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Introduction

The aim of this Ph.D. thesis is to present new results concerning the study of nonlinear elliptic problems in the context of the Heisenberg group and it is mainly based on [13, 14, 15]. We deal with different problems, but the common thread consists in extending to a more general setting, the Heisenberg group, results proved in the Euclidean case. This generalization process in the Heisenberg framework implies a series of technical difficulties, that force the use of new key theorems.

In Chapter 2 we first prove existence of nontrivial nonnegative solutions of a Schrödinger–Hardy system in the Heisenberg group, driven by two possibly different Laplacian operators. Then, we discuss and prove existence even for systems in the Heisenberg group, including critical nonlinear terms. These results are based on the already submitted paper [15], and the main originality of these studies is to work in the Heisenberg group. In fact several new theorems have to be proved in order to overcome the difficulties arising in the new framework, also due to the presence of the Hardy terms and the fact that the nonlinearities do not necessarily satisfy the *Ambrosetti–Rabinowitz* condition. Several authors tried to drop the *Ambrosetti–Rabinowitz* condition since the pioneering work of *Jeanjean*. We refer also to [40] and the references therein for further historical details.

Let us now introduce the Schrödinger–Hardy system, which consists in the main problem of Chapter 2,

$$(\mathcal{P}_1) \qquad \begin{cases} -\Delta^m_{\mathbb{H}^n} u + a(q)|u|^{m-2}u - \mu\psi^m \dfrac{|u|^{m-2}u}{r(q)^m} = H_u(q,u,v) & \text{in } \mathbb{H}^n, \\[2ex] -\Delta^p_{\mathbb{H}^n} v + b(q)|v|^{p-2}v - \sigma\psi^p \dfrac{|v|^{p-2}v}{r(q)^p} = H_v(q,u,v) & \text{in } \mathbb{H}^n, \end{cases}$$

where μ and σ are real parameters, $Q = 2n+2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$, $1 < p < Q$, $1 < m \leq p < m^* = mQ/(Q-m)$ and $\Delta^{\wp}_{\mathbb{H}^n}$ is the $\wp$–Laplacian operator on $\mathbb{H}^n$, $\wp > 1$, which is defined by

$$\Delta^{\wp}_{\mathbb{H}^n}\phi = \operatorname{div}_H(|D_{\mathbb{H}^n}\phi|^{\wp-2}_{\mathbb{H}^n} D_{\mathbb{H}^n}\phi)$$

along any $\phi \in C_0^\infty(\mathbb{H}^n)$. Moreover r is the Heisenberg norm

$$r(q) = r(z,t) = (|z|^4 + t^2)^{1/4}, \quad z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n, \quad t \in \mathbb{R},$$

$|\cdot|$ the Euclidean norm in $\mathbb{R}^{2n}$,

$$D_{\mathbb{H}^n} u = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u)$$

the horizontal gradient, $\{X_j, Y_j\}_{j=1}^n$ the basis of left invariant vector fields on $\mathbb{H}^n$, that is

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$. Futhermore, the weight function ψ is defined as

$$\psi = |D_{\mathbb{H}^n} r|_{\mathbb{H}^n}.$$

Thus, ψ is constantly equal to 1 in the Euclidean canonical case and we refer to Section 1.1 for further details.

Moreover the weight functions a and b and the nonlinearity H verify natural conditions in this context, discussed in Section 2.1.

A similar problem was recently studied in [40] and [93], for the fractional $\wp$–Laplacian operator, in the context of the Euclidean space. In [93], the Hardy terms were not considered. For Schrödinger–Hardy systems including critical nonlinearities in the Heisenberg group case, we mention the latest paper [82].

The main results of Chapter 2 stand on the validity of the Hardy–Sobolev inequality. Assume that $\wp$ is a general fixed exponent, with $1 < \wp < Q$ and $\wp^* = \wp Q/(Q - \wp)$, then the best Hardy–Sobolev constant $\mathcal{H}_\wp = \mathcal{H}(\wp, Q)$ is given by

$$\mathcal{H}_\wp = \inf_{\substack{u \in S^{1,\wp}(\mathbb{H}^n) \\ u \neq 0}} \frac{\|D_{\mathbb{H}^n} u\|_\wp^\wp}{\|u\|_{\mathcal{H}_\wp}^\wp}, \qquad \|u\|_{\mathcal{H}_\wp}^\wp = \int_{\mathbb{H}^n} \psi^\wp \frac{|u|^\wp}{r^\wp} dq,$$

where $S^{1,\wp}(\mathbb{H}^n)$ is the Folland–Stein space, defined as the completion of $C_0^\infty(\mathbb{H}^n)$, with respect to the norm

$$\|D_{\mathbb{H}^n} u\|_\wp = \left(\int_{\mathbb{H}^n} |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^\wp dq \right)^{1/\wp}.$$

We stress that Hardy–Sobolev inequalities are a fundamental tool for the study of $(\mathcal{P}_1)$ and similar problems including Hardy terms.

Furthermore system ($\mathcal{P}_1$) has a variational structure, thus, in order to prove the existence of solutions, we use the celebrated mountain pass theorem of *Ambrosetti* and *Rabinowitz*, in the version given in [34].

Taking inspiration from [40], we treat critical nonlinear terms and also the nonlinear terms proposed in [25]. That is, we consider finally the system in $\mathbb{H}^n$

$$(\mathcal{P}_2)\quad \begin{cases} -\Delta^m_{\mathbb{H}^n} u + a(q)|u|^{m-2}u - \mu\psi^m \dfrac{|u|^{m-2}u}{r(q)^m} = H_u(q,u,v) + |u|^{m^*-2}u^+ \\ \qquad\qquad\qquad\qquad + \dfrac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta} + h(q), \\ -\Delta^p_{\mathbb{H}^n} v + b(q)|v|^{p-2}v - \sigma\psi^p \dfrac{|v|^{p-2}v}{r(q)^p} = H_v(q,u,v) + |v|^{p^*-2}v^+ \\ \qquad\qquad\qquad\qquad + \dfrac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1} + g(q), \end{cases}$$

where $\theta > 1$, $\vartheta > 1$ and $\theta + \vartheta = m^*$. Moreover, h is a nonnegative perturbation of class $L^{\mathfrak{m}}(\mathbb{H}^n)$, with $\mathfrak{m}$ the Hölder conjugate of m^*, that is $\mathfrak{m} = m'Q/(Q+m')$. While, g is a nonnegative perturbation of class $L^{\mathfrak{p}}(\mathbb{H}^n)$, with $\mathfrak{p}$ the Hölder conjugate of p^*, i.e. $\mathfrak{p} = p'Q/(Q+p')$.

In Chapter 3 we study the existence and the asymptotic behavior of nontrivial solutions of a series of problems in general open subsets Ω of the Heisenberg group $\mathbb{H}^n$, possibly unbounded or even $\mathbb{H}^n$. The problems involve the p–Laplacian operator on $\mathbb{H}^n$, a Hardy coefficient and different critical nonlinearities. All the results that we present in Chapter 3, are based on [14].

In the past years, a great deal of interest has been paid to semilinear problems with critical nonlinearities arising in the context of Stratified groups [12, 49, 51, 60, 61, 65]. Recently, more complex nonlinear elliptic problems involving critical nonlinearities, have been studied. Specifically we refer to [66, 75] for the case $p = 2$ on Carnot groups, see also [74]. Moreover, recent results have been also produced by many authors in the Euclidean elliptic setting. We mention [23, 39] and related references cited there, since [14] is an extension to the Heisenberg setting of [23, 39]. Furthermore, always in the context of the Heisenberg group, in [5] we find comparison and maximum principles, while in [7, 18, 19, 59] existence results related to the Yamabe equation.

We start Chapter 3 by treating the following

$$(\mathcal{P}_3)\qquad \begin{cases} -\Delta^p_{\mathbb{H}^n} u - \gamma\psi^p \cdot \dfrac{|u|^{p-2}u}{r^p} = \sigma w(q)|u|^{s-2}u + k(q)|u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where γ and σ are real parameters, Q is the homogeneous dimension of $\mathbb{H}^n$, $1 < p < Q$ and the exponent s is in the open interval $(p,\, p^*)$, with

$$p^* = pQ/(Q-p).$$

Moreover, the weight functions w and k satisfy natural restrictions listed in Section 3.1.

Another significant problem of Chapter 3 is given by the following

$$(\mathcal{P}_4) \qquad \begin{cases} -\Delta^p_{\mathbb{H}^n} u - \gamma \|u\|^{p-p^*(\alpha)}_{\mathcal{H}_{\alpha,\Omega}} \psi^\alpha \dfrac{|u|^{p^*(\alpha)-2}u}{r^\alpha} \\ \qquad\qquad\qquad = \lambda a(q)|u|^{p-2}u + \sigma f(q,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where we assume that Ω is a bounded Poincaré–Sobolev domain of $\mathbb{H}^n$. Furthermore $q = (z,t) \in \Omega \subset \mathbb{H}^n$, $1 < p < Q$, $0 \le \alpha \le p$ and

$$p^*(\alpha) = p(Q-\alpha)/(Q-p),$$

with Q the homogeneous dimension. The parameters γ, λ, σ and the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ verify natural conditions in this context.

We point out that the equation of problem $(\mathcal{P}_4)$ contains the Hardy–Sobolev constant $\mathcal{H}_{\alpha,\Omega}$, in a more general formulation than the one introduced in the context of problem $(\mathcal{P}_1)$. Below we report the general Hardy–Sobolev type inequality, which will be discussed and used in the following

$$\mathcal{H}_{\alpha,\Omega} = \inf_{\substack{u \in S^{1,p}_0(\Omega) \\ u \neq 0}} \frac{\|D_{\mathbb{H}^n} u\|^p_p}{\|u\|^p_{\mathcal{H}_{\alpha,\Omega}}}, \qquad \|u\|^{p^*(\alpha)}_{\mathcal{H}_{\alpha,\Omega}} = \int_\Omega \psi^\alpha \frac{|u|^{p^*(\alpha)}}{r^\alpha} dq,$$

where $S^{1,p}_0(\Omega)$ is the Folland–Stein space defined in Ω. The above formulation is obtained combining properly the Sobolev and Hardy inequalities on the Heisenberg group $\mathbb{H}^n$. We specify that Sobolev type inequalities in the Heisenberg Group were introduced by *Folland* and *Stein* in 1975 in [41, 42], see also [88, 89]. Hardy type inequalities on the Heisenberg Group could be find in [80, 48, 28]. For further details we refer to Section 1.2.

Finally in Chapter 4, based on the published paper [13], we give sufficient conditions both for existence and for nonexistence of nontrivial, nonnegative, entire solutions of nonlinear elliptic inequalities with gradient terms on the Heisenberg group. These criteria are related to the validity or not validity of the Keller Osserman condition, which in our case it is generalized, due to the presence of the gradient term. Indeed, since 1957, it is well known that for semilinear coercive inequalities in the Euclidean setting, existence of solutions, as well as nonexistence, involves the Keller–Osserman condition, cf. [56, 81]. For further generalization to quasilinear inequalities, possibly with singular of degenerate weights, we refer to [32], [35]–[37], [73, 79]. The first result in this direction, but in the Heisenberg group setting, can be found in [68, 17]. This has been extended to the Carnot groups in [16], adding further restrictions due to the presence of a new term which arises since the norm is not ∞–harmonic in that setting. Recently in [2] *Albanese, Mari and Rigoli* produce another improvement in these studies, investigating the role of gradient terms in coercive quasilinear differential inequalities on Carnot groups.

We first study existence of nonnegative nontrivial radial stationary entire solutions u of

$$(\mathcal{E}) \qquad \Delta^{\varphi}_{\mathbb{H}^n} u = f(u)\ell(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}),$$

where $\Delta^{\varphi}_{\mathbb{H}^n} u$ is the φ–Laplacian on the Heisenberg group $\mathbb{H}^n$, whose rigorous definition is given in Section 4.2, and then for

$$(\mathscr{I}) \qquad \Delta^{\varphi}_{\mathbb{H}^n} u \geq f(u)\ell(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n})$$

Liouville type theorems, that is non–existence of nonnegative nontrivial entire solutions u.

The operator $\Delta^{\varphi}_{\mathbb{H}^n}$ includes as main prototype the well known *Kohn–Spencer Laplacian in* $\mathbb{H}^n$. Moreover, f, ℓ and φ satisfy general conditions, introduced in Section 4.1.

The picture is completed with the presentation of a uniqueness result of ($\mathcal{E}$) which is, as far as we know, the first attempt for general equations with gradient terms on the Heisenberg group.

The thesis is organized as follows. In Chapter 1, we recall the main features and notations of the Heisenberg group $\mathbb{H}^n$. In particular in Section 1.1 we give the preliminary definitions related to the Heisenberg setting, that is the Korányi norm, the Haar measure, the Kohn–Spencer Laplacian and the horizontal p–Laplacian operator. In Section 1.2 we first introduce Hardy–Sobolev inequalities, which will be used in the following, then we characterize the Folland–Stein space. We also present crucial embeddings concerning the horizontal Sobolev space. In Chapter 2 we deal with the existence of nontrivial nonnegative solutions of the Schrödinger–Hardy system ($\mathcal{P}_1$). Moreover, we discuss the existence even for ($\mathcal{P}_2$), which includes critical nonlinear terms. The chapter is based on the paper [15]. In the first part of Chapter 3 we study the existence and the asymptotic behavior of nontrivial solutions of

($\mathcal{P}_3$). Then we introduce a concentration–compactness result, with the related applications in bounded domains, treating in particular problem ($\mathcal{P}_4$). The chapter is based on the paper [14]. In Chapter 4 we report some results already appeared in [13], concerning the existence and the uniqueness of nonnegative nontrivial radial stationary entire solutions of ($\mathcal{E}$). Secondly we analyze Liouville type theorems, that is non–existence of nonnegative nontrivial entire solutions for ($\mathscr{I}$). Finally, in Chapter 5 we present some open problems arising from the papers listed above, which can be useful for future research.

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Contents

Chapter 1

Preliminaries

1.1 The Heisenberg group

In this section, we present some preliminary definitions and results which will be used in the following. In particular we recall the relevant definitions and notations related to the Heisenberg group functional setting. For a complete treatment, we refer to [26, 48, 62, 64].

Let $\mathbb{H}^n$ be the Heisenberg group of topological dimension $2n+1$, that is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$, endowed with the non–Abelian group law

$$q \circ q' = \left(z + z', t + t' + 2\sum_{i=1}^{n}(y_i x'_i - x_i y'_i) \right)$$

for all q, $q' \in \mathbb{H}^n$, with

$$q = (z,t) = (x_1, \dots, x_n, y_1, \dots, y_n, t), \quad q' = (z', t') = (x'_1, \dots, x'_n, y'_1, \dots, y'_n, t').$$

In $\mathbb{H}^n$ the natural origin is denoted by $O = (0,0)$. Define

$$r(q) = r(z,t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } q = (z,t) \in \mathbb{H}^n,$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^{2n}$.

The *Korányi norm* is homogeneous of degree 1, with respect to the dilations $\delta_R : (z,t) \mapsto (Rz, R^2 t)$, $R > 0$. Indeed, for all $q = (z,t) \in \mathbb{H}^n$

$$r(\delta_R(q)) = r(Rz, R^2 t) = (|Rz|^4 + R^4 t^2)^{1/4} = R\, r(q).$$

Hence, the *Korányi distance*, is

$$d_K(q, q') = r(q^{-1} \circ q') \quad \text{for all } (q, q') \in \mathbb{H}^n \times \mathbb{H}^n,$$

and the *Korányi open ball* of radius R centered at q_0 is

$$B_R(q_0) = \{q \in \mathbb{H}^n \,:\, d_K(q, q_0) < R\}.$$

For simplicity B_R denotes the ball of radius R centered at $q_0 = O$.

The Jacobian determinant of δ_R is R^{2n+2}. The natural number $Q = 2n+2$, which is the so-called *homogeneous dimension* of $\mathbb{H}^n$, plays a role analogous to the topological dimension in the Euclidean context, see [64].

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n}\times\mathbb{R}$. It is invariant under left translations and Q–homogeneous with respect to dilations. Hence, as noted in [62], the topological dimension $2n + 1$ of $\mathbb{H}^n$ is strictly less than its Hausdorff dimension $Q = 2n + 2$. We denote by $|E|$ the Lebesgue measure of any measurable set $E \subset \mathbb{H}^n$. Then

$$|\delta_R(E)| = R^Q|E|, \qquad d(\delta_R q) = R^Q dq.$$

In particular, if $E = B_R$, then $|B_R| = |B_1|R^Q$.

The vector fields for $j = 1, \dots, n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t}, \tag{1.1.1}$$

constitute a basis for the real Lie algebra of left–invariant vector fields on $\mathbb{H}^n$. This basis satisfies the Heisenberg canonical commutation relations for position and momentum $[X_j, Y_k] = -4\delta_{jk}\partial/\partial t$, all other commutators being zero.

From now on, a vector field in the span of $\{X_j, Y_j\}_{j=1}^n$ will be called *horizontal*.

Let $u \in C^1(\mathbb{H}^n)$ be fixed. The *horizontal gradient* $D_{\mathbb{H}^n}u$ is

$$D_{\mathbb{H}^n}u = \sum_{j=1}^{n} \left[(X_j u)X_j + (Y_j u)Y_j\right], \tag{1.1.2}$$

that is it is an element of the span of $\{X_j, Y_j\}_{j=1}^n$. Furthermore, if $f \in C^1(\mathbb{R})$, then $D_{\mathbb{H}^n}f(u) = f'(u)D_{\mathbb{H}^n}u$.

The natural inner product in the span of $\{X_j, Y_j\}_{j=1}^n$

$$\left(\mathcal{W}, \mathcal{Z}\right)_{\mathbb{H}^n} = \sum_{j=1}^{n} \left(w^j z^j + \widetilde{w}^j\widetilde{z}^j\right)$$

for $\mathcal{W} = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$ and $\mathcal{Z} = \{z^j X_j + \widetilde{z}^j Y_j\}_{j=1}^n$ produces the Hilbertian norm

$$|D_{\mathbb{H}^n}u|_{\mathbb{H}^n} = \sqrt{\left(D_{\mathbb{H}^n}u, D_{\mathbb{H}^n}u\right)_{\mathbb{H}^n}}$$

for the horizontal vector field $D_{\mathbb{H}^n}u$. Moreover, if also $v \in C^1(\mathbb{H}^n)$ then the Cauchy–Schwarz inequality

$$\left|\big(D_{\mathbb{H}^n}u, D_{\mathbb{H}^n}v\big)_{\mathbb{H}^n}\right|_{\mathbb{H}^n} \le |D_{\mathbb{H}^n}u|_{\mathbb{H}^n}|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}$$

continues to be valid.

Then, *the horizontal divergence* is defined, for horizontal vector fields

$$\mathcal{W} = \{w^j X_j + \widetilde{w}^j Y_j\}_{j=1}^n$$

of class $C^1(\mathbb{H}^n;\mathbb{R}^{2n})$, by

$$\mathrm{div}_H \mathcal{W} = \sum_{j=1}^n [X_j(w^j) + Y_j(\widetilde{w}^j)].$$

If furthermore $g \in C^1(\mathbb{R})$, then the *Leibnitz formula* holds, namely

$$\mathrm{div}_H(g\mathcal{W}) = g\,\mathrm{div}_H(\mathcal{W}) + \big(D_{\mathbb{H}^n}g, \mathcal{W}\big)_{\mathbb{H}^n}.$$

Similarly, if $u \in C^2(\mathbb{H}^n)$, then the *Kohn–Spencer Laplacian*, or equivalently the *horizontal Laplacian in* $\mathbb{H}^n$, of u is defined as follows

$$\begin{aligned}\Delta_{\mathbb{H}^n}u &= \sum_{j=1}^n (X_j^2 + Y_j^2)u \\ &= \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j\frac{\partial^2}{\partial x_j \partial t} - 4x_j\frac{\partial^2}{\partial y_j \partial t}\right)u + 4|z|^2\frac{\partial^2 u}{\partial t^2},\end{aligned}$$

and $\Delta_{\mathbb{H}^n}$ is *hypoelliptic* according to the celebrated Theorem 1.1 due to *Hörmander* in [53]. In particular, $\Delta_{\mathbb{H}^n}u = \mathrm{div}_H D_{\mathbb{H}^n}u$ for each $u \in C^2(\mathbb{H}^n)$.

The main geometrical function ψ is defined by

$$\psi(q) = |D_{\mathbb{H}^n}r|_{\mathbb{H}^n} = \frac{|z|}{r(q)} \quad \text{for all } q = (z,t) \in \mathbb{H}^n, \text{ with } q \ne O, \tag{1.1.3}$$

with $0 \le \psi \le 1$, $\psi(0,t) \equiv 0$, $\psi(z,0) \equiv 1$. Furthermore, ψ^2 is the *density function*, which is homogeneous of degree 0, with respect to the dilatation δ_R.

Direct calculations show

$$\Delta_{\mathbb{H}^n}r = \frac{2n+1}{r}\psi^2 \quad \text{in } \mathbb{H}^n \setminus \{O\}.$$

For further details we refer to Section 2.1 of [68].

A well known generalization of the Kohn–Spencer Laplacian is the *horizontal p–Laplacian* on the Heisenberg group defined by

$$\Delta_{\mathbb{H}^n}^p u = \mathrm{div}_H(|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}u), \qquad p \in (1,\infty).$$

1.2 Hardy–Sobolev inequalities

In this section, we introduce Hardy–Sobolev inequalities, which are crucial in order to handle the problems treated in the following chapters. In particular, we first introduce separately the Sobolev inequality and the Hardy inequality, in the context of the Heisenberg group. Then we get a more general formulation, which is commonly known as the Hardy–Sobolev inequality.

From now on let p be a general fixed exponent, with $1 < p < Q$ and denote by $p^* = pQ/(Q-p)$ the critical Sobolev exponent. By [41, 42, 88, 89], the *Sobolev inequality* asserts that

$$\|u\|_{p^*} \leq C_{Q,p} \|D_{\mathbb{H}^n} u\|_p \tag{1.2.1}$$

for all $u \in C_0^\infty(\mathbb{H}^n)$, where $C_{Q,p}$ is a positive constant depending only on Q and p. By theorem 1 of [80], the *Hardy inequality* states that

$$\int_{\mathbb{H}^n} \psi^p \frac{|u|^p}{r^p} dq \leq \left(\frac{p}{Q-p}\right)^p \int_{\mathbb{H}^n} |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^p dq \tag{1.2.2}$$

for all $u \in C_0^\infty(\mathbb{H}^n \setminus \{O\})$. The above inequality was obtained in [48] when $p = 2$ and in another version in [28] for all $p > 1$.

Assume that $0 \leq \alpha \leq p$ and put $p^*(\alpha) = p(Q-\alpha)/(Q-p)$, which is the corresponding critical exponent. Indeed, $p^*(0) = p^*$ if $\alpha = 0$ and $p^*(p) = p$, that is the Hardy exponent, if $\alpha = p$. The *best Hardy–Sobolev constant* $\mathcal{H}_\alpha = \mathcal{H}(p, Q, \alpha)$ is given by

$$\mathcal{H}_\alpha = \inf_{\substack{u \in S^{1,p}(\mathbb{H}^n) \\ u \neq 0}} \frac{\|D_{\mathbb{H}^n} u\|_p^p}{\|u\|_{\mathcal{H}_\alpha}^p}, \qquad \|u\|_{\mathcal{H}_\alpha}^{p^*(\alpha)} = \int_{\mathbb{H}^n} \psi^\alpha \frac{|u|^{p^*(\alpha)}}{r^\alpha} dq, \tag{1.2.3}$$

where $S^{1,p}(\mathbb{H}^n)$ is the Folland–Stein space, defined as the completion of $C_0^\infty(\mathbb{H}^n)$, with respect to the norm

$$\|D_{\mathbb{H}^n} u\|_p = \left(\int_{\mathbb{H}^n} |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^p dq\right)^{1/p}.$$

Moreover, let Ω be an open subset of $\mathbb{H}^n$, we denote by $\mathcal{H}_{\alpha,\Omega} = \mathcal{H}(p, Q, \alpha, \Omega)$, the best Hardy–Sobolev constant defined in Ω, that is

$$\mathcal{H}_{\alpha,\Omega} = \inf_{\substack{u \in S_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\|D_{\mathbb{H}^n} u\|_p^p}{\|u\|_{\mathcal{H}_{\alpha,\Omega}}^p}, \qquad \|u\|_{\mathcal{H}_{\alpha,\Omega}}^{p^*(\alpha)} = \int_\Omega \psi^\alpha \frac{|u|^{p^*(\alpha)}}{r^\alpha} dq,$$

where $S_0^{1,p}(\Omega)$ is the Folland–Stein space defined in Ω.

Taking inspiration from Lemma 2.1 of [39], we prove the Hardy–Sobolev inequality in $\mathbb{H}^n$.

Lemma 1.2.1. *Assume that $0 \le \alpha \le p(< Q)$. Then, there exists a positive constant $\mathcal{C}$, possibly depending on Q, p and α, such that*

$$\|u\|_{\mathcal{H}_\alpha} \le \mathcal{C}\|D_{\mathbb{H}^n}u\|_p$$

for all $u \in S^{1,p}(\mathbb{H}^n)$.

Proof. By (1.2.1) and (1.2.2) it is enough to prove the lemma only when $0 < \alpha < p$. Thus $p < p^*(\alpha) < p^*$. By (1.2.1), (1.2.2) and the Hölder inequality for all $u \in S^{1,p}(\mathbb{H}^n)$

$$\begin{aligned}
\|u\|_{\mathcal{H}_\alpha}^{p^*(\alpha)} &= \int_{\mathbb{H}^n} \psi^\alpha \frac{|u|^\alpha}{r^\alpha}|u|^{p^*(\alpha)-\alpha}dq \\
&\le \left(\int_{\mathbb{H}^n} \psi^p \frac{|u|^p}{r^p}dq\right)^{\alpha/p}\left(\int_{\mathbb{H}^n} |u|^{(p^*(\alpha)-\alpha)\frac{p}{p-\alpha}}dq\right)^{(p-\alpha)/p} \\
&= \left(\int_{\mathbb{H}^n} \psi^p \frac{|u|^p}{r^p}dq\right)^{\alpha/p}\left(\int_{\mathbb{H}^n} |u|^{p^*}dq\right)^{(p-\alpha)/p} \\
&\le \left(\frac{p}{Q-p}\right)^\alpha \|D_{\mathbb{H}^n}u\|_p^\alpha (C_{Q,p}\|D_{\mathbb{H}^n}u\|_p)^{p^*(p-\alpha)/p} = \mathcal{C}\|D_{\mathbb{H}^n}u\|_p^{p^*(\alpha)}
\end{aligned}$$

as required. □

From Lemma 1.2.1 it is clear that the Sobolev embedding

$$S^{1,p}(\mathbb{H}^n) \hookrightarrow L^{p^*}(\mathbb{H}^n)$$

and the Hardy–Sobolev embedding

$$S^{1,p}(\mathbb{H}^n) \hookrightarrow L^{p^*(\alpha)}\left(\mathbb{H}^n, \psi^\alpha r^{-\alpha}\right)$$

are continuous, but not compact. However, we are able to introduce the best Hardy–Sobolev constant $\mathcal{H}_\alpha = \mathcal{H}(p, Q, \alpha)$, as stated in (1.2.3). Of course, the number $\mathcal{H}_\alpha$ is well defined, strictly positive and it coincides with the best Sobolev constant when $\alpha = 0$. Indeed in (1.2.1) we get $C_{Q,p} = \mathcal{H}_0^{-1/p}$, where $\mathcal{H}_0$ is the constant given in (1.2.3) when $\alpha = 0$, so that $\|\cdot\|_{\mathcal{H}_0} = \|\cdot\|_{p^*}$. We refer to [45] for the best constants in various Sobolev inequalities in $\mathbb{H}^n$. In particular, in the paper [45] it is given explicitly the sharp constant in the case $\alpha = 0$ and $p = 2$. We also mention [55] for similar results.

Let $HW^{1,p}(\mathbb{H}^n)$ denote the *horizontal Sobolev space* consisting of functions $u \in L^p(\mathbb{H}^n)$ such that $D_{\mathbb{H}^n}u$ exists in the sense of distributions and $|D_{\mathbb{H}^n}u|_{\mathbb{H}^n} \in L^p(\mathbb{H}^n)$. Endow $HW^{1,p}(\mathbb{H}^n)$ with the natural norm

$$\|u\|_{HW^{1,p}(\mathbb{H}^n)} = \left(\int_{\mathbb{H}^n} |u|^p dq + \int_{\mathbb{H}^n} |D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^p dq\right)^{1/p}.$$

The embedding

$$HW^{1,p}(\mathbb{H}^n) \hookrightarrow L^\nu(\mathbb{H}^n) \tag{1.2.4}$$

is continuous for any $\nu \in [p, p^*]$ by (1.2.1) and the Hölder inequality. Furthermore, if Ω is a bounded Poincaré–Sobolev domain in $\mathbb{H}^n$ the embedding

$$HW^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^\nu(\Omega) \tag{1.2.5}$$

is compact, when $1 \le \nu < p^*$ by [50, 54, 91]. From now on a Poincaré–Sobolev domain is briefly called as PS domain. We emphasize that the class of PS domains is very large and we refer to [50] for further details. By [43, 54, 91] the property (1.2.5) holds in Carnot–Carathéodory balls, which are special bounded PS domains of $\mathbb{H}^n$. Since the Carnot–Carathéodory distance and the Korányi distance are equivalent on $\mathbb{H}^n$ by [6, 69], then (1.2.5) can be applied when Ω is any Korányi ball $B_R(q_0)$, $q_0 \in \mathbb{H}^n$, and $R > 0$.

Observe that, in Theorem 1.1 of [6] we find a compactness result similar to (1.2.5) for the case $p = 2$, which holds for symmetric unbounded domains of the Heisenberg group.

Until the end of the section let Ω be a fixed open set of $\mathbb{H}^n$. If $u \in S_0^{1,p}(\Omega)$ let $\tilde{u}$ denote the zero extension of u outside Ω. The next result shows that the mapping $u \to \tilde{u}$ takes $S_0^{1,p}(\Omega)$ into $S^{1,p}(\mathbb{H}^n)$. Its proof is based on Lemma 3.22 in [1], but now in the Folland–Stein Sobolev context on the Heisenberg group $\mathbb{H}^n$.

Lemma 1.2.2. *Let $u \in S_0^{1,p}(\Omega)$. Then $D_{\mathbb{H}^n}\tilde{u} = (D_{\mathbb{H}^n}u)^\sim$ in $\mathbb{H}^n$ in the distributional sense and $\tilde{u} \in S^{1,p}(\mathbb{H}^n)$.*

Proof. Let $(\varphi_k)_k$ be a sequence in $C_0^\infty(\Omega)$ converging to u in $S_0^{1,p}(\Omega)$. Clearly, $\varphi_k \to u$ in $L^{p^*}(\Omega)$ by (1.2.1). Hence, for all $\rho \in C_0^\infty(\mathbb{H}^n)$, the dominated convergence Lebesgue theorem, applied twice, and integration by parts give

$$\begin{aligned}
-\int_{\mathbb{H}^n} \tilde{u}(q) \cdot D_{\mathbb{H}^n}\rho(q)\,dq &= -\int_\Omega u(q) \cdot D_{\mathbb{H}^n}\rho(q)\,dq \\
&= -\lim_{k\to\infty} \int_\Omega \varphi_k(q) \cdot D_{\mathbb{H}^n}\rho(q)\,dq \\
&= \lim_{k\to\infty} \int_\Omega D_{\mathbb{H}^n}\varphi_k(q) \cdot \rho(q)\,dq \\
&= \int_\Omega D_{\mathbb{H}^n}u(q) \cdot \rho(q)\,dq \\
&= \int_{\mathbb{H}^n} D_{\mathbb{H}^n}u(q)^\sim \cdot \rho(q)\,dq.
\end{aligned}$$

Consequently $D_{\mathbb{H}^n}\tilde{u} = (D_{\mathbb{H}^n}u)^\sim$ in the distributional sense in $\mathbb{H}^n$, which means that these locally integrable functions are equal a.e. in $\mathbb{H}^n$. □

Lemma 1.2.2 and (1.2.1) imply that

$$S_0^{1,p}(\Omega) \subset \{u \in L^{p^*}(\Omega) : \tilde{u} \in S^{1,p}(\mathbb{H}^n)\} \tag{1.2.6}$$

and equality holds when $\Omega = \mathbb{H}^n$.

Clearly, the embedding $S_0^{1,p}(\Omega) \hookrightarrow HW^{1,p}(\Omega)$ is continuous, when Ω is bounded.

By (1.2.5) it is also apparent that if Ω is a bounded PS domain in $\mathbb{H}^n$, then the embedding

$$S_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{\nu}(\Omega) \tag{1.2.7}$$

is compact for all ν, with $1 \leq \nu < p^*$.

Chapter 2

Schrödinger–Hardy systems involving two Laplacian operators in the Heisenberg group

2.1 Introduction

In this chapter we deal with the Schrödinger–Hardy system, discussed in [15],

$$(\mathcal{P}_1) \qquad \begin{cases} -\Delta^m_{\mathbb{H}^n} u + a(q)|u|^{m-2}u - \mu\psi^m \dfrac{|u|^{m-2}u}{r(q)^m} = H_u(q,u,v) & \text{in } \mathbb{H}^n, \\ -\Delta^p_{\mathbb{H}^n} v + b(q)|v|^{p-2}v - \sigma\psi^p \dfrac{|v|^{p-2}v}{r(q)^p} = H_v(q,u,v) & \text{in } \mathbb{H}^n, \end{cases}$$

where μ and σ are real parameters, $Q = 2n+2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$, $1 < p < Q$, $1 < m \leq p < m^* = mQ/(Q-m)$. Moreover, the Heisenberg norm r and the weight function $\psi = |D_{\mathbb{H}^n} r|_{\mathbb{H}^n}$ are defined in Section 1.1. We recall that ψ is constantly equal to 1 in the Euclidean canonical case.

The nonlinearities H_u and H_v denote the partial derivatives of H with respect to the second variable and the third variable, respectively, and H satisfies assumptions (H_1)–(H_4) listed below.

Let $1 < \wp < Q$. In order to handle system $(\mathcal{P}_1)$, we briefly recall the best Hardy–Sobolev constant $\mathcal{H}_\wp = \mathcal{H}(\wp, Q)$, as defined in (1.2.3), in the subcase

$\alpha = \wp$,

$$\mathcal{H}_\wp = \inf_{\substack{u \in S^{1,\wp}(\mathbb{H}^n) \\ u \neq 0}} \frac{\|D_{\mathbb{H}^n} u\|_\wp^\wp}{\|u\|_{\mathcal{H}_\wp}^\wp}, \qquad \|u\|_{\mathcal{H}_\wp}^\wp = \int_{\mathbb{H}^n} \psi^\wp \frac{|u|^\wp}{r^\wp} dq, \tag{2.1.1}$$

where $S^{1,\wp}(\mathbb{H}^n)$ is the Folland–Stein space. Clearly, $\mathcal{H}_\wp > 0$ thanks to (1.2.2).

In the following the set $\mathscr{V}(\mathbb{H}^n)$ *consists of all functions* $V \in C(\mathbb{H}^n)$ *and verifying conditions*

(V_1) *V is bounded from below by a positive constant;*

(V_2) *there exists $\mathfrak{h} > 0$ such that* $\lim_{r(q_0)\to\infty} |\{q \in B_\mathfrak{h}(q_0) : V(q) \leq \mathfrak{c}\}| = 0$ *for any* $\mathfrak{c} > 0$,

where $B_\mathfrak{h}(q_0)$ denotes any open ball of $\mathbb{H}^n$ centered at q_0 and of radius $\mathfrak{h} > 0$, while $|\cdot|$ is the $(2n+1)$–dimensional Lebesgue measure on $\mathbb{H}^n$.

Indeed, as noted in Section 1.1, statements involving measure theory are always understood to be with respect to the Haar measure on $\mathbb{H}^n$, which coincides with the $(2n+1)$–dimensional Lebesgue measure.

Let us introduce some notation and assume that $1 < \wp < \infty$ and that $V \in C(\mathbb{H}^n)$ satisfies (V_1). Define

$$E_{\wp,V} = \left\{ u \in S^{1,\wp}(\mathbb{H}^n) : \int_{\mathbb{H}^n} V(q)|u(q)|^\wp dq < \infty \right\},$$

endowed with the norm

$$\|u\|_{E_{\wp,V}} = \left(\|D_{\mathbb{H}^n} u\|_\wp^\wp + \|u\|_{\wp,V}^\wp \right)^{1/\wp},$$

where $\|u\|_{\wp,V} = \left(\int_{\mathbb{H}^n} V(q)|u|^\wp dq \right)^{1/\wp}$. Now let the weight functions a and b satisfy (V_1) and from now on we assume that they are continuous in $\mathbb{H}^n$, without further mentioning. The natural solution space for ($\mathcal{P}_1$) is

$$W = E_{m,a} \times E_{p,b},$$

endowed with the norm

$$\|(u,v)\| = \|u\|_{E_{m,a}} + \|v\|_{E_{p,b}}.$$

By the continuous embedding $W \hookrightarrow L^\nu(\mathbb{H}^n) \times L^\nu(\mathbb{H}^n)$ for all $\nu \in [p, m^*]$, see Lemma 2.3.4 below, since $1 < m \leq p < m^*$, we can define

$$\lambda_\nu = \inf \left\{ \|u\|_{E_{m,a}}^\nu + \|v\|_{E_{p,b}}^\nu : \int_{\mathbb{H}^n} |(u,v)|^\nu dq = 1 \right\}, \tag{2.1.2}$$

and deduce that $\lambda_\nu > 0$.

Moreover, we suppose that the nonlinearity H satisfies the following mild conditions.

(H_1) *$H : \mathbb{H}^n \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and admits partial derivatives H_u and H_v of class $C(\mathbb{H}^n \times \mathbb{R}^2)$, $H \geq 0$ in $\mathbb{H}^n \times \mathbb{R}^2$, $H(q,0,0) = 0$ in $\mathbb{H}^n$ and $H_u(q,u,v) = 0$ if $q \in \mathbb{H}^n$ and $u \leq 0$, $v \in \mathbb{R}$, while $H_v(q,u,v) = 0$ if $q \in \mathbb{H}^n$, $u \in \mathbb{R}$ and $v \leq 0$;*

(H_2) *There are an exponent $s \in (p, m^*)$ and a number $\lambda \in [0, \lambda_p)$ such that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which the inequality*

$$|H_\omega(q,\omega)| \leq (\lambda + \varepsilon)|\omega|^{p-1} + C_\varepsilon |\omega|^{s-1}, \quad \omega = (u,v), \ |\omega| = \sqrt{u^2+v^2},$$

holds for all $(q,\omega) \in \mathbb{H}^n \times \mathbb{R}^2$, where λ_p is introduced in (2.1.2) and $H_\omega = (H_u, H_v)$;

(H_3) $\displaystyle\lim_{\substack{|\omega|\to\infty \\ u>0 \vee v>0}} \frac{H(q,\omega)}{|\omega|^p} = \infty$, *uniformly in* $\mathbb{H}^n$;

(H_4) *There exists a nonnegative function $\mathcal{G}$ of class $L^1(\mathbb{H}^n)$ and a constant $C_{\mathcal{F}} \geq 1$ such that*

$$\mathcal{F}(q, \tau\omega) \leq C_{\mathcal{F}} \mathcal{F}(q,\omega) + \mathcal{G}(q),$$

for a.e $q \in \mathbb{H}^n$ and all $u \in \mathbb{R}_0^+$, $v \in \mathbb{R}_0^+$ and $0 < \tau < 1$, where

$$\mathcal{F}(q,\omega) = H_\omega(q,\omega) \cdot \omega - pH(q,\omega).$$

As noted in [40] in the Euclidean setting, when $\mathcal{F}$ does not depend on q the function $\mathcal{G}$ should be identically zero in (H_4) and $\mathcal{F} = \mathcal{F}(u,v) \geq 0$ by (H_1). For simple examples of function $H = H(u,v)$ verifying (H_1)–(H_4) we refer to [40]. Before stating the first main result, we recall in passing that a nonnegative solution (u,v) of $(\mathcal{P}_1)$ is a vector function with all the components nonnegative in $\mathbb{H}^n$.

Theorem 2.1.1. *Under the assumptions (H_1)–(H_4) and for a and b of class $\mathscr{V}(\mathbb{H}^n)$, system $(\mathcal{P}_1)$ has at least one nontrivial nonnegative entire solution $(u,v) \in W$ for any $\mu \in (-\infty, \mathcal{H}_m)$ and for any $\sigma \in (-\infty, \mathcal{H}_p)$ such that*

$$1 - \frac{\mu^+}{\mathcal{H}_m} - \frac{\sigma^+}{\mathcal{H}_p} - 2^{p-1}\frac{\lambda}{\lambda_p} > 0, \tag{2.1.3}$$

being $\lambda \in [0, \lambda_p)$ given in (H_2).

The proof of Theorem 2.1.1 follows [40] somehow, but there are some technical difficulties due to the more general setting considered, as well as to the presence of the Hardy terms and the fact that the nonlinearities do not necessarily satisfy the *Ambrosetti–Rabinowitz* condition. The approach is based on the application of the mountain pass theorem in the version given in [34]. Thus, the Euler–Lagrange functional related to $(\mathcal{P}_1)$ has to satisfy the *Cerami* compactness condition, which is derived here from the use of the key new Theorem 2.3.2, based on the crucial Lemma 2.3 of [94].

The main compactness result, Theorem 2.3.2, continues to hold without the (V_2) condition, that is under the solely (V_1) on the potentials a and b, provided that assumption (H_2) is replaced by

$(H_2)'$ *There are exponents* $\mathfrak{s}$, s, *with* $p < \mathfrak{s} < s < m^*$, *and a number* $\lambda \in [0, \lambda_{\mathfrak{s}})$ *such that for every* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *for which the inequality*

$$|H_\omega(q,\omega)| \le (\lambda+\varepsilon)|\omega|^{\mathfrak{s}-1} + C_\varepsilon|\omega|^{s-1}$$

holds for all $(q,\omega) \in \mathbb{H}^n \times \mathbb{R}^2$.

This follows using the argument of Lemma 2.2 in [24] given for the Euclidean case. We refer also to Theorem 2.1 in [70]. In particular, we have

Theorem 2.1.2. *Under the assumptions* (H_1), $(H_2)'$, (H_3), (H_4) *and with* a *and* b *satisfying* (V_1), *system* $(\mathcal{P}_1)$ *has at least one nontrivial nonnegative entire solution* $(u,v) \in W$ *for any* $\mu \in (-\infty, \mathcal{H}_m)$ *and for any* $\sigma \in (-\infty, \mathcal{H}_p)$ *such that*

$$1 - \frac{\mu^+}{\mathcal{H}_m} - \frac{\sigma^+}{\mathcal{H}_p} - 2^{p-1}\frac{\lambda}{\lambda_{\mathfrak{s}}} > 0, \tag{2.1.4}$$

being $\lambda \in [0, \lambda_{\mathfrak{s}})$ *given in* $(H_2)'$.

Taking inspiration from [40], we treat critical nonlinear terms and also the nonlinear terms proposed in [25]. That is, as in [15], we consider finally the system in $\mathbb{H}^n$

$$(\mathcal{P}_2)\qquad \begin{cases} -\Delta^m_{\mathbb{H}^n} u + a(q)|u|^{m-2}u - \mu\psi^m \dfrac{|u|^{m-2}u}{r(q)^m} = H_u(q,u,v) + |u|^{m^*-2}u^+ \\ \qquad\qquad\qquad + \dfrac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta} + h(q), \\ -\Delta^p_{\mathbb{H}^n} v + b(q)|v|^{p-2}v - \sigma\psi^p \dfrac{|v|^{p-2}v}{r(q)^p} = H_v(q,u,v) + |v|^{p^*-2}v^+ \\ \qquad\qquad\qquad + \dfrac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1} + g(q), \end{cases}$$

where $\theta > 1$, $\vartheta > 1$ and $\theta + \vartheta = m^*$. Moreover, h is a nonnegative perturbation of class $L^{\mathfrak{m}}(\mathbb{H}^n)$, with $\mathfrak{m}$ the Hölder conjugate of m^*, that is $\mathfrak{m} = m'Q/(Q+m')$. While, g is a nonnegative perturbation of class $L^{\mathfrak{p}}(\mathbb{H}^n)$, with $\mathfrak{p}$ the Hölder conjugate of p^*, i.e. $\mathfrak{p} = p'Q/(Q+p')$.

Theorem 2.1.3. *Under the assumptions* (H_1), (H_2), *with* a *and* b *of class* $\mathscr{V}(\mathbb{H}^n)$, *there exists a number* $\delta > 0$ *such that for all nonnegative perturbations* h *and* g, *with* $0 < \|h\|_{\mathfrak{m}} + \|g\|_{\mathfrak{p}} < \delta$, *system* $(\mathcal{P}_2)$ *has at least one nontrivial nonnegative entire solution* $(u, v) \in W$ *for any* $\mu \in (-\infty, \mathcal{H}_m)$ *and for any* $\sigma \in (-\infty, \mathcal{H}_p)$ *satisfying* (2.1.4), *provided that either* $m < p$ *and* $\mu \le 0$, *or* $m = p$.

This chapter is organized as follows. In Section 2.2 we present a classical mountain pass Theorem 2.2.1, useful in the proof of Theorem 2.1.1. In Section 2.3, we construct the natural solution space W for $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ and prove the key compactness theorems, particularly helpful for the next sections. In Section 2.4, using the mountain pass Theorem 2.2.1, we obtain the existence of solutions for $(\mathcal{P}_1)$, that is we prove Theorem 2.1.1. Section 2.5 is devoted to the proof of Theorem 2.1.3.

2.2 Preliminaries

System $(\mathcal{P}_1)$ has a variational structure and to prove Theorem 2.1.1 we use the celebrated mountain pass theorem of *Ambrosetti* and *Rabinowitz*, stated in terms of the existence of Cerami's sequences as a direct consequence of Corollaries 4 and 9 of [34].

Let us recall some classical facts. Let $X = (X, \|\cdot\|)$ be a real Banach space, with its dual space X'. A functional $J : X \to \mathbb{R}$ of class $C^1(X)$ is said to satisfy the *Cerami condition* (C) if any *Cerami sequence* associated with J has a strongly convergent subsequence in X. A sequence $(u_k)_k$ in X is called a *Cerami sequence*, if $(J(u_k))_k$ is bounded and $(1 + \|u_k\|) \cdot \|J'(u_k)\|_{X'} \to 0$ as $k \to \infty$. In particular, $\|J'(u_k)\|_{X'} \to 0$ as $k \to \infty$.

Let us return to the mountain pass theorem, which we present here in the stronger form as given in Theorem I of [27] and refer to [27, 34] for further comments.

Theorem 2.2.1. *Let* X *be a real Banach space and let* $J \in C^1(X)$ *satisfy*

$$\max\{J(0), J(e)\} \le \beta < \delta \le \inf_{\|u\|=\rho} J(u),$$

for some $\beta < \delta$, $\rho > 0$ *and* $e \in X$ *with* $\|e\| > \rho$. *Let* $c \ge \delta$ *be characterized by*

$$c = \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} J(\gamma(\tau))$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$ *is the set of continuous paths joining 0 and* e. *Then there exists a Cerami sequence* $(u_k)_k$ *in* X *such that* $J(u_k) \to c \ge \delta$ *as* $k \to \infty$.

Finally, if the functional J satisfies the Cerami condition (C) at the mini–max level c, then c is a critical value of J in X.

We refer also to [40] and the references therein for interesting historical details on Theorem 2.2.1. It is worth noting that also in [57, 58] the Cerami condition was used in the study of variational systems on unbounded domains in the Euclidean setting. Finally, again for systems in the Euclidean setting, we mention [46].

The Euler–Lagrange functional associated to $(\mathcal{P}_1)$ is

$$I_{\mu,\sigma}(u,v) = \frac{1}{m}\|u\|^m_{E_{m,a}} + \frac{1}{p}\|v\|^p_{E_{p,b}} - \frac{\mu}{m}\|u\|^m_{\mathcal{H}_m} - \frac{\sigma}{p}\|v\|^p_{\mathcal{H}_p} - \int_{\mathbb{H}^n} H(q,u,v)dq.$$

Clearly, the functional $I_{\mu,\sigma}$ is well–defined in W. Under conditions (H_1)–(H_2), it is easy to see that the functional $I_{\mu,\sigma}$ is of class $C^1(W)$, and for $(u,v) \in W$

$$\begin{aligned}\langle I'_{\mu,\sigma}(u,v),(\Phi,\Psi)\rangle =& \langle u,\Phi\rangle_{E_{m,a}} + \langle v,\Psi\rangle_{E_{p,b}} - \mu\langle u,\Phi\rangle_{\mathcal{H}_m} - \sigma\langle v,\Psi\rangle_{\mathcal{H}_p} \\ &- \int_{\mathbb{H}^n} [H_u(q,u,v)\Phi + H_v(q,u,v)\Psi]dq\end{aligned} \tag{2.2.1}$$

for all $(\Phi,\Psi) \in W$. From here on in this chapter, $\langle\cdot,\cdot\rangle$ simply denotes the dual pairing between W and its dual space W', that is $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{W',W}$. Hence, the critical points of $I_{\mu,\sigma}$ in W are exactly the (weak) solutions of $(\mathcal{P}_1)$.

Indeed, we say that $(u,v) \in W$ is an *entire* (weak) *solution* of problem $(\mathcal{P}_1)$ if

$$\begin{aligned}&\langle u,\Phi\rangle_{E_{m,a}} + \langle v,\Psi\rangle_{E_{p,b}} - \mu\langle u,\Phi\rangle_{\mathcal{H}_m} - \sigma\langle v,\Psi\rangle_{\mathcal{H}_p} \\ &\qquad = \int_{\mathbb{H}^n} H_u(q,u,v)\Phi(q)dq + \int_{\mathbb{H}^n} H_v(q,u,v)\Psi(q)dq\end{aligned} \tag{2.2.2}$$

for any $(\Phi,\Psi) \in W$, where

$$\begin{aligned}\langle u,\Phi\rangle_m &= \int_{\mathbb{H}^n} \left(|D_{\mathbb{H}^n}u|^{m-2}_{\mathbb{H}^n} D_{\mathbb{H}^n}u, D_{\mathbb{H}^n}\Phi\right)_{\mathbb{H}^n} dq,\\ \langle v,\Psi\rangle_p &= \int_{\mathbb{H}^n} \left(|D_{\mathbb{H}^n}v|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n}v, D_{\mathbb{H}^n}\Psi\right)_{\mathbb{H}^n} dq,\\ \langle u,\Phi\rangle_{E_{m,a}} &= \langle u,\Phi\rangle_m + \int_{\mathbb{H}^n} a(q)|u(q)|^{m-2}u(q)\Phi(q)dq,\\ \langle v,\Psi\rangle_{E_{p,b}} &= \langle v,\Psi\rangle_p + \int_{\mathbb{H}^n} b(q)|v(q)|^{p-2}v(q)\Psi(q)dq,\\ \langle u,\Phi\rangle_{\mathcal{H}_m} &= \int_{\mathbb{H}^n} \psi^m \frac{|u(q)|^{m-2}u(q)}{r(q)^m}\Phi(q)dq,\end{aligned}$$

$$\langle v, \Psi\rangle_{\mathcal{H}_p} = \int_{\mathbb{H}^n} \psi^p \frac{|v(q)|^{p-2}v(q)}{r(q)^p}\Psi(q)dq.$$

The simplified notation is reasonable, since $\langle u, \cdot\rangle_m$, $\langle v, \cdot\rangle_p$, $\langle u, \cdot\rangle_{E_{m,a}}$, $\langle v, \cdot\rangle_{E_{p,b}}$, $\langle u, \cdot\rangle_{\mathcal{H}_m}$ and $\langle v, \cdot\rangle_{\mathcal{H}_p}$ are linear bounded functionals on W for all $(u, v) \in W$.

2.3 Properties of the space W

In this section, we give some basic results of the solution space W that will be used in the next section.

From now on, $\wp$ is a general fixed exponent, with $1 < \wp < Q$. We also recall that

$$1 < p < Q \quad \text{and} \quad 1 < m \leq p < m^* = mQ/(Q - m).$$

First, $E_{m,a} = (E_{m,a}, \|\cdot\|_{E_{m,a}})$ and $E_{p,b} = (E_{p,b}, \|\cdot\|_{E_{p,b}})$ are two separable, reflexive Banach spaces, by Lemma 10 of [86]. Hence, by Theorem 1.22 of [1], the main solution space $W = E_{m,a} \times E_{p,b}$, with

$$\|(u, v)\| = \|u\|_{E_{m,a}} + \|v\|_{E_{p,b}},$$

is a separable and reflexive Banach space.

The next result is an adaptation of Lemma 4.1 of [23] and Lemma 1 of [86], where the Euclidean space $\mathbb{R}^n$ is replaced by the more general case of the Heisenberg group $\mathbb{H}^n$.

Lemma 2.3.1. *Let V satisfy (V_1). If $\nu \in [\wp, \wp^*]$, then the embeddings*

$$E_{\wp,V} \hookrightarrow HW^{1,\wp}(\mathbb{H}^n) \hookrightarrow L^\nu(\mathbb{H}^n)$$

are continuous. In particular, there exists a constant C_ν such that

$$\|u\|_{L^\nu(\mathbb{H}^n)} \leq C_\nu \|u\|_{E_{\wp,V}} \text{ for all } u \in E_{\wp,V}. \tag{2.3.1}$$

If $\nu \in [1, \wp^*)$, then the embedding $E_{\wp,V} \hookrightarrow\hookrightarrow L^\nu(B_R)$ is compact for any $R > 0$.

Proof. The embeddings of the chain $E_{\wp,V} \hookrightarrow HW^{1,\wp}(\mathbb{H}^n) \hookrightarrow L^\nu(\mathbb{H}^n)$ are obviously continuous by (1.2.4), the definition of $\|\cdot\|_{E_{\wp,V}}$ and (V_1). Consequently, (2.3.1) follows at once. By (1.2.5) the embedding

$$HW^{1,\wp}(B_R) \hookrightarrow\hookrightarrow L^\nu(B_R)$$

is compact for all $\nu \in [1, \wp^*)$. Therefore, also the embedding

$$E_{\wp,V} \hookrightarrow\hookrightarrow L^\nu(B_R)$$

is compact for all $\nu \in [1, \wp^*)$ by the first part of the lemma. □

As explained in Section 1.1, we denote by $|\cdot|$ the Lebesgue measure on the σ–algebra of $\mathbb{H}^n$. Extending Theorem 2.1 in [86] and Lemma 4.4 in [23] for the Heisenberg case, we give the following result.

Theorem 2.3.2. *Suppose that $V \in \mathscr{V}(\mathbb{H}^n)$. Let $\nu \in [\wp, \wp^*)$ be a fixed exponent and let $(v_k)_k$ be a bounded sequence in $E_{\wp,V}$. Then there exists $v \in E_{\wp,V}$ such that up to a subsequence, still denoted by $(v_k)_k$,*

$$v_k \to v \text{ strongly in } L^\nu(\mathbb{H}^n) \text{ as } k \to \infty.$$

Proof. We first consider the case $\nu = \wp$. Fix $\mathfrak{c} > 0$ and set

$$A_{\mathfrak{c}}(q_0) = \{q \in \mathbb{H}^n : V(q) \le \mathfrak{c}\} \bigcap B_{\mathfrak{h}}(q_0),$$

where $\mathfrak{h} > 0$ is the number independent of $\mathfrak{c}$ given by (V_2). Since $(v_k)_k$ is a bounded sequence in $E_{\wp,V}$ there exists a function $v \in E_{\wp,V}$ such that, up to a subsequence, $v_k \rightharpoonup v$ weakly in $E_{\wp,V}$. Moreover, there exists a positive constant C such that

$$\|v_k\|_{E_{\wp,V}} + \|v\|_{E_{\wp,V}} \le C \quad \text{for all } k \in \mathbb{N}. \tag{2.3.2}$$

By virtue of Lemma 2.3, given in [94] there exists a sequence $(q_j)_j \subset \mathbb{H}^n$ such that $\mathbb{H}^n = \bigcup_{j=1}^\infty B_{\mathfrak{h}}(q_j)$ for all $\mathfrak{h} > 0$ and each $q \in \mathbb{H}^n$ is covered by at most 24^Q balls $B_{\mathfrak{h}}(q_j)$. Setting

$$C_{\mathfrak{h}}(q_j) = \{q \in \mathbb{H}^n : V(q) > \mathfrak{c}\} \bigcap B_{\mathfrak{h}}(q_j),$$

for all $R > 0$, we have

$$\begin{aligned}\int_{\mathbb{H}^n \setminus B_R} |v_k(q) - v(q)|^\wp dq &\le \sum_{r(q_j) \ge R - \mathfrak{h}} \int_{B_{\mathfrak{h}}(q_j)} |v_k(q) - v(q)|^\wp dq \\ &= \sum_{r(q_j) \ge R - \mathfrak{h}} \Bigg[\int_{C_{\mathfrak{h}}(q_j)} |v_k(q) - v(q)|^\wp dq \\ &\qquad + \int_{A_{\mathfrak{h}}(q_j)} |v_k(q) - v(q)|^\wp dq\Bigg].\end{aligned}$$

Then, the definitions of $C_{\mathfrak{h}}(q_j)$ and $A_{\mathfrak{c}}(q_j)$ yield

$$\begin{aligned}&\int_{C_{\mathfrak{h}}(q_j)} |v_k(q) - v(q)|^\wp dq \le \frac{1}{\mathfrak{c}} \int_{B_{\mathfrak{h}}(q_j)} V(q)|v_k(q) - v(q)|^\wp dq, \\ &\int_{A_{\mathfrak{c}}(q_j)} |v_k(q) - v(q)|^\wp dq \le \|v_k - v\|^\wp_{L^{\wp^*}(A_{\mathfrak{c}}(q_j))} |A_{\mathfrak{c}}(q_j)|^{(\wp^* - \wp)/\wp^*}\end{aligned}$$

$$\leq \|v_k - v\|^{\wp}_{L^{\wp^*}(B_{\mathfrak{h}}(q_j))} |A_{\mathfrak{c}}(q_j)|^{(\wp^*-\wp)/\wp^*},$$

in virtue of the Hölder inequality. Hence, being $(\wp^* - \wp)/\wp^* = \wp/Q$, for all $R > 0$ we obtain

$$\begin{aligned}
\int_{\mathbb{H}^n \setminus B_R} |v_k(q) - v(q)|^{\wp} dq &\leq \sum_{r(q_j) \geq R - \mathfrak{h}} \Bigg[\frac{1}{\mathfrak{c}} \int_{B_{\mathfrak{h}}(q_j)} V(q) |v_k(q) - v(q)|^{\wp} dq \\
&\quad + |A_{\mathfrak{c}}(q_j)|^{\wp/Q} \|v_k - v\|^{\wp}_{L^{\wp^*}(B_{\mathfrak{h}}(q_j))} \Bigg] \\
&\leq \Bigg[\frac{24^Q}{\mathfrak{c}} \int_{\mathbb{H}^n} V(q) |v_k(q) - v(q)|^{\wp} dq \\
&\quad + \sup_{r(q_j) \geq R - \mathfrak{h}} |A_{\mathfrak{c}}(q_j)|^{\wp/Q} \sum_{r(q_j) \geq R - \mathfrak{h}} \|v_k - v\|^{\wp}_{L^{\wp^*}(B_{\mathfrak{h}}(q_j))} \Bigg] \\
&\leq \frac{24^Q}{\mathfrak{c}} \|v_k - v\|^{\wp}_{\wp, V} \\
&\quad + 24^Q C^{\wp}_{\wp^*} \sup_{r(q_j) \geq R - \mathfrak{h}} |A_{\mathfrak{c}}(q_j)|^{\wp/Q} \|v_k - v\|^{\wp}_{E_{\wp,V}},
\end{aligned}$$

where the last inequality follows from Lemma 2.3.1. Consequently, using (2.3.2), we get

$$\begin{aligned}
\int_{\mathbb{H}^n \setminus B_R} |v_k(q) - v(q)|^{\wp} dq &\leq \frac{24^Q}{\mathfrak{c}} (\|v_k\|_{E_{\wp,V}} + \|v\|_{E_{\wp,V}})^{\wp} \\
&\quad + 24^Q C^{\wp}_{\wp^*} \sup_{r(q_j) \geq R - \mathfrak{h}} |A_{\mathfrak{c}}(q_j)|^{\wp/Q} (\|v_k\|_{E_{\wp,V}} + \|v\|_{E_{\wp,V}})^{\wp} \\
&\leq \frac{24^Q C^{\wp}}{\mathfrak{c}} + 24^Q (C_{\wp^*} C)^{\wp} \sup_{r(q_j) \geq R - \mathfrak{h}} |A_{\mathfrak{c}}(q_j)|^{\wp/Q}.
\end{aligned}$$

Now, we choose $\mathfrak{c} > 0$ in (V_2) so large that $3 \cdot 24^Q C^{\wp} < \varepsilon \cdot \mathfrak{c}$. Then, there exists $R_{\mathfrak{c}} > 0$ such that

$$24^Q (C_{\wp^*} C)^{\wp} \sup_{r(q_j) \geq R - \mathfrak{h}} |A_{\mathfrak{c}}(q_j)|^{\wp/Q} < \frac{\varepsilon}{3},$$

since

$$\sup_{r(q_j) \geq R - \mathfrak{h}} |A_{\mathfrak{c}}(q_j)|^{\wp/Q} \to 0 \quad \text{as} \quad R \to \infty,$$

by (V_2). Furthermore, thanks to the fact that $v_k \to v$ strongly in $L^{\wp}(B_{R_{\mathfrak{c}}})$ by Lemma 2.3.1, there exists $k_0 \in \mathbb{N}$ such that

$$\int_{B_{R_{\mathfrak{c}}}} |v_k(q) - v(q)|^{\wp} dq < \frac{\varepsilon}{3} \quad \text{for all } k \geq k_0.$$

Consequently, $v_k \to v$ strongly in $L^{\wp}(\mathbb{H}^n)$, as claimed.

Finally, consider the case $\nu \in (\wp, \wp^*)$. Then, there exists $\sigma \in (0,1)$ such that

$$\frac{1}{\nu} = \frac{\sigma}{\wp} + \frac{1-\sigma}{\wp^*} \quad \text{and} \quad \|v_k - v\|_\nu \le \|v_k - v\|_{\wp}^{\sigma} \|v_k - v\|_{\wp^*}^{1-\sigma} \to 0$$

as $k \to \infty$, since $v_k \to v$ in $L^{\wp}(\mathbb{H}^n)$ and $(v_k)_k$ is bounded in $L^{\wp^*}(\mathbb{H}^n)$ by Lemma 2.3.1. This completes the proof. □

As already noted in Section 2.1, Theorem 2.3.2 continues to hold under the solely condition (V_1) on the weight function V, provided that $\nu \in (\wp, \wp^*)$. The proof relies on Lemma 2.2 in [24], see also Theorem 2.1 in [70], for the Euclidean case. The extension to the Heisenberg setting can be derived proceeding as in [24, 70], with obvious changes.

Following Proposition A.8 (i) in [3], we get the proposition below, which will be crucial in the proof of Lemma 2.4.3.

Proposition 2.3.3. *Assume $w \in L^1_{\text{loc}}(\mathbb{H}^n)$. Let $(u_k)_k$ and u be in $L^{\wp}(\mathbb{H}^n, w)$. If $(u_k)_k$ is bounded in $L^{\wp}(\mathbb{H}^n, w)$ and $u_k \to u$ a.e. in $\mathbb{H}^n$, then $u_k \rightharpoonup u$ in $L^{\wp}(\mathbb{H}^n, w)$ and $|u_k|^{\wp-2}u_k \rightharpoonup |u|^{\wp-2}u$ in $L^{\wp'}(\mathbb{H}^n, w)$.*

Observe that the proof given in Proposition A.8 (i) in [3], related to the Euclidean space, continues to be valid for the Heisenberg group. Indeed, the space $(L^{\wp}(\mathbb{H}^n; w), \|\cdot\|_{\wp,w})$ is uniformly convex, by virtue of Proposition A.6 in [3].

The next result on the space W is a consequence of Lemma 2.3.1 and it extends Lemma 2.2 in [93] to the Heisenberg group case.

Lemma 2.3.4. *Let a and b satisfy (V_1). Then the embedding*

$$W \hookrightarrow L^\nu(\mathbb{H}^n) \times L^\nu(\mathbb{H}^n)$$

is continuous if $\nu \in [p, m^*]$, and

$$\|(u,v)\|_\nu \le C_\nu \|(u,v)\| \quad \text{for all } (u,v) \in W. \tag{2.3.3}$$

Proof. By (2.3.1), there exists $C_\nu > 0$ such that

$$\|u\|_\nu \le C_\nu \|u\|_{E_{m,a}} \quad \text{and} \quad \|v\|_\nu \le C_\nu \|v\|_{E_{p,b}} \quad \text{for all } (u,v) \in W.$$

Thus,

$$\begin{aligned} \|(u,v)\|_\nu &= \left\| \sqrt{u^2+v^2} \right\|_\nu \le \|u+v\|_\nu \le \|u\|_\nu + \|v\|_\nu \\ &\le C_\nu (\|u\|_{E_{m,a}} + \|v\|_{E_{p,b}}) = C_\nu \|(u,v)\|. \end{aligned}$$

This completes the proof. □

The upcoming result follows from Proposition A.10 of [3], which holds still in the Heisenberg case, by virtue of (1.2.5).

Lemma 2.3.5. *Assume that* (V_1) *holds. Let* $\{(u_k, v_k)\}_k \subset W$ *be such that* $(u_k, v_k) \rightharpoonup (u, v)$ *weakly in* W *as* $k \to \infty$. *Then, up to a subsequence,* $(u_k, v_k) \to (u, v)$ *a.e. in* $\mathbb{H}^n$ *as* $k \to \infty$.

By Theorem 2.3.2 and recalling that $1 < m \le p < m^*$, we have the following compact embeddings.

Lemma 2.3.6. *Suppose that* $a, b \in \mathscr{V}(\mathbb{H}^n)$. *Let* $\nu \in [p, m^*)$ *be a fixed exponent. Then the embeddings* $E_{m,a} \hookrightarrow\hookrightarrow L^\nu(\mathbb{H}^n)$ *and* $E_{p,b} \hookrightarrow\hookrightarrow L^\nu(\mathbb{H}^n)$ *are compact. Moreover, the embedding*

$$W \hookrightarrow\hookrightarrow L^\nu(\mathbb{H}^n) \times L^\nu(\mathbb{H}^n)$$

is compact.

Lemma 2.3.6 continues to hold under the solely condition (V_1) on the weight functions a and b, provided that $\nu \in (p, m^*)$, as a consequence of Theorem 2.3.2 and related observations.

Lemma 2.3.7. *Suppose that* (V_1) *holds. Let* $\nu \in (p, m^*)$ *be a fixed exponent. Then the embeddings* $E_{m,a} \hookrightarrow\hookrightarrow L^\nu(\mathbb{H}^n)$ *and* $E_{p,b} \hookrightarrow\hookrightarrow L^\nu(\mathbb{H}^n)$ *are compact. Moreover, the embedding*

$$W \hookrightarrow\hookrightarrow L^\nu(\mathbb{H}^n) \times L^\nu(\mathbb{H}^n)$$

is compact.

2.4 Existence of solutions

To prove Theorem 2.1.1, we shall apply Theorem 2.2.1 to the functional $I_{\mu,\sigma}$ introduced in Section 2.2. Thanks to the key results of Section 2.3 we are now able to extend the main arguments given in the recent paper [40] in our general framework.

Lemma 2.4.1. *Any Cerami sequence associated with the functional* $I_{\mu,\sigma}$ *is bounded in* W, *provided that* $\mu < \mathcal{H}_m$ *and* $\sigma < \mathcal{H}_p$.

Proof. Let $\{(u_k, v_k)\}_k \subset W$ be a Cerami sequence associated with $I_{\mu,\sigma}$. Then there exists $L > 0$ independent of k such that

$$\begin{gathered} |I_{\mu,\sigma}(u_k, v_k)| \le L \quad \text{for all } k \quad \text{and} \\ (1 + \|(u_k, v_k)\|) I'_{\mu,\sigma}(u_k, v_k) \to 0 \text{ as } k \to \infty. \end{gathered} \tag{2.4.1}$$

Thus there exists $\varepsilon_k > 0$, with $\varepsilon_k \to 0$ as $k \to \infty$, such that

$$|\langle I'_{\mu,\sigma}(u_k, v_k), (\Phi, \Psi)\rangle| \leq \frac{\varepsilon_k \|(\Phi, \Psi)\|}{1 + \|(u_k, v_k)\|} \tag{2.4.2}$$

for all $(\Phi, \Psi) \in W$ and $k \in \mathbb{N}$. Taking $(\Phi, \Psi) = (u_k, v_k)$ in (2.4.2), we have

$$\begin{aligned}\Big| \langle u_k, u_k\rangle_{E_{m,a}} &+ \langle v_k, v_k\rangle_{E_{p,b}} - \mu\langle u_k, u_k\rangle_{\mathcal{H}_m} - \sigma\langle v_k, v_k\rangle_{\mathcal{H}_p} \\ &- \int_{\mathbb{H}^n} [H_u(q, u_k, v_k)u_k + H_v(q, u_k, v_k)v_k]\, dq \Big| \\ &= |\langle I'_{\mu,\sigma}(u_k, v_k), (u_k, v_k)\rangle| \leq \frac{\varepsilon_k \|(u_k, v_k)\|}{1 + \|(u_k, v_k)\|} \leq \varepsilon_k \leq C,\end{aligned}$$

for all k and some appropriate $C > 0$. Hence,

$$\begin{aligned} -\|u_k\|^m_{E_{m,a}} - \|v_k\|^p_{E_{p,b}} + \mu\|u_k\|^m_{\mathcal{H}_m} + \sigma\|v_k\|^p_{\mathcal{H}_p} \\ + \int_{\mathbb{H}^n} [H_u(q, u_k, v_k)u_k + H_v(q, u_k, v_k)v_k]\, dq \leq C. \end{aligned} \tag{2.4.3}$$

Let us prove that $\{(u_k, v_k)\}_k$ is bounded in W. Otherwise, arguing by contradiction and without loss of generality, we assume that $\|(u_k, v_k)\| \to \infty$ as $k \to \infty$ and that $\|(u_k, v_k)\| \geq 1$ for all $k \in \mathbb{N}$. Define

$$(\mathcal{X}_k, \mathcal{Y}_k) = \frac{(u_k, v_k)}{\|(u_k, v_k)\|}.$$

It easily follows $\|(\mathcal{X}_k, \mathcal{Y}_k)\| = 1$. Then there exists $(\mathcal{X}, \mathcal{Y}) \in W$ such that, up to a subsequence,

$$(\mathcal{X}_k, \mathcal{Y}_k) \rightharpoonup (\mathcal{X}, \mathcal{Y}) \text{ in } W, \qquad (\mathcal{X}_k, \mathcal{Y}_k) \to (\mathcal{X}, \mathcal{Y}) \text{ a.e. in } \mathbb{H}^n,$$

by Lemma 2.3.5. Considering Lemma 2.3.6, we can assume that, up to a subsequence,

$$(\mathcal{X}_k, \mathcal{Y}_k) \to (\mathcal{X}, \mathcal{Y}) \text{ in } L^\nu(\mathbb{H}^n) \times L^\nu(\mathbb{H}^n) \text{ for any } \nu \in [p, m^*).$$

Set $\mathcal{X}_k^- = \min\{0, \mathcal{X}_k\}$ and $\mathcal{Y}_k^- = \min\{0, \mathcal{Y}_k\}$. Of course, $\{(\mathcal{X}_k^-, \mathcal{Y}_k^-)\}_k$ is also bounded in W. Choosing $(\Phi, \Psi) = (\mathcal{X}_k^-, \mathcal{Y}_k^-)$ in (2.4.2), by the fact that $\|(u_k, v_k)\| \to \infty$, we deduce

$$o(1) = \frac{\langle I'_{\mu,\sigma}(u_k, v_k), (\mathcal{X}_k^-, \mathcal{Y}_k^-)\rangle}{\|(u_k, v_k)\|^{m-1}}.$$

Then,

$$
\begin{aligned}
o(1) &= \frac{1}{\|(u_k,v_k)\|^{m-1}}\Big[\langle u_k,\mathcal{X}_k^-\rangle_{E_{m,a}} + \langle v_k,\mathcal{Y}_k^-\rangle_{E_{p,b}} \\
&\qquad - \mu\langle u_k,\mathcal{X}_k^-\rangle_{\mathcal{H}_m} - \sigma\langle v_k,\mathcal{Y}_k^-\rangle_{\mathcal{H}_p}\Big] \\
&\qquad - \int_{\mathbb{H}^n} \frac{H_u(q,u_k,v_k)\mathcal{X}_k^- + H_v(q,u_k,v_k)\mathcal{Y}_k^-}{\|(u_k,v_k)\|^{m-1}}dq \\
&= \frac{1}{\|(u_k,v_k)\|^{m}}\Big[\langle u_k,u_k^-\rangle_{E_{m,a}} + \langle v_k,v_k^-\rangle_{E_{p,b}} \\
&\qquad - \mu\langle u_k,u_k^-\rangle_{\mathcal{H}_m} - \sigma\langle v_k,v_k^-\rangle_{\mathcal{H}_p}\Big] \\
&\qquad - \int_{\mathbb{H}^n} \frac{H_u(q,u_k,v_k)u_k^- + H_v(q,u_k,v_k)v_k^-}{\|(u_k,v_k)\|^{m}}dq \\
&= \frac{1}{\|(u_k,v_k)\|^{m}}\Big[\langle u_k,u_k^-\rangle_{E_{m,a}} + \langle v_k,v_k^-\rangle_{E_{p,b}} \\
&\qquad - \mu\|u_k^-\|_{\mathcal{H}_m}^m - \sigma\|v_k^-\|_{\mathcal{H}_p}^p\Big] \\
&\ge \frac{1}{\|(u_k,v_k)\|^{m}}\left(\|u_k^-\|_{E_{m,a}}^m + \|v_k^-\|_{E_{p,b}}^p - \mu\|u_k^-\|_{\mathcal{H}_m}^m - \sigma\|v_k^-\|_{\mathcal{H}_p}^p\right) \\
&\ge \left(1-\frac{\mu^+}{\mathcal{H}_m}\right)\|\mathcal{X}_k^-\|_{E_{m,a}}^m + \left(1-\frac{\sigma^+}{\mathcal{H}_p}\right)\frac{\|v_k^-\|_{E_{p,b}}^p}{\|(u_k,v_k)\|^m} \\
&\ge \left(1-\frac{\mu^+}{\mathcal{H}_m}\right)\|\mathcal{X}_k^-\|_{E_{m,a}}^m.
\end{aligned} \tag{2.4.4}
$$

The first inequality in (2.4.4) follows from the elementary inequality

$$|\xi^- - \eta^-|^{\wp} \le |\xi-\eta|^{\wp-2}(\xi-\eta)(\xi^- - \eta^-) \quad \text{for } \xi,\eta\in\mathbb{R} \tag{2.4.5}$$

valid for all $\wp > 1$. While the second inequality in (2.4.4) follows from (1.2.2) and the fact that $\mu < \mathcal{H}_m$ and $\sigma < \mathcal{H}_p$. Thus, by (2.4.4) we find that as $k\to\infty$

$$\|\mathcal{X}_k^-\|_{E_{m,a}} \to 0.$$

In the same way, by

$$o(1) = \frac{\langle I'_{\mu,\sigma}(u_k,v_k),(\mathcal{X}_k^-,\mathcal{Y}_k^-)\rangle}{\|(u_k,v_k)\|^{p-1}},$$

we deduce that as $k\to\infty$,

$$\|\mathcal{Y}_k^-\|_{E_{p,b}} \to 0.$$

Consequently, we get $(\mathcal{X}_k^-, \mathcal{Y}_k^-) \to (0,0)$ in W as $k \to \infty$. Hence we obtain $(\mathcal{X}^-, \mathcal{Y}^-) = (0,0)$ a.e. in $\mathbb{H}^n$, that is $\mathcal{X} \geq 0$ and $\mathcal{Y} \geq 0$ a.e. in $\mathbb{H}^n$.

Let us now define

$$\Omega^+ = \{q \in \mathbb{H}^n : \mathcal{X} > 0 \text{ or } \mathcal{Y} > 0\} \quad \text{and}$$
$$\Omega^0 = \{q \in \mathbb{H}^n : (\mathcal{X}, \mathcal{Y}) = (0,0)\}.$$

Assume Ω^+ has a positive Haar measure. Since $\|(u_k, v_k)\| \to \infty$, we get

$$|(u_k, v_k)| = \big\|(u_k, v_k)\big\| \cdot |(\mathcal{X}_k, \mathcal{Y}_k)| \to \infty \quad \text{a.e. in} \quad \Omega^+.$$

Then, by (H_3)

$$\lim_{k\to\infty} \frac{H(q, u_k, v_k)}{\|(u_k, v_k)\|^p} = \lim_{k\to\infty} \frac{H(q, u_k, v_k)|(\mathcal{X}_k, \mathcal{Y}_k)|^p}{|(u_k, v_k)|^p} = \infty$$

a.e. in Ω^+. From Fatou's lemma it follows that

$$\text{(2.4.6)} \quad \lim_{k\to\infty} \int_{\mathbb{H}^n} \frac{H(q, u_k, v_k)}{\|(u_k, v_k)\|^p} dq = \lim_{k\to\infty} \int_{\mathbb{H}^n} \frac{H(q, u_k, v_k)|(\mathcal{X}_k, \mathcal{Y}_k)|^p}{|(u_k, v_k)|^p} dq = \infty.$$

Using (2.4.1), we obtain

$$\text{(2.4.7)} \quad \begin{aligned} \int_{\mathbb{H}^n} H(q, u_k, v_k) dq \leq \frac{1}{m}\|u_k\|^m_{E_{m,a}} + \frac{1}{p}\|v_k\|^p_{E_{p,b}} - \frac{\mu}{m}\|u_k\|^m_{\mathcal{H}_m} \\ - \frac{\sigma}{p}\|v_k\|^p_{\mathcal{H}_p} + L \end{aligned}$$

for all $k \in \mathbb{N}$. Clearly

$$\|v_k\|^p_{E_{p,b}} \leq \|(u_k, v_k)\|^p \quad \text{and}$$

$$\|u_k\|^m_{E_{m,a}} \leq \|(u_k, v_k)\|^m \leq \|(u_k, v_k)\|^p,$$

since $\|(u_k, v_k)\| \geq 1$ and $m \leq p$. Hence, using also (1.2.2) we have

$$\int_{\mathbb{H}^n} H(q, u_k, v_k) dq \leq \frac{2}{m}\|(u_k, v_k)\|^p + \frac{|\mu^-|}{m\mathcal{H}_m}\|(u_k, v_k)\|^p + \frac{|\sigma^-|}{p\mathcal{H}_p}\|(u_k, v_k)\|^p + L,$$

where as before $\tau^- = \min\{0, \tau\}$ for any $\tau \in \mathbb{R}$. Dividing by $\|(u_k, v_k)\|^p \geq 1$ for all k, we obtain that

$$\limsup_{k\to\infty} \int_{\mathbb{H}^n} \frac{H(q, u_k, v_k)}{\|(u_k, v_k)\|^p} dq \leq \frac{2}{m} + \frac{|\mu^-|}{m\mathcal{H}_m} + \frac{|\sigma^-|}{p\mathcal{H}_p} + \frac{L}{\|(u_k, v_k)\|^p},$$

which contradicts (2.4.6). Consequently, Ω^+ has zero measure, that is, $(\mathcal{X}, \mathcal{Y}) = (0,0)$ a.e. in $\mathbb{H}^n$.

Let τ_k be the smallest value of $\tau \in [0,1]$ such that

$$I_{\mu,\sigma}(\tau_k u_k, \tau_k v_k) = \max_{0\le\tau\le 1} I_{\mu,\sigma}(\tau u_k, \tau v_k).$$

Define

$$(U_k, V_k) = (2\mathfrak{a})^{1/m}(\mathcal{X}_k, \mathcal{Y}_k) = (2\mathfrak{a})^{1/m}\frac{(u_k, v_k)}{\|(u_k, v_k)\|} \in W,$$

with $\mathfrak{a} > 1/2$. By Lemma 2.3.6, $(U_k, V_k) \to (0,0)$ in $L^\nu(\mathbb{H}^n) \times L^\nu(\mathbb{H}^n)$ for any $\nu \in [p, m^*)$. Using (H_1) and (H_2), with $\varepsilon = 1$, we get

$$\begin{aligned}(2.4.8)\qquad 0 \le \int_{\mathbb{H}^n} H(q, U_k, V_k)dq &\le \int_{\mathbb{H}^n} [(\lambda+1)|(U_k, V_k)|^p + C_1|(U_k, V_k)|^s]\, dq \\ &\le (\lambda+1)\|(U_k, V_k)\|_p^p + C_1\|(U_k, V_k)\|_s^s \to 0,\end{aligned}$$

as $k \to \infty$, being $s \in (p, m^*)$. Consequently,

$$(2.4.9)\qquad \lim_{k\to\infty}\int_{\mathbb{H}^n} H(q, U_k, V_k)dq = 0.$$

Being $\|(u_k, v_k)\| \to \infty$ and $\|(u_k, v_k)\| \ge 1$ for all k, there exists k_0 large enough such that $(2\mathfrak{a})^{1/m}/\|(u_k, v_k)\| \in (0,1)$ for all $k \ge k_0$. Hence, by $m \le p$, $\mathfrak{a} > 1/2$ and $\|\mathcal{X}_k\|_{E_{m,a}} \le \|\mathcal{X}_k\|_{E_{p,a}} + \|\mathcal{Y}_k\|_{E_{m,b}} = 1$, we deduce for all $k \ge k_0$

$$\begin{aligned}I_{\mu,\sigma}(\tau_k u_k, \tau_k v_k) &\ge I_{\mu,\sigma}\left((2\mathfrak{a})^{1/m}u_k/\|(u_k, v_k)\|,\ (2\mathfrak{a})^{1/m}v_k/\|(u_k, v_k)\|\right) \\ &= \frac{2\mathfrak{a}}{m}\|\mathcal{X}_k\|^m_{E_{m,a}} + \frac{(2\mathfrak{a})^{p/m}}{p}\|\mathcal{Y}_k\|^p_{E_{p,b}} \\ &\quad - \mu\frac{2\mathfrak{a}}{m}\|\mathcal{X}_k\|^m_{\mathcal{H}_m} - \sigma\frac{(2\mathfrak{a})^{p/m}}{p}\|\mathcal{Y}_k\|^p_{\mathcal{H}_p} - \int_{\mathbb{H}^n} H(q, U_k, V_k)dq \\ &\ge \frac{2\mathfrak{a}}{m}\left(1 - \frac{\mu^+}{\mathcal{H}_m}\right)\|\mathcal{X}_k\|^m_{E_{m,a}} + \frac{(2\mathfrak{a})^{p/m}}{p}\left(1 - \frac{\sigma^+}{\mathcal{H}_p}\right)\|\mathcal{Y}_k\|^p_{E_{p,b}} \\ &\quad - \int_{\mathbb{H}^n} H(q, U_k, V_k)dq \\ &\ge 2\frac{\mathfrak{a}\kappa}{p}\left(\|\mathcal{X}_k\|^p_{E_{m,a}} + \|\mathcal{Y}_k\|^p_{E_{p,b}}\right) - \int_{\mathbb{H}^n} H(q, U_k, V_k)dq \\ &\ge 2\frac{\mathfrak{a}\kappa}{p2^{p-1}} - \int_{\mathbb{H}^n} H(q, U_k, V_k)dq,\end{aligned}$$

with $\kappa = \min\{1 - \mu^+/\mathcal{H}_m,\ 1 - \sigma^+/\mathcal{H}_p\} > 0$.

Thanks to (2.4.9), there exists $k_1 \geq k_0$ such that

$$\int_{\mathbb{H}^n} H(q, U_k, V_k)dq \leq \frac{a\kappa}{p2^{p-1}} \quad \text{for all } k \geq k_1.$$

Hence

$$I_{\mu,\sigma}(\tau_k u_k, \tau_k v_k) \geq \frac{a\kappa}{p2^{p-1}} \quad \text{for all } k \geq k_1.$$

In particular, the arbitrariness of $a > 1/2$ implies

$$\lim_{k\to\infty} I_{\mu,\sigma}(\tau_k u_k, \tau_k v_k) = \infty. \tag{2.4.10}$$

From $0 \leq \tau_k \leq 1$ and (H_4), we obtain

$$\int_{\mathbb{H}^n} \mathcal{F}(q, \tau_k u_k, \tau_k v_k)dq \leq C_{\mathcal{F}} \int_{\mathbb{H}^n} \mathcal{F}(q, u_k, v_k)dq + \int_{\mathbb{H}^n} \mathcal{G}(q)dq. \tag{2.4.11}$$

As in [4, Lemma 7.3], thanks to $I_{\mu,\sigma}(0,0) = 0$ and $I_{\mu,\sigma}(u_k, v_k) \to c \in \mathbb{R}$, by (2.4.10) we can assume that $\tau_k \in (0,1)$ for all k sufficiently large and in turn

$$\begin{aligned} 0 &= \tau_k \frac{d}{d\tau} I_{\mu,\sigma}(\tau u_k, \tau v_k)\Big|_{\tau=\tau_k} = \langle I'_{\mu,\sigma}(\tau_k u_k, \tau_k v_k), (\tau_k u_k, \tau_k v_k)\rangle \\ &= \|\tau_k u_k\|^m_{E_{m,a}} - \mu\|\tau_k u_k\|^m_{\mathcal{H}_m} + \|\tau_k v_k\|^p_{E_{p,b}} - \sigma\|\tau_k v_k\|^p_{\mathcal{H}_p} \\ &\quad - \int_{\mathbb{H}^n} \left[H_u(q, \tau_k u_k, \tau_k v_k)\tau_k u_k + H_v(q, \tau_k u_k, \tau_k v_k)\tau_k v_k\right] dq. \end{aligned} \tag{2.4.12}$$

Combining (2.4.11) and (2.4.12), we find

$$\begin{aligned} \|\tau_k u_k\|^m_{E_{m,a}} &- \mu\|\tau_k u_k\|^m_{\mathcal{H}_m} + \|\tau_k v_k\|^p_{E_{p,b}} - \sigma\|\tau_k v_k\|^p_{\mathcal{H}_p} \\ &= p\int_{\mathbb{H}^n} H(q, \tau_k u_k, \tau_k v_k)dq + \int_{\mathbb{H}^n} \mathcal{F}(q, \tau_k u_k, \tau_k v_k)dq \\ &\leq p\int_{\mathbb{H}^n} H(q, \tau_k u_k, \tau_k v_k)dq + C_{\mathcal{F}}\int_{\mathbb{H}^n} \mathcal{F}(q, u_k, v_k)dq \\ &\quad + \int_{\mathbb{H}^n} \mathcal{G}(q)dq \end{aligned}$$

for k sufficiently large. Then,

$$\begin{aligned} pI_{\mu,\sigma}(\tau_k u_k, \tau_k v_k) = \frac{p}{m}\left(\|\tau_k u_k\|^m_{E_{m,a}} - \mu\|\tau_k u_k\|^m_{\mathcal{H}_m}\right) + \|\tau_k v_k\|^p_{E_{p,b}} - \sigma\|\tau_k v_k\|^p_{\mathcal{H}_p} \\ - p\int_{\mathbb{H}^n} H(q, \tau_k u_k, \tau_k v_k)dq \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{p}{m}-1\right)\left(\|\tau_k u_k\|^m_{E_{m,a}} - \mu\|\tau_k u_k\|^m_{\mathcal{H}_m}\right) + \|\tau_k u_k\|^m_{E_{m,a}} \\
&\qquad - \mu\|\tau_k u_k\|^m_{\mathcal{H}_m} + \|\tau_k v_k\|^p_{E_{p,b}} - \sigma\|\tau_k v_k\|^p_{\mathcal{H}_p} \\
&\qquad\qquad - p\int_{\mathbb{H}^n} H(q,\tau_k u_k,\tau_k v_k)dq \\
&\le \left(\frac{p}{m}-1\right)\left(\|\tau_k u_k\|^m_{E_{m,a}} - \mu\|\tau_k u_k\|^m_{\mathcal{H}_m}\right) \\
&\qquad + C_{\mathcal{F}}\int_{\mathbb{H}^n}\mathcal{F}(q,u_k,v_k)dq + \int_{\mathbb{H}^n}\mathcal{G}(q)dq \\
&\le \left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{\mathcal{H}_m}\right) \\
&\qquad + C_{\mathcal{F}}\int_{\mathbb{H}^n}\mathcal{F}(q,u_k,v_k)dq + \int_{\mathbb{H}^n}\mathcal{G}(q)dq,
\end{aligned}
$$

since $\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{\mathcal{H}_m} \ge 0$ by (1.2.2) and the facts that $\mu < \mathcal{H}_m$ and $\tau_k \in (0,1)$. Hence (2.4.10) implies in particular that

$$
\begin{aligned}
\frac{1}{C_{\mathcal{F}}}\left(\frac{p}{m}-1\right)&\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{\mathcal{H}_m}\right) \\
&+ \int_{\mathbb{H}^n}\mathcal{F}(q,u_k,v_k)dq \to \infty \quad \text{as } k\to\infty.
\end{aligned}
\tag{2.4.13}
$$

On the other hand, by (2.4.1), (2.4.3) and the definition of $\mathcal{F}$ in (H_4), it follows that

$$
\begin{aligned}
\tilde{C} \ge pI_{\mu,\sigma}(u_k,v_k) &= \frac{p}{m}\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{\mathcal{H}_m}\right) + \|v_k\|^p_{E_{p,b}} - \sigma\|v_k\|^p_{\mathcal{H}_p} \\
&\qquad - p\int_{\mathbb{H}^n} H(q,u_k,v_k)dq \\
&= \left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{\mathcal{H}_m}\right) + \|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{\mathcal{H}_m} + \|v_k\|^p_{E_{p,b}} \\
&\qquad - \sigma\|v_k\|^p_{\mathcal{H}_p} - p\int_{\mathbb{H}^n} H(q,u_k,v_k)dq \\
&= \left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{\mathcal{H}_m}\right) + \|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{\mathcal{H}_m} + \|v_k\|^p_{E_{p,b}} \\
&\qquad - \sigma\|v_k\|^p_{\mathcal{H}_p} - \int_{\mathbb{H}^n}\left[H_u(q,u_k,v_k)u_k + H_v(q,u_k,v_k)v_k\right]dq \\
&\qquad + \int_{\mathbb{H}^n}\mathcal{F}(q,u_k,v_k)dq \\
&\ge -C + \left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}} - \mu\|u_k\|^m_{\mathcal{H}_m}\right) + \int_{\mathbb{H}^n}\mathcal{F}(q,u_k,v_k)dq.
\end{aligned}
$$

In particular, we obtain

$$\begin{aligned}\frac{1}{C_{\mathcal{F}}}\left(\frac{p}{m}-1\right)&\left(\|u_k\|^m_{E_{m,a}}-\mu\|u_k\|^m_{\mathcal{H}_m}\right)+\int_{\mathbb{H}^n}\mathcal{F}(q,u_k,v_k)dq\\&\le\left(\frac{p}{m}-1\right)\left(\|u_k\|^m_{E_{m,a}}-\mu\|u_k\|^m_{\mathcal{H}_m}\right)+\int_{\mathbb{H}^n}\mathcal{F}(q,u_k,v_k)dq\le \text{Const.},\end{aligned}$$

being $C_{\mathcal{F}}\ge 1$ by (H_4), $1<m\le p$, and $\|u_k\|^m_{E_{m,a}}-\mu\|u_k\|^m_{\mathcal{H}_m}\ge 0$ in virtue of (1.2.2) and the fact that $\mu<\mathcal{H}_m$. This contradicts (2.4.13) and proves the claim.

In conclusion, $\{(u_k,v_k)\}_k$ is bounded in W, as stated. □

Before verifying that $I_{\mu,\sigma}$ satisfies the Cerami condition at level c, we shall prove an essential lemma. We refer to Lemma 3.3 of [82] in the Heisenberg context, see also Lemma 3.8 of [3] in the Euclidean case.

Lemma 2.4.2. *Let μ and σ be two fixed parameters and let $(u_k,v_k)_k$ be a bounded sequence in W. Put*

$$\begin{aligned}&k\mapsto g_k(q)=-a(q)|u_k|^{m-2}u_k+\mu\,\psi(q)^m\cdot\frac{|u_k|^{m-2}u_k}{r^m}+H_u(q,u_k,v_k)\\&k\mapsto h_k(q)=-b(q)|v_k|^{p-2}v_k+\sigma\,\psi(q)^p\cdot\frac{|v_k|^{p-2}v_k}{r^p}+H_v(q,u_k,v_k).\end{aligned}\tag{2.4.14}$$

For all compact set $\mathcal{K}$ of $\mathbb{H}^n$ there exists $C_{\mathcal{K}}>0$ such that

$$\sup_k\int_{\mathcal{K}}\big(|g_k(q)|+|h_k(q)|\big)dq\le C_{\mathcal{K}}.$$

Proof. Fix μ, σ and $(u_k,v_k)_k$ as in the statement. Let $\mathcal{K}$ be a compact set of $\mathbb{H}^n$. Concerning the first and the fourth term, by Hölder's inequality

$$\begin{aligned}\int_{\mathcal{K}}\big(a(q)|u_k|^{m-1}+b(q)|v_k|^{p-1}\big)dq\le\|a\|_{x,\mathcal{K}}\sup_k\|u_k\|^{m-1}_{m^*}\\+\|a\|_{y,\mathcal{K}}\sup_k\|v_k\|^{p-1}_{m^*}=c_1,\end{aligned}$$

where $c_1=c_1(\mathcal{K})$, $x=m^*/(m^*-m+1)$ and $y=m^*/(m^*-p+1)$, since a, $b\in C(\mathbb{H}^n)$ and Lemma 2.3.1 can be applied, being $1<m\le p<m^*$. The second and the fifth term can be similarly evaluated, as

$$\begin{aligned}\int_{\mathcal{K}}\left[\left(\frac{\psi}{r}\right)^m|u_k|^{m-1}+\left(\frac{\psi}{r}\right)^p|v_k|^{p-1}dq\right]\le\|\psi/r\|_m\sup_k\|u_k\|^{m-1}_{\mathcal{H}_m}\\+\|\psi/r\|_p\sup_k\|v_k\|^{p-1}_{\mathcal{H}_p}=c_2,\end{aligned}$$

and $c_2 = c_2(\mathcal{K})$ by (1.2.2). Indeed, $\psi^m r^{-m}$ and $\psi^p r^{-p}$ are of class $L^1_{\mathrm{loc}}(\mathbb{H}^n)$, since $\psi = |\psi| \le 1$, the Jacobian determinant is r^Q and $1 < m \le p < Q$. Finally, elementary inequalities and (H_2), with $\varepsilon = 1$, give

$$\begin{aligned}\int_{\mathcal{K}} &\big|H_u(q,u_k,v_k) + H_v(q,u_k,v_k)\big| dq \\ &\le \sqrt{2}\int_{\mathcal{K}} \big|(\lambda+1)|(u_k,v_k)|^{p-1} + C_1|(u_k,v_k)|^{s-1}\big| dq \\ &\le \sqrt{2}\big[(\lambda+1)|\mathcal{K}|^{1/y}\sup_k \|(u_k,v_k)\|_{m^*}^{p-1} \\ &\quad + C_1|\mathcal{K}|^{(m^*-s+1)/m^*}\sup_k \|(u_k,v_k)\|_{m^*}^{s-1}\big] = c_3\end{aligned}$$

where $c_3 = c_3(\mathcal{K})$ and $y = m^*/(m^*-p+1) > 1$ is the Lebesgue exponent, since $p < s < m^*$ by (H_2). This completes the proof. □

Lemma 2.4.3. *The functional $I_{\mu,\sigma}$ satisfies the Cerami condition (C) in W for all $\mu < \mathcal{H}_m$ and for all $\sigma < \mathcal{H}_p$.*

Proof. Assume $\{(u_k,v_k)\}_k$ is a Cerami sequence for $I_{\mu,\sigma}$ in W. Then there exists $L > 0$ independent of k such that (2.4.1) holds and $\{(u_k,v_k)\}_k$ is bounded in W by Lemma 2.4.1. Thus, we can assume that up to a subsequence, still denoted by $\{(u_k,v_k)\}_k$, there exist $(u,v) \in W$, two vector field functions Θ and Λ in $\mathbb{H}^n$, with $\Theta \in L^{m'}(\mathbb{H}^n;\mathbb{R}^{2n})$ and $\Lambda \in L^{p'}(\mathbb{H}^n;\mathbb{R}^{2n})$, and four nonnegative numbers $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$, $\mathfrak{l}$ such that

$$\begin{gathered}(u_k,v_k) \rightharpoonup (u,v) \text{ in } W,\\ u_k \rightharpoonup u \text{ in } L^m(\mathbb{H}^n,\psi^m r^{-m}),\ v_k \rightharpoonup v \text{ in } L^p(\mathbb{H}^n,\psi^p r^{-p}),\\ (u_k,v_k) \to (u,v) \text{ in } L^\nu(\mathbb{H}^n)\times L^\nu(\mathbb{H}^n),\\ (u_k,v_k) \to (u,v) \text{ a.e. in } \mathbb{H}^{2n},\\ D_{\mathbb{H}^n}u_k \rightharpoonup D_{\mathbb{H}^n}u \text{ in } L^m(\mathbb{H}^n;\mathbb{R}^{2n}),\\ D_{\mathbb{H}^n}v_k \rightharpoonup D_{\mathbb{H}^n}v \text{ in } L^p(\mathbb{H}^n;\mathbb{R}^{2n}),\\ |D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k \rightharpoonup \Theta \text{ in } L^{m'}(\mathbb{H}^n;\mathbb{R}^{2n}),\\ |D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k \rightharpoonup \Lambda \text{ in } L^{p'}(\mathbb{H}^n;\mathbb{R}^{2n}),\\ \|u_k-u\|_{E_{m,a}} \to \mathfrak{i},\ \|u_k-u\|_{\mathcal{H}_m} \to \mathfrak{j},\ \|v_k-v\|_{E_{p,b}} \to \mathfrak{k},\ \|v_k-v\|_{\mathcal{H}_p} \to \mathfrak{l},\end{gathered} \tag{2.4.15}$$

for any $\nu \in [p,m^*)$ by Lemmas 2.3.5, 2.3.6 and (1.2.2). By (2.4.15), it follows that $|\omega_k - \omega| \to 0$ in $L^\nu(\mathbb{H}^n)$ for all $\nu \in [p,m^*)$, where $\omega_k = (u_k,v_k)$ and $\omega = (u,v)$.

Now $|u_k|^{m-2}u_k \rightharpoonup |u|^{m-2}u$ in $L^{m'}(\mathbb{H}^n, a)$, applying Proposition 2.3.3 thanks to (2.4.15). Consequently,

$$\int_{\mathbb{H}^n} a\,|u_k|^{m-2}u_k\Phi dq \to \int_{\mathbb{H}^n} a\,|u|^{m-2}u\Phi dq \tag{2.4.16}$$

for any $\Phi \in E_{m,a}$, since $\Phi \in L^m(\mathbb{H}^n, a)$. Analogously, by (2.4.15) and again Proposition 2.3.3, we obtain $|u_k|^{m-2}u_k \rightharpoonup |u|^{m-2}u$ in $L^{m'}(\mathbb{H}^n, \psi^m r^{-m})$ as $k \to \infty$. Indeed, Proposition 2.3.3 can be applied by (2.4.15) and by the fact that the weight function $\psi^m r^{-m}$ is of class $L^1_{\text{loc}}(\mathbb{H}^n)$, as explained in the proof of Lemma 2.4.2. Therefore,

$$\langle u_k, \Phi\rangle_{\mathcal{H}_m} \to \langle u, \Phi\rangle_{\mathcal{H}_m} \tag{2.4.17}$$

for any $\Phi \in E_{m,a}$, since $\Phi \in L^m(\mathbb{H}^n, \psi^m r^{-m})$.

A similar argument shows that

$$\int_{\mathbb{H}^n} b\,|v_k|^{p-2}v_k\Psi dq \to \int_{\mathbb{H}^n} b\,|v|^{p-2}v\Psi dq, \qquad \langle v_k, \Psi\rangle_{\mathcal{H}_p} \to \langle v, \Psi\rangle_{\mathcal{H}_p} \tag{2.4.18}$$

for all $\Psi \in E_{p,b}$.

By (H_2), with $\varepsilon = 1$, (2.4.15) and the Hölder inequality, there exists a suitable $C_\lambda > 0$

$$\begin{aligned}
\int_{\mathbb{H}^n} &|\big(H_u(q,u_k,v_k) - H_u(q,u,v)\big)(u_k - u) \\
&\qquad + \big(H_v(q,u_k,v_k) - H_v(q,u,v)\big)(v_k - v)|dq \\
&= \int_{\mathbb{H}^n} |H_\omega(q,\omega_k)(\omega_k - \omega) - H_\omega(q,\omega)(\omega_k - \omega)|dq \\
&\le \int_{\mathbb{H}^n} \big[(\lambda+1)\big(|\omega_k|^{p-1} + |\omega|^{p-1}\big)|\omega_k - \omega| \\
&\qquad + C_1\big(|\omega_k|^{s-1} + |\omega|^{s-1}\big)|\omega_k - \omega|\big]dq \\
&\le C_\lambda\big(\|\omega_k - \omega\|_p + \|\omega_k - \omega\|_s\big) \to 0,
\end{aligned} \tag{2.4.19}$$

as $k \to \infty$, applying Theorem 2.3.2, with $\wp = p$ and $\wp = s$, being $p < s < m^*$.

Since $I'_{\mu,\sigma}(u_k, v_k) \to 0$ in W', then as $k \to \infty$

$$\begin{aligned}
\langle I'_{\mu,\sigma}(u_k,v_k), (\Phi,\Psi)\rangle = &\int_{\mathbb{H}^n} \big(|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n}u_k, D_{\mathbb{H}^n}\Phi\big)_{\mathbb{H}^n} dq \\
&+ \int_{\mathbb{H}^n} \big(|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n}v_k, D_{\mathbb{H}^n}\Psi\big)_{\mathbb{H}^n} dq \\
&- \int_{\mathbb{H}^n} \big(g_k\Phi + h_k\Psi\big) dq
\end{aligned} \tag{2.4.20}$$

for any $(\Phi,\Psi)\in W$, where the sequences $(g_k)_k$ and $(h_k)_k$, defined in (2.4.14), are in $L^1_{\rm loc}(\mathbb{H}^n)$ by Lemma 2.4.2, being $(\|(u_k,v_k\|)_k$ bounded.

We claim that, up to a subsequence if necessary,

$$D_{\mathbb{H}^n}u_k\to D_{\mathbb{H}^n}u \quad\text{and}\quad D_{\mathbb{H}^n}v_k\to D_{\mathbb{H}^n}v \quad \text{a.e. in } \mathbb{H}^n. \tag{2.4.21}$$

To show the claim, we shall follow the proofs of Lemma 4.3 of [15] and of Lemma 3.4 of [82] for the Heisenberg setting. We also refer to the proofs of Theorem 2.1 of [9], of Lemma 2 of [33] and of Step 1 of Theorem 4.4 of [3] in the Euclidean setting.

Fix $R>0$. Let $\varphi_R\in C_0^\infty(\mathbb{H}^n)$ be such that $0\le\varphi_R\le 1$ in $\mathbb{H}^n$ and $\varphi_R\equiv 1$ in B_R. Given $\varepsilon>0$ define for each $\tau\in\mathbb{R}$

$$\eta_\varepsilon(\tau)=\begin{cases}\tau, & \text{if } |\tau|<\varepsilon,\\ \varepsilon\dfrac{\tau}{|\tau|}, & \text{if } |\tau|\ge\varepsilon.\end{cases}$$

Put $\phi_k=\varphi_R\eta_\varepsilon\circ(u_k-u)$ and $\psi_k=\varphi_R\eta_\varepsilon\circ(v_k-v)$ so that $\phi_k\in HW^{1,m}(\mathbb{H}^n)$ and similarly $\psi_k\in HW^{1,p}(\mathbb{H}^n)$ by Lemma 2.3.1. Taking $\Phi=\phi_k$ and $\Psi=\psi_k$ in (2.4.20), we get

$$\begin{aligned}
&\int_{\mathbb{H}^n}\varphi_R\big(|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k-|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u,\\
&\qquad\qquad D_{\mathbb{H}^n}(\eta_\varepsilon\circ(u_k-u))\big)_{\mathbb{H}^n}dq\\
&+\int_{\mathbb{H}^n}\varphi_R\big(|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k-|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v,\\
&\qquad\qquad D_{\mathbb{H}^n}(\eta_\varepsilon\circ(v_k-v))\big)_{\mathbb{H}^n}dq\\
&=-\int_{\mathbb{H}^n}\eta_\varepsilon\circ(u_k-u)\big(|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k,D_{\mathbb{H}^n}\varphi_R\big)_{\mathbb{H}^n}dq\\
&\quad-\int_{\mathbb{H}^n}\varphi_R\big(|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u,D_{\mathbb{H}^n}(\eta_\varepsilon\circ(u_k-u))\big)_{\mathbb{H}^n}dq\\
&\quad-\int_{\mathbb{H}^n}\eta_\varepsilon\circ(v_k-v)\big(|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k,D_{\mathbb{H}^n}\varphi_R\big)_{\mathbb{H}^n}dq\\
&\quad-\int_{\mathbb{H}^n}\varphi_R\big(|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v,D_{\mathbb{H}^n}(\eta_\varepsilon\circ(v_k-v))\big)_{\mathbb{H}^n}dq\\
&\quad+\langle I'_{\mu,\sigma}(u_k,v_k),(\phi_k,\psi_k)\rangle+\int_{\mathbb{H}^n}\big(g_k\phi_k+h_k\psi_k\big)dq.
\end{aligned}\tag{2.4.22}$$

Observe now that as $k\to\infty$

$$\int_{\mathbb{H}^n}\eta_\varepsilon\circ(u_k-u)\big(|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k,D_{\mathbb{H}^n}\varphi_R\big)_{\mathbb{H}^n}dq\to 0,$$

$$\int_{\mathbb{H}^n} \eta_\varepsilon \circ (v_k - v)\big(|D_{\mathbb{H}^n} v_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v_k, D_{\mathbb{H}^n}\varphi_R\big)_{\mathbb{H}^n} dq \to 0,$$

since

$$|\eta_\varepsilon \circ (u_k - u) D_{\mathbb{H}^n}\varphi_R|_{\mathbb{H}^n} \to 0 \text{ in } L^m(\operatorname{supp}\varphi_R),$$
$$|\eta_\varepsilon \circ (v_k - v) D_{\mathbb{H}^n}\varphi_R|_{\mathbb{H}^n} \to 0 \text{ in } L^p(\operatorname{supp}\varphi_R),$$

by (1.2.5), being $\operatorname{supp}\varphi_R$ contained in a suitable ball of $\mathbb{H}^n$. Moreover,

$$|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u_k \rightharpoonup \Theta \text{ in } L^{m'}(\mathbb{H}^n;\mathbb{R}^{2n}),$$

$$|D_{\mathbb{H}^n} v_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v_k \rightharpoonup \Lambda \text{ in } L^{p'}(\mathbb{H}^n;\mathbb{R}^{2n})$$

by (2.4.15).

Furthermore,

$$D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u)) \rightharpoonup 0$$

in $L^m(\mathbb{H}^n;\mathbb{R}^{2n})$ and

$$D_{\mathbb{H}^n}(\eta_\varepsilon \circ (v_k - v)) \rightharpoonup 0$$

in $L^p(\mathbb{H}^n;\mathbb{R}^{2n})$, since $u_k \rightharpoonup u$ in $HW^{1,m}(\mathbb{H}^n)$, $v_k \rightharpoonup v$ in $HW^{1,p}(\mathbb{H}^n)$ by Lemma 2.3.1. Consequently, as $k \to \infty$

$$\int_{\mathbb{H}^n} \varphi_R\big(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u, D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u))\big)_{\mathbb{H}^n} dq \to 0,$$
$$\int_{\mathbb{H}^n} \varphi_R\big(|D_{\mathbb{H}^n} v|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v, D_{\mathbb{H}^n}(\eta_\varepsilon \circ (v_k - v))\big)_{\mathbb{H}^n} dq \to 0,$$

since $|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u \in L^{m'}(\mathbb{H}^n;\mathbb{R}^{2n})$ and $|D_{\mathbb{H}^n} v|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v \in L^{p'}(\mathbb{H}^n;\mathbb{R}^{2n})$. Moreover,

$$\langle I'_{\mu,\sigma}(u_k, v_k), (\phi_k, \psi_k)\rangle \to 0$$

as $k \to \infty$, since $I'_{\mu,\sigma}(u_k, v_k) \to 0$ in W' and $(\phi_k, \psi_k) \rightharpoonup 0$ in W as $k \to \infty$.

In conclusion, the first five terms in the right hand side of (2.4.22) go to zero as $k \to \infty$. Now, recalling that $0 \le \varphi_R \le 1$ in $\mathbb{H}^n$, we have

$$\int_{\mathbb{H}^n} (g_k\phi_k + h_k\psi_k) dq \le \int_{\operatorname{supp}\varphi_R} \big(|g_k| \cdot |\eta_\varepsilon \circ (u_k - u)| + |h_k| \cdot |\eta_\varepsilon \circ (v_k - v)|\big) dq$$
$$\le \varepsilon \int_{\operatorname{supp}\varphi_R} \big(|g_k| + |h_k|\big) dq \le \varepsilon\, C_R,$$

since $(g_k)_k$ and $(h_k)_k$ are bounded in $L^1_{\mathrm{loc}}(\mathbb{H}^n)$ by Lemma 2.4.2. By the definitions of φ_R and η_ε,

$$\varphi_R\big(|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u, D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u))\big)_{\mathbb{H}^n} \ge 0,$$

$$\varphi_R\big(|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k-|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v,D_{\mathbb{H}^n}(\eta_\varepsilon\circ(v_k-v))\big)_{\mathbb{H}^n}\geq 0$$

a.e. in $\mathbb{H}^n$, and in turn

$$\begin{aligned}\int_{B_R}&\varphi_R\big(|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k-|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u,D_{\mathbb{H}^n}(\eta_\varepsilon\circ(u_k-u))\big)_{\mathbb{H}^n}dq\\&+\int_{B_R}\varphi_R\big(|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k-|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v,D_{\mathbb{H}^n}(\eta_\varepsilon\circ(v_k-v))\big)_{\mathbb{H}^n}dq\\&\leq\int_{\mathbb{H}^n}\varphi_R\big(|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k-|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u,D_{\mathbb{H}^n}(\eta_\varepsilon\circ(u_k-u))\big)_{\mathbb{H}^n}dq\\&+\int_{\mathbb{H}^n}\varphi_R\big(|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k-|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v,D_{\mathbb{H}^n}(\eta_\varepsilon\circ(v_k-v))\big)_{\mathbb{H}^n}dq.\end{aligned}$$

Combining all these facts with (2.4.22), we find that

$$\begin{aligned}\limsup_{k\to\infty}\Bigg(\int_{B_R}&\big(|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k-|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u,\\&\qquad D_{\mathbb{H}^n}(\eta_\varepsilon\circ(u_k-u))\big)_{\mathbb{H}^n}dq\\+\int_{B_R}&\big(|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k-|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v,\\&\qquad D_{\mathbb{H}^n}(\eta_\varepsilon\circ(v_k-v))\big)_{\mathbb{H}^n}dq\Bigg)\leq\varepsilon\,C_R,\end{aligned}\tag{2.4.23}$$

since $\varphi_R\equiv 1$ in B_R. Define the function $e_k=e_k(q)$ by

$$\begin{aligned}e_k&=e_{u,k}+e_{v,k},\\e_{u,k}&=\big(|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k-|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u,D_{\mathbb{H}^n}(u_k-u)\big)_{\mathbb{H}^n}\\e_{v,k}&=\big(|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k-|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v,D_{\mathbb{H}^n}(v_k-v)\big)_{\mathbb{H}^n}.\end{aligned}$$

Clearly, e_k is nonnegative a.e. in $\mathbb{H}^n$ for all k. Moreover, $(e_k)_k$ is bounded in $L^1(\mathbb{H}^n)$. Indeed,

$$\begin{aligned}0&\leq\int_{\mathbb{H}^n}e_k(q)\,dq\\&\leq\||D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k-|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u\|_{m'}\|D_{\mathbb{H}^n}u_k-D_{\mathbb{H}^n}u\|_m\\&\quad+\||D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k-|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v\|_{p'}\|D_{\mathbb{H}^n}v_k-D_{\mathbb{H}^n}v\|_p\\&\leq C_0,\end{aligned}\tag{2.4.24}$$

where C_0 is an appropriate constant, independent of k, since $(D_{\mathbb{H}^n}u_k)_k$ is bounded in $L^m(\mathbb{H}^n;\mathbb{R}^{2n})$, $(|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k)_k$ is bounded in $L^{m'}(\mathbb{H}^n;\mathbb{R}^{2n})$

and similarly $(D_{\mathbb{H}^n}v_k)_k$ is bounded in $L^p(\mathbb{H}^n;\mathbb{R}^{2n})$ and $(|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k)_k$ is bounded in $L^{p'}(\mathbb{H}^n;\mathbb{R}^{2n})$ by (2.4.15).

Fix $\theta\in(0,1)$. Split the ball B_R into four sets

$$S^\varepsilon_{u,k}(R)=\{q\in B_R\,:\,|u_k(q)-u(q)|\le\varepsilon\},\quad G^\varepsilon_{u,k}(R)=B_R\setminus S^\varepsilon_{u,k}(R),$$
$$S^\varepsilon_{v,k}(R)=\{q\in B_R\,:\,|v_k(q)-v(q)|\le\varepsilon\},\quad G^\varepsilon_{v,k}(R)=B_R\setminus S^\varepsilon_{v,k}(R).$$

By Hölder's inequality,

$$\begin{aligned}\int_{B_R}e_k^\theta dq &\le \int_{B_R}e_{u,k}^\theta dq+\int_{B_R}e_{v,k}^\theta dq\\
&=\int_{S^\varepsilon_{u,k}(R)}e_{u,k}^\theta dq+\int_{G^\varepsilon_{u,k}(R)}e_{u,k}^\theta dq+\int_{S^\varepsilon_{v,k}(R)}e_{v,k}^\theta dq+\int_{G^\varepsilon_{v,k}(R)}e_{v,k}^\theta dq\\
&\le\left(\int_{S^\varepsilon_{u,k}(R)}e_{u,k}dq\right)^\theta|S^\varepsilon_{u,k}(R)|^{1-\theta}+\left(\int_{G^\varepsilon_{u,k}(R)}e_k dq\right)^\theta|G^\varepsilon_{u,k}(R)|^{1-\theta}\\
&\quad+\left(\int_{S^\varepsilon_{v,k}(R)}e_{v,k}dq\right)^\theta|S^\varepsilon_{v,k}(R)|^{1-\theta}+\left(\int_{G^\varepsilon_{v,k}(R)}e_k dq\right)^\theta|G^\varepsilon_{v,k}(R)|^{1-\theta}\\
&\le(\varepsilon C_R)^\theta\big(|S^\varepsilon_{u,k}(R)|^{1-\theta}+|S^\varepsilon_{v,k}(R)|^{1-\theta}\big)+C_0^\theta\big(|G^\varepsilon_{u,k}(R)|^{1-\theta}+|G^\varepsilon_{v,k}(R)|^{1-\theta}\big),\end{aligned}$$

by (2.4.23), since

$$D_{\mathbb{H}^n}(\eta_\varepsilon\circ(u_k-u))=D_{\mathbb{H}^n}(u_k-u)\ \text{in}\ S^\varepsilon_{u,k}(R),$$
$$D_{\mathbb{H}^n}(\eta_\varepsilon\circ(v_k-v))=D_{\mathbb{H}^n}(v_k-v)\ \text{in}\ S^\varepsilon_{v,k}(R),$$

and by (2.4.24). Moreover, $|G^\varepsilon_{u,k}(R)|$ and $|G^\varepsilon_{v,k}(R)|$ tend to zero as $k\to\infty$. Hence

$$0\le\limsup_{k\to\infty}\int_{B_R}e_k^\theta\,dq\le(\varepsilon C_R)^\theta|B_R|^{1-\theta}.$$

Letting ε tend to 0^+ we find that $e_k^\theta\to0$ in $L^1(B_R)$ and so, thanks to the arbitrariness of R, we deduce that

$$e_k\to0\quad\text{a.e. in }\mathbb{H}^n$$

up to a subsequence. From Lemma 3 of [33] it follows the validity of (2.4.21). The claim is so proved.

In particular,

$$|D_{\mathbb{H}^n}u_k|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u_k\to|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}^{m-2}D_{\mathbb{H}^n}u,$$
$$|D_{\mathbb{H}^n}v_k|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v_k\to|D_{\mathbb{H}^n}v|_{\mathbb{H}^n}^{p-2}D_{\mathbb{H}^n}v$$

a.e. in $\mathbb{H}^n$. Hence, Proposition A.7 of [3] implies that

$$\Theta = |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u \quad \text{and} \quad \Lambda = |D_{\mathbb{H}^n} v|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v \quad \text{a.e. in} \quad \mathbb{H}^n.$$

Consequently for all $(\Phi, \Psi) \in W$

$$\begin{aligned} &\int_{\mathbb{H}^n} \left(|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u_k, D_{\mathbb{H}^n} \Phi\right)_{\mathbb{H}^n} dq \\ &\qquad\qquad \to \int_{\mathbb{H}^n} \left(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} \Phi\right)_{\mathbb{H}^n} dq \\ &\int_{\mathbb{H}^n} \left(|D_{\mathbb{H}^n} v_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v_k, D_{\mathbb{H}^n} \Psi\right)_{\mathbb{H}^n} dq \\ &\qquad\qquad \to \int_{\mathbb{H}^n} \left(|D_{\mathbb{H}^n} v|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v, D_{\mathbb{H}^n} \Psi\right)_{\mathbb{H}^n} dq \end{aligned} \tag{2.4.25}$$

as $k \to \infty$, since $|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u_k \rightharpoonup |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^{m-2} D_{\mathbb{H}^n} u$ in $L^{m'}(\mathbb{H}^n; \mathbb{R}^{2n})$ and $|D_{\mathbb{H}^n} v_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v_k \rightharpoonup |D_{\mathbb{H}^n} v|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v$ in $L^{p'}(\mathbb{H}^n; \mathbb{R}^{2n})$.

Finally, by (2.4.16)–(2.4.19) and (2.4.25) we derive that the weak limit $\omega = (u, v)$ is a critical point of $I'_{\mu,\sigma}$ in W. Therefore, (2.4.1), (2.4.15)–(2.4.19) and (2.4.25) we get as $k \to \infty$

$$\begin{aligned} o(1) &= \langle I'_{\mu,\sigma}(\omega_k) - I'_{\mu,\sigma}(\omega), \omega_k - \omega \rangle \\ &= \|u_k\|_{E_{m,a}}^m + \|u\|_{E_{m,a}}^m + \|v_k\|_{E_{p,b}}^p + \|v\|_{E_{p,b}}^p \\ &\quad - \langle u_k, u \rangle_{E_{m,a}} - \langle u, u_k \rangle_{E_{m,a}} - \langle v_k, v \rangle_{E_{p,b}} - \langle v, v_k \rangle_{E_{p,b}} \\ &\quad - \mu\big(\|u_k\|_{\mathcal{H}_m}^m + \|u\|_{\mathcal{H}_m}^m - \langle u_k, u \rangle_{\mathcal{H}_m} - \langle u, u_k \rangle_{\mathcal{H}_m}\big) \\ &\quad - \sigma\big(\|v_k\|_{\mathcal{H}_p}^p + \|v\|_{\mathcal{H}_p}^p - \langle v_k, v \rangle_{\mathcal{H}_p} - \langle v, v_k \rangle_{\mathcal{H}_p}\big) + o(1) \\ &= \|u_k\|_{E_{m,a}}^m - \|u\|_{E_{m,a}}^m + \|v_k\|_{E_{p,b}}^p - \|v\|_{E_{p,b}}^p \\ &\quad - \mu\big(\|u_k\|_{\mathcal{H}_m}^m - \|u\|_{\mathcal{H}_m}^m\big) - \sigma\big(\|v_k\|_{\mathcal{H}_p}^p - \|v\|_{\mathcal{H}_p}^p\big) + o(1). \end{aligned} \tag{2.4.26}$$

The last equality is a consequence of the facts that $\langle u_k, u \rangle_{E_{m,a}} \to \|u\|_{E_{m,a}}^m$ and $\langle v_k, v \rangle_{E_{p,b}} \to \|v\|_{E_{p,b}}^p$ by (2.4.25) and (2.4.16), similarly $\langle v_k, v \rangle_{E_{p,b}} \to \|v\|_{E_{p,p}}^p$. Finally, (2.4.26) yields as $k \to \infty$

$$o(1) = \|u_k - u\|_{E_{m,a}}^m - \mu \|u_k - u\|_{\mathcal{H}_m}^m + \|v_k - v\|_{E_{p,b}}^p - \sigma \|v_k - v\|_{\mathcal{H}_p}^p + o(1),$$

since (2.4.15), the key property (2.4.21) and the celebrated Brézis & Lieb lemma, see [21], give

$$\begin{aligned} \|D_{\mathbb{H}^n} u_k\|_m^m &= \|D_{\mathbb{H}^n}(u_k - u)\|_m^m + \|D_{\mathbb{H}^n} u\|_m^m + o(1), \\ \|D_{\mathbb{H}^n} v_k\|_p^p &= \|D_{\mathbb{H}^n}(v_k - v)\|_p^p + \|D_{\mathbb{H}^n} v\|_p^p + o(1), \end{aligned}$$

$$\begin{aligned}
\|u_k\|_{m,a}^m &= \|u_k - u\|_{m,a}^m + \|u\|_{m,a}^m + o(1),\\
\|v_k\|_{p,b}^p &= \|v_k - v\|_{p,b}^p + \|v\|_{p,b}^p + o(1),\\
\|u_k\|_{\mathcal{H}_m}^m &= \|u_k - u\|_{\mathcal{H}_m}^m + \|u\|_{\mathcal{H}_m}^m + o(1),\\
\|v_k\|_{\mathcal{H}_p}^p &= \|v_k - v\|_{\mathcal{H}_p}^p + \|v\|_{\mathcal{H}_p}^p + o(1).
\end{aligned}$$

This, together with (2.4.15), implies the main formula

$$\begin{aligned}
\mathfrak{i}^m + \mathfrak{k}^p &= \lim_{k\to\infty} \|u_k - u\|_{E_{m,a}}^m + \lim_{k\to\infty} \|v_k - v\|_{E_{p,b}}^p \\
&= \mu \lim_{k\to\infty} \|u_k - u\|_{\mathcal{H}_m}^m + \sigma \lim_{k\to\infty} \|v_k - v\|_{\mathcal{H}_p}^p = \mu \mathfrak{j}^m + \sigma \mathfrak{l}^p.
\end{aligned} \tag{2.4.27}$$

Clearly (2.4.27) gives at once that $(u_k, v_k) \to (u, v)$ in W as $k \to \infty$, when either $\mu^+ + \sigma^+ = 0$ or $\mathfrak{j} + \mathfrak{l} = 0$ and we are done. At this point, as in Lemma 3.2 in [40], let us therefore assume by contradiction that $\mu^+ + \sigma^+ > 0$ and $\mathfrak{j} + \mathfrak{l} > 0$. If either $\mu^+ + \mathfrak{l} = 0$ or $\sigma^+ + \mathfrak{j} = 0$, then either $\mathfrak{j} > 0$ and $\mathfrak{i} = 0$ or $\mathfrak{l} > 0$ and $\mathfrak{k} = 0$ by (2.4.27). Both cases are impossible by (1.2.2). Now, if either $\mu^+ + \mathfrak{j} = 0$ or $\sigma^+ + \mathfrak{l} = 0$, then either $\mathfrak{l} > 0$, $\sigma^+ > 0$ and $\mathfrak{k}^p \le \sigma^+ \mathfrak{l}^p < \mathcal{H}_p \mathfrak{l}^p \le \mathfrak{k}^p$ or $\mathfrak{j} > 0$, $\mu^+ > 0$ and $\mathfrak{i}^m \le \mu^+ \mathfrak{j}^m < \mathcal{H}_m \mathfrak{j}^m \le \mathfrak{i}^m$ by (2.4.27) and (1.2.2). Both cases give a contradiction. Finally, it remains to consider the case $\mu^+ > 0$, $\sigma^+ > 0$, $\mathfrak{j} > 0$ and $\mathfrak{l} > 0$, for which (2.4.27) and (1.2.2) imply that

$$\mathfrak{i}^m + \mathfrak{k}^p = \mu \mathfrak{j}^m + \sigma \mathfrak{l}^p < \mathcal{H}_m \mathfrak{j}^m + \mathcal{H}_p \mathfrak{l}^p \le \mathfrak{i}^m + \mathfrak{k}^p,$$

which is again the desired contradiction. Therefore, we conclude that

$$\mathfrak{j} + \mathfrak{l} = 0$$

in all cases and so $(u_k, v_k) \to (u, v)$ in W as $k \to \infty$ by (2.4.27), as claimed. □

Now thanks to Lemmas 2.4.1 and 2.4.3, we proceed with the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. The first step is to prove that the functional $I_{\mu,\sigma}$ satisfies a mountain pass geometry. By the main restriction (2.1.4), it is possible to choose $\varepsilon > 0$, with $2^p\varepsilon/\lambda_p = \kappa - 2^{p-1}\lambda/\lambda_p$, where

$$\kappa = \min\{1 - \mu^+/\mathcal{H}_m,\ 1 - \sigma^+/\mathcal{H}_p\} > 0.$$

By (2.1.2), (2.3.3) and (H_2), we obtain for all $(u, v) \in W$, with $\|(u, v)\| \le 1$,

$$I_{\mu,\sigma}(u, v) \ge \frac{1}{m}\left(1 - \frac{\mu^+}{\mathcal{H}_m}\right)\|u\|_{E_{m,a}}^m + \frac{1}{p}\left(1 - \frac{\sigma^+}{\mathcal{H}_p}\right)\|v\|_{E_{p,b}}^p$$

$$
\begin{aligned}
&\quad - \frac{1}{p}\int_{\mathbb{H}^n} \big(\lambda+\varepsilon\big)|(u,v)|^p dq - C_\varepsilon \int_{\mathbb{H}^n} |(u,v)|^s dq \\
&\geq \frac{\kappa}{p}\big(\|u\|^p_{E_{m,a}} + \|v\|^p_{E_{p,b}}\big) - \frac{1}{p}(\lambda+\varepsilon)\|(u,v)\|_p^p \\
&\quad - C_\varepsilon C_s^s \|(u,v)\|^s \\
&\geq \frac{1}{2^{p-1}p}\left(\kappa - 2^{p-1}\frac{\lambda+\varepsilon}{\lambda_p}\right)\|(u,v)\|^p - C_\varepsilon C_s^s\|(u,v)\|^s \\
&= \frac{1}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p} - 2^p p C_\varepsilon C_s^s \|(u,v)\|^{s-p}\right)\|(u,v)\|^p.
\end{aligned} \tag{2.4.28}
$$

Now take $\rho \in (0,1]$ so small that $\kappa - 2^{p-1}\lambda/\lambda_p - 2^p p C_\varepsilon C_s^s \rho^{s-p} > 0$, in virtue of (2.1.4). It follows, for all $(u,v) \in W$, with $\|(u,v)\| = \rho$,

$$
I_{\mu,\sigma}(u,v) \geq \frac{\rho^p}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p} - 2^p p C_\varepsilon C_s^s \rho^{s-p}\right) = \beta > 0.
$$

Let u^*, $v^* \in C_0^\infty(B_1)$ be two nonnegative nontrivial radial functions, such that $\|\,|(u^*,v^*)|\,\|_{L^p(B_1)} > 0$, where B_1 is the unit ball in $\mathbb{H}^n$ centered at 0. Let $\tilde{u}$ and $\tilde{v}$ be the natural extensions of u^* and v^*, respectively, to the entire $\mathbb{H}^n$, defining $\tilde{u}(q) = 0$ and $\tilde{v}(q) = 0$ in $\mathbb{H}^n \setminus \overline{B_1}$. Clearly, $(\tilde{u},\tilde{v}) \in W$, with $\|\tilde{u}\|_{E_{m,a}} > 0$ and $\|\tilde{v}\|_{E_{p,b}} > 0$.

By (H_3) for any positive constant $A > 0$ there exists $\delta_A > 0$ such that $H(q,u,v) \geq A\,|(u,v)|^p/p$ for all $(q,u,v) \in \mathbb{H}^n \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$, with $u > \delta_A$ and $v > \delta_A$. Clearly,

$$
\min_{(q,u,v)\in\overline{B_1}\times[0,\delta_A]^2}\left(H(q,u,v) - \frac{A}{p}\,|(u,v)|^p\right) \in \mathbb{R},
$$

so that there exists $C_A \geq 0$ such that

$$
H(q,u,v) \geq \frac{A}{p}\,|(u,v)|^p - C_A \quad \text{for all } (q,u,v) \in \overline{B_1} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+.
$$

Then, for $\tau \geq 1$ we find

$$
\begin{aligned}
I_{\mu,\sigma}(\tau\tilde{u},\tau\tilde{v}) &= \frac{\tau^m}{m}\|\tilde{u}\|^m_{E_{m,a}} - \frac{\mu\tau^m}{m}\|\tilde{u}\|^m_{\mathcal{H}_m} + \frac{\tau^p}{p}\|\tilde{v}\|^p_{E_{p,b}} - \frac{\sigma\tau^p}{p}\|\tilde{v}\|^p_{\mathcal{H}_p} \\
&\qquad - \int_{B_1} H(q,\tau\tilde{u},\tau\tilde{v})dq \\
&\leq \frac{\tau^p}{m}\Big(\|\tilde{u}\|^m_{E_{m,a}} + |\mu^-|\cdot\|\tilde{u}\|^m_{\mathcal{H}_m} + \|\tilde{v}\|^p_{E_{p,b}} + |\sigma^-|\cdot\|\tilde{v}\|^p_{\mathcal{H}_p} \\
&\qquad - A\|\,|(\tilde{u},\tilde{v})|\,\|_p^p\Big) + C_A|B_1|,
\end{aligned}
$$

where $\varsigma^- = \min\{0, \varsigma\}$ for all $\varsigma \in \mathbb{R}$. Choosing A so large that

$$0 < \|\tilde{u}\|_{E_{m,a}}^m + |\mu^-| \cdot \|\tilde{u}\|_{\mathcal{H}_m}^m + \|\tilde{v}\|_{E_{p,b}}^p + |\sigma^-| \cdot \|\tilde{v}\|_{\mathcal{H}_p}^p < A \| \, |(\tilde{u}, \tilde{v})| \, \|_p^p,$$

we have $I_{\mu,\sigma}(\tau\tilde{u}, \tau\tilde{v}) \to -\infty$ as $\tau \to \infty$. As a consequence, there exists $(u_0, v_0) = (\tau_0\tilde{u}, \tau_0\tilde{v}) \in W$ such that $\|(u_0, v_0)\| \geq 2 > \rho$ and $I_{\mu,\sigma}(u_0, v_0) < 0$.

We finally proved that $I_{\mu,\sigma}$ satisfies a mountain pass geometry. Considering also Lemma 2.4.3 and applying Theorem 2.2.1, we have the existence of $(u, v) \in W$, with $(u, v) \neq (0, 0)$, verifying

$$\begin{aligned} \langle u, \Phi \rangle_{E_{m,a}} + \langle v, \Psi \rangle_{E_{p,b}} - \mu \langle u, \Phi \rangle_{\mathcal{H}_m} - \sigma \langle v, \Psi \rangle_{\mathcal{H}_p} \\ = \int_{\mathbb{H}^n} \big[H_u(q, u, v)\Phi + H_v(q, u, v)\Psi \big] dq \end{aligned}$$

for all $(\Phi, \Psi) \in W$. Taking $\Phi = u^- = \min\{0, u\}$ and $\Psi = v^- = \min\{0, v\}$, by (H_1), (2.1.4) and (2.4.5) we get

$$\begin{aligned} 0 &= \int_{\mathbb{H}^n} \big[H_u(q, u, v)u^- + H_v(q, u, v)v^- \big] dq \\ &\geq \left(1 - \frac{\mu^+}{\mathcal{H}_m}\right) \|u^-\|_{E_{m,a}}^m + \left(1 - \frac{\sigma^+}{\mathcal{H}_p}\right) \|v^-\|_{E_{p,b}}^p \geq 0. \end{aligned}$$

In conclusion, $u^- = 0$ and $v^- = 0$ a.e. in $\mathbb{H}^n$, that is, $u \geq 0$ and $v \geq 0$ a.e. in $\mathbb{H}^n$. In other words, any solution of $(\mathcal{P}_1)$ is nonnegative, component by component. □

2.5 Existence of solutions of $(\mathcal{P}_2)$

Let us now consider $(\mathcal{P}_2)$. The system has a variational structure and the associated Euler–Lagrange functional is $J_{\mu,\sigma} : W \to \mathbb{R}$ defined as

$$\begin{aligned} J_{\mu,\sigma}(u, v) = & \frac{1}{m}\|u\|_{E_{m,a}}^m + \frac{1}{p}\|v\|_{E_{p,b}}^p - \frac{\mu}{m}\|u\|_{\mathcal{H}_m}^m - \frac{\sigma}{p}\|v\|_{\mathcal{H}_p}^p \\ & - \int_{\mathbb{H}^n} H(q, u, v)dq - \frac{1}{m^*}\|u^+\|_{m^*}^{m^*} - \frac{1}{p^*}\|v^+\|_{p^*}^{p^*} \\ & - \frac{1}{m^*}\int_{\mathbb{H}^n} (u^+)^\theta (v^+)^\vartheta dq - \int_{\mathbb{H}^n} h(q)u dq - \int_{\mathbb{H}^n} g(q)v dq. \end{aligned}$$

The functional $J_{\mu,\sigma}$ is well–defined and of class $C^1(W)$, by Lemma 2.3.6 and the choice of θ and ϑ. Then we have

$$\langle J'_{\mu,\sigma}(u, v), (\Phi, \Psi) \rangle = \langle u, \Phi \rangle_{E_{m,a}} + \langle v, \Psi \rangle_{E_{p,b}} - \mu \langle u, \Phi \rangle_{\mathcal{H}_m} - \sigma \langle v, \Psi \rangle_{\mathcal{H}_p}$$

$$
\begin{aligned}
&- \int_{\mathbb{H}^n} [H_u(q,u,v)\Phi + H_v(q,u,v)\Psi] dq \\
&- \langle u^+, \Phi\rangle_{m^*} - \langle v^+, \Psi\rangle_{p^*} \\
&- \int_{\mathbb{H}^n} \left[\frac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta}\Phi + \frac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1}\Psi \right] dq \\
&- \int_{\mathbb{H}^n} h(q)\Phi dq - \int_{\mathbb{H}^n} g(q)\Psi dq,
\end{aligned}
$$

for any $(\Phi, \Psi) \in W$, where

$$
\langle u^+, \Phi\rangle_{m^*} = \int_{\mathbb{H}^n} |u(q)|^{m^*-2} u^+(q)\Phi(q) dq \quad \text{and}
$$
$$
\langle v^+, \Psi\rangle_{p^*} = \int_{\mathbb{H}^n} |v(q)|^{p^*-2} v^+(q)\Psi(q) dq.
$$

Taking inspiration from [40], we first prove that $(\mathcal{P}_2)$ presents a suitable geometry for existence of local minima provided that the perturbations h and g are sufficiently small in their norms, as shown in [23] for general equations in a different framework.

Lemma 2.5.1. *Under* (2.1.4) *there exist numbers* β, ρ *and* $\delta > 0$ *such that* $J_{\mu,\sigma}(u,v) \geq \beta$ *for all* $(u,v) \in W$, *with* $\|(u,v)\| = \rho$, *and for all* $h \in L^{\mathfrak{m}}(\mathbb{H}^n)$ *and* $g \in L^{\mathfrak{p}}(\mathbb{H}^n)$, *with* $\|h\|_{\mathfrak{m}} + \|g\|_{\mathfrak{p}} \leq \delta$.

Proof. Thanks to the main restriction (2.1.4), we choose $\varepsilon > 0$ such that $2^p\varepsilon/\lambda_p = \kappa - 2^{p-1}\lambda/\lambda_p$. By (2.1.2), (2.3.3), (H_2) and the Hölder inequality, we have for all $(u,v) \in W$, with $\|(u,v)\| \leq 1$,

$$
\begin{aligned}
J_{\mu,\sigma}(u,v) \geq\ & \frac{1}{2^{p-1}p}\left(\kappa - 2^{p-1}\frac{\lambda+\varepsilon}{\lambda_p}\right)\|(u,v)\|^p - C_\varepsilon C_s^s \|(u,v)\|^s \\
& - \frac{C_{m^*}^{m^*}}{m^*}\|(u,0)\|^{m^*} - \frac{C_{p^*}^{p^*}}{p^*}\|(0,v)\|^{p^*} - \frac{1}{m^*}\|u\|_{m^*}^{\theta}\|v\|_{m^*}^{\vartheta} \\
& - C_{m^*}\|h\|_{\mathfrak{m}}\|(u,0)\| - C_{p^*}\|g\|_{\mathfrak{p}}\|(0,v)\| \\
\geq\ & \left[\frac{1}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p}\right)\|(u,v)\|^{p-1} - C_\varepsilon C_s^s\|(u,v)\|^{s-1} \right.\\
& - \frac{C_{m^*}^{m^*}}{m^*}\|(u,v)\|^{m^*-1} - \frac{C_{p^*}^{p^*}}{p^*}\|(u,v)\|^{p^*-1} \\
& \left. - \frac{C_{m^*}^{m^*}}{m^*}\|(u,v)\|^{m^*-1} - C_{m^*}\|h\|_{\mathfrak{m}} - C_{p^*}\|g\|_{\mathfrak{p}}\right]\|(u,v)\|,
\end{aligned}
$$

since $\|u\|_{m^*}^{\theta}\|v\|_{m^*}^{\vartheta} \le C_{m^*}^{m^*}\|u\|_{E_{m,a}}^{\theta}\|v\|_{E_{p,b}}^{\vartheta} \le C_{m^*}^{m^*}\|(u,v)\|^{m^*}$, being $\theta+\vartheta=m^*$. Thus, setting for all $\tau\in[0,1]$

$$\eta_{\mu,\sigma}(\tau) = \frac{1}{2^p p}\left(\kappa - 2^{p-1}\frac{\lambda}{\lambda_p}\right)\tau^{p-1} - C_\varepsilon C_s^s \tau^{s-1} - 2\frac{C_{m^*}}{m^*}\tau^{m^*-1} - \frac{C_{p^*}}{p^*}\tau^{p^*-1},$$

we find some $\rho\in(0,1]$ so small that $\max_{\tau\in[0,1]}\eta_{\mu,\sigma}(\tau)=\eta_{\mu,\sigma}(\rho)>0$, since $1<m\le p<s<m^*$. Choosing $\delta=\eta_{\mu,\sigma}(\rho)/2(C_{m^*}+C_{p^*})$ we obtain $J_{\mu,\sigma}(u,v)\ge\beta=\rho\eta_{\mu,\sigma}(\rho)/2$ for all $(u,v)\in W$, with $\|(u,v)\|=\rho$, and for all $h\in L^{\mathfrak{m}}(\mathbb{H}^n)$ and $g\in L^{\mathfrak{p}}(\mathbb{H}^n)$, with $\|h\|_{\mathfrak{m}}+\|g\|_{\mathfrak{p}}\le\delta$. □

Lemma 2.5.2. *Let ρ be given as in Lemma 2.5.1. Set*

$$m_{\mu,\sigma} = \inf\left\{J_{\mu,\sigma}(u,v):\ (u,v)\in\overline{B}_\rho\right\},$$

where $\overline{B}_\rho=\{(u,v)\in W:\ \|(u,v)\|\le\rho\}$. Then $m_{\mu,\sigma}<0$ for all nonnegative perturbations h in $L^{\mathfrak{m}}(\mathbb{H}^n)$ and g in $L^{\mathfrak{p}}(\mathbb{H}^n)$, with $\|h\|_{\mathfrak{m}}+\|g\|_{\mathfrak{p}}>0$.

Proof. Take $h\in L^{\mathfrak{m}}(\mathbb{H}^n)$ and $g\in L^{\mathfrak{p}}(\mathbb{H}^n)$, with $\|h\|_{\mathfrak{m}}+\|g\|_{\mathfrak{p}}>0$. We show that there exists a nonnegative function $\hat\varphi\in C_0^\infty(\mathbb{H}^n)$ such that

$$\int_{\mathbb{H}^n}\hat\varphi(q)(h+g)dq>0. \tag{2.5.1}$$

The functions

$$\hat h(q)=\begin{cases}h(q)^{\mathfrak{m}-1}, & \text{if } h(q)\ne0\\ 0, & \text{if } h(q)=0\end{cases}\in L^{m^*}(\mathbb{H}^n),$$

$$\hat g(q)=\begin{cases}g(q)^{\mathfrak{p}-1}, & \text{if } g(q)\ne0\\ 0, & \text{if } g(q)=0\end{cases}\in L^{p^*}(\mathbb{H}^n),$$

by the assumptions on h and g. Since $C_0^\infty(\mathbb{H}^n)$ is dense in $L^{m^*}(\mathbb{H}^n)$ and in $L^{p^*}(\mathbb{H}^n)$, there exist two sequences $(h_k)_k$ and $(g_k)_k$ in $C_0^\infty(\mathbb{H}^n)$ such that $h_k\to\hat h$ strongly in $L^{m^*}(\mathbb{H}^n)$ and a.e. in $\mathbb{H}^n$, while $g_k\to\hat g$ strongly in $L^{p^*}(\mathbb{H}^n)$ and a.e. in $\mathbb{H}^n$. Taking k_0 and k_1 in $\mathbb{N}$ large enough we obtain

$$h_{k_0},\ g_{k_1}\ge0 \text{ a.e. in } \mathbb{H}^n,\quad \|h_{k_0}-\hat h\|_{m^*}\le\frac12\|h\|_{\mathfrak{m}}^{\mathfrak{m}-1},\quad \|g_{k_1}-\hat g\|_{p^*}\le\frac12\|g\|_{\mathfrak{p}}^{\mathfrak{p}-1}.$$

Setting $\hat\varphi=h_{k_0}+g_{k_1}$, of course we have $\hat\varphi\in C_0^\infty(\mathbb{H}^n)$, $\hat\varphi\ge0$ a.e. in $\mathbb{H}^n$, and $(\hat\varphi,\hat\varphi)\in W$. Then, using the Hölder inequality

$$\int_{\mathbb{H}^n}\hat\varphi(q)\big(h+g)dq\ge\int_{\mathbb{H}^n}h_{k_0}h\,dq+\int_{\mathbb{H}^n}g_{k_1}g\,dq$$

$$\geq -\|h_{k_0} - \hat{h}\|_{m^*}\|h\|_{\mathfrak{m}} + \|h\|_{\mathfrak{m}}^{\mathfrak{m}} - \|g_{k_1} - \hat{g}\|_{p^*}\|g\|_{\mathfrak{p}} + \|g\|_{\mathfrak{p}}^{\mathfrak{p}}$$
$$\geq \frac{1}{2}\|h\|_{\mathfrak{m}}^{\mathfrak{m}} + \frac{1}{2}\|g\|_{\mathfrak{p}}^{\mathfrak{p}} > 0,$$

by assumption. Consequently (2.5.1) is proved.

Finally taking $\tau \in (0,1)$ sufficiently small, by (2.5.1) and (H_1)

$$\begin{aligned} J_{\mu,\sigma}(\tau\hat{\varphi},\tau\hat{\varphi}) \leq \frac{\tau^m}{m}\|\hat{\varphi}\|_{E_{m,a}}^m + \frac{\tau^p}{p}\|\hat{\varphi}\|_{E_{p,b}}^p - \frac{\mu\tau^m}{m}\|\hat{\varphi}\|_{\mathcal{H}_m}^m - \frac{\sigma\tau^p}{p}\|\hat{\varphi}\|_{\mathcal{H}_p}^p \\ - 2\frac{\tau^{m^*}}{m^*}\|\hat{\varphi}\|_{m^*}^{m^*} - \frac{\tau^{p^*}}{p^*}\|\hat{\varphi}\|_{p^*}^{p^*} - \tau\int_{\mathbb{H}^n}\hat{\varphi}(q)\big[h(q)+g(q)\big]dq \\ < 0, \end{aligned}$$

as claimed. □

Proof of Theorem 2.1.3. Fix $\mu \in (-\infty, \mathcal{H}_m)$ and $\sigma \in (-\infty, \mathcal{H}_p)$ so that condition (2.1.4) is verified. Observe that if (u,v) is a solution of $(\mathcal{P}_2)$, then

$$\begin{aligned} \langle u,\Phi\rangle_{E_{m,a}} - \mu\langle u,\Phi\rangle_{\mathcal{H}_m} + \langle v,\Psi\rangle_{E_{p,b}} - \sigma\langle v,\Psi\rangle_{\mathcal{H}_p} \\ = \int_{\mathbb{H}^n}[H_u(q,u,v)\Phi + H_v(q,u,v)\Psi]dq + \langle u^+,\Phi\rangle_{m^*} + \langle v^+,\Psi\rangle_{p^*} \\ + \int_{\mathbb{H}^n}\left[\frac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta}\Phi + \frac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1}\Psi\right]dq \\ + \int_{\mathbb{H}^n}h(q)\Phi dq + \int_{\mathbb{H}^n}g(q)\Psi dq \end{aligned}$$

for all $(\Phi,\Psi) \in W$. Taking $\Phi = u^- = \min\{0,u\}$ and $\Psi = v^- = \min\{0,v\}$, by (H_1), (2.1.4) and (2.4.5) we obtain

$$\begin{aligned} 0 &\geq \int_{\mathbb{H}^n}h(q)u^-dq + \int_{\mathbb{H}^n}g(q)v^-dq \\ &= \int_{\mathbb{H}^n}[H_u(q,u,v)u^- + H_v(q,u,v)v^-]dq + \langle u^+,u^-\rangle_{m^*} \\ &\quad + \langle v^+,v^-\rangle_{p^*} + \int_{\mathbb{H}^n}\left[\frac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta}u^- + \frac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1}v^-\right]dq \\ &\quad + \int_{\mathbb{H}^n}h(q)u^-dq + \int_{\mathbb{H}^n}g(q)v^-dq \\ &\geq \left(1-\frac{\mu^+}{\mathcal{H}_m}\right)\|u^-\|_{E_{m,a}}^m + \left(1-\frac{\sigma^+}{\mathcal{H}_p}\right)\|v^-\|_{E_{p,b}}^p \geq 0. \end{aligned}$$

Consequently, $u^- = 0$ and $v^- = 0$ a.e. in $\mathbb{H}^n$, that is, $u \geq 0$ and $v \geq 0$ a.e. in $\mathbb{H}^n$. Thus any solution of $(\mathcal{P}_2)$ is nonnegative, component by component.

Let $\delta > 0$ be the number determined in Lemma 2.5.1 and take $h \in L^{\mathfrak{m}}(\mathbb{H}^n)$ and $g \in L^{\mathfrak{p}}(\mathbb{H}^n)$, with $0 < \|h\|_{\mathfrak{m}} + \|g\|_{\mathfrak{p}} \le \delta$. By Lemmas 2.5.1, 2.5.2 and the Ekeland variational principle, in $\overline{B}_\rho$ there exists a sequence $\{(u_k, v_k)\}_k$ in B_ρ such that

$$
\begin{aligned}
m_{\mu,\sigma} \le J_{\mu,\sigma}(u_k, v_k) \le m_{\mu,\sigma} + \frac{1}{k} \quad \text{and} \\
J_{\mu,\sigma}(u, v) \ge J_{\mu,\sigma}(u_k, v_k) - \frac{1}{k}\|(u - u_k, v - v_k)\|
\end{aligned}
\tag{2.5.2}
$$

for all $k \in \mathbb{N}$ and for any $(u, v) \in \overline{B}_\rho$. Fixed $k \in \mathbb{N}$, for all $(\mathfrak{u}, \mathfrak{v}) \in S_W$, where $S_W = \{(\mathfrak{u}, \mathfrak{v}) \in W : \|(\mathfrak{u}, \mathfrak{v})\| = 1\}$, and for all $\varepsilon > 0$ so small that $(u_k + \varepsilon\, \mathfrak{u}, v_k + \varepsilon \mathfrak{v}) \in \overline{B}_\rho$, by (2.5.2) we get

$$
J_{\mu,\sigma}(u_k + \varepsilon\, \mathfrak{u}, v_k + \varepsilon \mathfrak{v}) - J_{\mu,\sigma}(u_k, v_k) \ge -\frac{\varepsilon}{k}.
$$

In particular, we have

$$
\langle J'_{\mu,\sigma}(u_k, v_k), (\mathfrak{u}, \mathfrak{v})\rangle = \lim_{\varepsilon \to 0} \frac{J_{\mu,\sigma}(u_k + \varepsilon\, \mathfrak{u}, v_k + \varepsilon \mathfrak{v}) - J_{\mu,\sigma}(u_k, v_k)}{\varepsilon} \ge -\frac{1}{k}
$$

for all $(\mathfrak{u}, \mathfrak{v}) \in S_W$, being $J_{\mu,\sigma}$ Gâteaux differentiable in W. Thus

$$
\left|\langle J'_{\mu,\sigma}(u_k, v_k), (\mathfrak{u}, \mathfrak{v})\rangle\right| \le \frac{1}{k},
$$

since $(\mathfrak{u}, \mathfrak{v}) \in S_W$ is arbitrary. Consequently, $J'_{\mu,\sigma}(u_k, v_k) \to 0$ in W' as $k \to \infty$. Obviously the bounded sequence $\{(u_k, v_k)\}_k$, up to a subsequence, weakly converges to some $(u, v) \in \overline{B}_\rho$. By Lemmas 2.3.6, 2.3.5 and (1.2.2) the following properties hold

$$
\begin{aligned}
&(u_k, v_k) \rightharpoonup (u, v) \text{ in } W, \qquad (u_k, v_k) \to (u, v) \text{ a.e. in } \mathbb{H}^n, \\
&(u_k, v_k) \to (u, v) \text{ in } L^\nu(\mathbb{H}^n) \times L^\nu(\mathbb{H}^n), \\
&u_k \rightharpoonup u \text{ in } L^m(\mathbb{H}^n, \psi^m r^{-m}), \\
&\|u_k\|_{E_{m,a}} \to \mathfrak{u}, \ \|v_k\|_{E_{p,b}} \to \mathfrak{v}, \qquad \|u_k\|_{\mathcal{H}_m} \to \mathfrak{z}, \ \|v_k\|_{\mathcal{H}_p} \to \mathfrak{k}, \\
&v_k \rightharpoonup v \text{ in } L^p(\mathbb{H}^n, \psi^p r^{-p}), \\
&u_k^+ \rightharpoonup u^+ \text{ in } L^{m^*}(\mathbb{H}^n), \qquad v_k^+ \rightharpoonup v^+ \text{ in } L^{p^*}(\mathbb{H}^n), \\
&\|u_k^+\|_{m^*} \to \mathfrak{i}, \quad \|v_k^+\|_{p^*} \to \mathfrak{j}, \\
&(u_k^+)^{\theta-1}(v_k^+)^{\vartheta} \rightharpoonup (u^+)^{\theta-1}(v^+)^{\vartheta} \text{ in } L^{m^*/(m^*-1)}(\mathbb{H}^n), \\
&(u_k^+)^{\theta}(v_k^+)^{\vartheta-1} \rightharpoonup (u^+)^{\theta}(v^+)^{\vartheta-1} \text{ in } L^{m^*/(m^*-1)}(\mathbb{H}^n),
\end{aligned}
\tag{2.5.3}
$$

for any $\nu \in [p, m^*)$. It results,

$$\begin{aligned} \lim_{k\to\infty} \int_{\mathbb{H}^n} (u_k^+)^{\theta-1}(v_k^+)^{\vartheta} u^+ dq &= \int_{\mathbb{H}^n} (u^+)^{\theta}(v^+)^{\vartheta} dq, \\ \lim_{k\to\infty} \int_{\mathbb{H}^n} (u_k^+)^{\theta}(v_k^+)^{\vartheta-1} v^+ dq &= \int_{\mathbb{H}^n} (u^+)^{\theta}(v^+)^{\vartheta} dq, \end{aligned} \tag{2.5.4}$$

since $(u^+, v^+) \in W$. Then the Fatou lemma yields

$$\int_{\mathbb{H}^n} (u^+)^{\theta}(v^+)^{\vartheta} dq \leq \liminf_{k\to\infty} \int_{\mathbb{H}^n} (u_k^+)^{\theta}(v_k^+)^{\vartheta} dq. \tag{2.5.5}$$

Moreover, by (H_2), (2.5.3) and the Lebesgue dominated convergence theorem we obtain

$$\begin{aligned} \lim_{k\to\infty} \int_{\mathbb{H}^n} &[H_u(q, u_k, v_k)u + H_v(q, u_k, v_k)v]dq \\ &= \int_{\mathbb{H}^n} [H_u(q, u, v)u + H_v(q, u, v)v]dq \end{aligned} \tag{2.5.6}$$

and analogously

$$\begin{aligned} \lim_{k\to\infty} \int_{\mathbb{H}^n} &[H_u(q, u_k, v_k)u_k + H_v(q, u_k, v_k)v_k]dq \\ &= \int_{\mathbb{H}^n} [H_u(q, u, v)u + H_v(q, u, v)v]dq, \\ \lim_{k\to\infty} \int_{\mathbb{H}^n} &H(q, u_k, v_k)dq = \int_{\mathbb{H}^n} H(q, u, v)dq. \end{aligned} \tag{2.5.7}$$

Furthermore, by (2.5.3)

$$\begin{aligned} \lim_{k\to\infty} \int_{\mathbb{H}^n} h(q)u_k dq &= \int_{\mathbb{H}^n} h(q)u dq, \\ \lim_{k\to\infty} \int_{\mathbb{H}^n} g(q)v_k dq &= \int_{\mathbb{H}^n} g(q)v dq, \end{aligned} \tag{2.5.8}$$

being $h \in L^{\mathfrak{m}}(\mathbb{H}^n)$ and $g \in L^{\mathfrak{p}}(\mathbb{H}^n)$.

Next we show that (u, v), given in (2.5.3), is actually in B_ρ, so that (u, v) is a critical point of $J_{\mu,\sigma}$ at level $m_{\mu,\sigma} < 0$. It implies that (u, v) is a nontrivial solution of $(\mathcal{P}_2)$. Obviously, $J_{\mu,\sigma}(u, v) \geq m_{\mu,\sigma}$, since $(u, v) \in \overline{B}_\rho$ by (2.5.3). Moreover, by (2.5.3) and (2.5.6) we have as $k \to \infty$

$$0 = \langle J'_{\mu,\sigma}(u_k, v_k), (u, v)\rangle + o(1)$$

$$
\begin{aligned}
&= \langle u_k, u\rangle_{E_{m,a}} - \mu\langle u_k, u\rangle_{\mathcal{H}_m} + \langle v_k, v\rangle_{E_{p,b}} - \sigma\langle v_k, v\rangle_{\mathcal{H}_p} \\
&\quad - \int_{\mathbb{H}^n} [H_u(q, u_k, v_k)u + H_v(q, u_k, v_k)v]dq \\
&\quad - \langle u_k^+, u\rangle_{m^*} - \langle v_k^+, v\rangle_{p^*} \\
&\quad - \int_{\mathbb{H}^n} \left[\frac{\theta}{m^*}(u_k^+)^{\theta-1}(v_k^+)^{\vartheta}u^+ + \frac{\vartheta}{m^*}(u_k^+)^{\theta}(v_k^+)^{\vartheta-1}v^+\right] dq \\
&\quad - \int_{\mathbb{H}^n} h(q)udq - \int_{\mathbb{H}^n} g(q)vdq + o(1) \\
&= \|u\|^m_{E_{m,a}} - \mu\|u\|^m_{\mathcal{H}_m} + \|v\|^p_{E_{p,b}} - \sigma\|v\|^p_{\mathcal{H}_p} \\
&\quad - \int_{\mathbb{H}^n} [H_u(q, u, v)u + H_v(q, u, v)v]dq - \|u^+\|^{m^*}_{m^*} - \|v^+\|^{p^*}_{p^*} \\
&\quad - \int_{\mathbb{H}^n} (u^+)^{\theta}(v^+)^{\vartheta}dq - \int_{\mathbb{H}^n} h(q)udq - \int_{\mathbb{H}^n} g(q)vdq.
\end{aligned} \tag{2.5.9}
$$

Now we divide the proof in two cases.

Case $m < p$ and $\mu \le 0$. Multiplying the expression (2.5.9) by $1/p$ and subtracting it below, by (2.5.3), (2.5.5) and (2.5.7)–(2.5.8) we get as $k \to \infty$

$$
\begin{aligned}
m_{\mu,\sigma} \le J_{\mu,\sigma}(u, v) &= \frac{1}{m}\|u\|^m_{E_{m,a}} - \frac{\mu}{m}\|u\|^m_{\mathcal{H}_m} + \frac{1}{p}\|v\|^p_{E_{p,b}} - \frac{\sigma}{p}\|v\|^p_{\mathcal{H}_p} \\
&\quad - \int_{\mathbb{H}^n} H(q, u, v)dq - \frac{1}{m^*}\|u^+\|^{m^*}_{m^*} - \frac{1}{p^*}\|v^+\|^{p^*}_{p^*} \\
&\quad - \frac{1}{m^*}\int_{\mathbb{H}^n} (u^+)^{\theta}(v^+)^{\vartheta}dq - \int_{\mathbb{H}^n} h(q)u(q)dq - \int_{\mathbb{H}^n} g(q)v(q)dq \\
&= \left(\frac{1}{m} - \frac{1}{p}\right)\left(\|u\|^m_{E_{m,a}} + |\mu|\cdot\|u\|^m_{\mathcal{H}_m}\right) \\
&\quad + \frac{1}{p}\int_{\mathbb{H}^n} [H_u(q, u, v)u + H_v(q, u, v)v]dq \\
&\quad - \int_{\mathbb{H}^n} H(q, u, v)dq + \left(\frac{1}{p} - \frac{1}{m^*}\right)\|u^+\|^{m^*}_{m^*} + \left(\frac{1}{p} - \frac{1}{p^*}\right)\|v^+\|^{p^*}_{p^*} \\
&\quad + \left(\frac{1}{p} - \frac{1}{m^*}\right)\int_{\mathbb{H}^n} (u^+)^{\theta}(v^+)^{\vartheta}dq - \left(1 - \frac{1}{p}\right)\int_{\mathbb{H}^n} h(q)udq \\
&\quad - \left(1 - \frac{1}{p}\right)\int_{\mathbb{H}^n} g(q)vdq \\
&\le \left(\frac{1}{m} - \frac{1}{p}\right)\left(\|u_k\|^m_{E_{m,a}} + |\mu|\cdot\|u_k\|^m_{\mathcal{H}_m}\right) \\
&\quad + \frac{1}{p}\int_{\mathbb{H}^n} [H_u(q, u_k, v_k)u_k + H_v(q, u_k, v_k)v_k]dq
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathbb{H}^n} H(q,u_k,v_k)dq + \left(\frac{1}{p}-\frac{1}{m^*}\right)\|u_k^+\|_{m^*}^{m^*} + \left(\frac{1}{p}-\frac{1}{p^*}\right)\|v_k^+\|_{p^*}^{p^*}\\
&+\left(\frac{1}{p}-\frac{1}{m^*}\right)\int_{\mathbb{H}^n}(u_k^+)^\theta(v_k^+)^\vartheta dq - \left(1-\frac{1}{p}\right)\int_{\mathbb{H}^n} h(q)u_k dq\\
&-\left(1-\frac{1}{p}\right)\int_{\mathbb{H}^n} g(q)v_k dq + o(1)\\
\le\ & J_{\mu,\sigma}(u_k,v_k) - \frac{1}{p}\langle J'_{\mu,\sigma}(u_k,v_k),(u_k,v_k)\rangle + o(1) = m_{\mu,\sigma},
\end{aligned}
$$

since $\|u\|_{E_{m,a}} \le \mathfrak{u}$, $\|u\|_{\mathcal{H}_m} \le \mathfrak{z}$, $\|v\|_{E_{p,b}} \le \mathfrak{v}$, $\|v\|_{\mathcal{H}_p} \le \mathfrak{k}$ and $1 < m < p < m^*$.
Case $m = p$. As in the previous case, we multiply the expression (2.5.9) by $1/p$ and, subtracting it below, by (2.5.3), (2.5.5) and (2.5.7)–(2.5.8) we find that as $k \to \infty$

$$
\begin{aligned}
m_{\mu,\sigma} \le J_{\mu,\sigma}(u,v) &= \frac{1}{p}\|u\|_{E_{p,a}}^p - \frac{\mu}{p}\|u\|_{\mathcal{H}_p}^p + \frac{1}{p}\|v\|_{E_{p,b}}^p - \frac{\sigma}{p}\|v\|_{\mathcal{H}_p}^p\\
&\quad - \int_{\mathbb{H}^n} H(q,u,v)dq - \frac{1}{p^*}\|u^+\|_{p^*}^{p^*} - \frac{1}{p^*}\|v^+\|_{p^*}^{p^*}\\
&\quad - \frac{1}{p^*}\int_{\mathbb{H}^n}(u^+)^\theta(v^+)^\vartheta dq - \int_{\mathbb{H}^n} h(q)u(q)dq - \int_{\mathbb{H}^n} g(q)v(q)dq\\
&\le \frac{1}{p}\int_{\mathbb{H}^n}[H_u(q,u_k,v_k)u_k + H_v(q,u_k,v_k)v_k]dq - \int_{\mathbb{H}^n} H(q,u_k,v_k)dq\\
&\quad + \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(\|u_k^+\|_{p^*}^{p^*} + \|v_k^+\|_{p^*}^{p^*}\right) - \left(1-\frac{1}{p}\right)\int_{\mathbb{H}^n} h(q)u_k dq\\
&\quad + \left(\frac{1}{p}-\frac{1}{p^*}\right)\int_{\mathbb{H}^n}(u_k^+)^\theta(v_k^+)^\vartheta dq - \left(1-\frac{1}{p}\right)\int_{\mathbb{H}^n} g(q)v_k dq + o(1)\\
&= J_{\mu,\sigma}(u_k,v_k) - \frac{1}{p}\langle J'_{\mu,\sigma}(u_k,v_k),(u_k,v_k)\rangle + o(1) = m_{\mu,\sigma}.
\end{aligned}
$$

In both cases, it results that (u,v) is a minimizer of $J_{\mu,\sigma}$ in $\overline{B}_\rho$ and $J_{\mu,\sigma}(u,v) = m_{\mu,\sigma} < 0 < \beta \le J_{\mu,\sigma}(\mathfrak{u},\mathfrak{v})$ for all $(\mathfrak{u},\mathfrak{v}) \in \partial B_\rho$ by Lemma 2.5.1. Hence $(u,v) \in B_\rho$, so that $J'_{\mu,\sigma}(u,v) = 0$. In conclusion (u,v) is a nontrivial solution of $(\mathcal{P}_2)$, as claimed. □

Chapter 3

Existence problems involving Hardy and critical terms in the Heisenberg group

3.1 Introduction

We start the chapter by treating elliptic problems in general open subsets Ω, with possibly $\Omega = \mathbb{H}^n$, as studied in [14]. In the latter case, the condition $u = 0$ on $\partial\Omega$, simply disappears thanks to the functional Folland–Stein setting. The first problem considered is

$$(\mathcal{P}_3) \qquad \begin{cases} -\Delta^p_{\mathbb{H}^n} u - \gamma\psi^p \cdot \dfrac{|u|^{p-2}u}{r^p} = \sigma w(q)|u|^{s-2}u + \cancel{k}(q)|u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where γ and σ are real parameters, $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$, $1 < p < Q$, the exponent s is in the open interval (p, p^*), with $p^* = pQ/(Q-p)$ and $\Delta^p_{\mathbb{H}^n}$ is the p–Laplacian operator on $\mathbb{H}^n$, which is defined by

$$\Delta^p_{\mathbb{H}^n}\phi = \mathrm{div}_H(|D_{\mathbb{H}^n}\phi|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n}\phi)$$

along any $\phi \in C^\infty_0(\mathbb{H}^n)$, that is $\Delta^p_{\mathbb{H}^n}$ is the familiar horizontal p–Laplacian operator. Moreover we recall that r denotes the Heisenberg norm

$$r(q) = r(z,t) = (|z|^4 + t^2)^{1/4}, \quad z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n, \quad t \in \mathbb{R},$$

$|z|$ the Euclidean norm in $\mathbb{R}^{2n}$, $D_{\mathbb{H}^n}u = (X_1u, \cdots, X_nu, Y_1u, \cdots, Y_nu)$ the horizontal gradient as in (1.1.2), $\{X_j, Y_j\}^n_{j=1}$ the basis of left invariant vector fields on $\mathbb{H}^n$, that is

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$. As noted in Chapter 1.1, the weight function ψ appearing in $(\mathcal{P}_3)$ is defined as $\psi = |D_{\mathbb{H}^n} r|_{\mathbb{H}^n}$. We emphasize that ψ is identically 1 in the Euclidean canonical case.

Concerning *the weights* w *and* $\mathcal{k}$*, we suppose*

(w) $w > 0$ *a.e. in* $\mathbb{H}^n$ *and* $w \in L^{\wp}(\mathbb{H}^n)$, *with* $\wp = p^*/(p^* - s)$ *and* $1 < s < p^*$,

$(\mathcal{k})$ $\mathcal{k} \geq 0$ *a.e. in* $\mathbb{H}^n$ *and* $\mathcal{k} \in L^\infty(\mathbb{H}^n)$.

When dealing with $(\mathcal{P}_3)$ we assume (w) and $(\mathcal{k})$, without further mentioning. Condition (w) first appears in [23] in another context.

We look for (weak) solutions of $(\mathcal{P}_3)$ in the Folland–Stein space $S_0^{1,p}(\Omega)$, which is defined as the completion of $C_0^\infty(\Omega)$, with respect to the norm

$$\|D_{\mathbb{H}^n} u\|_p = \left(\int_\Omega |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^p dq \right)^{1/p}.$$

When $\Omega = \mathbb{H}^n$, we shall simply denote $S_0^{1,p}(\mathbb{H}^n)$ by $S^{1,p}(\mathbb{H}^n)$.

It is crucial now to introduce the formulation of the Hardy–Sobolev inequality defined in $\Omega \subset \mathbb{H}^n$, that is used in the study of the problems within this chapter. As seen in Section 1.2, assume that $0 \leq \alpha \leq p$ and put $p^*(\alpha) = p(Q - \alpha)/(Q - p)$. The *best Hardy–Sobolev constant* $\mathcal{H}_{\alpha,\Omega} = \mathcal{H}(p, Q, \alpha, \Omega)$ is given by

$$\mathcal{H}_{\alpha,\Omega} = \inf_{\substack{u \in S_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\|D_{\mathbb{H}^n} u\|_p^p}{\|u\|_{\mathcal{H}_{\alpha,\Omega}}^p}, \qquad \|u\|_{\mathcal{H}_{\alpha,\Omega}}^{p^*(\alpha)} = \int_\Omega \psi^\alpha \frac{|u|^{p^*(\alpha)}}{r^\alpha} dq, \tag{3.1.1}$$

which is well defined, strictly positive by combining properly the Sobolev and Hardy inequalities on the Heisenberg group $\mathbb{H}^n$, as a direct consequence of Lemma 1.2.1. However, the Hardy embedding

$$S_0^{1,p}(\Omega) \hookrightarrow L^{p^*(\alpha)}(\Omega, \psi^\alpha r^{-\alpha})$$

is continuous, but not compact. This is one of the reason why problem $(\mathcal{P}_3)$ is fairly delicate to manage.

We observe that (weak) solutions of $(\mathcal{P}_3)$ are exactly the critical points of the underlying functional $\mathcal{J}_{\gamma,\sigma}$ introduced in Section 3.2, which satisfies the geometry of the mountain pass lemma under the above structural assumptions. The critical points $u_{\gamma,\sigma}$ of $\mathcal{J}_{\gamma,\sigma}$ in $S_0^{1,p}(\Omega)$ are found at special mountain pass levels $c_{\gamma,\sigma}$ and these solutions of $(\mathcal{P}_3)$ are simply called *mountain pass solutions.*

Theorem 3.1.1. *For every $\gamma \in (-\infty, \mathcal{H}_{p,\Omega})$ problem $(\mathcal{P}_3)$ admits a nontrivial mountain pass solution $u_{\gamma,\sigma}$ in $S_0^{1,p}(\Omega)$ for any $\sigma > 0$. Moreover*

$$\lim_{\sigma\to\infty} \|D_{\mathbb{H}^n} u_{\gamma,\sigma}\|_p = 0. \tag{3.1.2}$$

In the second part of the chapter, we assume that Ω is a bounded PS domain of $\mathbb{H}^n$. This seems necessary for deriving the exact behavior of weakly convergent sequences of $S_0^{1,p}(\Omega)$ in the space of measures.

Theorem 3.1.2. *Let Ω be a bounded PS domain of $\mathbb{H}^n$ and let $\alpha \in [0,p]$. Let $(u_k)_k$ be a weakly convergent sequence in $S_0^{1,p}(\Omega)$, with weak limit u. Then there exist two finite measures μ and ν in $\mathbb{H}^n$ such that*

$$|D_{\mathbb{H}^n} u_k(q)|^p dq \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |u_k|^{p^*(\alpha)} \psi^\alpha \frac{dq}{r^\alpha} \overset{*}{\rightharpoonup} \nu \qquad \text{in } \mathcal{M}(\Omega). \tag{3.1.3}$$

Furthermore, if $\alpha = 0$ there exist a denumerable index set Λ, points $q_j \in \overline{\Omega}$, numbers $\nu_j \geq 0$, $\mu_j \geq 0$, with $\nu_j + \mu_j > 0$, for all $j \in \Lambda$, such that

$$d\mu \geq |D_{\mathbb{H}^n} u(q)|^p_{\mathbb{H}^n} dq + \sum_{j\in\Lambda} \mu_j \delta_{q_j}, \quad \mu_j = \mu(\{q_j\}), \tag{3.1.4}$$

$$d\nu = |u(q)|^{p^*} dq + \sum_{j\in\Lambda} \nu_j \delta_{q_j}, \quad \nu_j = \nu(\{q_j\}), \tag{3.1.5}$$

$$\mu_j \geq \mathcal{H}_{0,\Omega} v_j^{p/p^*}. \tag{3.1.6}$$

While, if $0 < \alpha \leq p$, there exist two nonnegative numbers μ_0, ν_0 such that

$$\nu = |u(q)|^{p^*(\alpha)} \psi^\alpha \frac{dq}{r^\alpha} + \nu_0 \delta_0, \tag{3.1.7}$$

$$\mu \geq |D_{\mathbb{H}^n} u(q)|^p_{\mathbb{H}^n} dq + \mu_0 \delta_0, \qquad 0 \leq \mathcal{H}_{\alpha,\Omega} \nu_0^{p/p^*(\alpha)} \leq \mu_0, \tag{3.1.8}$$

where $\mathcal{H}_{\alpha,\Omega}$ is the Hardy constant defined in (3.1.1).

In Theorem 3.1.2 the assumption that Ω is bounded seems to play an essential rule to get the compact embedding (1.2.5). However, observe that Theorem 1.1 in [6] is a compactness embedding result in the case $p = 2$ and $\alpha = 0$ for symmetric unbounded domains of the Heisenberg group. This could be an interesting starting point for further studies.

The proof of the case $\alpha = 0$ of Theorem 3.1.2 follows the arguments given in Theorem 2.5 in [78] for the Euclidean setting, see also [63]. Lemma 6.3 of [51] and Lemma 3.2.5 of [90] in the Carnot groups are other versions of the case $\alpha = 0$ of Theorem 3.1.2. While the proof of the case $0 < \alpha \leq p$ of Theorem 3.1.2 is based on the arguments of Theorem 1.1 of [39] for the Euclidean frame, cf. also [77].

As a direct consequence of Theorem 3.1.2, we are able to study in Section 3.4 some nonlinear elliptic problems in bounded PS domains Ω of $\mathbb{H}^n$, since the underlying functional $\mathcal{H}_{\gamma,\lambda}$, which is the basis of the elliptic part, is weakly lower semi–continuous and coercive in $S_0^{1,p}(\Omega)$, when the parameters γ and λ verify suitable natural restrictions.

The chapter is organized as follows. Section 3.2 contains the proof of Theorem 3.1.1, while Section 3.3 the proof of Theorem 3.1.2. Finally, in Section 3.4 we provide some applications of Theorem 3.1.2.

3.2 Critical problems in general open sets Ω

As said in Section 3.1, the best solution space for problem $(\mathcal{P}_3)$ is $S_0^{1,p}(\Omega)$, where Ω is any open subset of $\mathbb{H}^n$, possibly the entire $\mathbb{H}^n$ itself and so $S_0^{1,p}(\mathbb{H}^n) = S^{1,p}(\mathbb{H}^n)$. Moreover, s is any fixed Lebesgue exponent, with $p < s < p^*$, and w, $\mathcal{k}$ satisfy conditions (w) and $(\mathcal{k})$ in $(\mathcal{P}_3)$, without further mentioning.

Let $L^s(\Omega, w) = (L^s(\Omega, w), \|u\|_{s,w})$ be the weighted Lebesgue space, endowed with the norm

$$\|u\|_{s,w} = \left(\int_\Omega w(q)|u(q)|^s dq \right)^{1/s}.$$

By Proposition A.6 in [3], which still holds in the context of the Heisenberg group, the Banach space $L^s(\Omega, w)$ is uniformly convex. Futhermore, condition (w) guaranties that the embedding $S_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega, w)$ is compact, even when Ω is the entire $\mathbb{H}^n$. As proved in Lemma 4.1 of [39] and Lemma 2.1 in [23], the following result holds also in our context.

Lemma 3.2.1. *The embedding $S_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^s(\Omega, w)$ is compact and in particular*

$$\|u\|_{s,w} \leq C_w \|D_{\mathbb{H}^n} u\|_p \qquad \text{for all } u \in S_0^{1,p}(\Omega), \tag{3.2.1}$$

with $C_w = \mathcal{H}_{0,\Omega}^{-1/p} \|w\|_{\wp}^{1/s} > 0$.

Proof. By (w), the Hölder inequality and (3.1.1), for all $u \in S_0^{1,p}(\Omega)$

$$\|u\|_{s,w} \le \left(\int_\Omega w(q)^p dq\right)^{1/ps} \cdot \left(\int_\Omega |u|^{p^*} dq\right)^{1/p^*} \le \mathcal{H}_{0,\Omega}^{-1/p} \|w\|_p^{1/s} \|D_{\mathbb{H}^n} u\|_p,$$

that is the embedding $S_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega, w)$ is continuous and (3.2.1) holds.

To complete the proof, we need to show that if $u_k \rightharpoonup u$ in $S_0^{1,p}(\Omega)$, then $u_k \to u$ in $L^s(\Omega, w)$ as $k \to \infty$. We denote by $\tilde{u}_k$ and $\tilde{u}$, respectively, the natural extension of u_k and u. As a consequence of Lemma 1.2.2 and (1.2.6), we know that $\tilde{u}_k \rightharpoonup \tilde{u}$ in $S^{1,p}(\mathbb{H}^n)$. In the same way, we denote $\tilde{w}$ the natural extension of the weight function w to $\mathbb{H}^n$. Using the Hölder inequality, we get

$$\int_{\mathbb{H}^n \setminus B_R} \tilde{w}(q) |\tilde{u}_k - \tilde{u}|^s dq \le L \left(\int_{\mathbb{H}^n \setminus B_R} \tilde{w}(q)^p dq\right)^{1/p} = o(1) \tag{3.2.2}$$

as $R \to \infty$, being $\tilde{w} \in L^p(\mathbb{H}^n)$ and $\sup_k \|\tilde{u}_k - \tilde{u}\|_{p^*}^s = L < \infty$ by (3.1.1). Moreover, for all $R > 0$ the embedding $S^{1,p}(\mathbb{H}^n) \hookrightarrow HW^{1,p}(B_R)$ is continuous and so the embedding $S^{1,p}(\mathbb{H}^n) \hookrightarrow\hookrightarrow L^\nu(B_R)$ is compact for all $\nu \in [1, p^*)$, by (1.2.5) and the subsequent comments.

Fix $\varepsilon > 0$. There exists $R_\varepsilon > 0$ so large that $\int_{\mathbb{H}^n \setminus B_{R_\varepsilon}} \tilde{w}(q)|\tilde{u}_k - \tilde{u}|^s dq < \varepsilon$ by (3.2.2). Take a subsequence $(\tilde{u}_{k_j})_j \subset (\tilde{u}_k)_k$. Since $\tilde{u}_{k_j} \to \tilde{u}$ in $L^\nu(B_{R_\varepsilon})$ for all $\nu \in [1, p^*)$, then up to a further subsequence, still denoted by $(\tilde{u}_{k_j})_j$, we get that $\tilde{u}_{k_j} \to \tilde{u}$ a.e. in B_{R_ε}. It follows $\tilde{w}(q)|\tilde{u}_{k_j} - \tilde{u}|^s \to 0$ a.e. in B_{R_ε}. Furthermore, for each measurable subset $E \subset B_{R_\varepsilon}$, by the Hölder inequality we obtain

$$\int_E \tilde{w}(q)|\tilde{u}_{k_j} - \tilde{u}|^s dq \le L \left(\int_E \tilde{w}(q)^p dq\right)^{1/p}.$$

Consequently, $(\tilde{w}\,|\tilde{u}_{k_j} - \tilde{u}|^s)_j$ is equi–integrable and uniformly bounded in $L^1(B_{R_\varepsilon})$, being $\tilde{w} \in L^p(\mathbb{H}^n)$ by (w). Then, the Vitali convergence theorem yields

$$\lim_{j\to\infty} \int_{B_{R_\varepsilon}} \tilde{w}(q)|\tilde{u}_{k_j} - \tilde{u}|^s dq = 0$$

and so $\tilde{u}_k \to \tilde{u}$ in $L^s(B_{R_\varepsilon}, \tilde{w})$, since the sequence $(\tilde{u}_{k_j})_j$ is arbitrary.

It follows, $\int_{B_{R_\varepsilon}} \tilde{w}(q)|\tilde{u}_k - \tilde{u}|^s dq = o(1)$ as $k \to \infty$. In conclusion, as $k \to \infty$

$$\|\tilde{u}_k - \tilde{u}\|_{s,\tilde{w}}^s = \int_{\mathbb{H}^n \setminus B_{R_\varepsilon}} \tilde{w}(q)|\tilde{u}_k - \tilde{u}|^s dq + \int_{B_{R_\varepsilon}} \tilde{w}(q)|\tilde{u}_k - \tilde{u}|^s dq \le \varepsilon + o(1),$$

that is, $\tilde{u}_k \to \tilde{u}$ in $L^s(\mathbb{H}^n, \tilde{w})$ as $k \to \infty$, being $\varepsilon > 0$ arbitrary. In particular, $u_k \to u$ in $L^s(\Omega, w)$ as $k \to \infty$ and this completes the proof. □

We now turn back to problem ($\mathcal{P}_3$). Observe that (weak) solutions of ($\mathcal{P}_3$) correspond to critical points of the associated Euler–Lagrange functional $\mathcal{J}_{\gamma,\sigma}$, with $\mathcal{J}_{\gamma,\sigma} : S_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{J}_{\gamma,\sigma}(u) = \frac{1}{p}\|D_{\mathbb{H}^n} u\|_p^p - \frac{\gamma}{p}\|u\|_{\mathcal{H}_{p,\Omega}}^p - \frac{\sigma}{s}\|u\|_{s,w}^s - \frac{1}{p^*}\|u\|_{p^*,\mathcal{k}}^{p^*},$$

where $\|u\|_{p^*,\mathcal{k}} = \left(\int_\Omega \mathcal{k}(q)|u(q)|^{p^*} dq\right)^{1/p^*}$. Note that $\mathcal{J}_{\gamma,\sigma}$ is a $C^1(S_0^{1,p}(\Omega))$ functional and for any $u, \phi \in S_0^{1,p}(\Omega)$

$$\langle \mathcal{J}_{\gamma,\sigma}'(u), \phi\rangle = \langle u, \phi\rangle_p - \gamma\langle u, \phi\rangle_{\mathcal{H}_p} - \sigma\langle u, \phi\rangle_{s,w} - \langle u, \phi\rangle_{p^*,\mathcal{k}}. \tag{3.2.3}$$

From here on in this chapter, $\langle\cdot,\cdot\rangle$ simply denotes the dual pairing between $S_0^{1,p}(\Omega)$ and its dual space $[S_0^{1,p}(\Omega)]'$, that is $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{[S_0^{1,p}(\Omega)]', S_0^{1,p}(\Omega)}$. Moreover

$$\begin{aligned}
\langle u, \phi\rangle_p &= \int_{\mathbb{H}^n} \left(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u, D_{\mathbb{H}^n}\phi\right)_{\mathbb{H}^n} dq,\\
\langle u, \phi\rangle_{\mathcal{H}_p} &= \int_\Omega |u(q)|^{p-2} u(q)\phi(q)\psi^p \frac{dq}{r^p},\\
\langle u, \phi\rangle_{s,w} &= \int_\Omega w(q)|u(q)|^{s-2} u(q)\phi(q)\, dq,\\
\langle u, \phi\rangle_{p^*,\mathcal{k}} &= \int_\Omega \mathcal{k}(q)|u(q)|^{p^*-2} u(q)\phi(q)\, dq.
\end{aligned}$$

The simplified notation is reasonable, since $\langle u,\cdot\rangle_p$, $\langle u,\cdot\rangle_{\mathcal{H}_p}$, $\langle u,\cdot\rangle_{s,w}$, $\langle u,\cdot\rangle_{p^*,\mathcal{k}}$ are linear bounded functionals on $S_0^{1,p}(\Omega)$ for all $u \in S_0^{1,p}(\Omega)$. In order to find the critical points of $\mathcal{J}_{\gamma,\sigma}$, we intend to apply the mountain pass theorem, by checking that $\mathcal{J}_{\gamma,\sigma}$ possesses a suitable geometrical structure and that it satisfies the Palais–Smale compactness condition. Throughout the chapter we put $\tau = \tau^+ - \tau^-$, $\tau^+ = \max\{\tau,\ 0\}$ and $\tau^- = \max\{-\tau,\ 0\}$ for all $\tau \in \mathbb{R}$.

Lemma 3.2.2. *For every $\gamma \in (-\infty, \mathcal{H}_{p,\Omega})$ and $\sigma > 0$ there exists a nonnegative function $e \in S_0^{1,p}(\Omega)$, with $\|D_{\mathbb{H}^n} e\|_p \geq 2$ and $\mathcal{J}_{\gamma,\sigma}(e) < 0$, and furthermore there exist $\rho \in (0,1]$ and $\jmath > 0$ such that $\mathcal{J}_{\gamma,\sigma}(u) \geq \jmath$ for any $u \in S_0^{1,p}(\Omega)$, with $\|D_{\mathbb{H}^n} u\|_p = \rho$. The function e depends only on γ^-, when $\mathcal{k} > 0$ a.e. in $\mathbb{H}^n$.*

Proof. Fix $\gamma \in (-\infty, \mathcal{H}_{p,\Omega})$ and $\sigma > 0$.

Take a nonnegative function $v \in S_0^{1,p}(\Omega)$, with $\|D_{\mathbb{H}^n} v\|_p = 1$. Since $p < s < p^*$, we get as $\tau \to \infty$

$$\mathcal{J}_{\gamma,\sigma}(\tau v) \leq \frac{1}{p}\tau^p + \gamma^- \frac{\|v\|_{\mathcal{H}_{p,\Omega}}^p}{p}\tau^p - \sigma\frac{\|v\|_{s,w}^s}{s}\tau^s - \frac{\|v\|_{p^*,\mathcal{k}}^{p^*}}{p^*}\tau^{p^*} \to -\infty. \tag{3.2.4}$$

Hence, taking $e = \tau_* v$, with $\tau_* > 0$ large enough, we obtain that $\|D_{\mathbb{H}^n} e\|_p \geq 2$ and $\mathcal{J}_{\gamma,\sigma}(e) < 0$. In particular, e depends on γ^- and, if $\mathcal{k} > 0$ a.e. in $\mathbb{H}^n$, the function e can be taken independent of σ.

Now, fix any $u \in S_0^{1,p}(\Omega)$, with $\|D_{\mathbb{H}^n} u\|_p \leq 1$. By (3.1.1), ($\mathcal{k}$) and (3.2.1) there exists a positive constant $S_{\mathcal{k}}$ such that

$$\mathcal{J}_{\gamma,\sigma}(u) \geq \frac{1}{p}\|D_{\mathbb{H}^n} u\|_p^p - \frac{\gamma^+}{p\mathcal{H}_{p,\Omega}}\|D_{\mathbb{H}^n} u\|_p^p - \frac{C_w^s \sigma}{s}\|D_{\mathbb{H}^n} u\|_p^s - S_{\mathcal{k}}\|D_{\mathbb{H}^n} u\|_p^{p^*}.$$

Thus, setting

$$\eta_{\gamma,\sigma}(\tau) = \left(\frac{1}{p} - \frac{\gamma^+}{p\mathcal{H}_{p,\Omega}}\right)\tau^p - \frac{C_w^s \sigma}{s}\tau^s - S_{\mathcal{k}}\tau^{p^*}, \tag{3.2.5}$$

we find some $\rho \in (0, 1]$ so small that $\max_{\tau\in[0,1]} \eta_{\gamma,\sigma}(\tau) = \eta_{\gamma,\sigma}(\rho) > 0$, since $p < s < p^*$ and $\gamma < \mathcal{H}_{p,\Omega}$. It follows, $\mathcal{J}_{\gamma,\sigma}(u) \geq \jmath = \eta_{\gamma,\sigma}(\rho) > 0$ for any $u \in S_0^{1,p}(\Omega)$, with $\|D_{\mathbb{H}^n} u\|_p = \rho$. □

Note that the function e, obtained in Lemma 3.2.2 at some $\gamma \in (-\infty, \mathcal{H}_{p,\Omega})$ and $\sigma_0 > 0$, is such that $\mathcal{J}_{\gamma,\sigma}(e) < 0$ and $\|D_{\mathbb{H}^n} e\|_p \geq 2 > \rho = \rho(\gamma, \sigma)$ for all $\sigma \geq \sigma_0$, since $\rho \in (0, 1]$.

In order to study the compactness property for the functional $\mathcal{J}_{\gamma,\sigma}$, we use the Palais–Smale condition at a suitable mountain pass level $c_{\gamma,\sigma}$. For this, we fix $\gamma \in (-\infty, \mathcal{H}_{p,\Omega})$, $\sigma > 0$ and put

$$c_{\gamma,\sigma} = \inf_{\xi\in\Gamma} \max_{\tau\in[0,1]} \mathcal{J}_{\gamma,\sigma}(\xi(\tau)), \tag{3.2.6}$$

where

$$\Gamma = \{\xi \in C([0, 1], S_0^{1,p}(\Omega)) \,:\, \xi(0) = 0,\ \xi(1) = e\}. \tag{3.2.7}$$

By Lemma 3.2.2, we clearly have $c_{\gamma,\sigma} > 0$. We recall that $(u_k)_k \subset S_0^{1,p}(\Omega)$ is a *Palais–Smale sequence, briefly* (PS) *sequence, for* $\mathcal{J}_{\gamma,\sigma}$ *at level* $c_{\gamma,\sigma} \in \mathbb{R}$ if

$$\mathcal{J}_{\gamma,\sigma}(u_k) \to c_{\gamma,\sigma} \quad \text{and} \quad \mathcal{J}'_{\gamma,\sigma}(u_k) \to 0 \quad \text{as } k \to \infty. \tag{3.2.8}$$

We say that $\mathcal{J}_{\gamma,\sigma}$ *satisfies the* (PS) *condition at level* $c_{\gamma,\sigma}$ if any (PS) sequence $(u_k)_k$ at level $c_{\gamma,\sigma}$ admits a convergent subsequence in $S_0^{1,p}(\Omega)$.

Before proving the relatively compactness of the (PS) sequences of $\mathcal{J}_{\gamma,\sigma}$, we introduce an asymptotic property for the level $c_{\gamma,\sigma}$. This specific technique was also used in [23, 38] in the study of elliptic problems, involving critical Hardy nonlinearities, in the Euclidean setting. Lemma 3.2.3, similar to Lemma 2.3 in [23] and Lemma 4.3 in [38], is a key tool to obtain (3.1.2) and to overcome the lack of compactness due to the presence of the Hardy term and the critical nonlinearity.

Lemma 3.2.3. *For all* $\gamma \in (-\infty, \mathcal{H}_{p,\Omega})$ *it results*

$$\lim_{\sigma\to\infty} c_{\gamma,\sigma} = 0.$$

Proof. Fix $\gamma \in (-\infty, \mathcal{H}_{p,\Omega})$ and $\sigma_0 > 0$. Let $e \in S_0^{1,p}(\Omega)$ be the function obtained by Lemma 3.2.2, depending on γ^- and possibly on σ_0. Hence $\mathcal{J}_{\gamma,\sigma}$ satisfies the mountain pass geometry at 0 and e for all $\sigma \geq \sigma_0$. Thus there exists $\tau_{\gamma,\sigma} > 0$ such that $\mathcal{J}_{\gamma,\sigma}(\tau_{\gamma,\sigma} e) = \max_{\tau\geq 0} \mathcal{J}_{\gamma,\sigma}(\tau e)$ and so

$$\langle \mathcal{J}'_{\gamma,\sigma}(\tau_{\gamma,\sigma} e), e\rangle = 0.$$

Hence, by (3.2.3)

$$\tag{3.2.9} \begin{aligned} \tau_{\gamma,\sigma}^{p-1}(\|D_{\mathbb{H}^n} e\|_p^p - \gamma\|e\|_{\mathcal{H}_{p,\Omega}}^p) &= \sigma\tau_{\gamma,\sigma}^{s-1}\|e\|_{s,w}^s + \tau_{\gamma,\sigma}^{p^*-1}\|e\|_{p^*,\mathcal{k}}^{p^*} \\ &\geq \sigma_0\tau_{\gamma,\sigma}^{s-1}\|e\|_{s,w}^s, \end{aligned}$$

being $\sigma \geq \sigma_0$. Then, using (3.1.1), (3.2.9) yields

$$\left(1 + \frac{\gamma^-}{\mathcal{H}_{p,\Omega}}\right)\|D_{\mathbb{H}^n} e\|_p^p \geq \sigma_0\tau_{\gamma,\sigma}^{s-p}\|e\|_{s,w}^s.$$

Consequently $\{\tau_{\gamma,\sigma}\}_{\sigma\geq\sigma_0}$ is bounded in $\mathbb{R}$, since $p < s$, $\|e\|_{s,w} > 0$ and e depends only on γ^- and σ_0 by Lemma 3.2.2.

Take now a sequence $(\sigma_k)_k \subset [\sigma_0, \infty)$ such that $\sigma_k \to \infty$ as $k \to \infty$. Clearly $(\tau_{\gamma,\sigma_k})_k$ is bounded in $\mathbb{R}$. Thus, there exist a number $\ell \geq 0$ and a subsequence, still relabeled $(\sigma_k)_k$, such that

$$\lim_{k\to\infty} \tau_{\gamma,\sigma_k} = \ell.$$

From (3.2.9) there exists $\mathcal{L}_{\gamma^-}$ such that

$$\tag{3.2.10} \sigma_k\tau_{\gamma,\sigma_k}^{s-1}\|e\|_{s,w}^s + \tau_{\gamma,\sigma_k}^{p^*-1}\|e\|_{p^*,\mathcal{k}}^{p^*} \leq \mathcal{L}_{\gamma^-}$$

for any $k \in \mathbb{N}$. We claim that $\ell = 0$. Indeed, if $\ell > 0$ we obtain

$$\lim_{k\to\infty}\left(\sigma_k\tau_{\gamma,\sigma_k}^{s-1}\|e\|_{s,w}^s + \tau_{\gamma,\sigma_k}^{p^*-1}\|e\|_{p^*,\mathcal{k}}^{p^*}\right) = \infty,$$

which contradicts (3.2.10). Hence $\ell = 0$ and

$$\tag{3.2.11} \lim_{\sigma\to\infty} \tau_{\gamma,\sigma} = 0,$$

since the sequence $(\sigma_k)_k$ is arbitrary.

Consider now the path $\xi(\tau) = \tau e$, $\tau \in [0,1]$, belonging to Γ. By Lemma 3.2.2 and (3.2.11) we get

$$0 < c_{\gamma,\sigma} \le \max_{\tau\in[0,1]} \mathcal{J}_{\gamma,\sigma}(\tau e) \le \mathcal{J}_{\gamma,\sigma}(\tau_{\gamma,\sigma} e) \le \frac{1}{p}\tau_{\gamma,\sigma}^p \left(1 + \frac{\gamma^-}{\mathcal{H}_{p,\Omega}}\right) \|D_{\mathbb{H}^n} e\|_p^p \to 0$$

as $\sigma \to \infty$. This completes the proof of the lemma, since e depends only on γ^- and σ_0. □

Before verifying that $\mathcal{J}_{\gamma,\sigma}$ satisfies the (PS) condition at level $c_{\gamma,\sigma}$, we shall prove an essential lemma, inspired by Lemma 4.2 of [15] and Lemma 3.3 of [82] for systems in the Heisenberg context. We also refer to Lemma 3.8 of [3] for general quasilinear problems in the Euclidean setting.

Lemma 3.2.4. *Let γ and σ be two fixed parameters and let $(u_k)_k$ be a bounded sequence in $S_0^{1,p}(\Omega)$. Put*

$$k \mapsto g_k(q) = \gamma\,\psi(q)^p \cdot \frac{|u_k|^{p-2}u_k}{r^p} + \sigma w(q)|u_k|^{s-2}u_k + \mathcal{k}(q)|u_k|^{p^*-2}u_k. \tag{3.2.12}$$

For all compact set $\mathcal{K} \subset \Omega$ there exists $C_{\mathcal{K}} > 0$ such that

$$\sup_k \int_{\mathcal{K}} |g_k(q)|\,dq \le C_{\mathcal{K}}.$$

Proof. Fix γ, σ and $(u_k)_k$ as in the statement. Let $\mathcal{K} \subset \Omega$ be a compact set. Concerning the first term,

$$\int_{\mathcal{K}} \left(\frac{\psi}{r}\right)^p |u_k|^{p-1} dq \le \|\psi/r\|_p \sup_k \|u_k\|_{\mathcal{H}_{p,\Omega}}^{p-1} = C_1,$$

and $C_1 = C_1(\mathcal{K})$ thanks to (1.2.2). Indeed, $\psi^p r^{-p}$ is of class $L^1_{\text{loc}}(\mathbb{H}^n)$, since $\psi = |\psi| \le 1$ and $1 < p < Q$. Similarly, by Hölder's inequality and (w), we obtain

$$\int_{\mathcal{K}} w(q)|u_k|^{s-1} dq \le |\mathcal{K}|^{1/p^*} \|w\|_p \sup_k \|u_k\|_{p^*}^{s-1} = C_2,$$

and $C_2 = C_2(\mathcal{K})$. Finally, since $\mathcal{k} \in L^\infty(\mathbb{H}^n) \subset L^1_{\text{loc}}(\mathbb{H}^n)$ and $(\|u_k\|_{p^*,\mathcal{k}})_k$ is bounded by (1.2.1) and $(\mathcal{k})$, then

$$\int_{\mathcal{K}} \mathcal{k}(q)|u_k|^{p^*-1} dq \le \left(\int_{\mathcal{K}} \mathcal{k}(q) dq\right)^{1/p^*} \sup_k \|u_k\|_{p^*,\mathcal{k}}^{p^*-1} = C_3,$$

with $C_3 = C_3(\mathcal{K})$. This completes the proof. □

Now, we are ready to show the validity of the (PS) condition for $\mathcal{J}_{\gamma,\sigma}$ at level $c_{\gamma,\sigma}$. This is the key point, since in problem $(\mathcal{P}_3)$ compactness is not guaranteed a priori.

Lemma 3.2.5. *Let $\gamma \in (-\infty, \mathcal{H}_{p,\Omega})$ be fixed. If $\|\mathcal{K}\|_\infty = 0$, then the functional $\mathcal{J}_{\gamma,\sigma}$ satisfies the (PS) condition at level $c_{\gamma,\sigma}$ for all $\sigma > 0$. While if $\|\mathcal{K}\|_\infty > 0$, then there exists $\sigma^* = \sigma^*(\gamma) > 0$ such that $\mathcal{J}_{\gamma,\sigma}$ satisfies the (PS) condition at level $c_{\gamma,\sigma}$ for any $\sigma \geq \sigma^*$.*

Proof. Fix $\gamma \in (-\infty, \mathcal{H}_{p,\Omega})$, take $\sigma > 0$ and let $(u_k)_k \subset S_0^{1,p}(\Omega)$ be a (PS) sequence for $\mathcal{J}_{\gamma,\sigma}$ at level $c_{\gamma,\sigma}$. By (3.1.1), (3.2.3)

$$
\begin{aligned}
\mathcal{J}_{\gamma,\sigma}(u_k) &- \frac{1}{s}\langle \mathcal{J}'_{\gamma,\sigma}(u_k), u_k\rangle \\
&= \left(\frac{1}{p} - \frac{1}{s}\right)\|D_{\mathbb{H}^n} u_k\|_p^p + \gamma\left(\frac{1}{s} - \frac{1}{p}\right)\|u_k\|_{\mathcal{H}_{p,\Omega}}^p \\
&\quad + \left(\frac{1}{s} - \frac{1}{p^*}\right)\|u_k\|_{p^*,\mathcal{K}}^{p^*} \\
&\geq \left(\frac{1}{p} - \frac{1}{s}\right)\|D_{\mathbb{H}^n} u_k\|_p^p - \gamma^+\left(\frac{1}{p} - \frac{1}{s}\right)\|u_k\|_{\mathcal{H}_{p,\Omega}}^p \\
&\quad + \left(\frac{1}{s} - \frac{1}{p^*}\right)\|u_k\|_{p^*,\mathcal{K}}^{p^*} \\
&\geq \left(\frac{1}{p} - \frac{1}{s}\right)\left(1 - \frac{\gamma^+}{\mathcal{H}_{p,\Omega}}\right)\|D_{\mathbb{H}^n} u_k\|_p^p,
\end{aligned}
\tag{3.2.13}
$$

since $p < s < p^*$. From (3.2.8) and (3.2.13), there exists $\beta_{\gamma,\sigma}$ such that as $k \to \infty$

$$
\begin{gathered}
c_{\gamma,\sigma} + \beta_{\gamma,\sigma}\|D_{\mathbb{H}^n} u_k\|_p + o(1) \geq \mu_\gamma \|D_{\mathbb{H}^n} u_k\|_p^p, \\
\mu_\gamma = \left(\frac{1}{p} - \frac{1}{s}\right)\left(1 - \frac{\gamma^+}{\mathcal{H}_{p,\Omega}}\right) > 0,
\end{gathered}
\tag{3.2.14}
$$

being $\gamma < \mathcal{H}_{p,\Omega}$. Therefore, $(u_k)_k$ is bounded in $S_0^{1,p}(\Omega)$, so that (3.2.8) and (3.2.13) yield at once

$$
c_{\gamma,\sigma} + o(1) \geq \mu_\gamma \|D_{\mathbb{H}^n} u_k\|_p^p + \left(\frac{1}{s} - \frac{1}{p^*}\right)\|u_k\|_{p^*,\mathcal{K}}^{p^*},
\tag{3.2.15}
$$

where $\mu_\gamma > 0$ is defined in (3.2.14).

Lemma 3.2.1 gives the existence of $u_{\gamma,\sigma} \in S_0^{1,p}(\Omega)$ such that, up to a subsequence, still relabeled $(u_k)_k$,

$$
u_k \rightharpoonup u_{\gamma,\sigma} \text{ in } S_0^{1,p}(\Omega), \qquad \|D_{\mathbb{H}^n} u_k\|_p \to \kappa_{\gamma,\sigma},
$$

$$
(3.2.16)\quad
\begin{aligned}
&u_k \rightharpoonup u_{\gamma,\sigma} \text{ in } L^{p^*}(\Omega), && \|u_k - u_{\gamma,\sigma}\|_{p^*,\mathcal{K}} \to \ell_{\gamma,\sigma},\\
&u_k \rightharpoonup u_{\gamma,\sigma} \text{ in } L^{p}(\Omega, \psi^p r^{-p}), && \|u_k - u_{\gamma,\sigma}\|_{\mathcal{H}_{p,\Omega}} \to \imath_{\gamma,\sigma},\\
&u_k \to u_{\gamma,\sigma} \text{ in } L^{s}(\Omega, w), && u_k \to u_{\gamma,\sigma} \text{ a.e. in } \Omega,\\
&D_{\mathbb{H}^n} u_k \rightharpoonup D_{\mathbb{H}^n} u_{\gamma,\sigma} \text{ in } L^{p}(\Omega;\mathbb{R}^{2n}),\\
&|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_k \rightharpoonup \Theta \text{ in } L^{p'}(\Omega;\mathbb{R}^{2n})
\end{aligned}
$$

hold for some $\Theta \in L^{p'}(\Omega;\mathbb{R}^{2n})$. Clearly $\kappa_{\gamma,\sigma} > 0$ since $c_{\gamma,\sigma} > 0$. First, we claim that

$$
(3.2.17)\qquad \lim_{\sigma\to\infty} \kappa_{\gamma,\sigma} = 0.
$$

Otherwise, $\limsup_{\sigma\to\infty} \kappa_{\gamma,\sigma} = \kappa_\gamma > 0$. Consequently there is a sequence, say $k \to \sigma_k \uparrow \infty$ such that $\kappa_{\gamma,\sigma_k} \to \kappa_\gamma$ as $k \to \infty$. Thus, letting $k \to \infty$ in (3.2.15) from Lemma 3.2.3 we get that

$$
0 \geq \mu_\gamma \kappa_\gamma^p > 0,
$$

which is the desired contradiction and proves the assertion (3.2.17).

Now, $\|D_{\mathbb{H}^n} u_{\gamma,\sigma}\|_p \leq \lim_{k\to\infty} \|D_{\mathbb{H}^n} u_k\|_p = \kappa_{\gamma,\sigma}$ since $u_k \rightharpoonup u_{\gamma,\sigma}$ in $S_0^{1,p}(\Omega)$, so that (3.1.1), ($\mathcal{K}$) and (3.2.17) imply at once

$$
(3.2.18)\qquad \lim_{\sigma\to\infty} \|u_{\gamma,\sigma}\|_{p^*,\mathcal{K}} = \lim_{\sigma\to\infty} \|u_{\gamma,\sigma}\|_{\mathcal{H}_{p,\Omega}} = \lim_{\sigma\to\infty} \|D_{\mathbb{H}^n} u_{\gamma,\sigma}\|_p = 0.
$$

By (3.2.8) we have as $k \to \infty$

$$
(3.2.19)\qquad o(1) = \langle u_k, \phi\rangle_p - \gamma\langle u_k, \phi\rangle_{\mathcal{H}_p} - \sigma\langle u_k, \phi\rangle_{s,w} - \langle u_k, \phi\rangle_{p^*,\mathcal{K}}
$$

for any $\phi \in S_0^{1,p}(\Omega)$. By Proposition A.8 in [3], that still holds in the Heisenberg group, we obtain

$$
\begin{aligned}
|u_k|^{s-2}u_k &\rightharpoonup |u_{\gamma,\sigma}|^{s-2}u_{\gamma,\sigma} && \text{in } L^{s'}(\Omega, w),\\
|u_k|^{p-2}u_k &\rightharpoonup |u_{\gamma,\sigma}|^{p-2}u_{\gamma,\sigma} && \text{in } L^{p'}(\Omega, \psi^p r^{-p}) \quad \text{and}\\
|u_k|^{p^*-2}u_k &\rightharpoonup |u_{\gamma,\sigma}|^{p^*-2}u_{\gamma,\sigma} && \text{in } L^{{p^*}'}(\Omega, \mathcal{K}).
\end{aligned}
$$

Indeed, Proposition A.8 in [3] can be applied, since the weight function $\psi^p r^{-p}$ is of class $L^1_{\mathrm{loc}}(\mathbb{H}^n)$, being $\psi = |\psi| \leq 1$ and $1 < p < Q$.

We claim that, up to a subsequence if necessary,

$$
(3.2.20)\qquad D_{\mathbb{H}^n} u_k \to D_{\mathbb{H}^n} u_{\gamma,\sigma} \quad \text{a.e. in } \Omega.
$$

To show the claim, we shall follow the proofs of Lemma 4.3 of [15] and of Lemma 3.4 of [82] for systems in the Heisenberg context. See also the proofs of Theorem 2.1 of [9], of Lemma 2 of [33] and of Step 1 of Theorem 4.4 of [3] in the Euclidean setting.

Fix $R > 0$. Let $\phi_R \in C_0^\infty(\Omega)$ be such that $0 \le \phi_R \le 1$ in Ω and $\phi_R \equiv 1$ in $B_R \cap \Omega$. Given $\varepsilon > 0$ define for each $\tau \in \mathbb{R}$

$$\eta_\varepsilon(\tau) = \begin{cases} \tau, & \text{if } |\tau| < \varepsilon, \\ \varepsilon \dfrac{\tau}{|\tau|}, & \text{if } |\tau| \ge \varepsilon. \end{cases}$$

Put $v_k = \phi_R \eta_\varepsilon \circ (u_k - u_{\gamma,\sigma})$, so that $v_k \in S_0^{1,p}(\Omega)$. Taking $\phi = v_k$ in (3.2.19), we get

$$\begin{aligned} \int_\Omega \phi_R \big(|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u_{\gamma,\sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma,\sigma}, \\ D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma})) \big)_{\mathbb{H}^n} dq \\ = - \int_\Omega \eta_\varepsilon \circ (u_k - u_{\gamma,\sigma}) \big(|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_k, D_{\mathbb{H}^n} \phi_R \big)_{\mathbb{H}^n} dq \\ - \int_\Omega \phi_R \big(|D_{\mathbb{H}^n} u_{\gamma,\sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma,\sigma}, \\ D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma})) \big)_{\mathbb{H}^n} dq \\ + \langle \mathcal{J}'_{\gamma,\sigma}(u_k), v_k \rangle + \int_{\mathbb{H}^n} g_k v_k dq. \end{aligned} \tag{3.2.21}$$

Observe now that

$$\int_\Omega \eta_\varepsilon \circ (u_k - u_{\gamma,\sigma}) \big(|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_k, D_{\mathbb{H}^n} \phi_R \big)_{\mathbb{H}^n} dq \to 0 \quad \text{as } k \to \infty,$$

since $|\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma}) D_{\mathbb{H}^n} \phi_R|_{\mathbb{H}^n} \to 0$ in $L^p(\operatorname{supp} \phi_R)$ and by the fact that $|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_k \rightharpoonup \Theta$ in $L^{p'}(\Omega; \mathbb{R}^{2n})$ by (3.2.16). Furthermore, we have $D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma})) \rightharpoonup 0$ in $L^p(\Omega; \mathbb{R}^{2n})$, since $u_k \rightharpoonup u_{\gamma,\sigma}$ in $S_0^{1,p}(\Omega)$, and consequently

$$\int_\Omega \phi_R \big(|D_{\mathbb{H}^n} u_{\gamma,\sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma,\sigma}, D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma})) \big)_{\mathbb{H}^n} dq \to 0 \quad \text{as } k \to \infty,$$

being $|D_{\mathbb{H}^n} u_{\gamma,\sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma,\sigma} \in L^{p'}(\Omega; \mathbb{R}^{2n})$. Moreover

$$\langle \mathcal{J}'_{\gamma,\sigma}(u_k), v_k \rangle \to 0$$

as $k \to \infty$, since $\mathcal{J}'_{\gamma,\sigma}(u_k) \to 0$ in $[S_0^{1,p}(\Omega)]'$ and $v_k \rightharpoonup 0$ in $S_0^{1,p}(\Omega)$ as $k \to \infty$.

In conclusion, the first three terms in the right hand side of (3.2.21) go to zero as $k \to \infty$. Now, recalling that $0 \le \phi_R \le 1$ in Ω, we have

$$\int_\Omega g_k v_k dq \le \int_{\operatorname{supp}\phi_R} |g_k| \cdot |\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma})|\, dq \le \varepsilon \int_{\operatorname{supp}\phi_R} |g_k|\, dq \le \varepsilon\, C_R,$$

since $(g_k)_k$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$ by Lemma 3.2.4. By the definitions of ϕ_R and η_ε,

$$\phi_R\big(|D_{\mathbb{H}^n} u_k|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u_{\gamma,\sigma}|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_{\gamma,\sigma}, D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma}))\big)_{\mathbb{H}^n} \ge 0$$

a.e. in Ω, and in turn

$$\begin{aligned}\int_{B_R\cap\Omega} &\phi_R\big(|D_{\mathbb{H}^n} u_k|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u_{\gamma,\sigma}|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_{\gamma,\sigma},\\ &\qquad D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma}))\big)_{\mathbb{H}^n} dq\\ &\le \int_\Omega \phi_R\big(|D_{\mathbb{H}^n} u_k|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u_{\gamma,\sigma}|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_{\gamma,\sigma},\\ &\qquad D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma}))\big)_{\mathbb{H}^n} dq.\end{aligned}$$

Combining all these facts with (3.2.21), we find that

$$\begin{aligned}\limsup_{k\to\infty} \int_{B_R\cap\Omega} &\phi_R\big(|D_{\mathbb{H}^n} u_k|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u_{\gamma,\sigma}|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_{\gamma,\sigma},\\ &\qquad D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma}))\big)_{\mathbb{H}^n} dq \le \varepsilon\, C_R.\end{aligned} \tag{3.2.22}$$

Define the nonnegative function e_k by

$$e_k(q) = \big(|D_{\mathbb{H}^n} u_k|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u_{\gamma,\sigma}|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_{\gamma,\sigma}, D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})\big)_{\mathbb{H}^n}.$$

Note that $(e_k)_k$ is bounded in $L^1(\Omega)$. Indeed,

$$\begin{aligned}0 \le \int_\Omega e_k(q)\, dq \le \big\||D_{\mathbb{H}^n} u_k|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u_{\gamma,\sigma}|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_{\gamma,\sigma}\big\|_{p'} \cdot\\ \times \|D_{\mathbb{H}^n} u_k - D_{\mathbb{H}^n} u_{\gamma,\sigma}\|_p \le C_0,\end{aligned} \tag{3.2.23}$$

where C_0 is an appropriate constant, independent of k, which derives from the boundedness of $(D_{\mathbb{H}^n} u_k)_k$ in $L^p(\Omega; \mathbb{R}^{2n})$ and of $(|D_{\mathbb{H}^n} u_k|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u_k)_k$ in $L^{p'}(\Omega; \mathbb{R}^{2n})$ as shown above.

Fix $\theta \in (0,1)$. Split the ball B_R into

$$S^\varepsilon_k(R) = \{q \in B_R \cap \Omega \,:\, |u_k(q) - u_{\gamma,\sigma}(q)| \le \varepsilon\}, \quad G^\varepsilon_k(R) = (B_R \cap \Omega) \setminus S^\varepsilon_k(R).$$

By Hölder's inequality,

$$\int_{B_R\cap\Omega} e_k^\theta dq \le \left(\int_{S_k^\varepsilon(R)} e_k dq\right)^\theta |S_k^\varepsilon(R)|^{1-\theta} + \left(\int_{G_k^\varepsilon(R)} e_k dq\right)^\theta |G_k^\varepsilon(R)|^{1-\theta}$$
$$\le (\varepsilon C_R)^\theta |S_k^\varepsilon(R)|^{1-\theta} + C_0^\theta |G_k^\varepsilon(R)|^{1-\theta},$$

by (3.2.22), since $\phi_R \equiv 1$ and $D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u_{\gamma,\sigma})) = D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})$ in $S_k^\varepsilon(R)$, and by (3.2.23). Moreover, $|G_k^\varepsilon(R)|$ tends to zero as $k \to \infty$. Hence

$$0 \le \limsup_{k\to\infty} \int_{B_R\cap\Omega} e_k^\theta\, dq \le (\varepsilon C_R)^\theta |B_R|^{1-\theta}.$$

Letting ε tend to 0^+ we find that $e_k^\theta \to 0$ in $L^1(B_R \cap \Omega)$ and so, thanks to the arbitrariness of R, we deduce

$$e_k \to 0 \quad \text{a.e. in } \Omega$$

up to a subsequence. From Lemma 3 of [33] it follows the validity of (3.2.20), proving the claim.

In particular, $|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_k \to |D_{\mathbb{H}^n} u_{\gamma,\sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma,\sigma}$ a.e. in Ω. Hence, Proposition A.7 of [3] implies

$$\Theta = |D_{\mathbb{H}^n} u_{\gamma,\sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma,\sigma} \quad \text{a.e. in} \quad \Omega.$$

Consequently for all $\phi \in S_0^{1,p}(\Omega)$

$$\int_\Omega \left(|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_k, D_{\mathbb{H}^n}\phi\right)_{\mathbb{H}^n} dq \to \int_\Omega \left(|D_{\mathbb{H}^n} u_{\gamma,\sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma,\sigma}, D_{\mathbb{H}^n}\phi\right)_{\mathbb{H}^n} dq \tag{3.2.24}$$

as $k \to \infty$, since $|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_k \rightharpoonup |D_{\mathbb{H}^n} u_{\gamma,\sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma,\sigma}$ in $L^{p'}(\Omega; \mathbb{R}^{2n})$.

Finally, by (3.2.19) and (3.2.24) we derive that the weak limit $u_{\gamma,\sigma}$ is a critical point of the $C^1(S_0^{1,p}(\Omega))$ functional $\mathcal{J}_{\gamma,\sigma}$.

Therefore, (3.2.8), (3.2.16) and (3.2.24) imply that as $k \to \infty$

$$\begin{aligned} o(1) &= \langle \mathcal{J}'_{\gamma,\sigma}(u_k) - \mathcal{J}'_{\gamma,\sigma}(u_{\gamma,\sigma}), u_k - u_{\gamma,\sigma}\rangle \\ &= \|D_{\mathbb{H}^n} u_k\|_p^p + \|D_{\mathbb{H}^n} u_{\gamma,\sigma}\|_p^p - 2\langle u_k, u_{\gamma,\sigma}\rangle_p \\ &\quad - \gamma \int_\Omega (|u_k|^{p-2}u_k - |u_{\gamma,\sigma}|^{p-2}u_{\gamma,\sigma})(u_k - u_{\gamma,\sigma})\psi^p \frac{dq}{r^p} \\ &\quad - \sigma \int_\Omega w(q)(|u_k|^{s-2}u_k - |u_{\gamma,\sigma}|^{s-2}u_{\gamma,\sigma})(u_k - u_{\gamma,\sigma})dq \end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \mathcal{K}(q)(|u_k|^{p^*-2}u_k - |u_{\gamma,\sigma}|^{p^*-2}u_{\gamma,\sigma})(u_k - u_{\gamma,\sigma})dq \\
&= \kappa^p_{\gamma,\sigma} - \|D_{\mathbb{H}^n} u_{\gamma,\sigma}\|^p_p - \gamma\|u_k\|^p_{\mathcal{H}_{p,\Omega}} + \gamma\|u_{\gamma,\sigma}\|^p_{\mathcal{H}_{p,\Omega}} \\
&\quad - \|u_k\|^{p^*}_{p^*,\mathcal{K}} + \|u_{\gamma,\sigma}\|^{p^*}_{p^*,\mathcal{K}} + o(1) \\
&= \|D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})\|^p_p - \gamma\|u_k - u_{\gamma,\sigma}\|^p_{\mathcal{H}_{p,\Omega}} - \|u_k - u_{\gamma,\sigma}\|^{p^*}_{p^*,\mathcal{K}} + o(1).
\end{aligned}
$$

Indeed, thanks to (3.2.16) it results

$$
\lim_{k\to\infty} \int_\Omega w(q)(|u_k|^{s-2}u_k - |u_{\gamma,\sigma}|^{s-2}u_{\gamma,\sigma})(u_k - u_{\gamma,\sigma})dq = 0,
$$

and futhermore, by (3.2.16), (3.2.20) and the celebrated Brézis & Lieb lemma, see [21],

$$
\begin{aligned}
\|D_{\mathbb{H}^n} u_k\|^p_p &= \|D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})\|^p_p + \|D_{\mathbb{H}^n} u_{\gamma,\sigma}\|^p_p + o(1), \\
\|u_k\|^{p^*}_{p^*,\mathcal{K}} &= \|u_k - u_{\gamma,\sigma}\|^{p^*}_{p^*,\mathcal{K}} + \|u_{\gamma,\sigma}\|^{p^*}_{p^*,\mathcal{K}} + o(1), \\
\|u_k\|^p_{\mathcal{H}_{p,\Omega}} &= \|u_k - u_{\gamma,\sigma}\|^p_{\mathcal{H}_{p,\Omega}} + \|u_{\gamma,\sigma}\|^p_{\mathcal{H}_{p,\Omega}} + o(1),
\end{aligned}
$$

as $k \to \infty$. Finally, we have used the fact that $\|D_{\mathbb{H}^n} u_k\|_p \to \kappa_{\gamma,\sigma}$ by (3.2.16). Therefore, we have proved the crucial formula

$$
\begin{aligned}
\lim_{k\to\infty} \|D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})\|^p_p &= \kappa^p_{\gamma,\sigma} - \|D_{\mathbb{H}^n} u_{\gamma,\sigma}\|^p_p \\
&= \lim_{k\to\infty} \|u_k - u_{\gamma,\sigma}\|^{p^*}_{p^*,\mathcal{K}} + \gamma \lim_{k\to\infty} \|u_k - u_{\gamma,\sigma}\|^p_{\mathcal{H}_{p,\Omega}} \\
&= \ell^{p^*}_{\gamma,\sigma} + \gamma \imath^p_{\gamma,\sigma}.
\end{aligned} \tag{3.2.25}
$$

Let us divide the proof in two parts.

Case $\|\mathcal{K}\|_\infty = 0$. Obviously $\ell_{\gamma,\sigma} = 0$ in (3.2.25). Suppose by contradiction that $\imath_{\gamma,\sigma} > 0$. By (3.1.1) and (3.2.16), we have

$$
\begin{aligned}
\lim_{k\to\infty} \|D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})\|^p_p = \gamma \lim_{k\to\infty} \|u_k - u_{\gamma,\sigma}\|^p_{\mathcal{H}_{p,\Omega}} &< \mathcal{H}_{p,\Omega} \lim_{k\to\infty} \|u_k - u_{\gamma,\sigma}\|^p_{\mathcal{H}_{p,\Omega}} \\
&\le \lim_{k\to\infty} \|D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})\|^p_p,
\end{aligned}
$$

which is impossible. Hence, $\imath_{\gamma,\sigma} = 0$ for all $\sigma > 0$. By (3.2.25), we get

$$
\lim_{k\to\infty} \|D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})\|^p_p = \lim_{k\to\infty} \|u_k - u_{\gamma,\sigma}\|^p_{\mathcal{H}_{p,\Omega}} = 0.
$$

Thus, (3.2.25) yields $u_k \to u_{\gamma,\sigma}$ in $S^{1,p}_0(\Omega)$ as $k \to \infty$ for all $\sigma > 0$.

Case $\|\mathcal{K}\|_\infty > 0$. By (3.2.15) and the Brézis & Lieb lemma, we have as $k \to \infty$

$$c_{\gamma,\sigma} + o(1) \geq \left(\frac{1}{s} - \frac{1}{p^*}\right) \|u_k\|_{p^*,\mathcal{K}}^{p^*} = \left(\frac{1}{s} - \frac{1}{p^*}\right) \left[\ell_{\gamma,\sigma}^{p^*} + \|u_{\gamma,\sigma}\|_{p^*,\mathcal{K}}^{p^*}\right] + o(1).$$

Then, Lemma 3.2.3 and (3.2.18) imply that

$$\lim_{\sigma\to\infty} \ell_{\gamma,\sigma} = 0. \tag{3.2.26}$$

Since $\gamma < \mathcal{H}_{p,\Omega}$ there exists $\mathfrak{c} \in [0, 1)$ such that $\gamma^+ = \mathfrak{c}\mathcal{H}_{p,\Omega}$. Thus, (3.2.25) can be rewritten as

$$(1 - \mathfrak{c}) \lim_{k\to\infty} \|D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})\|_p^p + \mathfrak{c} \lim_{k\to\infty} \|D_{\mathbb{H}^n}(u_k - u_{\gamma,\sigma})\|_p^p = \ell_{\gamma,\sigma}^{p^*} + \gamma \imath_{\gamma,\sigma}^p.$$

Now, for all $\sigma > 0$ we have $\ell_{\gamma,\sigma}^{p^*} + \gamma^+ \imath_{\gamma,\sigma}^p \geq (1 - \mathfrak{c})\mathcal{H}_{0,\Omega}\|\mathcal{K}\|_\infty^{-p/p^*} \ell_{\gamma,\sigma}^p + \mathfrak{c}\mathcal{H}_{p,\Omega}\imath_{\gamma,\sigma}^p$ by $(\mathcal{K})$ and (3.1.1), being $\mathfrak{c} \in [0, 1)$. Therefore, since $\gamma^+ = \mathfrak{c}\mathcal{H}_{p,\Omega}$,

$$\ell_{\gamma,\sigma}^{p^*} \geq (1 - \mathfrak{c})\mathcal{H}_{0,\Omega}\|\mathcal{K}\|_\infty^{-p/p^*} \ell_{\gamma,\sigma}^p. \tag{3.2.27}$$

Consequently (3.2.26) and (3.2.27) imply that there exists $\sigma^* = \sigma^*(\gamma) > 0$ such that $\ell_{\gamma,\sigma} = 0$ for all $\sigma \geq \sigma^*$. In other words,

$$\lim_{k\to\infty} \|u_k - u_{\gamma,\sigma}\|_{p^*,\mathcal{K}} = 0$$

for all $\sigma \geq \sigma^*$. From now on we can proceed as in the first case, and prove that $\imath_{\gamma,\sigma} = 0$ for all $\sigma \geq \sigma^*$. Thus, using also (3.2.25), we get $u_k \to u_{\gamma,\sigma}$ in $S_0^{1,p}(\Omega)$ as $k \to \infty$ for all $\sigma \geq \sigma^*$ as desired, and the proof is complete. □

Proof of Theorem 3.1.1. Fix $\gamma < \mathcal{H}_{p,\Omega}$. Lemmas 3.2.2 and 3.2.5 guarantee that the functional $\mathcal{J}_{\gamma,\sigma}$ satisfies all the assumptions of the mountain pass theorem for any $\sigma > 0$ when $\|\mathcal{K}\|_\infty = 0$ and if $\|\mathcal{K}\|_\infty > 0$ for any $\sigma \geq \sigma^*$, with $\sigma^* = \sigma^*(\gamma) > 0$. Hence, there exists a critical point $u_{\gamma,\sigma} \in S_0^{1,p}(\Omega)$ for $\mathcal{J}_{\gamma,\sigma}$ at level $c_{\gamma,\sigma}$. Since $\mathcal{J}_{\gamma,\sigma}(u_{\gamma,\sigma}) = c_{\gamma,\sigma} > 0 = \mathcal{J}_{\gamma,\sigma}(0)$ we have that $u_{\gamma,\sigma} \not\equiv 0$. Moreover the asymptotic behavior (3.1.2) holds thanks to (3.2.18). □

3.3 A concentration–compactness result

From now until the end of this chapter assume that Ω is *a bounded PS domain* of $\mathbb{H}^n$. This section is devoted to the proof of Theorem 3.1.2, which concerns the delicate study of the exact behavior of weakly convergent sequences of $S_0^{1,p}(\Omega)$ in the space of measures. The proof follows the arguments given in the Euclidean setting in Theorem 2.5 of [78] for the case $\alpha = 0$, and in Theorem 1.1 of [39] for the case $0 < \alpha \le p$. The technique is based on the tightness of the sequence $(|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n})_k$ and on the application of Phrokorov theorem, see Theorem 8.6.2 in [10].

Proof of Theorem 3.1.2 Fix a sequence $(u_k)_k$ and u in $S_0^{1,p}(\Omega)$, with $u_k \rightharpoonup u$ in $S_0^{1,p}(\Omega)$. Clearly $u_k \to u$ in $L^p(\Omega)$, being Ω bounded. We divide the proof into two cases.

Case $\alpha = 0$. Since Ω is bounded, the sequences $(|u_k|^{p^*})_k$ and $(|D_{\mathbb{H}^n} u_k|^p_{\mathbb{H}^n})_k$ are tight, ensuring the existence of μ and ν such that (3.1.3), with $\alpha = 0$, holds by Phrokorov theorem. We come to the proof of (3.1.4)–(3.1.6), following Lemma I.1 in [63]. To this aim, fix $\varepsilon > 0$ and $\phi \in C_0^\infty(\Omega)$. Then, there exists $C_\varepsilon > 0$ such that $|\xi + \eta|^p \le (1+\varepsilon)|\xi|^p + C_\varepsilon |\eta|^p$ for all numbers $\xi, \eta \in \mathbb{R}$. Hence, the Leibnitz formula gives for all k

$$\begin{aligned} \int_\Omega |D_{\mathbb{H}^n}(v_k\phi)(q)|^p_{\mathbb{H}^n} dq \le &(1+\varepsilon)\int_\Omega |D_{\mathbb{H}^n} v_k(q)|^p_{\mathbb{H}^n} |\phi(q)|^p dq \\ &+ C_\varepsilon \int_\Omega |D_{\mathbb{H}^n}\phi(q)|^p_{\mathbb{H}^n} |v_k(q)|^p dq. \end{aligned} \tag{3.3.1}$$

The Sobolev inequality (3.1.1) and (3.3.1) yield

$$\begin{aligned} \mathcal{H}_{0,\Omega} \|u_k\phi\|_{p^*}^{p/p^*} \le (1+\varepsilon) &\int_\Omega |D_{\mathbb{H}^n} u_k|^p_{\mathbb{H}^n} |\phi|^p dq \\ + C_\varepsilon &\int_\Omega |D_{\mathbb{H}^n}\phi|^p_{\mathbb{H}^n} |u_k|^p dq \end{aligned} \tag{3.3.2}$$

for any $\phi \in C_0^\infty(\Omega)$.

Let us first suppose that $u \equiv 0$. Letting $k \to \infty$ in (3.3.2) and using that $u_k \to 0$ in $L^p(\Omega)$ and $|D_{\mathbb{H}^n}\phi|^p_{\mathbb{H}^n} \in L^\infty(\Omega)$, we have

$$\mathcal{H}_{0,\Omega} \left(\int_\Omega |\phi|^{p^*} d\nu \right)^{p/p^*} \le (1+\varepsilon) \int_\Omega |\phi|^p d\mu \tag{3.3.3}$$

for any $\phi \in C_0^\infty(\Omega)$. Inequality (3.3.3) immediately implies (3.1.4) and (3.1.5), thanks to Lemma I.1 in [63]. To obtain (3.1.6), fix q_j, and for any $\delta > 0$, let $\phi_\delta \in C_0^\infty(B_{2\delta}(q_j))$ satisfy

$$0 \le \phi_\delta \le 1, \quad \phi\big|_{B_\delta} = 1, \quad |D_{\mathbb{H}^n}\phi_\delta|_\infty \le C/\delta. \tag{3.3.4}$$

The choice $\phi = \phi_\delta$ in (3.3.3) immediately gives that for any $j \in \Lambda$

$$\mathcal{H}_{0,\Omega} \left(\nu(B_\delta(q_j))\right)^{p/p^*} \le (1+\varepsilon)\mu(B_{2\delta}(q_j)). \tag{3.3.5}$$

Then, letting first $\delta \to 0^+$ and then $\varepsilon \to 0^+$ in (3.3.5), we get (3.1.6).

To prove the case $u \neq 0$, let us first note that (3.3.2) as $k \to \infty$ reduces to

$$(3.3.6) \quad \mathcal{H}_{0,\Omega} \left(\int_\Omega |\phi|^{p^*} d\nu \right)^{p/p^*} \leq (1+\varepsilon) \int_\Omega |\phi|^p d\mu + C_\varepsilon \int_\Omega |D_{\mathbb{H}^n} \phi|^p_{\mathbb{H}^n} |u|^p dq.$$

Now we can proceed as in [63] and obtain (3.1.5).

Concerning (3.1.4), we first claim that $d\mu \geq |D_{\mathbb{H}^n} u|^p_{\mathbb{H}^n}\, dq$. Indeed, for any $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, the functional

$$v \mapsto \int_\Omega |D_{\mathbb{H}^n} v|^p_{\mathbb{H}^n} \phi \, dq$$

is convex and continuous, therefore the fact that $u_k \rightharpoonup u$ in $S_0^{1,p}(\Omega)$ implies that for any $\phi \in C_0^\infty(\Omega)$, with $\phi \geq 0$,

$$\int_\Omega \phi \, d\mu = \lim_{k\to\infty} \int_\Omega |D_{\mathbb{H}^n} u_k|^p_{\mathbb{H}^n} \phi \, dq \geq \int_\Omega |D_{\mathbb{H}^n} u|^p_{\mathbb{H}^n} \phi \, dq.$$

On the other hand (3.3.6) gives $\mu(\{q_j\}) > 0$, whenever $\nu(\{q_j\}) > 0$. Hence (3.1.4) holds.

Finally, let us prove (3.1.6). Clearly, $u \in L^{p^*}(\Omega)$ so that the Hölder inequality provides at once that

$$\lim_{\delta\to 0^+} \int_\Omega |u|^p |D_{\mathbb{H}^n} \phi_\delta|^p_{\mathbb{H}^n} dq = 0,$$

where ϕ_δ is the cutoff function introduced in (3.3.4). Since $\phi_\delta = 1$ in B_δ, we obtain

$$\mathcal{H}_{0,\Omega}(\nu(B_\delta))^{p/p^*} \leq (1+\varepsilon)\mu(B_{2\delta}) + C_\varepsilon o(1).$$

Thus, we have shown the validity of (3.1.6), by letting first $\delta \to 0$ and then $\varepsilon \to 0$.

Case $0 < \alpha \leq p$. As noted above in Lemma 1.2.1, the given sequence $(u_k)_k$ converges weakly to u also in $L^{p^*(\alpha)}(\Omega, \psi^\alpha r^{-\alpha})$. In particular, there exist two positive measures μ and ν in $\mathbb{H}^n$ such that (3.1.3) holds, being the measures in $\mathbb{H}^n$

$$k \mapsto |D_{\mathbb{H}^n} u_k(q)|^p_{\mathbb{H}^n} dq, \qquad k \mapsto |u_k(q)|^{p^*(\alpha)} \psi^\alpha \frac{dq}{r^\alpha}$$

uniformly tight in k.

Put $v_k = u_k - u$. Clearly, $v_k \rightharpoonup 0$ in $S_0^{1,p}(\Omega)$ as $k \to \infty$. As observed above, we get the existence of two positive measures $\hat\mu$ and $\hat\nu$ on $\mathbb{H}^n$ such that

$$(3.3.7) \qquad |D_{\mathbb{H}^n} v_k(q)|^p_{\mathbb{H}^n} dq \overset{*}{\rightharpoonup} \hat\mu \quad \textit{and} \quad |v_k(q)|^{p^*(\alpha)} \psi^\alpha \frac{dq}{r^\alpha} \overset{*}{\rightharpoonup} \hat\nu \qquad \text{in } \mathcal{M}(\Omega).$$

By (1.2.7), being $0 < \alpha \le p$ the sequence $(u_k)_k$ strongly converges to u in $L^p(\Omega)$, being Ω a bounded PS domain. Thus Theorem 4.9 of [20] implies that there exists $h \in L^p(\Omega)$ such that, up to a subsequence, still named $(u_k)_k$,

$$u_k \to u \text{ a.e. in } \Omega, \quad |u_k| \le h \text{ a.e. in } \Omega \text{ and for all } k. \tag{3.3.8}$$

Hence, for any $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned}\int_\Omega |\phi(q)|^{p^*(\alpha)} d\nu - \|\phi u\|^{p^*(\alpha)}_{\mathcal{H}_{\alpha,\Omega}} &= \lim_{k\to\infty} \|\phi u_k\|^{p^*(\alpha)}_{\mathcal{H}_{\alpha,\Omega}} - \|\phi u\|^{p^*(\alpha)}_{\mathcal{H}_{\alpha,\Omega}} \\ &= \lim_{k\to\infty} \|\phi v_k\|^{p^*(\alpha)}_{\mathcal{H}_{\alpha,\Omega}} = \int_\Omega |\phi(q)|^{p^*(\alpha)} d\hat{\nu}\end{aligned}$$

by the Brézis & Lieb lemma, see [21]. This yields that

$$\nu = \hat{\nu} + |u(q)|^{p^*(\alpha)} \psi^\alpha \frac{dq}{r^\alpha},$$

since $\phi \in C_0^\infty(\Omega)$ is arbitrary.

Let us first prove (3.1.7). To this aim, fix $\phi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$. Then, as shown in case $\alpha = 0$, the Leibnitz formula (3.3.1) is valid for all k. Thus, the Hardy inequality (3.1.1) along the sequence $(\phi v_k)_k$ of $S_0^{1,p}(\Omega)$ yields

$$\begin{aligned}\mathcal{H}_{\alpha,\Omega} \|\phi v_k\|^p_{\mathcal{H}_{\alpha,\Omega}} \le \|D_{\mathbb{H}^n}(v_k\phi)\|_p^p \le (1+\varepsilon) \int_\Omega |D_{\mathbb{H}^n} v_k(q)|^p_{\mathbb{H}^n} |\phi(q)|^p dq \\ + C_{\varepsilon,\phi} \|v_k\|_p^p,\end{aligned} \tag{3.3.9}$$

for an appropriate constant $C_{\varepsilon,\phi} > 0$. By (3.3.7), (3.3.9) and the fact that $v_k = u_k - u \to 0$ in $L^p(\Omega)$ as $k \to \infty$, we obtain at once that

$$\left(\int_\Omega |\phi(q)|^{p^*(\alpha)} d\hat{\nu} \right)^{p/p^*(\alpha)} \le \frac{1+\varepsilon}{\mathcal{H}_{\alpha,\Omega}} \int_\Omega |\phi(q)|^p d\hat{\mu},$$

that is $\hat{\nu}$ is absolutely continuous with respect to $\hat{\mu}$. Hence, by Lemma 1.2 of [63] the measure $\hat{\nu}$ is decomposed as sum of Dirac masses.

It remains to show that $\hat{\nu}$ is concentrated at O. Here we assume that $O \notin \operatorname{supp}\phi$, so that $|\phi(q)|^{p^*(\alpha)}\psi^\alpha / r^\alpha$ is in $L^\infty(\operatorname{supp}\phi)$. In turn, (1.2.7) yields

$$\|\phi v_k\|^{p^*(\alpha)}_{\mathcal{H}_{\alpha,\Omega}} = \int_{\operatorname{supp}\phi} \psi^\alpha \frac{|\phi(q)|^{p^*(\alpha)}}{r^\alpha} |v_k(q)|^{p^*(\alpha)} dq \le C \int_{\operatorname{supp}\phi} |v_k(q)|^{p^*(\alpha)} dq \to 0$$

as $k \to \infty$, since $0 < \alpha \le p$, so that $p \le p^*(\alpha) < p^*$. This, combined with (3.3.7), gives $\int_\Omega |\phi(q)|^{p^*(\alpha)} d\hat{\nu} = 0$. In other words, $\hat{\nu}$ is a measure concentrated in O. Hence $\hat{\nu} = \nu_0 \delta_0$, and (3.1.7) is so proved.

In order to show (3.1.8), arguing as in (3.3.9), replacing v_k by u_k and letting $k \to \infty$, we have

$$\mathcal{H}_{\alpha,\Omega}\left(\int_\Omega |\phi(q)|^{p^*(\alpha)}d\nu\right)^{p/p^*(\alpha)} \le (1+\varepsilon)\int_\Omega |\phi(q)|^p d\mu + C_\varepsilon \int_\Omega |D_{\mathbb{H}^n}\phi(q)|^p_{\mathbb{H}^n}|u(q)|^p dq \tag{3.3.10}$$

by (3.1.3) and (3.3.8).

Let now $\phi \in C_0^\infty(\Omega)$, with $0 \le \phi \le 1$, $\phi(0) = 1$, $\mathrm{Supp}(\phi) = B_1$ and put $\phi_{\tilde\varepsilon}(q) = \phi(q/\tilde\varepsilon)$ for $\tilde\varepsilon > 0$ sufficiently small. Since $\nu \ge \nu_0\delta_0$, choosing $\phi_{\tilde\varepsilon}$ as test function in (3.3.10), we obtain

$$0 \le \mathcal{H}_{\alpha,\Omega}\nu_0^{p/p^*(\alpha)} \le (1+\varepsilon)\mu(B_{\tilde\varepsilon}) + C_\varepsilon \int_\Omega |u(q)|^p |D_{\mathbb{H}^n}\phi_{\tilde\varepsilon}(q)|^p_{\mathbb{H}^n} dq. \tag{3.3.11}$$

By Hölder inequality, the last term of the right–hand side of (3.3.11) goes to 0 as $\tilde\varepsilon \to 0^+$. Hence, letting $\tilde\varepsilon \to 0^+$ and $\varepsilon \to 0^+$ in (3.3.11), we have $0 \le \mathcal{H}_{\alpha,\Omega}\nu_0^{p/p^*(\alpha)} \le \mu_0$. By the Fatou lemma $\mu \ge |D_{\mathbb{H}^n}u(q)|^p_{\mathbb{H}^n}dq$ and this concludes the proof of (3.1.8), since $|D_{\mathbb{H}^n}u(q)|^p_{\mathbb{H}^n}dq$ and $\mu_0\delta_0$ are orthogonal. □

In what follows, consider *a positive weight a satisfying*

$$(a)\ \ a \in L^\theta(\Omega),\ \textit{with}\ \ \theta > Q/p.$$

Denote with $\|u\|_{p,a}$, the weighted p–norm, namely

$$\|u\|_{p,a} = \left(\int_\Omega a(q)|u(q)|^p dq\right)^{1/p}.$$

As a direct consequence of Theorem 3.1.2 we prove that the functional

$$\mathcal{H}_{\gamma,\lambda}(u) = \frac{1}{p}\left(\|D_{\mathbb{H}^n}u\|_p^p - \gamma\|u\|^p_{\mathcal{H}_{\alpha,\Omega}} - \lambda\|u\|^p_{p,a}\right) \tag{3.3.12}$$

is weakly lower semi–continuous and coercive in $S_0^{1,p}(\Omega)$, provided that γ and λ verify suitable restrictions. The embedding

$$S_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{\theta' p}(\Omega)$$

is compact by (1.2.7) and the fact that $\theta' p < p^*$. Moreover, the embedding $L^{\theta' p}(\Omega) \hookrightarrow L^p(\Omega, a)$ is continuous, since

$$\|u\|^p_{p,a} \le \|a\|_\theta \|u\|^p_{\theta' p} \quad \text{for all } u \in L^{\theta' p}(\Omega),$$

by Hölder inequality. Consequently, the embedding

$$S_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega, a) \tag{3.3.13}$$

is compact.

For the following result we also use the variational characterization of the first eigenvalue of the p–Laplacian. Let λ_1 be the first eigenvalue of the problem

$$\begin{cases} -\Delta_{\mathbb{H}^n}^p u = \lambda a(q)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3.14}$$

in $S_0^{1,p}(\Omega)$, that is λ_1 is defined by

$$\lambda_1 = \inf_{\substack{u \in S_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^p dq}{\displaystyle\int_\Omega a(q)|u|^p dq}. \tag{3.3.15}$$

The following result is similar to Proposition 2.1 in [83].

Proposition 3.3.1. *The infimum λ_1 in (3.3.15) is positive and attained at a certain function $u_1 \in S_0^{1,p}(\Omega)$, with $\|u_1\|_{p,a} = 1$ and $\|D_{\mathbb{H}^n} u_1\|_p^p = \lambda_1 > 0$. Moreover, u_1 is a solution of (3.3.14) when $\lambda = \lambda_1$.*

Proof. For any $u \in S_0^{1,p}(\Omega)$ define the functionals $\mathscr{E}(u) = \|D_{\mathbb{H}^n} u\|_p^p$ and $\mathscr{J}(u) = \|u\|_{p,a}^p$. Let

$$\lambda_0 = \inf\{\mathscr{E}(u)/\mathscr{J}(u) \,:\, u \in S_0^{1,p}(\Omega) \setminus \{0\}, \|u\|_{p,a} \leq 1\}.$$

Observe that $\mathscr{E}$ and $\mathscr{J}$ are continuously Fréchet differentiable and convex in $S_0^{1,p}(\Omega)$. Clearly $\mathscr{E}'(0) = \mathscr{J}'(0) = 0$. Moreover, $\mathscr{J}'(u) = 0$ implies $u = 0$. In particular, $\mathscr{E}$ and $\mathscr{J}$ are weakly lower semi–continuous on $S_0^{1,p}(\Omega)$. Actually, $\mathscr{J}$ is weakly sequentially continuous on $S_0^{1,p}(\Omega)$. Indeed, if $(u_k)_k$ and u are in $S_0^{1,p}(\Omega)$ and $u_k \rightharpoonup u$ in $S_0^{1,p}(\Omega)$, then $u_k \to u$ in $L^p(\Omega, a)$ by (3.3.13). This implies at once that $\mathscr{J}(u_k) = \|u_k\|_{p,a}^p \to \|u\|_{p,a}^p = \mathscr{J}(u)$, as claimed.

Now, either $\mathscr{W} = \{u \in S_0^{1,p}(\Omega) \,:\, \mathscr{J}(u) \leq 1\}$ is bounded in $S_0^{1,p}(\Omega)$, or not. In the first case we are done, while in the latter $\mathscr{E}$ is coercive in $\mathscr{W}$, being coercive in $S_0^{1,p}(\Omega)$. Therefore, all the assumptions of Theorem 6.3.2 of [8] are fulfilled, being $S_0^{1,p}(\Omega)$ a reflexive Banach space, so that λ_0 is attained at a point $u_1 \in S_0^{1,p}(\Omega)$, with $\|u_1\|_{p,a}^p = 1$. We claim now that $\lambda_0 = \lambda_1$. Indeed,

$$\lambda_1 = \inf_{\substack{u \in S_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\|D_{\mathbb{H}^n} u\|_p^p}{\|u\|_{p,a}^p} = \inf_{\substack{u \in S_0^{1,p}(\Omega) \\ \|u\|_{p,a} = 1}} \|D_{\mathbb{H}^n} u\|_p^p \geq \inf_{\substack{u \in S_0^{1,p}(\Omega) \\ 0 < \|u\|_{p,a} \leq 1}} \frac{\|D_{\mathbb{H}^n} u\|_p^p}{\|u\|_{p,a}^p} = \lambda_0 \geq \lambda_1.$$

In particular, $\lambda_1 = \|D_{\mathbb{H}^n} u_1\|_p^p > 0$ and $\mathscr{E}'(u_1) = \lambda_1 \mathscr{J}'(u_1)$ again by Theorem 6.3.2 of [8]. Hence u_1 is a solution of (3.3.14) when $\lambda = \lambda_1$. $\square$

Taking inspiration from Theorem 2.2 of [39], we establish a similar preliminary result in the Heisenberg setting.

Theorem 3.3.2. *Let Ω be a bounded PS domain. For all $\gamma \in [0, \mathcal{H}_{\alpha,\Omega})$ and $\lambda \in (-\infty, m_\gamma \lambda_1)$, with $m_\gamma = 1 - \gamma/\mathcal{H}_{\alpha,\Omega}$ and $\mathcal{H}_{\alpha,\Omega}$ given in (3.1.1), the functional $\mathscr{H}_{\gamma,\lambda} : S_0^{1,p}(\Omega) \to \mathbb{R}$, defined by (3.3.12), is weakly lower semi–continuous and coercive in $S_0^{1,p}(\Omega)$.*

Proof. Fix γ and λ as in the statement. Let $(u_k)_k$ be a sequence such that $u_k \rightharpoonup u$ in $S_0^{1,p}(\Omega)$. Since Ω is bounded, $u_k \to u$ in $L^p(\Omega)$, and by Theorem 3.1.2 there exist two positive measures, verifying (3.1.3). Let us divide the proof that $\mathscr{H}_{\gamma,\lambda}$ is weakly lower semi–continuous in $S_0^{1,p}(\Omega)$ into two parts.
Case $\alpha = 0$. Theorem 3.1.2 guarantees the existence of an at most denumerable set of index Λ, $q_j \in \overline{\Omega}$, $\mu_j \geq 0$, $\nu_j \geq 0$, with $\mu_j + \nu_j > 0$ for all $j \in \Lambda$, such that (3.1.4)–(3.1.6) hold, where $\mathcal{H}_{0,\Omega}$ is the Sobolev constant defined in (3.1.1), being $\alpha = 0$. Since $1 < p < p^*$ and $\gamma \in [0, \mathcal{H}_{0,\Omega})$, then

$$
\begin{aligned}
\liminf_{k\to\infty} \mathscr{H}_{\gamma,\lambda}(u_k) &= \liminf_{k\to\infty} \frac{1}{p}\left(\|D_{\mathbb{H}^n} u_k\|_p^p - \gamma \|u_k\|_{p^*}^p - \lambda \|u_k\|_{p,a}^p\right)\\
&\geq \frac{1}{p}\left[\|D_{\mathbb{H}^n} u\|_p^p + \sum_{j\in\Lambda}\mu_j - \gamma\left(\|u\|_{p^*}^{p^*} + \sum_{j\in\Lambda}\nu_j\right)^{p/p^*} - \lambda\|u\|_{p,a}^p\right]\\
&\geq \frac{1}{p}\left[\|D_{\mathbb{H}^n} u\|_p^p + \sum_{j\in\Lambda}\mu_j - \gamma\left(\|u\|_{p^*}^{p} + \sum_{j\in\Lambda}\nu_j^{p/p^*}\right) - \lambda\|u\|_{p,a}^p\right] \qquad (3.3.16)\\
&= \mathscr{H}_{\gamma,\lambda}(u) + \frac{1}{p}\sum_{j\in\Lambda}\left(\mu_j - \gamma\nu_j^{p/p^*}\right)\\
&\geq \mathscr{H}_{\gamma,\lambda}(u) + \frac{1}{p}\left(1 - \frac{\gamma}{\mathcal{H}_{0,\Omega}}\right)\sum_{j\in\Lambda}\mu_j = \mathscr{H}_{\gamma,\lambda}(u) + \frac{m_\gamma}{p}\sum_{j\in\Lambda}\mu_j,
\end{aligned}
$$

where the last inequality follows from (3.1.6).
Case $\alpha \in (0, p]$. Since $1 < p < p^*(\alpha)$, an application of Theorem 3.1.2 gives

$$
\begin{aligned}
\liminf_{k\to\infty} \mathscr{H}_{\gamma,\lambda}(u_k) &= \liminf_{k\to\infty} \frac{1}{p}\left(\|D_{\mathbb{H}^n} u_k\|_p^p - \gamma \|u_k\|_{\mathcal{H}_{\alpha,\Omega}}^p - \lambda \|u_k\|_{p,a}^p\right)\\
&\geq \frac{1}{p}\left[\|D_{\mathbb{H}^n} u\|_p^p + \mu_0 - \gamma\left(\|u\|_{\mathcal{H}_{\alpha,\Omega}}^{p^*(\alpha)} + \nu_0\right)^{p/p^*(\alpha)} - \lambda\|u\|_{p,a}^p\right]\\
&\geq \frac{1}{p}\left[\|D_{\mathbb{H}^n} u\|_p^p + \mu_0 - \gamma\left(\|u\|_{\mathcal{H}_{\alpha,\Omega}}^{p} + \nu_0^{p/p^*(\alpha)}\right) - \lambda\|u\|_{p,a}^p\right] \qquad (3.3.17)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{H}_{\gamma,\lambda}(u) + \frac{1}{p}\left(\mu_0 - \gamma\nu_0^{p/p^*(\alpha)}\right)\\
&\geq \mathcal{H}_{\gamma,\lambda}(u) + \frac{\mu_0}{p}\left(1 - \frac{\gamma}{\mathcal{H}_{\alpha,\Omega}}\right) = \mathcal{H}_{\gamma,\lambda}(u) + \frac{\mu_0 m_\gamma}{p},
\end{aligned}
$$

where last inequality follows from $(3.1.8)_2$.

Hence, the weak lower semi–continuity of $\mathcal{H}_{\gamma,\lambda}$ in $S_0^{1,p}(\Omega)$ is given at once in both cases by (3.3.16), (3.3.17) and the fact that $m_\gamma > 0$, being $\gamma < \mathcal{H}_{\alpha,\Omega}$. Now, (3.1.1) and (3.3.15) yield for all $u \in S_0^{1,p}(\Omega)$

$$
\begin{aligned}
\mathcal{H}_{\gamma,\lambda}(u) &\geq \frac{1}{p}\left(\|D_{\mathbb{H}^n}u\|_p^p - \frac{\gamma}{\mathcal{H}_{\alpha,\Omega}}\|D_{\mathbb{H}^n}u\|_p^p - \frac{\lambda^+}{\lambda_1}\|D_{\mathbb{H}^n}u\|_p^p\right)\\
&\geq \frac{1}{p}\left(1 - \frac{\gamma}{\mathcal{H}_{\alpha,\Omega}} - \frac{\lambda^+}{\lambda_1}\right)\|D_{\mathbb{H}^n}u\|_p^p\\
&= \frac{1}{p}\left(m_\gamma - \frac{\lambda^+}{\lambda_1}\right)\|D_{\mathbb{H}^n}u\|_p^p.
\end{aligned}
\tag{3.3.18}
$$

Thus, $\mathcal{H}_{\gamma,\lambda}(u) \to \infty$ as $\|D_{\mathbb{H}^n}u\|_p \to \infty$, provided that $\lambda < m_\gamma\lambda_1$, as required. This completes the proof. □

3.4 Some applications in bounded domains

Following [39], we present some applications of Theorem 3.3.2 in the Heisenberg setting. Hence, throughout the section we assume that $1 < p < Q$ as always, that Ω *is a bounded PS domain* of $\mathbb{H}^n$, and that a satisfies (a). Finally, we recall that statements involving measure theory are always understood to be with respect to the Haar measure on $\mathbb{H}^n$, which coincides with the $(2n+1)$–dimensional Lebesgue measure.

Theorem 3.4.1 (Superlinear f)**.** *Suppose that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions*

(f_1) $\sup\{|f(q,\tau)|$*: a.e.* $q \in \Omega, \tau \in [0,\mathfrak{C}]\} < \infty$ *for any* $\mathfrak{C} > 0$,

(f_2) $f(q,\tau) = o(|\tau|^{p^*-1})$ *as* $|\tau| \to \infty$ *uniformly a.e. in* $q \in \Omega$,

(f_3) *there exist a non–empty open set $A \subseteq \Omega$ and a set $B \subseteq A$ of positive Haar measure such that*

$$
\limsup_{\tau\to0^+}\frac{\operatorname{ess\,inf}_{q\in B}\mathcal{F}(q,\tau)}{\tau^p} = \infty \quad \textit{and} \quad \liminf_{\tau\to0^+}\frac{\operatorname{ess\,inf}_{q\in A}\mathcal{F}(q,\tau)}{\tau^p} > -\infty,
$$

where $\mathcal{F}(q,\tau)=\int_0^\tau f(q,\zeta)d\zeta$. *Then for all* $\gamma\in[0,\mathcal{H}_{\alpha,\Omega})$ *and* $\lambda\in(-\infty,m_\gamma\lambda_1)$, *where* $m_\gamma=1-\gamma/\mathcal{H}_{\alpha,\Omega}$, *there exists a positive constant* $\overline{\sigma}=\overline{\sigma}(\lambda,\gamma)$ *such that for any* $\sigma\in(0,\overline{\sigma})$ *problem*

$$(\mathcal{P}_4)\qquad \begin{cases} -\Delta^p_{\mathbb{H}^n}u-\gamma\|u\|^{p-p^*(\alpha)}_{\mathcal{H}_{\alpha,\Omega}}\psi^\alpha\dfrac{|u|^{p^*(\alpha)-2}u}{r^\alpha} \\ \qquad\qquad =\lambda a(q)|u|^{p-2}u+\sigma f(q,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

has a nontrivial solution $u_{\gamma,\lambda,\sigma}\in S^{1,p}_0(\Omega)$.

Moreover,

$$\lim_{\sigma\to0^+}\|D_{\mathbb{H}^n}u_{\gamma,\lambda,\sigma}\|_p=0 \tag{3.4.1}$$

Proof. Fix $\gamma\in[0,\mathcal{H}_{\alpha,\Omega})$ and $\lambda\in(-\infty,m_\gamma\lambda_1)$. Problem ($\mathcal{P}_4$) can be seen as the Euler–Lagrange equation of the functional $\mathcal{J}_{\gamma,\lambda,\sigma}$ defined by

$$\mathcal{J}_{\gamma,\lambda,\sigma}(u)=\mathscr{H}_{\gamma,\lambda}(u)-\sigma\Psi(u),\quad u\in S^{1,p}_0(\Omega),$$

where $\mathscr{H}_{\gamma,\lambda}$ is the functional given in (3.3.12), while

$$\Psi(u)=\int_\Omega\mathcal{F}(q,u(q))dq.$$

Clearly, the functional $\mathscr{H}_{\gamma,\lambda}$ and Ψ are Fréchet differentiable in $S^{1,p}_0(\Omega)$, and so $\mathcal{J}_{\gamma,\lambda,\sigma}$ is of class $C^1(S^{1,p}_0(\Omega))$.

Moreover, by Theorem 3.3.2 we know that $\mathscr{H}_{\gamma,\lambda}$ is weakly lower semi–continuous and coercive in $S^{1,p}_0(\Omega)$. While, from (f_1) and (f_2) for any $\varepsilon>0$ there exists $\delta_\varepsilon=\delta(\varepsilon)>0$ such that

$$|\mathcal{F}(q,\tau)|\le\varepsilon|\tau|^{p^*}+\delta_\varepsilon|\tau|\quad\text{for a.a. } q\in\Omega \text{ and all } \tau\in\mathbb{R}. \tag{3.4.2}$$

Then, the Vitali convergence theorem ensures that Ψ is continuous in the weak topology of $S^{1,p}_0(\Omega)$.

Now, in order to prove the existence of a nontrivial solution $u_{\gamma,\lambda,\sigma}$ for any $\sigma\in(0,\overline{\sigma})$, we argue essentially as in the proof of Theorem 1.1 of [38], but in the functional space $S^{1,p}_0(\Omega)$. Indeed, let $\phi_{\gamma,\lambda}$ be defined by

$$\phi_{\gamma,\lambda}(\mathfrak{b}):=\inf_{u\in\mathscr{H}^{-1}_{\gamma,\lambda}(I_\mathfrak{b})}\frac{\sup\limits_{v\in\mathscr{H}^{-1}_{\gamma,\lambda}(I_\mathfrak{b})}\Psi(v)-\Psi(u)}{\mathfrak{b}-\mathscr{H}_{\gamma,\lambda}(u)},\quad I_\mathfrak{b}=(-\infty,\mathfrak{b}),$$

for any $\mathfrak{b} > 0$. By virtue of Theorem 2.1 in [11] for any $\mathfrak{b} > 0$ and any $\sigma \in (0, 1/\phi_{\gamma,\lambda}(\mathfrak{b}))$, the restriction of the functional $\mathcal{J}_{\gamma,\lambda,\sigma}$ to $\mathcal{H}_{\gamma,\lambda}^{-1}(I_{\mathfrak{b}})$ admits a global minimum $u_{\sigma,\mathfrak{b}}$.

Let us define

$$\sigma_\lambda = \sup_{\mathfrak{b}>0} \frac{1}{\phi_{\gamma,\lambda}(\mathfrak{b})} > 0.$$

Now, we fix $\overline{\sigma} \in (0, \sigma_\lambda)$. By construction of σ_λ, there exists $\mathfrak{b}_{\overline{\sigma}} > 0$ such that $\overline{\sigma} \leq 1/\phi_{\gamma,\lambda}(\mathfrak{b}_{\overline{\sigma}})$. Thus a further application of Theorem 2.1 in [11] yields that for any $\sigma < \overline{\sigma} \leq 1/\phi_{\gamma,\lambda}(\mathfrak{b}_{\overline{\sigma}})$ there exists a $u_{\gamma,\lambda,\sigma}$ which is a global minimum for $\mathcal{J}_{\gamma,\lambda,\sigma}$ in the open set $\mathcal{H}_{\gamma,\lambda}^{-1}(I_{\mathfrak{b}_{\overline{\sigma}}})$. Then, $u_{\gamma,\lambda,\sigma}$ is a critical point for $\mathcal{J}_{\gamma,\lambda,\sigma}$ in the open set $\mathcal{H}_{\gamma,\lambda}^{-1}(I_{\mathfrak{b}_{\overline{\sigma}}})$, that is a solution for ($\mathcal{P}_4$). The fact that $u_{\gamma,\lambda,\sigma}$ is nontrivial follows as in the proof of Theorem 4 in [76], with light modifications.

Futhermore, the family $\{\|D_{\mathbb{H}^n} u_{\gamma,\lambda,\sigma}\|_p\}_{\sigma\in(0,\overline{\sigma})}$ is uniformly bounded in σ. Indeed, by (3.1.1) and (3.3.18) , and since $u_{\gamma,\lambda,\sigma}$ is in $\mathcal{H}_{\gamma,\lambda}^{-1}(I_{\mathfrak{b}_{\overline{\sigma}}})$,

$$\frac{1}{p}\left(m_\gamma - \frac{\lambda^+}{\lambda_1}\right)\|D_{\mathbb{H}^n} u_{\gamma,\lambda,\sigma}\|_p^p \leq \mathcal{H}_{\gamma,\lambda}(u_{\gamma,\lambda,\sigma}) < \mathfrak{b}_{\overline{\sigma}}.$$

Thus, $\|D_{\mathbb{H}^n} u_{\gamma,\lambda,\sigma}\|_p < k_{\gamma,\lambda}$ for some positive constant $k_{\gamma,\lambda}$ independent of σ.

To complete the proof, we need to show the asymptotic behavior (3.4.1). Arguing as above, by (f_1) and (f_2) for any $\varepsilon > 0$ there exists $\delta_\varepsilon = \delta(\varepsilon) > 0$ such that

$$|f(q,\tau)| \leq \varepsilon|\tau|^{p^*-1} + \delta_\varepsilon \quad \text{ for a.a. } q \in \Omega \text{ and all } \tau \in \mathbb{R}.$$

Choosing in the above inequality $\varepsilon = 1$, and using (1.2.1) we have

$$\left|\int_\Omega f(q, u_{\gamma,\lambda,\sigma})u_{\gamma,\lambda,\sigma}(q)dq\right| \leq \mathcal{H}_{0,\Omega}^{-p^*/p}\|D_{\mathbb{H}^n} u_{\gamma,\lambda,\sigma}\|_p^{p^*} + \delta_1 C_1\|D_{\mathbb{H}^n} u_{\gamma,\lambda,\sigma}\|_p \leq C_{\gamma,\lambda}, \tag{3.4.3}$$

with $C_{\gamma,\lambda}$ independent of σ, since $\{\|D_{\mathbb{H}^n} u_{\gamma,\lambda,\sigma}\|_p\}_{\sigma\in(0,\overline{\sigma})}$ is uniformly bounded in σ.

Fix $\gamma \in [0, \mathcal{H}_{\alpha,\Omega})$ and $\lambda \in (-\infty, m_\gamma\lambda_1)$. Since

$$\langle \mathcal{J}'_{\gamma,\lambda,\sigma}(u_{\gamma,\lambda,\sigma}), u_{\gamma,\lambda,\sigma}\rangle = 0 \text{ for any } \sigma \in (0, \overline{\sigma}),$$

we get

$$\begin{aligned}\|D_{\mathbb{H}^n} u_{\gamma,\lambda,\sigma}\|_p^p - \gamma\|u_{\gamma,\lambda,\sigma}\|_{\mathcal{H}_{\alpha,\Omega}}^p - \lambda\|u_{\gamma,\lambda,\sigma}\|_{p,a}^p &\\ = \langle \mathcal{H}'_{\gamma,\lambda,\sigma}(u_{\gamma,\lambda,\sigma}), u_{\gamma,\lambda,\sigma}\rangle &\\ = \sigma\int_\Omega f(q, u_{\gamma,\lambda,\sigma}(q))u_{\gamma,\lambda,\sigma}(q)dq.&\end{aligned}$$

This, combined with (3.1.1), (3.3.15) and (3.4.3), gives

$$\left(m_\gamma - \frac{\lambda^+}{\lambda_1}\right) \|D_{\mathbb{H}^n} u_{\gamma,\lambda,\sigma}\|_p^p \le \langle \mathcal{H}'_{\gamma,\lambda,\sigma}(u_{\gamma,\lambda,\sigma}), u_{\gamma,\lambda,\sigma}\rangle \le \sigma C_{\gamma,\lambda}.$$

Letting $\sigma \to 0^+$, we get (3.4.1) by the choices of γ and λ. □

Theorem 3.4.2 (Sublinear f). *Suppose that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions*

(f_4) *There exist $s \in (1, p)$ and $\mathfrak{c} \in L^{\frac{p^*}{p^*-s}}(\Omega)$ such that*

$$|f(q, \tau)| \le \mathfrak{c}(q)(1 + |\tau|^{s-1}) \text{ for all } (q, \tau) \in \Omega \times \mathbb{R}.$$

(f_5) *There exist $\tilde{s} \in (1, p), \delta > 0, c_0 > 0$ and a nonempty open subset $\mathcal{E}$ of Ω such that*

$$\mathcal{F}(q, \tau) \ge c_0\tau^{\tilde{s}} \text{ for all } (q, \tau) \in \mathcal{E} \times (0, \delta).$$

For all $\gamma \in [0, \mathcal{H}_{\alpha,\Omega})$ and $\lambda \in (-\infty, m_\gamma\lambda_1)$, where $m_\gamma = 1 - \gamma/\mathcal{H}_{\alpha,\Omega}$, and $\sigma > 0$ problem ($\mathcal{P}_4$) has a nontrivial solution $u_{\gamma,\lambda,\sigma} \in S^{1,p}_0(\Omega)$. Moreover, (3.4.1) holds.

Proof. Fix $\gamma \in [0, \mathcal{H}_{\alpha,\Omega})$, $\lambda \in (-\infty, m_\gamma\lambda_1)$ and $\sigma > 0$. Using the notation of the proof of Theorem 3.4.1, by (3.3.18), (f_4), (3.1.1) and the Hölder inequality, for all $u \in S^{1,p}_0(\Omega)$ we find

$$\begin{aligned}
\mathcal{J}_{\gamma,\lambda,\sigma}(u) &\ge \frac{1}{p}\left(m_\gamma - \frac{\lambda^+}{\lambda_1}\right) \|D_{\mathbb{H}^n} u\|_p^p - \sigma \int_\Omega \mathfrak{c}(q)|u|^s\,dq \\
&\quad - \sigma\|\mathfrak{c}\|_{(p^*)'}\|u\|_{p^*} \\
&\ge \frac{1}{p}\left(m_\gamma - \frac{\lambda^+}{\lambda_1}\right) \|D_{\mathbb{H}^n} u\|_p^p - \sigma\|\mathfrak{c}\|_{\frac{p^*}{p^*-s}}\|u\|_{p^*}^s \\
&\quad - \sigma\|\mathfrak{c}\|_{(p^*)'}\|u\|_{p^*} \\
&\ge \frac{1}{p}\left(m_\gamma - \frac{\lambda^+}{\lambda_1}\right) \|D_{\mathbb{H}^n} u\|_p^p - \sigma\mathcal{H}_{0,\Omega}^{-s/p}\|\mathfrak{c}\|_{\frac{p^*}{p^*-s}}\|D_{\mathbb{H}^n} u\|_p^s \\
&\quad - \sigma\|\mathfrak{c}\|_{(p^*)'}\|u\|_{p^*},
\end{aligned} \tag{3.4.4}$$

since $(p^*)' < p^*/(p^* - s)$ and Ω is bounded. Then, $\mathcal{J}_{\gamma,\lambda,\sigma}$ is coercive and bounded below on $S^{1,p}_0(\Omega)$.

Theorem 3.3.2 gives that $\mathcal{H}_{\gamma,\lambda}$ is weakly lower semi–continuous in $S^{1,p}_0(\Omega)$ and Ψ is weakly continuous in $S^{1,p}_0(\Omega)$ by (f_4). Consequently,

$$\mathcal{J}_{\gamma,\lambda,\sigma} = \mathcal{H}_{\gamma,\lambda} - \sigma\Psi$$

is weakly lower semi–continuous in $S_0^{1,p}(\Omega)$. Thus there exists $u_{\gamma,\lambda,\sigma} \in S_0^{1,p}(\Omega)$ such that

$$\mathcal{J}_{\gamma,\lambda,\sigma}(u_{\gamma,\lambda,\sigma}) = \inf\{\mathcal{J}_{\gamma,\lambda,\sigma}(u) : u \in S_0^{1,p}(\Omega)\}.$$

We now assert that $u_{\gamma,\lambda,\sigma} \neq 0$. Take $q_0 \in \mathcal{E}$ and let $R > 0$ such that $B_R(q_0) \subset \mathcal{E}$. Fix $\phi \in C_0^\infty(B_R(q_0))$ with $0 \leq \phi \leq 1$, $\|D_{\mathbb{H}^n}\phi\|_p \leq C_R$ and $\|\phi\|_{L^s(B_R(q_0))} > 0$. Then, by (f_5) for all $\tau \in (0, \delta)$

$$\mathcal{J}_{\gamma,\lambda,\sigma}(\tau\phi) \leq \frac{1}{p}\left[(\delta C_R)^p - \gamma\tau^p\|\phi\|^p_{\mathcal{H}_{\alpha,\Omega}} - \lambda\tau^p\|\phi\|^p_{p,a}\right] - \sigma\tau^{\tilde{s}}a_0\|\phi\|_{L^{\tilde{s}}(B_R(q_0))} < 0$$

by choosing $\tau > 0$ sufficiently small, since $1 < \tilde{s} < p$. The claim is so proved, which means that, the nontrivial critical point $u_{\gamma,\lambda,\sigma}$ of $\mathcal{J}_{\gamma,\lambda,\sigma}$ in $S_0^{1,p}(\Omega)$ is a nontrivial solution of $(\mathcal{P}_4)$.

It remains to prove (3.4.1). Fix $\gamma \in [0, \mathcal{H}_{\alpha,\Omega})$ and $\lambda \in (-\infty, m_\gamma\lambda_1)$. Thanks to (3.4.4), the family of nontrivial critical points $\{u_{\gamma,\lambda,\sigma}\}_{\sigma\in(0,1]}$, constructed above, is clearly uniformly bounded in $S_0^{1,p}(\Omega)$. From this point, for any $\gamma \in [0, \mathcal{H}_{\alpha,\Omega})$ and $\lambda \in (-\infty, m_\gamma\lambda_1)$, with $m_\gamma = 1 - \gamma/\mathcal{H}_{\alpha,\Omega}$, arguing exactly as in the last part of the proof of Theorem 3.4.1, we get (3.4.1). □

Chapter 4

Nonlinear elliptic inequalities with gradient terms in the Heisenberg group

4.1 Introduction

In this chapter, as in [13], we first study existence and uniqueness of nonnegative nontrivial radial stationary entire solutions u of

$$(\mathcal{E}) \qquad \Delta^{\varphi}_{\mathbb{H}^n} u = f(u)\ell(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}),$$

where $\Delta^{\varphi}_{\mathbb{H}^n} u$ is the φ–Laplacian on the Heisenberg group $\mathbb{H}^n$, whose rigorous definition is given in Section 4.2, and then for

$$(\mathscr{I}) \qquad \Delta^{\varphi}_{\mathbb{H}^n} u \geq f(u)\ell(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n})$$

Liouville type theorems, that is non–existence of nonnegative nontrivial entire solutions u.

Moreover, f, ℓ and φ *satisfy throughout the chapter*

$$f,\, \ell \in C(\mathbb{R}^+_0), \quad f > 0 \quad \textit{and} \quad \ell > 0 \textit{ in } \mathbb{R}^+, \tag{$\mathscr{H}$}$$

$$\begin{gathered} \varphi \in C(\mathbb{R}^+_0) \cap C^1(\mathbb{R}^+), \quad \varphi(0) = 0, \quad \varphi' > 0 \quad \textit{in } \mathbb{R}^+, \\ \lim_{s\to\infty} \varphi(s) = \varphi(\infty) = \infty. \end{gathered} \tag{ϕ}$$

In particular, in the case of the p–Laplacian, that is when $\varphi(s) = s^{p-1}$, $p > 1$, we simply write $\Delta^{p}_{\mathbb{H}^n} u$.

Since we are interested in nonnegative entire solutions of elliptic coercive inequalities in all the space, as in [37, 68, 17] we make use of an appropriate *generalized Keller–Osserman condition for inequality* ($\mathscr{I}$). To this aim we also *assume throughout the chapter that*

$$\int_{0^+} \frac{t\varphi'(t)}{\ell(t)}\,dt < \infty, \qquad \int^{\infty} \frac{t\varphi'(t)}{\ell(t)}\,dt = \infty, \tag{ϕL}$$

holds. Consequently *the function* $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *given by*

$$K(s) = \int_0^s \frac{t\varphi'(t)}{\ell(t)}\,dt \tag{4.1.1}$$

is a C^1*–diffeomorphism from* $\mathbb{R}_0^+$ *to* $\mathbb{R}_0^+$, with

$$K'(s) = \frac{s\varphi'(s)}{\ell(s)} > 0 \qquad \text{in } \mathbb{R}^+, \tag{4.1.2}$$

thanks to (ϕ) and ($\mathscr{H}$). Thus K has increasing inverse $K^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and denoting by $F(s) = \int_0^s f(t)\,dt$ we say that *the generalized Keller–Osserman condition holds for* ($\mathscr{I}$) if

$$\int^{\infty} \frac{ds}{K^{-1}(F(s))} < \infty. \tag{KO}$$

If $\ell \equiv 1$, then K coincides with the function

$$\mathcal{H}(s) = s\varphi(s) - \int_0^s \varphi(t)\,dt, \qquad s \geq 0,$$

which represents the Legendre trasform of $\Phi(s) = \int_0^s \varphi(t)\,dt$ for all $s \in \mathbb{R}_0^+$. Furthermore, in the case of the p–Laplacian, $\mathcal{H}(s) = (p-1)s^p/p$, so that if $\ell \equiv 1$, then (KO) reduces to the well known Keller–Osserman condition for the p–Laplacian, that is $\int^{\infty} F(s)^{-1/p} ds < \infty$.

At this point we roughly recall that the nonexistence of entire solutions for coercive problems is connected with the validity of condition (KO), while the failure of (KO) gives existence of entire solutions. In particular, in the latter case Theorem 1.5 of [35], relative to the Euclidean case, shows that we can expect only unbounded solutions or equivalently *large solutions*. We are now in a position to extend and to generalize in several directions the core of Corollary 1.4 of [37], without requiring any monotonicity on ℓ.

Theorem 4.1.1. *Let $f(0) = 0$ and $\ell(0) > 0$ in $(\mathscr{H})$. Then $(\mathcal{E})$ admits a nonnegative local radial stationary C^1 solution. If furthermore f is nondecreasing in $\mathbb{R}_0^+$ and*

$$\int^{\infty} \frac{dt}{K^{-1}(F(t))} = \infty \qquad (VsKO)$$

holds, then $(\mathcal{E})$ possesses a nonnegative entire large radial stationary solution u of class $C^1(\mathbb{H}^n)$. Finally, if in addition

$$\int_{0^+} \frac{dt}{K^{-1}(F(t))} = \infty \qquad (4.1.3)$$

is valid, then $u > 0$ in $\mathbb{H}^n$.

The requests of Theorem 4.1.1 are fairly natural and general. Theorem 4.1.1 can be applied not only in the p–Laplacian case, $\varphi(s) = s^{p-1}$, $p > 1$, but also in the generalized mean curvature case, $\varphi(s) = s(1+s^2)^{(p-2)/2}$, $p \in (1, 2)$. For other elliptic operators we refer to [85] and [17].

The next result concerns uniqueness of radial stationary solutions of $(\mathcal{E})$, as in Theorem 4.1.1 we do not require any monotonicity assumption on ℓ in $\mathbb{R}_0^+$.

Theorem 4.1.2. *Assume that f and ℓ are locally Lipschitz continuous in $\mathbb{R}_0^+$, that $\ell(0) > 0$ and finally that $\varphi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$. Then, for each fixed $u_0 > 0$ equation $(\mathcal{E})$ admits a unique radial stationary solution u, with $u(O) = u_0$, where O is the natural origin in $\mathbb{H}^n$, in the open maximal ball B_R of $\mathbb{H}^n$.*

When φ is the p–Laplacian operator Theorem 4.1.2 is applicable if and only if $1 < p \le 2$. The remaining case $p > 2$ seems to be fairly delicate. Theorem 4.1.2 is valid under general assumptions, so that in principle we cannot assert that the solution is entire. For existence of entire solutions we refer the interested reader to Theorem 4.4.2, which yields to the proof of Theorem 4.1.1.

In what follows we assume monotonicity on f. In particular in the next theorem we require strict monotonicity on f, similarly to [35, 68, 16, 17]. Indeed, this assumption is due to the technique used, that is to an argument involving a comparison theorem.

For the first Liouville type theorem we assume that ℓ is *$\mathscr{b}$–monotone nonincreasing on* $\mathbb{R}_0^+$, that is there exists $\mathscr{b} \in (0, 1]$ such that

$$\inf_{t\in[0,s]} \ell(t) \ge \mathscr{b}\, \ell(s) \quad \text{for all } s \in \mathbb{R}_0^+.$$

Clearly, if ℓ is monotone nonincreasing in $\mathbb{R}_0^+$, then ℓ is 1–monotone nonincreasing on the same set, furthermore the above condition allows a controlled oscillatory behavior of ℓ on $\mathbb{R}_0^+$. Similar results when ℓ is monotone nonincreasing can be found earlier in [37].

Theorem 4.1.3. *Suppose that f is strictly increasing in $\mathbb{R}_0^+$ and that ℓ is $\mathscr{b}$–monotone nonincreasing in $\mathbb{R}_0^+$. Assume that there exist an exponent $\tau < 1$ and a constant $\theta \geq 1$ such that*

$$(\phi_1) \qquad s^\tau \varphi'(st) \leq \theta \varphi'(t) \quad \textit{for all } s \in (0,1],\ t \in \mathbb{R}^+.$$

Then every nonnegative bounded C^1–solution u of $(\mathscr{I})$ is constant in $\mathbb{H}^n$.

The restriction that the solutions are assumed bounded in Theorem 4.1.3 is essential. Indeed, the simple inequality

$$\Delta_{\mathbb{H}^n} u \geq \ell(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}) \cdot u,$$

with $\ell(s) = 4m/(s^2+1)$, admits the regular nonnegative unbounded entire solution

$$u(q) = w(|z|) = |z|^2 + 1,\ \ q = (z,t) \in \mathbb{H}^n.$$

The restriction (ϕ_1) implies in particular that $\varphi(\infty) = \infty$, as required in the main assumption (ϕ). Furthermore, (ϕ_1) is satisfied with $\tau = 2 - p$ and $\theta = 1$ whenever φ is homogeneous, that is $\varphi(s) = s^{p-1}$, $p > 1$. Clearly, if φ' is nondecreasing in $\mathbb{R}^+$, again (ϕ_1) is automatic for every $\tau \in [0,1)$ and $\theta = 1$. Of course there are cases in which φ' is nonincreasing in $\mathbb{R}^+$ and (ϕ_1) holds, as for instance in the case of the generalized mean curvature operator, $\varphi(s) = s(1+s^2)^{(p-2)/2}$, $p \in (1,2)$, for which (ϕ_1) holds with $\tau = 2-p \in (0,1)$. Finally, the exponent τ in (ϕ_1) can be negative only if $\varphi'(s) \to 0$ as $s \to 0^+$ and $\varphi'(s) \to \infty$ as $s \to \infty$, as for the p–Laplacian operator when $p > 2$.

Under the assumptions of Theorem 4.1.3, then $\ell(0) > 0$ by $(\mathscr{H})$ and the $\mathscr{b}$–monotonicity. If furthermore $\ell(\infty) = \lim_{s\to\infty} \ell(s) > 0$, then the corresponding nonexistence results can be deduced from inequalities including no gradient terms, since

$$\Delta^{\varphi}_{\mathbb{H}^n} u \geq f(u)\ell(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}) \geq \ell(\infty) f(u).$$

Thus the truly significant new case for Theorem 4.1.3 is when $\ell(\infty) = 0$.

For quasilinear elliptic inequalities of the type $\Delta^{\varphi}_{\mathbb{H}^n} u \geq f(u)$ we refer to the pioneering work of *Mitidieri* and *Pohozaev* in the Euclidean setting, see i.e. [71, 72, 73], and to recent contributions due to *D'Ambrosio* and *Mitidieri*, see for instance [30, 31] and the references therein. In [30, 31] the results are also obtained for a wide class of degenerate elliptic operators in the Heisenberg group. More recently, *D'Ambrosio*, *Farina*, *Mitidieri* and *Serrin* proved in [29] comparison principles, uniqueness, regularity and symmetry results for p–regular distributional solutions of quasilinear very weak elliptic equations of coercive type and for related inequalities. Finally, *D'Ambrosio* and *Mitidieri* presented in [32] Liouville theorems and applications to general systems, which include the celebrated Allen–Cahn equation, Ginzburg–Landau systems, Gross–Pitaevskii systems and Lichnerowicz type equations.

Recently, in [68, 17] results similar to Theorem 4.1.3 are given when ℓ is $\mathcal{C}$*–monotone nondecreasing in* $\mathbb{R}_0^+$, that is there exists $\mathcal{C} \geq 1$ such that

$$\sup_{t\in[0,s]} \ell(t) \leq \mathcal{C}\,\ell(s) \quad \text{for all } s \in \mathbb{R}_0^+.$$

In the next result we extend Theorem 1.3–(i) of [68] from the p–Laplacian inequality in $\mathbb{H}^n$ to the $\Delta^{\varphi}_{\mathbb{H}^n}$ operator.

Theorem 4.1.4. *Suppose that f is also nondecreasing in $\mathbb{R}_0^+$, and that ℓ is also $\mathcal{C}$–monotone nondecreasing in $\mathbb{R}_0^+$. If $(VsKO)$ holds, then there exists a nonnegative large solution $u \in C^1(\mathbb{H}^n)$ of inequality $(\mathscr{I})$.*

Theorem 4.1.4 extends also the existence Theorem 6.1 of [17], where (ϕL) is replaced by a stronger condition. More details are given in Section 4.7.

Furthermore, we recall that the converse of Theorem 4.1.4, that is nonexistence of nonnegative entire solutions of inequality $(\mathscr{I})$ when (KO) is valid, has been established in Theorem 1.1 of [68]. In particular, Theorem 1.1 of [68] is the generalization of Theorem 1.3–(ii) of [68] and is given under the further requests that $\ell(0) > 0$ and that f is strictly increasing in $\mathbb{R}_0^+$. These two conditions appear also in [35, 68, 16] and are used in the main proofs when a general solution u of $(\mathscr{I})$ is compared with an appropriate radial stationary solution v of the reverse inequality, in order to overcome the difficulty at points in which $D_{\mathbb{H}^n}u = D_{\mathbb{H}^n}v = 0$. Lately, Theorem 1.1 of [68] has been further extended to the case $\ell(0) = 0$ in the nonexistence Theorems 5.1 and 5.2 of [17], but under more stringent conditions on the regularity of solutions due to the necessity of a deep analysis on the set where the horizontal gradient vanishes.

The chapter is organized as follows. In Section 4.2 we recall some preliminary notions related to the operator $\Delta^{\varphi}_{\mathbb{H}^n}$ on the Heisenberg group, as well as regularity properties of weak solutions. Section 4.3 deals with the radial version of $\Delta^{\varphi}_{\mathbb{H}^n}$. In Section 4.4 we prove Theorem 4.1.1, the main existence theorem of the chapter, where no monotonicity assumptions on ℓ are required. Furthermore, in Section 4.5 we present a uniqueness result which is, as far as we know, the first attempt for general equations with gradient terms on the Heisenberg group $\mathbb{H}^n$. The proof of Theorem 4.1.3, which is a Liouville type result for bounded solutions of ($\mathscr{I}$), is given in Section 4.6 under the nonincreasing $\mathscr{b}$–monotonicity on ℓ. Finally, in Section 4.7 we give the proof of the existence Theorem 4.1.4 assuming the nondecreasing $\mathcal{C}$–monotonicity on ℓ.

4.2 Preliminaries

We introduce a further generalization of the horizontal p–Laplacian

$$\Delta^p_{\mathbb{H}^n} u = \mathrm{div}_H(|D_{\mathbb{H}^n} u|^{p-2}_{\mathbb{H}^n} D_{\mathbb{H}^n} u), \qquad p \in (1, \infty),$$

that is the φ–Laplacian on the Heisenberg group $\mathbb{H}^n$, defined as

$$\Delta^\varphi_{\mathbb{H}^n} u = \mathrm{div}_H(A(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}) D_{\mathbb{H}^n} u),$$

where $A(s) = \varphi(s)/s$ with φ satisfying (ϕ). Of course, the horizontal p–Laplacian follows by the choice $\varphi(s) = s^{p-1}$, $p > 1$ and $s \in \mathbb{R}^+_0$, so that $A(s) = s^{p-2}$ is defined in $\mathbb{R}^+$ and satisfies the required properties (ϕ).

From [22] and [67] we know that weak solutions of the equation $\Delta^p_{\mathbb{H}^n} u = 0$ satisfy Harnack inequality and, as a consequence, up to a modification on a set of Lebesgue measure zero, they are locally Hölder continuous of some exponent $\alpha \in (0, 1)$. However, in [47] *Garofalo* emphasizes the fundamental question whether the horizontal gradient $\nabla_{\mathbb{H}^n} u$ of such a weak solution is also continuous (or Hőlder continuous), with respect to the intrinsic distance attached to the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$. Substantial progress in that direction can be found in [69].

Furthermore, $C^{1,\alpha}_{\mathrm{loc}}$ regularity has been proved for solutions with special symmetries in [47], for instance in the first Heisenberg group $\mathbb{H}^1$ he obtains such regularity for all weak solutions of the horizontal p–Laplacian, with $p \geq 2$ which are of the form $u(z, t) = u(|z|, t)$. For the case $1 < p < 2$ and other remarks we refer to [92].

As in [68] we write the φ–Laplacian in Euclidean divergence form by making use of the following matrix $B = B(q)$, defined by

$$B(q) = B(z,t) = \begin{bmatrix} I_{2n} & \begin{matrix} 2y \\ -2x \end{matrix} \\ 2y^t - 2x^t & 4|z|^2 \end{bmatrix}, \tag{4.2.1}$$

where $x^t = (x_1, x_2, \dots, x_n)$ and $y^t = (y_1, y_2, \dots, y_n)$. Throughout the section we denote by div, D, and $<,>$ respectively the ordinary Euclidean divergence, the gradient and the scalar product in $\mathbb{R}^{2n+1}$. Consequently,

$BDu = D_{\mathbb{H}^n}u$, where BDv is the vector in $\mathbb{R}^{2n+1}$ whose components in the standard basis $\{\partial_{x_j}, \partial_{y_j}, \partial_t\}_{j=1}^n$ are given by the matrix multiplication B with the components of Du in the same basis. With this in mind we deduce the required expression

$$\Delta^{\varphi}_{\mathbb{H}^n} u = \mathrm{div}_H(A(|D_{\mathbb{H}^n}u|_{\mathbb{H}^n})D_{\mathbb{H}^n}u) = \mathrm{div}(A(|D_{\mathbb{H}^n}u|_{\mathbb{H}^n})BDu). \tag{4.2.2}$$

In particular

$$< Du, BDv >= \big(D_{\mathbb{H}^n}u, D_{\mathbb{H}^n}v\big)_{\mathbb{H}^n}.$$

If $\varphi(s) = s$, $s \in \mathbb{R}^+_0$, then (4.2.2) reduces to *the well known formula for the Kohn–Spencer Laplacian*, that is $\Delta^{\varphi}_{\mathbb{H}^n} u = \Delta_{\mathbb{H}^n} u = \mathrm{div}(BDu)$.

Multiplying (4.2.2) by $\phi \in C^\infty_0(\mathbb{H}^n)$, we get

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} \phi\,\Delta^{\varphi}_{\mathbb{H}^n} u &= \int_{\mathbb{R}^{2n+1}} \phi\,\mathrm{div}(A(|D_{\mathbb{H}^n}u|_{\mathbb{H}^n})BDu)\\ &= -\int_{\mathbb{R}^{2n+1}} A(|D_{\mathbb{H}^n}u|_{\mathbb{H}^n}) < BDu, D\phi >\\ &= -\int_{\mathbb{R}^{2n+1}} \big(A(|D_{\mathbb{H}^n}u|_{\mathbb{H}^n})D_{\mathbb{H}^n}u, D_{\mathbb{H}^n}\phi\big)_{\mathbb{H}^n}.\end{aligned}$$

Hence *the weak formulation of* $(\mathscr{I})$ is given by

$$-\int_{\mathbb{R}^{2n+1}} \big(A(|D_{\mathbb{H}^n}u|_{\mathbb{H}^n})D_{\mathbb{H}^n}u, D_{\mathbb{H}^n}\phi\big)_{\mathbb{H}^n} \ge \int_{\mathbb{R}^{2n+1}} f(u)\ell(|D_{\mathbb{H}^n}u|_{\mathbb{H}^n})\phi, \tag{4.2.3}$$

for all $\phi \in C^\infty_0(\mathbb{H}^n)$, $\phi \ge 0$.

In conclusion, we say that $u \in C^1(\mathbb{H}^n)$ is an *entire (weak) classical solution* of $(\mathscr{I})$ if (4.2.3) is satisfied for all $\phi \in C^\infty_0(\mathbb{H}^n)$, with $\phi \ge 0$.

Later we make use of the next comparison theorem given in Proposition 2.1 of [68], in the extended version stated in Proposition 4.2 of [17].

Proposition 4.2.1. *Let $\Omega \subset\subset \mathbb{H}^n$ be a relatively compact domain. If u and v are of class $C(\overline{\Omega}) \cap C^1(\Omega)$ and satisfy*

$$\begin{cases} \Delta^{\varphi}_{\mathbb{H}^n} u \ge \Delta^{\varphi}_{\mathbb{H}^n} v & \text{in } \ \Omega,\\ u \le v & \text{on } \partial\Omega,\end{cases} \tag{4.2.4}$$

then $u \le v$ in Ω.

Finally, we report the strong maximum principle given in Proposition 2.2 of [68].

Proposition 4.2.2. *Let $\Omega \subset \mathbb{H}^n$ be a domain and let φ satisfy (ϕ_1). Assume that u is a solution of class $C(\overline{\Omega}) \cap C^1(\Omega)$ of the inequality*

$$\Delta^{\varphi}_{\mathbb{H}^n} u \geq 0 \quad in \quad \Omega \tag{4.2.5}$$

and that $u(q_M) = \sup_{\Omega} u = u^*$ for some $q_M \in \Omega$. Then $u \equiv u^*$ in Ω.

4.3 Radial version of the φ–Laplacian

Let v be a *radial regular* function, that is for all $q = (z, t) \in \mathbb{H}^n$

$$v(q) = a(r(q)), \qquad r(q) = r(z,t) = (|z|^4 + t^2)^{1/4}, \tag{4.3.1}$$

where $a : \mathbb{R}^+_0 \to \mathbb{R}$, $a \in C(\mathbb{R}^+_0) \cap C^2(\mathbb{R}^+)$. From (1.1.3),

$$|D_{\mathbb{H}^n} r|_{\mathbb{H}^n} = \psi,$$

so that

$$|D_{\mathbb{H}^n} v(q)|_{\mathbb{H}^n} = |a'(r)| \cdot |D_{\mathbb{H}^n} r|_{\mathbb{H}^n} = |a'(r)|\psi. \tag{4.3.2}$$

Thus

$$\begin{aligned} \Delta^{\varphi}_{\mathbb{H}^n} v = \psi \, [\psi \, \varphi'(|a'(r)|\psi) a''(r) & \\ & + \frac{2n+1}{r} \operatorname{sgn}(a'(r)) \varphi(|a'(r)|\psi) \Big] , \end{aligned} \tag{4.3.3}$$

which is the *radial version of* $\Delta^{\varphi}_{\mathbb{H}^n} v$. As noted in [68], it is possible to shift the origin for the Korányi distance from O to any other point q_0, indeed if we denote with $\bar{r}(q) = d_K(q_0, q) = r(q_0^{-1} \circ q)$, a direct calculation shows

$$[X_j(\bar{r})](q) = [X_j(r)](q_0^{-1} \circ q), \qquad [Y_j(\bar{r})](q) = [Y_j(r)](q_0^{-1} \circ q).$$

Hence *the invariance with respect to the left multiplication holds*, namely

$$\Delta^{\varphi}_{\mathbb{H}^n}(a \circ \bar{r})(q) = \Delta^{\varphi}_{\mathbb{H}^n}(a \circ r)(q_0^{-1} \circ q). \tag{4.3.4}$$

This property will be useful in what follows.

A further particular radial case of $(\mathscr{I})$ is the subcase of *radial stationary solutions*, that is solutions of the form

$$v(q) = w(|z|), \qquad q = (z,t) \in \mathbb{H}^n, \tag{4.3.5}$$

where $w : \mathbb{R}_0^+ \to \mathbb{R}$, $w \in C(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$. This case is morally the case $t = 0$ of (4.3.1), with $r(q) = |z|$ and $|\cdot|$ the Euclidean norm in $\mathbb{R}^{2n}$. Consequently, the density function ψ, given in (1.1.3), is identically 1. In particular

$$D_{\mathbb{H}^n}|z| = \sum_{j=1}^{n}(X_j|z|)X_j + (Y_j|z|)Y_j = \sum_{j=1}^{n} \frac{\partial|z|}{\partial x_j}X_j + \frac{\partial|z|}{\partial y_j}Y_j = \frac{z}{|z|},$$

so that

$$\begin{aligned}\Delta_{\mathbb{H}^n}|z| &= \sum_{j=1}^{n} X_j^2|z| + Y_j^2|z| = \sum_{j=1}^{n} \frac{\partial^2|z|}{\partial x_j^2} + \frac{\partial^2|z|}{\partial y_j^2} \\ &= \sum_{j=1}^{n} \frac{1}{|z|} - \frac{x_j^2}{|z|^3} + \frac{1}{|z|} - \frac{y_j^2}{|z|^3} = \frac{2n-1}{|z|}.\end{aligned}$$

In turn $|D_{\mathbb{H}^n}|z||_{\mathbb{H}^n} \equiv 1$, that is $\psi \equiv 1$. Consequently,

$$\Delta^{\varphi}_{\mathbb{H}^n} v = \varphi'(|w'(|z|)|)w''(|z|) + \frac{2n-1}{|z|}\operatorname{sgn}(w'(|z|))\varphi(|w'(|z|)|). \tag{4.3.6}$$

Hence, as noted above, *radial stationary functions in the Heisenberg group* $\mathbb{H}^n$ *behave as Euclidean radial functions in* $\mathbb{R}^{2n}$.

4.4 Proof of the existence Theorem 4.1.1

The next result can be proved using some of the main ideas of the proof of Proposition 3.1 in [37], see also Chapter 4 in [85], but with notable improvements in several directions. We recall in passing that $(\mathscr{H})$, (ϕ) *and* (ϕL) *are supposed to hold throughout the chapter*, without further mentioning. We point out that no monotonicity assumptions are required on ℓ. For simplicity in notation we put $|z| = r$ in what follows. Furthermore we assume that $\mathfrak{p}$ is sufficiently smooth, just for simplicity. For the results of this section, the case $\varphi(\infty) < \infty$, not covered in this work, could be treated as in Chapters 4 and 8 of [85], where $\ell \equiv 1$.

Theorem 4.4.1. *Assume that* $\mathfrak{p} \in C^1(\mathbb{R}_0^+)$, *with* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *nondecreasing in* $\mathbb{R}_0^+$, *and* $\mathfrak{p} > 0$ *in* $\mathbb{R}^+$. *Suppose furthermore that* $f(0) = 0$ *and* $\ell(0) > 0$. *Then for all* $\eta > 0$ *and* $r_0, r_1 \in \mathbb{R}_0^+$, *with* $0 < r_0 < r_1$, *problem*

$$\begin{cases} [\mathfrak{p}\, A(|w'|)w']' = \mathfrak{p} f(w)\ell(|w'|) \quad \text{in } (r_0, r_1], \quad 0 < r_0 < r_1, \\ w \geq 0, \quad w' \geq 0, \quad w'(r_0) = 0, \\ w(r_1) = \eta, \quad w < \eta \quad \text{in } [r_0, r_1), \end{cases} \tag{4.4.1}$$

admits a C^1 solution w in $[r_0, r_1]$, with the property that there exists $s_1 \in [r_0, r_1)$ such that $w(r) \equiv w(r_0) \geq 0$ in $[r_0, s_1]$, $w' > 0$ in $(s_1, r_1]$ and w' is differentiable in $(s_1, r_1]$, so that w satisfies the equation

$$\frac{\varphi'(w')}{\ell(w')} w'' = \sigma f(w) - \frac{\mathfrak{p}'}{\mathfrak{p}} \cdot \frac{\varphi(w')}{\ell(w')} \tag{4.4.2}$$

in $(s_1, r_1]$.

If $w(r_0) = 0$, then

$$\int_{0^+} \frac{du}{K^{-1}(F(u))} < \infty. \tag{4.4.3}$$

If $r_0 = 0$ the same conclusions hold provided that

$$\limsup_{r \to 0^+} \frac{r\mathfrak{p}'(r)}{\mathfrak{p}(r)} < \infty. \tag{4.4.4}$$

Proof. For the purpose of this proof, we shall redefine f so that $f(u) = f(\eta)$ for all $u \geq \eta$, and $f(u) = 0$ when $u \leq 0$. This will not affect the conclusion of the proposition, since clearly any ultimate solution w of (4.4.1), with $w \geq 0$, $w' \geq 0$ in $[r_0, r_1]$, satisfies $0 \leq w \leq \eta$.

We shall make use of the Leray–Schauder fixed point theorem. Denote by X the Banach space $X = C^1[r_0, r_1]$, endowed with the usual norm

$$\|w\| = \|w\|_\infty + \|w'\|_\infty.$$

Let $\mathcal{T}$ be the mapping from X to X, defined pointwise for all $w \in X$ and $r \in [r_0, r_1]$ by

$$\mathcal{T}[w](r) = \eta - \int_r^{r_1} \varphi^{-1}\left(\frac{1}{\mathfrak{p}(s)} \int_{r_0}^s \mathfrak{p}(\tau) f(w(\tau)) \ell(|w'(\tau)|) d\tau\right) ds. \tag{4.4.5}$$

Obviously, $\mathcal{T}[w](r_1) = \eta$. Furthermore, for each $r \in (r_0, r_1]$

$$\mathcal{T}[w]'(r) = \varphi^{-1}\left(\frac{1}{\mathfrak{p}(r)} \int_{r_0}^r \mathfrak{p}(\tau) f(w(\tau)) \ell(|w'(\tau)|) d\tau\right). \tag{4.4.6}$$

Clearly $\mathcal{T}[w]'$ is continuous and nonnegative in $(r_0, r_1]$, since $0 \leq f(w) \leq f_\eta$ for all $w \in X$, where $f_\eta = \max_{u \in [0,\eta]} f(u) > 0$, and $\ell > 0$ in $\mathbb{R}^+$ by $(\mathscr{H})$. As a matter of fact

$$\begin{aligned} 0 \leq \frac{1}{\mathfrak{p}(r)} \int_{r_0}^r \mathfrak{p}(\tau) f(w(\tau)) \ell(|w'(\tau)|) d\tau &\leq f_\eta \max_{r \in [r_0, r_1]} \ell(|w'(r)|)(r - r_0) \\ &= C_w(r - r_0), \end{aligned}$$

with $C_w = f_\eta \max_{r\in[r_0,r_1]} \ell(|w'(r)|)$. Then $\mathcal{T}[w]'(r)$ approaches 0 as $r \to r_0^+$, and in turn $\mathcal{T}[w] \in X$, with $\mathcal{T}[w]'(r_0) = 0$.

Let w be a fixed point of $\mathcal{T}$ in X. We claim that $w(r_0) \geq 0$. Otherwise $w(r_0) < 0$, while $w(r_1) = \eta > 0$. Thus there exists a first point $s_1 \in (r_0, r_1)$ such that $w(r) < 0$ in $[r_0, s_1)$ and $w(s_1) = 0$. Consequently $f(w(r)) = 0$ in $[r_0, s_1]$ and so $w' \equiv 0$ for $r \in [r_0, s_1]$ by (4.4.6). Hence, $w(s_1) = w(r_0) < 0$ which is impossible, proving the claim. Therefore, $w \geq 0$ and $w' \geq 0$ in $[r_0, r_1]$ by (4.4.6). Moreover, we assert that $w < \eta$ in $[r_0, r_1)$. Indeed, from the fact that $f > 0$ in $(0, \eta]$ and $\ell > 0$ in $\mathbb{R}_0^+$, it follows that for all $r \in [r_0, r_1)$

$$\begin{aligned}\int_r^{r_1} \varphi^{-1}\left(\frac{1}{p(s)}\int_{r_0}^s p(\tau)f(w(\tau))\ell(|w'(\tau)|)d\tau\right)ds \\ \geq \int_{\max\{\tau_0,r\}}^{r_1} \varphi^{-1}\left(\frac{1}{p(s)}\int_{r_0}^s p(\tau)f(w(\tau))\ell(|w'(\tau)|)d\tau\right)ds > 0,\end{aligned}$$

where τ_0 is a point in $[r_0, r_1)$ such that $f(w(r)) > 0$ for all $r \in (\tau_0, r_1]$, which exists since $f \circ w \in C[r_0, r_1]$, $f(w(r_1)) = \eta > 0$ and $\ell(|w'(r)|) > 0$ for all $r \in [\tau_0, r_1]$. The assertion now follows from (4.4.5).

Define the homotopy $\mathcal{H} : X \times [0,1] \to X$ by

$$\mathcal{H}[w,\sigma](r) = \sigma\eta - \int_r^{r_1} \varphi^{-1}\left(\frac{\sigma}{p(s)}\int_{r_0}^s p(\tau)f(w(\tau))\ell(|w'(\tau)|)d\tau\right)ds. \tag{4.4.7}$$

By the above argument, any fixed point $w_\sigma = \mathcal{H}[w_\sigma, \sigma]$ is in X and has the properties that $w_\sigma \geq 0$, $w'_\sigma \geq 0$ in $[r_0, r_1]$ and $w_\sigma(r_1) = \sigma\eta$. Additionally, by (4.4.6) we find that $\varphi(w'_\sigma)$ is of class $C^1[r_0, r_1]$, and then from (4.4.7) that w_σ is a classical distribution solution of the problem

$$\begin{cases} [p\, A(|w'_\sigma|)w'_\sigma]' = \sigma p f(w_\sigma)\ell(|w'_\sigma|) & \text{in } (r_0, r_1], \\ w'_\sigma(r_0) = 0, \quad w_\sigma(r_1) = \sigma\eta. \end{cases} \tag{4.4.8}$$

In turn, it is evident that any function w_1 which is a fixed point of $\mathcal{H}[w, 1]$ (that is $w_1 = \mathcal{H}[w_1, 1]$) is a nonnegative distribution solution of (4.4.1), with $w'_1(r_0) = 0$, $w_1 \geq 0$ and $w'_1 \geq 0$ in $[r_0, r_1]$, and $w_1 < \eta$ in $[r_0, r_1)$, as shown above.

We assert that such a fixed point $w = w_1$ exists, using the Browder version of the Leray–Schauder theorem (see Theorem 11.6 of [52]).

To begin with, obviously $\mathcal{H}[w, 0] \equiv 0$ for all $w \in X$, that is $\mathcal{H}[w, 0]$ maps X into the single point $w_0 = 0$ in X. (This is the first hypothesis required in the application of the Leray–Schauder theorem.) We next show that $\mathcal{H}$ is compact from $X \times [0,1]$ into X. First, $\mathcal{H}$ is continuous on $X \times [0,1]$. Indeed, let $(w_j,\sigma_j)_j \in X\times[0,1]$, with $w_j\to w$ in X, that is $w_j\to w$ and $w_j'\to w'$ uniformly in $[r_0,r_1]$, and $\sigma_j\to\sigma$. Clearly $\sigma_j f(w_j)\ell(|w_j'|)\to\sigma f(w)\ell(|w'|)$, since the modified function f is continuous in $\mathbb{R}$, and so $\mathcal{H}[w_j,\sigma_j]\to\mathcal{H}[w,\sigma]$ by (4.4.7) and the dominated convergence theorem, as required.

Next let $(w_k,\sigma_k)_k$ be a bounded sequence in $X\times[0,1]$, say $\|w_k'\|_\infty\le L$ for some $L>0$ and for all $k\in\mathbb{N}$. Put $\ell_L=\max_{\tau\in[0,L]}\ell(\tau)$. It is clear from (4.4.7) that

$$\|\mathcal{H}[w_k,\sigma_k]'\|_\infty\le\varphi^{-1}(c),\quad c=f_\eta\ell_L(r_1-r_0), \tag{4.4.9}$$

since φ^{-1} is strictly increasing in $\mathbb{R}^+$ by (ϕ) and p is assumed to be nondecreasing in $\mathbb{R}_0^+$. Consequently, $(\mathcal{H}[w_k,\sigma_k])_k$ is equi–bounded in X and equi–Lipschitz continuous in $[r_0,r_1]\times[0,1]$. Define

$$\mathcal{I}_k(r_0,r)=\int_{r_0}^{r}p(\tau)f(w_k(\tau))\ell(|w_k'(\tau)|)d\tau\quad\text{and}\quad\mathcal{J}_k(r_0,r)=\frac{\mathcal{I}_k(r_0,r)}{p(r)}.$$

Then for all r, with $0<r_0\le r\le r_1$,

$$0\le\mathcal{J}_k(r_0,r)\le c\quad\text{and}\quad\lim_{r\to r_0^+}\mathcal{J}_k(r_0,r)=0,$$

where c is given in (4.4.9).

Now, fix $\varepsilon>0$ and let $\delta=\delta(\varphi^{-1},\varepsilon)>0$ be the corresponding number of the uniform continuity of φ^{-1} in $[0,c]$. Take r, s, with $0<r_0\le r<s\le r_1$ and $|s-r|<\delta/C$, where

$$C=f_\eta\ell_L\left(1+\kappa\right),\quad\text{where }\kappa=\max_{t\in[r_0,r_1]}\frac{tp'(t)}{p(t)}.$$

This is possible since $r_0>0$ and $p(t)\ge p(r_0)>0$. Now, for some $\xi\in(r,s)$

$$\frac{|p(s)-p(r)|}{p(s)}(r-r_0)=\frac{p'(\xi)|s-r|}{p(s)}\,s\,\frac{r-r_0}{s}\le\frac{sp'(s)}{p(s)}\,|s-r|,$$

since p' is nondecreasing in $\mathbb{R}_0^+$. Therefore for all k

$$\begin{aligned}\Big|\sigma_k\mathcal{J}_k(r_0,r)-\sigma_k\mathcal{J}_k(r_0,s)\Big|&\le\left|\frac{p(s)-p(r)}{p(r)p(s)}\mathcal{I}_k(r_0,r)-\frac{1}{p(s)}\mathcal{I}_k(r,s)\right|\\&\le\frac{1}{p(s)}\,|\mathcal{I}_k(r,s)|+\frac{|p(s)-p(r)|}{p(r)p(s)}\,|\mathcal{I}_k(r_0,r)|\\&\le f_\eta\ell_L\left(|s-r|+\frac{|p(s)-p(r)|}{p(r)p(s)}\int_{r_0}^{r}p(\tau)d\tau\right)\end{aligned} \tag{4.4.10}$$

$$\le f_\eta \ell_L \left(|s-r| + \frac{|p(s)-p(r)|}{p(s)}(r-r_0) \right)$$
$$\le f_\eta \ell_L \left(1 + \frac{sp'(s)}{p(s)} \right) |s-r| \le C|s-r| < \delta.$$

In conclusion, we have for all r, s, with $0 < r_0 \le r < s \le r_1$ and $|s-r| < \delta/C$

$$\left|\mathcal{H}[w_k,\sigma_k]'(r) - \mathcal{H}[w_k,\sigma_k]'(s)\right| = \left|\varphi^{-1}\left(\sigma_k \mathcal{J}_k(r_0,r)\right) - \varphi^{-1}\left(\sigma_k \mathcal{J}_k(r_0,s)\right)\right| < \varepsilon,$$

uniformly in k.

As an immediate consequence of the Ascoli–Arzelà theorem $\mathcal{H}$ then maps bounded sequences into relatively compact sequences in X, so $\mathcal{H}$ is compact.

To apply the Leray–Schauder theorem it is now enough to show that there is a constant $M > 0$ such that

$$(4.4.11) \qquad \|w\| \le M \quad \text{for all } (w,\sigma) \in X \times [0,1], \text{ with } \quad \mathcal{H}[w,\sigma] = w.$$

Let (w,σ) be a pair of type (4.4.11). But, as observed above, $w \ge 0$, $w' \ge 0$ in $[r_0,r_1]$, being $w = \mathcal{H}[w,\sigma]$, so that $\|w\|_\infty = w(r_1) \le \sigma\eta \le \eta$. We claim that there exists $s_1 = s_1(w,\eta)$, with $r_0 \le s_1 < r_1$, such that $w' > 0$ in $(s_1,r_1]$ and $w' \equiv 0$ in $[r_0,s_1]$. Indeed, the set $W_+ = \{r \in [r_0,r_1] \,:\, w'(r) > 0\}$ is nonempty, being $0 \le w(r_0) < \eta$ and $w(r_1) = \eta$, and (relatively) open in $[r_0,r_1]$, being $w \in C^1[r_0,r_1]$. Put $s_1 = \inf W_+$. Clearly $s_1 \in [r_0,r_1)$ and $w \equiv w(r_0)$ in $[r_0,s_1]$, since we already proved that $w \ge w(r_0)$ and $w' \ge 0$ in $[r_0,r_1]$. Now, for any fixed $r \in (s_1,r_1]$ there exists $s \in (s_1,r)$ such that $w'(s) > 0$ and integrating the equation in (4.4.8) on $[s,r]$ we get

$$\int_s^r [p\,A(|w'|)w']'d\tau = \sigma \int_s^r p\,f(w)\,\ell(|w'|)d\tau \ge 0,$$

that is $p(r)A(|w'(r)|)w'(r) \ge p(s)A(|w'(s)|)w'(s) > 0$. Hence, $w' > 0$ in $(s_1,r_1]$, $w'(s_1) = 0$ being $s_1 \ge r_0$ and $w'(r_0) = 0$. Then, $w > w(r_0) \ge 0$ in $(s_1,r_1]$ and $w < \eta$ in $[r_0,r_1)$, as shown above.

Moreover, w' is differentiable in $(s_1,r_1]$ and by the equation in (4.4.8)

$$[p\,\varphi(w')]' = \sigma p f(w)\ell(|w'|),$$

which is equivalent in $(s_1,r_1]$ to (4.4.2). By (4.4.2) and the fact that p is nondecreasing in $\mathbb{R}_0^+$, we get at once that in $(s_1,r_1]$

$$\frac{\varphi'(w')}{\ell(w')}\,w'' \le f(w).$$

Multiplying by $w' > 0$, integrating on $[s_1, r]$, $r \in (s_1, r_1]$, we have

$$(4.4.12) \qquad \begin{aligned} K(w'(r)) &= \int_0^{w'(r)} \frac{s\varphi'(s)}{\ell(s)}\, ds = \int_{s_1}^{r} \frac{w'\varphi'(w')}{\ell(w')}\, w'' ds \\ &\le F(w(r)) - F(w(s_1)) \le F(w(r)) \le F(\eta). \end{aligned}$$

Since $w \equiv w(r_0)$ in $[r_0, s_1]$, we have shown the important *a priori estimate* for w'

$$(4.4.13) \qquad 0 \le w'(r) \le K^{-1}(F(\eta)) = \mathcal{W} \quad \text{for all } r \in [r_0, r_1].$$

Hence, by (4.4.13) also $\|w'\|_\infty \le \mathcal{W}$. Thus we can take $M = \eta + \mathcal{W}$ in (4.4.11).

The Leray–Schauder theorem implies that the mapping $\mathcal{T}[w] = \mathcal{H}[w, 1]$ has a fixed point $w \in X$, which is the required solution of (4.4.1), proving the assertion above.

If $w(r_0) = 0$, that is $w \equiv w(r_0) = 0$ in $[r_0, s_1]$, then (4.4.13) and integration on $[s_1, r_1]$ give

$$\int_0^{\eta} \frac{du}{K^{-1}(F(u))} = \int_{s_1}^{r_1} \frac{w'(r)dr}{K^{-1}(F(w(r)))} \le r_1 - s_1 < \infty,$$

that is (4.4.3) holds.

Finally, if $r_0 = 0$ and (4.4.4) holds, then we can proceed word by word as in the case $r_0 > 0$. The only change occurs at the end of (4.4.10) where now

$$\kappa = \sup_{t \in (0, r_1]} \frac{t\mathfrak{p}'(t)}{\mathfrak{p}(t)},$$

which is finite by (4.4.4). □

In particular, we have shown under the assumptions of Theorem 4.4.1, with also (4.4.4) when $r_0 = 0$, that *for all* r_0, r_1, *with* $0 \le r_0 < r_1$, *problem* (4.4.1) *admits a classical maximal solution* w *in* $[r_0, R)$, *where* R *is defined by*

$$R = \sup\{\tau \ge r_1 \;:\; w \text{ can be defined in } [r_0, \tau] \text{ as a solution of } (4.4.1)\}.$$

Of course, $R > r_1$, by the use of the standard initial value problem theory, being $w(r_1) = \eta$, $w'(r_1) > 0$. Furthermore, *there exists* $s_1 \in [r_0, r_1)$ *such that* $w(r) \equiv w(r_0) \ge 0$ *in* $[r_0, s_1]$ *and*

$$(4.4.14) \qquad w' > 0 \quad \text{in} \quad (s_1, R).$$

In particular, when $r_0 = 0$, the function $v = v(|z|) = w(r)$, $r = |z|$, is a radial stationary solution of $(\mathcal{E})$ when $\mathfrak{p}(r) = r^{2n-1}$ in the open ball B_R of $\mathbb{H}^n$.

Theorem 4.4.2. *Assume that $\mathfrak{p} \in C^1(\mathbb{R}_0^+)$, with $\mathfrak{p}$ and $\mathfrak{p}'$ nondecreasing in $\mathbb{R}_0^+$, and $\mathfrak{p} > 0$ in $\mathbb{R}^+$. Suppose furthermore that $f(0) = 0$, $\ell(0) > 0$ and $(VsKO)$ holds. Then any maximal solution v, constructed in Theorem 4.4.1, is a C^1 maximal solution of*

$$[\mathfrak{p}\ A(|v'|)v']' = \mathfrak{p} f(v)\ell(|v'|) \tag{4.4.15}$$

in (r_0, R), and v has the property that $R = \infty$. If furthermore (4.1.3) holds and

$$\limsup_{r\to\infty} \frac{1}{\mathfrak{p}(r)} \int_{r_1}^{r} \mathfrak{p}(s)ds = \infty, \tag{4.4.16}$$

then v has also the property that $v^* = \lim_{r\to\infty} v(r) = \infty$ and $v > 0$ in $I = (r_0, \infty)$, $v' > 0$ in I and $v \in C^2(I)$.

In particular, v is a positive entire large radial stationary solution of $(\mathcal{E})$ when $r_0 = 0$ and $\mathfrak{p}(r) = r^{2n-1}$.

Proof. Let v be a classical maximal solution of (4.4.15) in $[r_0, R)$, constructed as in Theorem 4.4.1. We want to show that v is global, namely that $R = \infty$. Suppose by contradiction that $R < \infty$. We claim that, if $R < \infty$, then necessarily

$$\lim_{r\to R^-} v(r) = v^* = \infty, \tag{4.4.17}$$

where the existence of the limit is guaranteed by the monotonicity of v, that is by (4.4.14). To prove (4.4.17), assume by contradiction that the limit is finite, that is $v^* \in (\eta, \infty)$. Since $v' > 0$ in (s_1, R), from (4.4.15) it follows that $[\mathfrak{p}\ \varphi(v')]' > 0$ in (s_1, R), therefore the function $\mathfrak{p}\,\varphi(v')$ is monotone increasing and approaches a limit as $r \to R^-$. Consequently, being $\mathfrak{p}$ positive and continuous in $r = R$, also $\varphi(v'(r))$ approaches a limit as $r \to R^-$. In turn, since $\varphi : \mathbb{R}_0^+ \to [0, a)$, $0 < a \le \infty$, is a homeomorphism, then v' approaches a limit v'_R as $r \to R^-$, with $v'_R \in [0, a)$. As shown in (4.4.12)

$$K(v'(r)) \le \int_{s_1}^{r} f(v)v'\,ds \le F(v(r)) \le F(v^*). \tag{4.4.18}$$

By the invertibility of K and the definition of v we have $0 \le v'(r) \le V^*$ for all $r \in [r_0, R)$, where $V^* = K^{-1}(F(v^*))$. It follows at once that $v'_R < \infty$, contradicting the maximality of R. Hence the claim (4.4.17).

Now we prove that if $v^* = \infty$, then $R = \infty$, obtaining the required contradiction. By (4.4.18), as noted above, $K(v'(r)) \le F(v(r))$ in (s_1, R).

Consequently, $v'(r) \le K^{-1}(F(v(r)))$ in $[s_1, R)$, and by integration on $[s_1, r]$, with $r \in (s_1, R)$, we obtain

$$\int_{v(s_1)}^{v(r)} \frac{ds}{K^{-1}(F(s))} = \int_{s_1}^{r} \frac{v'(s)}{K^{-1}(F(v(s)))}\, ds \le R - s_1.$$

By $(VsKO)$ and the fact that $v^* = \infty$, we get a contradiction by letting $r \to R^-$, because the left hand side term goes to infinity. In conclusion the case $R < \infty$ cannot occur, and so $R = \infty$, as stated.

Now we prove the second part of the theorem, namely that $v^* = \infty$, under conditions (4.1.3) and (4.4.16). Assume by contradiction that $v^* < \infty$. By (4.4.14) and (4.4.18) we have $0 < v'(r) \le V^*$ for all $r \in I = (r_0, \infty)$, where $V^* = K^{-1}(F(v^*))$, as defined above. Furthermore, $\ell_* = \min_{s\in[0,V^*]} \ell(s) > 0$ by $(\mathscr{H})$ and the assumption $\ell(0) > 0$.

Moreover, (4.4.15) is valid in $I = (r_0, \infty)$, since $v(r_1) = \eta$ by (4.4.1). Now, $v > \eta$ in (r_1, ∞) by (4.4.14), f is nondecreasing in $\mathbb{R}_0^+$ and $f(\eta) > 0$ by $(\mathscr{H})$, so that $[\mathtt{p}\, A(|v'|)v']' \ge c\, \mathtt{p}$ in $[r_1, \infty)$, where $c = f(\eta)\ell_* > 0$. Thus, using that $0 < v'(r) \le V^* < \infty$ in (r_1, ∞) and integrating on $[r_1, r]$ for all $r > r_1$, we get

$$\varphi(V^*) \ge \varphi(v'(r)) \ge \frac{\mathtt{p}(r_1)}{\mathtt{p}(r)}\varphi(v'(r_1)) + \frac{c}{\mathtt{p}(r)}\int_{r_1}^{r} \mathtt{p}(s)ds \ge \frac{c}{\mathtt{p}(r)}\int_{r_1}^{r} \mathtt{p}(s)ds$$

by (ϕ). By letting $r \to \infty$, assumption (4.4.16) gives the obvious contradiction $\varphi(V^*) = \infty$. Therefore, $v^* = \infty$, as stated.

Since v solves (4.4.1), clearly $v(r_0) \ge 0$, but the case $v(r_0) = 0$ cannot occur by Theorem 4.4.1 thanks to assumption (4.1.3). Since $v' \ge 0$ in $[r_0, \infty)$, it then follows that $v > 0$ in $[r_0, \infty)$. Integrating (4.4.15) in $[r_0, r]$, by $(\mathscr{H})$ and being $\ell(0) > 0$, we get

$$\mathtt{p}(r)\varphi(v'(r)) = \int_{r_0}^{r} \mathtt{p}(s) f(v)\ell(v')ds > 0.$$

Thus (ϕ) yields that $v'(r) > 0$ for all $r > r_0$ and

$$v'(r) = \varphi^{-1}\left(\frac{1}{\mathtt{p}(r)}\int_{r_0}^{r} \mathtt{p}(s) f(v)\ell(v')ds\right).$$

Hence v' is differentiable in I, with

$$v'' = \frac{\ell(v')}{\varphi'(v')}\left[f(v) - \frac{\mathtt{p}'}{\mathtt{p}}\frac{\varphi(v')}{\ell(v')}\right] \quad \text{in } I. \tag{4.4.19}$$

In particular, $v \in C^2(I)$.

The last part of the theorem is a consequence of the fact that $p(r) = r^{2n-1}$ verifies (4.4.4) and (4.4.16), taking $r_0 = 0$ in Theorem 4.4.1. Thus the maximal solution $v = v(r)$, $r = |z|$, is a positive entire large radial stationary solution of $(\mathcal{E})$. □

Proof of Theorem 4.1.1. It is enough to apply Theorems 4.4.1 and 4.4.2, with $r_0 = 0$ and $p(r) = r^{2n-1}$, $n \geq 1$, to the radial stationary version of $(\mathcal{E})$. □

4.5 Qualitative properties and uniqueness

We now turn to the radial stationary version of equation $(\mathcal{E})$ and assume throughout the section that $(\mathscr{H})$ *and* (ϕ) *hold, with* $\ell(0) > 0$, without further mentioning.

Proposition 4.5.1. *Problem*

$$\begin{aligned} &[r^{2n-1}A(|v'|)v']' = r^{2n-1}f(v)\ell(|v'|) \qquad \text{in } \mathbb{R}^+, \\ &v(0) = v_0 > 0, \qquad v'(0) = 0. \end{aligned} \tag{4.5.1}$$

has a solution on some interval $[0, r_0]$, $r_0 > 0$.

Proof. Any local solution of (4.5.1), for small $r > 0$, must be a fixed point of the operator

$$\mathcal{T}[v](r) = v_0 + \int_0^r \varphi^{-1}\left(\frac{1}{s^{2n-1}}\int_0^s \tau^{2n-1}f(v(\tau))\ell(|v'(\tau)|)d\tau\right) ds. \tag{4.5.2}$$

Fix $\varepsilon > 0$ so small that $[v_0 - \varepsilon, v_0 + \varepsilon] \subset \mathbb{R}^+$, so that by $(\mathscr{H})$

$$\begin{gathered} 0 < i = \min_{[v_0-\varepsilon, v_0+\varepsilon]} f(u) \leq \max_{[v_0-\varepsilon, v_0+\varepsilon]} f(u) = M < \infty, \\ 0 < l = \min_{[0,\varepsilon]} \ell(t) \leq \max_{[0,\varepsilon]} \ell(t) = L < \infty. \end{gathered}$$

Let $r_0 = r_0(\varepsilon)$ be so small that

$$r_0\varphi^{-1}(r_0LM) + \varphi^{-1}(r_0LM) \leq \varepsilon. \tag{4.5.3}$$

This can be done since $\varphi^{-1}(0) = 0$ by (ϕ). Denote by $C^1[0, r_0]$ the usual Banach space of real functions of class C^1 in $[0, r_0]$, endowed with the norm $u \mapsto \|u\| = \|u\|_\infty + \|u'\|_\infty$. Put $v_0(r) \equiv v_0 \in C^1[0, r_0]$ and let

$$\mathscr{C} = \{v \in C^1[0, r_0] \;:\; \|v - v_0\| \leq \varepsilon\},$$

that is $v \in \mathscr{C}$ if and only if $\|v - v_0\|_\infty + \|v'\|_\infty \le \varepsilon$. Clearly $\mathscr{C}$ is the closed ball in $C^1[0, r_0]$ of center v_0 and radius $\varepsilon > 0$, so that $\mathscr{C}$ is closed, convex and bounded in $C^1[0, r_0]$. If $v \in \mathscr{C}$ then $v([0, r_0]) \subset [v_0 - \varepsilon, v_0 + \varepsilon]$ and $v'([0, r_0]) \subset [-\varepsilon, \varepsilon]$, and in turn $0 < f(v(r)) \le M$ and $0 < \ell(|v'(r)|) \le L$ for all $r \in [0, r_0]$. Furthermore,

$$0 \le \int_0^s \left(\frac{\tau}{s}\right)^{2n-1} f(v(\tau))\ell(|v'(\tau)|)d\tau \le \int_0^s f(v(\tau))\ell(|v'(\tau)|)d\tau, \quad 0 < s \le r_0,$$

where the last integral approaches 0 as $s \to 0^+$ by $(\mathscr{H})$. Thus the operator $\mathcal{T}$ in (4.5.2) is well defined.

We show that $\mathcal{T} : \mathscr{C} \to \mathscr{C}$ and that $\mathcal{T}$ is compact. Indeed for $v \in \mathscr{C}$ we have

$$\begin{aligned}\|\mathcal{T}[v] - v_0\|_\infty &= \int_0^{r_0} \varphi^{-1}\left(\int_0^s \left(\frac{\tau}{s}\right)^{2n-1} f(v(\tau))\ell(|v'(\tau)|)d\tau\right) ds \\ &\le r_0 \varphi^{-1}(r_0 LM) \\ \|\mathcal{T}[v]'\|_\infty &\le \varphi^{-1}\left(\int_0^{r_0} f(v(\tau))\ell(|v'(\tau)|)d\tau\right) \le \varphi^{-1}(r_0 LM).\end{aligned}$$

Thus $\mathcal{T}[v] \in \mathscr{C}$ and so $\mathcal{T}(\mathscr{C}) \subset \mathscr{C}$ by (4.5.3). Let $(v_k)_k$ be a sequence in $\mathscr{C}$ and fix r, t be two points in $[0, r_0]$. Then

$$\begin{aligned}|\mathcal{T}[v_k](r) - \mathcal{T}[v_k](t)| &= \left|\int_r^t \varphi^{-1}\left(\int_0^s \left(\frac{\tau}{s}\right)^{2n-1} f(v_k(\tau))\ell(|v_k'(\tau)|)d\tau\right) ds\right| \\ &\le \varphi^{-1}(LM)|r - t|.\end{aligned}$$

Furthermore, as in (4.4.10), we compute

$$\left|\frac{\mathcal{I}_k(r)}{r^{2n-1}} - \frac{\mathcal{I}_k(t)}{t^{2n-1}}\right| \le LM\Big(|r - t| + (2n-1)|r - t|\Big) = 2nLM\,|r - t|\,,$$

where as in Theorem 4.4.1

$$\mathcal{I}_k(r) = \int_0^r \tau^{2n-1} f(v_k(\tau))\ell(|v_k'(\tau)|)d\tau. \tag{4.5.4}$$

Now for all $\sigma > 0$ there exists $\delta = \delta(\varphi^{-1}, \sigma) > 0$, by the uniform continuity of φ^{-1} in $[0, r_0 LM]$, such that for all r, $t \in [0, r_0]$, with $|r - t| < \delta/2nLM$, we have for all k

$$|\mathcal{T}[v_k]'(r) - \mathcal{T}[v_k]'(t)| = \left|\varphi^{-1}\left(\frac{\mathcal{I}_k(r)}{r^{2n-1}}\right) - \varphi^{-1}\left(\frac{\mathcal{I}_k(t)}{t^{2n-1}}\right)\right| \le \sigma.$$

Therefore, by the Ascoli–Arzelà theorem $\mathcal{T}$ maps bounded sequences into relatively compact sequences, with limit points in $\mathscr{C}$, since $\mathscr{C}$ is closed.

Finally $\mathcal{T}$ is continuous, because if $v \in \mathscr{C}$ and $(v_k)_k \subset \mathscr{C}$ are such that $\|v_k - v\|$ tends to 0 as $k \to \infty$, then by the Lebesgue dominated convergence theorem, we can pass under the sign of integrals twice in (4.5.2), and so $\mathcal{T}[v_k]$ tends to $\mathcal{T}[v]$ pointwise in $[0, r_0]$ as $k \to \infty$. By the above argument, it is obvious that $\|\mathcal{T}[v_k] - \mathcal{T}[v]\| \to 0$ as $k \to \infty$ as claimed.

By the Schauder fixed point theorem, $\mathcal{T}$ possesses a fixed point v in $\mathscr{C}$. Clearly, $v \in C^1[0, r_0]$ by the representation formula (4.5.2), that is

$$v(r) = v_0 + \int_0^r \varphi^{-1}\left(\int_0^s \left(\frac{\tau}{s}\right)^{2n-1} f(v(\tau))\ell(|v'(\tau)|)d\tau\right) ds, \tag{4.5.5}$$

as desired. □

Once it is known that a solution v of (4.5.1) exists, then v necessarily obeys to (4.5.5). In particular, problem (4.5.1) admits a classical maximal solution v in $[0, R)$, where R is defined by

$$R = \sup\{r \geq r_0 \ : \ v \text{ can be defined in } [0, r] \text{ as a solution of (4.5.1)}\}.$$

Of course, $R > r_0$, by the use of the standard initial value problem theory, being $v(r_0) > 0$, $v'(r_0) > 0$. Furthermore, the solution $v = v(r) = v(|z|)$, $r = |z|$, is a radial stationary solution of $(\mathcal{E})$ in the open ball B_R of $\mathbb{H}^n$.

Proof of Theorem 4.1.2. Let v_1 and v_2 be two C^1 solutions of (4.5.1), and $[0, \tilde{R})$ be the maximal interval in which both v_1 and v_2 exist. Assume by contradiction that there exists $\rho_0 \in (0, \tilde{R})$ such that $v_1(\rho_0) \neq v_2(\rho_0)$. Let R, with $\rho_0 < R < \tilde{R}$, be fixed. Then $v_1' > 0$ and $v_2' > 0$ in $(0, R]$ by (4.5.5). Put $\mathcal{V} = \max\{v_1(R), v_2(R)\}$ and

$$\mathcal{V}' = \max\left\{\max_{r\in[0,R]} v_1'(r),\ \max_{r\in[0,R]} v_2'(r)\right\}.$$

We denote by L and $L_{\varphi^{-1}}$ the Lipschitz constants of ℓ and φ^{-1} in $[0, \mathcal{V}']$, respectively, and by M the Lipschitz constant of f in $[v_0, \mathcal{V}]$. Set

$$f_1 = \max_{t\in[v_0,\mathcal{V}]} f(t), \qquad l_1 = \max_{t\in[0,\mathcal{V}']} \ell(t).$$

Fix $r \in [0, R]$. Then

$$\begin{aligned} |f(v_1)\ell(v_1') - f(v_2)\ell(v_2')| &\leq \ell(v_1')|f(v_1) - f(v_2)| \\ &\quad + f(v_2)|\ell(v_1') - \ell(v_2')| \\ &\leq l_1 M|v_1 - v_2| + f_1 L|v_1' - v_2'| \\ &\leq l_1 M \int_0^r |v_1' - v_2'|\, ds + f_1 L|v_1' - v_2'|. \end{aligned} \tag{4.5.6}$$

Choose $\delta > 0$ so small that

$$L_{\varphi^{-1}} c_\delta < 1 \quad \text{where} \quad c_\delta = \frac{l_1 M \delta^2}{2} + f_1 L \delta. \tag{4.5.7}$$

Since, for all $r \in (0, \delta]$

$$\begin{aligned}
\left| \frac{\mathcal{I}_1(r)}{r^{2n-1}} - \frac{\mathcal{I}_2(r)}{r^{2n-1}} \right| &\leq \int_0^r \left(\frac{s}{r}\right)^{2n-1} |f(v_1)\ell(v_1') - f(v_2)\ell(v_2')| ds \\
&\leq \int_0^r |f(v_1)\ell(v_1') - f(v_2)\ell(v_2')| ds \\
&\leq l_1 M \int_0^r ds \int_0^s |v_1' - v_2'| \, d\tau + f_1 L \int_0^r |v_1' - v_2'| \, ds \\
&\leq \frac{l_1 M \delta^2}{2} \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)| + f_1 L\delta \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)| \\
&= c_\delta \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)|,
\end{aligned}$$

then

$$\begin{aligned}
|v_1'(r) - v_2'(r)| &= \left| \varphi^{-1}\left(\frac{\mathcal{I}_1(r)}{r^{2n-1}}\right) - \varphi^{-1}\left(\frac{\mathcal{I}_2(r)}{r^{2n-1}}\right) \right| \\
&\leq L_{\varphi^{-1}} \left| \frac{\mathcal{I}_1(r)}{r^{2n-1}} - \frac{\mathcal{I}_2(r)}{r^{2n-1}} \right| \\
&\leq L_{\varphi^{-1}} c_\delta \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)|,
\end{aligned}$$

Therefore

$$\max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)| \leq L_{\varphi^{-1}} c_\delta \max_{r\in[0,\delta]} |v_1'(r) - v_2'(r)|, \tag{4.5.8}$$

so that $v_1' \equiv v_2'$ on $[0, \delta]$ by (4.5.7). Hence, $v_1 \equiv v_2$ on $[0, \delta]$, since we have $v_1(0) = v_2(0) = v_0$. Repeating the argument a finite number of times, being $[0, R]$ compact, we get that $v_1 \equiv v_2$ on $[0, R]$. This is impossible since $\rho_0 \in [0, R]$ and completes the proof. □

Remark 4.5.1. Clearly Theorem 4.1.2 can be applied both in the p–Laplacian case, $\varphi(s) = s^{p-1}$ when $p \in (1, 2]$ and in the generalized mean curvature case, $\varphi(s) = s(1 + s^2)^{(p-2)/2}$, $p \in (1, 2)$. Finally, Theorem 4.1.2 cannot be applied in the p–Laplacian case when $p > 2$, since φ^{-1} fails to be of class $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$.

4.6 Nonexistence under nonincreasing $\mathscr{b}$–monotonicity on ℓ

We recall that conditions (ϕ), (ϕL) and $(\mathscr{H})$ are assumed throughout the chapter.

Lemma 4.6.1. *Assume that (ϕ_1) holds. Let ℓ be $\mathscr{b}$–nonincreasing in $\mathbb{R}^+$ and f nondecreasing in $\mathbb{R}_0^+$. Fix*

$$0<\varepsilon<\eta<a<\infty, \quad \textit{and} \quad 0<r_0<r_1<\infty.$$

Then, there exist a finite radius $R>r_1$ and a strictly increasing, convex function $\mathscr{a}:[r_0,R)\longrightarrow[\varepsilon,a)$, $\mathscr{a}\in C^2[r_0,R)$, such that for every $q\in\mathbb{H}^n$ the radial function $v=\mathscr{a}\circ d_q$ satisfies

$$\text{(4.6.1)}\qquad \begin{cases} \Delta^{\varphi}_{\mathbb{H}^n} v \le f(v)\ell(|D_{\mathbb{H}^n} v|_{\mathbb{H}^n}) & \textit{in } B_R(q)\setminus\overline{B_{r_0}(q)} \\ v=\varepsilon & \textit{on } \partial B_{r_0}(q), \\ v=a & \textit{on } \partial B_R(q), \\ \varepsilon\le v\le\eta & \textit{on } B_{r_1}(q)\setminus B_{r_0}(q). \end{cases}$$

Proof. Fix ε,η, a, r_0 and r_1 as in the statement. Let $\sigma\in(0,1]$ be a parameter to be determined later and choose $R_\sigma>r_0$ such that

$$\text{(4.6.2)}\qquad R_\sigma-r_0=\int_\varepsilon^a\frac{ds}{K^{-1}(\sigma F(s))}.$$

Clearly R_σ is uniquely determined and *finite*, being a finite. Then, since the right hand side diverges as $\sigma\to0^+$, there exists σ so small that $R=R_\sigma>r_1$. We implicitly define the function $\mathscr{a}_\sigma$ for all $r\in[r_0,R)$ by

$$R=r+\int_{\mathscr{a}_\sigma(r)}^a\frac{ds}{K^{-1}(\sigma F(s))}.$$

By construction, $\mathscr{a}_\sigma(r_0)=\varepsilon$ by (4.6.2). Moreover, since $K^{-1}(\sigma F)>0$ and the integral in (4.6.2) is finite, then $\mathscr{a}_\sigma(r)\uparrow a$ as $r\to R^-$. A first differentiation yields

$$\mathscr{a}_\sigma'=K^{-1}(\sigma F(\mathscr{a}_\sigma)).$$

Hence $\mathscr{a}_\sigma$ is monotone increasing and $\sigma F(\mathscr{a}_\sigma)=K(\mathscr{a}_\sigma')$ in $[r_0,R)$. Differentiating once more we get

$$\sigma f(\mathscr{a}_\sigma)\mathscr{a}_\sigma'=K'(\mathscr{a}_\sigma')\mathscr{a}_\sigma''=\frac{\mathscr{a}_\sigma'\varphi'(\mathscr{a}_\sigma')}{\ell(\mathscr{a}_\sigma')}\,\mathscr{a}_\sigma''.$$

Thus a_σ is strictly convex, being $a_\sigma > 0$ and $a''_\sigma > 0$ by $(\mathscr{H})$, so that

$$[\varphi(a'_\sigma)]' = \varphi'(a'_\sigma)a''_\sigma = \sigma f(a_\sigma)\ell(a'_\sigma). \tag{4.6.3}$$

Now set $v = a \circ d_q$, so that v is a radial function in $\mathbb{H}^n$ and $v \in C^2_H$ radial function on $B_{R_\sigma}(q)\setminus B_{r_0}(q)$, where $v \in C^2_H$ means that the horizontal gradient of v is well defined and continuous. For further details we refer to [17] and [44].

We claim that there exists $\sigma \in (0,1]$, σ sufficiently small and independent of q, such that v is the required solution of (4.6.2). For simplicity in what follows we write a in place of a_σ. Hence, considering $(\mathscr{H})$, the positivity of a', the radial expression (4.3.3), (4.3.2), and (ϕ_1) with $s = \psi \in (0,1]$, by (1.1.3), we have

$$\begin{aligned}
\frac{\Delta^{\varphi}_{\mathbb{H}^n} v}{f(v)\ell(|D_{\mathbb{H}^n} v|_{\mathbb{H}^n})} &= \frac{\psi^2\varphi'(a'(r)\psi)a''(r)}{f(a)\ell(a'(r)\psi)} + \frac{2n+1}{r}\cdot\frac{\psi\varphi(a'(r)\psi)}{f(a)\ell(a'(r)\psi)}\\
&\le \frac{\theta}{\mathscr{b}}\cdot\psi^{2-\tau}\cdot\frac{\varphi'(a'(r))a''(r)}{f(a)\ell(a'(r))} + \frac{2n+1}{r}\cdot\frac{\psi\varphi(a'(r)\psi)}{\mathscr{b}\,f(a)\ell(a'(r))}\\
&\le \frac{\theta}{\mathscr{b}}\sigma + \frac{2n+1}{r}\cdot\frac{\psi\varphi(a'(r)\psi)}{\mathscr{b}\,f(a)\ell(a'(r))},
\end{aligned}$$

where in the last two inequalities we have used that $\ell(a'(r)\psi) \ge \mathscr{b}\ell(a'(r))$ by the nonincreasing $\mathscr{b}$–monotonicity of ℓ, (4.6.3) and the fact that $2-\tau > 0$ being $\tau < 1$ by (ϕ_1). Furthermore, $s^{\tau-1}\varphi(st) \le \theta\varphi(t)$ for all $s \in (0,1]$ and $t \in \mathbb{R}^+_0$, integrating (ϕ_1) with respect to the variable t. Hence,

$$\frac{\psi\varphi(a'(r)\psi)}{f(a)\ell(a'(r))} \le \psi^{2-\tau}\cdot\frac{\theta\,\varphi(a'(r))}{f(a)\ell(a'(r))} \le \frac{\theta\,\varphi(a'(r))}{f(a)\ell(a'(r))}$$

as above.

On the other hand, (4.6.3) and an integration over $[r_0, r]$, $r_0 < r < R$, yield

$$\varphi(a'(r)) = \varphi(a'(r_0)) + \sigma\int_{r_0}^{r} f(a(s))\ell(a'(s))ds.$$

In turn, using the monotonicity of f and the $\mathscr{b}$–monotonicity of ℓ we deduce

$$\begin{aligned}
\frac{\varphi(a'(r))}{f(a)\ell(a'(r))} &= \frac{\varphi(a'(r_0))}{f(a)\ell(a'(r))} + \frac{\sigma}{f(a)\ell(a'(r))}\int_{r_0}^{r} f(a(s))\ell(a'(s))ds\\
&\le \frac{\varphi(a'(r_0))}{\mathscr{b}f(a(r_0))\ell(a'(R))} + \sigma\frac{f(a(r))\int_{r_0}^{r}\ell(a'(s))\,ds}{\mathscr{b}f(a(r))\ell(a'(R))}\\
&\le \frac{\varphi(a'(r_0))}{\mathscr{b}f(a(r_0))\ell(a'(R))} + \sigma\frac{\ell(0)}{\mathscr{b}^2\ell(a'(R))}(r-r_0).
\end{aligned}$$

Combining all the above estimates we get for all r with $r_0 < r < R$

$$\begin{aligned}\frac{\Delta^{\varphi}_{\mathbb{H}^n} v}{f(v)\ell(|D_{\mathbb{H}^n} v|_{\mathbb{H}^n})} &\le \frac{\theta\sigma}{\mathscr{b}} + \frac{2n+1}{\mathscr{b}r}\left[\frac{\varphi(\mathscr{a}'(r_0))}{\mathscr{b}f(\mathscr{a}(r_0))\ell(\mathscr{a}'(R))} + \sigma\frac{\ell(0)}{\mathscr{b}^2\ell(\mathscr{a}'(R))}(r-r_0)\right]\\ &\le \frac{\sigma}{\mathscr{b}}\left[\theta + \frac{2n+1}{\mathscr{b}^2}\frac{\ell(0)}{\ell(\mathscr{a}'(R))}\right] + \frac{2n+1}{\mathscr{b}^2 r_0}\frac{\varphi(\mathscr{a}'(r_0))}{f(\mathscr{a}(r_0))\ell(\mathscr{a}'(R))}\end{aligned}$$

Since $K(0) = 0$ and $\mathscr{a}(r_0) = \varepsilon$, by (4.6.3) we have $\mathscr{a}'(r_0) = K^{-1}(\sigma F(\varepsilon)) \to 0$ as $\sigma \to 0$. We take σ so small, say $\sigma \le \bar{\sigma}$, in order to satisfy

$$\frac{\sigma}{\mathscr{b}}\left[\theta + \frac{2n+1}{\mathscr{b}^2}\frac{\ell(0)}{\ell(\mathscr{a}'(R))}\right] + \frac{2n+1}{\mathscr{b}^2 r_0}\frac{\varphi(\mathscr{a}'(r_0))}{f(\mathscr{a}(r_0))\ell(\mathscr{a}'(R))} \le 1.$$

This can be done, since $\mathscr{a}'(R) = K^{-1}(\sigma F(\mathscr{a}(R))) = K^{-1}(\sigma F(a)) \to 0$ as $\sigma \to 0^+$ and $\ell(0) > 0$.

In turn the claim is proved being v a radial solution of

$$\Delta^{\varphi}_{\mathbb{H}^n} v \le f(v)\ell(|D_{\mathbb{H}^n} v|_{\mathbb{H}^n})$$

in $B_R(q) \setminus \overline{B_{r_0}(q)}$, with $r_0 < R < \infty$, by $(\mathscr{H})$.

It remains to show that $\varepsilon \le v \le \eta$ on $B_{r_1}(q) \setminus B_{r_0}(q)$. To this aim, by the monotonicity of $\mathscr{a}$, it is enough to verify that $\mathscr{a}(r_1) = \mathscr{a}_\sigma(r_1) \le \eta$ for a certain σ, even smaller if necessary. Hence, from the trivial identity

$$\begin{aligned}\int_{\mathscr{a}(r_1)}^{a} \frac{ds}{K^{-1}(\sigma F(s))} &= R - r_1 = (R - r_0) + (r_0 - r_1)\\ &= \int_{\varepsilon}^{a} \frac{ds}{K^{-1}(\sigma F(s))} + r_0 - r_1\end{aligned}$$

and the fact that $\mathscr{a}(r_1) > \varepsilon$, we deduce

$$\int_{\varepsilon}^{\mathscr{a}(r_1)} \frac{ds}{K^{-1}(\sigma F(s))} = r_1 - r_0.$$

On the other hand, taking $\sigma > 0$ so small that $\int_{\varepsilon}^{\eta} ds/K^{-1}(\sigma F(s)) > r_1 - r_0$, then $\mathscr{a}(r_1) \le \eta$. This completes the proof of the lemma. □

Proof of Theorem 4.1.3. Let u be a nonnegative bounded entire solution of $(\mathscr{I})$. We denote $u^* = \sup_{\mathbb{H}^n} u(q)$. Assume by contradiction that $u \not\equiv u^*$. By the strong maximum principle, Proposition 4.2.2 as given in [68], we have $u < u^*$ on $\mathbb{H}^n$. Choose $r_0 > 0$ and define

$$u_0^* = \sup_{\overline{B_{r_0}}} u < u^*.$$

We now choose $\eta > 0$ so small that $u^* - u_0^* > 2\eta$. Next take $\tilde{q} \in \Omega_{r_0} = \mathbb{H}^n \setminus \overline{B_{r_0}}$, such that $u(\tilde{q}) > u^* - \eta$. Take also ε and a in such a way that $0 < \varepsilon < \eta$ and $a > 2\eta + \varepsilon$, obviously $a > \eta$. Put $r_1 = r(\tilde{q})$ so that $r_1 > r_0$. For such a choice of r_0, r_1, a, ε, η by Lemma 4.6.1 we can construct the radial function $v(q) = \boldsymbol{a}(r(q))$ on $B_R \setminus B_{r_0}$, with $\boldsymbol{a}$ and $R > r_1$, which is a solution of (4.6.1).

Being $v(\tilde{q}) \leq \eta$, it follows that

$$u(\tilde{q}) - v(\tilde{q}) > u^* - \eta - v(\tilde{q}) > u^* - \eta - \eta = u^* - 2\eta.$$

Since $u(q) - v(q) \leq u_0^* - \varepsilon < u^* - 2\eta - \varepsilon$ for all $q \in \partial B_{r_0}$ and

$$u(q) - v(q) \leq u^* - a < u^* - 2\eta - \varepsilon \quad \text{for all } q \in \partial B_R,$$

we deduce that the function $u - v$ attains a positive maximum μ on $B_R \setminus \overline{B_{r_0}}$. Let Γ_μ be a connected component of the set

$$\{q \in B_R \setminus \overline{B_{r_0}} : u(q) - v(q) = \mu\}.$$

For any $\xi \in \Gamma_\mu$, we have

$$u(\xi) > v(\xi), \quad |D_{\mathbb{H}^n} u(\xi)|_{\mathbb{H}^n} = |D_{\mathbb{H}^n} v(\xi)|_{\mathbb{H}^n}.$$

As a consequence in Γ_μ

$$\Delta^{\varphi}_{\mathbb{H}^n} u(\xi) \geq f(u(\xi))\ell(|D_{\mathbb{H}^n} u(\xi)|_{\mathbb{H}^n}) > f(v(\xi))\ell(|D_{\mathbb{H}^n} v(\xi)|_{\mathbb{H}^n}) \geq \Delta^{\varphi}_{\mathbb{H}^n} v(\xi),$$

since $f(u(\xi)) > f(v(\xi))$, by the strict monotonicity of f and since $\ell > 0$ in $\mathbb{R}_0^+$ by assumption. Hence by the C^1 regularity of u and v, in a sufficiently small neighborhood $\mathcal{N}$ of Γ_μ, the functions u and v satisfy

$$\Delta^{\varphi}_{\mathbb{H}^n} u \geq \Delta^{\varphi}_{\mathbb{H}^n} v \tag{4.6.4}$$

weakly in $\mathcal{N}$. Fix now a point $\xi \in \Gamma_\mu$, and for any $\varrho \in (0, \mu)$, denote by $\Omega_{\xi,\varrho}$ the connected component containing ξ of the set

$$\{q \in B_R \setminus \overline{B_{r_0}} : u(q) > v(q) + \varrho\}.$$

Let us now choose ϱ so close to μ that $\overline{\Omega_{\xi,\varrho}} \subset \mathcal{N}$. This can be shown by a compactness argument, for further details we refer to the proof of Theorem 4.3 of [17, page 702]. On $\partial\Omega_{\xi,\varrho}$ we have $u(q) = v(q) + \varrho$. Since $v(q) + \varrho$ solves

$$\Delta^{\varphi}_{\mathbb{H}^n}(v + \varrho) = \Delta^{\varphi}_{\mathbb{H}^n} v \leq f(v)\ell(|D_{\mathbb{H}^n} v|_{\mathbb{H}^n}) \leq f(v + \varrho)\ell(|D_{\mathbb{H}^n}(v + \varrho)|_{\mathbb{H}^n}),$$

thanks to the monotonicity of f and the fact that ℓ is nonnegative in $\mathbb{R}_0^+$, we get by Proposition 4.2.1, namely Proposition 4.2 of [17], that

$$u(q) \leq v(q) + \varrho.$$

But $u(\xi) = v(\xi) + \mu$. This contradicts the fact that $\xi \in \Omega_{\xi,\varrho}$ and shows that $u \equiv c$, where c is a nonnegative constant. $\square$

4.7 Existence under nondecreasing $\mathcal{C}$–monotonicity on ℓ

In this section we extend to the $\Delta^{\varphi}_{\mathbb{H}^n}$ operator Theorem 1.3–(i) of [68] given for the p–Laplacian in the Heisenberg group as well as the existence Theorem 6.1 of [17].

In particular, in [17], the proof of Theorem 6.1, relative to the existence of entire large solutions of $(\mathscr{I})$, uses the same main argument developed in [68]. We are planning to adapt the same construction in our context. It should be pointed out that Theorem 6.1 of [17] is proved under stronger conditions than (ϕL), namely assuming

$$\int_{0^+} \frac{\varphi'(t)}{\ell(t)}\,dt < \infty, \qquad \int^{\infty} \frac{\varphi'(t)}{\ell(t)}\,dt = \infty.$$

Proof of Theorem 4.1.4. Let $(VsKO)$ hold. We are going to construct a large entire radial stationary C^1 solution $u = u(|z|)$ of inequality $(\mathscr{I})$, that is u is of the form (4.3.5).

First, let us define implicitly the function w on $\mathbb{R}^+_0$ by setting

$$r = \int_1^{w(r)} \frac{ds}{K^{-1}(F(s))}. \tag{4.7.1}$$

Hence, w is well defined, $w(0) = 1$ and $w(r) > 1$ for all $r > 0$ because of the positivity of the left hand side of (4.7.1) and of the function $K^{-1} \circ F$ in $\mathbb{R}^+$. Clearly, $w(r) \to \infty$ for $r \to \infty$ by $(VsKO)$. Differentiating (4.7.1) in $\mathbb{R}^+$, we obtain

$$w'(r) = K^{-1}(F(w(r))) > 0, \tag{4.7.2}$$

so that $K(w') = F(w)$ and differentiating again

$$K'(w')w'' = f(w)w',$$

that is in $\mathbb{R}^+$ by (4.1.2) and $(\mathscr{H})$,

$$w''\varphi'(w') = f(w)\ell(w'). \tag{4.7.3}$$

Fix $\rho > 0$ and define $A_\rho = \{(z,t) \in \mathbb{H}^n : |z| < \rho\}$. Let u_1 be the radial stationary function defined on $\mathbb{H}^n \setminus A_\rho$ by the formula

$$u_1(z,t) = w(|z|), \quad |z| = r, \quad \text{in } \mathbb{H}^n \setminus A_\rho.$$

Of course, $|D_{\mathbb{H}^n}u_1|_{\mathbb{H}^n} = w'$ by (4.3.2), being $\psi \equiv 1$ and $w' > 0$. Using (4.3.6), (ϕ) and (4.7.3), we see that u_1 satisfies

$$\Delta^{\varphi}_{\mathbb{H}^n} u_1 = \varphi'(w')w'' + \frac{2n-1}{|z|}\varphi(w') \geq f(u_1)\ell(|D_{\mathbb{H}^n}u_1|_{\mathbb{H}^n})$$

in $\mathbb{H}^n \setminus A_\rho$. Hence u_1 is a large radial stationary C^1 solution of $(\mathscr{I})$ in $\mathbb{H}^n \setminus A_\rho$.

To produce a solution of $(\mathscr{I})$ in A_ρ, fix $v_0 > 0$, $\Theta > 0$ which are numbers to be chosen later. Put

$$v(r) = v_0 + \frac{1}{\Theta}\int_0^{r\Theta} \varphi^{-1}(\tau)d\tau, \tag{4.7.4}$$

obviously v is well defined since $\varphi^{-1}(0) = 0$ and by (ϕ). Define

$$u_2(z,t) = v(|z|), \quad |z| = r, \quad \text{in } A_\rho.$$

From

$$v'(r) = \varphi^{-1}(r\Theta), \quad r = |z|, \tag{4.7.5}$$

we have $v'(0) = 0$, and the function u_2 is of class C^1 in $\mathbb{H}^n$ with $D_{\mathbb{H}^n}u_2(0) = 0$. Using (4.3.6) along v, we get

$$\Delta^{\varphi}_{\mathbb{H}^n} u_2 = \varphi'(v')v'' + \frac{2n-1}{|z|}\varphi(v') = \Theta + \frac{2n-1}{|z|}\Theta|z| = 2n\Theta, \tag{4.7.6}$$

since $\varphi(v'(|z|)) = \Theta|z|$ by (4.7.5). If

$$2n\Theta \geq \mathcal{C} f(v(\rho))\ell(v'(\rho)), \tag{4.7.7}$$

where $\mathcal{C}$ is the constant of the $\mathcal{C}$–monotonicity of ℓ, then by virtue of v', $v'' > 0$ in $\mathbb{R}^+$, the monotonicity of f and the $\mathcal{C}$–monotonicity of ℓ, we obtain

$$\Delta^{\varphi}_{\mathbb{H}^n} u_2 \geq f(v(|z|))\ell(v'(|z|)) = f(u_2)\ell(|D_{\mathbb{H}^n}u_2|_{\mathbb{H}^n})$$

in A_ρ. In turn, assuming the validity of (4.7.7), we get that u_2 is a solution of inequality $(\mathscr{I})$ in A_ρ.

The next step is to join u_1, u_2 so that the resulting function is C^1. To this aim we choose the positive parameters ρ, Θ, v_0 in such a way that (4.7.7) and

$$v(\rho) = w(\rho), \qquad v'(\rho) = w'(\rho) \tag{4.7.8}$$

are verified. In other words, by (4.7.2) and (4.7.4) we need to prove that the following conditions hold

$$(i)\quad v_0 + \frac{1}{\Theta}\int_0^{\rho\Theta} \varphi^{-1}(\tau)d\tau = w(\rho), \qquad (ii)\quad \varphi^{-1}(\rho\Theta) = K^{-1}(F(w(\rho))),$$
$$(iii)\quad 2n\Theta \geq \mathcal{C} f(v(\rho))\ell(v'(\rho)).$$

Let $w(\rho) = \mu$. Then by (4.7.1) we have $\mu > 1$. Furthermore, by performing the change of variables $t = \varphi^{-1}(\tau)$ in the integral of (i) so that $d\tau = \varphi'(t)dt$ and $v'(\rho) = \varphi^{-1}(\rho\Theta)$ by (4.7.5), we have to verify

$$(i)\quad v_0 + \frac{1}{\Theta}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt = \mu, \qquad (ii)\quad \rho\,\Theta = \varphi(K^{-1}(F(\mu))),$$
$$(iii)\quad \Theta \geq \frac{\mathcal{C}}{2n} f(\mu)\ell(K^{-1}(F(\mu))).$$

Toward this aim, let μ be such that $1 < \mu \leq 2$ and define

$$\rho = \int_1^{\mu} \frac{ds}{K^{-1}(F(s))} > 0. \tag{4.7.9}$$

Since $K^{-1} \circ F$ is monotone increasing in $\mathbb{R}_0^+$ and positive in $\mathbb{R}^+$, then

$$\frac{\mu - 1}{K^{-1}(F(2))} \leq \rho \leq \frac{\mu - 1}{K^{-1}(F(1))}, \tag{4.7.10}$$

being $1 < \mu \leq 2$. Consequently $\rho \to 0$ as $\mu \to 1^+$. Thus we can choose μ so close to 1 that

$$\rho \leq \min\left\{\frac{1}{K^{-1}(F(2))}, \frac{2n\ \varphi(K^{-1}(F(1)))}{\mathcal{C}^2 f(2)\ell(K^{-1}(F(2)))}\right\} \tag{4.7.11}$$

With this choice of ρ we immediately obtain that Θ defined in (ii), satisfies (iii). Indeed, by (ii) and (4.7.11),

$$\begin{aligned}\Theta &= \frac{\varphi(K^{-1}(F(\mu)))}{\rho} \geq \frac{\mathcal{C}^2 f(2)\ell(K^{-1}(F(2)))}{2n} \cdot \frac{\varphi(K^{-1}(F(\mu)))}{\varphi(K^{-1}(F(1)))} \\ &\geq \frac{\mathcal{C} f(\mu)\ell(K^{-1}(F(\mu)))}{2n},\end{aligned}$$

where in the last inequality we have used that $\ell(K^{-1}(F(\mu))) \leq \mathcal{C}\ell(K^{-1}(F(2)))$ by the nondecreasing $\mathcal{C}$–monotonicity of ℓ, and the increasing monotonicity of f and of $\varphi \circ K^{-1} \circ F$.

Now it remains to prove the validity of (i). First observe that (ii) yields

$$\begin{aligned}\frac{1}{\Theta}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt &= \frac{\rho}{\varphi(K^{-1}(F(\mu)))}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt \\ &\leq \frac{\rho K^{-1}(F(\mu))}{\varphi(K^{-1}(F(\mu)))}\int_0^{K^{-1}(F(\mu))} \varphi'(t)dt \\ &= \rho K^{-1}(F(\mu)) < \rho K^{-1}(F(2)),\end{aligned}$$

being $\varphi' > 0$ in $\mathbb{R}^+$, $K^{-1} \circ F$ strictly increasing in $\mathbb{R}_0^+$ and $1 < \mu \leq 2$. In particular, by (4.7.11) and the above inequality, it follows

$$\frac{1}{\Theta}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt < 1,$$

so that it is possible to choose $v_0 > 0$ in such a way that (i) holds, precisely

$$v_0 = \mu - \frac{1}{\Theta}\int_0^{K^{-1}(F(\mu))} t\varphi'(t)dt > 0,$$

being $1 < \mu \leq 2$.

Hence, we conclude that, if μ is close enough to 1, the function

$$u(z) = \begin{cases} u_1(z) & \text{in} \quad \mathbb{H}^n \setminus A_\rho, \\ u_2(z) & \text{in} \quad A_\rho \end{cases}$$

is a large radial stationary C^1 solution of $(\mathscr{I})$. □

Chapter 5

Conclusions and open problems

In this chapter we present some open problems arising from the papers [13, 14, 15], which can be useful for future research. We divide this chapter in sections, each one related to a particular problem.

5.1 Schrödinger–Hardy systems involving two Laplacian operators in the Heisenberg group

In Chapter 2 we deal with the following Schrödinger–Hardy system

$$
(\mathcal{P}_1) \qquad \begin{cases} -\Delta^m_{\mathbb{H}^n} u + a(q)|u|^{m-2}u - \mu\psi^m \dfrac{|u|^{m-2}u}{r(q)^m} = H_u(q,u,v) & \text{in } \mathbb{H}^n, \\[2mm] -\Delta^p_{\mathbb{H}^n} v + b(q)|v|^{p-2}v - \sigma\psi^p \dfrac{|v|^{p-2}v}{r(q)^p} = H_v(q,u,v) & \text{in } \mathbb{H}^n, \end{cases}
$$

where μ and σ are real parameters, $Q = 2n+2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$, $1 < p < Q$, $1 < m \leq p < m^* = mQ/(Q-m)$.

In Theorem 2.1.1, which is the main result of the chapter, under the assumptions (H_1)–(H_4) on the nonlinearity H, for a and b of class $\mathscr{V}(\mathbb{H}^n)$, and with μ and σ satisfying a specific condition, system $(\mathcal{P}_1)$ has at least one nontrivial nonnegative entire solution $(u, v) \in W$.

An interesting result to analyze, would be the radial version of Theorem 2.1.1 under the solely condition $(V1)$ on a and b. In this way it is possible to cover the interesting case $a \equiv \text{Constant} > 0$ and $b \equiv \text{Constant} > 0$. For the radial stationary subcase, observe that, as noted in Section 4.3 radial stationary functions in the Heisenberg group $\mathbb{H}^n$ behave as Euclidean radial functions in $\mathbb{R}^{2n}$.

In Chapter 2 we treat the existence of entire solutions of the following system in $\mathbb{H}^n$, which includes also critical nonlinear terms and the nonnegative perturbations h and g

$$(\mathcal{P}_2) \quad \begin{cases} -\Delta^m_{\mathbb{H}^n} u + a(q)|u|^{m-2}u - \mu\psi^m \dfrac{|u|^{m-2}u}{r(q)^m} = H_u(q,u,v) + |u|^{m^*-2}u^+ \\ \qquad\qquad + \dfrac{\theta}{m^*}(u^+)^{\theta-1}(v^+)^{\vartheta} + h(q), \\ -\Delta^p_{\mathbb{H}^n} v + b(q)|v|^{p-2}v - \sigma\psi^p \dfrac{|v|^{p-2}v}{r(q)^p} = H_v(q,u,v) + |v|^{p^*-2}v^+ \\ \qquad\qquad + \dfrac{\vartheta}{m^*}(u^+)^{\theta}(v^+)^{\vartheta-1} + g(q), \end{cases}$$

where $\theta > 1$, $\vartheta > 1$ and $\theta + \vartheta = m^*$.

Analogously to the previous case, also for problem $(\mathcal{P}_2)$, it would be interesting to study the radial version of the existence theorem.

5.2 Existence problems involving Hardy and critical terms in the Heisenberg group

In Chapter 3 we treat the following problem

$$(\mathcal{P}_3) \quad \begin{cases} -\Delta^p_{\mathbb{H}^n} u - \gamma\psi^p \cdot \dfrac{|u|^{p-2}u}{r^p} = \sigma w(q)|u|^{s-2}u + \mathcal{k}(q)|u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where γ and σ are real parameters and Ω is a general open subset of $\mathbb{H}^n$, possibly $\Omega = \mathbb{H}^n$. The main result is Theorem 3.1.1, where we study the existence and the asymptotic behavior of nontrivial solutions of $(\mathcal{P}_3)$.

We tried to study a more complex problem, considering a different and more general exponent in the Hardy term. The problem is the following

$$\begin{cases} -\Delta^p_{\mathbb{H}^n} u - \gamma\psi^\alpha \cdot \dfrac{|u|^{p^*(\alpha)-2}u}{r^\alpha} = \sigma w(q)|u|^{s-2}u + \mathcal{k}(q)|u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \alpha \leqslant p$ and $p^*(\alpha) = p(Q-\alpha)/(Q-p)$ be the corresponding critical exponent. However, in the proof of the main Lemma 3.2.5, we did not get the desired contradiction, because of the difficulties arising from the new Hardy term formulation. Hence it would be interesting to analyze this kind of equation, obtained by a generalization of the Hardy term, in a general open subset Ω of the Heisenberg group.

In the second part of the chapter we study the existence of solutions of the following problem

$$(\mathcal{P}_4) \qquad \begin{cases} -\Delta^p_{\mathbb{H}^n} u - \gamma \|u\|^{p-p^*(\alpha)}_{\mathcal{H}_{\alpha,\Omega}} \psi^\alpha \dfrac{|u|^{p^*(\alpha)-2}u}{r^\alpha} \\ \qquad\qquad = \lambda a(q)|u|^{p-2}u + \sigma f(q,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω, differently from what we treat in the first problem of Chapter 3, is a bounded PS domain. This particular condition is crucial in order to get the concentration–compactness result (3.1.2). Thus it could be interesting to consider a more general class of domains for this kind of problems. Moreover we could investigate, at least in the case $p = 2$, the multiplicity results given in [6] for special unbounded domains having some symmetry.

5.3 Nonlinear elliptic inequalities with gradient terms in the Heisenberg group

In Chapter 4 we first study existence and uniqueness of nonnegative nontrivial radial stationary entire solutions u of

$$(\mathcal{E}) \qquad \Delta^\varphi_{\mathbb{H}^n} u = f(u)\ell(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}),$$

where $\Delta^\varphi_{\mathbb{H}^n} u$ is the φ–Laplacian on the Heisenberg group $\mathbb{H}^n$.

The main result related to this kind of equation, consists in Theorem 4.1.1, where we prove the existence of entire solutions of $(\mathcal{E})$, without requiring any monotonicity on ℓ. We emphasize that the hypotheses are fairly natural and general. Indeed Theorem 4.1.1 can be applied not only in the p–Laplacian case, $\varphi(s) = s^{p-1}$, $p > 1$, but also in the generalized mean curvature case, $\varphi(s) = s(1 + s^2)^{(p-2)/2}$, $p \in (1, 2)$. However the case $\varphi(\infty) < \infty$ is not covered, so it could consequently be an interesting starting point for further research.

Another significant result is given by Theorem 4.1.2, which concerns the uniqueness of radial stationary solutions of $(\mathcal{E})$. As in Theorem 4.1.1 we do not require any monotonicity assumption on ℓ in $\mathbb{R}_0^+$. The argument of the theorem is applicable when φ is the p–Laplacian operator with $1 < p \leq 2$. The remaining case $p > 2$ seems to be fairly delicate, since φ^{-1} fails to be of class $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_0^+)$ and so could be considered for further investigation.

Finally, we study for the following inequality

$$(\mathscr{I}) \qquad \Delta^{\varphi}_{\mathbb{H}^n} u \geq f(u)\ell(|D_{\mathbb{H}^n} u|_{\mathbb{H}^n})$$

Liouville type theorems, that is non–existence of nonnegative nontrivial entire solutions u. For this object, we recall the main theorems, that is Theorem 4.1.3 and Theorem 4.1.4. In the first one, ℓ is $\mathcal{b}$–monotone nonincreasing in $\mathbb{R}_0^+$, while in the second ℓ is assumed to be $\mathcal{C}$–monotone nondecreasing in $\mathbb{R}_0^+$.

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