_1 \oplus W_2)$, a contradiction (since (H, W) is indecomposable).

Fix $w := (w', w'') \in W_1 \oplus W_2$, pick $h \in H$ and write $h = h_{\mathrm{s}} z$ for some $h_{\mathrm{s}} \in H_{\mathrm{s}}$, $z \in Z^\circ$; then

$$(h_{\mathrm{s}}, z) \cdot w = (\mathrm{s} \cdot w', z \cdot w'') = h \cdot w,$$

where we have used the fact that H_{s} acts trivially on W_2, and our hypothesis $Z^\circ \subseteq \ker \rho_1$. This shows that $H \cdot w \subseteq (H_{\mathrm{s}} \times Z^\circ) \cdot w$. Conversely, let $(h_{\mathrm{s}}, z) \in H_{\mathrm{s}} \times Z^\circ$, and set $h := h_{\mathrm{s}} z$. We have, for the same reasons as above,

$$h \cdot w = (h_{\mathrm{s}} z \cdot w', h_{\mathrm{s}} z \cdot w'') = (h_{\mathrm{s}} \cdot w', z \cdot w'') = (h_{\mathrm{s}}, z) \cdot w,$$

and we are done. □

Lemma 3.2.7. *$\rho_1(H)$ has 1-dimensional centre.*

Proof. Clearly $\rho_1(Z^\circ) \subseteq Z(\rho_1(H))^\circ$, and the latter is either 0 or 1-dimensional. If $\dim Z(\rho_1(H)) = 0$ we would have $Z^\circ \subseteq \ker \rho_1$, contradicting Lemma 3.2.6. □

Lemma 3.2.8. *Set $K := \rho_1(H)$, and write $K = S^1 \cdot K_{\mathrm{s}}$ where K_{s} is the semisimple part of K. Then (K, W_1) is not orbit-equivalent to the induced representation (K_{s}, W_1).*

Proof. Representations (K, W_1) and $\rho_1 = (H, W_1)$ are orbit-equivalent, and so are (K_{s}, W_1) and $\rho_{\mathrm{s}} := (H_{\mathrm{s}}, W_1)$ since $K_{\mathrm{s}} = \rho_1(H_{\mathrm{s}})$. It is then enough to prove that ρ_1 and ρ_{s} are not orbit-equivalent.

Assuming that ρ_1 and ρ_{s} are orbit-equivalent we shall see that

$$(H, W) \quad \text{and} \quad (H_{\mathrm{s}} \times Z^\circ, W_1 \oplus W_2)$$

have the same orbits, a contradiction. Recall that the action of H_{s} on W_2 is trivial. Choose now $w' \in W_1$, $w'' \in W_2$, $h_{\mathrm{s}} \in H_{\mathrm{s}}$, $z \in Z^\circ$. By hypothesis, there exists $\tilde{h}_{\mathrm{s}} \in H_{\mathrm{s}}$ such that $h_{\mathrm{s}} z^{-1} \cdot w' = \tilde{h}_{\mathrm{s}} \cdot w'$, so

$$h_{\mathrm{s}} \cdot w' = z\tilde{h}_{\mathrm{s}} \cdot w'.$$

Set $h := z\tilde{h}_{\mathrm{s}}$ so that $h_{\mathrm{s}} \cdot w' = h \cdot w'$; we have $h \cdot w'' = z \cdot w''$ hence

$$h \cdot (w', w'') = (h_{\mathrm{s}} \cdot w', z \cdot w'') = (h_{\mathrm{s}}, z) \cdot (w', w'').$$

Conversely, choose $w' \in W_1$, $w'' \in W_2$, $h \in H$. By hypothesis we have $h \cdot w' = \tilde{h}_{\mathrm{s}} \cdot w'$ for some $\tilde{h}_{\mathrm{s}} \in H_{\mathrm{s}}$. We can also write $h = h_s z$ for suitable $h_{\mathrm{s}} \in H_{\mathrm{s}}$, $z \in Z^\circ$. Then

$$(\tilde{h}_{\mathrm{s}}, z) \cdot (w', w'') = (h \cdot w', z \cdot w'') = (h \cdot w', h \cdot w'') = h \cdot (w', w''),$$

and we are done. □

Lemma 3.2.7 and the main theorem in [16] imply that ρ_1 is orbit-equivalent to the isotropy representation of a rank 2 irreducible Hermitian symmetric space, i.e. one of the following (cf. [26, p.518-520]):

- AIII($p \geq 3$, $q = 2$): $\mathbf{SU}(p+2)/\mathrm{S}(\mathbf{U}(p) \times \mathbf{U}(q))$,
- DIII($n = 5$): $\mathbf{SO}(10)/\mathbf{U}(5)$,
- BDI($p \geq 3,\ q = 2$): $\mathbf{SO}(p+2)/(\mathbf{SO}(p) \times \mathbf{SO}(2))$,
- EIII: $\mathbf{E}_6/\mathbf{SO}(10) \cdot \mathbf{SO}(2)$.

Only the isotropy representation of BDI($p \geq 3,\ q = 2$) satisfies condition in Lemma 3.2.8 (cf. [16]), and the first part of Theorem 3.2.5 is finally proved.

Remark 3.2.9. Assuming ρ faithful, we have $a = \dim Z(H) = 1$. Indeed, since $\rho_1(H) = K = K_{\mathrm{s}} \cdot S^1$ and $\rho_2(H) = S^1$, if $a \geq 3$ then $Z^\circ \simeq \mathbb{T}^a$ contains a subgroup of positive dimension acting trivially on W. So $a \leq 2$. Suppose now $a = 2$. Since $\rho_1(Z^\circ) \simeq \rho_2(Z^\circ) \simeq S^1$ and ρ is faithful, we may write $Z^\circ = S^1_1 \times S^1_2$ for suitable subgroups S^1_i of Z° isomorphic to S^1 in such a way that S^1_1 acts trivially on W_2 and S^1_2 acts trivially on W_1. Hence $H_{\mathrm{s}} \cdot S^1_1$ acts trivially on W_2 and we conclude that ρ is orbit-equivalent to the product representation $((H_{\mathrm{s}} \times S^1_1) \times S^1_2, W_1 \oplus W_2)$. Once again this is a contradiction.

We now turn to the proof of the second part of Theorem 3.2.5; in what follows we shall the use notation introduced in its statement. In addition, we define $H := \mathbf{SO}(2) \times \mathbf{SO}(n)$, $W_1 := \mathbb{R}^2 \otimes \mathbb{R}^n$, and we let W_2 be the representation space of ρ_2. In this way we can write

$$\rho = (H, W), \qquad \rho_1 = (H, W_1), \qquad \rho_2 = (H, W_2),$$

where $W := W_1 \oplus W_2$. Here are some preliminary results:

Lemma 3.2.10. *Copolarity and abstract copolarity of ρ are both ≤ 2.*

Proof. The proof consists in computing the Luna-Richardson-Straume reduction of $\rho = \rho_1 \oplus \rho_2$. It is easily seen that the principal isotropy group K_1 of ρ_1 is isomorphic to $\mathbb{Z}_2 \times \mathbf{SO}(n-2)$, so the principal isotropy group K of ρ is isomorphic either to $\mathbf{SO}(n-2)$ or to $\mathbb{Z}_2 \times \mathbf{SO}(n-2)$. In both cases $N_H(K)^\circ \simeq \mathbb{T}^2 \times \mathbf{SO}(n-2)$ and $\left(\frac{N_H(K)}{K}\right)^\circ \simeq \mathbb{T}^2$, as claimed. □

Lemma 3.2.11. *ρ is indecomposable.*

Proof. Suppose this is not true; we may assume without loss of generality that ρ is orbit-equivalent to

$$\tilde{\rho} := (H' \times H'', W' \oplus W''),$$

where $W_1 \oplus W_2 = W' \oplus W''$ and W', W'' are G-invariant with positive dimension. Since W_1 is an irreducible isotypical component of ρ, we may assume also that $W_1 \subseteq W'$, and that

$$W' = W_1 \oplus \bar{W}_1.$$

Here $\bar{W}_1$ is a sum of irreducible components of ρ and is H'-invariant, while $W_2 = \bar{W}_1 \oplus W''$ is a sum of isotypical components of ρ. Using that ρ and $\tilde{\rho}$ have the same orbits, we easily deduce that the product representation $(H' \times H'', \bar{W}_1 \oplus W'')$ and (H, W_2) are orbit-equivalent; since principal orbits of (H, W_2) have dimension 1, this forces one of the two representations $(H', \bar{W}_1)$, (H'', W'') to be trivial. Since $\dim W'' \geq 1$ and (H, W_2) has no non-trivial fixed points, this implies $\bar{W}_1 = \{O\}$, and thus (H', W'), (H'', W'') are orbit-equivalent to ρ_1, ρ_2 respectively. It follows that

$$\rho \simeq_{\text{o.e.}} (H' \times H'', W' \oplus W'') \simeq_{\text{o.e.}} ((\mathbf{SO}(2) \times \mathbf{SO}(n)) \times S^1, (\mathbb{R}^2 \otimes \mathbb{R}^n) \oplus W_2),$$

where here $\simeq_{\text{o.e.}}$ denotes the relation of orbit-equivalence between representations. This is absurd since principal orbits of $((\mathbf{SO}(2) \times \mathbf{SO}(n)) \times S^1, (\mathbb{R}^2 \otimes \mathbb{R}^n) \oplus W_2)$ and of ρ don't have the same dimension. □

Lemma 3.2.12. *We have* $\mathrm{chm}(\rho) \geq 6$.

Proof. Clearly $\mathrm{chm}(\rho) = 2 + \dim W_2$. Since ρ_2 is not polar and has no non-trivial fixed points, $\dim W_2 \geq 4$ and we are done. □

We may finally prove the second part of Theorem 3.2.5:

Proof. Using Lemmas 3.2.11 and 3.2.12 it is enough to prove that $\mathrm{c}(\rho) = \mathrm{ac}(\rho) = 2$.

First we observe that ρ is not polar (since otherwise both ρ_1 and ρ_2 would be polar, cf. Lemma 3.1.3), and its abstract copolarity cannot be 1 (otherwise we would have $\mathrm{chm}(\rho) = 3$ by Theorem 3.2.1 and Lemma 3.2.11). Hence, the abstract copolarity of ρ has to be at least 2, which, together with Lemma 3.2.10, implies that the abstract copolarity (and of course the copolarity) of ρ is exactly 2. □

Remark 3.2.13. Theorems 3.2.1, 3.2.5 continue to hold if in their statement we change any occurrence of the words *abstract copolarity* with the word *copolarity*. Indeed, we have seen that a representation has copolarity 1 if and only if it has abstract copolarity 1 (Corollary 3.2.4). Moreover, for this reason and by Proposition 2.2.1, any representation of copolarity 2 must have abstract copolarity 2.

3.3 The reducible case

Irreducible representations admitting a toric reduction have been classified by Claudio Gorodski and Alexander Lytchak in [20]. Precisely, their classification states that non-polar effective irreducible representations $\rho = (H, W)$ of a connected compact Lie group H admitting a toric reduction are exactly those belonging to one of the following three disjoint families:

1. ρ is one if the non-polar irreducible representations of cohomogeneity 3:

H	W	Conditions
$\mathbf{SO}(2) \times \mathbf{Spin}(9)$	$\mathbb{R}^2 \otimes \mathbb{R}^{16}$	–
$\mathbf{U}(2) \times \mathbf{Sp}(n)$	$\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^n$	$n \geq 2$
$\mathbf{Sp}(1) \times \mathbf{Sp}(n)$	$S^3\mathbb{C}^2 \otimes_{\mathbb{H}} \mathbb{C}^{2n}$	$n \geq 2$

2. the group H is the semisimple factor of an irreducible polar representation of hermitian type such that the action of H is not orbit-equivalent to the polar representation:

H	W	Conditions
$\mathbf{SU}(n)$	$S^2\mathbb{C}^n$	$n \geq 3$
$\mathbf{SU}(n)$	$\Lambda^2\mathbb{C}^n$	$n = 2p \geq 6$
$\mathbf{SU}(n) \times \mathbf{SU}(n)$	$\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^n$	$n \geq 3$
$\mathbf{E}_6$	$\mathbb{C}^{27}$	–

3. ρ is one of the exceptions $(\mathbf{SO}(3)\times\mathbf{G}_2, \mathbb{R}^3\otimes\mathbb{R}^7)$, $(\mathbf{SO}(4)\times\mathbf{Spin}(7), \mathbb{R}^4\otimes\mathbb{R}^8)$.

The goal of this Section is to classify reducible representations of simple Lie groups admitting a toric reduction. Namely, we shall prove the following:

Theorem 3.3.1. *Assume that H is a compact, connected, simple Lie group, and let (H, W) be an effective, reducible, indecomposable, non-reduced representation of H that admits a toric reduction. If (H, W) is non-polar, then it is one of the following representations:*

H	W	ac	chm
$\mathbf{SO}(n)$	$\mathbb{R}^n \oplus \mathbb{R}^n$	1	3
$\mathbf{G}_2$	$\mathbb{R}^7 \oplus \mathbb{R}^7$	1	3
$\mathbf{Spin}(7)$	$\mathbb{R}^8 \oplus \mathbb{R}^8$	1	3
$\mathbf{Sp}(2)$	$\mathbb{R}^5 \oplus \mathbb{C}^4$	1	3
$\mathbf{Spin}(9)$	$\mathbb{R}^9 \oplus \mathbb{R}^{16}$	1	3
$\mathbf{Spin}(8)$	$\mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-$	2	4

Table 3.1: Representations of a simple Lie group admitting a toric reduction

Here $\mathbb{R}^8_0$ denotes the standard representation of $\mathbf{Spin}(8)$*, while $\mathbb{R}^8_+$, $\mathbb{R}^8_-$ denote its half spin representations.*

We begin with an observation wich allows us to find all double representations which admits a toric reduction.

Proposition 3.3.2. *Let $\rho = (H, W)$ be an irreducible, effective representation so that the double representation $2\rho = (H, W \oplus W)$ is non-reduced. Then 2ρ does not admit a toric reduction, unless ρ is one of the following representations:*

H	W	*Conditions*
$\mathbf{SO}(n)$	$\mathbb{R}^n$	$n \geq 3$
$\mathbf{G}_2$	$\mathbb{R}^7$	–
$\mathbf{Spin}(7)$	$\mathbb{R}^8$	–

In each case 2ρ has $(\mathbf{O}(2), \mathbb{R}^2 \oplus \mathbb{R}^2)$ as a minimal reduction.

Proof. We claim that $(H, W \oplus W)$ is indecomposable. Indeed assume by contradiction that $(H, W \oplus W)$ is decomposable. Then it is orbit-equivalent to a product representation of the form $(H_1 \times H_2, W_1 \oplus W_2)$, where $W \oplus W = W_1 \oplus W_2$, and the W_i's are H-invariant (cf. Proposition 2.4.2). Clearly we may assume $W_1 = W$, $W_2 = W$. Let $\rho_i := (H, W_i)$. Since $H_i = \rho_i(H)$, we see that $(H_1 \times H_2, W_1 \oplus W_2)$ has the same orbits as $(H \times H, W \oplus W)$. So also $(H, W \oplus W)$ and $(H \times H, W \oplus W)$ have the same orbits. This is impossible: indeed, there exist $w \in W$ and $h \in H$ so that $h \cdot w \neq w$. If the above representations had the same orbits, there would be $\bar{h} \in H$ so that

$$(e, h) \cdot (w, w) = \bar{h} \cdot (w, w) \quad \text{i.e.} \quad (w, h \cdot w) = (\bar{h} \cdot w, \bar{h} \cdot w).$$

Then $w = \bar{h} \cdot w = h \cdot w$, contradiction.

Now, $(H, W \oplus W)$ is not polar (cf. [12, 3]); assume then that it admits a minimal toric reduction of positive dimension, say $(G, V \oplus V)$ where $G^\circ = \mathbb{T}$, $\dim \mathbb{T} \geq 1$. Clearly all the isotypical components of the induced representation $(\mathbb{T}, V \oplus V)$ are reducible. This contradicts Propositions 3.1.9, 3.1.12 if $\dim \mathbb{T} \geq 2$.

If instead $\dim \mathbb{T} = 1$, then $(H, W \oplus W)$ has cohomogeneity 3 (cf. Theorem 3.2.1); so it must appear in [40, Table II] (cf. also [41, Table III]), and is therefore one of the representations listed above in the statement.

Finally we prove the assertion concerning the minimal reductions (cf. [40]). Observe that $(\mathbf{G}_2, 2\mathbb{R}^7)$ is orbit-equivalent to $(\mathbf{SO}(7), 2\mathbb{R}^7)$ and that $(\mathbf{Spin}(7), 2\mathbb{R}^8)$ is orbit-equivalent to $(\mathbf{SO}(8), 2\mathbb{R}^8)$, so it is enough to find a minimal reduction for the representations in the first row. The principal isotropy group of $(\mathbf{SO}(n), 2\mathbb{R}^n)$, $n \geq 3$, is $K := \mathbf{SO}(n-2)$, and its fixed point space is $\mathbb{R}^2 \oplus \mathbb{R}^2$. Now it is clear that $N_{\mathbf{SO}(n)}(K) = \mathrm{S}(\mathbf{O}(2) \times \mathbf{O}(n-2))$, so $N_{\mathbf{SO}(n)}(K)/K \simeq \mathbf{O}(2)$. □

Remark 3.3.3. In Proposition 3.3.2 we do *not* require the group H to be simple.

In order to classify reducible representations admitting a toric reduction, we need an argument which will allow us to determine whether a given representation does *not* admit a toric reduction. We therefore prove the following:

Lemma 3.3.4. *Let $\rho_1 = (H, W_1)$, $\rho_2 = (H, W_2)$ be two representations of the connected compact Lie group H, and set $\rho := (H, W_1 \oplus W_2)$. Define $c_1 := \mathrm{chm}(\rho_1)$, $c_2 := \mathrm{chm}(\rho_2)$, $c := \mathrm{chm}(\rho)$. Assume that ρ admits a minimal toric reduction (G, V) with $G^\circ = \mathbb{T}$ a torus. Then*

$$2c - c_1 - c_2 \leq \dim V \leq 2(c_1 + c_2).$$

Proof. By Proposition 2.2.11, we have $V = V_1 \oplus V_2$, where V_1, V_2 are G-invariant subspaces of V such that $V_i/G \simeq W_i/H$, $i = 1, 2$. We observe that we can find a principal point $v = v_1 + v_2 \in V$ for (G, V), with $v_1 \in V_1$, $v_2 \in V_2$, such that v_i is principal for (G, V_i), $i = 1, 2$. Then we have

$$c = \mathrm{chm}(G, V) = \mathrm{chm}(G, V_1) + \mathrm{chm}(G_{v_1}, V_2) = c_1 + \mathrm{chm}(G_{v_1}, V_2).$$

Now, the stabilizer of v_2 in G_{v_1} is contained in G_v, and the latter is trivial since (G, V) is reduced. Thus $\mathrm{chm}(G_{v_1}, V_2) = \dim V_2 - \dim G_{v_1}$ and it follows that

$$\dim V_2 = c - c_1 + \dim G_{v_1} \geq c - c_1.$$

Similarly we deduce

$$\dim V_1 = c - c_2 + \dim G_{v_2} \geq c - c_2,$$

therefore

$$\dim V = \dim V_1 + \dim V_2 \geq 2c - c_1 - c_2,$$

as claimed.

On the other hand, we decompose V_1 and V_2 into irreducible components with respect to the representations $(\mathbb{T}, V_1)$, $(\mathbb{T}, V_2)$:

$$V_1 = V_1' \oplus \cdots \oplus V_l', \qquad V_2 = V_1'' \oplus \cdots \oplus V_m''.$$

Let W be one of these $\mathbb{T}$-irreducible subspaces; then $\dim W \in \{1, 2\}$, and $\dim W = 1$ if and only if $\mathbb{T}$ acts trivially on W. In any case, $\mathrm{chm}(\mathbb{T}, W) = 1$. Thus

$$c_1 = \mathrm{chm}(G, V_1) = \mathrm{chm}(\mathbb{T}, V_1) \geq \sum_{i=1}^{l} \mathrm{chm}(\mathbb{T}, V_i') = \sum_{i=1}^{l} 1 = l,$$

and similarly

$$c_2 = \mathrm{chm}(G, V_2) = \mathrm{chm}(\mathbb{T}, V_2) \geq \sum_{i=1}^{m} \mathrm{chm}(\mathbb{T}, V_i'') = \sum_{i=1}^{m} 1 = m,$$

therefore

$$\dim V = \dim V_1 + \dim V_2 = 2(l + m) \leq 2(c_1 + c_2),$$

as claimed. □

Remark 3.3.5. Tipically we shall prove that a reducible representation

$$\rho = (H, W_1 \oplus W_2)$$

does not have a toric reduction as follows. By contradiction we shall assume that ρ admits a minimal toric reduction (G, V). Then, by Lemma 3.3.4, we have

$$2c - c_1 - c_2 \leq \dim V \leq 2(c_1 + c_2),$$

where $c_i := \mathrm{chm}(H, W_i)$, $i = 1, 2$, and $c := \mathrm{chm}(\rho)$. We shall give a bound for c of the form $c \geq c_0$. So also

$$2c_0 - c_1 - c_2 \leq \dim V \leq 2(c_1 + c_2). \tag{3.5}$$

In our applications, most of the times we shall find $2c_0 - c_1 - c_2 > 2(c_1 + c_2)$, thus obtaining a contradiction with (3.5). This will prove that the original representation ρ cannot admit a toric reduction.

Now we explain how we obtain the bound c_0 for c. In some cases we shall set $c_0 := c$. However we observe that, if H^1_{pr} denotes the principal isotropy group of (H, W_1), then

$$c = \mathrm{chm}(H, W_1 \oplus W_2) = \mathrm{chm}(H, W_1) + \mathrm{chm}(H^1_{\mathrm{pr}}, W_2) \geq c_1 + \dim W_2 - \dim H^1_{\mathrm{pr}},$$

so in other cases we shall set $c_0 := c_1 + \dim W_2 - \dim H^1_{\mathrm{pr}}$.

A first application of the method described in Remark 3.3.5 is given in the proof of the following:

Proposition 3.3.6. *Let H be a compact, connected, simple Lie group, and let (H, W_1), (H, W_2) be irreducible representations admitting a toric reduction. If at most one of them is polar, then $(H, W_1 \oplus W_2)$ does not admit a toric reduction.*

Proof. The non-polar irreducible representations (H, W), with H compact, connected and simple, which admit a toric reduction are the following (cf. [20]):

H	W	Conditions
$\mathbf{SU}(n)$	$S^2\mathbb{C}^n$	$n \geq 3$
$\mathbf{SU}(n)$	$\Lambda^2\mathbb{C}^n$	$n = 2p \geq 6$
$\mathbf{E}_6$	$\mathbb{C}^{27}$	–

Therefore, assuming for instance (H_1, W_1) non-polar, we may suppose that (H, W_1) and (H, W_2) are listed in the following table:

H	Conditions	W_1	W_2	c_1	c_2	c_0	$2c_0 - c_1 - c_2$	$2(c_1 + c_2)$
$\mathbf{SU}(n)$	$n \geq 3$	$S^2\mathbb{C}^n$	$S^2\mathbb{C}^n$	$n+1$	$n+1$	$(n+1)^2$	$2n^2 + 2n$	$4n + 4$
$\mathbf{SU}(n)$	n even ≥ 6	$S^2\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n$	$n+1$	$\frac{n}{2}+1$	$n^2 + 1$	$\frac{4n^2-3n}{2}$	$3n + 4$
$\mathbf{SU}(4)$	–	$S^2\mathbb{C}^4$	$\Lambda^2\mathbb{C}^4$	5	1	11	16	12
$\mathbf{SU}(n)$	n odd ≥ 3	$S^2\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n$	$n+1$	$\frac{n-1}{2}$	$n^2 + 1$	$\frac{4n^2-3n+3}{2}$	$3n + 1$
$\mathbf{SU}(n)$	$n \geq 3$	$S^2\mathbb{C}^n$	Ad	$n+1$	$n-1$	$n^2 + n$	$2n^2$	$4n$
$\mathbf{SU}(n)$	$n \geq 3$	$S^2\mathbb{C}^n$	$\mathbb{C}^n$	$n+1$	1	$3n + 1$	$5n$	$2n + 4$
$\mathbf{SU}(8)$	–	$S^2\mathbb{C}^8$	$\Lambda^4\mathbb{C}^8$	9	7	79	142	32
$\mathbf{SU}(n)$	n even ≥ 6	$\Lambda^2\mathbb{C}^n$	$\Lambda^2\mathbb{C}^n$	$\frac{n}{2}+1$	$\frac{n}{2}+1$	$(n-1)^2$	$2n^2 - 5n$	$2n + 4$
$\mathbf{SU}(n)$	n even ≥ 6	$\Lambda^2\mathbb{C}^n$	Ad	$\frac{n}{2}+1$	$n-1$	$n^2 - n$	$\frac{4n^2-7n}{2}$	$3n$
$\mathbf{SU}(n)$	n even ≥ 6	$\Lambda^2\mathbb{C}^n$	$\mathbb{C}^n$	$\frac{n}{2}+1$	1	$n + 1$	$\frac{3n}{2}$	$n + 4$
$\mathbf{SU}(8)$	–	$\Lambda^2\mathbb{C}^8$	$\Lambda^4\mathbb{C}^8$	5	7	63	114	24
$\mathbf{E}_6$	–	$\mathbb{C}^{27}$	$\mathbb{C}^{27}$	4	4	30	52	16
$\mathbf{E}_6$	–	$\mathbb{C}^{27}$	Ad	4	6	54	98	20

Here $c_i := \mathrm{chm}(H, W_i)$, $i = 1, 2$, and c_0 is a lower bound for $\mathrm{chm}(H, W_1 \oplus W_2)$ as explained in Remark 3.3.5. We note that $2c_0 - c_1 - c_2 > 2(c_1 + c_2)$ (and thus the corresponding representation cannot have a toric reduction) in all cases except the following two:

$$(\mathbf{SU}(6), \mathbb{C}^6 \oplus \Lambda^2\mathbb{C}^6), \qquad (\mathbf{SU}(8), \mathbb{C}^8 \oplus \Lambda^2\mathbb{C}^8).$$

We shall prove that the first representation does not have a toric reduction; one can argue similarly to prove that the same is true for the second representation.

Let $O \neq v \in \mathbb{C}^6$. The isotropy subgroup of $(\mathbf{SU}(6), \mathbb{C}^6 \oplus \Lambda^2\mathbb{C}^6)$ at (v, O) is $\mathbf{SU}(5)$, while the slice representation is

$$(\mathbf{SU}(5), \mathbb{R} \oplus \mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5).$$

By Lemma 3.1.3 it is enough to prove that $(\mathbf{SU}(5), \mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5)$ does not admit a toric reduction. Indeed, in this case

$$c_1 := \mathrm{chm}(\mathbf{SU}(5), \mathbb{C}^5) = 1, \qquad c_2 := \mathrm{chm}(\mathbf{SU}(5), \Lambda^2\mathbb{C}^5) = 2,$$
$$c_0 := \mathrm{chm}(\mathbf{SU}(5), \mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5) = 6,$$

so $2c_0 - c_1 - c_2 = 9 > 6 = 2(c_1 + c_2)$ and the claim follows by Lemma 3.3.4. □

The following is a direct consequence of Proposition 3.3.6 and Lemma 3.1.3:

Corollary 3.3.7. *Let $\rho = (H, W)$ be a reducible representation of the compact, connected, simple Lie group H admitting a toric reduction. Then all irreducible summands of ρ are polar.*

Next we prove the following:

Proposition 3.3.8. *Let H be a compact, connected, simple Lie group, and let (H, W) be a reducible, non-reduced, indecomposable representation of abstract copolarity $k \geq 2$ admitting a toric reduction. Then W is a sum of pairwise non-equivalent irreducible polar representations of H.*

Proof. Let (G, V) a minimal reduction of (H, W) such that $G^\circ = \mathbb{T}$ is a k-dimensional torus. By Proposition 3.1.13 we have a $\mathbb{T}$-invariant decomposition

$$V = W_1 \oplus \cdots \oplus W_r$$

where the W_i's are 2-dimensional and pairwisely $\mathbb{T}$-inequivalent. We now consider the action of $G/\mathbb{T}$ on the set $\{W_1, \ldots, W_r\}$, and let $\mathcal{O}_i$, $i = 1, \ldots, \ell$ be its orbits. Call V^α the sum of all W_i belonging the the orbit $\mathcal{O}_\alpha$ for $\alpha = 1, \ldots, \ell$. Then

$$V = V^1 \oplus \cdots \oplus V^\ell$$

is a decomposition of V into G-irreducible summands. If, for some $\alpha, \beta \in \{1, \dots, \ell\}$, $\alpha \neq \beta$, V^α were G-equivalent to V^β, then V^α, V^β would also be $\mathbb{T}$-equivalent, so there would exist $a, b \in \{1, \dots, r\}$, $a \neq b$, so that W_a is $\mathbb{T}$-equivalent to W_b, contradiction. This proves that the G-isotypical components of (G, V), and hence those of (H, W) (cf. Proposition 2.2.13), are irreducible. The fact that they all are polar follows from Corollary 3.3.7. □

Given a compact, connected Lie group H we shall denote by ℓ_H the number of non-equivalent irreducible polar representations of H. For H simple (and simply connected), ℓ_H is given in the following table (cf. [26, 16]).

H	ℓ_H	H	ℓ_H
$\mathbf{SU}(n)$, n odd	3	$\mathbf{Spin}(n)$, $n \neq 7, 8, 9, 10, 16$	3
$\mathbf{SU}(n)$, n even, $n \neq 2, 4, 8$	2	$\mathbf{Spin}(n)$, $n = 7, 9$	4
$\mathbf{SU}(2)$, $\mathbf{SU}(4)$, $\mathbf{SU}(8)$	3	$\mathbf{Spin}(n)$, $n = 8, 10, 16$	5
$\mathbf{Sp}(n)$, $n \neq 1, 4$	3	$\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$	1
$\mathbf{Sp}(1)$	2	$\mathbf{F}_4$	2
$\mathbf{Sp}(4)$	4	$\mathbf{G}_2$	2

From Proposition 3.3.8 we immediately deduce the following:

Corollary 3.3.9. *Let (H, W) be a reducible, non-reduced, indecomposable representation of a simple Lie group admitting a toric reduction. Suppose its abstract copolarity is at least 2. Then (H, W) has at most ℓ_H irreducible summands.*

We shall now analyze, for each compact, connected, simply connected, simple Lie group H, whether a sum of inequivalent irreducible polar representations of H can admit a toric reduction; our basic tool is the method described in Remark 3.3.5. Observe that if a sum $(H, W_1 \oplus W_2)$ of two irreducible inequivalent polar representations of H does not admit a toric reduction, then, by Lemma 3.1.3, also any sum which contains $(H, W_1 \oplus W_2)$ does not admit a toric reduction.

The case $H = \mathbf{SU}(n)$

For any n, $\mathbf{SU}(n)$ has (at least) two inequivalent polar representations: the adjoint representation Ad, and the standard representation $\mathbb{C}^n$. We have

$$c_1 := \mathrm{chm}(\mathbf{SU}(n), \mathrm{Ad}) = n - 1, \qquad c_2 := \mathrm{chm}(\mathbf{SU}(n), \mathbb{C}^n) = 1,$$
$$c_0 := \mathrm{chm}(\mathbf{SU}(n), \mathrm{Ad} \oplus \mathbb{C}^n) = 2n.$$

Therefore

$$2c_0 - c_1 - c_2 = 3n, \qquad 2(c_1 + c_2) = 2n.$$

Since $3n > 2n$ for all n, we deduce that $(\mathbf{SU}(n), \mathrm{Ad} \oplus \mathbb{C}^n)$ cannot have a toric reduction (cf. Remark 3.3.5).

If $n = 2$, $\mathbf{SU}(2) \simeq \mathbf{Sp}(1)$ has another polar representation, namely $S^2_0(\mathbb{R}^3) \simeq \mathbb{R}^5$, where here $\mathbf{SU}(2)$ is interpreted as the universal covering of $\mathbf{SO}(3)$. The fact that $(\mathbf{SU}(2), \mathbb{R}^3 \oplus \mathbb{R}^5)$ does not admit a toric reduction will be proved later in the paragraph concerning the groups $\mathbf{Spin}(n)$. Here we check that $(\mathbf{SU}(2), \mathbb{C}^2 \oplus \mathbb{R}^5)$ does not admit a toric reduction either. Indeed we have

$$c_1 := \mathrm{chm}(\mathbf{SU}(2), \mathbb{C}^2) = 1, \qquad c_2 := \mathrm{chm}(\mathbf{SU}(2), \mathbb{R}^5) = 2,$$
$$c_0 := \mathrm{chm}(\mathbf{SU}(2), \mathbb{C}^2 \oplus \mathbb{R}^5) = 6,$$

hence

$$2c_0 - c_1 - c_2 = 9 > 6 = 2(c_1 + c_2).$$

If $n \geq 3$ is odd (respectively, if $n = 4$, if $n = 8$), $\mathbf{SU}(n)$ has another irreducible polar representation, which is equivalent neither to Ad nor to $\mathbb{C}^n$, namely $\Lambda^2\mathbb{C}^n$ (respectively, $\Lambda^2\mathbb{C}^4$, $\Lambda^4\mathbb{C}^8$):

H	Conditions	W_1	W_2	c_1	c_2	c_0	$2c_0 - c_1 - c_2$	$2(c_1 + c_2)$
$\mathbf{SU}(n)$	n odd ≥ 3	$\Lambda^2\mathbb{C}^n$	Ad	$\frac{n-1}{2}$	$n-1$	$n^2 - n - 1$	$\frac{4n^2-7n-1}{2}$	$3n - 3$
$\mathbf{SU}(n)$	n odd ≥ 3	$\Lambda^2\mathbb{C}^n$	$\mathbb{C}^n$	$\frac{n-1}{2}$	1	$n+1$	$\frac{3n+3}{2}$	$n+1$
$\mathbf{SU}(8)$	–	$\Lambda^4\mathbb{C}^8$	Ad	7	7	70	126	28
$\mathbf{SU}(8)$	–	$\Lambda^4\mathbb{C}^8$	$\mathbb{C}^8$	7	1	23	38	16
$\mathbf{SU}(4)$	–	$\Lambda^2\mathbb{C}^4$	Ad	1	3	6	8	8
$\mathbf{SU}(4)$	–	$\Lambda^2\mathbb{C}^4$	$\mathbb{C}^4$	1	1	2	2	4

Here $c_i := \mathrm{chm}(H, W_i)$, $i = 1, 2$, while c_0 is a lower bound for $\mathrm{chm}(H, W_1 \oplus W_2)$. We have $2c_0 - c_1 - c_2 > 2(c_1 + c_2)$, and thus the corresponding representation does not admit a toric reduction (cf. Remark 3.3.5), in all cases, except those concerning $\mathbf{SU}(4)$.

First note that $(\mathbf{SU}(4), \Lambda^2\mathbb{C}^4 \oplus \mathbb{C}^4)$ does have a toric reduction; in fact, such a representation has cohomogeneity 2, is polar and decomposable (cf. also [3]).

Consider now $(\mathbf{SU}(4), \Lambda^2\mathbb{C}^4 \oplus \mathrm{Ad}) \simeq (\mathbf{SO}(6), \mathbb{R}^6 \oplus \mathbb{R}^{15})$. It is easily seen that, if $O \neq v \in \mathbb{R}^6$, the slice representation at (v, O) is

$$(\mathbf{SO}(5), \mathbb{R} \oplus \mathbb{R}^5 \oplus \mathbb{R}^{10}).$$

Using the same argument as in the last part of the proof of Proposition 3.3.6 we then see that $(\mathbf{SU}(4), \Lambda^2\mathbb{C}^4 \oplus \mathrm{Ad})$ cannot admit a toric reduction.

Finally observe that a sum τ of 3 pairwisely inequivalent irreducible polar representation of $\mathbf{SU}(n)$ (when possible) cannot admit a toric reduction: indeed τ contains at least two irreducible submodules whose sum cannot have a toric reduction by the discussion above.

The case $H = \mathbf{Sp}(n)$

For all $n \geq 1$, $\mathbf{Sp}(n)$ has at least 2 inequivalent irreducible polar representations: namely, the adjoint one Ad, and the standard one $\mathbb{C}^{2n}$. If $n \geq 2$, $\mathbf{Sp}(n)$ admits also the isotropy representation τ_1 of the irreducible symmetric space $\mathbf{SU}(2n)/\mathbf{Sp}(n)$. Finally, $\mathbf{Sp}(4)$ admits in addition the isotropy representation τ_2 of the irreducible symmetric space $\mathbf{E}_6/\mathbf{Sp}(4)$.

H	Conditions	W_1	W_2	c_1	c_2	c_0	$2c_0 - c_1 - c_2$	$2(c_1 + c_2)$
$\mathbf{Sp}(n)$	$n \geq 1$	Ad	$\mathbb{C}^{2n}$	n	1	$4n$	$7n - 1$	$2n + 2$
$\mathbf{Sp}(n)$	$n \geq 2$	τ_1	Ad	$n - 1$	n	$2n^2 - n - 1$	$4n^2 - 4n - 1$	$4n - 2$
$\mathbf{Sp}(n)$	$n \geq 2$	τ_1	$\mathbb{C}^{2n}$	$n - 1$	1	$2n - 1$	$3n - 2$	$2n$
$\mathbf{Sp}(4)$	–	τ_2	Ad	6	4	42	74	20
$\mathbf{Sp}(4)$	–	τ_2	$\mathbb{C}^8$	6	1	22	37	14
$\mathbf{Sp}(4)$	–	τ_2	τ_1	6	3	33	57	18

We have $2c_0 - c_1 - c_2 > 2(c_1 + c_2)$ (and thus the corresponding representation doesn't have a toric reduction) in all cases except

$$(\mathbf{Sp}(2), \mathbb{R}^5 \oplus \mathbb{C}^4) \simeq (\mathbf{Spin}(5), \mathbb{R}^5 \oplus \mathbb{C}^4).$$

This representation does admit a toric reduction: indeed it is non-polar, indecomposable and of cohomogeneity 3 therefore, by Corollary 3.2.2, it must have abstract copolarity 1. Notice that it has the same orbits as $(\mathbf{Sp}(1) \times \mathbf{Sp}(2), \mathbb{R}^5 \oplus \mathbb{C}^2 \otimes_{\mathbb{H}} \mathbb{C}^4)$ (cf. [41]), and the latter representation appears in the list of Corollary 3.2.2.

Now it is clear that the sum of at least 3 irreducible non-equivalent polar representations of $\mathbf{Sp}(n)$ (when possible) cannot admit a toric reduction, since it always contains the sum of two submodules which doesn't have a toric reduction.

The case of exceptional Lie groups

The groups $\mathbf{E}_6$ $\mathbf{E}_7$, $\mathbf{E}_8$ admit only one polar representation, namely the adjoint one, so they are out of our discussion.

The group $\mathbf{G}_2$ admits two irreducible polar representation, namely the adjoint one, Ad, and the 7-dimensional representation $\mathbb{R}^7$. Also the group $\mathbf{F}_4$ admits two polar representations: the adjoint one, Ad, and the isotropy representation of the irreducible symmetric space $\mathbf{E}_6/\mathbf{F}_4$.

H	W_1	W_2	c_1	c_2	c_0	$2c_0 - c_1 - c_2$	$2(c_1 + c_2)$
$\mathbf{G}_2$	Ad	$\mathbb{R}^7$	2	1	7	11	6
$\mathbf{F}_4$	Ad	$\mathbb{R}^{26}$	4	2	26	46	12

In both cases $2c_0 - c_1 - c_2 > 2(c_1 + c_2)$, hence the representations $(\mathbf{G}_2, \mathrm{Ad} \oplus \mathbb{R}^7)$, $(\mathbf{F}_4, \mathrm{Ad} \oplus \mathbb{R}^{26})$ do not admit a toric reduction.

The case $H = \mathbf{Spin}(n)$

If $n \geq 3$, $\mathbf{Spin}(n)$ has (at least) three non-equivalent irreducible polar representations: namely the adjoint one, Ad, the standard one, $\mathbb{R}^n$, and the isotropy representation of the irreducible symmetric space $\mathbf{SU}(n)/\mathbf{SO}(n)$ (which coincides with $S^2_0\mathbb{R}^n := S^2\mathbb{R}^n \ominus \mathbb{R}$). For $n = 7, 8, 9, 10, 16$ also the spin representations are polar.

First we observe that the representation $\mathbb{R}^n \oplus S^2_0\mathbb{R}^n$ does not admit a toric reduction. Indeed we have:

$$c_1 := \mathrm{chm}(\mathbf{Spin}(n), \mathbb{R}^n) = 1, \qquad c_2 := \mathrm{chm}(\mathbf{Spin}(n), S^2_0\mathbb{R}^n) = n-1,$$
$$c := \mathrm{chm}(\mathbf{Spin}(n), \mathbb{R}^n \oplus S^2_0\mathbb{R}^n) = 2n-1,$$

and $2c - c_1 - c_2 = 3n - 2 > 2n = 2(c_1 + c_2)$ for all $n \geq 3$ (cf. Lemma 3.3.4).

Now assume that $n = 2p + 1$ is odd. We have the following cases:

H	Cond.	W_1	W_2	c_1	c_2	c_0	$2c_0 - c_1 - c_2$	$2(c_1 + c_2)$
$\mathbf{Spin}(2p+1)$	$p \geq 1$	Ad	$S^2_0\mathbb{R}^{2p+1}$	p	$2p$	$2p^2 + 3p$	$4p^2 + 3p$	$6p$
$\mathbf{Spin}(2p+1)$	$p \geq 2$	Ad	$\mathbb{R}^{2p+1}$	p	1	$2p + 1$	$3p + 1$	$2p + 2$
$\mathbf{Spin}(7)$	–	$\mathbb{R}^8$	Ad	1	3	8	12	8
$\mathbf{Spin}(7)$	–	$\mathbb{R}^8$	$\mathbb{R}^7$	1	1	2	2	4
$\mathbf{Spin}(7)$	–	$\mathbb{R}^8$	$S^2_0\mathbb{R}^7$	1	6	14	21	14
$\mathbf{Spin}(9)$	–	$\mathbb{R}^{16}$	Ad	1	4	16	27	10
$\mathbf{Spin}(9)$	–	$\mathbb{R}^{16}$	$\mathbb{R}^9$	1	1	3	4	4
$\mathbf{Spin}(9)$	–	$\mathbb{R}^{16}$	$S^2_0\mathbb{R}^9$	1	8	24	39	18

We have $2c_0 - c_1 - c_2 > 2(c_1 + c_2)$ (and thus the corresponding representation cannot have a toric reduction) in all cases except:

$$(\mathbf{Spin}(7), \mathbb{R}^8 \oplus \mathbb{R}^7), \qquad (\mathbf{Spin}(9), \mathbb{R}^{16} \oplus \mathbb{R}^9).$$

These representations do have a toric reduction: indeed, the first one is polar and decomposable (cf. [3]), while the second one is indecomposable, has cohomogeneoty 3 and abstract copolarity 1 (cf. Corollary 3.2.2).

Now we assume that $n = 2p$ is even. First consider the following cases:

H	Cond.	W_1	W_2	c_1	c_2	c_0	$2c_0 - c_1 - c_2$	$2(c_1 + c_2)$
$\mathbf{Spin}(2p)$	$p \geq 3$	Ad	$S^2_0\mathbb{R}^{2p}$	p	$2p - 1$	$2p^2 + p - 1$	$4p^2 - p - 1$	$6p - 2$
$\mathbf{Spin}(2p)$	$p \geq 3$	Ad	$\mathbb{R}^{2p}$	p	1	$2p$	$3p - 1$	$2p + 2$
$\mathbf{Spin}(10)$	–	$\mathbb{C}^{16}_{\pm}$	Ad	2	5	32	57	14
$\mathbf{Spin}(10)$	–	$\mathbb{C}^{16}_{\pm}$	$\mathbb{R}^{10}$	2	1	5	7	6
$\mathbf{Spin}(10)$	–	$\mathbb{C}^{16}_{\pm}$	$S^2_0\mathbb{R}^{10}$	2	9	41	71	22
$\mathbf{Spin}(10)$	–	$\mathbb{C}^{16}_{+}$	$\mathbb{C}^{16}_{-}$	2	2	19	34	8
$\mathbf{Spin}(16)$	–	$\mathbb{R}^{128}_{\pm}$	Ad	8	8	128	240	32
$\mathbf{Spin}(16)$	–	$\mathbb{R}^{128}_{\pm}$	$\mathbb{R}^{16}$	8	1	24	39	18
$\mathbf{Spin}(16)$	–	$\mathbb{R}^{128}_{\pm}$	$S^2_0\mathbb{R}^{16}$	8	15	143	263	46
$\mathbf{Spin}(16)$	–	$\mathbb{R}^{128}_{+}$	$\mathbb{R}^{128}_{-}$	8	8	136	256	32

We have $2c_0 - c_1 - c_2 > 2(c_1 + c_2)$ (and thus the corresponding representation cannot have a toric reduction) in all cases except:

$$(\mathbf{Spin}(6), \mathbb{R}^6 \oplus \mathrm{Ad}) \simeq (\mathbf{SU}(4), \Lambda^2\mathbb{C}^4 \oplus \mathrm{Ad});$$

however we have already seen that this representation does not admit a toric reduction.

We finally study the case of $\mathbf{Spin}(8)$. First consider the following table, where $\mathbb{R}^8$ denotes any of the three irreducible, non-equivalent (polar) 8-dimensonal representations of $\mathbf{Spin}(8)$:

H	W_1	W_2	c_1	c_2	c_0	$2c_0 - c_1 - c_2$	$2(c_1 + c_2)$
$\mathbf{Spin}(8)$	$\mathbb{R}^8$	$S^2_0\mathbb{R}^8$	1	7	15	22	16
$\mathbf{Spin}(8)$	$\mathbb{R}^8$	Ad	1	4	8	11	10

Since in both cases $2c_0 - c_1 - c_2 > 2(c_1 + c_2)$, neither of the representations $(\mathbf{Spin}(8), \mathbb{R}^8 \oplus S^2_0\mathbb{R}^8)$, $(\mathbf{Spin}(8), \mathbb{R}^8 \oplus \mathrm{Ad})$ admits a toric reduction.

Next, we denote by $\mathbb{R}^8_0$, $\mathbb{R}^8_\pm$, respectively, the standard representation of $\mathbf{Spin}(8)$ and its half spin representations. It is known that

$$(\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_+), \qquad (\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_-), \qquad (\mathbf{Spin}(8), \mathbb{R}^8_+ \oplus \mathbb{R}^8_-)$$

are all polar (cf. [3]).

Now we show that $(\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-)$ does admit a minimal toric reduction of dimenson 2 (cf. [18]).

Indeed, the principal isotropy groups of $(\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-)$ are isomorphic to $\mathbf{SU}(3)$, and their fixed point space in $\mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-$ is given by $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$. Thus the identity component of the normalizer of a principal isotropy group is a 2-dimensional torus $\mathbb{T}^2$, and $(\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-)$ admits a reduction (G, V), where $G^\circ = \mathbb{T}^2$.

Note that $\mathrm{chm}(\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-) = 4$, so $(\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-)$ is indecomposable (cf. Lemma 2.4.13). In [18] it is proved moreover that such a representation is non-polar, hence its abstract copolarity is ≥ 1. If it were 1, $(\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-)$, which is indecomposable, would have cohomogeneity 3 by Theorem 3.2.1, contradiction. So $\mathrm{ac}(\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-) = 2$ and (G, V) is a minimal reduction.

At this point it is clear that a sum of at least 3 irreducible inequivalent polar representation of $\mathbf{Spin}(n)$ (when possible) never admits a toric reduction, unless it is $(\mathbf{Spin}(8), \mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-)$.

We can finally prove Theorem 3.3.1:

Proof. Let (H, W) be a representation as in the statement. If $\mathrm{ac}(H, W) = 1$, then $\mathrm{chm}(H, W) = 3$ by Theorem 3.2.1; going through the list of cohomogeneity

3 indecomposable representations given in [41], we determine the first five rows of Table 3.1.

Next assume $\mathrm{ac}(H, W) \geq 2$. By Propositon 3.3.8 (H, W) is a sum of irreducible non-equivalent polar representation of H, so the discussion above implies that $H = \mathbf{Spin}(8)$ and that $W = \mathbb{R}^8_0 \oplus \mathbb{R}^8_+ \oplus \mathbb{R}^8_-$. We have seen that this representation has cohomogeneity 4 and abstract copolarity 2. □

Bibliography

[1] Adams J.F.: *Lectures on Lie groups.* The University of Chicago Press (1969).

[2] Alexandrino M.M.: *Proofs of conjecture about singular Riemannian foliations.* Geom. Dedicata **119** (2006), 219-234.

[3] Bergmann I.: *Reducible polar representations.* Manuscripta Mathematica **104** (2001), 309-324.

[4] Berndt J., Console S., Olmos C.: *Submanifolds and holonomy.* Research Notes in Mathematics, Chapman & Hall/CRC, London (2003).

[5] Besse A.L.: *Einstein manifolds.* Springer-Verlag, Berlin Heidelberg (1987).

[6] Bourbaki N.: *Lie groups and Lie algebras.* Springer-Verlag Berlin Heidelberg (2005).

[7] Bridson M.R., Haefliger A.: *Metric spaces of non-positive curvature.* Springer-Verlag Berlin Heidelberg (1999).

[8] Bröcker T., tom Dieck T.: *Representations of compact Lie groups.* Springer-Verlag New York (1985).

[9] Burago D., Burago Y., Ivanov S.: *A course in Metric Geometry.* Graduate Studies in Mathematics, vol. 33. American Mathematical Society.

[10] Choi S.: *Geometric structures on orbifolds and holonomy representations.* Geom. Dedicata **104**(1) (2004), 161-199.

[11] Clerc J.-L.: *Geometry of the Shilov boundary of a bounded symmetric domain.* J. Geom. Symmetry Phys. **13** (2009), 25-74.

[12] Dadok J.: *Polar coordinates induced by actions of compact Lie groups.* Trans. Amer. Math. Soc. **288** (1985), 125-137.

[13] Davis M.: *Lectures on orbifolds and reflection groups.* Higher Education Press, pp. 63-93, Springer-Verlag (2010).

[14] Dragomir G.C.: *Closed geodesics on compact developable orbifolds.* Ph.D. Thesis, McMaster University (2011).

[15] Duistermaat J.J., Kolk J.A.C.: *Lie groups.* Universitext Springer-Verlag (2000).

[16] Eschenburg J.-H., Heintze E.: *On the classification of polar representations.* Math. Z. **232** (1999), 391-398.

[17] Foertsch T., Lytchak A.: *The de Rham decomposition theorem for metric spaces*. Geom. Funct. Anal. **18** (2008), 120-143.

[18] Gorodski C.: *Taut representations of compact simple Lie groups*. Illinois J. of Math. **52** (2008), 121-143.

[19] Gorodski C., Lytchak A.: *On orbit spaces of representations of compact Lie groups*. J. reine angew. Math. **691** (2014), 61-100.

[20] Gorodski C., Lytchak A.: *Representations whose minimal reduction has a toric identity component*. Proc. Amer. Math. Soc. **143**(1) (2015), 379-366.

[21] Gorodski C., Lytchak A.: *Isometric actions on spheres with an orbifold quotient*. Math. Ann. **365** (2016), 1041-1067.

[22] Gorodski C., Olmos C., Tojeiro R.: *Copolarity of isometric actions*. Trans. Amer. Math. Soc. **356**(4) (2004), 1585-1608.

[23] Gorodski C., Thorbergsson G.: *Representations of compact Lie groups and the osculating spaces of their orbits*. Available at `http://arxiv.org/abs/math/0203196`, preprint (2002).

[24] Grove K., Searle C.: *Global G-manifolds reductions and resolutions*. Ann. Global Anal. and Geom. **18** (2000), 437-446.

[25] Heintze E., Liu X., Olmos C.: *Isoparametric submanifolds and a Chevalley-type restriction theorem*. Integrable systems, geometry, and topology, AMS/IP Stud. Adv. math. **36**, International Press, Somerville (2006), 152-234.

[26] Helgason S.: *Differential geometry, Lie groups, and symmetric spaces*. Academic Press, New York (1978).

[27] Kobayashi S.: *Transformation groups in Differential Geometry*. Springer-Verlag (1995).

[28] Luna D., Richardson R.W.: *A generalization of the Chevalley restriction theorem*. Duke Math. J. **46** (1979), 487-496.

[29] Lytchak A.: *Geometric resolution of singular Riemannian foliations*. Geom. Dedicata **149** (2010), 379-395.

[30] Lytchak A.: *Polar foliations of symmetric spaces*. Geom. Funct. Anal. **24** (2014), 1298-1315.

[31] Lytchak A., Thorbergsson G.: *Curvature explosion in quotients and applications*. J. Diff. Geom. **85** (2010), 117-139.

[32] Magata F.: *Reductions, resolutions and the copolarity of isometric group actions*. Münstersches Informations- und Archivsystem für multimediale Inhalte (2008).

[33] Nomizu K., Ozeki H.: *The existence of complete Riemannian metrics*. Proc. Amer. Math. Soc. **12** (1961), 889-891.

[34] O'Neill B.: *The fundamental equations of a submersion*. Michigan Math. J. **13** (1966), 459-469.

[35] Palais R.S., Terng C.-L.: *A general theory of canonical forms*. Trans. Amer. Math. Soc. **300**(2) (1987), 771-789.

[36] Palais R.S., Terng C.L.: *Critical point theory and submanifold geometry.* Lecture Notes in Mathematics, vol 7. Springer-Verlag (1988).

[37] Panelli F., Pomrehn C.: *Representations of abstract copolarity one and two.* Diff. Geom. and its Appl. **58** (2018), 52-64.

[38] Radeschi M.: *Clifford algebras and new singular riemannian foliations in spheres.* Geom. Funct. Anal. **24** (2014), 1660-1682.

[39] Skjelbred T., Straume E.: *A note on the reduction principle for compact transformation groups.* Preprint (1995).

[40] Straume E.: *On the invariant theory and geometry of compact linear groups of cohomogeneity* ≤ 3. Diff. Geom. and its Appl. **4** (1994), 1-23.

[41] Straume E.: *Compact connected Lie transformation groups on spheres with low cohomogeneity, I.* Mem. Amer. Math. Soc. **569** (1996).

[42] Szenthe J.: *Orthogonally transversal submanifolds and the generalization of the Weyl group.* Period. Math. Hungar. **15** (1984), 281-299.

[43] Thurston W.P.: *The geometry and topology of three-manifolds.* Lecture notes, available at `http://library.msri.org/books/gt3m` (1980).