ON THE AUTOMORPHISM GROUP OF A CLOSED G_2-STRUCTURE

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ABSTRACT. We study the automorphism group of a compact 7-manifold M endowed with a closed non-parallel G_2-structure, showing that its identity component is abelian with dimension bounded by $\min\{6, b_2(M)\}$. This implies the non-existence of compact homogeneous manifolds endowed with an invariant closed non-parallel G_2-structure. We also discuss some relevant examples.

1. INTRODUCTION

A seven-dimensional smooth manifold M admits a G_2-structure if the structure group of its frame bundle can be reduced to the exceptional Lie group $\mathrm{G}_2 \subset \mathrm{SO}(7)$. Such a reduction is characterized by the existence of a global 3-form $\varphi \in \Omega^3(M)$ satisfying a suitable non-degeneracy condition and giving rise to a Riemannian metric g_φ and to a volume form dV_φ on M via the identity

$$g_\varphi(X, Y)\, dV_\varphi = \frac{1}{6}\, \iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi,$$

for all $X, Y \in \mathfrak{X}(M)$ (see e.g. [1, 11]).

By [9], the intrinsic torsion of a G_2-structure φ can be identified with the covariant derivative $\nabla^{g_\varphi}\varphi$, and it vanishes identically if and only if both $d\varphi = 0$ and $d *_\varphi \varphi = 0$, $*_\varphi$ being the Hodge operator defined by g_φ and dV_φ. On a compact manifold, this last fact is equivalent to $\Delta_\varphi\varphi = 0$, where $\Delta_\varphi = d^*d + dd^*$ is the Hodge Laplacian of g_φ. A G_2-structure φ satisfying any of these conditions is said to be *parallel* and its associated Riemannian metric g_φ has holonomy contained in G_2. Consequently, g_φ is Ricci-flat and the automorphism group $\mathrm{Aut}(M, \varphi) := \{f \in \mathrm{Diff}(M) \mid f^*\varphi = f\}$ of (M, φ) is finite when M is compact and $\mathrm{Hol}(g_\varphi) = \mathrm{G}_2$.

Parallel G_2-structures play a central role in the construction of compact manifolds with holonomy G_2, and known methods to achieve this result involve *closed* G_2-structures, i.e., those whose defining 3-form φ satisfies $d\varphi = 0$ (see [1, 2, 5, 12, 14, 17]).

Most of the known examples of 7-manifolds admitting closed G_2-structures consist of simply connected Lie groups endowed with a left-invariant closed G_2-form φ [4, 7, 8, 10, 15]. Compact locally homogeneous examples can be obtained considering the quotient of such groups by a co-compact discrete subgroup, whenever this exists. Further non-homogeneous closed G_2-structures on the 7-torus can be constructed starting from the symplectic half-flat $\mathrm{SU}(3)$-structure on $\mathbb{T}^6$ described in [6, Ex. 5.1] (see Example 2.4 for details).

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Up to now, the existence of compact homogeneous 7-manifolds admitting an invariant closed non-parallel G_2-structure is still not known (cf. [15, Question 3.1] and [16, 21]). Moreover, among the G_2-manifolds acted on by a cohomogeneity one simple group of automorphisms studied in [3] no compact examples admitting a closed G_2-structure occur.

In this short note, we investigate the properties of the automorphism group $\mathrm{Aut}(M,\varphi)$ of a compact 7-manifold M endowed with a closed non-parallel G_2-structure φ. Our main results are contained in Theorem 2.1, where we show that the identity component $\mathrm{Aut}(M,\varphi)^0$ is necessarily abelian with dimension bounded by $\min\{6, b_2(M)\}$. In particular, this answers negatively [15, Question 3.1] and explains why compact examples cannot occur in [3]. Moreover, we also prove some interesting properties of the automorphism group action. Finally, we describe some relevant examples.

Similar results hold for compact symplectic half-flat 6-manifolds, and they will appear in a forthcoming paper.

2. The automorphism group

Let M be a seven-dimensional manifold endowed with a closed G_2-structure φ, and consider its automorphism group

$$\mathrm{Aut}(M,\varphi) \coloneqq \{f \in \mathrm{Diff}(M) \mid f^*\varphi = f\}.$$

Notice that $\mathrm{Aut}(M,\varphi)$ is a closed Lie subgroup of $\mathrm{Iso}(M, g_\varphi)$, and that the Lie algebra of its identity component $\mathrm{G} \coloneqq \mathrm{Aut}(M,\varphi)^0$ is

$$\mathfrak{g} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X\varphi = 0\}.$$

In particular, every $X \in \mathfrak{g}$ is a Killing vector field for the metric g_φ (cf. [17, Lemma 9.3]).

When M is compact, the Lie group $\mathrm{Aut}(M,\varphi) \subset \mathrm{Iso}(M, g_\varphi)$ is also compact, and we can show the following.

Theorem 2.1. *Let M be a compact seven-dimensional manifold endowed with a closed non-parallel G_2-structure φ. Then, there exists an injective map*

$$F : \mathfrak{g} \to \mathscr{H}^2(M), \quad X \mapsto \iota_X\varphi,$$

where $\mathscr{H}^2(M)$ is the space of Δ_φ-harmonic 2-forms. As a consequence, the following properties hold:

1) $\dim(\mathfrak{g}) \leq b_2(M)$*;*
2) $\mathfrak{g}$ *is abelian with* $\dim(\mathfrak{g}) \leq 6$*;*
3) for every $p \in M$, the isotropy subalgebra $\mathfrak{g}_p$ has dimension $\dim(\mathfrak{g}_p) \leq 2$, with equality only when $\dim(\mathfrak{g}) = 2, 3$;
4) the G*-action is free when* $\dim(\mathfrak{g}) \geq 5$.

Proof. Let $X \in \mathfrak{g}$. Then, $0 = \mathcal{L}_X\varphi = d(\iota_X\varphi)$, as φ is closed. We claim that $\iota_X\varphi$ is co-closed (see also [17, Lemma 9.3]). Indeed, by [13, Prop. A.3] we have

$$\iota_X\varphi \wedge \varphi = -2 *_\varphi (\iota_X\varphi),$$

from which it follows that

$$0 = d(\iota_X\varphi \wedge \varphi) = -2d *_\varphi (\iota_X\varphi).$$

Consequently, the 2-form $\iota_X\varphi$ is Δ_φ-harmonic and F is the restriction of the injective map $Z \mapsto \iota_Z\varphi$ to $\mathfrak{g}$. From this 1) follows.

As for 2), we begin observing that $\mathcal{L}_Y(\iota_X\varphi) = 0$ for all $X, Y \in \mathfrak{g}$, since every Killing field on a compact manifold preserves every harmonic form. Hence, we have

$$0 = \mathcal{L}_Y(\iota_X\varphi) = \iota_{[Y,X]}\varphi + \iota_X(\mathcal{L}_Y\varphi) = \iota_{[Y,X]}\varphi.$$

This proves that $\mathfrak{g}$ is abelian, the map $Z \mapsto \iota_Z\varphi$ being injective. Now, G is compact abelian and it acts effectively on the compact manifold M. Therefore, the principal isotropy is trivial and $\dim(\mathfrak{g}) \leq 7$. When $\dim(\mathfrak{g}) = 7$, M can be identified with the 7-torus $\mathbb{T}^7$ endowed with a left-invariant metric, which is automatically flat. Hence, if φ is closed non-parallel, then $\dim(\mathfrak{g}) \leq 6$.

In order to prove 3), we fix a point p of M and we observe that the image of the isotropy representation $\rho : \mathrm{G}_p \to \mathrm{O}(7)$ is conjugated into G_2. Since G_2 has rank two and G_p is abelian, the dimension of $\mathfrak{g}_p$ is at most two. If $\dim(\mathfrak{g}_p) = 2$, then the image of ρ is conjugate to a maximal torus of G_2 and its fixed point set in T_pM is one-dimensional. As $T_p(\mathrm{G}\cdot p) \subseteq (T_pM)^{\mathrm{G}_p}$, the dimension of the orbit $\mathrm{G}\cdot p$ is at most one, which implies that $\dim(\mathfrak{g})$ is either two or three.

The last assertion 4) is equivalent to proving that G_p is trivial for every $p \in M$ whenever $\dim(\mathfrak{g}) \geq 5$. In this case, $\dim(\mathfrak{g}_p) \leq 1$ by 3). Assuming that the dimension is precisely one, then the dimension of the orbit $\mathrm{G}\cdot p$ is at least four. This means that the G_p-fixed point set in T_pM is at least four-dimensional. On the other hand, the fixed point set of a closed one-parameter subgroup of G_2 is at most three-dimensional. This gives a contradiction. □

The following corollary answers negatively a question posed by Lauret in [15].

Corollary 2.2. *There are no compact homogeneous 7-manifolds endowed with an invariant closed non-parallel* G_2*-structure.*

Proof. The assertion follows immediately from point 2) of Theorem 2.1. □

In contrast to the last result, it is possible to exhibit non-compact homogeneous examples. Consider for instance a six-dimensional non-compact homogeneous space H/K endowed with an invariant symplectic half-flat $\mathrm{SU}(3)$-structure, namely an $\mathrm{SU}(3)$-structure (ω, ψ) such that $d\omega = 0$ and $d\psi = 0$ (see [20] for the classification of such spaces when H is semisimple and for more information on symplectic half-flat structures). If (ω, ψ) is not torsion-free, i.e., if $d(J\psi) \neq 0$, then the non-compact homogeneous space $(\mathrm{H}\times\mathbb{S}^1)/\mathrm{K}$ admits an invariant closed non-parallel G_2-structure defined by the 3-form

$$\varphi := \omega \wedge ds + \psi,$$

where ds denotes the global 1-form on $\mathbb{S}^1$.

Remark 2.3. In [3], the authors investigated G_2-manifolds acted on by a cohomogeneity one simple group of automorphisms. Theorem 2.1 explains why compact examples in the case of closed non-parallel G_2-structures do not occur.

The next example shows that G can be non-trivial, that the upper bound on its dimension given in 2) can be attained, and that 4) is only a sufficient condition.

Example 2.4. In [6], the authors constructed a symplectic half-flat SU(3)-structure (ω, ψ) on the 6-torus $\mathbb{T}^6$ as follows. Let $(x^1, \ldots, x^6)$ be the standard coordinates on $\mathbb{R}^6$, and let $a(x^1)$, $b(x^2)$ and $c(x^3)$ be three smooth functions on $\mathbb{R}^6$ such that

$$\lambda_1 \coloneqq b(x^2) - c(x^3), \quad \lambda_2 \coloneqq c(x^3) - a(x^1), \quad \lambda_3 \coloneqq a(x^1) - b(x^2),$$

are $\mathbb{Z}^6$-periodic and non-constant. Then, the following pair of $\mathbb{Z}^6$-invariant differential forms on $\mathbb{R}^6$ induces an SU(3)-structure on $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$:

$$\begin{aligned} \omega &= dx^{14} + dx^{25} + dx^{36}, \\ \psi &= -e^{\lambda_3}\, dx^{126} + e^{\lambda_2}\, dx^{135} - e^{\lambda_1}\, dx^{234} + dx^{456}, \end{aligned}$$

where $dx^{ijk\cdots}$ is a shorthand for the wedge product $dx^i \wedge dx^j \wedge dx^k \wedge \cdots$. It is immediate to check that both ω and ψ are closed and that $d(J\psi) \neq 0$ whenever at least one of the functions $a(x^1)$, $b(x^2)$, $c(x^3)$ is not identically zero. Thus, the pair (ω, ψ) defines a symplectic half-flat SU(3)-structure on the 6-torus. The automorphism group of $(\mathbb{T}^6, \omega, \psi)$ is $\mathbb{T}^3$ when $a(x^1)\, b(x^2)\, c(x^3) \not\equiv 0$, while it becomes $\mathbb{T}^4$ ($\mathbb{T}^5$) when one (two) of them vanishes identically.

Now, we can consider the closed G_2-structure on $\mathbb{T}^7 = \mathbb{T}^6 \times \mathbb{S}^1$ defined by the 3-form $\varphi = \omega \wedge ds + \psi$. Depending on the vanishing of none, one or two of the functions $a(x^1)$, $b(x^2)$, $c(x^3)$, φ is a closed non-parallel G_2-structure and the automorphism group of $(\mathbb{T}^7, \varphi)$ is $\mathbb{T}^4$, $\mathbb{T}^5$ or $\mathbb{T}^6$, respectively.

Finally, we observe that there exist examples where the upper bound on the dimension of $\mathfrak{g}$ given in 1) is more restrictive than the upper bound given in 2).

Example 2.5. In [4], the authors obtained the classification of seven-dimensional nilpotent Lie algebras admitting closed G_2-structures. An inspection of all possible cases shows that the Lie algebras whose second Betti number is lower than seven are those appearing in Table 1.

nilpotent Lie algebra $\mathfrak{n}$	$b_2(\mathfrak{n})$
$(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34})$	3
$(0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34} + e^{25})$	3
$(0, 0, e^{12}, 0, e^{13} + e^{24}, e^{14}, e^{46} + e^{34} + e^{15} + e^{23})$	5
$(0, 0, e^{12}, 0, e^{13}, e^{24} + e^{23}, e^{25} + e^{34} + e^{15} + e^{16} - 3e^{26})$	6

Table 1.

Let $\mathfrak{n}$ be one of the Lie algebras in Table 1, and consider a closed non-parallel G_2-structure φ on it. Then, left multiplication extends φ to a left-invariant G_2-structure of the same type on the simply connected nilpotent Lie group N corresponding to $\mathfrak{n}$. Moreover, as the structure constants of $\mathfrak{n}$ are integers, there exists a co-compact discrete subgroup $\Gamma \subset \mathrm{N}$ giving rise to a compact nilmanifold $\Gamma\backslash\mathrm{N}$ [18]. The left-invariant 3-form φ on N passes to the quotient defining an invariant closed non-parallel G_2-structure on $\Gamma\backslash\mathrm{N}$. By Nomizu Theorem [19], the de Rham cohomology group $\mathrm{H}^k_{\mathrm{dR}}(\Gamma\backslash\mathrm{N})$ is isomorphic to the cohomology group $\mathrm{H}^k(\mathfrak{n}^*)$ of the Chevalley-Eilenberg complex of $\mathfrak{n}$. Hence, $b_2(\Gamma\backslash\mathrm{N}) = b_2(\mathfrak{n})$.

References

[1] R. L. Bryant. Some remarks on G_2-structures. In *Proceedings of Gökova Geometry-Topology Conference 2005*, pages 75–109. Gökova Geometry/Topology Conference (GGT), Gökova, 2006.

[2] R. L. Bryant and F. Xu. Laplacian flow for closed G_2-structures: Short time behavior. arXiv:1101.2004.

[3] R. Cleyton and A. Swann. Cohomogeneity-one G_2-structures. *J. Geom. Phys.*, **44**(2-3), 202–220, 2002.

[4] D. Conti and M. Fernández. Nilmanifolds with a calibrated G_2-structure. *Differential Geom. Appl.*, **29**(4), 493–506, 2011.

[5] A. Corti, M. Haskins, J. Nordström, and T. Pacini. G_2-manifolds and associative submanifolds via semi-Fano 3-folds. *Duke Math. J.*, **164**(10), 1971–2092, 2015.

[6] P. de Bartolomeis and A. Tomassini. On the Maslov index of Lagrangian submanifolds of generalized Calabi-Yau manifolds. *Internat. J. Math.*, **17**(8), 921–947, 2006.

[7] M. Fernández. An example of a compact calibrated manifold associated with the exceptional Lie group G_2. *J. Differential Geom.*, **26**(2), 367–370, 1987.

[8] M. Fernández. A family of compact solvable G_2-calibrated manifolds. *Tohoku Math. J. (2)*, **39**(2), 287–289, 1987.

[9] M. Fernández and A. Gray. Riemannian manifolds with structure group G_2. *Ann. Mat. Pura Appl. (4)*, **132**, 19–45, 1982.

[10] A. Fino and A. Raffero. Closed G_2-structures on non-solvable Lie groups. arXiv:1712.09664.

[11] N. Hitchin. Stable forms and special metrics. In *Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000)*, v. 288 of *Contemp. Math.*, pp. 70–89. Amer. Math. Soc., 2001.

[12] D. D. Joyce. Compact Riemannian 7-manifolds with holonomy G_2. I, II. *J. Differential Geom.*, **43**(2), 291–328, 329–375, 1996.

[13] S. Karigiannis. Flows of G_2-structures. I. *Q. J. Math.*, **60**(4), 487–522, 2009.

[14] A. Kovalev. Twisted connected sums and special Riemannian holonomy. *J. Reine Angew. Math.*, **565**, 125–160, 2003.

[15] J. Lauret. Laplacian solitons: questions and homogeneous examples. *Differential Geom. Appl.*, **54**(B), 345–360, 2017.

[16] H. V. Lê and M. Munir. Classification of compact homogeneous spaces with invariant G_2-structures. *Adv. Geom.*, **12**(2), 302–328, 2012.

[17] J. D. Lotay and Y. Wei. Laplacian flow for closed G_2 structures: Shi-type estimates, uniqueness and compactness. *Geom. Funct. Anal.*, **27**(1), 165–233, 2017.

[18] A. I. Malčev. On a class of homogeneous spaces. *Amer. Math. Soc. Translation*, **1951**(39), 33 pp., 1951.

[19] K. Nomizu. On the cohomology of compact homogeneous spaces of nilpotent Lie groups. *Ann. of Math. (2)*, **59**, 531–538, 1954.

[20] F. Podestà, A. Raffero. Homogeneous symplectic half-flat 6-manifolds. arXiv:1711.10346.

[21] F. Reidegeld. Spaces admitting homogeneous G_2-structures. *Differential Geom. Appl.*, **28**(3), 301–312, 2010.

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