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Borell-Brascamp-Lieb Inequalities: Rigidity and Stability

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Introduction

In this thesis we address some relevant generalizations of the famous Brunn-Minkowski inequality, and in particular we study its functional formulations, known in the literature as the *Borell-Brascamp-Lieb inequalities*.

Among the Borell-Brascamp-Lieb inequalities, the most famous one is known as the *Prékopa-Leindler inequality*, since it was previously proved by Prékopa [41] and Leindler [37] (later rediscovered by Brascamp and Lieb in [10]). It states that

$$\int_{\mathbb{R}^n} h \, dz \geq \left(\int_{\mathbb{R}^n} f \, dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \, dy \right)^{\lambda}$$

for any $\lambda \in (0,1)$ and every $f, g, h : \mathbb{R}^n \longrightarrow [0, +\infty)$ such that

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda} \qquad \text{for all} \ \ x, y \in \mathbb{R}^n. \qquad (0.1)$$

In other words, if h evaluated at the convex combination of any two points is greater than the (λ-weighted) geometric mean of f and g at those points, then the integral of h is not less than the (λ-weighted) geometric mean of the integrals of f and g. If the Prékopa-Leindler inequality reminds the reader of anything, it is probably Hölder's inequality with the inequality reversed. Recall that if $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ are nonnegative functions with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then Hölder's inequality claims that

$$\int_{\mathbb{R}^n} k \ dx \leq \left(\int_{\mathbb{R}^n} f^p \, dx \right)^{1/p} \left(\int_{\mathbb{R}^n} g^q \, dx \right)^{1/q},$$

where $k(x) = f(x)g(x)$. Let $\lambda \in (0,1)$, $1/p = 1 - \lambda$, $1/q = \lambda$: the latter can be rewritten as

$$\int_{\mathbb{R}^n} f^{1-\lambda} g^{\lambda} \, dx \leq \left(\int_{\mathbb{R}^n} f \, dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \, dx \right)^{\lambda}.$$

Thus the Prékopa-Leindler inequality can be seen in some sense as a reverse form of Hölder's inequality. On the other hand, there is no contradiction between the two inequalities, since condition (0.1) is rather strong and of course it implies

$$h(x) \geq f(x)^{1-\lambda} g(x)^{\lambda} \qquad \text{for all} \ \ x \in \mathbb{R}^n,$$

and in general with strict sign.

The Prékopa-Leindler inequality is largely recognized as the functional formulation of the multiplicative form of the *Brunn-Minkowski inequality*, that is

$$|(1 - \lambda)A + \lambda B| \geq |A|^{1-\lambda} |B|^{\lambda}, \tag{0.2}$$

where $\lambda \in (0, 1), |\cdot|$ denotes the Lebesgue measure, A, B are nonempty measurable subset of $\mathbb{R}^n$ and $(1 - \lambda)A + \lambda B$ is their *Minkowski combination*

$$(1 - \lambda)A + \lambda B = \{(1 - \lambda)a + \lambda b : \ a \in A, \ b \in B\}.$$

Indeed (0.2) can be easily deduced by applying the Prékopa-Leindler inequality to the characteristic functions $f = \chi_A$, $g = \chi_B$, $h = \chi_{(1-\lambda)A+\lambda B}$.

More generally, the Borell-Brascamp-Lieb inequalities constitute a larger family of integral inequalities, including the Prékopa-Leindler inequality as the special case $p = 0$. Let $\lambda \in (0, 1)$, $p \in [-1/n, +\infty]$. The Borell-Brascamp-Lieb inequality of index p states that

$$\int_{\mathbb{R}^n} h \, dx \geq \left[(1 - \lambda) \left(\int_{\mathbb{R}^n} f \, dx \right)^{\frac{p}{np+1}} + \lambda \left(\int_{\mathbb{R}^n} g \, dx \right)^{\frac{p}{np+1}} \right]^{\frac{np+1}{p}},$$

provided that the nonnegative functions f, g, h satisfy

$$h((1 - \lambda)x + \lambda y) \geq [(1 - \lambda)f(x)^p + \lambda g(y)^p]^{1/p} \qquad \text{for all} \ \ x, y \in \mathbb{R}^n.$$

The latter two inequalities have to be read as limit for the the index $p = 0$ (Prékopa-Leindler inequality), or $p = +\infty$ or $p = -1/n$: namely for $p = +\infty$

$$\int_{\mathbb{R}^n} h \, dx \geq \left[(1 - \lambda) \left(\int_{\mathbb{R}^n} f \, dx \right)^{1/n} + \lambda \left(\int_{\mathbb{R}^n} g \, dx \right)^{1/n} \right]^n,$$

if the nonnegative functions f, g, h satisfy

$$h((1 - \lambda)x + \lambda y) \geq \max \{f(x), g(y)\} \qquad \text{for all} \ \ x, y \in \mathbb{R}^n;$$

instead for $p = -1/n$ it becomes

$$\int_{\mathbb{R}^n} h \, dx \geq \min \left\{ \int_{\mathbb{R}^n} f \, dx, \int_{\mathbb{R}^n} g \, dx \right\},$$

provided that the nonnegative functions f, g, h satisfy

$$h((1 - \lambda)x + \lambda y) \geq \left[(1 - \lambda)f(x)^{-1/n} + \lambda g(y)^{-1/n} \right]^{-n} \qquad \text{for all} \ \ x, y \in \mathbb{R}^n.$$

All the Borell-Brascamp-Lieb inequalities can be interpreted as functional counterparts of the Brunn-Minkowski inequality, since they are all equivalent each other (and in particular equivalent to the Prékopa-Leindler inequality). The Borell-Brascamp-Lieb inequality

of index $p = +\infty$ (using again the same characteristic functions) leads immediately to the classical form of the Brunn-Minkowski inequality, i.e.

$$|(1-\lambda)A + \lambda B|^{1/n} \geq (1-\lambda)\,|A|^{1/n} + \lambda\,|B|^{1/n}\,, \qquad (0.3)$$

for every $\lambda \in (0,1)$, A, B nonempty measurable subset of $\mathbb{R}^n$. Moreover equality holds in (0.3) if and only if A, B are (essentially) homothetic convex sets.

Problems and topics inspired by the Brunn-Minkowski inequality and by the variety of its formulations have attracted a growing interest in recent decades. In particular the Borell-Brascamp-Lieb inequalities have been investigated in different areas of research, also owing to their connections with other fields of Mathematics, like probability theory and statistics, especially concerning concavity properties of certain functions and measures.

Between the various themes related to the Borell-Brascamp-Lieb inequalities, in this thesis we mainly address the *question of their stability*, which is an important aspect intrinsically related to their *equality conditions*.
Let us briefly explain what we mean by stability of an inequality. Suppose to known the inequality and also the precise characterization of the situations in which the equality occurs. The stability question is the following: assume that the involved objects (sets or functions, or something else) in the inequality almost get the equality; can we claim that these objects are near to the class which exactly attains the equality?
Moreover, in the affirmative, the stability of the inequality is called *quantitative* if we are able to estimate the closeness to the equality conditions in explicit terms of the distance from the equality in the original inequality.
Clearly, we should first formalize what we mean by *almost getting inequality*; then we should specify in which sense, that is under what suitable distance, we describe *nearness*.
In recent years there has been renewed interest in the research of quantitative versions for several inequalities. Such an interest finds its motivation in the following remark: quoting Gardner [28] *"If inequalities are silver currency in mathematics, those that come along with precise equality conditions are gold. Equality conditions are treasure boxes containing valuable information"*. Then a quantitative stability result is even more precious, because it provides additional valuable informations about the closeness to the equality conditions.

The purpose of this work is twofold. On one hand it is aimed to be a detailed guide regarding the world of the Borell-Brascamp-Lieb inequalities: we delve into the state of the art of this topic by means of various proofs, remarks, links, generalizations and refinements, including equality conditions. On the other hand we deal with questions related to the stability of these inequalities: after recalling the existing stability results concerning Borell-Brascamp-Lieb inequalities, we expose our original results.

Let us describe the organization of the dissertation. First of all we introduce in Chapter 1 some basic definitions and concepts. For the sake of brevity, this chapter does

not contain all the technical preliminaries needed in the thesis, which are given, when necessary, within each chapter.

In Chapter 2 we present the fundamental Brunn-Minkowski inequality, exhibiting some well known equivalent formulations and a simple proof, due to Hadwiger and Ohmann, basically derived by the arithmetic–geometric mean inequality. Furthermore we notice that the isoperimetric inequality for convex bodies is a direct consequence of the Brunn-Minkowski inequality. In the last part of Chapter 2 we recall the equality conditions of the Brunn-Minkowski inequality, and some related stability results, especially the crucial and recent one of Figalli and Jerison [25]. To get into the concept of stability, in the course of Chapter 2 we will quote further quantitative versions of the Brunn-Minkowski inequality, due to Groemer [31], Figalli-Maggi-Pratelli [26, 27], in addition to the mentioned result of Figalli and Jerison.

Chapter 3 is devoted to a detailed presentation of the Borell-Brascamp-Lieb inequalities, providing four different proofs and clarifying the relationship with the Brunn-Minkowski inequality. We underline that the third proof (Subsection 3.3.3) relies on an idea of Klartag [36] and it is crucial for our main result in [43], that is Theorem 7.1.1 in Chapter 7. On the other hand the fourth proof (Subsection 3.3.4) is original at some degree and based on a work of Uhrin [50]. At the end of Chapter 3 we show the relationship between the Borell-Brascamp-Lieb inequalities and another integral inequality, precisely Theorem 3.4.1.

In order to investigate the equality conditions for the Borell-Brascamp-Lieb inequalities, Chapter 4 contains the proof of a larger class of integral inequalities, consequence of Theorem 3.4.1, including their equality conditions provided by Dubuc [23]. These conditions, stated in Proposition 4.1.2, allow us to deduce, in Section 4.2 and Section 4.3, a precise characterization for the equality case of the Borell-Brascamp-Lieb inequalities. Although some of these equality conditions (surely the ones relative to the Prékopa-Leindler inequality, see [2, 3, 12] for example) are known in literature, their proofs can not be explicitly found, to our knowledge. Then we give detailed proofs in Chapter 4, and this can be considered an original contribution (see [42]).

In Chapter 5 we summarize the (few) known stability results for Borell-Brascamp-Lieb inequalities. We will describe stability results for Prékopa-Leindler inequality due to Ball-Böröczky [2, 3] and Bucur-Fragalà [12], and we will state a quantitative version of the Borell-Brascamp-Lieb inequalities of index $p > 0$ for power concave functions, proved by Ghilli and Salani [30].

Finally the last two chapters, Chapter 6 and Chapter 7, are aimed to present and describe the main original contributions of the thesis. While Chapter 6 concerns the stability of a strengthened one-dimensional Borell-Brascamp-Lieb inequality, Chapter 7 is devoted to a general stability version of the Borell-Brascamp-Lieb inequalities, for the first time without concavity assumptions on the involved functions. The results of these two chapters are respectively contained in the submitted paper [44] and of the published paper [43].

Chapter 1

Ingredients

1.1 Basic notations

Throughout this dissertation the symbol $|\cdot|$ is used to denote different things and we hope this is not going to cause confusion. In particular: for a real number a we denote by $|a|$ its absolute value, as usual; for a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we denote by $|x|$ its euclidean norm, that is $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$; for a set $A \subset \mathbb{R}^n$ we denote by $|A|$ its n-dimensional Lebesgue measure or, sometimes, its outer measure if A is not measurable. Let us recall that the outer measure of a subset $A \subset \mathbb{R}^n$ is defined as

$$\inf \left\{ \sum_{n \in \mathbb{N}} |P_n| : \ \{P_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}, \ A \subseteq \bigcup_{n \in \mathbb{N}} P_n \right\},$$

where $\mathcal{P}$ is the algebra of the finite unions of n-dimensional rectangles (with the term rectangle we mean the cartesian product of n real intervals, possibly unbounded). On the other hand, the inner Lebesgue measure of $A \subset \mathbb{R}^n$ is the supremum of the measures of its compact subsets. For any $A \subset \mathbb{R}^n$ consider the (perhaps infinite) outer and inner Lebesgue measure: if these two quantities coincide for the set A, the latter is said to be measurable and the common value of these quantities is its Lebesgue measure $|A|$.

Let $A \subset \mathbb{R}^n$. We denote by $\overline{A}$ its closure, by ∂A its boundary and indicate its surface area with $\mathcal{H}^{n-1}(\partial A)$. Its *convex hull*, denoted by $conv(A)$, is the smallest convex subset of $\mathbb{R}^n$ that contains A. We denote by $\mathcal{K}^n$ the family of n-dimensional *convex bodies*, i.e. compact, convex subsets of $\mathbb{R}^n$, with nonempty interior.

The support set of a nonnegative continuous function $f : \mathbb{R}^n \to [0, +\infty)$ is denoted by $\mathrm{Supp}(f)$, that is $\mathrm{Supp}(f) = \overline{\{x \in \mathbb{R}^n : f(x) > 0\}}$. In general (when f is not necessarily continuous), by $\mathrm{Supp}(f)$ we denote the (essential) support set, defined as follows:

$$\mathrm{Supp}(f) = \mathbb{R}^n \setminus \cup_{A \in \mathfrak{F}} A$$

where $\mathfrak{F} = \{A \subseteq \mathbb{R}^n : A \text{ is open and } f \text{ vanishes a.e. in } A\}$.

For further use, we recall the so-called *Cavalieri formula* in a simplified version: let f be a nonnegative function belonging to $L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} f(x)\ dx = \int_0^{+\infty} |\{x : f(x) \geq t\}|\ dt. \tag{1.1}$$

1.2 Minkowski combination

Let A be a subset of $\mathbb{R}^n$ and let $\alpha > 0$; we set

$$\alpha A = \{z \in \mathbb{R}^n :\ z = \alpha x,\ x \in A\};$$

The *Minkowski sum* of two subsets A and B of $\mathbb{R}^n$ is simply defined as

$$A + B = \{a + b :\ a \in A,\ b \in B\}.$$

It is trivial to check that these two operations, respectively rescaling and Minkowski sum, preserve convexity.

In particular, given $\lambda \in (0,1)$, we call *Minkowski combination* (of coefficient λ) of two nonempty sets $A, B \subseteq \mathbb{R}^n$ the set

$$(1 - \lambda)A + \lambda B = \{(1 - \lambda)a + \lambda b :\ a \in A,\ b \in B\}. \tag{1.2}$$

Being defined in terms of the previous operations, also the Minkowski combination preserve the convexity: if A, B are convex sets, then $(1 - \lambda)A + \lambda B$ is convex.

Exemple 1.2.1. In $\mathbb{R}^2$ the Minkowski combination (of coefficient $\lambda = 1/2$)

$$\frac{1}{2}Q + \frac{1}{2}B(0,1)$$

of a square and a disk is a rounded square.

Remark 1.2.2. If A is convex, then for every $\lambda \in (0,1)$ holds

$$(1 - \lambda)A + \lambda A = A.$$

It is interesting to note that if A is not convex, there exists $\lambda \in (0,1)$ for which the latter identity does not hold: indeed in general only the inclusion

$$A \subseteq (1-\lambda)A + \lambda A$$

holds, since any $a \in A$ can be trivially expressed by the sum $(1-\lambda)a + \lambda a$. We underline that the opposite inclusion characterizes the convex sets. In fact

$$(1-\lambda)A + \lambda A \subseteq A \qquad \text{for every } \lambda \in (0,1)$$

means that for any $a_1, a_2 \in A$ the convex combination $(1-\lambda)a_1 + \lambda a_2$ belongs to A: hence we recognize the definition of convexity. In other words, if A is not convex then there exists $\lambda \in (0,1)$ for which A is a proper subset of $(1-\lambda)A + \lambda A$ and vice versa.

1.3   Generalized means and p-concavity

It is useful to introduce the generalized means of two nonnegative numbers and the corresponding notion of power concavity for a nonnegative function.

Definition 1.3.1. Let $q \in [-\infty, +\infty]$ and $\lambda \in (0,1)$. Given two real numbers $a \geq 0$ and $b \geq 0$, the quantity

$$\mathcal{M}_q(a,b;\lambda) = \begin{cases} \max\{a,b\} & q = +\infty\,, \\ [(1-\lambda)a^q + \lambda b^q]^{\frac{1}{q}} & q \in \mathbb{R} \setminus \{0\}\,, \\ a^{1-\lambda}b^{\lambda} & q = 0\,, \\ \min\{a,b\} & q = -\infty\,, \end{cases} \quad \text{if } ab > 0\,; \qquad (1.3)$$

$$\mathcal{M}_q(a,b;\lambda) = 0 \quad \text{for every } q \in \mathbb{R} \cup \{\pm\infty\}\,, \qquad \text{if } ab = 0,$$

represents the (λ-weighted) q-mean of the nonnegative numbers a and b.

Observe that for every $\lambda \in (0,1)$ it holds

$$\lim_{q \to +\infty} \mathcal{M}_q(a,b;\lambda) = \max\{a,b\}\,, \ \lim_{q \to 0} \mathcal{M}_q(a,b;\lambda) = a^{1-\lambda}b^{\lambda}, \ \lim_{q \to -\infty} \mathcal{M}_q(a,b;\lambda) = \min\{a,b\}\,.$$

Some of these generalized means are well known: for example the cases $q = 1$ and $q = 0$ correspond respectively to the (λ-weighted) arithmetic mean and geometric mean. A simple consequence of Jensen's inequality is

$$\mathcal{M}_q(a,b;\lambda) \geq \mathcal{M}_p(a,b;\lambda) \quad \text{if } q > p, \qquad (1.4)$$

that is the p-means are monotone increasing with respect to the index p.
For every $q > p$, equality holds in (1.4) if and only if $a = b$ or $ab = 0$.
Choosing $q = 1$ and $p = 0$ in (1.4), we recognize the *arithmetic–geometric mean inequality*

$$(1-\lambda)a + \lambda b \geq a^{1-\lambda}b^{\lambda}.$$

for two nonnegative numbers a, b. Its classical (and symmetric) form occurs when $\lambda = 1/2$:

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

The latter can be generalized for three or more nonnegative numbers $a_1, ..., a_n$, becoming

$$\frac{1}{n}\sum_{j=1}^{n} a_j \geq \left[\prod_{j=1}^{n} a_j\right]^{1/n}. \tag{1.5}$$

Definition 1.3.2. A nonnegative function $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ is *p-concave* for some $p \in \mathbb{R} \cup \{\pm\infty\}$ if

$$u((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(u(x), u(y); \lambda) \qquad \forall\ x, y \in \mathbb{R}^n \quad \forall\ \lambda \in (0, 1).$$

Roughly speaking, u is p-concave if it has convex support Ω and:
(i) u^p is concave in Ω for $p > 0$;
(ii) u is *"log-concave"*, i.e. $\log u$ is concave in Ω for $p = 0$;
(iii) u^p is convex in Ω for $p < 0$;
(iv) u is *"quasiconcave"* (i.e. all its superlevel sets $\{z : u(z) \geq t\}$ are convex) for $p = -\infty$;
(v) u is a positive constant in Ω for $p = +\infty$.

Notice that $p = 1$ corresponds to usual concavity in Ω.
It follows from (1.4) that if u is q-concave, then u is p-concave for any $p \leq q$.
Hence quasiconcavity is the weakest conceivable concavity property. The *quasiconcave envelope* of a function f is the smallest quasiconcave function greater than or equal to f, that is the smallest function, greater than or equal to f, whose superlevel sets are convex.

The Brunn-Minkowski inequality (which will be presented in detail in Chapter 2)

$$|(1-\lambda)A + \lambda B|^{1/n} \geq (1-\lambda)\,|A|^{1/n} + \lambda\,|B|^{1/n}$$

states that the Lebesgue measure is a $(1/n)$-concave function in relation to the Minkowski combination in $\mathbb{R}^n$.

1.4   The (p, λ)-convolution of two functions

Let $\lambda \in (0, 1)$, $f, g : \mathbb{R}^n \longrightarrow [0, +\infty)$. The *supremal convolution* of f and g is the function $(1-\lambda)f \oplus_1 \lambda g : \mathbb{R}^n \longrightarrow [0, +\infty)$ defined as follows:

$$((1-\lambda)f \oplus_1 \lambda g)(z) = \sup\{\mathcal{M}_1(f(x), g(y); \lambda) :\ z = (1-\lambda)x + \lambda y\}.$$

This definition can be easily generalized.

Definition 1.4.1. Let $p \in [-\infty, +\infty], \lambda \in (0,1)$, $f, g : \mathbb{R}^n \longrightarrow [0, +\infty)$. The (p, λ)-*convolution* of f and g is the function $(1 - \lambda)f \oplus_p \lambda g : \mathbb{R}^n \longrightarrow [0, +\infty)$ given by

$$((1 - \lambda)f \oplus_p \lambda g)(z) = \sup \{ \mathcal{M}_p(f(x), g(y); \lambda) : z = (1 - \lambda)x + \lambda y \}. \qquad (1.6)$$

In other words, the (p, λ)-convolution of f and g, for $p > 0$ corresponds to the $(1/p)$-power of the supremal convolution (with coefficient λ) of f^p and g^p. Trivially

$$((1 - \lambda)f \oplus_p \lambda g)((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda) \quad \text{for all } x, y \in \mathbb{R}^n,$$

and the function $(1 - \lambda)f \oplus_p \lambda g$ is, by Definition 1.4.1, the smallest function satisfying this property. Indeed the condition

$$h((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda) \quad \text{for all } x, y \in \mathbb{R}^n \qquad (1.7)$$

is clearly equivalent to $h \geq (1 - \lambda)f \oplus_p \lambda g$. We underline that (1.7) is the key assumption of the Borell-Brascamp-Lieb inequalities, which represent the main topic of the thesis. From Definition 1.4.1 and (1.4), we get

$$(1 - \lambda)f \oplus_q \lambda g \geq (1 - \lambda)f \oplus_p \lambda g \qquad \text{if} \quad -\infty \leq p < q \leq +\infty.$$

Note that whenever f, g are p-concave, the function $(1 - \lambda)f \oplus_p \lambda g$ is also p-concave. Clearly the function $(1 - \lambda)f \oplus_p \lambda f$ is not less than f, since

$$[(1 - \lambda)f \oplus_p \lambda f](z) \geq \mathcal{M}_p(f(z), f(z); \lambda) = f(z).$$

Furthermore it holds

$$[(1 - \lambda)f \oplus_p \lambda f] = f$$

if and only if f is a p-concave function. This observation about the (p, λ)-convolution and the p-concavity is the functional counterpart of Remark 1.2.2.

Now we clarify the relation between the (p, λ)-convolution of two functions and the Minkowski combination of suitable sets, corresponding to the involved functions.

Definition 1.4.2. Let $p > 0$. Given $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ we consider the set which represents the subgraph of u^p, namely

$$A_u^{(p)} = \{ (x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{Supp} u, \ 0 \leq t \leq u(x)^p \}.$$

Let $p > 0$; then, roughly speaking, the subgraph of $((1 - \lambda)f \oplus_p \lambda g)^p$ is obtained as the Minkowski combination (with coefficient λ) of the subgraphs of f^p and g^p. Precisely we have

$$A_{(1-\lambda)f \oplus_p \lambda g}^{(p)} = (1 - \lambda)A_f^{(p)} + \lambda A_g^{(p)},$$

where these subsets of $\mathbb{R}^{n+1}$ represent respectively the graphs of $f^p, g^p, ((1 - \lambda)f \oplus_p \lambda g)^p$, according to Definition 1.4.2.

Throughout the thesis we use for simplicity the notation $h_{p,\lambda}$ in place of $(1-\lambda)f \oplus_p \lambda g$ to denote the (p, λ)-convolution of f and g, when there are no doubts about what the involved functions f, g are. More explicitly we mean:

$$h_{p,\lambda} = (1 - \lambda)f \oplus_p \lambda g.$$

Chapter 2

The Brunn-Minkowski inequality (BM)

2.1 Presentation and equivalent formulations

The classical form of the Brunn-Minkowski inequality (BM in the following) regards only convex bodies and it plays a crucial role in the related theory (see Schneider's excellent book [47], also for detailed remarks and references concerning its early history). The validity of the BM inequality has been extended later to the class of measurable sets. Providing a lower bound on the volume of a Minkowski combination in terms of the volume of the individual sets, for a long time it has been considered to belong only to geometry. Anyway, by the mid-twentieth century, Lusternik, Hadwiger and Ohmann, and Henstock and Macbeath gave considerable extensions of the inequality, which began to be viewed as an analytical result. When Prékopa [41] and Leindler [37] established the so-called *Prékopa-Leindler inequality*, a functional equivalent version of the Brunn-Minkowski inequality, its role as an analytical tool has been widely recognized. Since then it has been rightly considered one of the principal geometric-analytical inequalities.

We refer to the beautiful paper of Gardner [28] (please, note also the extended version [29]) for an exhaustive presentation of the historical development of the Brunn-Minkowski inequality, combined with a careful explanation of its several extensions and of its intriguing (sometimes unexpected) relationships with other meaningful inequalities. Among these, notice that the classical isoperimetric inequality for convex bodies can be derived as a quick consequence of BM inequality (see Section 2.3).

Let us now recall the BM inequality in its general form for the Euclidean spaces.

Theorem 2.1.1 (Brunn-Minkowski inequality)**.**
Given $\lambda \in (0,1)$, let $A, B \subseteq \mathbb{R}^n$ be nonempty measurable sets. Then

$$|(1-\lambda)A + \lambda B|^{1/n} \geq (1-\lambda)\,|A|^{1/n} + \lambda\,|B|^{1/n}, \tag{2.1}$$

where $|\cdot|$ possibly means outer measure if $(1-\lambda)A + \lambda B$ is not measurable. In other words, the Lebesgue measure in $\mathbb{R}^n$ is $1/n$-concave in relation to the Minkowski combination.

The theorem is clearly trivial when $|A| = 0 = |B|$ or $|(1-\lambda)A + \lambda B| = +\infty$, and also when only one among A, B has measure zero is trivial. For example, let $|B| = 0$. The set B is notempty, thus there exists an element b belonging to B. Hence the Minkowski combination $(1-\lambda)A + \lambda B$ contains $(1-\lambda)A + \lambda\{b\}$, which differs from the set $(1-\lambda)A$ only for the translation represented by λb. By the n-homogeneity of the Lebesgue measure in $\mathbb{R}^n$ and its invariance under translations, it follows

$$|(1-\lambda)A + \lambda B|^{1/n} \geq |(1-\lambda)A + \lambda\{b\}|^{1/n} = |(1-\lambda)A|^{1/n} = (1-\lambda)\,|A|^{1/n},$$

i.e. (2.1), being $|B| = 0$.

Remark 2.1.2. Moreover we can note that if B consists of only one point $B = \{b\}$, then equality holds in (2.1). A similar considerations hold if at least one among A, B has infinite measure, $|A| = +\infty$ say. In the same way

$$|(1-\lambda)A + \lambda B|^{1/n} \geq |(1-\lambda)A + \lambda\{b\}|^{1/n} = (1-\lambda)\,|A|^{1/n} = +\infty,$$

hence we get equality in (2.1), since the right and left hand sides coincide with $+\infty$.

Therefore the result (2.1) is really meaningful only if A, B have positive finite measure. Without loss of generality we can suppose

$$|A|,|B| \in (0, +\infty).$$

Remark 2.1.3. Another easy case happens if $B = \alpha A$;

$$(1-\lambda)A + \lambda B = (1-\lambda)A + \lambda\alpha A \supseteq [(1-\lambda) + \lambda\alpha]\,A; \qquad \text{thus}$$

$$|(1-\lambda)A + \lambda B|^{1/n} \geq |[(1-\lambda) + \lambda\alpha]\,A|^{1/n}$$

$$= [(1-\lambda) + \lambda\alpha]\,|A|^{1/n} = (1-\lambda)\,|A|^{1/n} + \lambda\,|\alpha A|^{1/n} = (1-\lambda)\,|A|^{1/n} + \lambda\,|B|^{1/n},$$

having repeatedly used the n-homogeneity of the volume in $\mathbb{R}^n$. Since the volume is invariant under rigid motions, the same property continue to hold if A and B are homothetic sets of the kind $B = \alpha A + v$, i.e. if they coincide up to dilatations and translations. In particular, if $\alpha = 1$ and $A = B$ is a convex set, then (Remark 1.2.2)

$$(1-\lambda)A + \lambda B = (1-\lambda)A + \lambda A = A.$$

In this way both the (right and left hand) sides of (2.1) coincide with the term $|A|^{1/n}$. This is the simplest (not trivial) case of equality in the BM inequality.
However, the other cases are strongly related to this one (Proposition 2.4.2).

Remark 2.1.4. Let us now focus on some observations regarding the measurability problem. Maybe unexpectedly, there exist measurable sets A, B whose Minkowski combination $(1 - \lambda)A + \lambda B$ is not measurable: see the examples of Sierpïnski [49]. But in rather general assumptions this problem does not occur: any convex set is measurable thus the Minkowski combination of two convex sets is measurable, being convex; moreover, if A and B are Borel sets then $(1 - \lambda)A + \lambda B$ is measurable. To remedy the possibly lack of measurability of one or more involved sets, one can simply replace the measure in the BM inequality (2.1) by the outer or inner Lebesgue measure. For a proof and a careful examination of these generalizations of the BM inequality (in the equivalent form (2.2)), we refer to [13] (Chapter 2, Theorem 8.3.1).

Although we usually consider the BM inequality in its classical form (2.1), since it behaves nicely with respect to the Borell-Brascamp-Lieb inequalities (and in particular with the key assumption (1.7)), there are several equivalent formulations of the BM inequality: for instance let us see (2.2), (2.3) and (2.4). Especially the latter may wrongly appear weaker than (2.1). The inequality (2.3) is known as the *multiplicative form* of the BM inequality.

Lemma 2.1.1 (Brunn-Minkowski inequality, equivalent statements)**.**
Given $\lambda \in (0, 1)$, let $A, B \subseteq \mathbb{R}^n$ be nonempty measurable. Then any of the following assertions is equivalent to the BM inequality (2.1):

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n} ; \tag{2.2}$$

$$|(1 - \lambda)A + \lambda B| \geq |A|^{1-\lambda} |B|^{\lambda} ; \tag{2.3}$$

$$|(1 - \lambda)A + \lambda B| \geq \min \{|A|, |B|\} . \tag{2.4}$$

Proof. Of course, (2.1) trivially implies (2.3), and (2.3) yields (2.4), since

$$|(1 - \lambda)A + \lambda B| \geq \left[(1 - \lambda) |A|^{1/n} + \lambda |B|^{1/n}\right]^n \geq |A|^{1-\lambda} |B|^{\lambda} \geq \min \{|A|, |B|\} ,$$

using (1.4) with the indices $1/n > 0 > -\infty$. For the implication (2.4) $\Rightarrow$ (2.2), suppose without loss of generality that $|A|, |B|$ are positive. Replace A and B in (2.4) by $|A|^{-1/n} A$ and $|B|^{-1/n} B$, respectively, and choose

$$\lambda = \frac{|B|^{1/n}}{|A|^{1/n} + |B|^{1/n}}.$$

In this way (2.4), by the n-homogeneity of Lebesgue measure, becomes

$$\left| \frac{|A|^{1/n}}{|A|^{1/n} + |B|^{1/n}} |A|^{-1/n} A + \frac{|B|^{1/n}}{|A|^{1/n} + |B|^{1/n}} |B|^{-1/n} B \right|$$

$$\geq \min\left\{\left|\,|A|^{-1/n}A\right|, \left|\,|B|^{-1/n}B\right|\right\} = 1,$$

i.e.

$$|A + B| \geq \left[|A|^{1/n} + |B|^{1/n}\right]^n, \qquad \text{that is (2.2)}.$$

Finally let us show the implication (2.2) $\Rightarrow$ (2.1), obtaining the equivalence of the three results. It suffices to apply (2.2) with $(1-\lambda)A$ and λB in place of A and B, deriving

$$|(1-\lambda)A + \lambda B|^{1/n} \geq |(1-\lambda)A|^{1/n} + |\lambda B|^{1/n},$$

that is (2.1), again by the n-homogeneity of the volume. $\qquad\qquad\square$

Remark 2.1.5. Noticeably, the $(1/n)$-concavity of the volume in the BM inequality (2.1) is the highest possible concavity property, thanks to the n-homogeneity of the volume in $\mathbb{R}^n$. In fact assume that

$$|(1-\lambda)A + \lambda B| \geq \left[(1-\lambda)\,|A|^p + \lambda\,|B|^p\right]^{1/p}.$$

Let us show that it necessarily implies $p \leq \frac{1}{n}$. Choosing $\lambda = 1/2$ and $A = \{0\}$ we get

$$\left|\frac{1}{2}\{0\} + \frac{1}{2}B\right| = \left|\frac{1}{2}B\right| = 2^{-n}\,|B|$$

$$\geq \left[\frac{1}{2}\,|\{0\}|^p + \frac{1}{2}\,|B|^p\right]^{1/p} = \left[\frac{1}{2}\,|B|^p\right]^{1/p} = 2^{-1/p}\,|B|.$$

It follows $2^{-n} \geq 2^{-1/p}$, that is $p \leq \frac{1}{n}$. In this sense, the BM inequality (2.1) is an optimal result.

We underline that the last observations concerning the optimal $(1/n)$-concavity of the volume in relation to the Minkowski combination in $\mathbb{R}^n$, and the equivalence between the BM inequality with its formulation (2.3) and (2.4), hold in the same way for any other operator having, just like the volume, the following properties: α-homogeneity in $\mathbb{R}^n$, increasing monotonicity with respect to set inclusion and invariance under rigid motions. Precisely one can prove

Lemma 2.1.2. *Let* $\lambda \in (0,1)$, $\alpha \neq 0$, $\mathcal{C}$ *be a subset of* $\mathcal{P}(\mathbb{R}^n)$ *(i.e the set of all subsets of* $\mathbb{R}^n$*) close under Minkowski combination* (1.2). *Let* $F : \mathcal{C} \longrightarrow [0, +\infty]$ *be a functional which is* α-*homogeneous, monotone increasing with respect to set inclusion and invariant under rigid motions. Suppose moreover that* $F^{1/\alpha}$ *is concave in relation to the Minkowski combination, i.e.*

$$F\left((1-\lambda)A + \lambda B\right)^{1/\alpha} \geq (1-\lambda)F(A)^{1/\alpha} + \lambda F(B)^{1/\alpha}, \qquad (2.5)$$

for all $A, B \in \mathcal{C}$. *Then* (2.5) *is equivalent to*

$$F\left((1-\lambda)A + \lambda B\right) \geq F(A)^{1-\lambda}F(B)^{\lambda} \quad \text{and} \quad F\left((1-\lambda)A + \lambda B\right) \geq \min\left\{F(A), F(B)\right\},$$

and $\frac{1}{\alpha}$ *is the highest possible concavity exponent for* F.

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The proof, which we omit, is very similar to those one provided for the BM inequality. Since (2.5) is a statement which clearly reminds the BM inequality (it coincides with the BM inequality choosing the volume as the functional F), we introduce the next definition, following Colesanti [16].

Definition 2.1.6. Let F be an operator which satisfies the assumptions of Lemma 2.1.2. If in addition (2.5) holds, then we say that F satisfies a Brunn-Minkowski inequality.

Many variational functionals satisfy a BM inequality: for instance see [6, 7, 9, 11, 16–18, 34, 45, 46].

2.2 A sketch of the proof

The BM inequality, in its full generality, can be obtained through several proofs, using different techniques: for instance it can be derived using the Steiner symmetrization. For the sake of clearness we reproduce a sketch of the proof given by Hadwiger and Ohmann in [32]. This proof, which provides the BM inequality for bounded measurable sets, is surprising simple and elegant. In fact only two tools are essentially required: the classical arithmetic–geometric mean inequality (1.5) to prove BM inequality for parallelepipeds, and a trick to deduce the BM inequality for finite unions of parallelepipeds, arguing by induction.

Proof of the BM inequality for bounded sets. As noticed in the previous paragraph, without loss of generality we can suppose that $|A|, |B| \in (0, +\infty)$. First we prove the result when $A, B \subset \mathbb{R}^n$ are parallelepipeds (i.e. n-dimensional rectangles) whose sides are parallel to the coordinate hyperplanes. Indicating with a_j and b_j, respectively, the lengths of the sides of A and B (in the j-th coordinate direction), trivially

$$|(1 - \lambda)A| = \prod_{j=1}^{n} (1 - \lambda)a_j, \qquad |\lambda B| = \prod_{j=1}^{n} \lambda b_j,$$

since $(1 - \lambda)A$ and λB are the corresponding parallelepipeds whose sides are rescaled by the factors $1 - \lambda$ and λ. On the other hand also their Minkowski convex combination $(1 - \lambda)A + \lambda B$ is a parallelepiped whose sides are parallel to the coordinate hyperplanes, this time having lengths $(1 - \lambda)a_j + \lambda b_j$. Therefore

$$|(1 - \lambda)A + \lambda B| = \prod_{j=1}^{n} \left[(1 - \lambda)a_j + \lambda b_j \right].$$

Now compare the volumes of these three parallelepipeds, noticing that (2.1), by n-homogeneity of the volume in $\mathbb{R}^n$, is equivalent to

$$\frac{|(1 - \lambda)A|^{1/n}}{|(1 - \lambda)A + \lambda B|^{1/n}} + \frac{|\lambda B|^{1/n}}{|(1 - \lambda)A + \lambda B|^{1/n}} \leq 1, \quad \text{i.e.}$$

$$\left[\prod_{j=1}^{n}\frac{(1-\lambda)a_j}{(1-\lambda)a_j+\lambda b_j}\right]^{1/n}+\left[\prod_{j=1}^{n}\frac{\lambda b_j}{(1-\lambda)a_j+\lambda b_j}\right]^{1/n}\leq 1.$$

Now the arithmetic–geometric mean inequality (1.5) yields

$$\left[\prod_{j=1}^{n}\frac{(1-\lambda)a_j}{(1-\lambda)a_j+\lambda b_j}\right]^{1/n}+\left[\prod_{j=1}^{n}\frac{\lambda b_j}{(1-\lambda)a_j+\lambda b_j}\right]^{1/n}$$

$$\leq\frac{1}{n}\sum_{j=1}^{n}\frac{(1-\lambda)a_j}{(1-\lambda)a_j+\lambda b_j}+\frac{1}{n}\sum_{j=1}^{n}\frac{\lambda b_j}{(1-\lambda)a_j+\lambda b_j}=\frac{1}{n}\sum_{j=1}^{n}\frac{(1-\lambda)a_j+\lambda b_j}{(1-\lambda)a_j+\lambda b_j}=1.$$

This gives the BM inequality for boxes (i.e. parallelepipeds whose sides are parallel to the coordinate hyperplanes).

If A and B are finite unions of boxes, we argue by induction on the number of boxes, using a trick called the *Hadwiger-Ohmann cut*, as follows. Without loss of generality (possibly translating A) we can assume that a coordinate hyperplane, say $\{x_n=0\}$, separates two boxes in A. With A_+ and A_- denote the unions of the boxes formed by intersecting the boxes in A with the half-spaces $\{x_n\geq 0\}$ and $\{x_n\leq 0\}$, respectively. Analogously define B_+ and B_-. Possibly translating B (if necessary) it holds

$$\frac{|A_+|}{|A|}=\frac{|B_+|}{|B|},\tag{2.6}$$

and consequently

$$\frac{|A_-|}{|A|}=\frac{|B_-|}{|B|},\tag{2.7}$$

since $|A_+|+|A_-|=|A|$ and $|B_+|+|B_-|=|B|$. Furthermore

$$(1-\lambda)A+\lambda B\supseteq[(1-\lambda)A_++\lambda B_+]\cup[(1-\lambda)A_-+\lambda B_-].\tag{2.8}$$

Observe that $A_++B_+\subseteq\{x_n\geq 0\}$, $A_-+B_-\subseteq\{x_n\leq 0\}$, and that the numbers of boxes in $A_+\cup B_+$ and $A_-\cup B_-$ are both smaller, by construction, than the number of boxes in $A\cup B$. By induction on the latter number of boxes, and by (2.6), (2.7) and (2.8) we derive

$$|(1-\lambda)A+\lambda B|\geq|(1-\lambda)A_++\lambda B_+|+|(1-\lambda)A_-+\lambda B_-|$$

$$\geq\left[(1-\lambda)|A_+|^{1/n}+\lambda|B_+|^{1/n}\right]^n+\left[(1-\lambda)|A_-|^{1/n}+\lambda|B_-|^{1/n}\right]^n$$

$$=\left[(1-\lambda)|A_+|^{1/n}+\lambda\frac{|B|^{1/n}}{|A|^{1/n}}|A_+|^{1/n}\right]^n+\left[(1-\lambda)|A_-|^{1/n}+\lambda\frac{|B|^{1/n}}{|A|^{1/n}}|A_-|^{1/n}\right]^n$$

$$=|A_+|\left[(1-\lambda)+\lambda\frac{|B|^{1/n}}{|A|^{1/n}}\right]^n+|A_-|\left[(1-\lambda)+\lambda\frac{|B|^{1/n}}{|A|^{1/n}}\right]^n$$

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$$= (|A_+| + |A_-|) \left[(1-\lambda) + \lambda \frac{|B|^{1/n}}{|A|^{1/n}} \right]^n = |A| \left[(1-\lambda) + \lambda \frac{|B|^{1/n}}{|A|^{1/n}} \right]^n$$

$$= \left[(1-\lambda) |A|^{1/n} + \lambda |B|^{1/n} \right]^n .$$

The BM inequality is established for finite unions of boxes.

The next step regards bounded measurable sets, which can be approximated by sequences of finite unions of boxes. Given A, B bounded measurable sets, it is well known that

$$|A| = \sup \left\{ |P| : \ P \text{ finite union of boxes such that } \overline{P} \subseteq A \right\},$$

and the corresponding property for $|B|$. In particular there exist two sequences $\{P_k\}_{k \in \mathbb{N}}$, $\{Q_k\}_{k \in \mathbb{N}}$ of finite unions of boxes such that $\overline{P_k} \subseteq A$, $\overline{Q_k} \subseteq B$ for every k, and

$$\lim_{k \to +\infty} |P_k| = |A|, \qquad \lim_{k \to +\infty} |Q_k| = |B|.$$

We have already derived the BM inequality for finite unions of boxes, thus for any $k \in \mathbb{N}$

$$(1-\lambda) |P_k|^{1/n} + \lambda |Q_k|^{1/n} \le |(1-\lambda)P_k + \lambda Q_k|^{1/n}$$

Passing to the limsup in the latter, for $k \to +\infty$, we obtain

$$(1-\lambda) |A|^{1/n} + \lambda |B|^{1/n} \le \limsup_{k \to +\infty} |(1-\lambda)P_k + \lambda Q_k|^{1/n}$$

$$\le \sup \left\{ |P|^{1/n} : \ P \text{ finite union of boxes such that } \overline{P} \subseteq (1-\lambda)A + \lambda B \right\}$$

$$= |(1-\lambda)A + \lambda B|^{1/n} ,$$

that is the desired BM inequality. $\qquad\square$

Finally (we omit this last part) the assumption that the sets are bounded can be removed (we refer to [13], Chapter 2). The complete generalization of the BM inequality regards nonempty sets not necessarily measurable: the inequality continues to hold considering the outer or inner Lebesgue measure in $\mathbb{R}^n$ in place of the usual Lebesgue measure.

2.3 A remarkable consequence: the isoperimetric inequality for convex bodies

The BM inequality has strong and unexpected relations with many other fundamental analytic and geometric inequalities: for instance with the isoperimeric inequality and Sobolev inequalities. Let us discuss how the BM inequality quickly yields the classical isoperimetric inequality for convex bodies. The well known isoperimetric principle states: *among all the sets of fixed surface area, the ball maximises the volume.* In rigorous terms, it can be formulated in the realm of convex bodies as

Proposition 2.3.1 (Isoperimetric inequality for convex bodies).
There exists a dimensional constant $c(n)$ such that for every $K \in \mathcal{K}^n$ holds

$$|K|^{\frac{n-1}{n}} \leq c(n) \cdot \mathcal{H}^{n-1}\left(\partial K\right); \tag{2.9}$$

in addition equality holds in (2.9) *if and only if K is a ball.*

Proof. Adopting the Minkowski's definition of the surface area, we know that

$$\mathcal{H}^{n-1}\left(\partial K\right) = \lim_{\epsilon \to 0} \frac{|K + \epsilon B_n| - |K|}{\epsilon}, \tag{2.10}$$

where B_n is the closed n-dimensional unit ball in $\mathbb{R}^n$. Notice that the BM inequality (2.2) yields

$$|K + \epsilon B_n|^{1/n} \geq |K|^{1/n} + |\epsilon B_n|^{1/n} = |K|^{1/n} + \epsilon\,|B_n|^{1/n}. \tag{2.11}$$

Consider the function

$$v(\epsilon) = |K + \epsilon B_n|.$$

By (2.10) and (2.11)

$$\left(\frac{d}{d\epsilon}v\right)(0) = \lim_{\epsilon \to 0} \frac{v(\epsilon) - v(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{|K + \epsilon B_n| - |K|}{\epsilon} = \mathcal{H}^{n-1}\left(\partial K\right),$$

$$\left(\frac{d}{d\epsilon}v^{1/n}\right)(0) = \lim_{\epsilon \to 0} \frac{v^{1/n}(\epsilon) - v^{1/n}(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{|K + \epsilon B_n|^{1/n} - |K|^{1/n}}{\epsilon} \geq |B_n|^{1/n}.$$

Consequently

$$\frac{1}{n} \cdot v(0)^{\frac{1}{n}-1} \cdot \left[\left(\frac{d}{d\epsilon}v\right)(0)\right] = \left(\frac{d}{d\epsilon}v^{1/n}\right)(0) \geq |B_n|^{1/n}, \qquad \text{i.e.}$$

$$\frac{1}{n}\,|K|^{\frac{1-n}{n}} \cdot \mathcal{H}^{n-1}\left(\partial K\right) \geq |B_n|^{1/n}.$$

Then every convex body K satisfies

$$|K|^{\frac{n-1}{n}} \leq \frac{\mathcal{H}^{n-1}\left(\partial K\right)}{n\,|B_n|^{1/n}},$$

i.e. (2.9) with $c(n) = \left(n\,|B_n|^{1/n}\right)^{-1}$.

Anticipating the equality conditions of the BM inequality, stated in Proposition 2.4.2 of the next section, we can claim that equality holds in (2.11) if and only if $K = B_n$ (up to homotheties). Since (2.11) is the unique inequality in this proof, if K is a ball then equality holds in (2.9). Vice versa the ball is the only convex body for which equality holds in (2.9) (we omit this technical part: see [29] for a complete proof). $\square$

In particular, for $n = 2$ in Proposition 2.3.1, we recognize the classical isoperimetric inequality in the plane:

$$L^2 \geq 4\pi A,$$

where A is the area of a (convex) domain enclosed by a curve of length L. In such a case $c(2) = (2\sqrt{\pi})^{-1}$, according to $c(n) = \left(n\,|B_n|^{1/n}\right)^{-1}$.

2.4 Equality conditions and the stability question

To investigate the stability question, it is important to know when the BM inequality (2.1) becomes an equality. A very careful examination of the proof provides the following equality conditions (first we state them in the assumption $|A|, |B| \in (0, +\infty)$, more interesting for us).

Proposition 2.4.1 (Equality case in BM inequality, when $|A|, |B| \in (0, +\infty)$)**.**
Let $\lambda \in (0,1)$, let $A, B \subset \mathbb{R}^n$ such that $|A|, |B| \in (0, +\infty)$ and $(1-\lambda)A + \lambda B$ is measurable. Then equality holds in the BM inequality, namely

$$|(1-\lambda)A + \lambda B|^{1/n} = (1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}, \qquad (2.12)$$

if and only if there exist a convex set $K \subseteq \mathbb{R}^n$, $v_1, v_2 \in \mathbb{R}^n$ and $\lambda_1, \lambda_2 > 0$ such that

$$\lambda_1 A + v_1 \subseteq K, \quad \lambda_2 B + v_2 \subseteq K, \quad |K \setminus (\lambda_1 A + v_1)| = |K \setminus (\lambda_2 B + v_2)| = 0. \quad (2.13)$$

We remark that Proposition 2.4.1 may be rephrased by saying that equality holds in (2.1) if and only if the involved sets are convex and homothetic sets from which sets of measure zero have been removed.
To clarify this characterization we exhibit two similar examples in $\mathbb{R}^2$, which have an opposite behaviour with respect to BM. Let $\lambda \in (0,1)$. If $A = B$ is a circle deprived of an internal point, then equality in (2.1) holds. Vice versa if $A = B$ is the union of a circle and an external point, then equality in (2.1) is not satisfied. In both situations the right hand side of (2.1) coincide with the volume of the circle raise to the power $1/n$, but only in the first situation the left hand side has the same value. In fact when $A = B$ is a circle deprived of an internal point, then $(1-\lambda)A + \lambda B$ is the whole circle and nothing more... while, in the second situation, $(1-\lambda)A + \lambda B$ contains also a small circle (a rescaling of the original circle of ratio $\max\{1-\lambda, \lambda\}$) having the external point as center. These so different behaviours intrinsically depend on the definition (1.2) of Minkowski combination.

In general assumption, there are only three situations in which equality holds in the BM inequality. We summarize these equality conditions and refer to [13], Theorem 8.3.1, in which the authors prove the following equality conditions for the BM inequality (2.2). In their statements $|\cdot|$ possibly means the inner Lebesgue measure of the involved sets; for simplicity, let's assume the measurability of the involved sets.

Proposition 2.4.2 (Equality conditions for BM inequality, Theorem 8.3.1 in [13])**.**
Given $\lambda \in (0,1)$, let $A, B \subset \mathbb{R}^n$ be nonempty measurable sets such that $(1-\lambda)A + \lambda B$ is measurable and $|(1-\lambda)A + \lambda B|$ is finite. Then

$$|(1-\lambda)A + \lambda B|^{1/n} = (1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}$$

holds only in these three cases:
(1) $|(1-\lambda)A + \lambda B| = 0$;

(2)   A or B consists of only one point;
(3)   A, B   are homothetic convex bodies from which sets of zero measure may have been removed.

Note that (3) is exactly the equality condition of Proposition 2.4.1, i.e. the one in the assumptions $|A|, |B| \in (0, +\infty)$, the meaningful case. Instead the conditions (1) and (2) trivially imply equality in the BM inequality (see Remark 2.1.2). About (2), note that if $|A| \in (0, +\infty)$ and B contains at least two elements, i.e. $\{b_1, b_2\} \subseteq B$, equality cannot hold in the BM inequality. Indeed in such a case $(1 - \lambda)A + \lambda B$ contains two different copies of $(1 - \lambda)A$, precisely $(1 - \lambda)A + \lambda b_1$ and $(1 - \lambda)A + \lambda b_2$. This yields the strict BM inequality. See [13] for more details and further comments.

When dealing with a rigid inequality, a natural question arises about the *stability of the equality case*; here the question at hand is the following: if $|A|, |B| \in (0, +\infty)$ and we are close to equality (2.12), i.e. if

$$\delta = |(1 - \lambda)A + \lambda B|^{1/n} - (1 - \lambda)|A|^{1/n} - \lambda|B|^{1/n} \geq 0$$

is sufficiently small, must the sets A, B be close (in some suitable sense) to satisfy (2.13)? And in such a case, is it possible to estimate, in terms of the perturbation δ, the closeness of A and B from the equality condition (2.13)?

Both the questions have affirmative answer. In the next sections we mention some stability results for the BM inequality, which will be useful in Chapter 5 and Chapter 7.

2.5   Some stability results for convex sets

The stability of BM inequality was first investigated and solved in the class of convex sets, see for instance [22, 24, 26, 27, 31, 48].
Let us recall the stability results due to Groemer [31] and Figalli, Maggi, Pratelli [26, 27]. Their statements represent quantitative versions of the BM inequality (2.1), which hold in the family $\mathcal{K}^n$ of the n-dimensional convex bodies in $\mathbb{R}^n$.
    In order to present these results, we introduce the following notations.
Given K_0, K_1 subsets of $\mathbb{R}^n$ having positive Lebesgue measure, we set

$$\nu_j = |K_j|^{1/n}, \ M = \max\{\nu_0, \nu_1\}, \ m = \min\{\nu_0, \nu_1\}, \ \tilde{d} = \max\left\{\frac{d(K_0)}{\nu_0}, \frac{d(K_1)}{\nu_1}\right\},$$

where $d(K_j)$ indicates the diameter of K_j. Recall that the Hausdorff distance $H(J, L)$ between two sets $J, L \subset \mathbb{R}^n$ is defined as follows:

$$H(K, L) = \inf\{r \geq 0 : \ J \subseteq L + rB_n, \ L \subseteq J + rB_n\},$$

where B_n is the closed unit ball in $\mathbb{R}^n$. Then we set

$$H_0(J, L) = H(\tau_0 J, \tau_1 L), \tag{2.14}$$

where τ_0, τ_1 are two homotheties (i.e. translation plus dilation) such that $|\tau_0 J| = |\tau_1 L| = 1$ and such that the centroids of $\tau_0 J$ and $\tau_1 L$ coincide.

We also recall that the relative asymmetry of two sets $J, L \subset \mathbb{R}^n$ is defined as follows:

$$\mathcal{A}(J, L) = \inf_{x \in \mathbb{R}^n} \left\{ \frac{|J \Delta (x + \mu L)|}{|J|}, \ \mu = \left(\frac{|J|}{|L|} \right)^{\frac{1}{n}} \right\}, \tag{2.15}$$

where Δ denotes the operation of symmetric difference, i.e. $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

The first result, due to Groemer, is written in terms of the Hausdorff distance between (two suitable homothetic copies of) A and B.

Proposition 2.5.1. *Let $n \geq 2$, $\lambda \in (0,1)$, $A, B \in \mathcal{K}^n$. Then*

$$|(1-\lambda)A + \lambda B| \geq \left[(1-\lambda) |A|^{1/n} + \lambda |B|^{1/n} \right]^n \left(1 + \eta_n H_0(A, B)^{n+1} \right),$$

where

$$\eta_n = \left(\gamma_n \left(\frac{M}{m} \frac{1}{\sqrt{\lambda(1-\lambda)}} + 2 \right) \tilde{d} \right)^{-n-1}, \tag{2.16}$$

H_0 *is defined as in (2.14) and*

$$\gamma_n = \left(1 + \frac{1}{3} 2^{-13} \right) 3^{\frac{n-1}{n}} 2^{\frac{n+2}{n+1}} n < 6.00025 n.$$

The second quantitative proposition of the BM inequality, due to Figalli, Maggi, Pratelli, is written in terms of the relative asymmetry of A and B.

Proposition 2.5.2. *Let $\lambda \in (0,1)$, $A, B \in \mathcal{K}^n$. Then*

$$|(1-\lambda)A + \lambda B| \geq \left[(1-\lambda) |A|^{1/n} + \lambda |B|^{1/n} \right]^n \left(1 + \frac{nm}{\Lambda M} \left(\frac{\mathcal{A}(A, B)}{\theta_n} \right)^2 \right),$$

where $\Lambda = \max \left\{ \frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda} \right\}$, $\mathcal{A}$ is defined in (2.15) and θ_n is a positive constant depending on dimension n with polynomial growth. In particular

$$\theta_n \leq \frac{362 n^7}{\left(2 - 2^{\frac{n-1}{n}} \right)^{3/2}}. \tag{2.17}$$

2.6 A meaningful stability result without convexity assumptions

Let us examine some general stability results for the BM inequality, which do not require the convexity of the involved sets.

In Chapter 6, the following sharp stability result for the BM inequality in $\mathbb{R}$ (see [25] for more details) will be useful.

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Proposition 2.6.1 (One-dimensional BM stability). *Let $A, B \subset \mathbb{R}$ be measurable. If*

$$|A + B| < |A| + |B| + \delta \qquad \text{for some} \quad \delta \leq \min\{|A|, |B|\}$$

(where $|\cdot|$ means outer measure if $A + B$ is not measurable), then there exist two intervals $I, J \subset \mathbb{R}$ such that

$$A \subseteq I, \qquad B \subseteq J, \qquad |I \setminus A| \leq \delta, \qquad |J \setminus B| \leq \delta.$$

In particular it holds $|conv(A) \setminus A| \leq \delta$ and $|conv(B) \setminus B| \leq \delta$.

Notice that it is not required the convexity of the sets A, B. It is also interesting to notice that the result holds only under the assumption that δ is sufficiently small, namely $\delta \leq \min\{|A|, |B|\}$. This smallness assumption on δ is necessary, as it can be easily seen by the following example: let $L \gg 1$ (i.e. greater than 1 and big enough) and

$$A = B = [0, 1] \cup \{L\}.$$

Then it is easily checked that

$$|A + B| = |[0, 2] \cup [L, L+1] \cup \{2L\}| = 3, \qquad |A| + |B| = 2,$$

while $|conv(A) \setminus A| = L - 1$ can be arbitrarily large. Hence the result is false for $\delta > 1 = \min\{|A|, |B|\}$, despite the assumption $|A + B| < |A| + |B| + \delta$ is satisfied.

Remark 2.6.2. Recall that the question we are trying to address, for $n \geq 2$, is the following: assume that (2.1) is almost an equality; is it true that both A and B are almost convex, and that actually they are close to the same convex set? Notice that this question has two statements in it. Indeed, we are wondering if the error in the BM inequality, namely

$$\delta = |(1 - \lambda)A + \lambda B|^{1/n} - (1 - \lambda)|A|^{1/n} - \lambda|B|^{1/n} \geq 0$$

is able to control how far A and B are from their respective convex hulls and how much the shapes of A and B differ each other.

Recently Christ [14, 15] addressed the investigation of the stability without convexity assumptions, and its qualitative results have been made quantitative by Figalli and Jerison in [25]; here is their result.

Proposition 2.6.3. *Let $n \geq 2$, and $A, B \subset \mathbb{R}^n$ be measurable sets with $|A| = |B| = 1$. Let $\lambda \in (0, 1)$, set $\tau = \min\{\lambda, 1 - \lambda\}$ and $S = (1 - \lambda)A + \lambda B$. If*

$$|S| \leq 1 + \delta \tag{2.18}$$

for some $\delta \leq e^{-M_n(\tau)}$, then there exists a convex $K \subset \mathbb{R}^n$ such that, up to a translation,

$$A \cup B \subseteq K \qquad \text{and} \qquad |K \setminus A| + |K \setminus B| \leq \tau^{-N_n} \delta^{\sigma_n(\tau)}.$$

The constant N_n can be explicitly computed and we can take

$$M_n(\tau) = \frac{2^{3^{n+2}} n^{3^n} |\log \tau|^{3^n}}{\tau^{3^n}}, \qquad \sigma_n(\tau) = \frac{\tau^{3^n}}{2^{3^{n+1}} n^{3^n} |\log \tau|^{3^n}}.$$

Notice that the result holds under the assumption that δ is sufficiently small, namely $\delta \le e^{-M_n(\tau)}$. A smallness assumption on δ is again crucial (see [25]). Anyway, this proposition successfully solves the question of Remark 2.6.2: if the error δ is sufficiently small, that is A and B are sufficiently close to satisfy equality in BM inequality, then the closeness of A and B from their respective convex hulls and the nearness between their shapes can be estimated in terms of δ.

For further use, we rewrite Proposition 2.6.3 without the normalization constraint $|A| = |B| = 1$.

Corollary 2.6.4. *Let $n \ge 2$ and $A, B \subset \mathbb{R}^n$ be measurable sets with $|A|, |B| \in (0, +\infty)$. Let $\lambda \in (0, 1)$, set $\tau = \min\{\lambda, 1 - \lambda\}$ and $S = (1 - \lambda)A + \lambda B$. If*

$$\frac{|S| - \left[(1 - \lambda)|A|^{1/n} + \lambda|B|^{1/n}\right]^n}{\left[(1 - \lambda)|A|^{1/n} + \lambda|B|^{1/n}\right]^n} \le \delta \tag{2.19}$$

for some $\delta \le e^{-M_n(\tau)}$, then there exist a convex $K \subset \mathbb{R}^n$ and two homothetic copies $\tilde{A}$ and $\tilde{B}$ of A and B such that

$$\tilde{A} \cup \tilde{B} \subseteq K \qquad and \qquad \left|K \setminus \tilde{A}\right| + \left|K \setminus \tilde{B}\right| \le \tau^{-N_n} \delta^{\sigma_n(\tau)}.$$

Proof. The proof is standard and we give it just for the sake of completeness. First set

$$\tilde{A} = \frac{A}{|A|^{1/n}}, \qquad \tilde{B} = \frac{B}{|B|^{1/n}}$$

so that $|\tilde{A}| = |\tilde{B}| = 1$. Then define

$$\tilde{S} := \mu\tilde{A} + (1 - \mu)\tilde{B} \qquad \text{with} \quad \mu = \frac{(1 - \lambda)|A|^{1/n}}{(1 - \lambda)|A|^{1/n} + \lambda|B|^{1/n}},$$

and observe that $|\tilde{S}| \ge 1$ by the Brunn-Minkowski inequality. It is easily seen that

$$\tilde{S} = \frac{S}{(1 - \lambda)|A|^{1/n} + \lambda|B|^{1/n}}.$$

Now we see that the hypothesis (2.18) holds for $\tilde{A}, \tilde{B}, \tilde{S}$, indeed by (2.19)

$$\left|\tilde{S}\right| - 1 = \frac{|S| - \left[(1 - \lambda)|A|^{1/n} + \lambda|B|^{1/n}\right]^n}{\left[(1 - \lambda)|A|^{1/n} + \lambda|B|^{1/n}\right]^n} \le \delta.$$

Proposition 2.6.3 applied to $\tilde{A}, \tilde{B}$ and $\tilde{S}$ implies the result and this concludes the proof. $\qquad \square$

Chapter 3

The Borell-Brascamp-Lieb inequalities (BBL)

3.1 Introduction

This thesis is primarily concerned with a family of inequalities called *Borell-Brascamp-Lieb inequalities*, which we recall hereafter. For simplicity of exposition we refer to them with the acronym "BBL" inequalities.

They can be viewed as functional or analytic versions of the Brunn-Minkoski inequality. In Section 3.2 we explain this link in details. To state the BBL inequalities we shall use the generalized means (1.3) introduced in Section 1.3.

Theorem 3.1.1 (BBL inequalities).
Let $0 < \lambda < 1$, $-\frac{1}{n} \leq p \leq +\infty$, and let $f, g, h : \mathbb{R}^n \longrightarrow [0, +\infty)$ be integrable functions such that

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda) \qquad \textit{for all } x, y \in \mathbb{R}^n. \tag{3.1}$$

Then

$$\int_{\mathbb{R}^n} h(x)\,dx \geq \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f(x)\,dx, \int_{\mathbb{R}^n} g(x)\,dx \,; \lambda\right). \tag{3.2}$$

Here the number $p/(np+1)$ has to be interpreted in the obvious way in the extremal cases, i.e. it is equal to $-\infty$ when $p = -1/n$ and to $1/n$ when $p = +\infty$.

The BBL inequalities were first proved (in a slightly different form) for $p > 0$ by Henstock and Macbeath (with $n = 1$) in [35] and by Dinghas in [21]. Then they were generalized by Brascamp and Lieb in [11] and by Borell in [8].

Surely (3.1) is relevant only when $f(x)$ and $g(y)$ are positive, otherwise their p-mean is zero and (3.1) is obviously satisfied, being h nonnegative. Hence (3.1) is equivalent to

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda) \quad \text{for all } x, y \text{ such that } f(x), g(y) > 0.$$

Though inequality (3.2) in itself is rather simple, the condition behind it, i.e. (3.1), is unusual as it is not a point-wise condition but involves the values of f, g and h at different points. We emphasize that the condition (3.1) is strong, since it requires that the function h is big enough compared to f and g: precisely h evaluated at the convex combination of any two points must be greater than or equal to the p-mean of f and g at those points. In other words (3.1) means

$$h \ \geq \ h_{p,\lambda}, \tag{3.3}$$

where $h_{p,\lambda}$ is the (p, λ)-convolution of f and g, defined in (1.6). Roughly speaking $h_{p,\lambda}$ is the smallest function satisfying (3.1): therefore it suffices to prove the theorem for $h = h_{p,\lambda}$, by monotony of the Lebesgue integral and (3.3).

Remark 3.1.2. First of all observe that (3.2) can be trivial in some special situations. A first case occurs when one among f and g is almost everywhere zero, i.e. $\int_{\mathbb{R}^n} f = 0$ or $\int_{\mathbb{R}^n} g = 0$, from which obviously follows (3.2) by the definition of generalized mean (a generalized mean of two nonnegative numbers is zero if at least one of them is zero, according to (1.3)). Another situation in which BBL inequality is trivial happens when $f = g$, because choosing $x = y$ in (3.1) we get $h(x) = h((1-\lambda)x + \lambda x) \geq \mathcal{M}_p(f(x), g(x); \lambda) = f(x)$, therefore by comparison $\int_{\mathbb{R}^n} h \geq \int_{\mathbb{R}^n} f = \mathcal{M}_q\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} f; \lambda\right)$ for every $q \in [-\infty, +\infty]$. Then, without loss of generality, suppose that f and g are not a.e. zero (i.e. their integral are positive) and that they are different from each other (i.e. the set $\{x : f(x) \neq g(x)\}$ has positive measure).

Also, if $\int_{\mathbb{R}^n} h = +\infty$, there is nothing to prove. Let us examine some cases in which the key assumption (3.1) forces $\int_{\mathbb{R}^n} h$ to diverge making (3.2) a trivial result. For example if $p > 0$ and at least one among $\int_{\mathbb{R}^n} f$ and $\int_{\mathbb{R}^n} g$ diverges, then $\int_{\mathbb{R}^n} h$ also diverges thanks to (3.1), as it is easily seen: let y_0 such that $g(y_0) = \epsilon > 0$; then (3.1) yields, for any x satisfying $f(x) > 0$,

$$h((1-\lambda)x + \lambda y_0) \geq \mathcal{M}_p(f(x), \epsilon; \lambda) \geq \max\left\{(1-\lambda)^{1/p} f(x), \ \lambda^{1/p}\epsilon\right\}, \tag{3.4}$$

where in the last inequality we use the positivity of the index p. Then, by comparison and using the change of variable $z = (1-\lambda)x + \lambda y_0$ (where y_0 is the element previously fixed), we get

$$\int_{\mathbb{R}^n} h(z) \ dz = (1-\lambda)^n \int_{\mathbb{R}^n} h\left((1-\lambda)x + \lambda y_0\right) \ dx$$

$$\geq (1-\lambda)^{\frac{np+1}{p}} \int_{\{x:\ f(x)>0\}} f(x) \ dx = (1-\lambda)^{\frac{np+1}{p}} \int_{\mathbb{R}^n} f(x) \ dx = +\infty.$$

Thus, for an index $p > 0$, we can reduce our study to the function $f, g \in L^1(\mathbb{R}^n) \setminus \{0\}$. Also with this restriction, if $|\{x : f(x) > 0\}| = +\infty$ or $|\{y : g(y) > 0\}| = +\infty$, then h necessarily diverges. Indeed (3.4) leads to

$$\int_{\mathbb{R}^n} h(z) \ dz = (1-\lambda)^n \int_{\mathbb{R}^n} h\left((1-\lambda)x + \lambda y_0\right) \ dx \geq (1-\lambda)^n \lambda^{1/p}\epsilon \, |\{x : f(x) > 0\}| = +\infty.$$

So the meaningful case of BBL inequality of index $p > 0$ occurs when $f \neq g \in L^1(\mathbb{R}^n) \backslash \{0\}$ and they are compactly supported functions. Instead if $p \leq 0$ the condition (3.1) does not ensure that $\int_{\mathbb{R}^n} h$ diverges, even if $\int_{\mathbb{R}^n} f$ (or $\int_{\mathbb{R}^n} g$) diverges or if their support sets have infinite measure.

Remark 3.1.3. When the functions h, f, g coincide, (3.1) clearly represents an assumption of p-concavity, becoming

$$f((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), f(y); \lambda)$$

for the fixed $\lambda \in (0, 1)$ of the assumptions. Actually the latter inequality does not force f to be p-concave, because it is required only for the fixed λ, not for every $\lambda \in (0, 1)$. However if f is continuous, then

$$f((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), f(y); \lambda)$$

for a fixed $\lambda \in (0, 1)$ yields the same inequality for every $\lambda \in (0, 1)$. Thus if f, g, h coincide with a continuous function, (3.1) ensures the p-concavity of the involved functions. Note that if $f = g = h$ almost everywhere, then (3.2) is trivially satisfied and it is an equality, since $\int_{\mathbb{R}^n} f = \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} f; \lambda\right)$. We emphasize that, up to homotheties of the involved function, this one is essentially the unique situation in which equality holds in the BBL inequality. Indeed we will able to prove, in Chapter 4, that equality holds in BBL of index p if and only if the three functions f, g, h coincide almost everywhere (and up to suitable homotheties) with a same p-concave function.

The case $p = 0$ in the BBL inequalities (Theorem 3.1.1) is known as *Prékopa-Leindler inequality*, as it was previously proved by Prékopa [41] and Leindler [37] (later rediscovered by Brascamp and Lieb in [10]). From now on, we often call it "PL" inequality. It is probably the most famous one among the BBL inequalities and the unique case in which the indices p and $p/(np+1)$ coincide (being both zero). Let us state it separately.

Proposition 3.1.4 (PL inequality)**.**
Let $0 < \lambda < 1$, f, g, h be nonnegative measurable functions defined in $\mathbb{R}^n$ such that

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)^\lambda \qquad for\ all\ \ x, y \in \mathbb{R}^n.$$

Then

$$\int_{\mathbb{R}^n} h\, dx \geq \left(\int_{\mathbb{R}^n} f\, dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\, dx\right)^\lambda. \tag{3.5}$$

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As explained in [29], in Theorem 3.1.1 we have to place greater emphasis on the negative values of p, and in particular on the critical value $p = -1/n$. Indeed is rather simple, starting with the BBL inequality of index $p = -1/n$, to derive all the other cases. On the contrary, at our knowledge, there is not a direct proof which allows to prove the BBL inequality of index $p = -1/n$ by means of another BBL inequality.

The proof of Corollary 1.1 in [19] shows that Theorem 3.1.1 for $p = -1/n$ implies Theorem 3.1.1 for all $p > -1/n$. This follows from a suitable rescaling of the involved functions f, g, h, together with the obvious observation that

$$\frac{1}{\mathcal{M}_p(a, b; \lambda)} = \mathcal{M}_{-p}\left(\frac{1}{a}, \frac{1}{b}; \lambda\right), \tag{3.6}$$

and the following technical lemma (for a proof we refer to [29], Lemma 10.1), consequence of Hölder's inequality (see [33], page 24).

Lemma 3.1.1. *Let $0 < \lambda < 1$, and let a, b, c, d be nonnegative numbers. Let $q \in \mathbb{R}$, $p \in \mathbb{R} \cup \{+\infty\}$. If $p + q \geq 0$, then*

$$\mathcal{M}_p(a, b; \lambda)\mathcal{M}_q(c, d; \lambda) \geq \mathcal{M}_s(ac, bd; \lambda),$$

where

$$s = \begin{cases} 0 & \text{if} \quad p = q = 0, \\ -\infty & \text{if} \quad p = -q \neq 0, \\ \frac{pq}{p+q} & \text{otherwise}. \end{cases}$$

For the sake of clarity we present the proof of Corollary 1.1 in [19], in a simplified version. Indeed the proofs in [19] are more general, because they concern not only the usual version of the BBL inequalities in Euclidean spaces, i.e. Theorem 3.1.1, but a generalization in the framework of Riemannian manifolds.

Proposition 3.1.5. *BBL inequality for $p = -1/n$ implies all the other BBL inequalities; in other words, Theorem 3.1.1 for $p = -1/n$ implies Theorem 3.1.1 for all $p \in (-1/n, +\infty]$.*

Proof. Beforehand notice that the latter proposition is obvious if $\int_{\mathbb{R}^n} f \, dx = \int_{\mathbb{R}^n} g \, dx = I$, since in this case every generalized mean of the two integrals is I, so (3.2) is the same for every p, namely $\int_{\mathbb{R}^n} h \, dx \geq I$, while the assumption (3.1) in the case $p = -1/n$ is clearly the weakest one, due to the the monotonicity (1.4) of p-means. Then suppose that the integrals of f and g, respectively denoted by F and G, are positive and different each other (if $F = 0$ or $G = 0$, then (3.2) is a trivial consequence of $\mathcal{M}_{\frac{p}{np+1}}(F, G; \lambda) = 0$).

Let f, g, h satisfy the assumptions of Theorem 3.1.1 for a fixed index $p \geq -1/n$. In particular it holds (3.1), i.e. $h((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda)$. Normalize $\overline{f} = f/F$, $\overline{g} = g/G$ and define

$$\overline{h} = \frac{h}{\mathcal{M}_{\frac{p}{np+1}}(F, G; \lambda)}.$$

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If we show that $\overline{f}, \overline{g}, \overline{h}$ satisfy (3.1) for $p = -1/n$, the desired inequality $\int_{\mathbb{R}^n} \overline{h}\, dx \geq 1$ (clearly equivalent to $\int_{\mathbb{R}^n} h\, dx \geq \mathcal{M}_{\frac{p}{np+1}}(F, G; \lambda)$), follows by the case $p = -1/n$, since (3.2) in such a case implies

$$\int_{\mathbb{R}^n} \overline{h}\, dx \geq \min\left\{ \int_{\mathbb{R}^n} \overline{f}\, dx, \int_{\mathbb{R}^n} \overline{g}\, dx \right\} = \min\{1, 1\} = 1$$

By (3.6) and (3.1)

$$\overline{h}((1-\lambda)x + \lambda y) = \mathcal{M}_{-\frac{p}{np+1}}\left(\frac{1}{F}, \frac{1}{G}; \lambda\right) h((1-\lambda)x + \lambda y)$$

$$\geq \mathcal{M}_{-\frac{p}{np+1}}\left(\frac{1}{F}, \frac{1}{G}; \lambda\right) \mathcal{M}_p(f(x), g(y); \lambda)$$

$$\geq \mathcal{M}_{-\frac{1}{n}}\left(\frac{f(x)}{F}, \frac{g(y)}{G}; \lambda\right) = \mathcal{M}_{-\frac{1}{n}}\left(\overline{f}(x), \overline{g}(y); \lambda\right),$$

where Lemma 3.1.1 has been applied in the last inequality with $q = -\frac{p}{np+1}$ and $s = -1/n$ if $p \neq 0$ (or $s = 0$ if $p = 0$: however $\mathcal{M}_0 \geq \mathcal{M}_{-\frac{1}{n}}$ by monotonicity of p-means). Therefore $\overline{f}, \overline{g}, \overline{h}$ satisfy (3.1) for $p = -1/n$, so we conclude that $\int_{\mathbb{R}^n} h\, dx \geq \mathcal{M}_{\frac{p}{np+1}}(F, G; \lambda)$, namely Theorem 3.1.1 holds for the index p. $\qquad\square$

Remark 3.1.6. In fact in the previous proof we have applied the BBL inequality of index $p = -1/n$ only to the functions $\overline{f}, \overline{g}, \overline{h}$, and let us emphasize that $\int_{\mathbb{R}^n} \overline{f}\, dx = 1 = \int_{\mathbb{R}^n} \overline{g}\, dx$ (since $\overline{f} = f/F$, $\overline{g} = g/G$). In particular the previous proof shows that it suffices to prove the BBL inequality for $p = -1/n$ and for $\overline{f}, \overline{g}, \overline{h}$ satisfying

$$\overline{h}((1-\lambda)x + \lambda y) \geq \mathcal{M}_{-1/n}(\overline{f}(x), \overline{g}(y); \lambda) \quad \text{and} \quad \int_{\mathbb{R}^n} \overline{f}\, dx = 1 = \int_{\mathbb{R}^n} \overline{g}\, dx,$$

in order to demonstrate all the BBL inequalities (including the extremal case $p = -1/n$), where f, g may have arbitrary integrals.

Let us summarize this fact, namely the strength of the normalized version of BBL for $p = -1/n$, as follows.

Corollary 3.1.7 (BBL inequality of index $-1/n$, with $\int f = \int g = 1$).
Let $0 < \lambda < 1$, and let f, g, h be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} f\, dx = 1 = \int_{\mathbb{R}^n} g\, dx, \qquad and$$

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_{-1/n}(f(x), g(y); \lambda) \qquad for\ all \quad x, y \in \mathbb{R}^n.$$

Then

$$\int_{\mathbb{R}^n} h\, dx \geq 1.$$

Furthermore this inequality implies all the BBL inequalities.

3.2 Equivalence with the BM inequality

One way to understand the importance of BBL inequalities is to set them in relation with the BM inequality, showing that the BBL inequalities represent its analytic counterparts. Let us clarify this intrinsic link.

The BM inequality follows immediately applying one of the BBL inequalities to the characteristic functions of the involved sets A, B and of their Minkowski combination. Denote by χ_A the characteristic function of a set A:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Precisely, Theorem 3.1.1 trivially implies (2.1) by applying (3.2) for $p = +\infty$ to the characteristic functions $f = \chi_A$, $g = \chi_B$, $h = \chi_{(1-\lambda)A+\lambda B}$, where A, B are subsets of $\mathbb{R}^n$ having positive measure (besides, the BM inequality is obvious if $|A| = 0$ or $|B| = 0$). Indeed $f = \chi_A$, $g = \chi_B$, $h = \chi_{(1-\lambda)A+\lambda B}$ satisfy (3.1) (in (3.1) the right hand side is not zero if and only if $x \in A, y \in B$, and in such a case the right and left side coincide with the value 1), and (3.2) for $p = +\infty$ claims

$$|(1-\lambda)A + \lambda B| = \int_{\mathbb{R}^n} h \geq \mathcal{M}_{1/n}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \,;\, \lambda\right) = \mathcal{M}_{1/n}\left(|A|, |B| \,;\, \lambda\right)$$

i.e. the BM inequality (2.1). Using the same characteristic functions, observe that an arbitrary BBL inequality of index $p \in \left[-\frac{1}{n}, +\infty\right]$ implies (2.1): again (3.1) is satisfied and (3.2), applied to these characteristic functions, claims the corresponding $\left(\frac{p}{np+1}\right)$-concavity of the volume in $\mathbb{R}^n$. In particular it yields its quasiconcavity (2.4), which is equivalent to the BM inequality (thanks to Lemma 2.1.1). The opposite implication, that is

$$\text{BM inequality yields BBL inequality} \tag{3.7}$$

is rather delicate. Note that this implication becomes intuitive as soon as one notices a correspondence between the graph of $h_{p,\lambda}^p$ (where $h_{p,\lambda}$ is (p, λ)-supremal convolution of f and g, defined in (1.6)) and the graphs of f^p and g^p: let $p \neq 0$; then, roughly speaking, the graph of $h_{p,\lambda}^p$ is obtained as the Minkowski combination (with coefficient λ) of the graphs of f^p and g^p; precisely we have (see Section 1.4)

$$A_{h_{p,\lambda}}^{(p)} = (1-\lambda)A_f^{(p)} + \lambda A_g^{(p)},$$

where these three subsets of $\mathbb{R}^{n+1}$ are defined as in Definition 1.4.2. In particular the BM inequality applied (in $\mathbb{R}^{n+1}$) to the sets $A_{h_{1,\lambda}}^{(1)}, A_f^{(1)}, A_g^{(1)}$ leads to the BBL inequality of index $p = 1$, because

$$\left|A_{h_{1,\lambda}}^{(1)}\right|^{\frac{1}{n+1}} \geq (1-\lambda)\left|A_f^{(1)}\right|^{\frac{1}{n+1}} + \lambda\left|A_g^{(1)}\right|^{\frac{1}{n+1}}$$

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clearly coincide with (3.2) for $p = 1$ (from which $\frac{n}{np+1} = \frac{1}{n+1}$), since

$$\left| A_f^{(1)} \right| = \int_{\mathbb{R}^n} f \; dx,$$

and the same property holds for $A_g^{(1)}$ and $A_{h_{1,\lambda}}^{(1)}$.

For the other indices $p \neq 1$ the implication (3.7) is far less direct and we refer to the next sections for this.

3.3 Four proofs of the BBL inequalities

Now our purpose is to show how so different techniques and approaches can lead to the BBL inequalities. We emphasize that all the four proofs have in common a key ingredient: the BM inequality. In fact it can be considered the seed, or the heart, of the BBL inequalities.

The first proof, described in Gardner's paper [29] (Theorem 10.2 therein), is by induction on the dimension n. It is based on the introduction of a suitable volume parameter t (in fact a 1-dimensional mass transportation) and on Lemma 3.1.1.

The second one is an application of optimal transportation theory, following the ideas of McCann [40] and Barthe [5] (and adopting a slighty different strategy). These authors (see the proofs 6.1.4 and 6.1.5 in [51]) proved PL inequality through arguments of mass transportation and using the arithmetic-geometric inequality. Replacing the arithmetic-geometric inequality with Lemma 3.1.1, these approaches can be generalized and allow us to prove BBL inequalities.

The third proof is less general, since it concerns only the BBL inequalities of indices $p \geq 0$ in a slightly different version, given in Proposition 3.3.2. In such a proof, due to Klartag [36], first we consider the cases $p = 1/s$ with $s \in \mathbb{N}$. The essential ingredient in Klartag's proof is the definition of suitable set $K_f, K_g, K_h \subset \mathbb{R}^{n+s}$, related to the involved functions f, g, h of Proposition 3.3.2, and the consequent application of the BM inequality to K_f, K_g, K_h. Then all the remaining cases (including $p = 0$) are treated by approximation. Let us point out that Klartag's proof, providing this direct link between BM and BBL inequalities, has been the crucial tool in order to establish our main result in [43], i.e. Theorem 7.1.1 in this thesis.

The last proof deals with a slightly different form of the BBL inequalities of index $p \leq 1$ for bounded compactly supported functions. It is an alternative and original proof. First we prove the PL inequality applying the BM inequality to certain sets, suitably related to the involved functions, as in the spirit of the third proof. Then a similar proof of the BBL inequality of index $p \leq 1$ relies on the *curvilinear extension of the Brunn-Minkowski inequality*, due to Uhrin [50].

3.3.1 Classical proof

First Proof of BBL inequalities, Theorem 3.1.1.
We argue by induction on the dimension n. First let $n = 1$. Without loss of generality

suppose that

$$\int_{\mathbb{R}} f(x)\ dx = F > 0 \qquad \text{and} \qquad \int_{\mathbb{R}} g(x)\ dx = G > 0,$$

otherwise (3.2) is trivial, since the right-hand side of (3.2) is zero if one among f and g has null integral. Define $u, v : [0, 1] \longrightarrow [-\infty, +\infty]$ such that $u(t)$ and $v(t)$ are the smallest numbers satisfying

$$\frac{1}{F} \int_{-\infty}^{u(t)} f(x)\ dx \ = \ \frac{1}{G} \int_{-\infty}^{v(t)} g(x)\ dx \ = t, \tag{3.8}$$

and observe that u, v are strictly increasing functions and so are differentiable almost everywhere. Taking the derivative of (3.8) with respect to t, it yields

$$\frac{f(u(t))u'(t)}{F} = \frac{g(v(t))v'(t)}{G} = 1 \qquad \text{for a.e.} \quad t.$$

Therefore for a.e. t

$$u'(t) = \frac{F}{f(u(t))}, \qquad v'(t) = \frac{G}{g(v(t))}.$$

Let

$$w(t) = (1 - \lambda)u(t) + \lambda v(t), \qquad t \in [0, 1].$$

For any t satisfying $f(u(t)) \cdot g(v(t)) \neq 0$, consider the derivative

$$w'(t) = (1 - \lambda)u'(t) + \lambda v'(t)$$

$$= (1 - \lambda)\frac{F}{f(u(t))} + \lambda \frac{G}{g(v(t))} = \mathcal{M}_1 \left(\frac{F}{f(u(t))}, \frac{G}{g(v(t))}; \lambda \right). \tag{3.9}$$

Then, by assumption (3.1), (3.9) and Lemma 3.1.1 with $q = 1$, we deduce

$$\int_{\mathbb{R}} h(x)\ dx \ \geq \ \int_0^1 h(w(t))w'(t)\ dt \ = \ \int_0^1 h\left((1 - \lambda)u(t) + \lambda v(t)\right) w'(t)\ dt$$

$$\geq \int_0^1 \mathcal{M}_p(f(u(t)), g(v(t)); \lambda) \cdot \mathcal{M}_1 \left(\frac{F}{f(u(t))}, \frac{G}{g(v(t))}; \lambda \right)\ dt$$

$$\geq \int_0^1 \mathcal{M}_{\frac{p}{p+1}} (F, G; \lambda)\ dt \ = \ \mathcal{M}_{\frac{p}{p+1}} (F, G; \lambda),$$

namely the desired result (3.2) when $n = 1$.

Now suppose that (3.2) is true for all natural numbers less than n. We want to prove (3.2) for the dimension n, so let f, g, h defined on $\mathbb{R}^n$ satisfying Theorem 3.1.1. For each $s \in \mathbb{R}$, define the function $h_s : \mathbb{R}^{n-1} \longrightarrow [0, \infty)$ as

$$h_s(z) = h(z, s) \qquad \text{for } z \in \mathbb{R}^{n-1}, \tag{3.10}$$

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and define f_s and g_s analogously. Let $x, y \in \mathbb{R}^{n-1}$, let $a, b \in \mathbb{R}$ and consider

$$c = (1 - \lambda)a + \lambda b.$$

Therefore, by definition (3.10) and (3.1),

$$h_c\left((1-\lambda)x + \lambda y\right) = h\left((1-\lambda)x + \lambda y, c\right) = h\left((1-\lambda)x + \lambda y, (1-\lambda)a + \lambda b\right)$$

$$= h\left((1-\lambda)(x,a) + \lambda(y,b)\right) \geq \mathcal{M}_p(f(x,a), g(y,b); \lambda) = \mathcal{M}_p(f_a(x), g_b(y); \lambda).$$

The inductive hypothesis (since h_c, f_a, g_b are nonnegative functions on $\mathbb{R}^{n-1}$ which satisfy (3.1)) implies

$$\int_{\mathbb{R}^{n-1}} h_c(x)\, dx \geq \mathcal{M}_{\frac{p}{(n-1)p+1}}\left(\int_{\mathbb{R}^{n-1}} f_a(x)\, dx, \int_{\mathbb{R}^{n-1}} g_b(x)\, dx \,; \lambda\right).$$

Finally define the following nonnegative functions $\mathcal{H}, \mathcal{F}, \mathcal{G}$ on $\mathbb{R}$, given by

$$\mathcal{H}(c) = \int_{\mathbb{R}^{n-1}} h_c(x)\, dx, \qquad \mathcal{F}(a) = \int_{\mathbb{R}^{n-1}} f_a(x)\, dx, \qquad \mathcal{G}(b) = \int_{\mathbb{R}^{n-1}} g_b(x)\, dx.$$

Then

$$\mathcal{H}(c) = \mathcal{H}((1-\lambda)a + \lambda b) \geq \mathcal{M}_{\frac{p}{(n-1)p+1}}\left(\mathcal{F}(a), \mathcal{G}(b); \lambda\right),$$

i.e. $\mathcal{H}, \mathcal{F}, \mathcal{G}$ satisfy (3.1) with the index

$$q = \frac{p}{(n-1)p+1}.$$

Then

$$\frac{q}{q+1} = \frac{p}{np+1},$$

and, by Fubini's Theorem and Theorem 3.1.1 applied to the one-dimensional functions $\mathcal{H}, \mathcal{F}, \mathcal{G}$, we conclude that

$$\int_{\mathbb{R}^n} h(x)\ dx = \int_{\mathbb{R}}\left(\int_{\mathbb{R}^{n-1}} h_c(z)\ dz\right) dc = \int_{\mathbb{R}} \mathcal{H}(c)\ dc$$

$$\geq \mathcal{M}_{\frac{q}{q+1}}\left(\int_{\mathbb{R}} \mathcal{F}(a)\ da, \int_{\mathbb{R}} \mathcal{G}(b)\ db; \lambda\right)$$

$$= \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}} \mathcal{F}(a)\ da, \int_{\mathbb{R}} \mathcal{G}(b)\ db; \lambda\right)$$

$$= \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^{n-1}} f_a(z)\ dz\right) da, \int_{\mathbb{R}}\left(\int_{\mathbb{R}^{n-1}} g_b(z)\ dz\right) db; \lambda\right)$$

$$= \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f(x)\, dx, \int_{\mathbb{R}^n} g(x)\, dx \,; \lambda\right).$$

$\square$

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3.3.2 A proof via optimal transportation theory

We recall basic notions and tools of the mass transportation theory, referring to Villani's book [51] for more details. Whenever T is a map from a measurable space X, equipped with a measure μ, to an arbitrary space Y, we denote by $T\#\mu$ the *push-forward* (or image measure) of μ by T. Explicitly,

$$(T\#\mu)[B] = \mu[T^{-1}(B)].$$

If $\nu = T\#\mu$, we say that T transports μ onto ν. Given two probability measure μ, ν, with respective support in measurable spaces X and Y, let $c(x,y)$ be a measurable *cost function* defined on $X \times Y$: the *Monge's optimal transportation problem* consists in

$$\text{minimizing} \quad \int_X c\left(x, T(x)\right) \, d\mu(x)$$

on the set of all measurable maps $T : X \longrightarrow Y$ such that $T\#\mu = \nu$.

The optimal T's, if they exist, are called *optimal transport*. From now on we consider $X = Y = \mathbb{R}^n$ and the quadratic transportation cost $c(x,y) = |x - y|^2$. The strict convexity of $c(x,y) = |x - y|^2$ guarantees existence and uniqueness of the solution to the Monge's problem, if μ, ν are absolutely continuous with respect to Lebesgue measure. Furthermore, one can geometrically characterize the optimal transport: it has to be the gradient of a convex function on $\mathbb{R}^n$. Precisely, Brenier's theorem states that there exists a unique optimal transport T, given by

$$T = \nabla\phi \qquad (\mu\text{-a.e.})$$

where ϕ is a convex function satisfying the *Monge-Ampère equation* (see Chapter 4 of [51])

$$f(x) = g(\nabla\phi(x)) \det(D^2\phi(x)), \tag{3.11}$$

where $d\mu(x) = f(x)dx$, $d\nu(y) = g(y)dy$ are the probability measures, absolutely continuous with respect to Lebesgue measure.

Second Proof of BBL inequalities, Theorem 3.1.1.
We use the strategy of Barthe (see the proof of the PL inequality in [51], Chapter 6) and adapt it to the extremal index $p = -1/n$. In fact it suffices to prove BBL inequality for $p = -1/n$, thanks to Proposition 3.1.5. Precisely, it is enough to prove Corollary 3.1.7, as explained in Remark 3.1.6. Therefore we may assume that $\int_{\mathbb{R}^n} f = 1 = \int_{\mathbb{R}^n} g$ and that h, f, g satisfy (3.1) for $p = -1/n$, namely

$$h((1 - \lambda)x + \lambda y) \geq \mathcal{M}_{-1/n}(f(x), g(y); \lambda) \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{3.12}$$

So we have just to prove that

$$\int_{\mathbb{R}^n} h \geq 1.$$

Identify the functions f and g with the corresponding probability densities and let $\mathcal{L}$ be the Lebesgue density on $[0, 1]^n$. The main idea consists in a linear interpolation between

the optimal transportation maps from f to $\mathcal{L}$ on one hand, from g to $\mathcal{L}$ on the other hand, combined with the Monge-Ampère equations (3.11) satisfied by these maps. Consider the convex maps ϕ_1 and ϕ_2, whose gradients transport respectively $\mathcal{L}$ onto f and $\mathcal{L}$ onto g. The theory of optimal transportation ensures that ϕ_1 and ϕ_2 satisfy (almost everywhere $x \in [0,1]^n$) the Monge-Ampère equations

$$f(\nabla\phi_1(x)) \det(D^2\phi_1(x)) = 1; \tag{3.13}$$

$$g(\nabla\phi_2(x)) \det(D^2\phi_2(x)) = 1. \tag{3.14}$$

The linear interpolation is given by

$$\phi(x) = (1-\lambda)\phi_1(x) + \lambda\phi_2(x). \tag{3.15}$$

The last necessary ingredient is the *Minkowki Determinant Theorem* (for a proof we mention [39], Section 4.1.8): let $\lambda \in (0,1)$, let A, B be $n \times n$ nonnegative symmetric matrices: then

$$\det\left((1-\lambda)A + \lambda B\right)^{1/n} \geq (1-\lambda)\det(A)^{1/n} + \lambda\det(B)^{1/n}. \tag{3.16}$$

The inequality (3.16) essentially relies on an application of the arithmetic-geometric mean inequality. Using sequentially the change of variables $z = \nabla\phi(x)$, (3.15), (3.12) (3.16), (3.13) and (3.14) we deduce

$$\int_{\mathbb{R}^n} h(z)\ dz \geq \int_{[0,1]^n} h(\nabla\phi(x)) \det\left(D^2\phi(x)\right)\ dx$$

$$= \int_{[0,1]^n} h\left((1-\lambda)\nabla\phi_1(x) + \lambda\nabla\phi_2(x)\right) \cdot \det\left((1-\lambda)D^2\phi_1(x) + \lambda D^2\phi_2(x)\right)\ dx$$

$$\geq \int_{[0,1]^n} \mathcal{M}_{-1/n}(f(\nabla\phi_1(x)), g(\nabla\phi_2(x)); \lambda)\ \cdot\ \mathcal{M}_{1/n}(\det(D^2\phi_1(x)), \det(D^2\phi_2(x)); \lambda)\ dx$$

$$\geq \int_{[0,1]^n} \mathcal{M}_{-\infty}(f(\nabla\phi_1(x))\det(D^2\phi_1(x)), g(\nabla\phi_2(x))\det(D^2\phi_1(x)); \lambda)\ dx$$

$$= \int_{[0,1]^n} \mathcal{M}_{-\infty}(1,1;\lambda)\ dx\ =\ |[0,1]^n| = 1,$$

where Lemma 3.1.1 has been used with $p = 1/n$, $q = -1/n$, $s = -\infty$. $\qquad\square$

Remark 3.3.1. Instead of the linear interpolation between the optimal transportation maps from f to $\mathcal{L}$ and from g to $\mathcal{L}$, McCann's argument [40] realizes the direct interpolation between the two probability densities f and g, by means of the so-called *displacement interpolant* $\{\rho_\lambda\}_{0\leq\lambda\leq1}$, defined as

$$\rho_\lambda = [(1-\lambda)Id + \lambda\nabla\phi]\,\#f, \qquad 0 \leq \lambda \leq 1,$$

where $\nabla\phi$ is (f-a.e.) the gradient of convex functions such that $\nabla\phi\#f = g$. This interpolation is the solution of a time-dependent transportation problem with a quadratic cost function. Of course $\{\rho_\lambda\}_{0\leq\lambda\leq 1}$ is a family of probability densities that interpolates between f and g, i.e. $\rho_0 = f$ and $\rho_1 = g$. Moreover this interpolation has several remarkable advantages; for example, some typical functionals F (defined on the sets of absolutely continuous probability measures on $\mathbb{R}^n$) are convex with respect to $\{\rho_\lambda\}_{0\leq\lambda\leq 1}$, meaning that

$$\lambda \longmapsto F(\rho_\lambda) \quad \text{is convex on } [0,1]. \tag{3.17}$$

If the functional F satisfies (3.17), it is said *displacement convex*. For instance, the Brunn-Minkowski inequality in $\mathbb{R}^n$ can be seen as a consequence (see [51], proof 1.2 in Chapter 6) of the displacement convexity of the functional

$$\mathcal{U}(\rho) = -\int_{\mathbb{R}^n} \left(\frac{d\rho}{dx}\right)^{1-1/n} dx.$$

3.3.3 Geometric proof

In certain cases the BBL inequalities can be seen as direct consequences of the BM inequality. In order to prove this fact, we present a proof due to Klartag [36], which is particularly useful for our future goals.

The version of BBL inequalities to which we are now interested, that is slightly weaker than the usual formulation in Theorem 3.1.1, regards the cases $p = 1/s > 0$.

Proposition 3.3.2. *Let* $0 < \lambda < 1$, $s > 0$, *and let* f, g, h *be nonnegative integrable functions on* $\mathbb{R}^n$ *satisfying*

$$h((1-\lambda)x + \lambda y) \geq \left[(1-\lambda)f(x)^{1/s} + \lambda g(y)^{1/s}\right]^s \tag{3.18}$$

for every $x \in \operatorname{Supp}(f)$, $y \in \operatorname{Supp}(g)$. *Then*

$$\int_{\mathbb{R}^n} h \geq \mathcal{M}_{\frac{1}{n+s}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g; \lambda\right). \tag{3.19}$$

Fixed $x \in \operatorname{Supp}(f)$ and $y \in \operatorname{Supp}(g)$ such that $f(x)$ and $g(y)$ are positive, notice that the assumption (3.18) coincides with (3.1) for $p = 1/s$, while the latter is weaker than the (3.18) if $f(x) = 0$ and $g(y) > 0$ or vice versa (because in our convention the generalized mean of two nonnegative numbers is zero if one of them is zero).

Also note that (3.18) is required for the couples (x,y) belonging to $\operatorname{Supp}(f) \times \operatorname{Supp}(g)$: one can easily check (arguing similarly to Remark 3.1.2) that in certain cases (for instance if f, g are compactly supported) to require (3.18) for every $x, y \in \mathbb{R}^n$ necessarily force $\int_{\mathbb{R}^n} h$ to diverge, making (3.19) trivial.

Notice that, given f and g, the smallest function satisfying (3.18) (hence the smallest function to which Proposition 3.3.2 possibly applies to) is their $(1/s)$-Minkowksi sum, defined as follows

$$h_{s,\lambda}(z) = \sup\left\{\left[(1-\lambda)f(x)^{1/s} + \lambda g(y)^{1/s}\right]^s : z = (1-\lambda)x + \lambda y\right\} \tag{3.20}$$

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for $z \in (1 - \lambda)\operatorname{Supp}(f) + \lambda\operatorname{Supp}(g)$ and $h_{s,\lambda}(z) = 0$ if $z \notin (1 - \lambda)\operatorname{Supp}(f) + \lambda\operatorname{Supp}(g)$.

To begin, given two positive integers n, s, let $f : \mathbb{R}^n \longrightarrow [0, +\infty)$ be an integrable function with nonempty support (to avoid the trivial case in which f is identically zero). Following Klartag's notations and ideas [36] (see also [1]), we associate with f the nonempty measurable set

$$K_{f,s} = \left\{ (x,y) \in \mathbb{R}^{n+s} = \mathbb{R}^n \times \mathbb{R}^s : \ x \in \operatorname{Supp}(f), \ |y| \leq f(x)^{1/s} \right\}, \qquad (3.21)$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^s$. In other words, $K_{f,s}$ is the subset of $\mathbb{R}^{n+s}$ obtained as union of the s-dimensional closed balls of center $(x, 0)$ and radius $f(x)^{1/s}$, for x belonging to the support of f, or, if you prefer, the set in $\mathbb{R}^{n+s}$ obtained by rotating with respect to $y = 0$ the $(n+1)$-dimensional set $\{(x,y) \in \mathbb{R}^{n+s} : 0 \leq y_1 \leq f(x)^{1/s}, \ y_2 = \cdots = y_s = 0\}$.
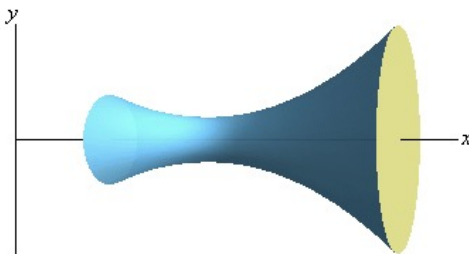

Figure 3.1: An example of $K_{f,s}$

Observe that $K_{f,s}$ is convex if and only if f is $(1/s)$-concave (that is for us a function f having compact convex support such that $f^{1/s}$ is concave on $\operatorname{Supp}(f)$). If $\operatorname{Supp}(f)$ is compact, then $K_{f,s}$ is bounded if and only if f is bounded.

Moreover, thanks to Fubini's Theorem, it holds

$$|K_{f,s}| = \int_{\operatorname{Supp}(f)} \omega_s \cdot \left(f(x)^{1/s} \right)^s \ dx = \omega_s \int_{\mathbb{R}^n} f(x) \ dx, \qquad (3.22)$$

where ω_s denotes the measure of the unit ball in $\mathbb{R}^s$. In this way, the integral of f coincides, up to the dimensional constant ω_s, with the volume of $K_{f,s}$. This simple identity allows to deduce Proposition 3.3.2 as a direct application of the BM inequality. Although of course the set $K_{f,s}$ depends heavily on s, for simplicity from now on we will remove the subindex s and just write K_f for $K_{f,s}$.

Let us begin with the simplest case, when $p = 1/s$ with s positive integer.

Third Proof of BBL inequalities, Proposition 3.3.2 with $s \in \mathbb{N}$.
Without loss of generality the integrals of f and g are positive, thus the sets K_f and K_g have positive measure. Let Ω_λ be the Minkowski combination (with coefficient λ) of $\Omega_0 = \operatorname{Supp}(f)$ and $\Omega_1 = \operatorname{Supp}(g)$. Consider the function $h_{s,\lambda}$ as defined by (3.20);

to simplify the notation, we will denote $h_{s,\lambda}$ by h_λ from now on. First notice that the support of h_λ is Ω_λ. Then it is easily seen that

$$K_{h_\lambda} = (1-\lambda)K_f + \lambda K_g. \tag{3.23}$$

Moreover, since $h \geq h_\lambda$ by assumption (3.18), we have $K_h \supseteq K_{h_\lambda}$. Applying BM inequality (Theorem 2.1.1) to K_{h_λ}, by means of (3.23),

$$|K_h|^{\frac{1}{n+s}} \geq |K_{h_\lambda}|^{\frac{1}{n+s}} \geq (1-\lambda)|K_f|^{\frac{1}{n+s}} + \lambda|K_g|^{\frac{1}{n+s}}, \tag{3.24}$$

where $|K_{h_\lambda}|$ possibly means the outer measure of the set K_{h_λ}. Finally (3.22) yields

$$|K_h| = \omega_s \int_{\mathbb{R}^n} h, \qquad |K_f| = \omega_s \int_{\mathbb{R}^n} f, \qquad |K_g| = \omega_s \int_{\mathbb{R}^n} g,$$

thus dividing (3.24) by $\omega_s^{\frac{1}{n+s}}$ we get (3.19). $\qquad\square$

Next we show how it is possible to generalize Proposition 3.3.2 to a positive index s. The case of a positive rational index s requires the following definition. Given $f : \mathbb{R}^n \longrightarrow [0,+\infty)$ integrable and a positive integer q (it will be the denominator of the rational number s) we consider the auxiliary function $\tilde{f} : \mathbb{R}^{nq} \longrightarrow [0,+\infty)$ defined as

$$\tilde{f}(x) = \tilde{f}(x_1,...,x_q) = \prod_{j=1}^{q} f(x_j), \tag{3.25}$$

where $x = (x_1,...,x_q) \in (\mathbb{R}^n)^q$. We observe that, by construction,

$$\int_{\mathbb{R}^{nq}} \tilde{f} = \left(\int_{\mathbb{R}^n} f \right)^q; \tag{3.26}$$

moreover $\operatorname{Supp} \tilde{f} = (\operatorname{Supp} f) \times ... \times (\operatorname{Supp} f) = (\operatorname{Supp} f)^q$.

The following lemma is an useful consequence of Hölder's inequality (see [33], Theorem 10, pag 21) for families of real numbers (in our case for two sets of q positive numbers).

Lemma 3.3.1. *Given a positive integer q, let $\{a_1,...,a_q\}$, $\{b_1,...,b_q\}$ be two sets of q real numbers. Then*

$$\left| \prod_{j=1}^{q} a_j \right| + \left| \prod_{j=1}^{q} b_j \right| \leq \left[\prod_{j=1}^{q} (|a_j|^q + |b_j|^q) \right]^{1/q}.$$

From this lemma we deduce the following.

Corollary 3.3.3. *Let $\lambda \in (0,1)$ and $s = \frac{t}{q}$ with positive integers t, q. Given $f, g : \mathbb{R}^n \longrightarrow [0,\infty)$, $x_1,...,x_q, y_1,...,y_q \in \mathbb{R}^n$, it holds*

$$(1-\lambda) \prod_{j=1}^{q} f(x_j)^{1/t} + \lambda \prod_{j=1}^{q} g(y_j)^{1/t} \leq \prod_{j=1}^{q} \left[(1-\lambda)f(x_j)^{1/s} + \lambda g(y_j)^{1/s} \right]^{1/q}.$$

Proof. Observing that

$$(1-\lambda)\prod_{j=1}^{q}f(x_j)^{1/t}+\lambda\prod_{j=1}^{q}g(y_j)^{1/t}=\prod_{j=1}^{q}(1-\lambda)^{1/q}f(x_j)^{1/t}+\prod_{j=1}^{q}\lambda^{1/q}g(y_j)^{1/t},$$

the result follows directly from Lemma 3.3.1 applied to $\{a_1,...,a_q\}$, $\{b_1,...,b_q\}$ with $a_j=(1-\lambda)^{1/q}f(x_j)^{1/t}$, $b_j=\lambda^{1/q}g(y_j)^{1/t}$, $j=1,...,q$. $\qquad\square$

Third Proof of BBL inequalities, Proposition 3.3.2 with generic $s>0$.
We have already demonstrated the case $s\in\mathbb{N}$. We show that this case suffices for all the other cases. Next, assume that $s=t/q$ is a rational number, where $t,q\in\mathbb{N}$. By Corollary 3.3.3 and (3.18), we have for any $x_1,...,x_q,y_1,...,y_q\in\mathbb{R}^n$

$$(1-\lambda)\prod_{j=1}^{q}f(x_j)^{1/t}+\lambda\prod_{j=1}^{q}g(y_j)^{1/t}\le\prod_{j=1}^{q}\left[(1-\lambda)f(x_j)^{1/s}+\lambda g(y_j)^{1/s}\right]^{1/q}$$

$$\le\prod_{j=1}^{q}h\left((1-\lambda)x_j+\lambda y_j\right)^{1/t},$$

i.e., by definition (3.25),

$$\tilde{h}((1-\lambda)x+\lambda y)\ge\left[(1-\lambda)\tilde{f}(x)^{1/t}+\lambda\tilde{g}(y)^{1/t}\right]^{t},$$

where $x=(x_1,...,x_q)\in\mathbb{R}^n$, $y=(y_1,...,y_q)\in\mathbb{R}^n$. Notice that $qs=t$ is integer, which is the case we already dealt with, and that the last inequality states that $\tilde{h},\tilde{f},\tilde{g}$ satisfy the assumptions of Proposition 3.3.2 with t in place of s. Hence, using (3.26) and (3.19) (for the index $t\in\mathbb{N}$ and for the functions $\tilde{h},\tilde{f},\tilde{g}$ defined on $\mathbb{R}^{nq}$) we derive

$$\left(\int_{\mathbb{R}^n}h\right)^{\frac{1}{n+s}}=\left(\int_{\mathbb{R}^{nq}}\tilde{h}\right)^{\frac{1}{q(n+s)}}=\left(\int_{\mathbb{R}^{nq}}\tilde{h}\right)^{\frac{1}{nq+t}}$$

$$\ge(1-\lambda)\left(\int_{\mathbb{R}^{nq}}\tilde{f}\right)^{\frac{1}{nq+t}}+\lambda\left(\int_{\mathbb{R}^{nq}}\tilde{g}\right)^{\frac{1}{nq+t}}=(1-\lambda)\left(\int_{\mathbb{R}^n}f\right)^{\frac{1}{n+s}}+\lambda\left(\int_{\mathbb{R}^n}g\right)^{\frac{1}{n+s}}$$

and we get the proposition for every rational positive index s.

By an approximation argument, we generalize it to the case of a real number $s>0$, as follows. Given $s>0$ there exists a decreasing sequence of positive rational numbers $\{s_j\}_{j\in\mathbb{N}}$ convergent to s. For every j, being $1/s>1/s_j$, (3.18) yields

$$h((1-\lambda)x+\lambda y)\ge\left[(1-\lambda)f(x)^{1/s_j}+\lambda g(y)^{1/s_j}\right]^{s_j}$$

with $s_j\in\mathbb{Q}^+$: then

$$\left(\int_{\mathbb{R}^n}h\right)^{\frac{1}{n+s_j}}\ge(1-\lambda)\left(\int_{\mathbb{R}^n}f\right)^{\frac{1}{n+s_j}}+\lambda\left(\int_{\mathbb{R}^n}g\right)^{\frac{1}{n+s_j}}.$$

Thus, passing to the limit $j\to+\infty$, the sequence $\left\{\frac{1}{n+s_j}\right\}$ converges increasing to $\frac{1}{n+s}$ and we finally have (3.19). $\qquad\square$

3.3.4 A proof through the curvilinear extension of BM inequality

The PL inequality, stated in Proposition 3.1.4 and corresponding to the case $p = 0$ of the BBL inequalities, is the simplest functional version of the Brunn-Minkowski inequality. In this section we want to derive the PL inequality applying the BM inequality to suitable sets related to the involved functions. We present a proof of the PL inequality in the spirit of Klartag's proof for the BBL inequalities (that is the proof given in the previous Section 3.3.3). We need a few concepts and notations, in order to present a generalization of the BM inequality, due to Uhrin [50]. He called it the *curvilinear extension of the Brunn-Minkowski inequality*. In what follows we denote with $\mathbb{R}_+^n$ the nonnegative orthant of $\mathbb{R}^n$, i.e. the subset of $\mathbb{R}^n$ in which every component is nonnegative. Let $0 < \lambda < 1$, $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n) \in \mathbb{R}_+^n$, $\alpha = (\alpha_1, ..., \alpha_n)$ with $\alpha_i \in [-\infty, +\infty]$ $i = 1, ..., n$.
We define and denote by

$$(1 - \lambda)a \, \triangle_\alpha \, \lambda b \tag{3.27}$$

the element of $\mathbb{R}^n$ whose components are

$$\mathcal{M}_{\alpha_i}(a_i, b_i; \lambda) \qquad i = 1, ..., n.$$

The operation (3.27) of two points $a, b \in \mathbb{R}_+^n$ is a sort of abstract "curvilinear" combination. If $\alpha_i = 1$ for any $i = 1, ..., n$ we can recognize the usual convex combination of two points in $\mathbb{R}^n$, i.e.

$$(1 - \lambda)a \, \triangle_{(1,1,...,1)} \, \lambda b \; = \; (1 - \lambda)a + \lambda b.$$

Through the previous definition, and extending the notion of Minkowski combination (case $\alpha = (1, 1, ..., 1)$), we introduce the concept of curvilinear combination of two sets. Namely, given two nonempty sets $A, B \subseteq \mathbb{R}_+^n$, we define

$$(1 - \lambda)A \, \triangle_\alpha \, \lambda B = \{(1 - \lambda)a \, \triangle_\alpha \, \lambda b : \; a \in A, \; b \in B\} \; \subseteq \mathbb{R}_+^n.$$

We notice that, similarly to the Minkowski combination of two measurable sets, it is possible that the set $(1 - \lambda)A \, \triangle_\alpha \, \lambda B$ is not measurable, so eventually we consider its inner Lebesgue measure (indicated again with $|\cdot|$).

Uhrin established the following extension of the BM inequality, giving a lower estimate for the volume of the curvilinear combination $(1 - \lambda)A \, \triangle_\alpha \, \lambda B$.

Theorem 3.3.4 (Theorem 2.1 in [50]). *Let $0 < \lambda < 1$, $\alpha = (\alpha_1, ..., \alpha_n) \in [0, 1]^n$, let $A, B \subset \mathbb{R}_+^n$ be bounded sets of positive measure. Then*

$$|(1 - \lambda)A \, \triangle_\alpha \, \lambda B| \; \geq \mathcal{M}_\gamma(|A|, |B|; \lambda) \tag{3.28}$$

where γ is the harmonic mean of $\alpha_1, ..., \alpha_n$, namely $\gamma = \left[\sum_{i=1}^n \alpha_i^{-1} \right]^{-1}$.

We observe that in the particular case of $\alpha = (1, 1, ..., 1)$ the set $(1 - \lambda)A \, \triangle_\alpha \, \lambda B$ corresponds to the Minkowski combination and the index γ coincide with $1/n$, thus we obtain the classic BM inequality (2.1).

Another special case, this time related to PL inequality, is given by $\alpha = (1, 1, ..., 1, 0)$ with each of the n components equal to 1, except the last one, that is 0: the corresponding index γ is 0. We denote this combination $(1-\lambda)A \,\triangle_{(1,1,...,1,0)}\, \lambda B$ with the symbol $A \,\blacktriangle\, B$: (3.28) in this case becomes

$$|A \,\blacktriangle\, B| \geq |A|^{1-\lambda} |B|^{\lambda}. \tag{3.29}$$

In other words, the volume of $A \,\blacktriangle\, B$ is greater or equal to than the (λ-weighted) geometric mean of the volumes of A and B. Notice that (3.29) is stronger than the multiplicative form (2.3) of the BM inequality since $A \,\blacktriangle\, B \subseteq (1 - \lambda)A + \lambda B$ and the inclusion is generally strict. Observing the inequalities (3.29) and (3.5), it is not difficult to imagine a link among them. Indeed, given a nonnegative measurable function f in the assumptions of PL inequality, the required link is provided by the corresponding measurable set given by

$$A_f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \; x \in \text{Supp}(f), \; 0 \leq t \leq f(x)\} \subseteq \mathbb{R}^n \times \mathbb{R}_+. \tag{3.30}$$

We can clearly recognize an analogy with the Klartag's set $K_{f,s}$ (see (3.21)), in fact A_f is exactly the upper half of $K_{f,1}$. Therefore A_f inherits the properties of $K_{f,1}$, in particular A_f is convex if and only if f is concave and

$$|A_f| = \int_{\mathbb{R}^n} f. \tag{3.31}$$

We are ready to prove the following version of PL inequality, which holds for bounded functions with compact supports.

Proposition 3.3.5 (PL inequality for bounded functions)**.**
Let $0 < \lambda < 1$, f, g, h be nonnegative measurable functions defined in $\mathbb{R}^n$. Assume that f and g are bounded with compact supports. If

$$h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)^{\lambda} \qquad \text{for all} \;\; x, y \in \mathbb{R}^n$$

then

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda}.$$

Proof. Without loss of generality assume that the integrals of f and g are positive (otherwise (3.32) holds obviously, being 0 the right hand side). Since the integrals of f and g are positive, the sets A_f and A_g have positive measure. Furthermore the assumptions on f and g (boundness and compact support) ensure the boundness of the corresponding A_f and A_g. Therefore we can apply Theorem 3.3.4 to A_f and A_g deducing

$$|A_f \,\blacktriangle\, A_g| \geq |A_f|^{1-\lambda} |A_g|^{\lambda}. \tag{3.32}$$

Then it is easily seen that (3.32) yields $A_h \supseteq [A_f \,\blacktriangle\, A_g]$, hence

$$|A_h| \geq |A_f \,\blacktriangle\, A_g| \geq |A_f|^{1-\lambda} |A_g|^{\lambda}. \tag{3.33}$$

Finally (3.31) implies

$$|A_h| = \int_{\mathbb{R}^n} h, \qquad |A_f| = \int_{\mathbb{R}^n} f, \qquad |A_g| = \int_{\mathbb{R}^n} g,$$

thus (3.33) is equivalent to the desired result. $\qquad\qquad\qquad\qquad\qquad$ □

The same kind of approach of Theorem 3.3.4, allow us to prove a slightly different form of the BBL inequalities of index $p \in (0,1]$. And actually, through a variant of Theorem 3.3.4, this can be done for every $p \in \left[-\frac{1}{n}, 1\right]$. The required variant of Theorem 3.3.4, stated in [50] (see (2.15) therein), is

Theorem 3.3.6. *Let $0 < \lambda < 1$, $A, B \subset \mathbb{R}^n_+$ be bounded sets of positive measure. Suppose that $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ satisfies*

$$\left[\sum_{i=1}^n \alpha_i^{-1} - n\right]^{-1} \geq -1/n \qquad and$$

$$\left[\sum_{i=1}^j \alpha_i^{-1} - j\right]^{-1} + \frac{\alpha_{j+1}}{1 - \alpha_{j+1}} > 0 \quad for\ every\ \ j = 1, ..., n-1.$$

Then

$$|(1-\lambda)A \triangle_\alpha \lambda B| \geq \mathcal{M}_\gamma(|A|, |B|; \lambda)$$

where γ is the harmonic mean of $\alpha_1, ..., \alpha_n$, namely $\gamma = \left[\sum_{i=1}^n \alpha_i^{-1}\right]^{-1}$.

For simplicity denote the set $(1-\lambda)A \triangle_{(1,1,...,1,p)} \lambda B$ with the symbol $A \triangle_p B$. Let $p \in \left[-\frac{1}{n-1}, 1\right]$. Theorem 3.3.4 and 3.3.6, applied to $\alpha = (1, ..., 1, p)$ respectively with $p \in [0,1]$ and $p \in \left[-\frac{1}{n-1}, 0\right]$, imply immediately the following generalization of (3.29).

Corollary 3.3.7. *Let $0 < \lambda < 1$, $n \geq 2$, $p \in \left[-\frac{1}{n-1}, 1\right]$, let $A, B \subset \mathbb{R}^n_+$ be bounded sets of positive measure. Then*

$$|A \triangle_p B| \geq \mathcal{M}_\gamma(|A|, |B|; \lambda)$$

where

$$\gamma = \begin{cases} -\infty & if \quad p = -\frac{1}{n-1}, \\ \frac{p}{p(n-1)+1} & otherwise. \end{cases}$$

We are ready to prove the BBL inequalities, in a slightly different form compared to Theorem 3.1.1, for every index $p \in \left[-\frac{1}{n}, 1\right]$. The idea is the same of the proof of Proposition 3.3.5 and we use again the sets A_f, A_g, A_h given by (3.30).

Proposition 3.3.8 (BBL inequality of index $p \in \left[-\frac{1}{n}, 1\right]$)**.**

Let $\lambda \in (0,1)$, $p \in \left[-\frac{1}{n}, 1\right]$ and let be f, g, h be nonnegative measurable functions defined in $\mathbb{R}^n$. Assume that f and g are bounded and compactly supported such that

$$h((1-\lambda)x + \lambda y) \geq [(1-\lambda)f(x)^p + \lambda g(y)^p]^{1/p} \ \text{ for all } x \in \mathrm{Supp}(f), y \in \mathrm{Supp}(g). \quad (3.34)$$

Then

$$\int_{\mathbb{R}^n} h \geq \mathcal{M}_{\frac{p}{np+1}} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g; \lambda \right).$$

Proof. Suppose $p \in (-1/n, 1] \setminus \{0\}$ (for the remaining cases $p = 0$ and $p = -1/n$ the proof repeats itself by interpreting the involved quantities by limits). Once again we can suppose without loss of generality that the integrals of f and g are positive. The assumptions on f and g (namely their boundness and the compactness of their support) ensure the boundness of the corresponding A_f and A_g. Then we apply Corollary 3.3.7 to the sets $A_f, A_g \subset \mathbb{R}^{n+1}$ associated with f, g, deriving

$$|A_f \triangle_p A_g| \geq \mathcal{M}_\gamma(|A_f|, |A_g|; \lambda),$$

where

$$\gamma = \frac{p}{p((n+1)-1)+1} = \frac{p}{pn+1}.$$

On the other hand, according to (3.34), it holds $A_h \supseteq [A_f \triangle_p A_g]$, hence

$$|A_h| \geq |A_f \triangle_p A_g| \geq \mathcal{M}_{\frac{p}{np+1}} (|A_f|, |A_g|; \lambda).$$

Consequently the latter means, by (3.31),

$$\int_{\mathbb{R}^n} h \geq \mathcal{M}_{\frac{p}{np+1}} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g; \lambda \right),$$

i.e. the desired result. $\square$

3.4 Relation with a wide class of integral inequalities

In [23], Dubuc investigated a family of integral inequalities in $\mathbb{R}^n$, equivalent to BBL inequalities, wondering when these inequalities were just equalities. We state the milestone of these integral inequalities, due to Borell [8], and called Theoreme B_n in [23]. First we focus our attention on the inequality in Theoreme B_n, then the next chapter will be devoted to its equality conditions, provided by Dubuc. These equality conditions will lead us to the precise characterization of the equality conditions for all the BBL inequalities.

Let us notice that, although this characterization seems to have been known by many mathematicians, at least in same cases, and it is in general ascribed to Dubuc, in fact in literature we cannot find any explicit precise and complete statement or proof. Then the content of the next chapter can be considered new at some degree.

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We will show the equivalence between the following result and the BBL inequalities, through Corollary 3.4.3. After that we will observe that BBL inequalities lead to a rather general family of integral inequalities, proved in Theoreme 11 of [23] using Theoreme B_n. Between these inequalities, we will mention those one due to Dancs and Uhrin (Proposition 3.4.5).

Proposition 3.4.1 (First part of Theoreme B_n, [23])**.**
Assume that the BBL inequality of index $p = -1/n$ holds. Let f_0, g_0, h_0 be nonnegative integrable functions defined in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} f_0 = \int_{\mathbb{R}^n} g_0 = 1,$$

and

$$h_0(x_0 + y_0) \geq \left[f_0(x_0)^{-1/n} + g_0(y_0)^{-1/n} \right]^{-n} \quad a.e. \ (x_0, y_0) \in \mathbb{R}^{2n}. \tag{3.35}$$

Then

$$\int_{\mathbb{R}^n} h_0 \geq 1.$$

Proof. Let $x_0 = (1 - \lambda)x, \ y_0 = \lambda y$. Setting

$$f(\cdot) = (1 - \lambda)^n f_0((1 - \lambda)\cdot), \quad g(\cdot) = \lambda^n g_0(\lambda \cdot),$$

we have (by simple changes of variable)

$$\int_{\mathbb{R}^n} f(x) \ dx = \int_{\mathbb{R}^n} f_0(x_0) \ dx_0 = 1, \quad \int_{\mathbb{R}^n} g(y) \ dy = \int_{\mathbb{R}^n} g_0(y_0) \ dy_0 = 1.$$

Then the condition (3.35) can be rewritten as

$$h_0((1 - \lambda)x + \lambda y) \geq \left[(1 - \lambda)f(x)^{-1/n} + \lambda g(y)^{-1/n} \right]^{-n} \geq \mathcal{M}_{-1/n}\left(f(x), \ g(y); \lambda \right),$$

that is exactly the assumption (3.1) with $p = -1/n$. Thus we can apply Theorem 3.1.1 to the function f, g, h_0, concluding that

$$\int_{\mathbb{R}^n} h_0 \geq \min \left\{ \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right\} = \min \left\{ 1, 1 \right\} = 1.$$

$\square$

Noticeably, by Proposition 3.4.1 and its corresponding equality conditions, Dubuc proved a wide class of integral inequalities, with a characterization of the equality case that we will examine in detail in the next chapter. Precisely we are referring to Theoreme 11 in [23], which we recall adopting the notations of the author. So we indicate with H_2^+ the set of the 1-homogeneous nonnegative functions $\mathfrak{p}$ defined in $[0, +\infty) \times [0, +\infty)$; in other words a function $\mathfrak{p} : [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ belong to H_2^+ if and only if

$$\mathfrak{p}(\alpha u, \alpha v) = \alpha \cdot \mathfrak{p}(u, v) \quad \text{for every} \ \alpha > 0, \ u, v \in [0, +\infty).$$

Given a function $\mathfrak{p} \in H_2^+$, we define the function $\mathfrak{p}_n$ defined in $[0, +\infty) \times [0, +\infty)$ as

$$\mathfrak{p}_n(u, v) = \inf_{0 < t < 1} \mathfrak{p}\left(\frac{u}{t^n}, \frac{v}{(1-t)^n}\right). \tag{3.36}$$

Proposition 3.4.2 (Theoreme 11 in [23])**.**
Let $\mathfrak{p} \in H_2^+$, f, g, h be nonnegative integrable functions defined in $\mathbb{R}^n$ such that

$$h(x + y) \geq \mathfrak{p}\left(f(x), g(y)\right) \quad a.e. \ (x, y) \in \mathbb{R}^{2n}. \tag{3.37}$$

Then

$$\int_{\mathbb{R}^n} h \geq \mathfrak{p}_n\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right). \tag{3.38}$$

Proof. We set $F = \int_{\mathbb{R}^n} f$, $G = \int_{\mathbb{R}^n} g$ (which without loss of generality we can suppose positive) and introduce the functions

$$f_0(\cdot) = \frac{f(\cdot)}{F}, \qquad g_0(\cdot) = \frac{g(\cdot)}{G}, \qquad h_0(\cdot) = \frac{h(\cdot)}{\mathfrak{p}_n(F, G)}.$$

The inequality (3.38) is trivially equivalent to $\int_{\mathbb{R}^n} h_0 \geq 1$. To prove the latter inequality it is sufficient to observe that $\int_{\mathbb{R}^n} f_0 = 1 = \int_{\mathbb{R}^n} g_0$ and that h_0, f_0, g_0 satisfy, by definition of $\mathfrak{p}_n$, the crucial condition (3.35). Therefore we can apply Proposition 3.4.1 to h_0, f_0, g_0 and conclude that $\int_{\mathbb{R}^n} h_0 \geq 1$. Let us check the property (3.35) for a couple $(x, y) \in \mathbb{R}^{2n}$ such that $f_0(x)$ and $g_0(y)$ are positive (otherwise (3.35) is obviously satisfied), in particular $f(x)$ and $g(y)$ are positive. For such a couple $(x, y) \in \mathbb{R}^n$ consider the value $t \in (0, 1)$ given by

$$t = \frac{f_0(x)^{-1/n}}{f_0(x)^{-1/n} + g_0(x)^{-1/n}}.$$

Then, by definition (3.36) and using the 1-homogeneity of $\mathfrak{p}$ and (3.37), we have

$$\mathfrak{p}_n(F, G) \leq \mathfrak{p}\left(Ft^{-n}, G(1-t)^{-n}\right)$$

$$= \mathfrak{p}\left(F \cdot \frac{\left[(F/f(x))^{1/n} + (G/g(y))^{1/n}\right]^n}{F/f(x)}, G \cdot \frac{\left[(F/f(x))^{1/n} + (G/g(y))^{1/n}\right]^n}{G/g(y)}\right)$$

$$= \left[(F/f(x))^{1/n} + (G/g(y))^{1/n}\right]^n \mathfrak{p}\left(f(x), g(y)\right)$$

$$\leq \left[(F/f(x))^{1/n} + (G/g(y))^{1/n}\right]^n h(x + y),$$

hence

$$\frac{h(x + y)}{\mathfrak{p}_n(F, G)} \geq \left[(F/f(x))^{1/n} + (G/g(y))^{1/n}\right]^{-n},$$

which is exactly (3.35). □

Next we prove that Proposition 3.4.2 implies, with suitable choices of the 1-homogeneous functions $\mathfrak{p} \in H_2^+$, all the BBL inequalities. Therefore we exhibit the right choices of $\mathfrak{p} \in H_2^+$ to obtain the BBL inequalities, and calculate the related functions $\mathfrak{p}_n$, distinguishing between the different types of index.

Lemma 3.4.1. *All the following functions $\mathfrak{p}$ belong to H_2^+:*

$$(i) \qquad \mathfrak{p}(u,v) = \mathcal{M}_{+\infty}(u,v;\lambda) = \begin{cases} \max\{u,v\} & \text{if } uv > 0, \\ 0 & \text{if } uv = 0; \end{cases}$$

$$(ii) \qquad \mathfrak{p}(u,v) = \begin{cases} [u^p + v^p]^{1/p} & \text{if } uv > 0, \\ 0 & \text{if } uv = 0, \end{cases} \quad \text{if } p \in (-1/n, +\infty) \setminus \{0\};$$

$$(iii) \qquad \mathfrak{p}(u,v) = \mathcal{M}_0(u,v;\lambda) = u^{1-\lambda}v^\lambda;$$

$$(iv) \qquad \mathfrak{p}(u,v) = \left[u^{-1/n} + v^{-1/n}\right]^{-n}.$$

They allow respectively to prove, by Proposition 3.4.2, the BBL inequality of index $+\infty$, $p \in (-1/n, +\infty) \setminus \{0\}$, 0, $-1/n$. The corresponding functions $\mathfrak{p}_n$ are respectively

$$(i) \qquad \mathfrak{p}_n(u,v) = \begin{cases} \left[u^{1/n} + v^{1/n}\right]^n & \text{if } uv > 0, \\ 0 & \text{if } uv = 0; \end{cases}$$

$$(ii) \qquad \mathfrak{p}_n(u,v) = \begin{cases} \left[u^{\frac{p}{np+1}} + v^{\frac{p}{np+1}}\right]^{\frac{np+1}{p}} & \text{if } uv > 0, \\ 0 & \text{if } uv = 0, \end{cases} \quad \text{if } p \in (-1/n, +\infty) \setminus \{0\};$$

$$(iii) \qquad \mathfrak{p}_n(u,v) = (1-\lambda)^{-n(1-\lambda)} \lambda^{-n\lambda} \cdot u^{1-\lambda}v^\lambda;$$

$$(iv) \qquad \mathfrak{p}_n(u,v) = \min\{u,v\}.$$

Observe that all the q-mean $\mathcal{M}_q(a,b;\lambda)$ belong to H_2^+, and we can consider directly them in place of the previous ones, to obtain BBL inequalities. However we prefer to use the functions $\mathfrak{p}$ of the latter lemma for the computation of the related $\mathfrak{p}_n$, because it is simpler since these functions $\mathfrak{p}$ (except $\mathfrak{p}(u,v) = u^{1-\lambda}v^\lambda$) are independent of the weight λ.

Proof. The property of 1-homogeneity is trivial for all the mentioned functions $\mathfrak{p}$, thus they belong to H_2^+. For example, given $p \in (-1/n, +\infty) \setminus \{0\}$, let us check the 1-homogeneity for the second case of $\mathfrak{p}$, namely

$$\mathfrak{p}(u,v) = \begin{cases} [u^p + v^p]^{1/p} & \text{if } uv > 0, \\ 0 & \text{if } uv = 0. \end{cases}$$

In this case, fixed $\alpha > 0$, it holds

$$\mathfrak{p}(\alpha u, \alpha v) = \begin{cases} [(\alpha u)^p + (\alpha v)^p]^{1/p} = \alpha [u^p + v^p]^{1/p} & \text{if } \alpha u \cdot \alpha v > 0 \\ 0 & \text{if } \alpha u \cdot \alpha v = 0 \end{cases}$$

hence it coincides with $\alpha \cdot \mathfrak{p}(u,v)$. Let us compute the functions $\mathfrak{p}_n$, case by case. First of all in every case $uv = 0$ clearly implies $\mathfrak{p}(u,v) = 0$, therefore $\mathfrak{p}_n(u,v) = 0$ (since, setting $t = 1/2$ in (3.36), it holds $\mathfrak{p}_n(u,v) \le 2^n \mathfrak{p}(u,v)$). Thus we suppose $u, v > 0$ from now on.

(i) Case $\mathfrak{p}(u,v) = \max\{u,v\}$. By definition (3.36) of $\mathfrak{p}_n$,

$$\mathfrak{p}_n(u,v) = \inf_{0<t<1} \ \max\left\{ \frac{u}{t^n}, \frac{v}{(1-t)^n} \right\}.$$

We begin to consider the simple case $u = v$:

$$\mathfrak{p}_n(u,u) = \inf_{0<t<1} \ \max\left\{ \frac{u}{t^n}, \frac{u}{(1-t)^n} \right\} = u \cdot \inf_{0<t<1} \max\left\{ t^{-n}, (1-t)^{-n} \right\} = 2^n u,$$

since this infimum is attained for $\bar{t} = 1/2$ ($\max\{t^{-n}, (1-t)^{-n}\} > 2^n$ for any other $t \in (0,1)$). This consideration can be generalized for $u \ne v$: the infimum is attained for $\bar{t} = \left[1 + (v/u)^{1/n}\right]^{-1}$ (again $\bar{t}$ is the value for which the terms $\frac{u}{t^n}$ and $\frac{v}{(1-t)^n}$ coincide): hence

$$\mathfrak{p}_n(u,v) = \max\left\{ \frac{u}{\bar{t}^n}, \frac{u}{(1-\bar{t})^n} \right\} = \left[u^{1/n} + v^{1/n} \right]^n.$$

(ii) Case $\mathfrak{p}(u,v) = [u^p + v^p]^{1/p}$ for $p \in (-1/n, +\infty) \setminus \{0\}$. By definition

$$\mathfrak{p}_n(u,v) = \inf_{0<t<1} \Lambda_p(t), \quad \text{where} \quad \Lambda_p(t) = \left[u^p t^{-np} + v^p (1-t)^{-np} \right]^{1/p}.$$

We calculate

$$\Lambda_p'(t) = \frac{1}{p} \Lambda_p(t)^{1-p} \left[u^p \cdot (-np) \cdot t^{-np-1} + v^p \cdot (-np) \cdot (1-t)^{-np-1} \cdot (-1) \right],$$

and we see that the infimum of $\mathfrak{p}_n(u,v)$ is attained for $\bar{t} = \left[1 + (v/u)^{\frac{p}{np+1}}\right]^{-1}$. After some calculation we get

$$\mathfrak{p}_n(u,v) = \Lambda_p(\bar{t}) = \left[u^{\frac{p}{np+1}} + v^{\frac{p}{np+1}} \right]^{\frac{np+1}{p}}.$$

(iii) Case $\mathfrak{p}(u,v) = u^{1-\lambda} v^\lambda$. Similarly it holds

$$\mathfrak{p}_n(u,v) = u^{1-\lambda} v^\lambda \cdot \inf_{0<t<1} \left[t^{-n(1-\lambda)} (1-t)^{-n\lambda} \right]$$

$$= u^{1-\lambda} v^\lambda \cdot \left[(1-\lambda)^{-n(1-\lambda)} (1-(1-\lambda))^{-n\lambda} \right] = (1-\lambda)^{-n(1-\lambda)} \lambda^{-n\lambda} \cdot u^{1-\lambda} v^\lambda.$$

(iv) Case $\mathfrak{p}(u,v) = \left[u^{-1/n} + v^{-1/n} \right]^{-n}$. By definition

$$\mathfrak{p}_n(u,v) = \inf_{0<t<1} \Lambda_{-1/n}(t), \quad \text{where} \quad \Lambda_{-1/n}(t) = \left[u^{-1/n} t + v^{-1/n}(1-t) \right]^{-n}.$$

It is trivial to check, by means of its derivative

$$\Lambda'_{-1/n}(t) = -n \cdot \left(\Lambda_{-1/n}(t)\right)^{\frac{n+1}{n}} \left[u^{-1/n} - v^{-1/n}\right],$$

that $\Lambda_{-1/n}$ is increasing if $u > v$, decreasing if $u < v$ (and constant if $u = v$). Therefore

$$\mathfrak{p}_n(u, v) = \inf_{0 < t < 1} \Lambda_{-1/n}(t) = \begin{cases} \Lambda_{-1/n}(0) & \text{if } u \geq v, \\ \Lambda_{-1/n}(1) & \text{otherwise} \end{cases} =$$

$$= \begin{cases} v & \text{if } u \geq v, \\ u & \text{otherwise} \end{cases} = \min\{u, v\}.$$

$\square$

Before showing that every BBL inequality is a consequence of Proposition 3.4.2 and Lemma 3.4.1 , we summarize the functions $\mathfrak{p}$ and the corresponding $\mathfrak{p}_n$.

	$\mathfrak{p}(u, v)$	$\mathfrak{p}_n(u, v)$
(i)	$\mathcal{M}_{+\infty}(u, v; \lambda)$	$\begin{cases} \left[u^{1/n} + v^{1/n}\right]^n & \text{if } uv > 0 \\ 0 & \text{if } uv = 0 \end{cases}$
(ii)	given $p \in (-1/n, +\infty) \setminus \{0\}$ with $\begin{cases} [u^p + v^p]^{1/p} & \text{if } uv > 0 \\ 0 & \text{if } uv = 0 \end{cases}$	$\begin{cases} \left[u^{\frac{p}{np+1}} + v^{\frac{p}{np+1}}\right]^{\frac{np+1}{p}} & \text{if } uv > 0 \\ 0 & \text{if } uv = 0 \end{cases}$
(iii)	$\mathcal{M}_0(u, v; \lambda)$	$(1 - \lambda)^{-n(1-\lambda)} \lambda^{-n\lambda} \cdot \mathcal{M}_0(u, v; \lambda)$
(iv)	$\left[u^{-1/n} + v^{-1/n}\right]^{-n}$	$\min\{u, v\}$

Proposition 3.4.3. *Proposition 3.4.2 implies all the BBL inequalities.*

Proof. Let $0 < \lambda < 1$, $p \in [-\frac{1}{n}, +\infty]$, and let f, g, h be nonnegative integrable functions on $\mathbb{R}^n$ satisfying

$$h((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda) \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{3.39}$$

We want to show that

$$\int_{\mathbb{R}^n} h \geq \mathcal{M}_{\frac{p}{np+1}} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g; \lambda\right). \tag{3.40}$$

Without loss of generality we may suppose that the integrals of f and g are positive (otherwise the result is obvious). Set

$$x_0 = (1 - \lambda)x, \qquad y_0 = \lambda y.$$

(i) Case $p = +\infty$, i.e. $\mathfrak{p}(u,v) = \mathcal{M}_{+\infty}(u,v;\lambda)$. Set

$$f_0(\cdot) = f\left(\frac{\cdot}{1-\lambda}\right), \qquad g_0(\cdot) = g\left(\frac{\cdot}{\lambda}\right), \tag{3.41}$$

hence

$$\int_{\mathbb{R}^n} f_0 = (1-\lambda)^n \int_{\mathbb{R}^n} f, \qquad \int_{\mathbb{R}^n} g_0 = \lambda^n \int_{\mathbb{R}^n} g. \tag{3.42}$$

On the other hand (3.39) with $p = +\infty$ becomes

$$h(x_0 + y_0) \geq \mathcal{M}_{+\infty}\left(f_0(x_0), g_0(y_0); \lambda\right) = \mathfrak{p}\left(f_0(x_0), g_0(y_0)\right).$$

Proposition 3.4.2 and Lemma 3.4.1 yield (3.40) for $p = +\infty$, indeed

$$\int_{\mathbb{R}^n} h \geq \mathfrak{p}_n\left(\int_{\mathbb{R}^n} f_0, \int_{\mathbb{R}^n} g_0\right) = \left[\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}\right]^n$$

$$= \left[(1-\lambda)\left(\int_{\mathbb{R}^n} f\right)^{1/n} + \lambda\left(\int_{\mathbb{R}^n} g\right)^{1/n}\right]^n = \mathcal{M}_{1/n}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g; \lambda\right).$$

(ii) Case $p \in (-1/n, 0) \cup (0, +\infty)$, i.e. $\mathfrak{p}(u,v) = \begin{cases} [u^p + v^p]^{1/p} & \text{if } uv > 0, \\ 0 & \text{if } uv = 0, \end{cases}$

Set

$$f_0(\cdot) = (1-\lambda)^{1/p} f\left(\frac{\cdot}{1-\lambda}\right), \qquad g_0(\cdot) = \lambda^{1/p} g\left(\frac{\cdot}{\lambda}\right), \tag{3.43}$$

hence

$$\int_{\mathbb{R}^n} f_0 = (1-\lambda)^{\frac{np+1}{p}} \int_{\mathbb{R}^n} f, \qquad \int_{\mathbb{R}^n} g_0 = \lambda^{\frac{np+1}{p}} \int_{\mathbb{R}^n} g. \tag{3.44}$$

Then (3.39) with $p \in (-1/n, 0) \cup (0, +\infty)$ can be rewritten as

$$h(x_0 + y_0) \geq \begin{cases} [(f_0(x_0))^p + (g_0(y_0))^p]^{1/p} & \text{if } f_0(x_0) \cdot g_0(y_0) > 0, \\ 0 & \text{if } f_0(x_0) \cdot g_0(y_0) = 0, \end{cases} = \mathfrak{p}\left(f_0(x_0), g_0(y_0)\right).$$

Proposition 3.4.2 and Lemma 3.4.1 lead to (3.40) for $p \in (-1/n, 0) \cup (0, +\infty)$, since

$$\int_{\mathbb{R}^n} h \geq \mathfrak{p}_n\left(\int_{\mathbb{R}^n} f_0, \int_{\mathbb{R}^n} g_0\right) = \left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{\frac{np+1}{p}}$$

$$= \left[(1-\lambda)\left(\int_{\mathbb{R}^n} f\right)^{\frac{p}{np+1}} + \lambda\left(\int_{\mathbb{R}^n} g\right)^{\frac{p}{np+1}}\right]^{\frac{np+1}{p}} = \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g; \lambda\right).$$

(iii) Case $p = 0$, i.e. $\mathfrak{p}(u,v) = \mathcal{M}_0(u,v;\lambda)$. With f_0, g_0 given by (3.41), condition (3.39) of index $p = 0$ is equivalent to

$$h(x_0 + y_0) \geq \mathcal{M}_0\left(f_0(x_0), g_0(y_0); \lambda\right) = \mathfrak{p}\left(f_0(x_0), g_0(y_0)\right).$$

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Therefore

$$\int_{\mathbb{R}^n} h \geq \mathfrak{p}_n \left(\int_{\mathbb{R}^n} f_0, \int_{\mathbb{R}^n} g_0 \right)$$

$$= \left(\frac{\int_{\mathbb{R}^n} f_0}{(1-\lambda)^n} \right)^{1-\lambda} \left(\frac{\int_{\mathbb{R}^n} g_0}{\lambda^n} \right)^{\lambda} = \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda},$$

by (3.42), Proposition 3.4.2 and Lemma 3.4.1. Thus we have obtained (3.40) for $p = 0$.

(iv) Case $p = -1/n$, i.e. $\mathfrak{p}(u,v) = \left[u^{-1/n} + v^{-1/n} \right]^{-n}$. Set

$$f_0(\cdot) = (1-\lambda)^{-n} f \left(\frac{\cdot}{1-\lambda} \right), \qquad g_0(\cdot) = \lambda^{-n} g \left(\frac{\cdot}{\lambda} \right), \tag{3.45}$$

hence

$$\int_{\mathbb{R}^n} f_0 = \int_{\mathbb{R}^n} f, \qquad \int_{\mathbb{R}^n} g_0 = \int_{\mathbb{R}^n} g.$$

Proposition 3.4.2 and Lemma 3.4.1 yield (3.40) for $p = -\infty$, being

$$\int_{\mathbb{R}^n} h \geq \mathfrak{p}_n \left(\int_{\mathbb{R}^n} f_0, \int_{\mathbb{R}^n} g_0 \right) = \min \left\{ \int_{\mathbb{R}^n} f_0, \int_{\mathbb{R}^n} g_0 \right\}$$

$$= \min \left\{ \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right\} = \mathcal{M}_{-\infty} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g; \lambda \right).$$

$\square$

Summarizing we have the following.

Corollary 3.4.4. *The BBL inequality of index $p = -1/n$ is equivalent to Proposition 3.4.1.*

Proof. The BBL inequality with $p = -1/n$ yields Proposition 3.4.1, which in turn implies Proposition 3.4.2 which finally implies every BBL inequality. $\square$

As a further application of Proposition 3.4.2, we mention and deduce the following integral inequalities, due to Dancs and Uhrin [20]. These inequalities complement in some sense the BBL inequalities.

Proposition 3.4.5 (Theorem 3.3 in [20])**.**
Let $0 < \lambda < 1$, $-\infty \leq p < -1/n$, $0 \leq f, g, h \in L^1(\mathbb{R}^n)$ such that

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda) \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{3.46}$$

Then

$$\int_{\mathbb{R}^n} h \geq \min \left\{ (1-\lambda)^{\frac{np+1}{p}} \int_{\mathbb{R}^n} f, \; \lambda^{\frac{np+1}{p}} \int_{\mathbb{R}^n} g \right\}. \tag{3.47}$$

Here the number $(np+1)/p$ has to be interpreted in the obvious way in the extremal case, i.e. it is equal to n when $p = -\infty$. Note that the latter claim is really similar to BBL inequalities: in fact they require the same kind of assumption, namely (3.46), this time for an index $p \in [-\infty, -1/n)$. Not surprising, being a condition weaker than (3.1), we get (3.47), which is in general a weaker result than (3.2).

Proof. Likewise the BBL inequalities (proof of Proposition 3.4.3), it is enough to apply Proposition 3.4.2 to the suitable function $\mathfrak{p}$, that in this case is

$$\mathfrak{p}(u,v) = [u^p + v^p]^{1/p} \quad \text{if} \quad p \in (-\infty, -1/n)$$

$$\text{or} \quad \mathfrak{p}(u,v) = \min\{u,v\} \quad \text{if} \quad p = -\infty.$$

Case $p \in (-\infty, -1/n)$: $\mathfrak{p}(u,v) = [u^p + v^p]^{1/p}$. By definition

$$\mathfrak{p}_n(u,v) = \inf_{0<t<1} \Lambda_p(t), \quad \text{where} \quad \Lambda_p(t) = [u^p t^{-np} + v^p(1-t)^{-np}]^{1/p}.$$

Let $\Lambda_p(0^+) = \lim_{t\to 0^+} \Lambda_p(t)$ and $\Lambda_p(1^-) = \lim_{t\to 1^-} \Lambda_p(t)$. It is easy to see that its derivative

$$\Lambda'_p(t) = \frac{1}{p}\Lambda_p(t)^{1-p}\left[u^p \cdot (-np) \cdot t^{-np-1} + v^p \cdot (-np) \cdot (1-t)^{-np-1} \cdot (-1)\right],$$

then (notice that p and $np+1$ are negative) $\Lambda_p(t)$ is increasing for $t < \left[1 + (v/u)^{\frac{p}{np+1}}\right]^{-1}$ and decreasing for $t > \left[1 + (v/u)^{\frac{p}{np+1}}\right]^{-1}$. Thus

$$\mathfrak{p}_n(u,v) = \inf_{0<t<1} \Lambda_p(t) = \min\left\{\Lambda_p(1^-), \Lambda_p(0^+)\right\} = \min\{u,v\}.$$

Fix $x_0 = (1-\lambda)x$, $y_0 = \lambda y$, and set

$$f_0(\cdot) = (1-\lambda)^{1/p} f\left(\frac{\cdot}{1-\lambda}\right), \qquad g_0(\cdot) = \lambda^{1/p} g\left(\frac{\cdot}{\lambda}\right),$$

from which

$$\int_{\mathbb{R}^n} f_0 = (1-\lambda)^{\frac{np+1}{p}} \int_{\mathbb{R}^n} f, \qquad \int_{\mathbb{R}^n} g_0 = \lambda^{\frac{np+1}{p}} \int_{\mathbb{R}^n} g. \tag{3.48}$$

Then (3.46) with $p \in (-\infty, -1/n)$ can be rewritten as

$$h(x_0 + y_0) = [(f_0(x_0))^p + (g_0(y_0))^p]^{1/p} = \mathfrak{p}(f_0(x_0), g_0(y_0)).$$

Applying Proposition 3.4.2 and (3.48) we get (3.47) for $p \in (-\infty, -1/n)$:

$$\int_{\mathbb{R}^n} h \geq \min\left\{\int_{\mathbb{R}^n} f_0, \int_{\mathbb{R}^n} g_0\right\} = \min\left\{(1-\lambda)^{\frac{np+1}{p}} \int_{\mathbb{R}^n} f, \lambda^{\frac{np+1}{p}} \int_{\mathbb{R}^n} g\right\}.$$

Case $p = -\infty$: $\mathfrak{p}(u,v) = \min\{u,v\}$. By definition (3.36) of $\mathfrak{p}_n$,

$$\mathfrak{p}_n(u,v) = \inf_{0<t<1} \min\left\{\frac{u}{t^n}, \frac{v}{(1-t)^n}\right\} = \min\{u,v\},$$

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since
$$\inf_{0<t<1} \frac{u}{t^n} = u, \qquad \inf_{0<t<1} \frac{v}{(1-t)^n} = v.$$

Fix $x_0 = (1-\lambda)x$, $y_0 = \lambda y$, and set

$$f_0(\cdot) = f\left(\frac{\cdot}{1-\lambda}\right), \qquad g_0(\cdot) = g\left(\frac{\cdot}{\lambda}\right),$$

from which

$$\int_{\mathbb{R}^n} f_0 = (1-\lambda)^n \int_{\mathbb{R}^n} f, \qquad \int_{\mathbb{R}^n} g_0 = \lambda^n \int_{\mathbb{R}^n} g. \tag{3.49}$$

Then (3.46) with $p \in (-\infty, -1/n)$ can be rewritten as

$$h(x_0 + y_0) = [(f_0(x_0))^p + (g_0(y_0))^p]^{1/p} = \mathfrak{p}\left(f_0(x_0), g_0(y_0)\right).$$

Applying Proposition 3.4.2 and (3.49) we have (3.47) for $p = -\infty$, since

$$\int_{\mathbb{R}^n} h \geq \min\left\{\int_{\mathbb{R}^n} f_0, \int_{\mathbb{R}^n} g_0\right\} = \min\left\{(1-\lambda)^n \int_{\mathbb{R}^n} f, \; \lambda^n \int_{\mathbb{R}^n} g\right\}.$$

$\square$

Chapter 4

Equality case in BBL inequalities

4.1 A key result

In [23], as already explained, Dubuc proved a general family of integral inequalities in $\mathbb{R}^n$, represented by Proposition 3.4.2, wondering when these inequalities were just equalities. To solve this question, it is useful to begin with the equality conditions of Theoreme B_n. Thus we restate it, this time including the equality conditions (without proof).

Theorem 4.1.1 (Theoreme B_n, [23]).
Let f, g, h be three nonnegative integrable functions defined in $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1,$$

and

$$h(x_0 + y_0) \geq \left[f(x_0)^{-1/n} + g(y_0)^{-1/n} \right]^{-n} \quad a.e. \ (x_0, y_0) \in \mathbb{R}^{2n}.$$

Then

$$\int_{\mathbb{R}^n} h \geq 1. \tag{4.1}$$

Moreover equality holds in (4.1), i.e. $\int_{\mathbb{R}^n} h(x) \, dx = 1$, if and only if there exist a convex function $\Phi : \mathbb{R}^n \longrightarrow (0, \infty]$ and a homothety in $\mathbb{R}^n$

$$x \to m_0 x + b_0 \qquad with \ \ m_0 \in (0, +\infty), \ b_0 \in \mathbb{R}^n$$

such that $f(x), g(x), h(x)$ coincide almost everywhere respectively with

$$\Phi(x)^{-n}, \quad m_0^n \Phi(m_0 x + b_0)^{-n}, \quad (m_0 + 1)^n \, \Phi((m_0 + 1)x + b_0)^{-n}. \tag{4.2}$$

In other words equality holds in (4.1) when the involved functions coincide (up to homotheties and a.e.) with the same $(-1/n)$-concave function $\varphi(x) = \Phi(x)^{-n}$.
One of the implication of the equality case is trivial, namely $\int_{\mathbb{R}^n} h(x) \, dx = 1$ follows

directly by (4.2), knowing for example that $\int_{\mathbb{R}^n} f(x)\ dx = 1$ and using the change of variable $z = (m_0 + 1)x + b_0$:

$$\int_{\mathbb{R}^n} h(x)\ dx = \int_{\mathbb{R}^n} (m_0 + 1)^n\ \Phi((m_0+1)x+b_0)^{-n}\ dx = \int_{\mathbb{R}^n} \Phi(z)^{-n}\ dz = \int_{\mathbb{R}^n} f(z)\ dz = 1.$$

Instead the reverse implication is a delicate question, solved by Dubuc through several technical lemmas and by an induction argument on the dimension n.

We underline that, by Theorem 4.1.1 and its corresponding equality conditions, Dubuc characterized, in Theoreme 12 of [23], the equality case for inequality (3.38) of Proposition 3.4.2.

Proposition 4.1.2 (Theoremes 11-12, [23])**.**
Let $\mathfrak{p} \in H_2^+$, f, g, h be three nonnegative integrable functions defined in $\mathbb{R}^n$ such that

$$h(x + y) \geq \mathfrak{p}\left(f(x), g(y)\right) \quad a.e.\ (x, y) \in \mathbb{R}^{2n}. \tag{4.3}$$

Then

$$\int_{\mathbb{R}^n} h(x)\ dx \geq \mathfrak{p}_n\left(\int_{\mathbb{R}^n} f,\ \int_{\mathbb{R}^n} g\right). \tag{4.4}$$

Moreover if equality holds in (4.4), then there exist $m \in (0, +\infty)$, $b \in \mathbb{R}^n$, and a $(-1/n)$-concave function $\varphi : \mathbb{R}^n \longrightarrow [0, +\infty)$ such that

$$f(x) = \left(\int_{\mathbb{R}^n} f\right)\varphi(x) \quad a.e., \tag{4.5}$$

$$m^n \cdot g(mx + b) = \left(\int_{\mathbb{R}^n} g\right)\varphi(x) \quad a.e., \tag{4.6}$$

$$(m + 1)^n \cdot h\left((m + 1)x + b\right) = \left(\int_{\mathbb{R}^n} h\right)\varphi(x) \quad a.e., \tag{4.7}$$

$$(m + 1)^{-n}\left(\int_{\mathbb{R}^n} h\right)\cdot\varphi\left(\frac{x_1 + mx_2}{1 + m}\right) \geq \mathfrak{p}\left(\left(\int_{\mathbb{R}^n} f\right)\cdot\varphi(x_1),\ m^{-n}\left(\int_{\mathbb{R}^n} g\right)\cdot\varphi(x_2)\right) \tag{4.8}$$

for every $x_1, x_2 \in \mathbb{R}^n$, and the coefficient $m \in (0, +\infty)$ satisfies

$$\mathfrak{p}_n\left(\int_{\mathbb{R}^n} f,\ \int_{\mathbb{R}^n} g\right) = \mathfrak{p}\left(\frac{\int_{\mathbb{R}^n} f}{\left(\frac{1}{m+1}\right)^n},\ \frac{\int_{\mathbb{R}^n} g}{\left(\frac{m}{m+1}\right)^n}\right). \tag{4.9}$$

In particular (see (3.36)) $\mathfrak{p}_n\left(\int_{\mathbb{R}^n} f,\ \int_{\mathbb{R}^n} g\right)$ is actually a minimum, which is attained for $t = \frac{1}{m+1}$. If this is the only value in which the minimum is attained, then the coefficient $m \in (0, +\infty)$ is uniquely determined.

For the sake of clearness, we give the proof of the latter result, regarding the equality case. We also emphasize that the necessary condition (4.9) is not part of the original statement of Dubuc and it seems to be new. We have deduced (4.9) as a simple consequence of the other equality conditions in Proposition 4.1.2. This additional condition will be very useful to characterize the equality case in the BBL inequalities, since it allow to determine uniquely the coefficient m and consequently to obtain the desired p-concavity of the involved functions.

Proof. We set $F = \int_{\mathbb{R}^n} f \ dx \ G = \int_{\mathbb{R}^n} g \ dx$, which without loss of generality we suppose positive. Introduce the functions

$$f_0(\cdot) = \frac{f(\cdot)}{F}, \qquad g_0(\cdot) = \frac{g(\cdot)}{G}, \qquad h_0(\cdot) = \frac{h(\cdot)}{\mathfrak{p}_n(F,G)}.$$

We have already seen that h_0, f_0, g_0 satisfy the assumptions of Theorem 4.1.1, concluding that $\int_{\mathbb{R}^n} h_0 \ dx \geq 1$. Assume that equality holds in (4.4), i.e. $\int_{\mathbb{R}^n} h_0 \ dx = 1$. Then, applying Theorem 4.1.1 (equality case) to the functions h_0, f_0, g_0 we can conclude that there exist $m_0 \in (0, +\infty)$, $b_0 \in \mathbb{R}^n$, and a convex function Φ such that $f_0(z)$, $g_0(z)$, $h_0(z)$ coincide almost everywhere respectively with

$$\varphi(z), \qquad m_0^n \varphi(m_0 z + b_0), \qquad (m_0 + 1)^n \varphi((m_0 + 1)z + b_0),$$

where

$$\varphi(z) = \Phi(z)^{-n}$$

is a $(-1/n)$-concave function (in particular φ is continuous in $\{z : \varphi(z) > 0\}$). Thus (4.5) holds. Setting $x = m_0 z + b_0$, $m = \frac{1}{m_0}$, $b = -\frac{b_0}{m_0}$, we get (4.6) and (4.7). Finally (4.3) is equivalent to (4.8), thanks to (4.5), (4.6) and (4.7). Indeed, setting $x_1 = x$ and $x_2 = \frac{y-b}{m}$, it holds

$$\mathfrak{p}\left(\left(\int_{\mathbb{R}^n} f\right) \cdot \varphi(x_1), m^{-n}\left(\int_{\mathbb{R}^n} g\right) \cdot \varphi(x_2)\right)$$

$$= \mathfrak{p}\left(\left(\int_{\mathbb{R}^n} f\right) \cdot \varphi(x), m^{-n}\left(\int_{\mathbb{R}^n} g\right) \cdot \varphi\left(\frac{y-b}{m}\right)\right)$$

$$= \mathfrak{p}\left(f(x), g(y)\right) \ \leq \ h(x+y) \ = \ (m+1)^{-n}\left(\int_{\mathbb{R}^n} h\right)\varphi\left(\frac{x+y-b}{m+1}\right)$$

$$= (m+1)^{-n}\left(\int_{\mathbb{R}^n} h\right) \cdot \varphi\left(\frac{x_1 + mx_2}{m+1}\right),$$

for every $x_1, x_2 \in \mathbb{R}^n$. Now set $x_1 = x_2 = x \in \{z : \varphi(z) > 0\}$. Using the 1-homogeneity of $\mathfrak{p}$, we get that

$$\varphi(x) \cdot \mathfrak{p}\left(\int_{\mathbb{R}^n} f, \ m^{-n}\left(\int_{\mathbb{R}^n} g\right)\right) \leq (m+1)^{-n}\left(\int_{\mathbb{R}^n} h\right) \cdot \varphi(x), \qquad \text{i.e.}$$

$$\mathfrak{p}\left(\int_{\mathbb{R}^n} f, \ m^{-n}\left(\int_{\mathbb{R}^n} g\right)\right) \leq (m+1)^{-n} \cdot \mathfrak{p}_n\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right).$$

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Again by homogeneity of $\mathfrak{p}$ it means

$$\mathfrak{p}_n \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right) \geq \mathfrak{p} \left(\frac{\int_{\mathbb{R}^n} f}{\left(\frac{1}{m+1} \right)^n}, \frac{\int_{\mathbb{R}^n} g}{\left(\frac{m}{m+1} \right)^n} \right).$$

On the other hand the opposite inequality follows trivially by definition (3.36) of $\mathfrak{p}_n$, (since $\frac{1}{m+1}$ belongs to $(0,1)$). Therefore we obtain (4.9).

$\square$

4.2 Equality conditions for Prékopa-Leindler inequality

We deal with the equality case of the Prékopa-Leindler inequality.Our aim is to discover how equality in PL inequality is naturally connected to log-concavity.

Let us check, using Proposition 4.1.2, the following well known characterization for the equality case in PL, mentioned in various works (Theorem 4.2 in [17], and also [2,12]) but not explicitly proved in any of them.

Proposition 4.2.1 (Theorem 4.2 in [17])**.**
Let $\lambda \in (0,1)$, $f, g, h : \mathbb{R}^n \longrightarrow [0,+\infty)$ *with positive finite integrals such that*

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda} \qquad \text{for all} \;\; x, y \in \mathbb{R}^n.$$

Then

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda}. \tag{4.10}$$

In addition, if equality holds in (4.10) *then f coincides a.e. with a log-concave function and there exist* $C \in \mathbb{R}$, $a > 0$ *and* $x_0 \in \mathbb{R}^n$ *such that*

$$g(x) = Cf(ax + x_0) \quad \text{for almost every} \;\; x \in \mathbb{R}^n.$$

Precisely, we prove

Proposition 4.2.2 (Equality conditions for PL inequality)**.**
Let $\lambda \in (0,1)$, $f, g, h : \mathbb{R}^n \longrightarrow [0,+\infty)$ *with positive finite integrals such that*

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda} \qquad \text{for all} \;\; x, y \in \mathbb{R}^n.$$

If equality holds in PL inequality i.e.

$$\int_{\mathbb{R}^n} h = \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda},$$

then there exist $b \in \mathbb{R}^n$, *and a* log*-concave function* φ *such that a.e.*

$$(1-\lambda)^n \cdot \varphi(x) \;=\; \frac{f\left(\frac{x}{1-\lambda} \right)}{\int_{\mathbb{R}^n} f} \;=\; \frac{g\left(\frac{x}{1-\lambda} + \frac{b}{\lambda} \right)}{\int_{\mathbb{R}^n} g} \;=\; \frac{h\left(\frac{x}{1-\lambda} + b \right)}{\int_{\mathbb{R}^n} h}. \tag{4.11}$$

Proof. It suffices to apply Proposition 4.1.2 to the functions h,

$$f_0(\cdot) = f\left(\frac{\cdot}{1-\lambda}\right), \qquad g_0(\cdot) = g\left(\frac{\cdot}{\lambda}\right) \tag{4.12}$$

with

$$\mathfrak{p}(u,v) = u^{1-\lambda}v^\lambda, \qquad \mathfrak{p}_n(u,v) = (1-\lambda)^{-n(1-\lambda)}\,\lambda^{-n\lambda}\,\mathfrak{p}(u,v).$$

Hence (4.9) reads

$$(1-\lambda)^{-n(1-\lambda)}\,\lambda^{-n\lambda}\cdot\left(\int_{\mathbb{R}^n} f_0\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g_0\right)^\lambda$$

$$= (m+1)^{n(1-\lambda)}\left(\int_{\mathbb{R}^n} f_0\right)^{1-\lambda}\left(\frac{m+1}{m}\right)^{n\lambda}\left(\int_{\mathbb{R}^n} g_0\right)^\lambda,$$

that is, denoting with t the fraction $1/(m+1)$,

$$(1-\lambda)^{-n(1-\lambda)}\cdot\lambda^{-n\lambda} = t^{-n(1-\lambda)}\cdot(1-t)^{-n\lambda}. \tag{4.13}$$

Consider the function $\Lambda:(0,1)\longrightarrow(0,+\infty)$ given by $\Lambda(z) = z^{-n(1-\lambda)}\cdot(1-z)^{-n\lambda}$. Then

$$\Lambda'(z) = -n(1-\lambda)\cdot z^{-n(1-\lambda)-1}\cdot(1-z)^{-n\lambda} + n\lambda\cdot(1-z)^{-n\lambda-1}\cdot z^{-n(1-\lambda)},$$

and it is easy to check that $\Lambda(z)$ is increasing if $z > 1-\lambda$, decreasing if $z < 1-\lambda$. Thus $\Lambda(z)$ admits minimum and this minimum is attained only for $z = 1-\lambda$, where

$$\Lambda(1-\lambda) = (1-\lambda)^{-n(1-\lambda)}\cdot\lambda^{-n\lambda}.$$

In other words (4.13) corresponds to $\Lambda(1-\lambda) = \Lambda(t) = \min_{z\in(0,1)}\Lambda(z)$, and it yields $t = 1-\lambda$, namely

$$m = \frac{\lambda}{1-\lambda}. \tag{4.14}$$

Then, replacing (4.14) and (4.12), equality (4.11) follows from (4.5), (4.6) and (4.7), where φ is $(-1/n)$-concave. Now we want to prove that φ is in fact log-concave. Indeed (4.8) holds:

$$(m+1)^{-n}\left(\int_{\mathbb{R}^n} h\right)\cdot\varphi\left(\frac{x_1+mx_2}{1+m}\right) \geq \mathfrak{p}\left(\left(\int_{\mathbb{R}^n} f_0\right)\cdot\varphi(x_1),\ m^{-n}\left(\int_{\mathbb{R}^n} g_0\right)\cdot\varphi(x_2)\right),$$

i.e.

$$(m+1)^{-n}\left(\int_{\mathbb{R}^n} f_0\right)^{1-\lambda}(1-\lambda)^{-n(1-\lambda)}\left(\int_{\mathbb{R}^n} g_0\right)^\lambda \lambda^{-n\lambda}\,\varphi\left(\frac{x_1+mx_2}{1+m}\right)$$

$$\geq \left(\int_{\mathbb{R}^n} f_0\right)^{1-\lambda}\varphi(x_1)^{1-\lambda}\left(\int_{\mathbb{R}^n} g_0\right)^\lambda m^{-n\lambda}\,\varphi(x_2)^\lambda.$$

Through (4.14) the latter is equivalent to

$$\varphi\left((1-\lambda)x_1 + \lambda x_2\right) \geq \varphi(x_1)^{1-\lambda}\varphi(x_2)^\lambda.$$

Moreover φ is a continuous function, being $(-1/n)-$concave (thanks to Proposition 4.1.2). This implies that φ is a log-concave function, thanks to the continuity of φ (ensured by its $(-1/n)$-concavity). $\qquad\square$

4.3 Equality conditions for the other BBL inequalities

4.3.1 Case $p \in (-1/n, +\infty) \setminus \{0\}$

We will prove that the equality in the BBL inequality of index p is essentially linked to the property of p-concavity. Precisely in the equality case the involved functions, under suitable assumptions, have to be p-concave (while Proposition 4.1.2 ensures only $(-1/n)$-concavity). Given $p \in (-1/n, 0) \cup (0, +\infty)$, we have demonstrated in Corollary 3.4.3 that Proposition 4.1.2 implies the BBL inequality of index p, by using the function $\mathfrak{p} \in H_2^+$ defined as

$$\mathfrak{p}(u, v) = \begin{cases} [u^p + v^p]^{1/p} & \text{if } uv > 0, \\ 0 & \text{if } uv = 0, \end{cases} \tag{4.15}$$

with the corresponding function

$$\mathfrak{p}_n(u, v) = \begin{cases} \left[u^{\frac{p}{np+1}} + v^{\frac{p}{np+1}} \right]^{\frac{np+1}{p}} & \text{if } uv > 0, \\ 0 & \text{if } uv = 0. \end{cases} \tag{4.16}$$

Let $\lambda \in (0, 1)$, h, f, g nonnegative (with positive finite integrals) satisfying (3.39) for a fixed $p \in (-1/n, +\infty) \setminus \{0\}$, i.e. h, f_0, g_0 satisfy (3.37), where f_0, g_0 are given by (3.43). Suppose that equality holds in the BBL of index p, i.e.

$$\int_{\mathbb{R}^n} h = \mathcal{M}_{\frac{p}{np+1}} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \, ; \lambda \right) \in (0, +\infty);$$

this is equivalent to (see the proof of Proposition 3.4.3)

$$\int_{\mathbb{R}^n} h \, dx = \mathfrak{p}_n \left(\int_{\mathbb{R}^n} f_0, \int_{\mathbb{R}^n} g_0 \right) = \left[\left(\int_{\mathbb{R}^n} f_0 \right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0 \right)^{\frac{p}{np+1}} \right]^{\frac{np+1}{p}},$$

so we can apply Proposition 4.1.2 to the functions h, f_0, g_0. The equality case in Proposition 4.1.2 states that h, f_0, g_0 coincide (almost everywhere and up to homotheties (4.5),(4.6),(4.7)) with a same function φ, at least $(-1/n)$-concave. Now we want to prove that φ is in fact p-concave.

Theorem 4.3.1 (Equality conditions for BBL inequality of index $p \in (-1/n, 0) \cup (0, +\infty)$)**.** *Let $\lambda \in (0, 1)$, $p \in (-1/n, 0) \cup (0, +\infty)$, $f, g, h : \mathbb{R}^n \longrightarrow [0, +\infty)$ with positive finite integrals such that*

$$h((1 - \lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda) \qquad \text{for all } x, y \in \mathbb{R}^n.$$

Then equality holds in BBL inequality, i.e.

$$\int_{\mathbb{R}^n} h = \mathcal{M}_{\frac{p}{np+1}} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g; \lambda \right)$$

if and only if there exist $b \in \mathbb{R}^n$, and a p-concave function φ such that a.e.

$$(1-\lambda)^{\frac{np+1}{p}} \left(\int_{\mathbb{R}^n} f \right) \cdot \varphi(x) = (1-\lambda)^{1/p} \cdot f\left(\frac{x}{1-\lambda} \right)$$

$$= \left(\frac{\lambda}{m} \right)^{1/p} \cdot g\left(\frac{mx+b}{\lambda} \right) = (m+1)^{-1/p} \cdot h\left((m+1)x+b \right), \qquad (4.17)$$

where

$$m = \frac{\lambda}{1-\lambda} \left(\frac{\int_{\mathbb{R}^n} g}{\int_{\mathbb{R}^n} f} \right)^{\frac{p}{np+1}}. \qquad (4.18)$$

Proof. One implication is trivial: if f, g, h satisfy (4.17) then equality holds in the corresponding BBL inequality. Indeed, considering the respective integrals, (4.17) yields

$$(1-\lambda)^{1/p} \cdot \int_{\mathbb{R}^n} f\left(\frac{x}{1-\lambda} \right) \, dx = \left(\frac{\lambda}{m} \right)^{1/p} \cdot \int_{\mathbb{R}^n} g\left(\frac{mx+b}{\lambda} \right) \, dx$$

$$= (m+1)^{-1/p} \cdot \int_{\mathbb{R}^n} h\left((m+1)x+b \right) \, dx,$$

i.e. (through simple changes of variables)

$$(1-\lambda)^{\frac{np+1}{p}} \left(\int_{\mathbb{R}^n} f \right) = \left(\frac{\lambda}{m} \right)^{\frac{np+1}{p}} \left(\int_{\mathbb{R}^n} g \right) = \left(\frac{1}{m+1} \right)^{\frac{np+1}{p}} \left(\int_{\mathbb{R}^n} h \right).$$

In particular

$$\int_{\mathbb{R}^n} f = \left(\frac{\lambda}{m(1-\lambda)} \right)^{\frac{np+1}{p}} \int_{\mathbb{R}^n} g, \qquad \int_{\mathbb{R}^n} h = \lambda^{\frac{np+1}{p}} \left(\frac{m+1}{m} \right)^{\frac{np+1}{p}} \left(\int_{\mathbb{R}^n} g \right), \qquad \text{and}$$

$$\mathcal{M}_{\frac{p}{np+1}} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g, ; \lambda \right) = \mathcal{M}_{\frac{p}{np+1}} \left(\left(\frac{\lambda}{m(1-\lambda)} \right)^{\frac{np+1}{p}} \int_{\mathbb{R}^n} g, \int_{\mathbb{R}^n} g; \lambda \right)$$

$$= \left[(1-\lambda)\frac{\lambda}{m(1-\lambda)} \left(\int_{\mathbb{R}^n} g \right)^{\frac{p}{np+1}} + \lambda \left(\int_{\mathbb{R}^n} g \right)^{\frac{p}{np+1}} \right]^{\frac{np+1}{p}}$$

$$= \lambda^{\frac{np+1}{p}} \left(\frac{m+1}{m} \right)^{\frac{np+1}{p}} \left(\int_{\mathbb{R}^n} g \right) = \int_{\mathbb{R}^n} h.$$

The other implication is based on the application of Proposition 4.1.2 to the functions h, f_0, g_0 mentioned above and given by (3.43), with $\mathfrak{p}$ and $\mathfrak{p}_n$ defined as (4.15) and (4.16). First we have to prove the necessary condition (4.18). In this case (4.9) reads:

$$\left[\left(\int_{\mathbb{R}^n} f_0 \right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0 \right)^{\frac{p}{np+1}} \right]^{\frac{np+1}{p}} = \left[\frac{\left(\int_{\mathbb{R}^n} f_0 \right)^p}{\left(\frac{1}{m+1} \right)^{np}} + \frac{\left(\int_{\mathbb{R}^n} g_0 \right)^p}{\left(\frac{m}{m+1} \right)^{np}} \right]^{1/p}.$$

The latter is equivalent to

$$\left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{np+1} = \Lambda\left(\frac{1}{m+1}\right), \tag{4.19}$$

where $\Lambda : (0,1) \longrightarrow (0,+\infty)$ is the function

$$\Lambda(z) = z^{-np}\left(\int_{\mathbb{R}^n} f_0\right)^p + (1-z)^{-np}\left(\int_{\mathbb{R}^n} g_0\right)^p.$$

First suppose $p \in (0,+\infty)$. We show that in this case Λ attains its minimum value only for

$$z = \frac{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}}}{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}}$$

and the minimum coincides with $\left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{np+1}$. Therefore (4.19) can be expressed as follows

$$\Lambda\left(\frac{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}}}{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}}\right) = \min_{z\in(0,1)} \Lambda(z) = \Lambda\left(\frac{1}{m+1}\right), \qquad \text{thus}$$

$$\frac{1}{m+1} = \frac{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}}}{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}}, \tag{4.20}$$

and consequently (see (3.44))

$$m = \left(\frac{\int_{\mathbb{R}^n} g_0}{\int_{\mathbb{R}^n} f_0}\right)^{\frac{p}{np+1}} = \frac{\lambda}{1-\lambda}\left(\frac{\int_{\mathbb{R}^n} g}{\int_{\mathbb{R}^n} f}\right)^{\frac{p}{np+1}}.$$

For the sake of completeness here we write briefly the computations to derive the behaviour of Λ, and its minimum value in particular. Its derivative is

$$\Lambda'(z) = -np \cdot z^{-np-1}\left(\int_{\mathbb{R}^n} f_0\right)^p + np \cdot (1-z)^{-np-1}\left(\int_{\mathbb{R}^n} g_0\right)^p.$$

Then Λ is clearly increasing (dividing by $np > 0$) if

$$(1-z)^{-np-1}\left(\int_{\mathbb{R}^n} g_0\right)^p > z^{-np-1}\left(\int_{\mathbb{R}^n} f_0\right)^p, \tag{4.21}$$

i.e.

$$\left(\frac{1-z}{z}\right)^{np+1} < \left(\frac{\int_{\mathbb{R}^n} g_0}{\int_{\mathbb{R}^n} f_0}\right)^p, \qquad \text{namely}$$

$$\frac{1}{z} < 1 + \left(\frac{\int_{\mathbb{R}^n} g_0}{\int_{\mathbb{R}^n} f_0}\right)^{\frac{p}{np+1}} = \frac{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}}{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}}}.$$

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Then Λ attains its minimum only for

$$z = \frac{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}}}{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}}.$$

Replacing this value we check that the minimum attained by Λ coincides with

$$\frac{\left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{np}}{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{np^2}{np+1}-p}} + \frac{\left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{np}}{\left(\int_{\mathbb{R}^n} g_0\right)^{\frac{np^2}{np+1}-p}}$$

$$= \left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{np} \cdot \left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]$$

$$= \left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{np+1}.$$

In the case $p \in (-1/n, 0)$ the same considerations hold, replacing "minimum" by "maximum" and "increasing" by "decreasing", since $p < 0$ implies (dividing by $np < 0$) that (4.21) holds if and only if Λ is decreasing. Therefore we get (4.18) again.

Moreover replacing (4.18) and (3.43), (4.17) follows from (4.5), (4.6) and (4.7). The p-concavity of φ is a consequence of (4.8) and (4.18). Indeed, for every $x_1, x_2 \in \mathbb{R}^n$ such that $\varphi(x_1) > 0$ and $\varphi(x_2) > 0$, (4.8) takes the form

$$\varphi\left(\frac{x_1 + mx_2}{1+m}\right) \geq \frac{\left[\left(\int_{\mathbb{R}^n} f_0\right)^p \varphi(x_1)^p + m^{-np}\left(\int_{\mathbb{R}^n} g_0\right)^p \varphi(x_2)^p\right]^{1/p}}{(m+1)^{-n} \cdot \left(\int_{\mathbb{R}^n} h\right)}$$

$$= \frac{\left[\left(\frac{1}{m+1}\right)^{-np}\left(\int_{\mathbb{R}^n} f_0\right)^p \varphi(x_1)^p + \left(\frac{m}{m+1}\right)^{-np}\left(\int_{\mathbb{R}^n} g_0\right)^p \varphi(x_2)^p\right]^{1/p}}{\left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{\frac{np+1}{p}}} \quad \text{i.e.,}$$

being $t = 1/(m+1)$, it holds

$$\varphi\left(tx_1 + (1-t)x_2\right) \geq \frac{\left[t^{-np}\left(\int_{\mathbb{R}^n} f_0\right)^p \varphi(x_1)^p + (1-t)^{-np}\left(\int_{\mathbb{R}^n} g_0\right)^p \varphi(x_2)^p\right]^{1/p}}{\left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{\frac{np+1}{p}}}.$$

Replacing (4.20) we conclude that

$$\varphi\left(tx_1 + (1-t)x_2\right) \geq \frac{\left[\left(\int_{\mathbb{R}^n} f_0\right)^{p-\frac{np^2}{np+1}} \varphi(x_1)^p + \left(\int_{\mathbb{R}^n} g_0\right)^{p-\frac{np^2}{np+1}} \varphi(x_2)^p\right]^{1/p}}{\left[\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}\right]^{1/p}}$$

$$= \left[\frac{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}}}{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}} \cdot \varphi(x_1)^p + \frac{\left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}}{\left(\int_{\mathbb{R}^n} f_0\right)^{\frac{p}{np+1}} + \left(\int_{\mathbb{R}^n} g_0\right)^{\frac{p}{np+1}}} \cdot \varphi(x_2)^p\right]^{1/p}$$

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$$= [t\varphi(x_1)^p + (1-t)\varphi(x_2)^p]^{1/p}.$$

Summarizing, φ satisfies

$$\varphi\left(tx_1 + (1-t)x_2\right) \geq \mathcal{M}_p\left(\varphi(x_1),\ \varphi(x_2);\ 1-t\right) \qquad \text{for every}\ \ x_1, x_2 \in \mathbb{R}^n.$$

Moreover φ is a continuous function, being $(-1/n)-$concave (thanks to Proposition 4.1.2). Therefore φ is a p-concave function and we have proved Theorem 4.3.1. □

4.3.2 Case $p = +\infty$

Let us analyse the equality case in the BBL inequality for the extremal index $p = +\infty$. We proceed in the same way as in the case $p \in (-1/n, +\infty) \setminus \{0\}$, obtaining the $+\infty$-concavity of the involved functions. We argue by applying again Proposition 4.1.2, this time for the functions $\mathfrak{p} \in H_2^+$ and $\mathfrak{p}_n$ given by

$$\mathfrak{p}(u,v) = \mathcal{M}_{+\infty}(u,v;\lambda), \qquad \mathfrak{p}_n(u,v) = \begin{cases} \left[u^{1/n} + v^{1/n}\right]^n & \text{if}\ \ uv > 0, \\ 0 & \text{if}\ \ uv = 0. \end{cases}$$

Let $\lambda \in (0,1)$, h, f, g nonnegative (with positive finite integrals) satisfying (3.35) for $p = +\infty$, i.e. h, f_0, g_0 satisfy (3.37), where f_0, g_0 are given by (3.41). Suppose that equality holds in the BBL of index $p = +\infty$, i.e.

$$\int_{\mathbb{R}^n} h = \mathcal{M}_{1/n}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\,;\lambda\right);$$

this is equivalent to (see the proof of Proposition 3.4.3)

$$\int_{\mathbb{R}^n} h = \mathfrak{p}_n\left(\int_{\mathbb{R}^n} f_0, \int_{\mathbb{R}^n} g_0\right) = \left[\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}\right]^n,$$

so we can apply Proposition 4.1.2 to the functions h, f_0, g_0. Let $m \in (0, +\infty)$ be the coefficient given by Proposition 4.1.2: the condition (4.9) states:

$$\left[\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}\right]^n = \max\left\{(m+1)^n \left(\int_{\mathbb{R}^n} f_0\right),\ \left(\frac{m+1}{m}\right)^n \left(\int_{\mathbb{R}^n} g_0\right)\right\}.$$

Setting $t = 1/(m+1)$ it trivially becomes

$$\left[\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}\right]^n = \max\left\{t^{-n} \left(\int_{\mathbb{R}^n} f_0\right),\ (1-t)^{-n} \left(\int_{\mathbb{R}^n} g_0\right)\right\}.$$

We show, arguing by contradiction, that the latter identity imply

$$t = \frac{\left(\int_{\mathbb{R}^n} f_0\right)^{1/n}}{\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}}, \tag{4.22}$$

whence, thanks to (3.42),

$$m = \frac{1-t}{t} = \left(\frac{\int_{\mathbb{R}^n} g_0}{\int_{\mathbb{R}^n} f_0}\right)^{1/n} = \frac{\lambda}{1-\lambda}\left(\frac{\int_{\mathbb{R}^n} g}{\int_{\mathbb{R}^n} f}\right)^{1/n}. \qquad (4.23)$$

Indeed if

$$t > \frac{\left(\int_{\mathbb{R}^n} f_0\right)^{1/n}}{\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}}, \quad \text{we have} \quad 1-t < \frac{\left(\int_{\mathbb{R}^n} g_0\right)^{1/n}}{\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}},$$

hence

$$(1-t)^{-n}\left(\int_{\mathbb{R}^n} g_0\right) > \frac{\left[\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}\right]^n}{\int_{\mathbb{R}^n} g_0} \cdot \left(\int_{\mathbb{R}^n} g_0\right).$$

Thus

$$\max\left\{t^{-n}\left(\int_{\mathbb{R}^n} f_0\right),\ (1-t)^{-n}\left(\int_{\mathbb{R}^n} g_0\right)\right\} > \left[\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}\right]^n,$$

which contradicts the previous identity. Likewise

$$t < \frac{\left(\int_{\mathbb{R}^n} f_0\right)^{1/n}}{\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}} \quad \text{yields} \quad t^{-n}\left(\int_{\mathbb{R}^n} f_0\right) > \left[\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}\right]^n,$$

leading to the same contradiction.

For every $x_1, x_2 \in \mathbb{R}^n$ such that $\varphi(x_1) > 0$ and $\varphi(x_2) > 0$, condition (4.8) means

$$\varphi\left(\frac{x_1 + mx_2}{1+m}\right) \geq \frac{\max\left\{\left(\int_{\mathbb{R}^n} f_0\right)\varphi(x_1), m^{-n}\left(\int_{\mathbb{R}^n} g_0\right)\varphi(x_2)\right\}}{(m+1)^{-n} \cdot \left(\int_{\mathbb{R}^n} h\right)}$$

$$= \frac{\max\left\{\left(\frac{1}{m+1}\right)^{-n}\left(\int_{\mathbb{R}^n} f_0\right)\varphi(x_1),\ \left(\frac{m}{m+1}\right)^{-n}\left(\int_{\mathbb{R}^n} g_0\right)\varphi(x_2)\right\}}{\left[\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}\right]^n} \quad \text{i.e.}$$

$$\varphi(tx_1 + (1-t)x_2) \geq \frac{\max\left\{t^{-n}\left(\int_{\mathbb{R}^n} f_0\right)\varphi(x_1),\ (1-t)^{-n}\left(\int_{\mathbb{R}^n} g_0\right)\varphi(x_2)\right\}}{\left[\left(\int_{\mathbb{R}^n} f_0\right)^{1/n} + \left(\int_{\mathbb{R}^n} g_0\right)^{1/n}\right]^n},$$

where $t = \frac{1}{m+1}$. Using (4.22) we deduce

$$\varphi(tx_1 + (1-t)x_2) \geq \max\{\varphi(x_1),\ \varphi(x_2)\}.$$

Moreover φ is a continuous function, being $(-1/n)-$concave (thanks to Proposition 4.1.2). Therefore φ is $(+\infty)$-concave, i.e. it is a positive constant in its support, which is a convex compact set of positive measure. Finally (4.5), (4.6) and (4.7) state that h, f_0, g_0 coincide (almost everywhere) up to homotheties, where m is given by (4.23). Recalling (3.41) we have proved the following claim.

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Proposition 4.3.2 (Equality conditions for BBL inequality of index $p = +\infty$).
Let $\lambda \in (0, 1)$, $f, g, h : \mathbb{R}^n \longrightarrow [0, +\infty)$ with positive finite integrals such that

$$h((1 - \lambda)x + \lambda y) \geq \max \{f(x), g(y)\} \qquad for\ all\ \ x, y \in \mathbb{R}^n.$$

Then equality holds in BBL inequality, i.e.

$$\int_{\mathbb{R}^n} h = \mathcal{M}_{1/n} \left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\,;\lambda \right),$$

if and only if there exist $b \in \mathbb{R}^n$, and a $(+\infty)$-concave function φ such that a.e.

$$(1 - \lambda)^n \left(\int_{\mathbb{R}^n} f \right) \varphi(x) = f \left(\frac{x}{1 - \lambda} \right) = g \left(\frac{mx + b}{\lambda} \right) = h \left((m + 1)x + b \right)$$

where

$$m = \frac{\lambda}{1 - \lambda} \left(\frac{\int_{\mathbb{R}^n} g}{\int_{\mathbb{R}^n} f} \right)^{1/n}.$$

In particular φ is constant in its convex compact support of positive measure, namely there exists a convex body K such that almost everywhere

$$\varphi(x) = \frac{\chi_K(x)}{|K|}.$$

Consequently also f, g, h are constant functions in their respective supports, which preserve the same properties of K.

4.3.3   Case $p = -1/n$

At last we deal with the equality case for the BBL inequality of index $-1/n$. This case is the simplest one with regard to the development of the conditions (4.9) and (4.8) of Proposition 4.1.2. We proceed in the same way of the previous cases, that is with the application of Proposition 4.1.2, this time for the function $\mathfrak{p} \in H_2^+$ given by

$$\mathfrak{p}(u, v) = \left[u^{-1/n} + v^{-1/n} \right]^{-n},$$

with the corresponding $\mathfrak{p}_n(u, v) = \min \{u, v\}$, as we have seen in the proof of Corollary 3.4.3, case (iv).

In the previous case of BBL inequalities, the equality in BBL, by means of (4.9), determines uniquely the coefficient m provided in Proposition 4.1.2. It depends on the fixed constants λ, n, p, $\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g$: see (4.14), (4.18), (4.23), respectively for $p = 0$, $p \in (+\infty, -1/n) \setminus \{0\}$, $p = +\infty$. Thanks to these formulas of m we were able to deduce the p-concavity of the involved functions.

Instead in the case $p = -1/n$ the corresponding condition (4.9) will not provide informations about m. Remarkably in this case the equality in BBL forces f and g to have the same integral. In other words one can easily prove that (4.9) in this case is equivalent to $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$. Precisely it holds

Proposition 4.3.3 (Equality conditions for BBL of index $-1/n$)**.**
Let $\lambda \in (0,1)$, $f, g, h : \mathbb{R}^n \longrightarrow [0, +\infty)$ with positive finite integrals such that

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_{-1/n}\left(f(x), g(y); \lambda\right) \qquad \text{for all} \ \ x, y \in \mathbb{R}^n. \tag{4.24}$$

Then equality holds in BBL inequality, i.e.

$$\int_{\mathbb{R}^n} h = \min\left\{ \int_{\mathbb{R}^n} f, \ \int_{\mathbb{R}^n} g \right\}, \tag{4.25}$$

if and only if

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$$

and simultaneously there exist a $(-1/n)$-concave function φ, $m \in (0, +\infty)$, $b \in \mathbb{R}^n$ such that almost everywhere

$$\left(\int_{\mathbb{R}^n} f \right) \cdot \varphi(x) \ = \ (1-\lambda)^{-n} \cdot f\left(\frac{x}{1-\lambda} \right) \tag{4.26}$$

$$= \ \left(\frac{m}{\lambda} \right)^n \cdot g\left(\frac{mx+b}{\lambda} \right) \ = \ (m+1)^n \cdot h\left((m+1)x + b \right).$$

Proof. One implication is trivial: if the function f, g, h satisfy (4.26) then $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = \int_{\mathbb{R}^n} h$, hence (4.25) holds. The opposite implication relies on the application of Proposition 4.1.2 to the functions h, f_0, g_0, where f_0, g_0 are defined as in (3.45). After showing that the integrals of f and g coincide, (4.26) follows immediately from (4.5), (4.6) and (4.7). Condition (4.9), setting $t = 1/(m+1)$, states:

$$\min\left\{ \int_{\mathbb{R}^n} f, \ \int_{\mathbb{R}^n} g \right\} = \left[t\left(\int_{\mathbb{R}^n} f \right)^{-1/n} + (1-t)\left(\int_{\mathbb{R}^n} g \right)^{-1/n} \right]^{-n}$$

i.e. $\qquad \mathcal{M}_{-\infty}\left(\int_{\mathbb{R}^n} f, \ \int_{\mathbb{R}^n} g; \ 1-t \right) \ = \ \mathcal{M}_{-1/n}\left(\int_{\mathbb{R}^n} f, \ \int_{\mathbb{R}^n} g; \ 1-t \right).$

Thus $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$. Finally the $(-1/n)$-concavity of φ is ensured by Proposition 4.1.2. $\quad\square$

Notice that the latter proposition may be seen as a generalization of Theorem 4.1.1 to the functions f and g whose integrals coincide with a value not necessarily equal to 1 (differently from Theorem 4.1.1).

About equality in BBL inequality of index $-1/n$, we note that the necessary condition

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$$

is not entirely new in literature. Indeed very recently Balogh and Kristály in the paper [4], concerning equality case in BBL in the generalized framework of Riemannian manifolds, proved (we refer to Theorem 3.1, condition (iii)) that equality in BBL of index

$p = -1/n$ implies $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$, assuming that $h, f, g : \mathbb{R}^n \longrightarrow [0, +\infty)$ are non-zero compactly supported integrable functions satisfying (4.24). The generality of their result is considerable, because it holds not only in Euclidean spaces but in the wide context of Riemannian manifolds. On the other hand their statement requires the compactness of the support sets of the involved functions, differently from Proposition 4.3.3. Their approach is based on an estimate for the *Borell-Brascamp-Lieb deficit*

$$\delta_\lambda^{(-1/n)}(f, g, h) = \frac{\int_{\mathbb{R}^n} h}{\min\left\{\int_{\mathbb{R}^n} f, \ \int_{\mathbb{R}^n} g\right\}} - 1 \ \geq 0,$$

which is achieved by using the theory of optimal mass transportation and a quantitative version of Lemma 3.1.1.

Chapter 5

Known stability results for BBL inequalities

The aim of this chapter is to present some stability results concerning BBL inequalities, in order to introduce our new stability results, proved in [43] and described in detail in Chapter 7.

To our knowledge, the existing literature before [43] consists of only four works. We are referring to [2, 3, 12], in which stability results of PL inequality for log-concave functions are established, and to [30], where the authors prove quantitative BBL inequalities in the class of p-concave functions for $p > 0$. The aim of the research presented in this thesis has been that of finding stability results for BBL inequalities which do not require any assumption about the power concavity of the involved functions, while instead the closeness to power concavity is one of the objective of our stability result.

Let us introduce, in chronological order, the main statements of [2, 3, 12, 30], giving a brief idea of their proofs.

5.1 Stability for PL inequality

The first stability results, due to Ball and Böröczky, regard the PL inequality. The authors in [2, 3] investigated the stability of the following version of PL inequality, which is simply Proposition 3.1.4 for $\lambda = 1/2$.

Proposition 5.1.1 (PL inequality, $\lambda = 1/2$)**.**
Let f, g, h be nonnegative measurable functions defined in $\mathbb{R}^n$ such that

$$h\left(\frac{x+y}{2}\right) \geq \sqrt{f(x)g(y)} \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{5.1}$$

Then

$$\int_{\mathbb{R}^n} h \geq \sqrt{\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g}. \tag{5.2}$$

In Section 4.2 we have characterized the equality case of the latter proposition if the integrals of h, f, g are finite and positive. Indeed Proposition 4.2.1 states that equality holds in (5.2) only if (see (4.11) for $\lambda = 1/2$) there exist $b \in \mathbb{R}^n$, and a log-concave function φ such that a.e.

$$2^{-n} \cdot \varphi(x) \; = \; \frac{f(2x)}{\int_{\mathbb{R}^n} f} \; = \; \frac{g(2x + 2b)}{\int_{\mathbb{R}^n} g} \; = \; \frac{h(2x + b)}{\sqrt{\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g}}. \tag{5.3}$$

In other words h is a log-concave function and it is easy to check that (5.3) gives

$$f(t) = ah(t + b), \qquad g(t) = a^{-1}h(t - b), \quad \text{where} \quad a = \sqrt{\frac{\int_{\mathbb{R}^n} f}{\int_{\mathbb{R}^n} g}}.$$

Ball and Böröczky proved in [2] the stability of PL inequality (5.2) for $n = 1$, in the special class of log-concave functions.

Theorem 5.1.1 (Theorem 1.2 in [2])**.** *There exists a positive absolute constant c with the following property. If h, f, g are* log*-concave functions defined in $\mathbb{R}$ with finite positive integrals satisfying* (5.1) *and such that*

$$\int_{\mathbb{R}} h \; \leq (1 + \epsilon) \sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g} \tag{5.4}$$

for a certain $\epsilon > 0$, then there exist $a > 0$, $b \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} |f(t) - ah(t + b)| \; dt \leq c \cdot \omega(\epsilon) \cdot a \cdot \int_{\mathbb{R}} h,$$

$$\int_{\mathbb{R}} \left| g(t) - a^{-1}h(t - b) \right| \; dt \leq c \cdot \omega(\epsilon) \cdot a^{-1} \cdot \int_{\mathbb{R}} h,$$

where
$$\omega(\epsilon) = \epsilon^{1/3} \left| \ln \epsilon \right|^{4/3}. \tag{5.5}$$

Remark 5.1.2. The estimate in latter theorem is probably not sharp: in fact the authors conjecture an optimal estimate of order ϵ. They also admit that it is not clear if the assumption of log-concavity is really necessary to derive a stability estimate of the PL inequality.

Let us summarize the idea to prove Theorem 5.1.1. It can be assumed that f, g are log-concave probability distributions with zero mean. After deducing some fine properties of log-concave probability distributions holding in dimension 1, the authors study the one-dimensional case by exploiting the mass transportation of the distributions. In particular they translate the condition (5.4) into an integral estimate for the transportation map $T : \text{Supp} f \longrightarrow \text{Supp} g$ defined by the identity

$$\int_{-\infty}^{x} f(t) \; dt = \int_{-\infty}^{T(x)} g(t) \; dt.$$

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This estimate, through other delicate estimates regarding the transportation map, allows to control the L^1 distance between f and g and hence to obtain Theorem 5.1.1.

Extending this kind of result to higher dimension is a rather delicate problem. In [3] Ball and Böröczky extended Theorem 5.1.1 to higher dimensions for *even* log-concave functions. The restriction to such a class of functions is crucial. Indeed the underlying idea is to apply a refined version of the BM inequality proved by Figalli, Maggi and Pratelli, to the convex and origin-symmetric (thanks to the evenness) superlevel sets of the involved functions. This version involves the notion of relative asymmetry (2.15) for convex bodies, which reduces to the Lebesgue measure of their symmetric difference, since these convex bodies are origin-symmetric. This is the heuristic reason why, after integration, such approach leads to the same kind of L^1 estimate between the initial functions as in dimension $n = 1$. Precisely they prove the following

Theorem 5.1.2 (Theorem 4.1 in [3]). *There exists a positive constant $c(n)$ (depending only on the dimension n) with the following property: if h, f, g are even* log-*concave functions on $\mathbb{R}^n$ with finite positive integrals satisfying* (5.1) *and*

$$\int_{\mathbb{R}^n} h \ \leq (1 + \epsilon) \sqrt{\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g},$$

for a certain $\epsilon > 0$, then there exists $a > 0$ such that

$$\int_{\mathbb{R}^n} |af(x) - h(x)| \ dx \leq c(n) \cdot \sqrt{\omega(\epsilon)} \cdot \int_{\mathbb{R}^n} h$$

$$\int_{\mathbb{R}^n} \left| a^{-1} g(x) - h(x) \right| \ dx \leq c(n) \cdot \sqrt{\omega(\epsilon)} \cdot \int_{\mathbb{R}^n} h$$

where $\omega(\epsilon)$ is defined as in (5.5).

In fact this generalization has been obtained arguing by slices, that is considering the corresponding superlevel sets of the functions f, g, h and applying consequently a stability version of the BM inequality. For $t > 0$, let

$$\Phi_t = \{x \in \mathbb{R}^n : \ f(x) \geq t\}, \quad \Psi_t = \{x \in \mathbb{R}^n : \ g(x) \geq t\}, \quad \Omega_t = \{x \in \mathbb{R}^n : \ h(x) \geq t\}.$$

Let $r, s > 0$; then the assumption (5.1) yields the inclusion

$$\frac{1}{2}\Phi_r + \frac{1}{2}\Psi_s \subseteq \Omega_{\sqrt{rs}},$$

provided that Φ_r, Ψ_s are nonempty sets. Hence the BM inequality states

$$\left| \Omega_{\sqrt{rs}} \right|^{1/n} \geq \frac{1}{2} \left| \Phi_r \right|^{1/n} + \frac{1}{2} \left| \Psi_s \right|^{1/n}.$$

By assumptions of Theorem 5.1.2, the superlevel sets Φ_t, Ψ_t, Ω_t are origin-symmetric convex bodies to whom we apply a variant of Proposition 2.5.2, i.e. a stability version of

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the BM inequality for convex bodies in terms of their homothetic distance $\mathcal{A}(\cdot, \cdot)$ given in (2.15). Since the functions are even, their superlevel sets are origin-symmetric: this convenient structure allow to simplify considerably the estimate provided. Indeed if $J = L \in \mathcal{K}^n$ are origin-symmetric and $|J| = |L|$ the infimum in (2.15) is attained for $x = o$ (the origin of the coordinate system) and $\mu = 1$, thus

$$\mathcal{A}(J, L) = \frac{|J\Delta L|}{|J|}.$$

In this way it is possible to estimate the closeness of the corresponding superlevel sets in terms of their symmetric difference. Further estimates concerning these superlevel sets, combined with the application of the one dimensional Theorem 5.1.1 to the functions

$$F(t) = |\Phi_t|, \qquad G(t) = |\Psi_t|, \qquad H(t) = |\Omega_t|,$$

lead to the proof of Theorem 5.1.2.

Bucur and Fragalà proved some refinements of the PL inequality in their work [12], which is linked to the mentioned papers of Ball and Böröczky. These refinements consist in the addition of an extra-term depending on a *distance modulo translations d* and hold on suitable classes $\mathcal{F}$ of functions on n variables. By this we mean that $d : \mathcal{F} \times \mathcal{F} \longrightarrow [0, +\infty)$ is a symmetric function satisfying the triangular inequality, and such that

$$d(f, g) = 0 \quad \text{if and only if} \quad f(x) = g(x + b) \text{ for some b } \in \mathbb{R}^n.$$

For example (see Proposition 3.2 in [12]) the function

$$d(f, g) = \inf_{b \in \mathbb{R}} \int_{\mathbb{R}} |f(x) - g(x + b)| \; dx \tag{5.6}$$

is a distance modulo translation on

$$\mathcal{A} = \left\{ f \in L^1\left(\mathbb{R}; [0, +\infty)\right) : \int_{\mathbb{R}} f = 1, \; f \text{ is log-concave} \right\}. \tag{5.7}$$

Among the improved versions of the PL inequality, we mention the quantitative version obtained for log-concave functions using the one-dimensional result by Ball and Böröczky, i.e. Theorem 5.1.1. By following Theorem 5.1.1 and the proof of the n-dimensional PL inequality by induction on n (cf. [29], Theorem 4.2), in [12] Bucur and Fragalà improved the PL inequality for f, g belonging to suitable classes of $L^1\left(\mathbb{R}^n; [0, +\infty)\right)$. One of the main result in in [12] can be written in the following way.

Proposition 5.1.3 (Theorem 3.1 and Proposition 3.2, [12]).
Let $0 < \lambda < 1$, f, g, h be nonnegative functions defined in $\mathbb{R}^n$ such that

$$h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda \qquad \text{for all } x, \; y \in \mathbb{R}^n.$$

If f, g belong to the subclass $\mathcal{A}_n$ of $L^1(\mathbb{R}^n; [0, +\infty))$, corresponding to $\mathcal{A}$ (5.7) (see Corollary 2.3 in [12]), then

$$\int_{\mathbb{R}^n} h\, dx \geq [1 + \psi_\lambda(d_n(f,g))] \left(\int_{\mathbb{R}^n} f\, dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\, dx\right)^\lambda,$$

where d_n is the distance modulo translations on $\mathcal{A}_n$, corresponding to d (5.6) (see Corollary 2.3 in [12]) and $\psi_\lambda : [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous increasing function such that $\psi_\lambda(0) = 0$. Precisely $\psi_\lambda(t)$ is the inverse function on $[0, +\infty)$ of the function

$$c(\lambda)t^{1/3} |\ln t|^{4/3} (1 + t),$$

being $c = c(\lambda)$ a suitable positive constant.

5.2 Stability for BBL inequalities of index $p > 0$

The stability of the BBL inequalities of index $p > 0$ has been recently investigated by Ghilli and Salani [30]. They strengthen, in two different ways, the BBL inequality (Theorem 3.1.1) of index $p > 0$ in the class of p-concave compactly supported functions. Their main result (Theorem 4.1) is a stability result for BBL inequality of index $p > 0$ and some consequent quantitative version of Theorem 3.1.1, provided that f, g are p-concave compactly supported functions. With "quantitative" it means that (3.2) can be strengthened in terms of some distance between the functions f, g: precisely in term of some distance between their support sets

$$\Omega_0 = \mathrm{Supp}(f), \qquad \Omega_1 = \mathrm{Supp}(g).$$

Here the used distance are: $H_0(\Omega_0, \Omega_1)$, given in (2.14), and $\mathcal{A}(\Omega_0, \Omega_1)$, introduced in (2.15).

The crucial idea relies on an estimate of the measures of Ω_0 and Ω_1, provided in the following Theorem 4.1 (which can be considered the main result in [30]).

Theorem 5.2.1 (Theorem 4.1 in [30]).
Let $0 < \lambda < 1$, $p > 0$, $0 \leq f, g, h \in L^1(\mathbb{R}^n)$ such that

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda) \qquad \text{for all } x, y \in \mathbb{R}^n.$$

Assume furthermore that f, g are p-concave functions with convex compact supports Ω_0 and Ω_1 respectively, and let $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$. If for some $0 < \epsilon < (2n)^{-\frac{p+1}{p}}$ it holds

$$\int_{\Omega_\lambda} h(x)\, dx \leq \mathcal{M}_{\frac{p}{np+1}}\left(\int_{\Omega_0} f(x)\, dx, \int_{\Omega_1} g(x)\, dx\,;\lambda\right) + \epsilon, \tag{5.8}$$

then

$$|\Omega_\lambda| \leq \mathcal{M}_{1/n}(|\Omega_0|, |\Omega_1|; \lambda)\left[1 + \eta^{\frac{p}{p+1}}\right] \tag{5.9}$$

where

$$\eta \leq 2 \left(n + \mathcal{M}_{\frac{p}{np+1}} \left(\int_{\Omega_0} f(x)\,dx, \int_{\Omega_1} g(x)\,dx \,;\, \lambda \right)^{-1} \right).$$

This statement is meaningful: if we are close to equality in the BBL inequality, i.e. condition (5.8) holds, then the measure of the Minkowski combination Ω_λ is close to $\mathcal{M}_{1/n}(|\Omega_0, \Omega_1; \lambda|)$. In other words the closeness (5.8) to equality in the BBL inequality (regarding the functions f, g) yields the closeness (5.9) to equality in the BM inequality (regarding the corresponding support sets Ω_0, Ω_1). Then the leading idea is to apply existing quantitative results for the classical BM inequality: precisely Propositions 2.5.1 and 2.5.2, described in Section 2.5 and actually written in terms of the two mentioned distances $H_0(\Omega_0, \Omega_1)$ and $\mathcal{A}(\Omega_0, \Omega_1)$. This leads to the following quantitative versions of Theorem 3.1.1.

Theorem 5.2.2 (Theorems 1.2-1.3 in [30])**.**
Let $0 < \lambda < 1$, $p > 0$, $0 \leq f, g, h \in L^1(\mathbb{R}^n)$ such that

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p(f(x), g(y); \lambda) \qquad \text{for all } x,\, y \in \mathbb{R}^n.$$

Assume furthermore that f, g are p-concave functions with convex compact supports Ω_0 and Ω_1 respectively, and let $\Omega_\lambda = (1-\lambda)\Omega_0 + \lambda\Omega_1$. Then, if $H_0(\Omega_0, \Omega_1)$ or $\mathcal{A}(\Omega_0, \Omega_1)$ is small enough, it holds respectively

$$\int_{\Omega_\lambda} h(x)\,dx \geq \mathcal{M}_{\frac{p}{np+1}} \left(\int_{\Omega_0} f(x)\,dx, \int_{\Omega_1} g(x)\,dx \,;\, \lambda \right) + \beta H_0(\Omega_0, \Omega_1)^{\frac{(n+1)(p+1)}{p}}$$

or

$$\int_{\Omega_\lambda} h(x)\,dx \geq \mathcal{M}_{\frac{p}{np+1}} \left(\int_{\Omega_0} f(x)\,dx, \int_{\Omega_1} g(x)\,dx \,;\, \lambda \right) + \delta \mathcal{A}(\Omega_0, \Omega_1)^{\frac{2(p+1)}{p}}$$

where β is a constant depending only on n, λ, p, $\int_{\Omega_0} f, \int_{\Omega_1} g$, and on the diameters and the measures of Ω_0 and Ω_1, and δ is a constant depending only on n, λ, p, $\int_{\Omega_0} f, \int_{\Omega_1} g$, and on the measures of Ω_0 and Ω_1.

Chapter 6

Stability for a strengthened one-dimensional BBL inequality

6.1 Introduction

We study stability issues for some one-dimensional integral inequalities, in particular for the extremal case $p = -1$ of the BBL inequalities. In this case, when near equality is realized, we prove that the involved functions must be L^1-close to be quasiconcave.

As motivated in Chapter 3 the Borell-Brascamp-Lieb inequalities (stated in Theorem 3.1.1) play a crucial role among all the integral inequalities, since they can be recognized as the functional versions of the Brunn-Minkowski inequality.
The equality conditions for (3.2) have been determined in Chapter 4 using Theorem 4.1.1 due to Dubuc [23].
In this chapter we deal only with the one-dimensional case, i.e. $n = 1$. Thus let us restate the BBL inequality for the extremal index $p = -1$, including its equality condition, provided in Proposition 4.3.3 with $n = 1$. The equality conditions in this case are particularly simple.

Proposition 6.1.1 (BBL inequality with $n = 1$ and $p = -1$). *Let $0 < \lambda < 1$, let $f, g, h : \mathbb{R} \longrightarrow [0, \infty)$ be integrable functions with nonempty compact supports such that*

$$h((1 - \lambda)x + \lambda y) \geq \mathcal{M}_{-1}(f(x), g(y); \lambda) \qquad \text{for every} \ \ x, y \in \mathbb{R}.$$

Then

$$\int_{\mathbb{R}} h \, dx \geq \min \left\{ \int_{\mathbb{R}} f \, dx, \int_{\mathbb{R}} g \, dx \right\}. \tag{6.1}$$

Moreover equality holds in (6.1) *if and only if*

$$\int_{\mathbb{R}} f \, dx = \int_{\mathbb{R}} g \, dx$$

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and there exist $m \in (0, +\infty), b \in \mathbb{R}$, and a (-1)-concave function φ such that a.e. $x \in \mathbb{R}$

$$\varphi(x) = \frac{1}{1-\lambda} \cdot f\left(\frac{x}{1-\lambda}\right) = \frac{m}{\lambda} \cdot g\left(\frac{mx+b}{\lambda}\right) = (m+1) \cdot h\left((m+1)x + b\right).$$

In [23] Dubuc proved also a strengthened one-dimensional version of Theorem 3.1.1 under the additional assumption that the involved functions have the same L^∞ norm.

Proposition 6.1.2 (Theoreme 9 in [23])**.**
Let $\varphi, \psi, \theta : \mathbb{R} \longrightarrow [0, +\infty)$ belong to $L^1(\mathbb{R})$ and satisfy

$$\theta(x+y) \geq \min\{\varphi(x), \psi(y)\} \qquad a.e. \ (x, y) \in \mathbb{R}^2. \tag{6.2}$$

Moreover suppose that φ and ψ have the same L^∞-norm. Then

$$\int_{\mathbb{R}} \theta(x)\,dx \geq \int_{\mathbb{R}} \varphi(x)\,dx + \int_{\mathbb{R}} \psi(x)\,dx. \tag{6.3}$$

Notice that the smallest function satisfying assumption (6.2) is the so-called *quasisup-convolution* of φ and ψ, namely the function denoted by $\theta_{-\infty}$ and given by

$$\theta_{-\infty}(z) = \sup\{\min\{\varphi(x), \psi(y)\} : \ x + y = z\}, \qquad z \in \mathbb{R}^n$$

(from a geometric point of view, $\theta_{-\infty}$ is the function whose superlevel sets are the Minkowski addition of the corresponding superlevel sets of φ and ψ). Then (6.2) and (6.3) can be condensed in

$$\int_{\mathbb{R}} \theta_{-\infty}(x)\,dx \geq \int_{\mathbb{R}} \varphi(x)\,dx + \int_{\mathbb{R}} \psi(x)\,dx. \tag{6.4}$$

As we said, Dubuc investigates the rigidity of BBL inequalities, and he gives explicit necessary and sufficient conditions for equality to hold in (6.4). Later on we will state precisely these conditions (see Proposition 6.1.3). For the moment, let us just say that equality in (6.4) holds only if the functions φ, ψ are essentially *quasiconcave*, i.e. if they coincide almost everywhere with two functions whose superlevel sets are convex.

When dealing with a rigid inequality, a natural question arises about the stability of the equality case; here the question at hand is the following: if we are close to equality in (6.4), must the functions φ and ψ be close (in some suitable sense) to quasiconcavity?

Regarding the BBL inequalities, the investigation of stability issues in the case $p = 0$ was started by Ball and Böröczky in [2, 3] and related results are in [12]. The case $p > 0$ has been faced in [30]. The results of [30], as well as the quoted results for $p = 0$, hold only in the restricted class of p-concave functions. New stability results for BBL inequalities in $\mathbb{R}^n$, without concavity assumptions, have been recently obtained in [43], again for a positive index $p > 0$.

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To our knowledge, no stability result has been yet proved in the case $p < 0$.

Here, as we said, we want to prove a stability result for Proposition 6.1.2, showing that near equality in (6.3) is possible only if the involved functions φ, ψ are nearly quasiconcave, with respect to L^1 distance. Precisely, the main result of this chapter is the following.

Theorem 6.1.1. *Let φ, ψ, θ as in Proposition 6.1.2 and let*

$$S = \|\varphi\|_\infty = \|\psi\|_\infty.$$

Assume that there exist $R > 0$, $\epsilon > 0$ such that $\operatorname{Supp}(\varphi) \cup \operatorname{Supp}(\psi) \subseteq [-R, R]$ and

$$\int_{\mathbb{R}} \theta(x)\, dx \le \int_{\mathbb{R}} \varphi(x)\, dx + \int_{\mathbb{R}} \psi(x)\, dx + \epsilon. \tag{6.5}$$

Then there exist two quasiconcave functions $\tilde\varphi_\epsilon, \tilde\psi_\epsilon : \mathbb{R} \longrightarrow [0, S]$ such that

$$\min\left\{\int_{\mathbb{R}} (\tilde\varphi_\epsilon(x) - \varphi(x))\ dx, \int_{\mathbb{R}} \left(\tilde\psi_\epsilon(x) - \psi(x)\right)\ dx\right\} \le 2\,(R + S)\,\sqrt{\epsilon}. \tag{6.6}$$

Moreover

$$\lim_{\epsilon \to 0}\ \max\left\{\int_{\mathbb{R}} (\tilde\varphi_\epsilon(x) - \varphi(x))\ dx, \int_{\mathbb{R}} \left(\tilde\psi_\epsilon(x) - \psi(x)\right)\ dx\right\} = 0. \tag{6.7}$$

The proof of the above theorem is based on the Cavalieri formula (1.1) and on the stability result for the BM inequality given in Proposition 2.6.1, properly applied to the superlevel sets of the involved functions. As a corollary we will get an asymptotic stability for the one-dimensional BBL inequality of index $p = -1$ (see Corollary 6.3.1 in Section 6.3).

The chapter is organized as follows: in Section 6.2 we prove Theorem 6.1.1, while Sections 6.3 and 6.4 contain some consequences of Theorem 6.1.1, in particular a quantitative stability result for a special class of functions.

We recall some notation from [23]. Let $x \in \mathbb{R}^n$ and A a measurable subset of $\mathbb{R}^n$: the *lower density* of A at the point x, denoted with $d_*(x, A)$, is the infimum of the collection of numbers

$$\liminf_{k \to +\infty} \frac{|A \cap P_k|}{|P_k|}$$

where $\{P_k\}_{k \in \mathbb{N}}$ is a sequence of n-dimensional rectangles (i.e. products of n-intervals of positive length) containing x such that $\operatorname{diam}(P_k)$ converges to 0 for $k \to +\infty$. The *lower density set* of A is given by

$$A_* = \{x \in \mathbb{R}^n :\ d_*(x, A) = 1\}. \tag{6.8}$$

By construction the set A_* is measurable with

$$|A_*| = |A|. \tag{6.9}$$

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Given a measurable function $\varphi : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$, the previous definition allows us to define a measurable function $\varphi_* : \mathbb{R}^n \longrightarrow [-\infty, +\infty]$, defined as

$$\varphi_*(x) = \sup \{t \in \mathbb{R} : (x,t) \in (End(\varphi))_*\}, \tag{6.10}$$

where $End(\varphi) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t \leq \varphi(x)\}$. Property (6.9) implies $\varphi = \varphi_*$ a.e..

Now we can precisely state the necessary and sufficient equality conditions for (6.3). Therefore we reformulate Proposition 6.1.2, giving the exact statement of Theoreme 9 in [23].

Proposition 6.1.3 (Theoreme 9 in [23]). *Let $\varphi, \psi, \theta : \mathbb{R} \longrightarrow [0, +\infty)$ belonging to $L^1(\mathbb{R})$ such that*

$$\theta(x + y) \geq \min \{\varphi(x), \psi(y)\} \qquad a.e. \ (x,y) \in \mathbb{R}^2. \tag{6.11}$$

Moreover suppose that

$$\|\varphi\|_\infty = \|\psi\|_\infty .$$

Then

$$\int_{\mathbb{R}} \theta(x) \, dx \geq \int_{\mathbb{R}} \varphi(x) \, dx + \int_{\mathbb{R}} \psi(x) \, dx. \tag{6.12}$$

In addition equality holds in (6.12) if and only if the functions φ_*, ψ_* and θ_* (defined as in (6.10)) are quasiconcave and θ_* is the quasisupconvolution of φ_*, ψ_*, i.e

$$\theta_*(z) = \sup \{\min \{\varphi_*(x), \psi_*(y)\} : x + y = z\} .$$

For the sake of completeness, let's see a sketch of the proof of Dubuc. The idea of his proof, namely the use of the Cavalieri formula and the consequent application of the one-dimensional BM inequality for the superlevel sets of the involved functions, is the first ingredient of the proof of Theorem 6.1.1. The second one will be a careful use of Proposition 2.6.1.

A sketch of the proof of Proposition 6.1.3. We use similar notations as Dubuc. Let S be the common value of the L^∞-norms of φ and ψ, i.e.

$$S = \|\varphi\|_\infty = \|\psi\|_\infty .$$

For every $t \in [0, S]$, we set

$$A(t) = \{x : \varphi(x) \geq t\}, \quad B(t) = \{x : \psi(x) \geq t\}, \quad C(t) = \{x : \theta(x) \geq t\}$$

and denote respectively with $F(t), G(t), H(t)$ the Lebesgue measure of the superlevel sets $A(t), B(t), C(t)$. As explained in Dubuc's proof, the assumption (6.11) implies

$$A(t)_* + B(t)_* \subseteq C(t)_* \qquad \forall \, t \in [0, S] \tag{6.13}$$

where $A(t)_*, B(t)_*, C(t)_*$ are the lower density sets (see (6.8)) of $A(t), B(t), C(t)$ (in particular they are measurable and have respectively the same measure of $A(t), B(t), C(t)$).

The assumption $\|\varphi\|_\infty = \|\psi\|_\infty$ ensures that the superlevel sets $A(t), B(t)$ are nonempty at least for $t < S$, so their Minkowski sum in (6.13) is well-defined and nonempty.

Then, using (6.13) and applying the Brunn-Minkowski inequality (2.2) ($n = 1$), we get

$$F(t) + G(t) = |A(t)_*| + |B(t)_*| \leq |A(t)_* + B(t)_*| \leq H(t) \qquad \forall\, t \in [0, S]. \qquad (6.14)$$

Since Cavalieri formula (1.1) implies

$$\int_{\mathbb{R}} \varphi(x)dx = \int_0^S F(t)dt, \quad \int_{\mathbb{R}} \psi(x)dx = \int_0^S G(t)dt, \quad \int_{\mathbb{R}} \theta(x)dx \geq \int_0^S H(t)dt, \qquad (6.15)$$

we deduce the thesis (6.12), thanks to (6.14).

Now suppose that equality holds in (6.12), i.e.

$$\int_{\mathbb{R}} \theta(z)\, dz = \int_{\mathbb{R}} \varphi(x)\, dx + \int_{\mathbb{R}} \psi(y)\, dy.$$

This means (thanks to (6.14) and (6.15)) that

$$F(t) + G(t) = H(t) \qquad \forall\, t \in [0, S].$$

Lusternik [38] showed that this condition necessarily implies that $A(t)_*, B(t)_*$ are intervals and

$$A(t)_* + B(t)_* = C(t)_*.$$

Therefore φ_*, ψ_* are quasiconcave and θ_* is their quasisupconvolution. Vice versa, if φ_*, ψ_* are quasiconcave and θ_* is their quasisupconvolution, then equality in (6.12) holds trivially. $\qquad\square$

6.2 The proof of Theorem 6.1.1

Let us use the same notation as in the proof of Proposition 6.1.3. The assumption (6.5) can be rewritten as follows:

$$\int_0^S H(t)\, dt \;\leq\; \int_0^S F(t)\, dt + \int_0^S G(t)\, dt + \epsilon,$$

or equivalently

$$\int_0^S [H(t) - F(t) - G(t)]\; dt \leq \epsilon. \qquad (6.16)$$

For every $t \in [0, S]$, we define the nonnegative (thanks to (6.14)) measurable function

$$E(t) = H(t) - F(t) - G(t).$$

Furthermore, we fix $\delta > 0$ and set $E_\delta = \{t \in [0, S] : E(t) \geq \delta\}$. Thus, thanks to (6.16)

$$\epsilon \geq \int_0^S E(t)\, dt \geq \int_{E_\delta} E(t)\, dt \geq \delta\, |E_\delta|,$$

whence

$$|E_\delta| \leq \frac{\epsilon}{\delta}. \qquad (6.17)$$

If ϵ is small enough ($\epsilon < S\delta$), obviously it holds $|E_\delta| < S$, i.e. E_δ is a proper subset of $[0, S]$ (i.e. $[0, S] \setminus E_\delta = E_\delta^C$ is non empty).

If $t \in E_\delta^C$ then $H(t) < F(t) + G(t) + \delta$, which means

$$|A(t)_* + B(t)_*| < |A(t)_*| + |B(t)_*| + \delta. \qquad (6.18)$$

Then we set

$$s(\delta) = \sup \{t \in [0, S] : \ \delta \leq \min\{|A(t)_*|, |B(t)_*|\}\},$$

and

$$t(\delta) = S - s(\delta), \qquad (6.19)$$

and observe that

$$\lim_{\delta \to 0} t(\delta) = 0\,.$$

If $t \in E_\delta^C \cap [0, s(\delta))$, then (6.18) holds with $\delta \leq \min\{|A(t)_*|, |B(t)_*|\}$, so we can apply Proposition 2.6.1 getting

$$|conv\,(A(t)_*) \setminus A(t)_*| \leq \delta, \quad |conv\,(B(t)_*) \setminus B(t)_*| \leq \delta. \qquad (6.20)$$

Now we are ready to define the nonnegative function $\tilde\varphi : \mathbb{R} \longrightarrow [0, S]$ (we omit the subindex ϵ for simplicity) by means of its superlevel sets $\{x : \tilde\varphi(x) \geq t\}$ (for $t \geq 0$), that we denote with $\tilde A(t)$. We set

$$\{x : \tilde\varphi(x) \geq t\} = \tilde A(t) := conv\,(A(t)_*) \qquad 0 \leq t \leq S.$$

Having defined all its superlevel sets $\tilde A(t)$, the function $\tilde\varphi$ is uniquely determined. Clearly $\tilde\varphi$ is quasiconcave, since $\tilde A(t)$ is convex for every t. Moreover $\tilde\varphi \geq \varphi$ and

$$\int_{\mathbb{R}} (\tilde\varphi(x) - \varphi_*(x)) \ dx = \int_{[A(s(\delta))_*]^C} (\tilde\varphi(x) - \varphi_*(x)) \ dx + \int_{A(s(\delta))_*} (\tilde\varphi(x) - \varphi_*(x)) \ dx.$$

By Cavalieri formula we have

$$\int_{[A(s(\delta))_*]^C} \tilde\varphi(x) \ dx = \int_0^{s(\delta)} \left| \tilde A(t) \cap [A(s(\delta))_*]^C \right| \ dt,$$

$$\int_{[A(s(\delta))_*]^C} \varphi_*(x) \ dx = \int_0^{s(\delta)} \left| A(t)_* \cap [A(s(\delta))_*]^C \right| \ dt,$$

so

$$\int_{[A(s(\delta))_*]^C} (\tilde\varphi(x) - \varphi_*(x)) \ dx$$

$$= \int_0^{s(\delta)} \left| \tilde A(t) \cap [A(s(\delta))_*]^C \right| - \left| A(t)_* \cap [A(s(\delta))_*]^C \right| dt$$

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$$\int_0^{s(\delta)} \left| \left(\tilde{A}(t) \setminus A(t)_* \right) \cap [A(s(\delta))_*]^C \right| \leq \int_0^{s(\delta)} \left| \tilde{A}(t) \setminus A(t)_* \right| \, dt$$

$$\int_{E_\delta \cap [0, s(\delta))} \left| \tilde{A}(t) \setminus A(t)_* \right| \, dt + \int_{E_\delta^C \cap [0, s(\delta))} \left| \tilde{A}(t) \setminus A(t)_* \right| \, dt.$$

In the first integral, we can use the information (6.17) on the measure of E_δ, and the obvious inclusions

$$\left(\tilde{A}(t) \setminus A(t)_* \right) \subseteq \tilde{A}(t) \subseteq \tilde{A}(0) \subseteq \mathrm{Supp}(\tilde{\varphi}) \subseteq [-R, R],$$

obtaining

$$\int_{E_\delta \cap [0, s(\delta))} \left| \tilde{A}(t) \setminus A(t)_* \right| \, dt \leq \int_{E_\delta} 2R \, dt \ \leq \ 2R \, |E_\delta| \leq 2R \frac{\epsilon}{\delta}.$$

For the second integral, thanks to (6.20), we deduce that

$$\int_{E_\delta^C \cap [0, s(\delta))} \left| \tilde{A}(t) \setminus A(t)_* \right| \, dt = \int_{E_\delta^C \cap [0, s(\delta))} \left| conv\left(A(t)_* \right) \setminus A(t)_* \right| \, dt$$

$$\leq \ \int_{E_\delta^C \cap [0, s(\delta))} \delta \, dt \ \leq \ \delta \left| E_\delta^C \right| \leq \delta S.$$

Then it holds

$$\int_{[A(s(\delta))_*]^C} \left(\tilde{\varphi}(x) - \varphi_*(x) \right) \, dx \ \leq \ 2R \frac{\epsilon}{\delta} + \delta S. \tag{6.21}$$

On the other hand we have

$$\int_{A(s(\delta))_*} \left(\tilde{\varphi}(x) - \varphi_*(x) \right) \, dx \ \leq \ |A(s(\delta))_*| \cdot t(\delta). \tag{6.22}$$

Indeed it holds

$$\int_{A(s(\delta))_*} \left(\tilde{\varphi}(x) - \varphi_*(x) \right) \, dx = \int_{A(s(\delta))_*} \left(\tilde{\varphi}(x) - s(\delta) \right) \, dx$$

$$\leq \int_{A(s(\delta))_*} \left(S - s(\delta) \right) \, dx = |A(s(\delta))_*| \cdot t(\delta).$$

Finally (6.21) and (6.22) imply

$$\int_{\mathbb{R}} \left(\tilde{\varphi}(x) - \varphi_*(x) \right) \, dx = \int_{[A(s(\delta))_*]^C} \left(\tilde{\varphi}(x) - \varphi_*(x) \right) \, dx + \int_{A(s(\delta))_*} \left(\tilde{\varphi}(x) - \varphi_*(x) \right) \, dx$$

$$\leq \ 2R \frac{\epsilon}{\delta} + \delta S + |A(s(\delta))_*| \cdot t(\delta).$$

Choosing $\delta = \sqrt{\epsilon}$, we get

$$\int_{\mathbb{R}} \left(\tilde{\varphi}(x) - \varphi_*(x) \right) \, dx \ \leq \ (2R + S) \sqrt{\epsilon} + |A(s(\sqrt{\epsilon}))_*| \cdot t(\sqrt{\epsilon}), \tag{6.23}$$

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which is an infinitesimal function for $\epsilon \to 0$.

Defining the quasiconcave function $\tilde{\psi}$ similarly, i.e.

$$\left\{ x : \tilde{\psi}(x) \geq t \right\} = \tilde{B}(t) := conv\left(B(t)_* \right) \qquad 0 \leq t \leq S.$$

we get the same L^1 estimate: then (6.7) holds. Finally observe that, by definition (6.19) of $t(\delta)$, at least one among $A(s(\delta))_*$ and $B(s(\delta))_*$ has measure less than or equal to δ. Without loss of generality suppose

$$|A(s(\delta))_*| \leq \delta, \tag{6.24}$$

then (6.22) gives

$$\int_{A(s(\delta))_*} (\tilde{\varphi}(x) - \varphi(x)_*) \ dx \leq \delta S. \tag{6.25}$$

Therefore we get (6.6) using (6.21) and (6.25) with the choice $\delta = \sqrt{\epsilon}$:

$$\int_{\mathbb{R}} (\tilde{\varphi}(x) - \varphi_*(x)) \ dx = \int_{[A(s(\delta))_*]^C} (\tilde{\varphi}(x) - \varphi_*(x)) \ dx + \int_{A(s(\delta))_*} (\tilde{\varphi}(x) - \varphi_*(x)) \ dx$$

$$\leq \ 2R\sqrt{\epsilon} + \sqrt{\epsilon} \cdot S + \sqrt{\epsilon} \cdot S.$$

6.3   Some corollaries

As you may have noticed, assumption (6.2) does not coincide (and it is not directly comparable) with (3.1) when $p = -\infty$.

On the other hand, fixed $\lambda \in (0,1)$, let φ, ψ, θ given by Proposition 6.1.2 and consider the related functions $f, g : \mathbb{R} \longrightarrow [0, +\infty)$ defined as

$$f(z) = \varphi\left((1-\lambda)z\right), \qquad g(z) = \psi\left(\lambda z\right).$$

Then assumption (6.2), through the change of variables

$$x = (1-\lambda)\overline{x}, \qquad y = \lambda\overline{y},$$

is equivalent to

$$\theta((1-\lambda)\overline{x} + \lambda\overline{y}) \geq \min\left\{f(\overline{x}), g(\overline{y})\right\} \quad \text{for every} \ \ \overline{x}, \overline{y} \in \mathbb{R},$$

namely θ, f, g satisfy exactly (3.1) with $p = -\infty$, and (6.3) becomes

$$\int_{\mathbb{R}} \theta(x) \, dx \geq (1-\lambda) \int_{\mathbb{R}} f(x) \, dx + \lambda \int_{\mathbb{R}} g(x) \, dx,$$

that is (3.2) with $p = +\infty$ (recall that here $n = 1$). Hence, Proposition 6.1.2 says:

$$\textit{if} \ \ \theta((1-\lambda)\overline{x} + \lambda\overline{y}) \geq \mathcal{M}_{-\infty}(f(\overline{x}), g(\overline{y})), \quad \textit{then} \quad \int_{\mathbb{R}} \theta \geq \mathcal{M}_1\left(\int_{\mathbb{R}} f, \int_{\mathbb{R}} g; \lambda\right), \tag{6.26}$$

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provided that $\|f\|_\infty = \|g\|_\infty$.

Thanks to the monotonicity of means, it is then clear as Proposition 6.1.2 is a refinement of Theorem 3.1.1: a weaker assumption, a stronger conclusion! But with the additional hypothesis $\|f\|_\infty = \|g\|_\infty$.

Moreover, under this extra assumption, we can trivially derive a weak stability result for the BBL inequalities. For example, for the extremal case $p = -1$, through the relation $\mathcal{M}_1(a, b; \lambda) \geq \mathcal{M}_{-1}(a, b; \lambda) \geq \min\{a, b\}$, we explicitly get the following.

Corollary 6.3.1. *Let* $\lambda \in (0, 1)$, $\Lambda = \min\{\lambda, 1 - \lambda\}$, $f, g, h : \mathbb{R} \longrightarrow [0, +\infty)$ *belonging to* $L^1(\mathbb{R})$ *such that*

$$h((1 - \lambda)\overline{x} + \lambda\overline{y}) \geq \mathcal{M}_{-1}(f(\overline{x}), g(\overline{y}); \lambda) \qquad a.e. \ (\overline{x}, \overline{y}) \in \mathbb{R}^2.$$

Assume

$$\|f\|_\infty = \|g\|_\infty = S > 0$$

and that there exist $R > 0$, $\epsilon > 0$ *such that* $\mathrm{Supp}(f) \cup \mathrm{Supp}(g) \subseteq [-R, R]$ *and*

$$\int_{\mathbb{R}} h(x)\,dx \leq \min\left\{\int_{\mathbb{R}} f(x)\,dx, \int_{\mathbb{R}} g(x)\,dx\right\} + \epsilon. \tag{6.27}$$

Then

$$\left|\int_{\mathbb{R}} f(x)\,dx - \int_{\mathbb{R}} g(x)\,dx\right| \leq \frac{\epsilon}{\Lambda} \tag{6.28}$$

and there exist two quasiconcave functions $\tilde{f}_\epsilon, \tilde{g}_\epsilon : \mathbb{R} \longrightarrow [0, S]$ *such that*

$$\min\left\{\int_{\mathbb{R}}\left(\tilde{f}_\epsilon - f\right), \int_{\mathbb{R}}(\tilde{g}_\epsilon - g)\right\} \leq \frac{2(R + S)}{\Lambda}\sqrt{\epsilon}.$$

Moreover

$$\lim_{\epsilon \to 0} \max\left\{\int_{\mathbb{R}}\left(\tilde{f}_\epsilon - f\right), \int_{\mathbb{R}}(\tilde{g}_\epsilon - g)\right\} = 0. \tag{6.29}$$

Observe that (6.28) is trivial, since by (6.26) it holds

$$\int_{\mathbb{R}} h\,dx \geq \mathcal{M}_1\left(\int_{\mathbb{R}} f\,dx, \int_{\mathbb{R}} g\,dx; \lambda\right),$$

so (6.27) implies

$$\mathcal{M}_1\left(\int_{\mathbb{R}} f(x)\,dx, \int_{\mathbb{R}} g(x)\,dx; \lambda\right) - \min\left\{\int_{\mathbb{R}} f(x)\,dx, \int_{\mathbb{R}} g(x)\,dx\right\} \leq \epsilon,$$

which easily leads to (6.28).

Corollary 6.3.1 is however not satisfying especially because, in relation to the equality conditions found by Dubuc, we expect in this case to obtain some closeness of the involved functions to (-1)-concavity.

6.4 A truly quantitative result

Because of (6.7) and (6.29), Theorem 6.1.1 and Corollary 6.3.1 are not truly quantitative results, but they give only an asymptotic stability. However, in certain cases, we can improve Theorem 6.1.1 (and consequently also Corollary 6.3.1), proving a genuine quantitative version of Proposition 6.1.2. To do this, we need to explicitly estimate the term $t(\sqrt{\epsilon})$ in (6.23). This is possible for instance when

$$\lim_{t \to S} \min\{|A(t)|, |B(t)|\} = \ell > 0.$$

In this case, indeed, we have $t(\delta) = 0$ for $\delta < \ell$ (see (6.19)).

More in general, to have a truly quantitative refinement of the inequality by Dubuc, it is sufficient that there exist $L \geq 1$ such that

$$\frac{1}{L}|B(t)| \leq |A(t)| \leq L|B(t)| \qquad \forall\, t \in [0, S]. \tag{6.30}$$

Indeed in this case (6.24) also implies

$$|B(s(\delta))_*| \leq L\delta,$$

whence we have

$$\int_{B(s(\delta))_*} \left(\tilde{\psi}(x) - \psi(x)_*\right)\, dx\ \leq \delta SL,$$

and finally, with the choice $\delta = \sqrt{\epsilon}$, we arrive to

$$\int_{\mathbb{R}} \left(\tilde{\psi}(x) - \psi(x)_*\right)\, dx \leq\ \ 2R\sqrt{\epsilon} + \sqrt{\epsilon} \cdot S + \sqrt{\epsilon} \cdot SL.$$

We have so proved the following.

Theorem 6.4.1. *In the same assumptions and notation of Theorem 6.1.1, assume furthermore that (6.30) holds for some $L \geq 1$ for $t \in [0, S]$. Then*

$$\max\left\{\int_{\mathbb{R}} \left(\tilde{\varphi}(x) - \varphi(x)\right)\, dx,\ \int_{\mathbb{R}} \left(\tilde{\psi}(x) - \psi(x)\right)\, dx\right\} \leq (2R + S + SL)\sqrt{\epsilon}, \tag{6.31}$$

where $\tilde{\varphi}$ and $\tilde{\psi}$ are respectively the quasiconcave envelopes of φ_* and ψ_*.

Notice that (6.30) is satisfied if

$$\psi(Lx + x_0) \leq \varphi(x) \leq \psi\left(\frac{x}{L} + x_1\right)$$

for some $x_0, x_1 \in \mathbb{R}^n$.

Moreover notice that we can obviously obtain from Theorem 6.4.1 a quantitative version of Corollary 6.3.1.

Finally, let us say that we conjecture that the quantitative estimate (6.31) holds in general, also without the extra assumption (6.30), and that the exponent $1/2$ of ϵ is optimal.

Chapter 7

New stability results for BBL inequalities of index $p > 0$

7.1 Introduction

In this final chapter we report and summarize the results obtained in [43].

In particular we investigate stability issues for the Borell-Brascamp-Lieb inequalities of index $p > 0$, proving that when near equality is realized, the involved functions must be L^1-close to be p-concave and to coincide up to homotheties of their graphs, according to Theorem 4.3.1 as regards the equality case (i.e. $\epsilon = 0$ in Theorem 7.1.1).

We deal only with the case $p > 0$ and, to avoid triviality, throughout this final chapter we will assume (if not otherwise explicitly declared) that $f, g \in L^1(\mathbb{R}^n)$ are nonnegative compactly supported functions (with supports $\mathrm{Supp}(f)$ and $\mathrm{Supp}(g)$) such that

$$F = \int_{\mathbb{R}^n} f \, dx > 0 \quad \text{and} \quad G = \int_{\mathbb{R}^n} g \, dx > 0 \, .$$

Let us restate a version of the BBL inequality including its equality condition (provided in Theorem 4.3.1) in the case

$$p = \frac{1}{s} > 0 \, ,$$

adopting a slightly different notation.

Proposition 7.1.1 (BBL inequalities, index $p = 1/s > 0$).
Let $s > 0$ and f, g be as said above. Let $\lambda \in (0, 1)$ and h be a nonnegative function belonging to $L^1(\mathbb{R}^n)$ such that

$$h((1 - \lambda)x + \lambda y) \geq \left[(1 - \lambda)f(x)^{1/s} + \lambda g(y)^{1/s} \right]^s \tag{7.1}$$

for every $x \in \mathrm{Supp}(f)$, $y \in \mathrm{Supp}(g)$. Then

$$\int_{\mathbb{R}^n} h \, dx \geq \mathcal{M}_{\frac{1}{n+s}} \left(F, G; \lambda \right) \, . \tag{7.2}$$

Moreover equality holds in (7.2) if and only if there exists a (1/s)-concave function φ such that

$$\varphi(x) = a_1 \, f(b_1 x - \overline{x}_1) = a_2 \, g(b_2 x - \overline{x}_2) = a_3 \, h(b_3 x - \overline{x}_3) \quad a.e. \ x \in \mathbb{R}^n,$$

for some $\overline{x}_1, \overline{x}_2, \overline{x}_3 \in \mathbb{R}^n$ and suitable $a_i, b_i > 0$ for $i = 1, 2, 3$, explicitly computable by (4.17) and (4.18) with $p = 1/s > 0$.

In Chapter 5 we have introduced the main stability results about BBL inequalities. The investigation of stability issues in the case $p = 0$ was started by Ball and Böröczky in [2, 3] and related results are in [12]. The general case $p > 0$ has been very recently faced in [30]. But, as already noticed Chapter 5, the results of [30], as well as the quoted results for $p = 0$, hold only in the restricted class of p-concave functions, hence answering only a half of the question. Here we want to remove this restriction, proving that near equality in (7.2) is possible if and only if the involved functions are close to coincide up to homotheties of their graphs and they are also nearly p-concave, in a suitable sense. But before stating our main result in detail, we need to introduce some notation: for $s > 0$, we say that two functions $v, \hat{v} : \mathbb{R}^n \to [0, +\infty)$ are *s-equivalent* if there exist $\mu_v > 0$ and $\overline{x} \in \mathbb{R}^n$ such that

$$\hat{v}(x) = \mu_v^s \, v\left(\frac{x - \overline{x}}{\mu_v}\right) \qquad a.e. \ x \in \mathbb{R}^n. \tag{7.3}$$

Now we are ready to state our main result, which regards the case $s = 1/p \in \mathbb{N}$. Later (see Section 7.3) we will extend the result to the case $0 < s \in \mathbb{Q}$ in Corollary 7.3.1 and finally (see Corollary 7.4.1) we will give a slightly weaker version, valid for every $s > 0$.

Theorem 7.1.1. *Let f, g, h as in Proposition 7.1.1 with*

$$0 < s \in \mathbb{N}.$$

Assume that

$$\int_{\mathbb{R}^n} h \, dx \leq \mathcal{M}_{\frac{1}{n+s}}(F, G ; \lambda) + \epsilon \tag{7.4}$$

for some $\epsilon > 0$ small enough.
Then there exist a $\frac{1}{s}$-concave function $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ and two functions $\hat{f}$ and $\hat{g}$, s-equivalent to f and g in the sense of (7.3) (with suitable μ_f and μ_g given in (7.20)) such that the following hold:

$$u \geq \hat{f}, \qquad u \geq \hat{g},$$

$$\int_{\mathbb{R}^n} (u - \hat{f}) \, dx \ + \ \int_{\mathbb{R}^n} (u - \hat{g}) \, dx \ \leq \ C_{n+s}\left(\frac{\epsilon}{\mathcal{M}_{\frac{1}{n+s}}(F, G ; \lambda)}\right), \tag{7.5}$$

where $C_{n+s}(\eta)$ is an infinitesimal function for $\eta \longrightarrow 0$ (whose explicit expression is given later, see (7.6)).

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Notice that the function u is bounded, hence as a byproduct of the proof we obtain that the functions f and g have to be bounded as well (see Remark 7.2.1).

The proof of the above theorem is based on the proof of the BBL inequality described in Section 3.3.3 and due to Klartag [36], which directly connects the BBL inequality to the Brunn-Minkowski inequality, and the consequent application of the recent stability result for the BM inequality by Figalli and Jerison [25], stated in Proposition 2.6.3, which does not require any convexity assumption of the involved sets. Indeed [25] is the first paper, at our knowledge, investigating on stability issues for the Brunn-Minkowski inequality outside the realm of convex bodies. Noticeably, Figalli and Jerison ask therein for a functional counterpart of their result, pointing out that *"at the moment some stability estimates are known for the Prékopa-Leindler inequality only in one dimension or for some special class of functions [2, 3], and a general stability result would be an important direction of future investigations."*. Since BBL inequality is the functional counterpart of the Brunn-Minkowksi inequality [43] can be considered a first answer to the question by Figalli and Jerison.

Remark 7.1.2. As already said, the proof of our main result is based on Proposition 2.6.3 and now we can give the explicit expression of the infinitesimal function C_{n+s} of Theorem 7.1.1:

$$C_{n+s}(\eta) = \frac{\eta^{\sigma_{n+s}(\tau)}}{\omega_s \, \tau^{N_{n+s}}} \, , \tag{7.6}$$

where ω_s denotes the measure of the unit ball in $\mathbb{R}^s$.

Instead of Proposition 2.6.3, we will usually apply Corollary 2.6.4, which does not require the normalization constraint about the measures of the involved sets A and B.

The link between BM and BBL inequalities is well known and it is the topic of Section 3.2. Briefly, every BBL inequality implies easily BM inequality applying the BBL inequality to the characteristic functions $f = \chi_A$, $g = \chi_B$, $h = \chi_{(1-\lambda)A+\lambda B}$.

The opposite implication, for the BBL inequalities of positive index, can be proved using the proof due to Klartag [36], which is particularly useful for our goals. This proof has been presented in Section 3.3.3. Let $n, s \in \mathbb{N}$, and $f : \mathbb{R}^n \longrightarrow [0, +\infty)$ a fixed function belonging to $L^1(\mathbb{R}^n)$ and with nonempty support. We recall the definition and the main properties ((3.21), (3.22), ecc.) of the fundamental set $K_{f,s}$, associated to the function f. $K_{f,s}$ is defined as

$$K_{f,s} = \left\{ (x,y) \in \mathbb{R}^{n+s} = \mathbb{R}^n \times \mathbb{R}^s : \ x \in \mathrm{Supp}(f), \ |y| \le f(x)^{1/s} \right\}.$$

Notice that $K_{f,s}$ is convex if and only if f is $(1/s)$-concave (that is for us a function f having compact convex support such that $f^{1/s}$ is concave on $\mathrm{Supp}(f)$). Moreover, a very useful property is (3.22) i.e.

$$|K_{f,s}| = \int_{\mathrm{Supp}(f)} \omega_s \cdot \left(f(x)^{1/s} \right)^s \ dx = \omega_s \int_{\mathbb{R}^n} f(x) \ dx,$$

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where ω_s is the volume of the s-dimensional ball of radius one. In this way, the integral of f coincides, up to the constant ω_s, with the volume of $K_{f,s}$.

Remember the definition (3.20) of the function $h_{s,\lambda}$ (to simplify the notation we will denote it with h_λ), that is the smallest function satisfying (7.1). Using the property (3.22) combined with the key identity (3.23) i.e.

$$K_{h_\lambda} = (1-\lambda)K_f + \lambda K_g,$$

in Section 3.3.3 we have proved Proposition 7.1.1 as a direct application of the BM inequality, first for the case $s \in \mathbb{N}$.

Then (see the second proof of Section 3.3.3) we generalized it to a positive rational index

$$s = \frac{t}{q}$$

with coprime integers $t, q > 0$, using the auxiliary function $\tilde{f} : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ defined in (3.25) as

$$\tilde{f}(x) = \tilde{f}(x_1, ..., x_q) = \prod_{j=1}^{q} f(x_j),$$

where $x = (x_1, ..., x_q) \in (\mathbb{R}^n)^q$. By construction, (3.26) holds i.e.

$$\int_{\mathbb{R}^{nq}} \tilde{f} \; dx = \left(\int_{\mathbb{R}^n} f \; dx \right)^q.$$

Furthermore $\quad \operatorname{Supp} \tilde{f} = (\operatorname{Supp} f) \times ... \times (\operatorname{Supp} f) = (\operatorname{Supp} f)^q$.

Next we show another way to generalize Proposition 7.1.1 to a positive rational index s. The idea is to apply again the Brunn-Minkowski inequality to sets that generalize those of the type (3.21). What follows, which will useful later, is a slight variant of the proof of Theorem 2.1 in [36], i.e. a slight variant of the second proof Section 3.3.3.

Remark 7.1.3. As just done, from now on we write A^q to indicate the Cartesian product of q copies of a set A. Let A, B be nonempty sets, $q > 0$ be an integer, μ a real. Clearly

$$(A + B)^q = A^q + B^q, \qquad\qquad (\mu A)^q = \mu A^q.$$

Recall that

$$s = \frac{t}{q}$$

with integers $t, q > 0$ that we can assume are coprime.

Given an integrable function $f : \mathbb{R}^n \longrightarrow [0, +\infty)$ not identically zero, we define the nonempty measurable subset of $\mathbb{R}^{nq+t}$

$$W_{f,s} = K_{\tilde{f},t} = \left\{ (x, y) \in (\mathbb{R}^n)^q \times \mathbb{R}^t : \; x \in \operatorname{Supp}(\tilde{f}), \; |y| \leq \tilde{f}(x)^{1/t} \right\} \qquad (7.7)$$

$$= \left\{ (x_1, ..., x_q, y) \in (\mathbb{R}^n)^q \times \mathbb{R}^t : x_j \in \text{Supp}(f) \ \forall \ j = 1, ..., q, \quad |y| \leq \prod_{j=1}^{q} f(x_j)^{1/t} \right\}.$$

We notice that this definition naturally generalizes (3.21), since in the case of an integer $s > 0$ it holds $s = t$, $q = 1$, so in this case $\tilde{f} = f$ and $W_{f,s} = K_f$.

As for $K_{f,s}$, for simplicity we will remove systematically the subindex s and write W_f in place of $W_{f,s}$ if there is no possibility of confusion. Clearly

$$|W_f| = \int_{\text{Supp}(\tilde{f})} \omega_t \cdot \left(\tilde{f}(x)^{1/t} \right)^t \ dx = \omega_t \int_{\mathbb{R}^{nq}} \tilde{f}(x) \ dx = \omega_t \left(\int_{\mathbb{R}^n} f(x) \ dx \right)^q \qquad (7.8)$$

where the last equality is given by (3.26).

Moreover we see that W_f is convex if and only if $\tilde{f}$ is $\frac{1}{t}$-concave (that is, if and only if f is $\frac{1}{s}$-concave, see Lemma 7.3.1 later on). Next we set

$$W = (1 - \lambda)W_f + \lambda W_g. \qquad (7.9)$$

Finally, we notice that, by (3.23), we have

$$W = K_{\tilde{h}_{t,\lambda}, t},$$

where $\tilde{h}_{t,\lambda}$ is the $(1/t, \lambda)$-supremal convolution of $\tilde{f}$ and $\tilde{g}$ as defined in (3.20). In other words, W is the set made by the elements $(z, y) \in (\mathbb{R}^n)^q \times \mathbb{R}^t$ such that $z \in (1 - \lambda) \text{Supp}(\tilde{f}) + \lambda \text{Supp}(\tilde{g})$ and

$$\begin{aligned} |y| \leq \quad & \sup \left\{ (1 - \lambda)\tilde{f}(x)^{1/t} + \lambda \tilde{g}(x')^{1/t} : \right. \\ & \left. z = (1 - \lambda)x + \lambda x', x \in \text{Supp}(\tilde{f}), x' \in \text{Supp}(\tilde{g}) \right\}. \end{aligned} \qquad (7.10)$$

Lemma 7.1.2. *With the notations introduced above, it holds*

$$W \subseteq W_{h_\lambda} \subseteq W_h,$$

where h_λ is the $(1/s, \lambda)$-supremal convolution of f,g, and h is as in Proposition 7.1.1.

Proof. The second inclusion is obvious, since $h \geq h_\lambda$ by assumption (7.1). Regarding the other inclusion, first we notice that (7.7) and Remark 7.1.3 yield

$$W_{h_\lambda} = \left\{ (z, y) \in (\mathbb{R}^n)^q \times \mathbb{R}^t : z \in \text{Supp}(\tilde{h_\lambda}), \ |y| \leq \tilde{h_\lambda}(z)^{1/t} \right\}$$

$$= \left\{ (z, y) \in (\mathbb{R}^n)^q \times \mathbb{R}^t : z \in ((1 - \lambda) \text{Supp}(f) + \lambda \text{Supp}(g))^q, \ |y| \leq \tilde{h_\lambda}(z)^{1/t} \right\}$$

$$= \left\{ (z, y) \in (\mathbb{R}^n)^q \times \mathbb{R}^t : z \in (1 - \lambda) \text{Supp}(\tilde{f}) + \lambda \text{Supp}(\tilde{g}), \ |y| \leq \tilde{h_\lambda}(z)^{1/t} \right\},$$

where $\tilde{h_\lambda}$ is the function associated to h_λ by (3.25). To conclude it is sufficient to compare this with the condition given by (7.10).

For every $z \in (1 - \lambda) \text{Supp}(\tilde{f}) + \lambda \text{Supp}(\tilde{g})$ consider

$$\sup \left\{ (1 - \lambda)\tilde{f}(x)^{1/t} + \lambda \tilde{g}(x')^{1/t} \right\} = \sup \left\{ (1 - \lambda) \prod_{j=1}^{q} f(x_j)^{1/t} + \lambda \prod_{j=1}^{q} g(x_j')^{1/t} \right\},$$

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where the supremum is made with respect to $x \in \operatorname{Supp}(\tilde{f}), \ x' \in \operatorname{Supp}(\tilde{g})$ such that $z = (1-\lambda)x + \lambda x'$. Corollary 3.3.3 then implies

$$\sup\left\{(1-\lambda)\tilde{f}(x)^{1/t} + \lambda\tilde{g}(x')^{1/t}\right\} \leq \sup\left\{\prod_{j=1}^{q}\left[(1-\lambda)f(x_j)^{1/s} + \lambda g(x_j')^{1/s}\right]^{1/q}\right\}$$

$$\leq \prod_{j=1}^{q}\left\{\sup\left[(1-\lambda)f(x_j)^{1/s} + \lambda g(x_j')^{1/s}\right]^{1/q}\right\} = \prod_{j=1}^{q}\left\{h_\lambda\left((1-\lambda)x_j + \lambda x_j'\right)^{1/qs}\right\}$$

$$= \tilde{h_\lambda}\left((1-\lambda)x + \lambda x'\right)^{1/t} = \tilde{h_\lambda}(z)^{1/t},$$

having used the definition (3.25) in the penultimate equality. Therefore if

$$|y| \leq \sup\left\{(1-\lambda)\tilde{f}(x)^{1/t} + \lambda\tilde{g}(x')^{1/t}\right\},$$

that is if $(z,y) \in W$ by (7.10), then

$$|y| \leq \tilde{h_\lambda}(z)^{1/t},$$

i.e. $(z,y) \in W_{h_\lambda}$. This concludes the proof. $\qquad\square$

We are ready to prove the BBL inequality in the version of Proposition 7.1.1, which holds for any positive real index s (and in fact also for $s=0$). This proof represent an alternative way to prove Theorem 2.1 in [36].

Proof of Proposition 7.1.1 for $s \geq 0$.
Assume first that $s > 0$ is rational and let $s = \frac{t}{q}$ with t, q coprime positive integers. Thanks to (7.9) we can apply Theorem 2.1.1 to W_f, W_g (that are nonempty measurable subsets of $\mathbb{R}^{nq+t}$), so

$$|W|^{\frac{1}{nq+t}} \geq (1-\lambda)\,|W_f|^{\frac{1}{nq+t}} + \lambda\,|W_g|^{\frac{1}{nq+t}},$$

where $|W|$ possibly means the outer measure of the set W. On the other hand Lemma 7.1.2 implies $|W_h| \geq |W|$, thus

$$|W_h|^{\frac{1}{nq+t}} \geq (1-\lambda)\,|W_f|^{\frac{1}{nq+t}} + \lambda\,|W_g|^{\frac{1}{nq+t}}.$$

Finally the latter inequality with the identity (7.8) is equivalent to

$$\omega_t^{\frac{1}{nq+t}}\left(\int_{\mathbb{R}^n} h\,dx\right)^{\frac{q}{nq+t}} \geq \omega_t^{\frac{1}{nq+t}}\left[(1-\lambda)\left(\int_{\mathbb{R}^n} f\,dx\right)^{\frac{q}{nq+t}} + \lambda\left(\int_{\mathbb{R}^n} g\,dx\right)^{\frac{q}{nq+t}}\right].$$

Dividing by $\omega_t^{\frac{1}{nq+t}}$ we get (7.2), since

$$\frac{q}{nq+t} = \frac{q}{q(n+s)} = \frac{1}{n+s}$$

is exactly the required index. The case of a real $s > 0$ (and also $s = 0$) follows in the same way of the second proof of Section 3.3.3.

$\qquad\square$

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7.2 The proof of the main result (Theorem 7.1.1)

The idea is to apply the result of Figalli-Jerison, more precisely Corollary 2.6.4, to the sets K_{h_λ}, K_f, K_g, and then translate the result in terms of the involved functions. We remember that with h_λ we denote the function $h_{s,\lambda}$ given by (3.20). We also recall that we set $F = \int f$ and $G = \int g$.

Since $h \geq h_\lambda$ by assumption (7.1), we have

$$K_h \supseteq K_{h_\lambda}. \qquad (7.11)$$

Thanks to (3.22), assumption (7.4) is equivalent to

$$\omega_s^{-1} |K_h| \leq \omega_s^{-1} \left[(1-\lambda) |K_f|^{\frac{1}{n+s}} + \lambda |K_g|^{\frac{1}{n+s}} \right]^{n+s} + \epsilon,$$

which, by (7.11), implies

$$|K_{h_\lambda}| \leq \left[(1-\lambda) |K_f|^{\frac{1}{n+s}} + \lambda |K_g|^{\frac{1}{n+s}} \right]^{n+s} + \epsilon \omega_s. \qquad (7.12)$$

If ϵ is small enough, by virtue of (3.23) we can apply Corollary 2.6.4 to the sets K_{h_λ}, K_f, K_g and from (7.12) we obtain that they satisfy assumption (2.19) with

$$\delta = \frac{\epsilon \omega_s}{\mathcal{M}_{\frac{1}{n+s}} (|K_f|, |K_g|; \lambda)} = \frac{\epsilon}{\mathcal{M}_{\frac{1}{n+s}} (F, G; \lambda)}.$$

Then, if $\delta \leq e^{-M_{n+s}(\tau)}$, there exist a convex $K \subset \mathbb{R}^{n+s}$ and two homothetic copies $\hat{K}_f$ and $\hat{K}_g$ of K_f and K_g such that

$$|\hat{K}_f| = |\hat{K}_g| = 1, \qquad \left(\hat{K}_f \cup \hat{K}_g \right) \subseteq K, \qquad (7.13)$$

and

$$\left| K \setminus \hat{K}_f \right| + \left| K \setminus \hat{K}_g \right| \leq \tau^{-N_{n+s}} \left(\frac{\epsilon}{\mathcal{M}_{\frac{1}{n+s}} (F, G; \lambda)} \right)^{\sigma_{n+s}(\tau)}. \qquad (7.14)$$

Remark 7.2.1. Since $|\hat{K}_f| = |\hat{K}_g| = 1$, (7.14) implies that the convex set K has finite positive measure. Then it is bounded (since convex), whence (7.13) yields the boundedness of K_f and K_g which in turn implies the boundedness of the functions f and g. For simplicity, we can assume the convex K is compact (possibly substituting it with its closure).

In what follows, we indicate with $(x, y) \in \mathbb{R}^n \times \mathbb{R}^s$ an element of $\mathbb{R}^{n+s}$. When we say (see just before (7.13)) that $\hat{K}_f$ and $\hat{K}_g$ are homothetic copies of K_f and K_g, we mean that there exist $z_0 = (x_0, y_0) \in \mathbb{R}^{n+s}$ and $z_1 = (x_1, y_1) \in \mathbb{R}^{n+s}$ such that

$$\hat{K}_f = |K_f|^{-\frac{1}{n+s}} (K_f + z_0) \quad \text{and} \quad \hat{K}_g = |K_g|^{-\frac{1}{n+s}} (K_g + z_1).$$

Clearly, without loss of generality we can take $z_0 = 0$.

To conclude the proof, we want now to show that, up to a suitable symmetrization, we can take $y_1 = 0$ (i.e. the translation of the homothetic copy $\hat{K}_g$ of K_g is horizontal) and that the convex set K given by Figalli and Jerison can be taken of the type K_u for some $\frac{1}{s}$-concave function u.

For this, let us introduce the following Steiner type symmetrization in $\mathbb{R}^{n+s}$ with respect to the n-dimensional hyperspace $\{y = 0\}$ (see for instance [13]). Let C be a bounded measurable set in $\mathbb{R}^{n+s}$, for every $\overline{x} \in \mathbb{R}^n$ we set

$$C(\overline{x}) = C \cap \{x = \overline{x}\} = \{y \in \mathbb{R}^s \,:\, (\overline{x}, y) \in C\}$$

and

$$r_C(\overline{x}) = \left(\omega_s^{-1}|C(\overline{x})|\right)^{1/s} . \tag{7.15}$$

Then we define the $\mathcal{S}$-*symmetrand* of C as follows

$$\mathcal{S}(C) = \left\{(\overline{x}, y) \in \mathbb{R}^{n+s} \,:\, C \cap \{x = \overline{x}\} \neq \emptyset, \ |y| \leq r_C(\overline{x})\right\}. \tag{7.16}$$

We notice that $\mathcal{S}(C)$ is obtained as union of the s-dimensional closed balls of center $(\overline{x}, 0)$ and radius $r_C(\overline{x})$, for $\overline{x} \in \mathbb{R}^n$ such that $C \cap \{x = \overline{x}\}$ is nonempty. Thus, fixed $\overline{x}$, the measure of the corresponding section of $\mathcal{S}(C)$ is

$$|\mathcal{S}(C) \cap \{x = \overline{x}\}| = \omega_s r_C(\overline{x})^s = |C(\overline{x})| . \tag{7.17}$$

We describe the main properties of $\mathcal{S}$-*symmetrization*, for bounded measurable susbsets of $\mathbb{R}^{n+s}$:
(i) if $C_1 \subseteq C_2$ then $\mathcal{S}(C_1) \subseteq \mathcal{S}(C_2)$ (obvious by definition);
(ii) $|C| = |\mathcal{S}(C)|$ (consequence of (7.17) and Fubini's Theorem) so the $\mathcal{S}$-symmetrization is measure preserving;
(iii) if C is convex then $\mathcal{S}(C)$ is convex (the proof is based on the BM inequality in $\mathbb{R}^s$ and, for the sake of completeness, is given in Appendix 1).

Now we symmetrize $K, \hat{K}_f, \hat{K}_g$ (and then replace them with $\mathcal{S}(K), \mathcal{S}(\hat{K}_f), \mathcal{S}(\hat{K}_g)$). Clearly

$$\mathcal{S}(\hat{K}_f) = \hat{K}_f,$$

$$\mathcal{S}(\hat{K}_g) = \mathcal{S}\left(|K_g|^{-\frac{1}{n+s}}(K_g + (x_1, y_1))\right) = |K_g|^{-\frac{1}{n+s}}(K_g + (x_1, 0)) .$$

Moreover, (iii) implies that $\mathcal{S}(K)$ is convex and by (i) and (7.13) we have

$$(\mathcal{S}(\hat{K}_f) \cup \mathcal{S}(\hat{K}_g)) \subseteq \mathcal{S}(K) .$$

The latter, (7.14) and (ii) imply

$$\left|\mathcal{S}(K) \setminus \mathcal{S}(\hat{K}_f)\right| + \left|\mathcal{S}(K) \setminus \mathcal{S}(\hat{K}_g)\right| \leq \tau^{-N_{n+s}} \left(\frac{\epsilon}{\mathcal{M}_{\frac{1}{n+s}}(F, G; \lambda)}\right)^{\sigma_{n+s}(\tau)} . \tag{7.18}$$

Finally we notice that $\mathcal{S}(K)$ is a compact convex set of the desired form.

Remark 7.2.2. Consider the set K_u associated to a function $u : \mathbb{R}^n \to [0, +\infty)$ by (3.21) and let $\bar{x} \in \mathbb{R}^n$, $\bar{z} = (\bar{x}, 0) \in \mathbb{R}^{n+s}$, $\mu > 0$ and

$$H = \mu \left(K_u + \bar{z} \right) .$$

Then

$$H = K_v$$

(the set associated to v by (3.21)) where

$$v(x) = \mu^s u \left(\frac{x - \bar{x}}{\mu} \right) . \tag{7.19}$$

From the previous remarks, we see that the sets $\mathcal{S}(\hat{K}_f)$ and $\mathcal{S}(\hat{K}_g)$ are in fact associated via (3.21) to two functions $\hat{f}$ and $\hat{g}$, such that

$$\mathcal{S}(\hat{K}_f) = K_{\hat{f}} , \quad \mathcal{S}(\hat{K}_g) = K_{\hat{g}} ,$$

and $\hat{f}$ and $\hat{g}$ are s-equivalent to f and g respectively, in the sense of (7.3), with

$$\mu_f = (\omega_s F)^{\frac{-1}{n+s}} , \qquad \mu_g = (\omega_s G)^{\frac{-1}{n+s}} . \tag{7.20}$$

We notice that the support sets Ω_0 and Ω_1 of $\hat{f}$ and $\hat{g}$ are given by

$$\Omega_0 = \{ x \in \mathbb{R}^n : (x, 0) \in \mathcal{S}(\hat{K}_f) \} , \qquad \Omega_1 = \{ x \in \mathbb{R}^n : (x, 0) \in \mathcal{S}(\hat{K}_g) \}$$

and that they are in fact homothetic copies of the support sets of the original functions f and g.

Now we want to find a $\frac{1}{s}$-concave function u such that $\mathcal{S}(K)$ is associated to u via (3.21). We define $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ as follows

$$u(x) = \begin{cases} r_K(x)^s & \text{if } x \in \mathbb{R}^n : (x, 0) \in \mathcal{S}(K), \\ 0 & \text{otherwise} , \end{cases}$$

and prove that

$$K_u = \mathcal{S}(K) . \tag{7.21}$$

First notice that

$$\text{Supp}(u) = \{ x \in \mathbb{R}^n : (x, 0) \in \mathcal{S}(K) \} . \tag{7.22}$$

Indeed we have $\{ z \in \mathbb{R}^n : u(z) > 0 \} \subseteq \{ x \in \mathbb{R}^n : (x, 0) \in \mathcal{S}(K) \}$, whence $\text{Supp}(u) = \overline{\{ z \in \mathbb{R}^n : u(z) > 0 \}} \subseteq \{ x \in \mathbb{R}^n : (x, 0) \in \mathcal{S}(K) \}$, since the latter is closed. Vice versa let x such that $(x, 0) \in \mathcal{S}(K)$. If $r_K(x) > 0$ (see (7.15)) then $x \in \text{Supp}(u)$ obviously. Otherwise suppose $r_K(x) = 0$, then, by the convexity of $\mathcal{S}(K)$ and the fact that $\mathcal{S}(K)$ is not contained in $\{ y = 0 \}$, evidently

$$[(U \setminus \{x\}) \cap \{ z \in \mathbb{R}^n : r_K(z) > 0 \}] \neq \emptyset$$

for every neighborhood U of x, i.e. $x \in \mathrm{Supp}(u)$.

By the definition of u and (3.21), using (7.22), we get

$$K_u = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^s : \ x \in \mathrm{Supp}(u), \ |y| \le u(x)^{1/s} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^s : \ (x,0) \in \mathcal{S}(K), \ |y| \le u(x)^{1/s} \right\}$$

$$= \{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^s : \ (x,0) \in \mathcal{S}(K), \ |y| \le r_K(x) \} = \mathcal{S}(K) \,.$$

Therefore we have shown (7.21) and from the convexity of K follows that u is a $\frac{1}{s}$-concave function. Being $K_u \supseteq \left(K_{\hat{f}} \cup K_{\hat{g}} \right)$, clearly

$$\mathrm{Supp}(u) \supseteq (\Omega_0 \cup \Omega_1) \,, \qquad u \ge \hat{f} \ \text{in} \ \Omega_0, \qquad u \ge \hat{g} \ \text{in} \ \Omega_1 \,.$$

The final estimate can be deduced from (7.18). Indeed, thanks to (3.22), we get

$$\left| K_u \setminus K_{\hat{f}} \right| = |K_u| - \left| K_{\hat{f}} \right| = \omega_s \int_{\mathbb{R}^n} (u - \hat{f}) \ dx,$$

and the same equality holds for $|K_u \setminus K_{\hat{g}}|$. So (7.18) becomes

$$\int_{\mathbb{R}^n} (u - \hat{f}) \ dx \ + \ \int_{\mathbb{R}^n} (u - \hat{g}) \ dx \ \le \omega_s^{-1} \tau^{-N_{n+s}} \left(\frac{\epsilon}{\mathcal{M}_{\frac{1}{n+s}}(F,G;\lambda)} \right)^{\sigma_{n+s}(\tau)},$$

that is the desired result.

7.3   A generalization to the case s positive rational

We explain how Theorem 7.1.1 can be generalized to a positive rational index s.
Given $f : \mathbb{R}^n \longrightarrow [0, +\infty)$ and a positive integer q, we consider the auxiliary function $\tilde{f} : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ given by (3.25), i.e.

$$\tilde{f}(x) = \tilde{f}(x_1, ..., x_q) = \prod_{j=1}^{q} f(x_j),$$

with $x = (x_1, ..., x_q) \in (\mathbb{R}^n)^q$. Clearly f is bounded if and only if $\tilde{f}$ is bounded. We study further properties of functions of type (3.25).

Lemma 7.3.1. *Let $q \in \mathbb{N}$, $\alpha > 0$, let $\tilde{u} : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ be a function of the type (3.25). Then $\tilde{u}$ is α-concave if and only if the function $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ is $(q\alpha)$-concave.*

Proof. Suppose first that $\tilde{u}^{\alpha}$ is concave. Fixed $\lambda \in (0,1)$, $x, x' \in \mathbb{R}^n$, we consider the element of $\mathbb{R}^{nq}$ which has all the q components identical to $(1-\lambda)x + \lambda x'$. From hypothesis it holds

$$\tilde{u}^{\alpha} \left((1-\lambda)x + \lambda x', ..., (1-\lambda)x + \lambda x'\right) \geq (1-\lambda)\tilde{u}^{\alpha}(x, ..., x) + \lambda\tilde{u}^{\alpha}(x', ..., x'),$$

i.e. (thanks to (3.25))

$$u^{q\alpha}\left((1-\lambda)x + \lambda x'\right) \geq (1-\lambda)u^{q\alpha}(x) + \lambda u^{q\alpha}(x').$$

Thus $u^{q\alpha}$ is concave. Vice versa assume that $u^{q\alpha}$ is concave, and fix $\lambda \in (0,1)$, $x = (x_1, ..., x_q)$, $x' = \left(x_1', ..., x_q'\right) \in (\mathbb{R}^n)^q$. We have

$$\tilde{u}^{\alpha}\left((1-\lambda)x + \lambda x'\right) = \prod_{j=1}^{q} u^{\alpha}\left((1-\lambda)x_j + \lambda x_j'\right) = \prod_{j=1}^{q}\left[u^{q\alpha}\left((1-\lambda)x_j + \lambda x_j'\right)\right]^{1/q}$$

$$\geq \prod_{j=1}^{q}\left[(1-\lambda)u^{q\alpha}(x_j) + \lambda u^{q\alpha}(x_j')\right]^{1/q} \geq \prod_{j=1}^{q}(1-\lambda)^{1/q}u^{\alpha}(x_j) + \prod_{j=1}^{q}\lambda^{1/q}u^{\alpha}(x_j')$$

$$= (1-\lambda)\prod_{j=1}^{q} u^{\alpha}(x_j) + \lambda\prod_{j=1}^{q} u^{\alpha}(x_j') = (1-\lambda)\tilde{u}^{\alpha}(x) + \lambda\tilde{u}^{\alpha}(x'),$$

where the first inequality holds by concavity of $u^{q\alpha}$, while in the second one we have used Corollary 3.3.3 with $a_j = (1-\lambda)^{1/q}u^{\alpha}(x_j)$, $b_j = \lambda^{1/q}u^{\alpha}(x_j')$. Hence u^{α} is concave. $\square$

Lemma 7.3.2. *Let $q > 0$ integer and $u \geq f \geq 0$ in $\mathbb{R}^n$. Then*

$$\tilde{u} - \tilde{f} \geq \widetilde{u - f}.$$

Proof. The proof is by induction on the integer $q \geq 1$. The case $q = 1$ is trivial, because in such case $\tilde{u} = u$, $\tilde{f} = f$, $\widetilde{u - f} = u - f$. For the inductive step assume that the result is true until the index q, and denote with $\tilde{\tilde{u}}, \tilde{\tilde{f}}, \widetilde{\widetilde{u-f}}$ the respective functions of index $q+1$. By the definition (3.25)

$$\left(\tilde{\tilde{u}} - \tilde{\tilde{f}}\right)(x_1, ..., x_{q+1}) = \tilde{u}(x_1, ..., x_q)u(x_{q+1}) - \tilde{f}(x_1, ..., x_q)f(x_{q+1}),$$

$$\widetilde{\widetilde{u - f}}(x_1, ..., x_{q+1}) = \widetilde{u - f}(x_1, ..., x_q) \cdot (u - f)(x_{q+1}).$$

These two equalities imply

$$\left(\tilde{\tilde{u}} - \tilde{\tilde{f}}\right)(x_1, ..., x_{q+1})$$

$$= \widetilde{\widetilde{u - f}}(x_1, ..., x_{q+1}) - \widetilde{u - f}(x_1, ..., x_q) \cdot [u(x_{q+1}) - f(x_{q+1})] +$$

$$+ \tilde{u}(x_1, ..., x_q)u(x_{q+1}) - \tilde{f}(x_1, ..., x_q)f(x_{q+1})$$

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$$\geq \widetilde{\widetilde{u-f}}(x_1,...,x_{q+1}) - \left(\tilde{u} - \tilde{f}\right)(x_1,...,x_q)\left[u(x_{q+1}) - f(x_{q+1})\right]$$

$$+ \tilde{u}(x_1,...,x_q)u(x_{q+1}) - \tilde{f}(x_1,...,x_q)f(x_{q+1})$$

$$= \widetilde{\widetilde{u-f}}(x_1,...,x_{q+1}) + f(x_{q+1})\left[\tilde{u}(x_1,...,x_q) - \tilde{f}(x_1,...,x_q)\right]$$

$$+ \tilde{f}(x_1,...,x_q)\left[u(x_{q+1}) - f(x_{q+1})\right]$$

$$\geq \widetilde{\widetilde{u-f}}(x_1,...,x_{q+1}),$$

having used the inductive hypothesis and the assumption $u \geq f \geq 0$. $\qquad\square$

Corollary 7.3.1. *Given an integer $n > 0$, $\lambda \in (0,1)$, $s = \frac{t}{q}$ with t,q positive integers, let $f, g \in L^1(\mathbb{R}^n)$ be nonnegative compactly supported functions such that*

$$F = \int_{\mathbb{R}^n} f \, dx > 0 \quad and \quad G = \int_{\mathbb{R}^n} g \, dx > 0.$$

Let $h : \mathbb{R}^n \longrightarrow [0, +\infty)$ satisfy assumption (7.1) and suppose there exists $\epsilon > 0$ small enough such that

$$\left(\int_{\mathbb{R}^n} h \, dx\right)^q \leq \left[\mathcal{M}_{\frac{1}{n+s}}(F, G\,;\lambda)\right]^q + \epsilon. \tag{7.23}$$

Then there exist a $\frac{1}{t}$-concave function $u' : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ and two functions $\hat{f}, \hat{g} : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$, t-equivalent to $\tilde{f}$ and $\tilde{g}$ (given by (3.25)) in the sense of (7.3) with

$$\mu_{\tilde{f}} = \omega_t^{\frac{-1}{nq+t}} F^{\frac{-1}{n+s}}, \qquad \mu_{\tilde{g}} = \omega_t^{\frac{-1}{nq+t}} G^{\frac{-1}{n+s}},$$

such that the following hold:

$$u' \geq \hat{f}, \qquad u' \geq \hat{g},$$

$$\int_{\mathbb{R}^{nq}} (u' - \hat{f})dx + \int_{\mathbb{R}^{nq}} (u' - \hat{g}) \, dx \leq C_{nq+t}\left(\frac{\epsilon}{\mathcal{M}_{\frac{1}{nq+t}}(F^q, G^q\,;\lambda)}\right). \tag{7.24}$$

Proof. We can assume $h = h_\lambda$. Since f and g are nonnegative compactly supported functions belonging to $L^1(\mathbb{R}^n)$, thus by (3.25) $\tilde{f}, \tilde{g}$ are nonnegative compactly supported functions belonging to $L^1(\mathbb{R}^{nq})$. The assumption (7.23) is equivalent, considering the corresponding functions $\tilde{f}, \tilde{g}, \tilde{h} : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ and using (3.26), to

$$\int_{\mathbb{R}^{nq}} \tilde{h} \, dx \leq \left[(1-\lambda)\left(\int_{\mathbb{R}^{nq}} \tilde{f} \, dx\right)^{\frac{1}{nq+qs}} + \lambda\left(\int_{\mathbb{R}^{nq}} \tilde{g} \, dx\right)^{\frac{1}{nq+qs}}\right]^{nq+qs} + \epsilon$$

$$\text{i.e.} \qquad \int_{\mathbb{R}^{nq}} \tilde{h} \, dx \leq \mathcal{M}_{\frac{1}{nq+t}}(F^q, G^q\,;\lambda) + \epsilon. \tag{7.25}$$

We notice that the index $qs = t$ is integer, while nq is exactly the dimension of the space in which $\tilde{f}, \tilde{g}, \tilde{h}$ are defined.

To apply Theorem 7.1.1, we have to verify that $\tilde{f}, \tilde{g}, \tilde{h}$ satisfy the corresponding inequality (7.1) of index qs. Given $x_1, ..., x_q \in \mathrm{Supp}(f)$, $x'_1, ..., x'_q \in \mathrm{Supp}(g)$, let $x = (x_1, ..., x_q) \in \mathrm{Supp}(\tilde{f})$, $x' = (x'_1, ..., x'_q) \in \mathrm{Supp}(\tilde{g})$. By hypothesis, we know that f, g, h satisfy (7.1), in particular for every $j = 1, ..., q$

$$h\left((1-\lambda)x_j + \lambda x'_j\right) \geq \left[(1-\lambda)f(x_j)^{1/s} + \lambda g(x'_j)^{1/s}\right]^s.$$

This implies

$$\prod_{j=1}^{q} h\left((1-\lambda)x_j + \lambda x'_j\right) \geq \left[\prod_{j=1}^{q}\left[(1-\lambda)f(x_j)^{1/s} + \lambda g(x'_j)^{1/s}\right]\right]^s$$

$$\geq \left[(1-\lambda)\left(\prod_{j=1}^{q} f(x_j)\right)^{1/qs} + \lambda\left(\prod_{j=1}^{q} g(x'_j)\right)^{1/qs}\right]^{qs}, \tag{7.26}$$

where the last inequality is due to Corollary 3.3.3. By definition of (3.25), (7.26) means that for every $x \in \mathrm{Supp}(\tilde{f})$, $x' \in \mathrm{Supp}(\tilde{g})$ we have

$$\tilde{h}\left((1-\lambda)x + \lambda x'\right) \geq \left[(1-\lambda)\tilde{f}(x)^{1/qs} + \lambda\tilde{g}(x')^{1/qs}\right]^{qs},$$

i.e. the functions $\tilde{f}, \tilde{g}, \tilde{h} : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ satisfy the hypothesis (7.1) with the required index $qs = t$. Therefore we can apply Theorem 7.1.1 and conclude that there exist a $\frac{1}{t}$-concave function $u' : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$ and two functions $\hat{f}, \hat{g}$, t-equivalent to $\tilde{f}$ and $\tilde{g}$, with the required properties. The estimate (7.5), applied to (7.25), implies

$$\int_{\mathbb{R}^{nq}} (u' - \hat{f})\ dx\ +\ \int_{\mathbb{R}^{nq}} (u' - \hat{g})\ dx\ \leq\ C_{nq+t}\left(\frac{\epsilon}{\mathcal{M}_{\frac{1}{nq+t}}(F^q, G^q; \lambda)}\right).$$

$\square$

Remark 7.3.2. Assume $F = G$ and, for simplicity, suppose that $\hat{f} = \tilde{f}$, $\hat{g} = \tilde{g}$ in Corollary 7.3.1 (as it is true up to a t-equivalence). Moreover assume that the $\frac{1}{t}$-concave function $u' : \mathbb{R}^{nq} \longrightarrow [0, +\infty)$, given by Corollary 7.3.1, is of the type (3.25), i.e. $u' = \tilde{u}$ where $u : \mathbb{R}^n \longrightarrow [0, +\infty)$ has to be $\frac{1}{s}$-concave by Lemma 7.3.1. In this case Corollary 7.3.1 assumes a simpler statement, which naturally extends the result of Theorem 7.1.1. Indeed (7.24), thanks to Lemma 7.3.2, becomes

$$\int_{\mathbb{R}^{nq}} \widetilde{u - f}\ dx\ +\ \int_{\mathbb{R}^{nq}} \widetilde{u - g}\ dx\ \leq\ C_{nq+t}\left(\frac{\epsilon}{\mathcal{M}_{\frac{1}{nq+t}}(F^q, G^q; \lambda)}\right), \qquad \text{i.e.}$$

$$\left[\int_{\mathbb{R}^n} (u-f)\ dx\right]^q + \left[\int_{\mathbb{R}^n} (u-g)\ dx\right]^q \leq C_{nq+t}\left(\frac{\epsilon}{\mathcal{M}_{\frac{1}{nq+t}}(F^q, G^q\,;\lambda)}\right). \qquad (7.27)$$

Unfortunately the function u' constructed in Theorem 7.1.1 is not necessarily of the desired form, that is in general we can not find a function $u : \mathbb{R}^n \longrightarrow [0,+\infty)$ such that $u' = \tilde{u}$ (a counterexample can be explicitly given). Then our proof can not be easily extended to the general case $s \in \mathbb{Q}$ to get (7.27).

7.4   A stability for $s > 0$

To complete the paper, we give a (weaker) version of our main stability result Theorem 7.1.1 which works for an arbitrary real index $s > 0$. For this, let us denote by $[s]$ the integer part of s, i.e. the largest integer not greater than s. Obviously $[s] + 1 > s \geq [s]$, whereby (by the monotonicity of p-means with respect to p, i.e. $\mathcal{M}_p(a,b;\lambda) \leq \mathcal{M}_q(a,b;\lambda)$ if $p \leq q$) for every $a, b \geq 0,\ \lambda \in (0,1)$

$$\left[(1-\lambda)a^{\frac{1}{s}} + \lambda b^{\frac{1}{s}}\right]^s \geq \left[(1-\lambda)a^{\frac{1}{[s]+1}} + \lambda b^{\frac{1}{[s]+1}}\right]^{[s]+1}, \qquad (7.28)$$

$$\left[(1-\lambda)a^{\frac{1}{n+s}} + \lambda b^{\frac{1}{n+s}}\right]^{n+s} \geq \left[(1-\lambda)a^{\frac{1}{n+[s]+1}} + \lambda b^{\frac{1}{n+[s]+1}}\right]^{n+[s]+1}.$$

We arrive to the following corollary for every index $s > 0$.

Corollary 7.4.1. *Given $s > 0,\ \lambda \in (0,1)$, let $f, g : \mathbb{R}^n \longrightarrow [0,+\infty)$ be integrable functions such that*

$$\int_{\mathbb{R}^n} f\ dx = \int_{\mathbb{R}^n} g\ dx = 1\,. \qquad (7.29)$$

Assume $h : \mathbb{R}^n \longrightarrow [0,+\infty)$ satisfies assumption (7.1) and there exists $\epsilon > 0$ small enough such that

$$\int_{\mathbb{R}^n} h\ dx \leq 1 + \epsilon. \qquad (7.30)$$

Then there exist a $\frac{1}{[s]+1}$-concave function $u : \mathbb{R}^n \longrightarrow [0,+\infty)$ and two functions $\hat{f}$ and $\hat{g}$, $([s]+1)$-equivalent to f and g in the sense of (7.19) (with $\mu_f = \mu_g = \left(\omega_{[s]+1}\right)^{\frac{-1}{n+[s]+1}}$) such that

$$u \geq \hat{f}, \qquad\qquad u \geq \hat{g},$$

and

$$\int_{\mathbb{R}^n} (u - \hat{f})\ dx + \int_{\mathbb{R}^n} (u - \hat{g})\ dx \leq C_{n+[s]+1}(\epsilon).$$

Proof. We notice that the assumption (7.1) (i.e. the hypothesis of BBL inequality of index $1/s$), through (7.28), implies that for every $x \in \operatorname{Supp}(f)$, $y \in \operatorname{Supp}(g)$

$$h\left((1-\lambda)x + \lambda y\right) \geq \left[(1-\lambda)f(x)^{\frac{1}{[s]+1}} + \lambda g(y)^{\frac{1}{[s]+1}}\right]^{[s]+1},$$

i.e. the corresponding hypothesis of BBL for the index $\frac{1}{[s]+1}$. Therefore, thanks to the assumptions (7.29) and (7.30), it holds $\int h \leq 1 + \epsilon = \mathcal{M}_{\frac{1}{n+[s]+1}}(\int f, \int g; \lambda) + \epsilon$, so we can apply directly Theorem 7.1.1 using the integer $[s]+1$ as index. This concludes the proof. $\square$

Remark 7.4.2. If we do not use the normalization (7.29) and want to write a result for generic unrelated $F = \int f$ and $G = \int g$, we can notice that assumption (7.30) should be replaced by

$$\int_{\mathbb{R}^n} h \, dx \leq \mathcal{M}_{\frac{1}{n+[s]+1}}(F, G; \lambda) + \epsilon.$$

On the other hand, thanks to assumption (7.1), we can apply the corresponding BBL inequality and obtain

$$\int_{\mathbb{R}^n} h \, dx \geq \mathcal{M}_{\frac{1}{n+s}}(F, G; \lambda).$$

Then we would have

$$\mathcal{M}_{\frac{1}{n+s}}(F, G; \lambda) \leq \mathcal{M}_{\frac{1}{n+[s]+1}}(F, G; \lambda) + \epsilon.$$

The latter inequality is possible only if F and G are close to each others, thanks to the stability of the monotonicity property of p-means, which states

$$\mathcal{M}_{\frac{1}{n+[s]+1}}(F, G; \lambda) \leq \mathcal{M}_{\frac{1}{n+s}}(F, G; \lambda),$$

with equality if and only if $F = G$. In this sense the normalization (7.29) cannot be completely avoided and the result obtained in Corollary 7.4.1 is weaker than what desired. Indeed notice in particular that it does not coincide with Theorem 7.1.1 even in the case when s is integer, since $[s]+1 > s$ in that case as well.

7.5 Appendix 1

To conclude our digression we show that the $\mathcal{S}$-symmetrization, introduced in Remark 7.2.1, preserves the convexity of the involved set (that is the property (iii) therein).
We use the notations of Remark 7.2.1, in particular we refer to (7.15) and (7.16), and remember that C is a bounded measurable set in $\mathbb{R}^{n+s}$. We need the following preliminary result, based on the Brunn-Minkowski inequality in $\mathbb{R}^s$.

Lemma 7.5.1. *If C is convex, then for every $t \in (0,1)$ and every $x_0, x_1 \in \mathbb{R}^n$ such that $C(x_0), C(x_1)$ are nonempty sets, it holds*

$$(1-t)r_C(x_0) + tr_C(x_1) \leq r_C((1-t)x_0 + tx_1).$$

Proof. By defintion of (7.15)

$$r_C(x_0) = \omega_s{}^{-1/s}|C(x_0)|^{1/s}, \quad r_C(x_1) = \omega_s{}^{-1/s}|C(x_1)|^{1/s}, \qquad \text{thus}$$

$$(1-t)r_C(x_0) + tr_C(x_1) = \omega_s^{-1/s}\left[(1-t)|C(x_0)|^{1/s} + t|C(x_1)|^{1/s}\right]. \tag{7.31}$$

Since C is convex, we notice that $C(x_0), C(x_1)$ are (nonempty) convex sets in $\mathbb{R}^s$ such that

$$(1-t)C(x_0) + tC(x_1) \subseteq C((1-t)x_0 + tx_1). \tag{7.32}$$

Applying BM inequality (i.e. Theorem 2.1.1) to the sets $C(x_0), C(x_1) \subset \mathbb{R}^s$, (7.31) implies

$$(1-t)r_C(x_0) + tr_C(x_1) \leq \omega_s^{-1/s}|(1-t)C(x_0) + tC(x_1)|^{1/s}$$

$$\leq \omega_s^{-1/s}|C((1-t)x_0 + tx_1)|^{1/s} = r_C((1-t)x_0 + tx_1),$$

where in the last inequality we use (7.32). $\qquad\square$

Proposition 7.5.1. *If C is convex then $\mathcal{S}(C)$ is convex.*

Proof. Let $t \in (0,1)$, and let $P = (x_0, y_0), Q = (x_1, y_1)$ be two distinct points belonging to $\mathcal{S}(C)$, i.e. $C(x_0), C(x_1)$ are nonempty sets and

$$|y_0| \leq r_C(x_0), \qquad |y_1| \leq r_C(x_1). \tag{7.33}$$

We prove that

$$(1-t)P + tQ = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1) \in \mathcal{S}(C).$$

By assumptions and (7.32) the set $C((1-t)x_0 + tx_1)$ is nonempty. Furthermore by the triangle inequality, (7.33) and Lemma 7.5.1 we obtain

$$|(1-t)y_0 + ty_1| \leq (1-t)|y_0| + t|y_1| \leq (1-t)r_C(x_0) + tr_C(x_1) \leq r_C((1-t)x_0 + tx_1).$$

Then $(1-t)P + tQ \in \mathcal{S}(C)$, i.e. $\mathcal{S}(C)$ is convex. $\qquad\square$

7.6 Appendix 2: stability of BBL inequalities for power concave functions

Let $n, s \in \mathbb{N}$ $f, g : \mathbb{R}^n \longrightarrow [0, \infty)$ bounded functions with nonempty compact supports that satisfy the assumptions of Theorem 7.1.1. The first fundamental step in its proof is the application of the Figalli and Jerison's stability result for the Brunn-Minkowski inequality (Proposition 2.6.3) to the measurable sets $K_f, K_g, K_{h_\lambda} \subset \mathbb{R}^{n+s}$. This choice is justified by quality of this BM stability result is its applicability to nonempty measurable sets, without any topological or convexity assumptions. On the other hand, we can similarly make use of the quantitative stability results of BM inequality, introduced in Section 2.5, that instead require the convexity of the involved sets. We are referring to the Propositions 2.5.1 and 2.5.2, the first due to Groemer [31] and the second due to Figalli, Maggi, Pratelli [26, 27]. They are respectively written in terms of the distance H_0 (2.14) and of the relative asymmetry $\mathcal{A}$ (2.15) of the involved convex bodies. From now on suppose that K_f, K_g are compact and convex sets having positive measure: thus they are the convex bodies to which we apply Propositions 2.5.1 and 2.5.2. Our aim in this paragraph is to use these two propositions to deduce two new stability results for BBL inequalities of index $1/s >$ which hold when the involved functions f, g are $1/s$-concave. As already observed in the first paragraph, K_f is convex if and only if f is $\frac{1}{s}$-concave (that is for us a function f having compact convex support such that $f^{1/s}$ is concave on Supp (f)). If Supp(f) is compact, then K_f is bounded if and only if f is bounded.

Let us apply the quantitative BM inequalities stated in 2.5.1 and 2.5.2 in order to deduce the announced stability results for the BBL inequality of index $1/s$ (with $s \in \mathbb{N}$).

Corollary 7.6.1. *Given n, s positive integers, $\lambda \in (0, 1)$, let $f, g : \mathbb{R}^n \longrightarrow [0, \infty)$ be $\frac{1}{s}$-concave functions with compact supports Ω_0 and Ω_1 respectively, such that*

$$\int_{\Omega_0} f \; dx > 0, \quad and \quad \int_{\Omega_1} g \; dx > 0.$$

Let $h : \mathbb{R}^n \longrightarrow [0, \infty)$ that satisfies (7.1), and suppose there exists $\epsilon \geq 0$ such that

$$\int_{\mathbb{R}^n} h \; dx \leq \left[(1-\lambda) \left(\int_{\mathbb{R}^n} f \; dx \right)^{\frac{1}{n+s}} + \lambda \left(\int_{\mathbb{R}^n} g \; dx \right)^{\frac{1}{n+s}} \right]^{n+s} + \epsilon. \qquad (7.34)$$

Then

$$H_0(K_f, K_g) \leq \left\{ \frac{\epsilon}{\eta_{n+s}} \left[(1-\lambda) \left(\int_{\mathbb{R}^n} f \; dx \right)^{\frac{1}{n+s}} + \lambda \left(\int_{\mathbb{R}^n} g \; dx \right)^{\frac{1}{n+s}} \right]^{-n-s} \right\}^{\frac{1}{n+s+1}},$$

where η_{n+s} is given as in (2.16).

Proof. Without loss of generality $h = h_\lambda$: indeed if h satisfies (7.1) then $h \geq h_\lambda$, so (7.34) holds for $h = h_\lambda$ (by monotonicity of integral). As already observed in the beginning of

the third paragraph, the assumption (7.34) is equivalent to the condition (7.12), i.e.

$$|K_{h_\lambda}| \leq \left[(1-\lambda) |K_f|^{\frac{1}{n+s}} + \lambda |K_g|^{\frac{1}{n+s}} \right]^{n+s} + \epsilon \omega_s.$$

The sets $K_f, K_g \subset \mathbb{R}^{n+s}$ are convex, since f and g are $\frac{1}{s}$-concave functions. Moreover by hypothesis f, g are also bounded, therefore K_f, K_g are bounded sets. We can assume that K_f, K_g are also closed (up to consider their closure, that are larger convex sets having the same measure), so K_f, K_g are compact convex sets (clearly with nonempty interior, because they have positive measure: see (3.22)). Thus we can apply Proposition 2.5.1 that, by means of (3.23), becomes

$$|K_{h_\lambda}| \geq \left[(1-\lambda) |K_f|^{\frac{1}{n+s}} + \lambda |K_g|^{\frac{1}{n+s}} \right]^{n+s} \left(1 + \eta_{n+s} H_0(K_f, K_g)^{(n+s+1)} \right).$$

The latter with (7.12) implies

$$\left[(1-\lambda) |K_f|^{\frac{1}{n+s}} + \lambda |K_g|^{\frac{1}{n+s}} \right]^{n+s} \cdot \eta_{n+s} H_0(K_f, K_g)^{(n+s+1)} \leq \epsilon \omega_s,$$

and using (3.22) we conclude. $\qquad \square$

Similarly, using this time Proposition 2.5.2 we deduce the following.

Corollary 7.6.2. *Given n, s positive integers, $\lambda \in (0,1)$, let $f, g : \mathbb{R}^n \longrightarrow [0, \infty)$ be $\frac{1}{s}$-concave functions with compact supports Ω_0 and Ω_1 respectively, such that*

$$\int_{\Omega_0} f \, dx > 0, \quad \text{and} \quad \int_{\Omega_1} g \, dx > 0.$$

Let $h : \mathbb{R}^n \longrightarrow [0, \infty)$ that satisfies (7.1), and suppose there exists $\epsilon \geq 0$ such that

$$\int_{\mathbb{R}^n} h \, dx \leq \left[(1-\lambda) \left(\int_{\mathbb{R}^n} f \, dx \right)^{\frac{1}{n+s}} + \lambda \left(\int_{\mathbb{R}^n} g \, dx \right)^{\frac{1}{n+s}} \right]^{n+s} + \epsilon.$$

Then

$$A(K_f, K_g) \leq \theta_{n+s} \sqrt{\frac{\epsilon \cdot C}{(n+s)} \left[(1-\lambda) \left(\int_{\mathbb{R}^n} f \, dx \right)^{\frac{1}{n+s}} + \lambda \left(\int_{\mathbb{R}^n} g \, dx \right)^{\frac{1}{n+s}} \right]^{-n-s}},$$

where

$$C = \max \left\{ \frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda} \right\} \cdot \frac{\max \left\{ \left(\int_{\mathbb{R}^n} f \, dx \right)^{\frac{1}{n+s}}, \left(\int_{\mathbb{R}^n} g \, dx \right)^{\frac{1}{n+s}} \right\}}{\min \left\{ \left(\int_{\mathbb{R}^n} f \, dx \right)^{\frac{1}{n+s}}, \left(\int_{\mathbb{R}^n} g \, dx \right)^{\frac{1}{n+s}} \right\}},$$

and θ_{n+s} is a positive constant depending on dimension $n + s$ with polynomial growth (see (2.17)).

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