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## SEVERI-BOULIGAND TANGENTS, FRENET FRAMES AND RIESZ SPACES

LEONARDO MANUEL CABRER AND DANIELE MUNDICI

**ABSTRACT.** A compact set  $X \subseteq \mathbb{R}^2$  has an outgoing Severi-Bouligand tangent unit vector  $u$  at some point  $x \in X$  iff some principal quotient of the Riesz space  $\mathcal{R}(X)$  of piecewise linear functions on  $X$  is not archimedean. To generalize this preliminary result, we extend the classical definition of Frenet  $k$ -frame to any sequence  $\{x_i\}$  of points in  $\mathbb{R}^n$  converging to a point  $x$ , in such a way that when the  $\{x_i\}$  arise as sample points of a smooth curve  $\gamma$ , the Frenet  $k$ -frames of  $\{x_i\}$  and of  $\gamma$  at  $x$  coincide. Our method of computation of Frenet frames via sample sequences of  $\gamma$  does not require the knowledge of any higher-order derivative of  $\gamma$ . Given a compact set  $X \subseteq \mathbb{R}^n$  and a point  $x \in \mathbb{R}^n$ , a Frenet  $k$ -frame  $u$  is said to be a *tangent* of  $X$  at  $x$  if  $X$  contains a sequence  $\{x_i\}$  converging to  $x$ , whose Frenet  $k$ -frame is  $u$ . We prove that  $X$  has an outgoing  $k$ -dimensional tangent of  $X$  iff some principal quotient of  $\mathcal{R}(X)$  is not archimedean. If, in addition,  $X$  is convex, then  $X$  has no outgoing tangents iff it is a polyhedron.

## 1. INTRODUCTION

In [10, §53, p.59 and p.392] and [11, §1, p.99], Severi defined (outgoing) tangents of arbitrary subsets of the euclidean space  $\mathbb{R}^n$ . Subsequently and independently, Bouligand defined the same notion [2, p.32], which today is widely known as “Bouligand tangent”. Throughout we will adopt the following equivalent definition, where  $\|\cdot\|$  denotes euclidean norm and  $\text{conv}(Y)$  is the convex hull of  $Y \subseteq \mathbb{R}^n$ :

**Definition 1.1.** [8, pp.14 and 133] Let  $\emptyset \neq X \subseteq \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ . A unit vector  $u \in \mathbb{R}^n$  is a *Severi-Bouligand tangent* of  $X$  at  $x$  if  $X$  contains a sequence  $\{x_i\}$  such that  $x_i \neq x$  for all  $i$ ,  $\lim_{i \rightarrow \infty} x_i = x$ , and  $\lim_{i \rightarrow \infty} (x_i - x)/\|x_i - x\| = u$ . If for some  $\mu > 0$ ,  $\text{conv}(x, x + \mu u) \cap X = \{x\}$ , we say that  $u$  is *outgoing*.

For an equivalent algebraic handling of tangents, in Section 4 we introduce the Riesz space (=vector lattice)  $\mathcal{R}(X)$  of piecewise linear functions on any nonempty compact set  $X \subseteq \mathbb{R}^n$ . When  $n = 2$ , the geometric properties of  $X$  are immediately linked to the algebraic properties of  $\mathcal{R}(X)$  by the following elementary result (Lemma 4.3): *If  $\mathcal{R}(X)$  has a non-archimedean principal quotient then  $X$  has an outgoing Severi-Bouligand tangent.*

In Theorem 5.1 we will extend this result, as well as its converse, to all  $n$ . To this purpose, in Section 2 we introduce the notion of a Frenet  $k$ -frame of a sequence  $\{x_i\}$  of points in  $\mathbb{R}^n$ , as the natural generalization of the classical Frenet (Jordan)  $k$ -frame [5, 4] of a curve  $\gamma$ . Specifically, if the  $x_i$  arise as sample points of a smooth curve  $\gamma$  accumulating at some point  $x$  of  $\gamma$ , then the Frenet  $k$ -frame of  $\{x_i\}$  coincides with the Frenet  $k$ -frame of  $\gamma$  at  $x$ . This is Theorem 2.2. The proof yields a method to calculate the Frenet  $k$ -frame of a  $C^{k+1}$  curve  $\gamma$  at a point  $x$  without knowing the derivatives of any parametrization of  $\gamma$ : one just takes a sampling sequence  $\{x_i\}$  of points of  $\gamma$  converging to  $x$ , and then makes the linear algebra calculations as in the proof of the theorem. To show the wide applicability of our method, Example 2.5 provides a curve  $\gamma$  having no Frenet

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$k$ -frame at a point  $x$ , but such that the Frenet  $k$ -frame of each sequence of points of  $\gamma$  converging to  $x$  exists and is independent of the parametrization of  $\gamma$ .

In Section 3 we deal with the relationship between the Frenet  $k$ -frame  $u = (u_1, \dots, u_k)$  of a sequence  $\{x_i\}$  in  $\mathbb{R}^n$  converging to  $x$ , and any simplex  $T \subseteq \mathbb{R}^n$  containing  $\{x_i\}$ . Theorem 3.3 shows that  $T$  automatically contains the simplex  $\text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k)$ , for some  $\lambda_1, \dots, \lambda_k > 0$ . This elementary result will find repeated use in the rest of the paper.

As a  $k$ -dimensional generalization of the classical Severi-Bouligand tangents, we then say that a Frenet  $k$ -frame  $u$  is *tangent* at  $x$  to a compact set  $X \subseteq \mathbb{R}^n$  if  $X$  contains a sequence  $\{x_i\}$  converging to  $x$ , whose Frenet  $k$ -frame is  $u$ . Then Theorem 5.1 provides the desired strengthening of Lemma 4.3, showing that  $X$  has no outgoing tangent iff every principal ideal of  $\mathcal{R}(X)$  is an intersection of maximal ideals. This latter property is considered in the literature for various classes of structures: For commutative noetherian rings it is known as “von Neumann regularity”; frames having this property are known as “Yosida frames”, [7, 2.1]; Chang MV-algebras with this property are said to be “strongly semisimple”, [3]. As a corollary of Stone representation ([6, 4.4]), every boolean algebra is strongly semisimple.

Since  $\{+, -, \wedge, \vee\}$ -reducts of Riesz spaces with strong unit are lattice-ordered abelian groups with strong unit, and the latter are categorically equivalent to MV-algebras, [9, 3.9], following [3] we say that a Riesz space  $R$  is *strongly semisimple* if every principal ideal of  $R$  is an intersection of maximal ideals of  $R$ . Equivalently, every principal quotient of  $R$  is archimedean. A large class of examples of strongly semisimple Riesz spaces with totally disconnected maximal spectrum is immediately provided by hyperarchimedean Riesz spaces, [1]. At the other extreme, when  $X$  is a polyhedron,  $\mathcal{R}(X)$  is strongly semisimple, (see Proposition 6.2).

Using Theorem 5.1, in Theorem 6.4 we prove that a nonempty compact *convex* subset  $X \subseteq \mathbb{R}^n$  has no outgoing tangent iff  $X$  has only finitely many extreme points iff  $X$  is a polyhedron. This shows the naturalness of Definition 4.1 of “outgoing tangent” as a  $k$ -dimensional extension of the classical Severi-Bouligand tangent. Counterexamples of Theorem 6.4 are easily found in case  $X$  is not convex (see Example 6.3).

The only prerequisite for this paper is a working knowledge of elementary polyhedral topology (as given, e.g., by the first chapters of [12]), and of the classical Yosida (Kakutani-Gelfand-Stone) correspondence between points of  $X$  and maximal ideals of the Riesz space  $\mathcal{R}(X)$ . See [6] for a comprehensive account.

## 2. THE FRENET FRAME OF A SEQUENCE $\{x_i\} \subseteq \mathbb{R}^n$

Given two sequences  $\{p_i\}, \{q_i\} \subseteq \mathbb{R}$ , by writing  $\lim_{i \rightarrow \infty} p_i/q_i = r$  we understand that  $q_i \neq 0$  for each  $i$ , and  $\lim_{i \rightarrow \infty} p_i/q_i$  exists and equals  $r$ .

For any vector  $y \in \mathbb{R}^n$  and linear subspace  $L$  of  $\mathbb{R}^n$ , the orthogonal projection of  $y$  onto  $L$  is denoted

$$\text{proj}_L(y).$$

For our generalization of Severi-Bouligand tangents we first extend Definition 1.1, replacing the unit vector  $u \in \mathbb{R}^n$  therein by a  $k$ -tuple  $\{u_1, \dots, u_k\}$  of pairwise orthogonal unit vectors in  $\mathbb{R}^n$ .

**Definition 2.1.** Given a sequence  $\sigma = \{x_i\}$  of points in  $\mathbb{R}^n$  converging to  $x$ , and a  $k$ -tuple  $(u_1, \dots, u_k)$  of pairwise orthogonal unit vectors in  $\mathbb{R}^n$ , we say:

- $u_1$  is the Frenet 1-frame of  $\sigma$  if  $u_1 = \lim_{i \rightarrow \infty} (x_i - x)/\|x_i - x\|$ ;
- $(u_1, \dots, u_k)$  is the Frenet  $k$ -frame of  $\sigma$  if  $(u_1, \dots, u_{k-1})$  is the Frenet  $(k-1)$ -frame of  $\sigma$ , and

$$u_k = \lim_{i \rightarrow \infty} \frac{x_i - x - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{k-1}}(x_i - x)}{\|x_i - x - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{k-1}}(x_i - x)\|}.$$

Following [5], for  $[a, b] \subseteq \mathbb{R}$  an interval, suppose  $\phi: [a, b] \rightarrow \mathbb{R}^n$  is a  $C^k$  function such that for all  $a \leq t < b$ , the  $k$ -tuple of vectors  $(\phi'(t), \phi''(t), \dots, \phi^{(k)}(t))$  forms a linearly independent set in  $\mathbb{R}^n$ . Then the Gram-Schmidt process yields an orthonormal  $k$ -tuple  $(v_1(t), \dots, v_k(t))$ , called the *Frenet  $k$ -frame* of  $\phi$  at  $\phi(t)$ .

The terminology of Definition 2.1 is justified by the following result:

**Theorem 2.2.** Suppose  $\phi: [a, b] \rightarrow \mathbb{R}^n$  is a  $C^{k+1}$  function. Let  $a \leq t_0 < b$  be such that the vectors  $\phi'(t_0), \phi''(t_0), \dots, \phi^{(k)}(t_0)$  are linearly independent. Then for every sequence  $t_1, t_2, \dots$  in  $[t_0, b] \setminus \{t_0\}$  converging to  $t_0$ , the Frenet  $k$ -frame of  $\{\phi(t_i)\}$  exists and is equal to the Frenet  $k$ -frame of  $\phi$  at  $\phi(t_0)$ .

*Proof.* We can write

$$\phi(t) = \phi(t_0) + \phi'(t_0)(t - t_0) + \frac{\phi''(t_0)}{2}(t - t_0)^2 + \dots + \frac{\phi^{(k)}(t_0)}{k!}(t - t_0)^k + R(t), \quad (1)$$

where the remainder  $R: [a, b] \rightarrow \mathbb{R}^n$  satisfies

$$\|R(t)\| \leq M(t - t_0)^{k+1} \quad \text{for some } 0 \leq M \in \mathbb{R}. \quad (2)$$

Let  $(v_1, \dots, v_k)$  be the Frenet  $k$ -frame of  $\phi$  at  $\phi(t_0)$ . Then  $v_1 = \phi'(t_0)/\|\phi'(t_0)\|$ , and for each  $1 < j \leq k$ ,

$$v_j = \frac{\phi^{(j)}(t_0) - \text{proj}_{\mathbb{R}v_1 + \dots + \mathbb{R}v_{j-1}}(\phi^{(j)}(t_0))}{\|\phi^{(j)}(t_0) - \text{proj}_{\mathbb{R}v_1 + \dots + \mathbb{R}v_{j-1}}(\phi^{(j)}(t_0))\|}.$$

By induction on  $1 \leq j \leq k$  we will prove that the Frenet  $j$ -frame  $(u_1, \dots, u_j)$  of the sequence  $\{\phi(t_i)\}$  (exists and) coincides with the Frenet  $j$ -frame  $(v_1, \dots, v_j)$  of  $\phi$  at  $\phi(t_0)$ .

*Basis:* Since  $\|\phi'(t_0)\| \neq 0$ , for all suitably large  $i$  we have  $\phi(t_i) \neq \phi(t_0)$  and

$$\begin{aligned} u_1 &= \lim_{i \rightarrow \infty} \frac{\phi(t_i) - \phi(t_0)}{\|\phi(t_i) - \phi(t_0)\|} \\ &= \lim_{i \rightarrow \infty} \frac{(\phi(t_i) - \phi(t_0))/(t_i - t_0)}{\|(\phi(t_i) - \phi(t_0))/(t_i - t_0)\|} \\ &= \frac{\lim_{i \rightarrow \infty} (\phi(t_i) - \phi(t_0))/(t_i - t_0)}{\|\lim_{i \rightarrow \infty} (\phi(t_i) - \phi(t_0))/(t_i - t_0)\|} \\ &= \frac{\phi'(t_0)}{\|\phi'(t_0)\|} \\ &= v_1. \end{aligned}$$

*Induction Step:* By induction hypothesis, for each  $1 \leq j < k$  the  $j$ -tuple  $(v_1, \dots, v_j)$  coincides with the Frenet  $j$ -frame  $(u_1, \dots, u_j)$  of the sequence  $\{\phi(t_i)\}$ . Let the linear subspace  $S_j$  of  $\mathbb{R}^n$  be defined by

$$S_j = \mathbb{R}u_1 + \dots + \mathbb{R}u_j = \mathbb{R}v_1 + \dots + \mathbb{R}v_j = \mathbb{R}\phi'(t_0) + \dots + \mathbb{R}\phi^{(j)}(t_0).$$

From (2) we have

$$\frac{\|R(t) - \text{proj}_{S_j}(R(t))\|}{(t - t_0)^{j+1}} \leq M(t - t_0)^{k-j}. \quad (3)$$

For each  $l = j+1, \dots, k$  let us define the vector  $\alpha_l \in \mathbb{R}^n$  by

$$\alpha_l = \frac{\phi^{(l)}(t_0) - \text{proj}_{S_j}(\phi^{(l)}(t_0))}{l!}, \quad (4)$$

whence in particular,

$$\|\alpha_{j+1}\| = \frac{\|\phi^{(j+1)}(t_0) - \text{proj}_{S_j}(\phi^{(j+1)}(t_0))\|}{(j+1)!} \neq 0.$$

By (1),

$$\begin{aligned} \phi(t_i) - \phi(t_0) - \text{proj}_{S_j}(\phi(t_i) - \phi(t_0)) &= \\ \alpha_{j+1}(t_i - t_0)^{j+1} + \dots + \alpha_k(t_i - t_0)^k + R(t_i) - \text{proj}_{S_j}(R(t_i)). \end{aligned} \quad (5)$$

From (3)-(5) we get

$$\begin{aligned}
u_{j+1} &= \lim_{i \rightarrow \infty} \frac{\phi(t_i) - \phi(t_0) - \text{proj}_{S_j}(\phi(t_i) - \phi(t_0))}{\|\phi(t_i) - \phi(t_0) - \text{proj}_{S_j}(\phi(t_i) - \phi(t_0))\|} \\
&= \lim_{i \rightarrow \infty} \frac{\alpha_{j+1}(t_i - t_0)^{j+1} + \cdots + \alpha_k(t_i - t_0)^k + R(t_i) - \text{proj}_{S_j}(R(t_i))}{\|\alpha_{j+1}(t_i - t_0)^{j+1} + \cdots + \alpha_k(t_i - t_0)^k + R(t_i) - \text{proj}_{S_j}(R(t_i))\|} \\
&= \lim_{i \rightarrow \infty} \frac{\sum_{l=j+1}^k \alpha_l(t_i - t_0)^{l-(j+1)} + (R(t_i) - \text{proj}_{S_j}(R(t_i))) \cdot (t_i - t_0)^{-(j+1)}}{\|\sum_{l=j+1}^k \alpha_l(t_i - t_0)^{l-(j+1)} + (R(t_i) - \text{proj}_{S_j}(R(t_i))) \cdot (t_i - t_0)^{-(j+1)}\|} \\
&= \frac{\alpha_{j+1}}{\|\alpha_{j+1}\|} = \frac{\phi^{(j+1)}(t_0) - \text{proj}_{S_j}(\phi^{(j+1)}(t_0))}{\|\phi^{(j+1)}(t_0) - \text{proj}_{S_j}(\phi^{(j+1)}(t_0))\|} = v_{j+1}.
\end{aligned}$$

This concludes the proof.  $\square$

**Remark 2.3.** The assumption  $\phi \in C^{k+1}$  can be relaxed to  $\phi \in C^k$ , so long as the  $k$ th Taylor remainder  $R(t)$  satisfies (2).

**Remark 2.4.** Theorem 2.2 yields a method to calculate the Frenet  $k$ -frame of a  $C^{k+1}$  curve, not involving higher-order derivatives, but taking instead a sampling sequence  $\{x_i\}$  of points on the curve, and then making the elementary linear algebra calculations in the proof above.

The wide applicability of this method is shown by the following example:

**Example 2.5.** Let  $\phi: [0, 1] \rightarrow \mathbb{R}^2$  be defined by  $\phi(x) = (x, x^3)$ . Then  $\phi'(0) = (1, 0)$  and  $\phi''(0) = (0, 0)$ . The Frenet 1-frame of  $\phi$  at  $(0, 0)$  is the vector  $(1, 0)$ , but  $\phi$  has no Frenet 2-frame at  $(0, 0)$ . And yet, letting  $\mathbb{R}(1, 0)$  denote the linear subspace of  $\mathbb{R}^2$  given by the  $x$ -axis, every sequence  $\{t_i\} \in [0, 1] \setminus \{0\}$  converging to 0 satisfies

$$\lim_{i \rightarrow \infty} \frac{\phi(t_i) - \phi(0) - \text{proj}_{\mathbb{R}(1,0)}(\phi(t_i) - \phi(0))}{\|\phi(t_i) - \phi(0) - \text{proj}_{\mathbb{R}(1,0)}(\phi(t_i) - \phi(0))\|} = \lim_{i \rightarrow \infty} \frac{(0, t_i^3)}{\|(0, t_i^3)\|} = (0, 1).$$

We have shown: *There exist a curve  $\gamma$  having no Frenet  $k$ -frame at a point  $x$ , but the Frenet  $k$ -frame of every sequence of points of  $\gamma$  converging to  $x$  exists and is independent of the parametrization of  $\gamma$ .*

**Example 2.6.** While under the hypotheses of Theorem 2.2 the Frenet  $k$ -frames of any two sampling sequences of a curve  $\gamma$  at a point  $x \in \gamma$  are equal, the map  $\psi(x) = (x, x^2 \sin(1/x)): [0, 1] \rightarrow \mathbb{R}^2$  (with the proviso that  $\psi(0) = (0, 0)$ ), yields an example of a curve  $\gamma$  that is not  $C^2$  and has two sequences  $\{x_i\}$  and  $\{y_i\}$  of points of  $\gamma$  both converging to the same point  $(0, 0)$  of  $\gamma$ , but having different Frenet 2-frames.

### 3. SIMPLEXES AND FRENET FRAMES

Fix  $n = 1, 2, \dots$ . For any subset  $E$  of the euclidean space  $\mathbb{R}^n$ , the *convex hull*  $\text{conv}(E)$  is the set of all *convex combinations* of elements of  $E$ . We say that  $E$  is *convex* if  $E = \text{conv}(E)$ . For any subset  $Y$  of  $\mathbb{R}^n$ , the *affine hull*  $\text{aff}(Y)$  of  $Y$  is the set of all *affine combinations* in  $\mathbb{R}^n$  of elements of  $Y$ . A set  $\{y_1, \dots, y_m\}$  of points in  $\mathbb{R}^n$  is said to be *affinely independent* if none of its elements is an affine combination of the remaining elements. The *relative interior*  $\text{relint}(C)$  of a convex set  $C \subseteq \mathbb{R}^n$  is the interior of  $C$  in the affine hull of  $C$ . For  $0 \leq d \leq n$ , a  *$d$ -simplex*  $T$  in  $\mathbb{R}^n$  is the convex hull  $\text{conv}(v_0, \dots, v_d)$  of  $d + 1$  affinely independent points in  $\mathbb{R}^n$ . The *vertices*  $v_0, \dots, v_d$  are uniquely determined by  $T$ . A *face* of  $T$  is the convex hull of a subset  $V$  of vertices of  $T$ . If the cardinality of  $V$  is  $d$ , then  $V$  is said to be a *facet* of  $T$ .

The *positive cone* of  $Y \subseteq \mathbb{R}^n$  at a point  $x \in Y$  is the set

$$\text{Cone}(Y, x) = \{y \in \mathbb{R}^n \mid x + \rho(y - x) \in Y \text{ for some } \rho > 0\}. \quad (6)$$

When  $T$  is a simplex,  $\text{Cone}(T, x)$  is closed. If  $F$  is a face of  $T$  and  $x \in \text{relint}(F)$  then for each  $y \in F$  we have

$$\text{Cone}(T, x) = \text{aff}(F) + \text{Cone}(T, y). \quad (7)$$

In particular, if  $x \in \text{relint}(T)$  then  $\text{Cone}(T, x) = \text{aff}(T)$ .

**Lemma 3.1.** *Suppose  $T \subseteq \mathbb{R}^n$  is a simplex and  $F$  is a face of  $T$ .*

- (a) *If  $S$  is an arbitrary simplex contained in  $T$ , and  $F \cap \text{relint}(S) \neq \emptyset$ , then  $S$  is contained in  $F$ .*
- (b) *A point  $z$  lies in  $\text{relint}(F)$  iff  $F$  is the smallest face of  $T$  containing  $z$ .*

*Proof.* (a) Let  $F_1, \dots, F_u$  be the facets of  $T$ , with their respective affine hulls  $H_1, \dots, H_u$ . Each  $H_j$  is the boundary of the closed half-space  $H_j^+ \subseteq T$  and of the other closed half-space  $H_j^-$ . Without loss of generality,  $F_1, \dots, F_t$  are the facets of  $T$  containing  $F$ . Then  $\text{aff}(F) = H_1 \cap \dots \cap H_t$  and  $F = (H_{t+1}^+ \cap \dots \cap H_u^+) \cap \text{aff}(F)$ . By way of contradiction, suppose  $x \in F \cap \text{relint}(S)$  and  $y \in S \setminus F$ . For some  $\epsilon > 0$  the segment  $\text{conv}(x + \epsilon(y - x), x - \epsilon(y - x))$  is contained in  $S$ . For some hyperplane  $H \in \{H_1, \dots, H_t\}$  the point  $y$  lies in the open half-space  $\text{int}(H^+) = \mathbb{R}^n \setminus H^-$ , where “int” denotes topological interior. Now  $x + \epsilon(y - x) \in \text{int}(H^+)$  and  $x - \epsilon(y - x) \in \text{int}(H^-)$ , whence  $x - \epsilon(y - x) \notin T$ , which contradicts  $S \subseteq T$ .

(b) This easily follows from (a).  $\square$

**Proposition 3.2.** *Let  $x \in \mathbb{R}^n$  and  $u_1, \dots, u_m$  be linearly independent vectors in  $\mathbb{R}^n$ . Let  $\lambda_1, \mu_1, \dots, \lambda_m, \mu_m > 0$ . Then the intersection of the two  $m$ -simplexes  $\text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_m u_m)$  and  $\text{conv}(x, x + \mu_1 u_1, \dots, x + \mu_1 u_1 + \dots + \mu_m u_m)$  is an  $m$ -simplex of the form  $\text{conv}(x, x + \nu_1 u_1, \dots, x + \nu_1 u_1 + \dots + \nu_m u_m)$  for uniquely determined real numbers  $\nu_1, \dots, \nu_m > 0$ .*

*Proof.* We argue by induction on  $t = 1, \dots, m$ . The cases  $t = 1, 2$  are trivial. Proceeding inductively, for any simplex  $W = \text{conv}(x, x + \theta_1 u_1, \dots, x + \theta_1 u_1 + \dots + \theta_t u_t)$ , let  $W' = \text{conv}(x, x + \theta_1 u_1, \dots, x + \theta_1 u_1 + \dots + \theta_{t-1} u_{t-1})$  and  $W'' = \text{conv}(x, x + \theta_1 u_1, \dots, x + \theta_1 u_1 + \dots + \theta_{t-2} u_{t-2})$ . By (7), for each  $y \in W' \setminus W''$  the half-line from  $y$  in direction  $u_t$  intersects  $W$  in a segment  $\text{conv}(y, y + \gamma u_t)$  for some  $\gamma > 0$  depending on  $y$ . Now let

$$U_t = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_t u_t),$$

$$V_t = \text{conv}(x, x + \mu_1 u_1, \dots, x + \mu_1 u_1 + \dots + \mu_t u_t).$$

We then have

$$U_{t-1} = U'_t = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_{t-1} u_{t-1}),$$

$$V_{t-1} = V'_t = \text{conv}(x, x + \mu_1 u_1, \dots, x + \mu_1 u_1 + \dots + \mu_{t-1} u_{t-1}),$$

and

$$U_{t-2} = U''_t = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_{t-2} u_{t-2}),$$

$$V_{t-2} = V''_t = \text{conv}(x, x + \mu_1 u_1, \dots, x + \mu_1 u_1 + \dots + \mu_{t-2} u_{t-2}).$$

By induction hypothesis, for uniquely determined  $\nu_1, \dots, \nu_{t-1} > 0$  we can write

$$U'_t \cap V'_t = \text{conv}(x, x + \nu_1 u_1, \dots, x + \nu_1 u_1 + \dots + \nu_{t-1} u_{t-1}).$$

The point  $z = x + \nu_1 u_1 + \dots + \nu_{t-1} u_{t-1}$  lies in  $U'_t \setminus U''_t$ . Let  $\eta_1$  be the largest  $\eta$  such that  $z + \eta u_t$  lies in  $U_t$ . Since  $z \in V'_t \setminus V''_t$ , let similarly  $\eta_2$  be the largest  $\eta$  such that  $z + \eta u_t$  lies in  $V_t$ . As already noted at the beginning of this proof, the real number  $\nu_t = \min(\eta_1, \eta_2)$  is  $> 0$ . Evidently,  $\nu_t$  is the largest  $\eta$  such that  $z + \eta u_t$  lies in  $U_t \cap V_t$ . We conclude that  $U_t \cap V_t = \text{conv}(x, x + \nu_1 u_1, \dots, x + \nu_1 u_1 + \dots + \nu_t u_t)$ .  $\square$

The following key result will find repeated use in the rest of this paper:

**Theorem 3.3.** *Let  $(u_1, \dots, u_k)$  be the Frenet  $k$ -frame of a sequence  $\{x_i\}$  in  $\mathbb{R}^n$  converging to  $x$ . Suppose a simplex  $T \subseteq \mathbb{R}^n$  contains  $\{x_i\}$ . Then  $T$  contains the simplex  $\text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k)$ , for some  $\lambda_1, \dots, \lambda_k > 0$ .*

*Proof.* We will prove the following stronger statement:

*Claim.* For each  $l \in \{1, \dots, k\}$  there exist  $\lambda_1, \dots, \lambda_l > 0$  such that:

- (i)  $\text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_l u_l) \subseteq T$  and
- (ii) letting  $F_l$  be the smallest face of  $T$  containing the point  $z_l = x + \lambda_1 u_1 + \dots + \lambda_l u_l$  (which by Lemma 3.1(b) is equivalent to  $z_l \in \text{relint}(F_l)$ ), we have the inclusion  $\text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_l u_l) \subseteq F_l$ .

The proof is by induction on  $l = 1, \dots, k$ .

*Basis Step* ( $l = 1$ ): Since each  $x_i$  is in  $T$  then  $x + (x_i - x)/\|x_i - x\| \in \text{Cone}(T, x)$ . Since  $\text{Cone}(T, x)$  is closed, then  $x + u_1 \in \text{Cone}(T, x)$ . From (6) we obtain an  $\epsilon > 0$  such that  $x + \epsilon u_1 \in T$ . Let  $\lambda_1 = \epsilon/2$ . Then  $\text{conv}(x, x + \lambda_1 u_1) \subseteq \text{conv}(x, x + \epsilon u_1) \subseteq T$ , and (i) follows. Let  $F_1$  be the smallest face of  $T$  containing the point  $z_1 = x + \lambda_1 u_1$ . Evidently,  $z_1 \in \text{relint}(\text{conv}(x, x + \epsilon u_1))$ . By Lemma 3.1(b),  $z_1 \in \text{relint}(F_1)$ . By Lemma 3.1(a),  $F_1 \supseteq \text{conv}(x, x + \epsilon u_1) \supseteq \text{conv}(x, x + \lambda_1 u_1)$ . This proves (ii) and concludes the proof of the basis step.

*Induction Step:* For  $1 \leq l < k$ , induction yields  $\lambda_1, \dots, \lambda_l > 0$  such that, letting  $C_l = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_l u_l)$  and  $z_l = x + \lambda_1 u_1 + \dots + \lambda_l u_l$ , we have  $C_l \subseteq T$ . Further, letting  $F_l$  be the smallest face of  $T$  containing  $z_l$ , we have  $C_l \subseteq F_l$ , whence  $\text{aff}(C_l) = x + \mathbb{R}u_1 + \dots + \mathbb{R}u_l \subseteq \text{aff}(F_l)$ . Since  $z_l \in \text{relint}(F_l)$  and  $x_i - x \in \text{Cone}(T, x)$ , from (7) we obtain

$$z_l + \frac{x_i - x - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_l}(x_i - x)}{\|x_i - x - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_l}(x_i - x)\|} \in \text{Cone}(T, z_l).$$

$\text{Cone}(T, z_l)$  is closed, because  $z_l + u_{l+1} \in \text{Cone}(T, z_l)$ . By (6), there exists  $\epsilon > 0$  such that  $z_l + \epsilon u_{l+1} \in T$ , whence  $\text{conv}(z_l, z_l + \epsilon u_{l+1}) \subseteq T$ . Setting now  $\lambda_{l+1} = \epsilon/2$  and  $z_{l+1} = z_l + \lambda_{l+1} u_{l+1}$ , condition (i) in the claim above follows from the identity

$$\text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_{l+1} u_{l+1}) = \text{conv}(C_l \cup \{z_{l+1}\}) \subseteq T.$$

Let  $F_{l+1}$  be the smallest face of  $T$  containing the point  $z_{l+1} \in \text{relint}(\text{conv}(z_l, z_l + \epsilon u_{l+1}))$ . By Lemma 3.1(b),  $z_{l+1} \in \text{relint}(F_{l+1})$ . By Lemma 3.1(a),

$$F_{l+1} \supseteq \text{conv}(z_l, z_l + \epsilon u_{l+1}) \supseteq \text{conv}(z_l, z_l + \lambda_{l+1} u_{l+1}).$$

The minimality property of  $F_l$  yields  $F_l \subseteq F_{l+1}$ . By induction hypothesis,  $C_l \subseteq F_{l+1}$ . In conclusion,  $\text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_{l+1} u_{l+1}) = \text{conv}(C_l \cup \{z_{l+1}\}) \subseteq F_{l+1}$ , as required to prove (ii) and to complete the proof.  $\square$

#### 4. TANGENTS OF $X$ , PRINCIPAL IDEALS OF $\mathcal{R}(X)$ : THE CASE $X \subseteq \mathbb{R}^2$

For  $k = 1$  the following definition boils down to Definition 1.1 of Severi-Bouligand tangent vector. As in Definition 1.1,  $X$  is an arbitrary nonempty subset of  $\mathbb{R}^n$ .

**Definition 4.1.** Let  $X \subseteq \mathbb{R}^n$ ,  $x \in \mathbb{R}^n$  and  $u = (u_1, \dots, u_k)$  be a  $k$ -tuple of pairwise orthogonal unit vectors in  $\mathbb{R}^n$ . Then  $u$  is said to be a *tangent of  $X$  at  $x$*  if  $X$  contains a sequence  $\{x_i\}$  converging to  $x$ , whose Frenet  $k$ -frame is  $u$ . We say that  $\{x_i\}$  *determines*  $u$ . We say that  $u$  is *outgoing* if, in addition, there are  $\lambda_1, \dots, \lambda_k > 0$  such that the simplex  $C = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k)$  and its facet  $C' = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_{k-1} u_{k-1})$  have the same intersection with  $X$ .

The following elementary material on piecewise linear topology [12] is necessary to introduce the Riesz space  $\mathcal{R}(X)$  of piecewise linear functions on  $X$ . In Theorem 5.1 below, the Frenet tangent frames of  $X$  will be related to the maximal and principal ideals of  $\mathcal{R}(X)$ .

A *polyhedron*  $P$  in  $\mathbb{R}^n$  is a finite union of simplexes in  $\mathbb{R}^n$ .  $P$  need not be convex or connected. Given a polyhedron  $P$ , a *triangulation* of  $P$  is an (always finite) simplicial complex  $\Delta$  such that  $P = \bigcup \Delta$ . Every polyhedron has a triangulation, [12, 2.1.5]. Given a rational polyhedron  $P$  and triangulations  $\Delta$  and  $\Sigma$  of  $P$ , we say that  $\Delta$  is a *subdivision* of  $\Sigma$  if every simplex of  $\Delta$  is contained in a simplex of  $\Sigma$ . Suppose an  $n$ -cube  $K \subseteq \mathbb{R}^n$  is contained in another  $n$ -cube  $K' \subseteq \mathbb{R}^n$ . Then every triangulation  $\Delta$  of  $K$  has an *extension*  $\Delta'$  to a triangulation of  $K'$ , in the sense that  $\Delta = \{T \in \Delta' \mid T \subseteq K\}$ . A continuous function  $f: K \rightarrow \mathbb{R}$  is  $\Delta$ -linear if it is linear (in the affine

sense) on each simplex of  $\Delta$ . Via the extension  $\Delta'$ ,  $f$  can be extended to a  $\Delta'$ -linear function on  $K'$ . A function  $g: K \rightarrow \mathbb{R}$  is *piecewise linear* if it is  $\Delta$ -linear for some triangulation  $\Delta$  of  $K$ . We denote by  $\mathcal{R}(K)$  the Riesz space of all piecewise linear functions on  $K$ , with the pointwise operations of the Riesz space  $\mathbb{R}$ .

More generally, let  $X$  be a nonempty compact subset of  $\mathbb{R}^n$ . Let  $K \subseteq \mathbb{R}^n$  be an (always closed)  $n$ -cube containing  $X$ . We momentarily denote by  $\mathcal{R}(K) \upharpoonright X$  the Riesz space of restrictions to  $X$  of the functions in  $\mathcal{R}(K)$ . If  $L \subseteq \mathbb{R}^n$  is an  $n$ -cube containing  $K$ , then  $\mathcal{R}(K) \upharpoonright X = \mathcal{R}(L) \upharpoonright X$ . (For the nontrivial direction, the above mentioned extension property of triangulations yields  $\mathcal{R}(L) \upharpoonright K = \mathcal{R}(K)$ .) Thus, if both  $n$ -cubes  $K$  and  $L$  contain  $X$ , letting  $M \subseteq \mathbb{R}^n$  be an  $n$ -cube containing both  $K$  and  $L$ , we obtain  $\mathcal{R}(K) \upharpoonright X = \mathcal{R}(L) \upharpoonright X = \mathcal{R}(M) \upharpoonright X$ , independently of the ambient cube  $K \supseteq X$ . Without fear of ambiguity we may then use the notation  $\mathcal{R}(X)$  for the Riesz space of functions thus obtained. Each  $f \in \mathcal{R}(X)$  is said to be a *piecewise linear function on  $X$* . It follows that  $f$  is continuous.

**Lemma 4.2.** *There is a one-one correspondence  $x \mapsto \mathfrak{m}_x$ ,  $\mathfrak{m} \mapsto x_{\mathfrak{m}}$  between maximal ideals  $\mathfrak{m}$  of  $\mathcal{R}(X)$  and points  $x$  of  $X$ . Specifically,  $\mathfrak{m}_x$  is the set of all functions in  $\mathcal{R}(X)$  vanishing at  $x$ ; conversely,  $x_{\mathfrak{m}}$  is the only element in the intersection of the zerosets  $Zh = h^{-1}(0)$  of all functions  $h \in \mathfrak{m}$ .*

*Proof.* The functions in  $\mathcal{R}(X)$  separate points, and the constant function 1 is a strong unit in  $\mathcal{R}(X)$ . Now apply [6, 27.7].  $\square$

The following elementary result deals with the special case  $X \subseteq \mathbb{R}^2$ . It is an adaptation to Riesz spaces of the MV-algebraic result [3, Theorem 3.1(ii)], and will have a key role in the proof of the much stronger Theorem 5.1.

**Lemma 4.3.** *Let  $X \subseteq \mathbb{R}^2$  be a nonempty compact set. If the Riesz space  $\mathcal{R}(X)$  has a principal ideal that is not an intersection of maximal ideals, then  $X$  has an outgoing Severi-Bouligand tangent at some point  $x \in X$ .*

*Proof.* For every element  $e$  of  $\mathcal{R}(X)$  let  $\langle e \rangle$  denote the principal ideal generated by  $e$ . Let  $g \in \mathcal{R}(X)$  be such that the ideal  $\mathfrak{p} = \langle g \rangle$  is not an intersection of maximal ideals of  $\mathcal{R}(X)$ . Lemma 4.2 yields an element  $f \in \mathcal{R}(X)$  such that  $f \notin \mathfrak{p}$  and  $Zg \subseteq Zf$ . Replacing, if necessary,  $f$  and  $g$  by their absolute values  $|f|$  and  $|g|$ , we may assume  $f \geq 0$  and  $g \geq 0$ . Let  $K \subseteq \mathbb{R}^2$  be a fixed but otherwise arbitrary closed square containing  $X$ . By definition of  $\mathcal{R}(X)$ , there are elements  $0 \leq \tilde{f} \in \mathcal{R}(K)$  and  $0 \leq \tilde{g} \in \mathcal{R}(K)$  such that  $\tilde{f} \upharpoonright X = f$  and  $\tilde{g} \upharpoonright X = g$ . Since  $\tilde{f} \upharpoonright X$  does not belong to  $\mathfrak{p}$  then for each  $m > 0$  there is a point  $x_m \in X$  such that

$$\tilde{f}(x_m) > m \cdot \tilde{g}(x_m). \quad (8)$$

Since  $X$  is compact, for some  $x \in X$  there is a subsequence  $\{x_{m_1}, x_{m_2}, \dots\}$  of  $\{x_1, x_2, \dots\}$  such that

$$x_i \neq x_j \text{ for all } i \neq j, \text{ and } \lim_{i \rightarrow \infty} x_{m_i} = x. \quad (9)$$

For each  $i = 1, 2, \dots$ , let the unit vector  $u_i$  be defined by

$$u_i = (x_{m_i} - x) / \|x_{m_i} - x\|.$$

Since the unit circumference  $S^1 = \{z \in \mathbb{R}^2 \mid \|z\| = 1\}$  is compact, it is no loss of generality to assume  $\lim_{i \rightarrow \infty} u_i = u$ , for some  $u \in S^1$ . Therefore,  $u$  is a tangent of  $X$  at  $x$ . There remains to be shown that  $u$  is outgoing. To this purpose we make the following

*Claim.* There is a real number  $\lambda > 0$  such that:

- (a)  $\tilde{f}$  is (affine) linear on the line segment  $\text{conv}(x, x + \lambda u)$ ;
- (b)  $\tilde{g}$  identically vanishes on  $\text{conv}(x, x + \lambda u)$ ;
- (c)  $\tilde{f}(x + \lambda u) \neq 0$ .

As a matter of fact, since each of  $x_{m_1}, x_{m_2}, \dots$  lies in  $K$ , by (9) there exists  $\delta > 0$  such that  $\text{conv}(x, x + \delta u) \subseteq K$ . An elementary result in polyhedral topology ([12, 2.2.4]) yields a triangulation  $\Delta$  of  $K$  such that both functions  $\tilde{f}$  and  $\tilde{g}$  are  $\Delta$ -linear and  $\text{conv}(x, x + \delta u) = \bigcup \{T \in$



$\Delta \mid T \subseteq \text{conv}(x, x + \delta u)\}$ . Therefore, there exists  $\lambda > 0$  such that  $\text{conv}(x, x + \lambda u) \in \Delta$ . We have proved that  $\tilde{f}$  is linear in  $\text{conv}(x, x + \lambda u)$ , and (a) is settled.

To settle (b), since both functions  $\tilde{g}$  and  $\tilde{f}$  are continuous, we can write

$$0 \geq \tilde{g}(x) = \lim_{i \rightarrow \infty} \tilde{g}(x_i) \leq \lim_{i \rightarrow \infty} \frac{\tilde{f}(x_i)}{m_i} = 0,$$

whence  $\tilde{g}(x) = g(x) = 0$ . From  $X \cap Z\tilde{g} \subseteq X \cap Z\tilde{f}$  we get  $\tilde{f}(x) = f(x) = 0$ . Since  $\Delta$  is finite set, there exists a 2-simplex  $S \in \Delta$  containing infinitely many elements  $x_{n_1}, x_{n_2}, \dots$  of the set  $\{x_{m_1}, x_{m_2}, \dots\}$ . By (9),  $x \in S$ . Further, from  $\lim_{i \rightarrow \infty} u_{n_i} = u$  and  $\text{conv}(x, x + \lambda u) \in \Delta$  it follows that  $\text{conv}(x, x + \lambda u) \subseteq S$ . Therefore,

$$S = \text{conv}(x, x + \lambda u, v) \text{ for some } v \in S. \quad (10)$$

For some  $2 \times 1$ -matrix  $A$  and vector  $b \in \mathbb{R}^2$  we can write  $\tilde{g}(z) = Az + b$  for each  $z \in S$ . Since  $\lim_{i \rightarrow \infty} u_{m_i} = u$  and  $\tilde{g}(x) = 0$ , we have the identities

$$\begin{aligned} \tilde{g}(x + \lambda u) &= \lambda Au + \tilde{g}(x) = \lim_{i \rightarrow \infty} \frac{\lambda(Ax_{n_i} - Ax)}{\|x_{n_i} - x\|} = \lim_{i \rightarrow \infty} \frac{\lambda(\tilde{g}(x_{n_i}) - \tilde{g}(x))}{\|x_{n_i} - x\|} \\ &= \lim_{i \rightarrow \infty} \frac{\lambda\tilde{g}(x_{n_i})}{\|x_{n_i} - x\|} = \lim_{i \rightarrow \infty} \frac{\lambda g(x_{n_i})}{\|x_{n_i} - x\|}. \end{aligned}$$

Similarly,

$$\tilde{f}(x + \lambda u) = \lim_{i \rightarrow \infty} \frac{\lambda f(x_{n_i})}{\|x_{n_i} - x\|},$$

whence

$$0 \leq \tilde{g}(x + \lambda u) = \lim_{i \rightarrow \infty} \frac{\lambda g(x_{n_i})}{\|x_{n_i} - x\|} \leq \lim_{i \rightarrow \infty} \frac{\lambda}{n_i} \frac{f(x_{n_i})}{\|x_{n_i} - x\|} = \tilde{f}(x + \lambda u) \lim_{i \rightarrow \infty} \frac{1}{n_i} = 0.$$

Since  $\tilde{g}$  is linear on  $\text{conv}(x, x + \lambda u)$  and  $\tilde{g}(x + \lambda u) = 0 = \tilde{g}(x)$ , then (b) follows.

To prove (c), by (8) we get  $\tilde{f}(x_{n_i}) \neq 0$  for all  $i$ , whence  $\tilde{g}(x_{n_i}) \neq 0$ , because  $Zg \subseteq Zf$ . Then our assumptions about  $S$ , together with (10), show that  $\tilde{g}(v) \neq 0$ . Let the integer  $m^*$  satisfy the inequality  $m^* \cdot \tilde{g}(v) \geq \tilde{f}(v)$ . If (absurdum hypothesis)  $\tilde{f}(x + \lambda u) = 0$  then  $m^* \cdot \tilde{g}(z) \geq \tilde{f}(z)$  for each  $z \in S$ . In view of (8), this contradicts the existence of infinitely many elements  $x_{n_i}$  in  $S$ . Having thus proved (c), our claim is settled.

In conclusion, from (a) and (c) it follows that  $\text{conv}(x, x + \lambda u) \cap Z\tilde{f} = \{x\}$ . Then from (b) we get

$$X \cap \text{conv}(x, x + \lambda u) = X \cap Z\tilde{g} \cap \text{conv}(x, x + \lambda u) \subseteq X \cap Z\tilde{f} \cap \text{conv}(x, x + \lambda u) = \{x\},$$

thus proving that  $u$  is an outgoing tangent of  $X$  at  $x$ .  $\square$

## 5. TANGENTS AND STRONG SEMISIMPLICITY

Recall that a Riesz space  $R$  is said to be *strongly semisimple* if for every principal ideal  $\langle g \rangle$  of  $R$  the quotient  $R/\langle g \rangle$  is *archimedean* (i.e., the intersection of the maximal ideals of  $R/\langle g \rangle$  is  $\{0\}$ ). Equivalently,  $\langle g \rangle$  is an intersection of maximal ideals of  $R$ . (This follows from the canonical one-to-one correspondence between ideals of  $R$  containing  $\langle g \rangle$ , and ideals of  $R/\langle g \rangle$ .) Since  $\{0\}$  is a principal ideal of  $R$ , if  $R$  is strongly semisimple then it is archimedean.

The following result is the promised strengthening of Lemma 4.3:

**Theorem 5.1.** *For any nonempty compact set  $X \subseteq \mathbb{R}^n$  the following conditions are equivalent:*

- (i)  *$X$  has an outgoing tangent at some point  $x \in X$ .*
- (ii) *The Riesz space  $\mathcal{R}(X)$  is not strongly semisimple, i.e., there exists a principal ideal of  $\mathcal{R}(X)$  that is not an intersection of maximal ideals.*

*Proof.* Without loss of generality,  $X \subseteq [0, 1]^n$ . (This trivially follows because any  $n$ -cube in  $\mathbb{R}^n$  is PL-homeomorphic to any other  $n$ -cube).

(i) $\Rightarrow$ (ii) By Definition 4.1, for some  $x \in \mathbb{R}^n$  and  $k$ -tuple  $u = (u_1, \dots, u_k)$  of pairwise orthogonal unit vectors in  $\mathbb{R}^n$ , there is a sequence  $\{x_i\}$  of points in  $\mathbb{R}^n$  converging to  $x$ , such that  $u$  is the Frenet  $k$ -frame of  $\{x_i\}$ . Further, there are reals  $\lambda_1, \dots, \lambda_k > 0$  such that the simplex  $C = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k)$  and its facet  $C' = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_{k-1} u_{k-1})$  satisfy  $C \cap X = C' \cap X$ .

Let  $f_1$  and  $f_2$  be piecewise linear functions defined on  $[0, 1]^n$ , taking their values in  $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$  and satisfying the conditions

$$Zf_1 = f_1^{-1}(0) = C, \quad Zf_2 = C', \quad \text{and} \quad f_2 \text{ is (affine) linear over } C. \quad (11)$$

The existence of  $f_1$  and  $f_2$  follows from [12, 2.2.4]. Both restrictions  $f_2 \upharpoonright X$  and  $f_1 \upharpoonright X$  are elements of  $\mathcal{R}(X)$ . By construction,

$$Zf_1 \cap X = Zf_2 \cap X. \quad (12)$$

We *claim* that the principal ideal  $\mathfrak{p} = \langle f_1 \upharpoonright X \rangle$  of  $\mathcal{R}(X)$  generated by  $f_1 \upharpoonright X$  does not coincide with the intersection of all maximal ideals of  $\mathcal{R}(X)$  containing  $\mathfrak{p}$ .

By (12) together with Lemma 4.2,  $f_2 \upharpoonright X$  belongs to all maximal ideals of  $\mathcal{R}(X)$  containing  $\mathfrak{p}$ . So our claim will be settled once we prove

$$f_2 \upharpoonright X \notin \mathfrak{p}. \quad (13)$$

To this purpose, arguing by way of contradiction, suppose  $f_2 \upharpoonright X \leq m f_1 \upharpoonright X$  for some  $m = 1, 2, \dots$ . Since  $f_1$  and  $f_2$  are (continuous) piecewise linear, the set  $L = \{x \in [0, 1]^n \mid f_2(x) \leq m f_1(x)\}$  is a union of simplexes  $T_1 \cup \dots \cup T_r$ . Necessarily for some  $j = 1, \dots, r$  the simplex  $T_j$  contains infinitely many points of the sequence  $\{x_i\}$ . This subsequence  $\{x_t\}$  still converges to  $x \in T_j$ , and  $u$  is its Frenet  $k$ -frame. Theorem 3.3 yields  $\mu_1, \dots, \mu_k > 0$  such that  $T_j$  contains the simplex  $M = \text{conv}(x, x + \mu_1 u_1, \dots, x + \mu_1 u_1 + \dots + \mu_k u_k)$ . Now Proposition 3.2 yields uniquely determined  $\nu_1, \dots, \nu_k > 0$  such that

$$C \cap M = \text{conv}(x, x + \nu_1 u_1, \dots, x + \nu_1 u_1 + \dots + \nu_k u_k).$$

By (11),  $f_1$  identically vanishes on  $C \cap M$ . Further, from  $L \supseteq T_j \supseteq M \supseteq C \cap M$  and  $f_2 \leq m f_1$  on  $L$ , it follows that  $f_2 = 0$  on  $C \cap M$ . The two simplexes  $C \cap M$  and  $C$  have the same dimension  $k$ , and  $f_2$  is (affine) linear on  $C \supseteq C \cap M$ . Therefore,  $f_2 = 0$  on  $C$ , which contradicts  $Zf_2 = C'$ . We have thus proved (13), settled our claim, and completed the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) By hypothesis, there is a function  $f_1 \in \mathcal{R}([0, 1]^n)$  such that the principal ideal  $\langle f_1 \upharpoonright X \rangle$  of  $\mathcal{R}(X)$  generated by the restriction  $f_1 \upharpoonright X$  is not an intersection of maximal ideals of  $\mathcal{R}(X)$ . Thus there is  $f_2 \in \mathcal{R}([0, 1]^n)$  whose restriction  $f_2 \upharpoonright X$  does not belong to the principal ideal  $\langle f_1 \upharpoonright X \rangle$  generated by  $f_1 \upharpoonright X$ , but belongs to all maximal ideals of  $\mathcal{R}(X)$  containing  $\langle f_1 \upharpoonright X \rangle$ . By Lemma 4.2,  $Zf_2 \upharpoonright X = Zf_1 \upharpoonright X$ , i.e.,  $X \cap Zf_2 = X \cap Zf_1$ .

Let the map  $g: X \rightarrow \mathbb{R}^2$  be defined by

$$g(x) = (f_1(x), f_2(x)) \text{ for all } x \in X. \quad (14)$$

Let  $\iota: \mathcal{R}(g(X)) \rightarrow \mathcal{R}(X)$  be defined by  $\iota(h) = h \circ g$  for all  $h \in \mathcal{R}(g(X))$ , where  $\circ$  denotes composition. It is easy to see that  $\iota$  is a Riesz space homomorphism of  $\mathcal{R}(g(X))$  into  $\mathcal{R}(X)$ . Letting  $\pi_1, \pi_2: \mathbb{R}^2 \rightarrow \mathbb{R}$  be the canonical projections (=coordinate functions), we have the identities  $f_1 \upharpoonright X = \iota(\pi_1 \upharpoonright g(X))$  and  $f_2 \upharpoonright X = \iota(\pi_2 \upharpoonright g(X))$ . Whenever  $h \in \mathcal{R}(g(X))$ ,  $\iota(h) = 0$  and  $z \in g(X)$ , there exists  $x \in X$  such that  $g(x) = z$ . Then  $h(z) = h(g(x)) = (\iota(h))(x) = 0$  and  $\iota$  is one-to-one. Actually,  $\iota$  is an isomorphism between  $\mathcal{R}(g(X))$  and the Riesz subspace of  $\mathcal{R}(X)$  generated by  $\{f_1 \upharpoonright X, f_2 \upharpoonright X\}$ . It follows that the principal ideal  $\mathfrak{p}$  of  $\mathcal{R}(g(X))$  generated by  $\pi_1 \upharpoonright g(X)$  is not an intersection of maximal ideals of  $\mathcal{R}(g(X))$ : specifically,  $\pi_2 \upharpoonright g(X)$  belongs to all maximal ideals containing  $\mathfrak{p}$ , but does not belong to  $\mathfrak{p}$ . By Lemma 4.3,

$$g(X) \text{ has a Severi-Bouligand outgoing tangent.} \quad (15)$$

There remains to be proved that  $X$  has an outgoing tangent. To help the reader, the long proof is subdivided into two parts.

Part 1: Construction of a tangent  $u$  of  $X$ .

By (15) and Definition 4.1 with  $k = 1$  (which is the same as Definition 1.1), for some point  $y^* \in \mathbb{R}^2$ , unit vector  $v^* \in \mathbb{R}^2$ , sequence  $\{y_i\} \subseteq \mathbb{R}^2$  converging to  $y^*$ , and  $\mu > 0$ , we can write

$$\lim_{i \rightarrow \infty} (y_i - y^*) / \|y_i - y^*\| = v^* \quad \text{and} \quad \text{conv}(y^*, y^* + \mu v^*) \cap g(X) = \{y^*\}. \quad (16)$$

By (14),  $g$  is the restriction to  $X$  of the function  $f = (f_1, f_2): [0, 1]^n \rightarrow \mathbb{R}^2$ . Since (each component of)  $f$  is piecewise linear, then  $f$  is continuous, and both sets  $f^{-1}(y^*)$  and  $f^{-1}(\text{conv}(y^*, y^* + \mu v^*))$  are polyhedra in  $[0, 1]^n$ . An elementary result in polyhedral topology ([12, 2.2.4]) yields a triangulation  $\Delta$  of  $[0, 1]^n$  having the following properties:

- $f$  is (affine) linear over each simplex of  $\Delta$ ,
- $f^{-1}(y^*) = \bigcup \{R \in \Delta \mid R \subseteq f^{-1}(y^*)\}$ , and
- $f^{-1}(\text{conv}(y^*, y^* + \mu v^*)) = \bigcup \{U \in \Delta \mid U \subseteq f^{-1}(\text{conv}(y^*, y^* + \mu v^*))\}$ .

For some  $n$ -simplex  $T \in \Delta$ , the set  $\{i \mid f^{-1}(y_i) \cap T \cap X\} = \{i \mid g^{-1}(y_i) \cap T\}$  is infinite. Let  $z_0, z_1, \dots$  be a converging sequence of elements of  $T$  such that  $f(z_0), f(z_1), \dots$  is a subsequence of  $y_0, y_1, \dots$ . Without loss of generality this subsequence coincides with the sequence  $\{y_i\}$ , and we can write

$$g(z_i) = y_i. \quad (17)$$

Letting  $z^* = \lim_{i \rightarrow \infty} z_i$  we have

$$z^* \in X \cap T \quad \text{and} \quad y^* = f(z^*) = g(z^*). \quad (18)$$

The linearity of  $f$  on  $T$  yields a  $2 \times n$  matrix  $A$ , together with a vector  $b \in \mathbb{R}^2$  such that for each  $t \in T$ ,  $f(t) = At + b$ .

*Claim.* For some  $k \in \{1, \dots, n\}$  there is a  $k$ -tuple of pairwise orthogonal unit vectors  $u_i \in \mathbb{R}^n$ , ( $1 \leq i \leq k$ ) such that:

- $Au_j = 0$  for each  $1 \leq j < k$ ,
- $Au_k \neq 0$ ,
- $u = (u_1, \dots, u_k)$  is a tangent of  $X$  at  $z^*$ , determined by a suitable subsequence of  $z_0, z_1, \dots$ , in the sense of Definition 4.1.

The vectors  $u_1, \dots, u_k$  are constructed by the following inductive procedure:

*Basis Step:* From  $Az_i + b = y_i \neq y^* = Az^* + b$  it follows that  $z_i \neq z^*$  for each  $i$ , and hence every vector  $z_i^1 = (z_i - z^*) / \|z_i - z^*\|$  is well defined. Since the  $(n-1)$ -dimensional unit sphere  $S^{n-1} \subseteq \mathbb{R}^n$  is compact, it is no loss of generality to assume that the sequence  $z_0^1, z_1^1, \dots$  converges to some unit vector  $u_1$ . It follows that  $u_1$  is a tangent of  $X$  at  $z^*$ . If  $Au_1 \neq 0$ , upon setting  $u = u_1$  the claim is proved. If  $Au_1 = 0$  we proceed inductively.

*Induction Step:* Having constructed a tangent  $u(l) = (u_1, \dots, u_l)$  of  $X$  at  $z^*$  with  $Au_i = 0$  for each  $i \in \{1, \dots, l\}$ , we first observe that  $l < n$ . (For otherwise, the  $u_j$  would constitute an orthonormal basis of  $\mathbb{R}^n$ , whence  $A$  is the zero matrix, and  $Ax + b = b$  for each  $x \in \mathbb{R}^n$ , which contradicts  $Az_i + b \neq Az^* + b$ .) Let  $\rho_1, \dots, \rho_l$  be arbitrary real numbers. From

$$A(z^* + \rho_1 u_1 + \dots + \rho_l u_l) + b = A(z^*) + b = g(z^*) \neq g(z_i) = A(z_i) + b, \quad (19)$$

it follows that no  $z_i$  lies in the affine space  $z^* + \mathbb{R}u_1 + \dots + \mathbb{R}u_l$ , i.e.,  $z_i - z^* \notin \mathbb{R}u_1 + \dots + \mathbb{R}u_l$ . For each  $i$ , the unit vector

$$z_i^{l+1} = \frac{z_i - z^* - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_l}(z_i - z^*)}{\|z_i - z^* - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_l}(z_i - z^*)\|}$$

is well defined. Without loss of generality, we can write  $\lim_{i \rightarrow \infty} z_i^{l+1} = u_{l+1}$  for some unit vector  $u_{l+1} \in \mathbb{R}^n$ . By construction,  $u_{l+1}$  is orthogonal to each of  $u_1, \dots, u_l$ , and the  $(l+1)$ -tuple

$u(l+1) = (u_1, \dots, u_l, u_{l+1})$  is a tangent of  $X$  at  $z^*$ . In case  $Au_{l+1} \neq 0$ , upon setting  $k = l+1$  and  $u = u(l+1)$  we are done. In case  $Au_{l+1} = 0$ , we proceed inductively, with  $(u_1, \dots, u_l, u_{l+1})$  in place of  $(u_1, \dots, u_l)$ . Our claim is settled, and so is the proof of Part 1.

Part 2:  $u$  is an outgoing tangent of  $X$ .

With the notation of Part 1, for some  $\lambda_1, \dots, \lambda_k > 0$  we prove the inclusion

$$\text{conv}(z^*, z^* + \lambda_1 u_1, \dots, z^* + \lambda_1 u_1 + \dots + \lambda_k u_k) \subseteq T \cap f^{-1}(\text{conv}(y^*, y^* + \mu v^*)). \quad (20)$$

As a matter of fact, by construction,  $u = (u_1, \dots, u_k)$  is a tangent of  $X \cap T$  at  $z^*$ . Theorem 3.3 yields real numbers  $\epsilon_1, \dots, \epsilon_k > 0$  such that

$$\text{conv}(z^*, z^* + \epsilon_1 u_1, \dots, z^* + \epsilon_1 u_1 + \dots + \epsilon_k u_k) \subseteq T. \quad (21)$$

Since  $Au_j = 0$  for each  $j = 1, \dots, k-1$ , from (18)-(19) we obtain the identities

$$y^* = g(z^*) = g(x) \text{ for all } x \in \text{conv}(z^*, z^* + \epsilon_1 u_1, \dots, z^* + \epsilon_1 u_1 + \dots + \epsilon_{k-1} u_{k-1}). \quad (22)$$

Recalling (17) we can write

$$\begin{aligned} 0 \neq Au_k &= \lim_{i \rightarrow \infty} Az_i^k = \lim_{i \rightarrow \infty} A \left( \frac{z_i - z^* - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{l-1}}(z_i - z^*)}{\|z_i - z^* - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{l-1}}(z_i - z^*)\|} \right) \\ &= \lim_{i \rightarrow \infty} \frac{A(z_i) - A(z^*)}{\|z_i - z^* - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{l-1}}(z_i - z^*)\|} \\ &= \lim_{i \rightarrow \infty} \frac{y_i - y^*}{\|z_i - z^* - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{l-1}}(z_i - z^*)\|} \cdot \frac{\|y_i - y^*\|}{\|y_i - y^*\|} \\ &= \lim_{i \rightarrow \infty} \frac{y_i - y^*}{\|y_i - y^*\|} \cdot \frac{\|y_i - y^*\|}{\|z_i - z^* - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{l-1}}(z_i - z^*)\|}. \end{aligned}$$

Since  $0 \neq v^* = \lim_{i \rightarrow \infty} (y_i - y^*)/\|y_i - y^*\|$ , for some  $\tau > 0$  we obtain

$$\tau = \lim_{i \rightarrow \infty} \frac{\|y_i - y^*\|}{\|z_i - z^* - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_{l-1}}(z_i - z^*)\|} \quad \text{and} \quad Au_k = \tau v^*.$$

Now the desired  $\lambda$ 's in (20) are given by setting  $\lambda_j = \epsilon_j$  for  $1 \leq j < k$ , and  $\lambda_k = \min\{\epsilon_k, \mu/\tau\}$ . Indeed, letting  $C = \text{conv}(z^*, z^* + \lambda_1 u_1, \dots, z^* + \lambda_1 u_1 + \dots + \lambda_k u_k)$ , from (21) we obtain

$$C \subseteq \text{conv}(z^*, z^* + \epsilon_1 u_1, \dots, z^* + \epsilon_1 u_1 + \dots + \epsilon_k u_k) \subseteq T. \quad (23)$$

Further, for every  $x \in C$  there exists  $0 \leq \omega \leq \lambda_k$  such that

$$Ax + b = Az^* + \omega Au_k + b = Az^* + b + \omega \tau v^* = y^* + \omega \tau v^*, \quad (24)$$

whence  $Ax + b \in \text{conv}(y^*, y^* + \mu v^*)$ , because  $\omega \leq \mu/\tau$ . The proof of (20) is complete.

To complete the proof that  $(u_1, \dots, u_k)$  is outgoing, letting  $C' = \text{conv}(z^*, z^* + \lambda_1 u_1, \dots, z^* + \lambda_1 u_1 + \dots + \lambda_{k-1} u_{k-1})$ , we must show  $C' \cap X = C \cap X$ . By way of contradiction, suppose  $x \in (X \cap C) \setminus (X \cap C')$ . Then for suitable  $\xi_1, \dots, \xi_{k-1} \geq 0$  and  $\xi_k > 0$ , we can write  $x = z^* + \xi_1 u_1 + \dots + \xi_k u_k$ . By (23),  $x \in X \cap T$ . Since  $\xi_k > 0$ , by (24) we have  $g(x) = f(x) = Ax + b = y^* + \xi_k \tau v^* \neq y^*$ . This contradicts the identity  $g(x) \in g(X) \cap \text{conv}(y^*, y^* + \mu v^*) = \{y^*\}$ , which follows from (16) and (22).

Having thus proved that the tangent  $u$  is outgoing, we have also completed the proof of Part 2, as well as the proof of the theorem.  $\square$

## 6. EXAMPLES AND FURTHER RESULTS

**Proposition 6.1.** *Let  $I = \text{conv}(a, b) \subseteq \mathbb{R}$  be an interval, and  $\phi: I \rightarrow \mathbb{R}^n$  a  $C^2$  function. Then the Riesz space  $\mathcal{R}(\phi(I))$  is strongly semisimple iff  $\phi$  is (affine) linear.*

*Proof.* The proof directly follows from Theorems 5.1 and 2.2.  $\square$

**Proposition 6.2.** *For every polyhedron  $P \subseteq \mathbb{R}^n$  the Riesz space  $\mathcal{R}(P)$  is strongly semisimple, and  $P$  has no outgoing tangent.*

*Proof.* For some finite set  $\{S_1, \dots, S_m\}$  of simplexes in  $\mathbb{R}^n$  we can write  $P = S_1 \cup \dots \cup S_m$ . If  $u$  is a tangent of  $P$  at some point  $x \in P$  then  $u$  is also a tangent of  $S_i$  at  $x$  for some  $i = 1, \dots, m$ . By Theorem 3.3,  $u$  is not an outgoing tangent of  $S_i$ . Thus  $u$  is not an outgoing tangent of  $P$ . Now apply Theorem 5.1.  $\square$

The following is an example of a strongly semisimple Riesz space  $\mathcal{R}(X)$ , where  $X$  is not a polyhedron:

**Example 6.3.** Let the set  $X \subseteq \mathbb{R}^2$  be defined by

$$X = \{(0, 0)\} \cup \{(1/n, 0) \mid n = 1, 2, \dots\} \cup \{(1/n, 1/n^2) \mid n = 1, 2, \dots\}.$$

The origin  $(0, 0)$  is the only accumulation point of  $X$ . The only tangents of  $X$  are given by the vector  $(1, 0)$  and the pair of vectors  $((1, 0), (0, 1))$ . Therefore,  $X$  has no outgoing tangents. By Theorem 5.1, the Riesz space  $\mathcal{R}(X)$  is strongly semisimple.

However, when the compact set  $X \subseteq \mathbb{R}^n$  is convex we have:

**Theorem 6.4.** *Let  $X \subseteq \mathbb{R}^n$  be a nonempty compact convex set. Then the following conditions are equivalent:*

- (I) *The Riesz space  $\mathcal{R}(X)$  is strongly semisimple.*
- (II)  *$X = \text{conv}(x_1, \dots, x_m)$  for some  $x_1, \dots, x_m \in \mathbb{R}^n$ , i.e.,  $X$  is a polyhedron.*
- (III)  *$X$  has no outgoing tangent.*

*Proof.* (III)  $\Leftrightarrow$  (I) This is a particular case of Theorem 5.1. (II)  $\Rightarrow$  (I) By Proposition 6.2. (I)  $\Rightarrow$  (II) Arguing by way of contradiction, assume  $\mathcal{R}(P)$  to be strongly semisimple, but  $X \neq \text{conv}(x_1, \dots, x_m)$  for any finite set  $\{x_1, \dots, x_m\} \subseteq \mathbb{R}^n$ . Letting  $\text{ext}(X)$  denote the set of extreme point of  $X$ , Minkowski theorem yields the identity  $X = \text{conv}(\text{ext}(X))$ . Since  $X$  is compact, there exists a point  $x \in X$  together with a sequence  $x_1, x_2, \dots$  of extreme points of  $X$  such that  $\lim_{i \rightarrow \infty} x_i = x$  and  $x_i \neq x_j$  for every  $i \neq j$ .  $\blacksquare$

*Claim 1.* There exists a subsequence  $x_{m_1}, x_{m_2}, \dots$  of the sequence  $x_1, x_2, \dots$ , together with a  $k$ -tuple  $(u_1, \dots, u_k)$  of pairwise orthogonal unit vectors in  $\mathbb{R}^n$  (for some  $k \in \{1, \dots, n\}$ ), having the following properties:

- (a)  $x_{m_1}, x_{m_2}, \dots$  determines the tangent  $(u_1, \dots, u_k)$  of  $X$  at  $x$ , in the sense of Definition 4.1.
- (b)  $\text{aff}(x_{m_1}, x_{m_2}, \dots) = x + \mathbb{R}u_1 + \dots + \mathbb{R}u_k$ .

The vectors  $u_1, u_2, \dots, u_k$  are constructed by the following inductive procedure:

*Basis:* Since  $x_i \neq x_j$  for each  $i \neq j$ , then each unit vector  $(x_i - x)/\|x_i - x\|$  is well defined. There is a subsequence  $x_{m_1^1}, x_{m_2^1}, \dots$  of  $x_1, x_2, \dots$  and a unit vector  $u_1 \in \mathbb{R}^n$  such that  $\lim_{i \rightarrow \infty} (x_{m_i^1} - x)/\|x_{m_i^1} - x\| = u_1$ . Then  $u_1$  is a tangent of  $X$  at  $x$  determined by  $x_{m_1^1}, x_{m_2^1}, \dots$ .  $\blacksquare$

*Induction Step:* Let  $l \geq 1$  and assume the subsequence  $x_{m_1^l}, x_{m_2^l}, \dots$  of  $x_1, x_2, \dots$  determines the tangent  $(u_1, \dots, u_l)$  of  $X$  at  $x$ . If there exists an integer  $r$  such that  $\text{aff}(x_{m_1^l}, x_{m_{r+1}^l}, \dots) = x + \mathbb{R}u_1 + \dots + \mathbb{R}u_l$ , then upon setting  $k = l$ , we are done. If no such  $r$  exists, infinitely many vectors in  $x_{m_1^l}, x_{m_2^l}, \dots$  do not belong to the affine space  $x + \mathbb{R}u_1 + \dots + \mathbb{R}u_l$ . Therefore, for some subsequence  $x_{m_1^{l+1}}, x_{m_2^{l+1}}, \dots$  and unit vector  $u_{l+1} \in \mathbb{R}^n$  we can write

$$u_{l+1} = \lim_{i \rightarrow \infty} \frac{x_{m_i^{l+1}} - x - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_l}(x_{m_i^{l+1}} - x)}{\|x_{m_i^{l+1}} - x - \text{proj}_{\mathbb{R}u_1 + \dots + \mathbb{R}u_l}(x_{m_i^{l+1}} - x)\|}. \quad (25)$$

We then proceed with  $(u_1, \dots, u_{l+1})$  in place of  $(u_1, \dots, u_l)$ . Since the affine space  $\text{aff}(x_{m_1}, x_{m_2}, \dots)$  is contained in  $\mathbb{R}^n$ , this procedure must terminate for some  $1 \leq k \leq n$ . Claim 1 is settled.

Let us now fix a subsequence  $x_{m_1}, x_{m_2}, \dots$  of  $x_1, x_2, \dots$ , together with a  $k$ -tuple  $(u_1, \dots, u_k)$  of pairwise orthogonal unit vectors satisfying conditions (a) and (b) in Claim 1.

*Claim 2.* There are  $\lambda_1, \dots, \lambda_k > 0$  such that the  $k$ -simplex  $C_k = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_k u_k)$  is contained in  $X$ .

We have already observed that  $x \in X$ . By Theorem 5.1, the tangent  $u_1$  of  $X$  at  $x$  is not outgoing. Hence  $\text{conv}(x, x + u_1) \cap X \neq \{x\}$ . Let  $y \in (\text{conv}(x, x + u_1) \cap X) \setminus \{x\}$ . Thus  $y = x + \lambda_1 u_1$  for some  $0 < \lambda_1 \leq 1$ . Since  $X$  is convex,  $\text{conv}(x, x + \lambda_1 u_1) \subseteq X$ .

Proceeding inductively, let us assume that  $\lambda_1, \dots, \lambda_l > 0$  are such that the  $l$ -simplex  $C_l = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_l u_l)$  is contained in  $X$ , for some  $l \in \{1, \dots, k\}$ . If  $l = k$  we are done. If  $l < k$  let  $C'_{l+1} = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_l u_l + u_{l+1})$ . By construction,  $(u_1, \dots, u_{l+1})$  is a tangent of  $X$  at  $x$ . Since by hypothesis  $\mathcal{R}(X)$  is strongly semisimple, by Theorem 5.1  $(u_1, \dots, u_{l+1})$  is not outgoing, whence there is  $y \in (C'_{l+1} \cap X) \setminus C_l$ . As a consequence, there are  $\lambda'_1, \dots, \lambda'_l > 0$  and  $\lambda_{l+1} > 0$  such that  $y = x + \lambda'_1 u_1 + \dots + \lambda'_l u_l + \lambda_{l+1} u_{l+1}$  and  $\lambda'_i \leq \lambda_i$  for each  $i \in \{1, \dots, l\}$ . Since  $X$  is convex, the set  $\text{conv}(x, x + \lambda'_1 u_1, \dots, x + \lambda'_1 u_1 + \dots + \lambda'_l u_l, y)$  is contained in  $X$ . Setting now (without loss of generality)  $\lambda_i = \lambda'_i$ , we obtain the inclusion  $C_{l+1} = \text{conv}(x, x + \lambda_1 u_1, \dots, x + \lambda_1 u_1 + \dots + \lambda_{l+1} u_{l+1}) \subseteq X$ , thus completing the inductive step. This procedure terminates after  $k$  steps. Claim 2 is settled.

Since the  $k$ -simplex  $C_k$  is contained in the affine space  $\text{aff}(x_{m_1}, x_{m_2}, \dots)$ , and  $(u_1, \dots, u_k)$  is the Frenet  $k$ -frame of the sequence  $x_{m_1}, x_{m_2}, \dots$ , there exists an integer  $r^* > 0$  such that  $x_{m_j} \in C_k$  for each  $j = r^*, r^* + 1, \dots$ . By definition,  $x_{m_1}, x_{m_2}, \dots \in \text{ext}(X)$ . By Claim 2,  $C_k \subseteq X$ . Thus  $x_{m_{r^*}}, x_{m_{r^*+1}}, \dots \in \text{ext}(C_k)$ . Since  $x_i \neq x_j$  for every  $i \neq j$ , then the set  $\text{ext}(C_k)$  must be infinite, a contradiction. The proof is complete.  $\square$

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