Dimensional estimates for singular sets in geometric variational problems with free boundaries

DIMENSIONAL ESTIMATES FOR SINGULAR SETS IN GEOMETRIC VARIATIONAL PROBLEMS WITH FREE BOUNDARIES

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Abstract. We show that singular sets of free boundaries arising in codimension one anisotropic geometric variational problems are $\mathcal{H}^{n-3}$-negligible, where n is the ambient space dimension. In particular our results apply to capillarity type problems, and establish everywhere regularity in the three-dimensional case.

1. Introduction

In [DPM14], having in mind applications to capillarity problems and to relative isoperimetric problems, we studied the regularity of free boundaries in anisotropic geometric variational problems. The main result contained in [DPM14] asserts that free boundaries are regular outside closed sets of vanishing $\mathcal{H}^{n-2}$-measure. In this paper we improve upon this result by showing $\mathcal{H}^{n-3}$-negligibility of singular sets, see Theorem 1.5 below.

The "interior part" of this statement dates back to [SSA77]. The boundary case is addressed here by combining the set of ideas introduced in [SSA77] with the $\mathcal{H}^{n-2}$-negligibility we have obtained in [DPM14] (see, in particular, Lemma 2.7 below).

We note that singular sets must necessarily be smaller than merely $\mathcal{H}^{n-3}$-negligible. Indeed, a general argument due to Almgren (and appeared in [Whi86, Lemma 5.1]) implies that the set of $s>0$ such that singular sets of minimizers of a given elliptic functional are $\mathcal{H}^{s}$-negligible is open. At the same time, the cone over $\mathbf{S}^1\times\mathbf{S}^1\subset\mathbb{R}^4$ minimizes a suitable elliptic anisotropic functional [Mor91]. This example may lead to conjecture that singular sets of arbitrary anisotropic functionals have Hausdorff dimension at most $n-4$, although we are not aware of further evidence supporting this possibility.

The $\mathcal{H}^{n-3}$-negligibility of the singular set, although not optimal, has two interesting consequences. Firstly, and obviously, it implies everywhere regularity in $\mathbb{R}^3$; secondly, it provides the needed regularity in order to exploit second variation arguments in the study of geometric properties of minimizers; see for example [SZ99] and Lemma 2.5 below (actually $\mathcal{H}^{n-3}$-locally finiteness of the singular set would be enough for this, see for instance [EG92, Section 4.7.2]).

We now define the class of functionals and the notion of minimizers that we shall use.

Definition 1.1 (Regular elliptic integrands)**.** Given an open set $A\subset\mathbb{R}^n$, $\lambda\ge 1$ and $\ell\ge 0$, we consider the family $\boldsymbol{\mathcal{E}}(A,\lambda,\ell)$ of functions $\Phi:\mathrm{cl}(A)\times\mathbb{R}^n\to[0,\infty]$ such that $\Phi(x,\cdot)$ is convex and positively one-homogeneous on $\mathbb{R}^n$ with $\Phi(x,\cdot)\in C^{2,1}(\mathbf{S}^{n-1})$ for every $x\in\mathrm{cl}(A)$, and such that the following properties hold for every $x\,,y\in\mathrm{cl}(A)$, $\nu,\nu'\in\mathbf{S}^{n-1}$, and $e\in\mathbb{R}^n$:

$$\frac{1}{\lambda}\le\Phi(x,\nu)\le\lambda\,,$$

$$|\Phi(x,\nu)-\Phi(y,\nu)|+|\nabla\Phi(x,\nu)-\nabla\Phi(y,\nu)|\le\ell\,|x-y|\,,$$

$$|\nabla\Phi(x,\nu)|+\|\nabla^2\Phi(x,\nu)\|+\frac{\|\nabla^2\Phi(x,\nu)-\nabla^2\Phi(x,\nu')\|}{|\nu-\nu'|}\le\lambda\,,$$

and

$$\nabla^2\Phi(x,\nu)[e]\cdot e\ge\frac{\big|e-(e\cdot\nu)\nu\big|^2}{\lambda}\,. \tag{1.1}$$

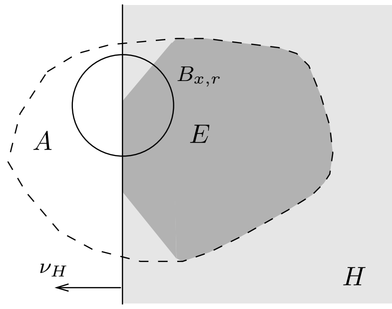

FIGURE 1.1. The situation in Definition 1.2: roughly speaking, E minimizes $\mathbf{\Phi}$ with respect to perturbations F which agree with E on $H \cap \partial B_{x,r}$ and are allowed to freely move the boundary of E close to $B_{x,r} \cap \partial H$. In other words, we impose a Dirichlet condition on $H \cap \partial B_{x,r}$ and a Neumann condition of $B_{x,r} \cap \partial H$.

In the above definition $\nabla\Phi$ and $\nabla^2\Phi$ stand for the gradient and Hessian of Φ in the ν-variable, $\|L\| = \sup\{Le : |e| = 1\}$ is the operator norm of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$, $L[e]$ is the action of L on $e \in \mathbb{R}^n$, and $\mathrm{cl}(A)$ is the closure of A. We also set

$$\boldsymbol{\mathcal{E}}_*(\lambda) = \boldsymbol{\mathcal{E}}(\mathbb{R}^n, \lambda, 0)\,,$$

for the class of *regular autonomous elliptic integrand* (indeed, $\ell = 0$ forces $\Phi(x, \nu) = \Phi(\nu)$). We shall regard $\boldsymbol{\mathcal{E}}_*(\lambda)$ as a subset of $C^{2,1}(\mathbf{S}^{n-1})$ by the obvious identification of a one-homogeneous function with its trace on the sphere. With this identification it is immediate to check that $\boldsymbol{\mathcal{E}}_*(\lambda)$ is a compact subset with respect to uniform convergence on $\mathbf{S}^{n-1}$. Finally, if $\Phi \in \boldsymbol{\mathcal{E}}(A, \lambda, \ell)$ and E is a set of locally finite perimeter in A, then we set

$$\mathbf{\Phi}(E; G) = \int_{G \cap \partial^* E} \Phi(x, \nu_E(x))\, d\mathcal{H}^{n-1}(x) \in [0, \infty]\,, \qquad \forall G \subset A\,.$$

Here $\partial^* E$ denotes the reduced boundary of E in A and ν_E is the measure-theoretic outer unit normal to E; see [Mag12, Chapter 15].

Definition 1.2 (Almost-minimizers)**.** Let an open set A and an open half-space H in $\mathbb{R}^n$ be given (possibly $H = \mathbb{R}^n$), together with $r_0 \in (0, \infty]$ and $\Lambda \geq 0$. Given $\Phi \in \boldsymbol{\mathcal{E}}(A \cap H, \lambda, \ell)$ and a set $E \subset H$ of locally finite perimeter in A, one says that E is a (Λ, r_0)*-minimizer of* $\mathbf{\Phi}$ *in* (A, H), if

$$\mathbf{\Phi}(E; H \cap W) \leq \mathbf{\Phi}(F; H \cap W) + \Lambda\, |E \Delta F|\,,$$

whenever $F \subset H$, $E \Delta F \subset\subset W$, and $W \subset\subset A$ is open with $\mathrm{diam}(W) < 2r_0$; see Figure 1.1. When $\Lambda = 0$, and $r_0 = +\infty$, one simply says that E is a *minimizer of* $\mathbf{\Phi}$ *in* (A, H).

Remark 1.3. As proved in [DPM14, Lemma 6.1], up to local diffeomorphisms, minimizers of capillarity-type problems fall in the framework of Definition 1.2. Other applications include relative isoperimetric problems in Riemannian and Finsler geometry.

Remark 1.4. Since the class $\boldsymbol{\mathcal{E}}(A \cap H, \lambda, \ell)$ is invariant by isometries of $\mathbb{R}^n$ (in the sense that, if $f(x) = x_0 + R[x]$, $R \in O(n)$, then E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A, H) if and only if $f(E)$ is a (Λ, r_0)-minimizer of $\mathbf{\Phi}^f$ in $(f(A), f(H))$ where $\Phi^f(x, \nu) = \Phi(f^{-1}(x), R^{-1}\nu)$ belongs to $\boldsymbol{\mathcal{E}}(f(A) \cap f(H), \lambda, \ell)$, see [DPM14, Lemma 2.18]) and we are interested in boundary regularity, in the sequel we can and do assume that H is a fixed half-space with $0 \in \partial H$.

Let now E be a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A, H) of some $\Phi \in \boldsymbol{\mathcal{E}}(A \cap H, \lambda, \ell)$, and set

$$M_A(E) = A \cap \mathrm{cl}(H \cap \partial E)\,.$$

The *regular set* $R_A(E)$ *of* E *in* A is defined by

$$R_A(E) = \left\{ x \in M_A(E) : \begin{array}{l} \text{there exists } r_x > 0 \text{ such that } M_A(E) \cap B_{x,r_x} \\ \text{is a } C^1\text{-manifold with boundary contained in } \partial H \end{array} \right\},$$

while $\Sigma_A(E) = M_A(E) \setminus R_A(E)$ is called the *singular set* $\Sigma_A(E)$ *of* E *in* A. In this way, $\Sigma_A(E)$ is relatively closed in A. We shall also set

$$R_G(E) = R_A(E) \cap G\,, \qquad \Sigma_G(E) = \Sigma_A(E) \cap G\,, \qquad \forall G \subset A\,.$$

By combining the results of [SSA77] for the interior situation with the ones of [DPM14] for the boundary situation, one sees that $E \cap A$ is (equivalent to) an open set, that $A \cap \partial E \cap \partial H$ is a set of finite perimeter in ∂H, and that

$$\mathcal{H}^{n-3}(\Sigma_{A\cap H}(E)) = 0\,, \qquad \text{by [SSA77]}\,, \tag{1.2}$$

$$\mathcal{H}^{n-2}(\Sigma_{A\cap \partial H}(E)) = 0\,, \qquad \text{by [DPM14]}\,, \tag{1.3}$$

with $\nabla\Phi(x, \nu_E) \cdot \nu_H = 0$ at every $x \in R_{A\cap\partial H}(E)$. Moreover, one has a characterization of the regular and singular sets in terms of the following notion of excess: given $x \in A$ and $r < \mathrm{dist}(x, \partial A)$ and denoting by $B_{x,r}$ the open ball centered at x and with radius r, we define *spherical excess of* E *at the point* x*, at scale* r*, relative to* H as

$$\mathbf{exc}^H(E, x, r) = \inf \left\{ \frac{1}{r^{n-1}} \int_{B_{x,r}\cap H \cap \partial^* E} \frac{|\nu_E - \nu|^2}{2}\, d\mathcal{H}^{n-1} : \nu \in \mathbf{S}^{n-1} \right\}.$$

Then, for positive constants $\varepsilon = \varepsilon(n, \lambda)$ and $c = c(n, \lambda)$, we have that

$$\mathbf{exc}^H(E, x, r) < \varepsilon \quad \Longrightarrow \quad M_A(E) \cap B_{x,cr} \subset R_A(E)\,, \tag{1.4}$$

see [DPM14, Theorem 3.1]. In particular

$$\Sigma_A(E) = \left\{ x \in M_A(E) : \liminf_{r\to 0^+} \mathbf{exc}^H(E, x, r) \ge \varepsilon(n, \lambda) \right\}\,, \tag{1.5}$$

Theorem 1.5. *If* $\Phi \in \boldsymbol{\mathcal{E}}(A, \lambda, \ell)$ *and* E *is a* (Λ, r_0)*-minimizer of* $\boldsymbol{\Phi}$ *in* (A, H)*, then*

$$\mathcal{H}^{n-3}(\Sigma_{A\cap\partial H}(E)) = 0\,.$$

We now describe the proof of Theorem 1.5. First of all, by a blow-up argument, Theorem 1.5 is seen to be equivalent to the following theorem.

Theorem 1.6. *If* $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$*,* $B = B_{0,1}$*, and* E *is a minimizer of* $\boldsymbol{\Phi}$ *in* (B, H)*, then*

$$\mathcal{H}^{n-3}(\Sigma_{B\cap\partial H}(E)) = 0\,. \tag{1.6}$$

We deduce Theorem 1.6 from the following two propositions, where we set

$$\boldsymbol{\mathcal{E}}_{**}(\lambda) = \left\{ \Phi \in \boldsymbol{\mathcal{E}}_*(\lambda) : \text{such that (1.6) holds true for every } E \text{ is a minimizer of } \boldsymbol{\Phi} \text{ in } (B, H) \right\}.$$

Proposition 1.7. *The set* $\boldsymbol{\mathcal{E}}_{**}(\lambda)$ *is open in* $\boldsymbol{\mathcal{E}}_*(\lambda)$ *in the uniform convergence on* $\mathbf{S}^{n-1}$.

Proposition 1.8. *The set* $\boldsymbol{\mathcal{E}}_{**}(\lambda)$ *is closed* $\boldsymbol{\mathcal{E}}_*(\lambda)$ *in the uniform convergence on* $\mathbf{S}^{n-1}$.

Proof of Theorem 1.6. Obviously, $\boldsymbol{\mathcal{E}}_*(\lambda)$ is convex, thus connected. By [Grü87] (or, alternatively, by [DPM14, Corollary 1.4]) the isotropic functional $\Phi(\nu) = |\nu|$ belongs to $\boldsymbol{\mathcal{E}}_{**}(\lambda)$ for all $\lambda \ge 1$. Propositions 1.7 and 1.8 thus imply $\boldsymbol{\mathcal{E}}_{**}(\lambda) = \boldsymbol{\mathcal{E}}_*(\lambda)$. □

In section 2 we prove Propositions 1.7 and 1.8 and show that Theorem 1.6 implies Theorem 1.5. Second variation formulas used in these arguments are collected in appendix.

We close this introduction by describing the main ideas behind the two key propositions. Proposition 1.7 is based on the idea that, roughly speaking, for every $s > 0$ the map

$$\Phi \mapsto \sup \left\{ \mathcal{H}^s(\Sigma_{B\cap\partial H}(E)) : \ E \text{ is a minimizer of } \boldsymbol{\Phi} \text{ in } (B, H) \right\}$$

is upper semi-continuous on $\boldsymbol{\mathcal{E}}_*(\lambda)$ with respect to the uniform convergence on $\mathbf{S}^{n-1}$. Concerning Proposition 1.8, one starts by observing that, if $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$, then $R_A(E)$ is a C^2-manifold with boundary. Denoting by II_E the second fundamental form of $R_A(E)$, we set

$$|\mathbf{II}_E|^2(G) = \int_{G\cap R_E(A)} |\mathrm{II}_E|^2 \, d\mathcal{H}^{n-1} \in [0,\infty] \,, \qquad \forall G \subset \mathbb{R}^n \,, \tag{1.7}$$

where $|\mathrm{II}_E|^2$ is the squared Hilbert-Schmidt norm of the tensor II_E, which equals the sum of the squared principal curvatures of $R_A(E)$. One then shows that $\Phi \in \boldsymbol{\mathcal{E}}_{**}(\lambda)$ if and only if

$$|\mathbf{II}_E|^2(B) \le C \qquad \text{for every minimizer } E \text{ of } \mathbf{\Phi} \text{ in } (B,H)\,,$$

for some $C = C(n,\lambda)$, and hence concludes by proving that the map

$$\Phi \mapsto \sup\Big\{ |\mathbf{II}_E|^2(B) : \ E \text{ is a minimizers of } \mathbf{\Phi} \text{ in } (B,H) \ \Big\}$$

is lower-semicontinuous on $\boldsymbol{\mathcal{E}}_*(\lambda)$ with respect to the uniform convergence on $\mathbf{S}^{n-1}$.

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2. Proofs

Here and in the following we say that $E_h \to E$ in A as $h \to \infty$ if $|(E_h \Delta E) \cap A| \to 0$ as $h \to \infty$, and that $E_h \to E$ locally in A as $h \to \infty$ if, for every $K \subset\subset A$, we have $E_h \to E$ in K as $h \to \infty$. Moreover, we set set $I_\varepsilon(S)$ for the ε-neighborhood of $S \subset \mathbb{R}^n$. We begin with a classical lemma concerning convergence of minimizers and of singular sets, see for instance [Mag12, Lemma 28.14]

Lemma 2.1. *Let $\{\Phi_h\}_{h\in\mathbb{N}} \subset \boldsymbol{\mathcal{E}}_*(\lambda)$ with $\Phi_h \to \Phi$ in $C^0(\mathbf{S}^{n-1})$ as $h \to \infty$, and let $\{E_h\}_{h\in\mathbb{N}}$ be such that E_h is a (Λ_h, r_h)-minimizer of $\mathbf{\Phi}_h$ in (A,H) with $\Lambda_h \to \Lambda < \infty$ and $r_h \to r_0 > 0$ as $h \to \infty$. Then there exists a (Λ, r_0)-minimizer E of $\mathbf{\Phi}$ in (A,H) such that, up to subsequences, $E_h \to E$ locally in A as $h \to \infty$. Moreover, for every $\varepsilon > 0$ and $K \subset\subset A$ there exists $h_0 > 0$ such that*

$$\Sigma_K(E_h) \subset I_\varepsilon(\Sigma_K(E))\,, \qquad \forall h \ge h_0\,. \tag{2.1}$$

In particular,

$$\mathcal{H}^s_\infty(\Sigma_K(E)) \ge \limsup_{h\to\infty} \mathcal{H}^s_\infty(\Sigma_K(E_h))\,, \qquad \forall s \in [0,n]\,, \tag{2.2}$$

where $\mathcal{H}^s_\infty$ is defined for every $G \subset \mathbb{R}^n$ as

$$\mathcal{H}^s_\infty(G) = \inf\Big\{ \sum_{i\in\mathbb{N}} \omega_s \Big(\frac{\mathrm{diam}(G_i)}{2}\Big)^s : G \subset \bigcup_{i\in\mathbb{N}} G_i\,,\ G_i \ \textit{open}\Big\} \quad \textit{with} \quad \omega_s = \frac{\pi^{s/2}}{\int_0^\infty t^{s/2}\, e^{-t}\, dt}\,.$$

Proof. The local convergence in A to a minimizer E of $\mathbf{\Phi}$ follows by [DPM14, Theorem 2.9]. Since $\mathbf{exc}^H(E_h, x, r) \to \mathbf{exc}^H(E, x, r)$ for a.e. $r > 0$ and for every $x \in A$ (cf. with [DPM14, Equation (3.10)]) and by (1.4) and (1.5), one proves (2.1). Finally, if $\{G_i\}_{i\in\mathbb{N}}$ is an open covering of $\Sigma_K(E)$, then there exists $\varepsilon > 0$ such that $\{G_i\}_{i\in\mathbb{N}}$ is a covering of $I_\varepsilon(\Sigma_K(E))$, and thus of $\Sigma_K(E_h)$ too, provided $h \ge h_0$: by minimizing on all the open coverings we obtain (2.2). □

We now prove Proposition 1.7 by using Lemma 2.1. To this end we recall some properties of $\mathcal{H}^s_\infty$. First of all, $\mathcal{H}^s_\infty \ge \mathcal{H}^s$, with

$$\mathcal{H}^s(G) = 0 \quad \text{if and only if} \quad \mathcal{H}^s_\infty(G) = 0\,. \tag{2.3}$$

Moreover, for every $G \subset \mathbb{R}^n$ and $s \in [0,n]$ we have

$$\limsup_{r\to 0} \frac{\mathcal{H}^s_\infty(G \cap B_{x,r})}{r^s} \ge c(s) > 0 \qquad \text{for } \mathcal{H}^s\text{-a.e. } x \in G\,, \tag{2.4}$$

see [Sim83, Theorem 3.26 (2)]. We now set

$$E^{x,r} = \frac{E-x}{r}\,, \qquad \forall x \in \mathbb{R}^n\,, r>0\,,$$

and we notice that, if $\Phi \in \boldsymbol{\mathcal{E}}(A,\lambda,\ell)$ and E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A,H), then $E^{x,r}$ is a $(\Lambda\, r, r_0/r)$-minimizer of $\mathbf{\Phi}^{x,r}$ in $(A^{x,r}, H^{x,r})$, where

$$\Phi^{x,r}(y,\nu) = \Phi(x + r\,y, \nu)\,, \qquad \forall y \in A^{x,r}\,, \nu \in \mathbf{S}^{n-1}\,.$$

We shall also frequently use the facts that if $x \in A \cap \partial H$ and $0 \in \partial H$ (see Remark 1.4), then $H^{x,r} = H$ for every $r > 0$ and $A^{x,r}$ eventually contains every compact set of $\mathbb{R}^n$ as $r \to 0$; and that if $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$, then $\Phi^{x,r} = \Phi$.

Proof of Proposition 1.7. Let $\Phi \in \boldsymbol{\mathcal{E}}_{**}(\lambda)$ and assume there exists $\{\Phi_h\}_{h\in\mathbb{N}} \subset \boldsymbol{\mathcal{E}}_*(\lambda)\setminus\boldsymbol{\mathcal{E}}_{**}(\lambda)$ such that $\Phi_h \to \Phi$ in $C^0(\mathbf{S}^{n-1})$ as $h \to \infty$. In particular, for every $h \in \mathbb{N}$ there exists a minimizer E_h of $\mathbf{\Phi}_h$ in (B,H) such that $\mathcal{H}^{n-3}(\Sigma_{B\cap\partial H}(E_h)) > 0$. By (2.4) there exist $x_h \in \Sigma_{B\cap\partial H}(E_h)$ and $r_h \to 0$ with

$$\frac{r_h}{\mathrm{dist}(x_h, \partial B)} \to 0 \qquad \text{as } h \to \infty, \tag{2.5}$$

such that

$$\mathcal{H}^{n-3}_\infty(\Sigma_{B\cap\partial H}(E_h) \cap B_{x_h,r_h}) \ge c(n)\, r_h^{n-3}\,.$$

Let us set $F_h = (E_h)^{x_h,r_h}$. Then F_h is a minimizer of $\mathbf{\Phi}_h$ in (B^{x_h,r_h}, H) and

$$\mathcal{H}^{n-3}_\infty(\Sigma_{B\cap\partial H}(F_h)) = \frac{\mathcal{H}^{n-3}_\infty(\Sigma_{B\cap\partial H}(E_h) \cap B_{x_h,r_h})}{r_h^{n-3}} \ge c(n) > 0\,.$$

By Lemma 2.1, there exist a minimizer F of $\mathbf{\Phi}$ in $(\mathbb{R}^n, H)$ (since $B^{x_h,r_h} \to \mathbb{R}^n$ by (2.5)) such that $\mathcal{H}^{n-3}_\infty(\Sigma_{B\cap\partial H}(F)) > 0$. By (2.3), this contradicts the fact that $\Phi \in \boldsymbol{\mathcal{E}}_{**}(\lambda)$. □

The same argument gives the following lemma.

Lemma 2.2. *If A is an open set, $\Phi \in \boldsymbol{\mathcal{E}}_{**}(\lambda)$ and E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A,H), then $\mathcal{H}^{n-3}(\Sigma_{A\cap\partial H}(E)) = 0$.*

Proof of Lemma 2.2. If E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A,H) with $\mathcal{H}^{n-3}(\Sigma_{A\cap\partial H}(E)) > 0$, then by arguing as in the proof of Proposition 1.7 we can find $r_h \to 0$ as $h \to \infty$ and $x \in \Sigma_{A\cap\partial H}(E)$ such that

$$\mathcal{H}^{n-3}_\infty(\Sigma_{A\cap\partial H}(E) \cap B_{x,r_h}) \ge c(n)\, r_h^{n-3}\,.$$

Hence $E_h = E^{x,r_h}$ is $(\Lambda\, r_h, r_0/r_h)$-minimizer of $\mathbf{\Phi}$ in $(B^{x,r}, H^{x,r})$. By Lemma 2.1 there exists a minimizer F of $\mathbf{\Phi}$ in $(\mathbb{R}^n, H)$ such that $\mathcal{H}^{n-3}_\infty(\Sigma_{B\cap\partial H}(F)) \ge c(n)$, against $\Phi \in \boldsymbol{\mathcal{E}}_{**}(\lambda)$. □

We now come to the proof of Lemma 1.8. Given $\Phi_h \to \Phi$ and a minimizer E of $\mathbf{\Phi}$, we shall need to approximate E by minimizers of $\mathbf{\Phi}_h$. This will be done by minimizing $\mathbf{\Phi}_h$ plus a suitable lower order perturbation.

Definition 2.3. Given $g \in L^\infty_{\mathrm{loc}}(A)$ one says that E *is a minimizer of* $\mathbf{\Phi} + \int g$ *on* (A,H) if $E \subset H$ is a set of locally finite perimeter in A, and

$$\mathbf{\Phi}(E; W\cap H) + \int_{W\cap H\cap E} g(x)\,dx \le \mathbf{\Phi}(F; W\cap H) + \int_{W\cap H\cap F} g(x)\,dx\,, \tag{2.6}$$

whenever $F \subset H$ and $E\Delta F \subset\subset W$ with $W \subset\subset A$ open.

Note that if E is a minimizer of $\mathbf{\Phi} + \int g$ on (A,H), then for every $A' \subset\subset A$ one has that E is a (Λ, ∞)-minimizer of $\mathbf{\Phi}$ in (A', H) with $\Lambda = \|g\|_{L^\infty(A')}$. In particular, $R_A(E)$ is always a C^1-manifold with boundary. Moreover, by exploiting the Euler-Lagrange equation associated to (2.6) (more precisely, we use the second order elliptic PDE satisfied by the first derivatives of any function u whose graph locally coincides with $R_A(E)$), one finds that, if in addition $g \in \mathrm{Lip}(\mathbb{R}^n)$, then $R_A(E)$ is actually a $C^{2,\alpha}$-manifold with boundary for every $\alpha < 1$, and hence the second fundamental form II_E is a continuous function on $R_A(E)$. It thus makes sense to define a Borel measure $|\mathbf{II}_E|^2$ on $\mathbb{R}^n$ by setting

$$|\mathbf{II}_E|^2 = |\mathrm{II}_E|^2\,\mathcal{H}^{n-1}\llcorner R_A(E)\,,$$

compare with (1.7). The continuity of II_E on $R_A(E)$ guarantees that $|\mathbf{II}_E|^2$ is a Radon measure on $A\setminus\Sigma_A(E)$.

Lemma 2.4. *Let $\{\Phi_h\}_{h\in\mathbb{N}}\subset\boldsymbol{\mathcal{E}}_*(\lambda)$ with $\Phi_h\to\Phi$ in $C^0(\mathbf{S}^{n-1})$ as $h\to\infty$, $\{g_h\}_{h\in\mathbb{N}}\subset\mathrm{Lip}(\mathbb{R}^n)$ with $\mathrm{Lip}\,g_h\le C$ and $g_h\to g$ locally uniformly on $\mathbb{R}^n$ as $h\to\infty$, and let E_h (resp., E) be a minimizer of $\mathbf{\Phi}_h+\int g_h$ (resp., $\mathbf{\Phi}+\int g$) on (A,H), with $E_h\to E$ locally in A as $h\to\infty$. Then,*

$$|\mathbf{II}_E|^2(A')\le\liminf_{h\to\infty}|\mathbf{II}_{E_h}|^2(A')\,,\tag{2.7}$$

for every open set $A'\subset A$.

Proof. The regularity, in particular [DPM14, Lemma 3.4] theory ensures that if $x\in R_{A\cap H}(E)$, then there exist $h_x\in\mathbb{N}$, $r_x>0$ and $\nu_x\in\mathbf{S}^{n-1}$ such that, if we set

$$\begin{aligned}\mathbf{C}_x&=x+\Big\{y\in\mathbb{R}^n:|y\cdot\nu_x|<r_x\,,\big|y-(y\cdot\nu_x)\nu_x\big|<r_x\Big\}\,,\\ \mathbf{D}_x&=x+\Big\{y\in\mathbb{R}^n:y\cdot\nu_x=0\,,\big|y-(y\cdot\nu_x)\nu_x\big|<r_x\Big\}\,,\end{aligned}$$

then $\mathbf{C}_x\subset\subset A\cap H$ and there exist $u_h,u\in C^{2,\alpha}(\mathbf{D}_x)$ with $u_h\to u$ in $C^{2,\alpha}(\mathbf{D}_x)$ as $h\to\infty$ and

$$\begin{aligned}\mathbf{C}_x\cap\partial E&=\mathbf{C}_x\cap R_A(E)=\Big\{z+u(z)\,\nu_x:z\in\mathbf{D}_x\Big\}\,,\\ \mathbf{C}_x\cap\partial E_h&=\mathbf{C}_x\cap R_A(E_h)=\Big\{z+u_h(z)\,\nu_x:z\in\mathbf{D}_x\Big\}\,,\end{aligned}$$

for every $h\ge h_x$. In particular, if $\varphi\in C^0(\mathbf{C}_x)$, then, as $h\to\infty$,

$$\varphi(z,u_h)\,\sqrt{1+|\nabla u_h|^2}\,|\mathrm{II}_{E_h}(z+u_h\nu_x)|^2\to\varphi(z,u)\,\sqrt{1+|\nabla u|^2}\,|\mathrm{II}_E(z+u\nu_x)|^2\,,$$

for every $z\in\mathbf{D}_x$, and, actually, locally uniformly on $z\in\mathbf{D}_x$. Thus, by the area formula for graphs one finds

$$\int_{\mathbb{R}^n}\varphi\,d|\mathbf{II}_E|^2=\lim_{h\to\infty}\int_{\mathbb{R}^n}\varphi\,d|\mathbf{II}_{E_h}|^2\,,\qquad\forall\varphi\in C^0(\mathbf{C}_x)\,.$$

By a covering argument we conclude that

$$\int_{\mathbb{R}^n}\varphi\,d|\mathbf{II}_E|^2=\lim_{h\to\infty}\int_{\mathbb{R}^n}\varphi\,d|\mathbf{II}_{E_h}|^2\,,\qquad\forall\varphi\in C^0_c((A\cap H)\setminus\Sigma_A(E))\,.\tag{2.8}$$

If now $A'\subset A$ is open, then by (2.8),

$$|\mathbf{II}_E|^2\Big((A'\cap H)\setminus\Sigma_A(E)\Big)\le\liminf_{h\to\infty}|\mathbf{II}_{E_h}|^2\Big((A'\cap H)\setminus\Sigma_A(E)\Big)\le\liminf_{h\to\infty}|\mathbf{II}_{E_h}|^2(A')\,.$$

We deduce (2.7) as $|\mathbf{II}_E|^2(A\cap\partial H)=0$ and $|\mathbf{II}_E|^2(\Sigma_A(E))=0$. □

We now exploit a second variation argument to show that the $\mathcal{H}^{n-3}$-negligibility of singular sets implies uniform L^2-estimates on second fundamental forms.

Lemma 2.5. *Let $\Phi\in\boldsymbol{\mathcal{E}}_{**}(\lambda)$, $g\in C^2(\mathbb{R}^n)$, A be a bounded open set, and E be a minimizer of $\mathbf{\Phi}+\int g$ on (A,H). Then,*

$$\frac{|\mathbf{II}_E|^2(B_{x,r})}{r^{n-3}}\le C_0(n,\lambda,\mathrm{Lip}(g))\,,\qquad\forall B_{x,2r}\subset\subset A\,.$$

Proof. By Lemma A.5 in the appendix, there exists a constant $C = C(n, \lambda, \mathrm{Lip}(g))$ such that

$$\int_{R_A(E)} |\mathrm{II}_E|^2 \, \zeta^2 \, d\mathcal{H}^{n-1} \le C \int_{R_A(E)} |\nabla\zeta|^2 + \zeta^2 \, d\mathcal{H}^{n-1} \,, \tag{2.9}$$

whenever $\zeta \in C^1_c(A)$ with $\mathrm{spt}\zeta \cap \Sigma_A(E) = \emptyset$. We shall now exploit $\Phi \in \boldsymbol{\mathcal{E}}_{**}(\lambda)$ to deduce that (2.9) holds true for every $\zeta \in C^1_c(A)$. To this end let us fix such a $\zeta \in C^1_c(A)$, and let us assume without loss of generality that $|\zeta| \le 1$ on $\mathbb{R}^n$. Since E is a (Λ, ∞)-minimizer of $\boldsymbol{\Phi}$ in (A, H), by Lemma 2.2 and by (1.2) one has $\mathcal{H}^{n-3}(\Sigma_A(E)) = 0$. In particular, given $\varepsilon > 0$ we can find a countable cover $\{F_k\}_{k\in\mathbb{N}}$ of $\Sigma_A(E)$ such that

$$\mathrm{diam}(F_k) < \varepsilon_k \,, \qquad \sum_{k\in\mathbb{N}} \varepsilon_k^{n-3} < \varepsilon \,. \tag{2.10}$$

By (2.10), for every $k \in \mathbb{N}$ we choose $x_k \in F_k$ so that $F_k \subset B_{x_k,2\varepsilon_k}$. Since $\{B_{x_k,2\varepsilon_k}\}_{k\in\mathbb{N}}$ is an open covering of $\Sigma_A(E)$, by compactness $\{B_{x_k,2\varepsilon_k}\}_{k=1}^N$ is an open covering of $\Sigma_A(E) \cap \mathrm{spt}\zeta$ for some $N \in \mathbb{N}$, and thus of $I_\delta(\Sigma_A(E) \cap \mathrm{spt}\zeta)$ for some $\delta > 0$ such that $\delta \to 0^+$ as $\varepsilon \to 0^+$. Correspondingly we consider $\psi_k \in C^1_c(B_{x_k,3\varepsilon_k}; [0,1])$ such that

$$\psi_k = 1 \text{ on } B_{x_k,2\varepsilon_k} \,, \qquad |\nabla\psi_k| \le \frac{2}{\varepsilon_k} \,, \tag{2.11}$$

and set $\psi = \max\{\psi_k : 1 \le k \le N\}$. In this way,

$$\psi = 1 \qquad \text{on } I_\delta(\Sigma_A(E) \cap \mathrm{spt}\zeta) \,.$$

This implies that $\zeta_0 = (1-\psi)\,\zeta$ is a Lipschitz function with $\mathrm{spt}\zeta_0 \cap \Sigma_A(E) = \emptyset$. By approximation, we can apply (2.9) to ζ_0 in order to find

$$\int_{R_A(E)\setminus I_\delta(\Sigma_A(E))} |\mathrm{II}_E|^2 \, \zeta^2 \, d\mathcal{H}^{n-1} \le C \int_{R_A(E)} |\nabla\zeta|^2 + |\nabla\psi|^2 + \zeta^2 \, d\mathcal{H}^{n-1} \,, \tag{2.12}$$

with $C = C(n, \lambda, \mathrm{Lip}(g))$. By the second conditions in (2.10) and (2.11) we easily find

$$\int_{R_A(E)} |\nabla\psi|^2 \le \sum_{k=1}^N \int_{R_A(E)\cap B_{x_k,3\varepsilon_k}} |\nabla\psi_k|^2 \le 4 \sum_{k=1}^N \frac{P(E; B_{x_k,3\varepsilon_k})}{\varepsilon_k^2} \le C \sum_{k\in\mathbb{N}} \varepsilon_k^{n-3} < C\,\varepsilon \,,$$

where we have used the upper density estimate $P(E; B_{x,r}) \le C(n, \lambda)\, r^{n-1}$, see [DPM14, Equation (2.47)]. By plugging this last estimate into (2.12), and then letting $\varepsilon \to 0^+$, we conclude as desired that (2.9) holds for every $\zeta \in C^1_c(A)$. Finally, for $B_{x,2r} \subset\subset A$ and $\zeta \in C^1_c(B_{x,2r})$ with $\zeta = 1$ on $B_{x,r}$ and $|\nabla\zeta| \le C/r$, (2.9) gives

$$|\mathbf{II}_E|^2(B_{x,r}) \le C \, \frac{P(E; B_{x,r})}{r^2} \le C\, r^{n-3} \,,$$

thanks again to the upper density estimate [DPM14, Equation (2.47)]. □

We finally prove that if $|\mathbf{II}_E|^2$ is a finite measure, then the singular set is $\mathcal{H}^{n-3}$-negligible. We start with the following lemma.

Lemma 2.6. *There exists $\delta = \delta(n, \lambda)$ such that if $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$, E is a minimizer of $\boldsymbol{\Phi}$ in (B, H), $0 \in \partial H$, and*

$$|\mathbf{II}_E|^2(B) \le \delta \,,$$

then $0 \in R_E(B)$.

Proof. We argue by contradiction. Let $\{\Phi_h\}_{h\in\mathbb{N}} \subset \boldsymbol{\mathcal{E}}_*(\lambda)$ be such that for each $h \in \mathbb{N}$ there exists a minimizer E_h of $\boldsymbol{\Phi}_h$ in (B, H) with $|\mathbf{II}_{E_h}|^2(B) \to 0$ as $h \to \infty$ and $0 \in \Sigma_B(E_h)$ for every $h \in \mathbb{N}$. By the compactness of $\boldsymbol{\mathcal{E}}_*(\lambda)$ and Lemma 2.1, there exist $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$ and E a minimizer of $\boldsymbol{\Phi}$ in (B, H) such that, up to subsequences, $E_h \to E$ locally in B as $h \to \infty$. Moreover, by

(1.4), (1.5) and the continuity of the excess, $0\in\Sigma_B(E)$. By (2.1), for every $\varepsilon>0$ and $r<1$ there exists h_0 such that $\Sigma_{B_r}(E_h)\subset I_\varepsilon(\Sigma_{B_r}(E))$ provided $h\ge h_0$. By Lemma 2.4,

$$\begin{aligned}|\mathbf{II}_E|^2\Big(B\setminus \mathrm{cl}\Big(I_\varepsilon(\Sigma_{B_r}(E))\Big)\Big) &\le \liminf_{h\to\infty}|\mathbf{II}_{E_h}|^2\Big(B\setminus \mathrm{cl}\Big(I_\varepsilon(\Sigma_{B_r}(E))\Big)\Big)\\ &\le \liminf_{h\to\infty}|\mathbf{II}_{E_h}|^2\Big(B\setminus \mathrm{cl}\Big(\Sigma_{B_r}(E_h)\Big)\Big)=0\,.\end{aligned}$$

By the arbitrariness of ε and r, $|\mathbf{II}_E|^2(B)=0$. We now show that this last fact implies the existence of *finitely* many hyperplanes L_i such that

$$M_{B_{1/2}}(E)\cap H=\bigcup_i L_i\cap B_{1/2}\cap H\,,\qquad L_i\cap L_j\cap B_{1/2}\cap H=\emptyset\qquad\forall i\ne j\,. \tag{2.13}$$

Indeed, by $|\mathbf{II}_E|^2(B)=0$ we have that $R_B(E)$ is contained into the union of at most *countably* many hyperplanes L_i. Let us set $A_i=B\cap H\cap L_i$ and $R_i=R_{B\cap H}(E)\cap L_i$. We claim that

$$A_i\cap\partial_{L_i}R_i\subset\Sigma_{B\cap H}(E)\,, \tag{2.14}$$

where $\partial_{L_i}R_i$ denotes the boundary of R_i as a subset of L_i. Indeed, $A_i\cap\partial_{L_i}R_i\subset M_B(E)\cap H$, so that if (2.14) fails, then there exists $x\in A_i\cap\partial_{L_i}R_i\cap R_{B\cap H}(E)$. By using the local C^1-graphicality of $R_{B\cap H}(E)$ at x, we immediately see that x belongs to the interior of R_i seen as a subset of L_i, in contradiction with $x\in\partial_{L_i}R_i$. By (2.14) and by (1.2), we find that $\mathcal{H}^{n-3}(A_i\cap\partial_{L_i}R_i)=0$, thus that $\mathcal{H}^{n-2}(A_i\cap\partial_{L_i}R_i)=0$. This implies that the distributional derivative of $1_{R_i}\in L^1_{\rm loc}(L_i)$ vanishes on the connected open set A_i: in other words, since $R_i\cap A_i\ne\emptyset$, it must be $R_i=A_i$. By the upper density estimate [DPM14, Equation (2.47)], there are finitely many hyperplanes L_i such that $L_i\cap B_{1/2}\ne\emptyset$. This proves (2.13). Since $0\in\Sigma_{B\cap\partial H}(E)$, there must be $i\ne j$ such that $0\in L_i\cap L_j\cap\partial H$: but then, by (2.13), $L_i\cap L_j\subset\Sigma_{B\cap\partial H}(E)$, against (1.3). □

Lemma 2.7. *If $\Phi\in\boldsymbol{\mathcal{E}}_*(\lambda)$, E is a minimizer of $\mathbf{\Phi}$ in (B,H), and*

$$|\mathbf{II}_E|^2(B)<\infty\,,$$

then $\mathcal{H}^{n-3}(\Sigma_B(E))=0$.

Proof. By Lemma 2.6 and by scaling

$$|\mathbf{II}_E|^2(B_{x,r})\ge\delta\,r^{n-3}\,,\qquad\forall x\in\Sigma_{B\cap\partial H}(E)\,,\quad r<\mathrm{dist}(x,\partial B)\,. \tag{2.15}$$

We now prove that, if we fix $s\in(0,1)$ and set $\Sigma_s=\Sigma_{B_s\cap\partial H}(E)$ for the sake of brevity, then

$$\lim_{r\to0^+}\frac{|I_r(\Sigma_s)|}{r^3}=0\,. \tag{2.16}$$

Let $r<1-s$ and let $\{x_i\}_{i=1}^{N(r)}\subset\Sigma_s$ be such that $|x_i-x_j|>2r$ for every $i\ne j$ and $\inf_i|x-x_i|\le 2r$ for every $x\in\Sigma_s$, i.e. $\{x_i\}_{i=1}^{N(r)}$ is a maximal $2r$-net on Σ_s. In this way, $\{B_{x_i,r}\}_{i=1}^{N(r)}$ is a finite disjoint family of balls to which we can apply (2.15), and such that $I_r(\Sigma_s)$ is covered by $B_{x_i,3r}$. Hence,

$$|I_r(\Sigma_s)|\le 3^n\,N(r)\,r^n\le\frac{3^n\,r^3}{\delta}\sum_{i=1}^{N(r)}|\mathbf{II}_E|^2(B_{x_i,r})\le\frac{3^n\,r^3}{\delta}\,|\mathbf{II}_E|^2(I_r(\Sigma_s))\,.$$

Since, by assumption, $|\mathbf{II}_E|^2(B)<\infty$, we have

$$\lim_{r\to0^+}|\mathbf{II}_E|^2(I_r(\Sigma_s))=|\mathbf{II}_E|^2(\Sigma_s)=0\,,$$

where in the last identity we have used the fact that $|\mathbf{II}_E|^2$ is concentrated on $R_B(E)$. This proves (2.16), which immediately implies $\mathcal{H}^{n-3}(\Sigma_s)=0$ (note that this could be directly inferred by the previous proof, however (2.16) provides a slightly stronger information). By the arbitrariness of s we complete the proof. □

Proof of Proposition 1.8. Let us consider a sequence $\{\Phi_h\}_{h\in\mathbb{N}} \subset \boldsymbol{\mathcal{E}}_{**}(\lambda)$ such that $\Phi_h \to \Phi$ in $C^0(\mathbf{S}^{n-1})$ as $h \to \infty$ for some $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$, and let E be a minimizer of $\mathbf{\Phi}$ in (B, H). We fix $s \in (0,1)$ and consider the variational problems

$$\inf\left\{\mathbf{\Phi}_h(F; H\cap B) + \int_F g_h(x)\,dx : F \subset H\,, F\Delta E \subset B_s\right\}, \tag{2.17}$$

where we have set

$$g_h = \varphi_h * \Big(\mathrm{dist}(\cdot, E) - \mathrm{dist}(\cdot, E^c)\Big)\,,$$

for a sequence of smooth mollifiers $\{\varphi_h\}_h$; in particular, $g_h \in C^\infty(\mathbb{R}^n)$ with $\mathrm{Lip} g_h \le 1$ for every $h \in \mathbb{N}$. Let now E_h be a minimizer in (2.17): we claim that $E_h \to E$ in B as $h \to \infty$. Indeed, by [DPM14, Theorem 2.9] there exists $G \subset H$ such that, up to subsequences, $E_h \to G$ locally in B_s as $h \to \infty$. By comparing E_h with E in (2.17), by lower semicontinuity (see [DPM14, Equation (2.64)]), and setting $g = \mathrm{dist}(\cdot, E) - \mathrm{dist}(\cdot, E^c)$, one has

$$\mathbf{\Phi}(G; H\cap B) + \int_G g \le \liminf_{h\to\infty} \mathbf{\Phi}_h(E_h; H\cap B) + \int_{E_h} g_h \le \mathbf{\Phi}(E; H\cap B_s) + \int_E g\,.$$

By minimality of E (note that $G\Delta E \subset B_s \subset\subset B$), $\mathbf{\Phi}(E; H\cap B) \le \mathbf{\Phi}(G; H\cap B)$, and thus

$$0 \ge \int_G g - \int_E g = \int_{G\setminus E} \mathrm{dist}(x, E)\,dx + \int_{E\setminus G} \mathrm{dist}(x, E^c)\,dx\,.$$

In particular, $|E\Delta G| = 0$, that is, $E_h \to E$ locally in B_s, thus in B by $E_h\Delta E \subset B_s$, as $h \to \infty$.
Since E_h is a minimizer for $\mathbf{\Phi}_h + \int g_h$ on (B_s, H), by Lemma 2.5 (and $\mathrm{Lip} g_h \le 1$) we find

$$\frac{|\mathbf{II}_{E_h}|^2(B_{x,r})}{r^{n-3}} \le C(n,\lambda)\,, \qquad \forall B_{x,2r} \subset\subset B_s\,.$$

Hence, by Lemma 2.4, one finds

$$|\mathbf{II}_E|^2(B_{x,r}) < \infty, \qquad \forall B_{x,2r} \subset\subset B_s\,.$$

By Lemma 2.7 we have $\mathcal{H}^{n-3}(\Sigma_{B_{x,r}\cap\partial H}(E)) = 0$ for every $B_{x,2r} \subset\subset B_s$. By covering and by the arbitrariness of s we find $\mathcal{H}^{n-3}(\Sigma_{B\cap\partial H}(E)) = 0$. This shows that $\Phi \in \boldsymbol{\mathcal{E}}_{**}(\lambda)$. □

As explained in the introduction, Propositions 1.7 and 1.8 imply Theorem 1.6. We finally deduce Theorem 1.5 from this last result.

Proof of Theorem 1.5. The proof is essentially the same as that of Lemma 2.2. Let us briefly sketch it: assume by contradiction that there exist constants $\lambda \ge 1$, $\ell \ge 0$, $\Lambda \ge 0$, $r_0 > 0$, an open set A, $\Phi \in \boldsymbol{\mathcal{E}}(A\cap H, \lambda, \ell)$ and E a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A, H) such that

$$\mathcal{H}^{n-3}(\Sigma_{A\cap\partial H}(E)) > 0\,.$$

According to (2.4) we can find $x_0 \in \Sigma_{A\cap\partial H}(E)$ and $r_h \to 0$ as $h \to \infty$ such that

$$\mathcal{H}^{n-3}_\infty(\Sigma_{A\cap\partial H}(E) \cap B_{x_0,r_h}) > c(n) r_h^{n-3}. \tag{2.18}$$

Let us set $F_h = E^{x_0,r_h}$ and notice that F_h are $(\Lambda r_h, r_0/r_h)$-minimizer of $\mathbf{\Phi}_h$ in (A^{x_0,r_h}, H) where $\Phi_h(x,\nu) = \Phi(x_0 + r_h x, \nu) \in \boldsymbol{\mathcal{E}}(A^{x_0,r_h} \cap H, \lambda, \ell r_h)$. According to Lemma 2.1 and arguing as in the proof of Lemma 2.2 one finds E_∞ a minimizer of $\mathbf{\Phi}_\infty$ in $(\mathbb{R}^n, H)$ where $\Phi_\infty(\nu) = \Phi(x_0,\nu) \in \boldsymbol{\mathcal{E}}_*(\lambda)$. However, by (2.18), (2.2) and (2.3), we find $\mathcal{H}^{n-3}(\Sigma_{B\cap\partial H}(E_\infty)) > 0$, a contradiction to Theorem 1.6. □

Appendix A. First and second variations of anisotropic functionals

Lemma 2.5 relies on the second variation formulas for anisotropic functionals. For the reader's convenience, and since this kind of computation is not so easily accessible in the literature, we include a derivation of these formulas.

We consider an open set with smooth boundary Ω in $\mathbb{R}^n$, a bounded open set A with $A\cap\Omega \neq \emptyset$, and a set $E \subset \Omega$ of finite perimeter in A. Given $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$ and $g \in C^2(\mathbb{R}^n)$, we compute the first and second variation of

$$(\boldsymbol{\Phi} + \textstyle\int g)(f_t(E)) = \displaystyle\int_{A\cap\Omega\cap\partial^* f_t(E)} \Phi(\nu_{f_t(E)})\, d\mathcal{H}^{n-1} + \int_{A\cap f_t(E)} g\,,$$

where $\{f_t\}_{|t|\le\varepsilon_0}$ is such that:

(i) $(x,t) \mapsto f_t(x)$ of class $C^1(\Omega\times(-\varepsilon_0,\varepsilon_0);\Omega)$ with $f_0 = \mathrm{Id}\,,\ f_t(\Omega) = \Omega$ for every $|t| < \varepsilon_0$, and $t \in (-\varepsilon_0,\varepsilon_0) \mapsto f_t(x)$ of class $C^3((-\varepsilon_0,\varepsilon_0);\Omega)$ uniformly with respect to $x \in \Omega$;

(ii) $\mathrm{spt}(f_t - \mathrm{Id}\,) \subset\subset A$.

These conditions imply that

$$\frac{d}{dt} f_t(x)\cdot\nu_\Omega(f_t(x)) = 0\,,\qquad x \in \partial\Omega\cap A\,,\quad |t| < \varepsilon_0\,. \tag{A.1}$$

We also notice that, if we define $T, Z \in C^1_c(\Omega;\mathbb{R}^n)$ by setting

$$T(x) = \frac{d}{dt}\Big|_{t=0} f(x) \qquad\text{and}\qquad Z(x) = \frac{d^2}{dt^2}\Big|_{t=0} f_t(x)\,, \tag{A.2}$$

then we have, uniformly on $x\in\mathbb{R}^n$ as $t\to 0^+$,

$$f_t = \mathrm{Id}\, + tT + \frac{t^2}{2} Z + O(t^3)\,. \tag{A.3}$$

By (A.1) we find

$$T\cdot\nu_\Omega = 0\,,\qquad \forall\, x\in\partial\Omega\,. \tag{A.4}$$

By differentiating (A.1) with respect to t we obtain that

$$Z\cdot\nu_\Omega = -T\cdot \mathrm{II}_\Omega[T]\,,\qquad \forall x\in\partial\Omega\,, \tag{A.5}$$

where $\mathrm{II}_\Omega : T_x\partial\Omega \to T_x\partial\Omega$ is the second fundamental form of $\partial\Omega$. (Note that $T(x)$ is a tangent vector to $\partial\Omega$ at $x\in\partial\Omega$ exactly by (A.4).) We now recall two basic facts. Lemma A.1 is consequence of the classical area formula, see for example [Mag12, Proposition 17.1], while Lemma A.2 is a standard Taylor expansion, see [Mag12, Lemma 17.4].

Lemma A.1. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz diffeomorphism with $\det(\nabla f) > 0$ on $\mathbb{R}^n$, then $f(E)$ is a set of finite perimeter in $f(A)$, with $f(\partial^* E) =_{\mathcal{H}^{n-1}} \partial^*(f(E))$ and*

$$\nu_{f(E)}(f(x)) = \frac{\mathrm{cof}\,(\nabla f(x))[\nu_E(x)]}{|\mathrm{cof}\,(\nabla f(x))[\nu_E(x)]|}\,,\qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in\partial^* f(E)\,,$$

where for any invertible linear map $L : \mathbb{R}^n\to\mathbb{R}^n$ one defines $\mathrm{cof}\, L = (\det L)\,(L^{-1})^*$. Moreover, for every $G\subset A$, one has

$$\int_{f(G\cap\partial^* E)} \Phi(\nu_{f(E)}(y))\, d\mathcal{H}^{n-1}(y) = \int_{G\cap\partial^* E} \Phi\big(\mathrm{cof}\,(\nabla f(x))\,[\nu_E(x)]\big)\, d\mathcal{H}^{n-1}(x)\,. \tag{A.6}$$

Lemma A.2. *If $X, Y : \mathbb{R}^n\to\mathbb{R}^n$ are linear maps, then*

$$\det\Big(\mathrm{Id}\, + tX + \frac{t^2}{2} Y + O(t^3)\Big) = 1 + t\,\mathrm{tr}X + \frac{t^2}{2}\Big((\mathrm{tr}X)^2 - \mathrm{tr}(X^2) + \mathrm{tr}Y\Big) + O(t^3)\,, \tag{A.7}$$

$$\Big(\mathrm{Id}\, + tX + \frac{t^2}{2} Y + O(t^3)\Big)^{-1} = \mathrm{Id}\, - tX + \frac{t^2}{2}\Big(2X^2 - Y\Big) + O(t^3)\,,$$

and thus

$$
\begin{aligned}
&\operatorname{cof}\Big(\mathrm{Id} + tX + \frac{t^2}{2}\,Y + O(t^3)\Big)\\
&\qquad\qquad = \mathrm{Id} + t\Big(\operatorname{tr}(X)\mathrm{Id} - X^*\Big)\\
&\qquad\qquad + \frac{t^2}{2}\Big[\Big(\operatorname{tr}(X)^2 - \operatorname{tr}(X^2) + \operatorname{tr}(Y)\Big)\mathrm{Id} + 2\,(X^*)^2 - 2\operatorname{tr}(X)\,X^* - Y^*\Big] + O(t^3)\,.
\end{aligned}
$$

We are now ready to compute the first and second variation of $\mathbf{\Phi} + \int g$.

Lemma A.3. *If $g \in C^2(A)$, then*

$$
\frac{d}{dt}\Big|_{t=0}\int_{A\cap f_t(E)} g = \int_{A\cap\Omega\cap\partial^* E} g\,(T\cdot\nu_E)\,d\mathcal{H}^{n-1}\,, \tag{A.8}
$$

and

$$
\begin{aligned}
\frac{d^2}{dt^2}\Big|_{t=0}\int_{A\cap f_t(E)} g &= \int_{A\cap\Omega\cap\partial^* E} g\,(Z\cdot\nu_E)\,d\mathcal{H}^{n-1}\\
&\quad + \int_{A\cap\Omega\cap\partial^* E} \operatorname{div}(g\,T)\,(T\cdot\nu_E) - g\,(\nabla T[T]\cdot\nu_E)\,d\mathcal{H}^{n-1}\,.
\end{aligned} \tag{A.9}
$$

Proof. Step one: We notice the validity of the following formula: if $S \in C^1_c(A;\mathbb{R}^n)$ and $E \subset \Omega$, then

$$
\begin{aligned}
&\int_{A\cap E} g\big[(\operatorname{div} S)^2 - \operatorname{tr}(\nabla S)^2\big] + 2\operatorname{div} S\,\nabla g\cdot S + \nabla^2 g[S]\cdot S\\
&\qquad\qquad = \int_{\Omega\cap A\cap\partial^* E} \operatorname{div}(g\,S)(S\cdot\nu_E) - g\,\nabla S[S]\cdot\nu_E\,d\mathcal{H}^{n-1}\\
&\qquad\qquad\quad + \int_{A\cap\partial\Omega\cap\partial^* E} \operatorname{div}(g\,S)(S\cdot\nu_\Omega) - g\,\nabla S[S]\cdot\nu_\Omega\,d\mathcal{H}^{n-1}\,,
\end{aligned}
$$

where $E^{(1)}$ is the set of points of density one of E. Indeed, if $S \in C^2_c(A;\mathbb{R}^n)$, then the assertion follow by the divergence theorem and by the identity

$$
g\big[(\operatorname{div} S)^2 - \operatorname{tr}(\nabla S)^2\big] + 2\operatorname{div} S\,\nabla g\cdot S + \nabla^2 g[S]\cdot S = \operatorname{div}(\operatorname{div}(gS)S) - \operatorname{div}(g\,\nabla S[S])\,.
$$

The case when $S \in C^1_c(A;\mathbb{R}^n)$ is then obtained by approximation.

Step two: Since $f_t(A) = A$, we find $f_t(E)\cap A = f_t(E\cap A)$. Hence by the area formula,

$$
\int_{A\cap f_t(E)} g(y)\,dy = \int_{A\cap E} g(f_t(x))\det\nabla f_t(x)dx\,.
$$

By (A.3), by (A.7) and by the Taylor expansion of g we get

$$
\begin{aligned}
&\int_{A\cap f_t(E)} g(y)\,dy = \int_{A\cap E} g + t\int_{A\cap E}\nabla g\cdot T + g\operatorname{div} T\\
&+ \frac{t^2}{2}\int_{A\cap E} g\big[\operatorname{div} Z + (\operatorname{div} T)^2 - \operatorname{tr}(\nabla T)^2\big] + 2\operatorname{div} T\,\nabla g\cdot T + \nabla^2 g[T]\cdot T + \nabla g\cdot Z + O(t^3)\,.
\end{aligned}
$$

Inasmuch, $\mathrm{div}\,(g\,T)=\nabla g\cdot T+g\,\mathrm{div}\,T$ and $\mathrm{div}\,(g\,Z)=\nabla g\cdot Z+g\,\mathrm{div}\,Z$, by step one and by (A.4), one finds (A.8) and

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0}\int_{A\cap f_t(E)} g = &\int_{A\cap\Omega\cap\partial^*E} g\,(Z\cdot\nu_E)\,d\mathcal{H}^{n-1}\\&+\int_{A\cap\Omega\cap\partial^*E}\mathrm{div}\,(g\,T)\,(T\cdot\nu_E)\,d\mathcal{H}^{n-1}\\&-\int_{A\cap\Omega\cap\partial^*E} g\,(\nabla T[T]\cdot\nu_E)\,d\mathcal{H}^{n-1}\\&+\int_{A\cap\partial\Omega\cap\partial^*E} g\,\big(Z\cdot\nu_\Omega-\nabla T[T]\cdot\nu_\Omega\big)\,d\mathcal{H}^{n-1}\,.\end{aligned}$$

We now complete the proof of (A.9) by showing that $\nabla T[T]\cdot\nu_\Omega=Z\cdot\nu_\Omega$. Indeed, by differentiating (A.4) along T one finds $0=\nabla T[T]\cdot\nu_\Omega+T\cdot\,\mathrm{II}_\Omega[T]$, and then conclude by (A.5). ☐

Lemma A.4. *We have*

$$\frac{d}{dt}\Big|_{t=0}\int_{A\cap\Omega\cap\partial^*f_t(E)}\Phi(\nu_{f_t(E)})\,d\mathcal{H}^{n-1}=\int_{A\cap\Omega\cap\partial^*E}\Phi(\nu_E)\mathrm{div}\,T-\nabla T^*[\nu_E]\cdot\nabla\Phi(\nu_E)\,d\mathcal{H}^{n-1}\,,\tag{A.10}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0}\int_{A\cap\Omega\cap\partial^*f_t(E)}\Phi(\nu_{f_t(E)})\,d\mathcal{H}^{n-1}=&\int_{A\cap\Omega\cap\partial^*E}\Phi(\nu_E)\mathrm{div}\,Z-\nabla Z^*[\nu_E]\cdot\nabla\Phi(\nu_E)\,d\mathcal{H}^{n-1}\\&+\int_{A\cap\Omega\cap\partial^*E}\Phi(\nu_E)\big\{(\mathrm{div}\,T)^2-\mathrm{tr}(\nabla T)^2\big\}\,d\mathcal{H}^{n-1}\\&+2\int_{A\cap\Omega\cap\partial^*E}(\nabla T^*)^2[\nu_E]\cdot\nabla\Phi(\nu_E)-\mathrm{div}\,T\nabla T^*[\nu_E]\cdot\nabla\Phi(\nu_E)\,d\mathcal{H}^{n-1}\\&+\int_{A\cap\Omega\cap\partial^*E}\nabla^2\Phi(\nu_E)\big[\nabla T^*[\nu_E]\big]\cdot\nabla T^*[\nu_E]\,d\mathcal{H}^{n-1}\,.\end{aligned}\tag{A.11}$$

Proof. By (A.3), Lemma A.2, and by the Taylor expansion of Φ at ν_E, we get

$$\begin{aligned}\Phi\big(\mathrm{cof}\,(\nabla f_t(x))[\nu_E]\big)=&\,\Phi(\nu_E)+t\Big\{\Phi(\nu_E)\mathrm{div}\,T-\nabla T^*[\nu_E]\cdot\nabla\Phi(\nu_E)\Big\}\\&+\frac{t^2}{2}\Big\{\Phi(\nu_E)\mathrm{div}\,Z-\nabla Z^*[\nu_E]\cdot\nabla\Phi(\nu_E)\\&\quad+\Phi(\nu_E)\big\{(\mathrm{div}\,T)^2-\mathrm{tr}(\nabla T)^2\big\}-2\mathrm{div}\,T\,\nabla T^*[\nu_E]\cdot\nabla\Phi(\nu_E)\\&\quad+2(\nabla T^*)^2[\nu_E]\cdot\nabla\Phi(\nu_E)+\nabla^2\Phi(\nu_E)\big[\nabla T^*[\nu_E]\big]\cdot\nabla T^*[\nu_E]\Big\}+O(t^3)\,,\end{aligned}$$

where we have also used $\Phi(\nu_E)=\nabla\Phi(\nu_E)\cdot\nu_E$ and $\nabla^2\Phi(\nu_E)[\nu_E]=0$. By (A.6) and by $f_t(A)=A$ we find (A.10) and (A.11). ☐

We now come to the lemma that was used in the proof of Lemma 2.5. In the following we define II_E^Φ by setting

$$\mathrm{II}_E^\Phi(x)=\nabla^2\Phi(\nu_E(x))\,\mathrm{II}_E(x)\qquad\forall\,x\in R_A(E)\,.$$

Note that, by one-homogeneity of Φ, $\nabla^2\Phi(\nu_E)[\nu_E]=0$; therefore, by symmetry of $\nabla^2\Phi(\nu_E)$, the tensor $\mathrm{II}_E^\Phi(x)$ is a well defined operator from $T_xR_A(E)$ into itself.

Lemma A.5. *Let $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$, $g \in C^2(\mathbb{R}^n)$, A be a bounded open set, H an open half-space and E be a minimizer of $\mathbf{\Phi} + \int g$ on (A, H). Then*

$$\int_{R_A(E)} \zeta^2 \Phi(\nu_E) \mathrm{tr}[(\mathrm{II}_E^\Phi)^2]\, d\mathcal{H}^{n-1} \\ \leq \int_{R_A(E)} \Phi(\nu_E)^2\, \nabla^2\Phi(\nu_E)[\nabla\zeta]\cdot\nabla\zeta + \zeta^2\Phi(\nu_E)(\nabla g\cdot\nabla\Phi(\nu_E))\, d\mathcal{H}^{n-1}\,, \tag{A.12}$$

for every $\zeta \in C^1_c(A)$ with $\mathrm{spt}\zeta \cap \Sigma_A(E) = \emptyset$. Moreover, there exists a constant $C = C(n, \lambda, \mathrm{Lip}(g))$ such that

$$\int_{R_A(E)} |\mathrm{II}_E|^2\, \zeta^2\, d\mathcal{H}^{n-1} \leq C \int_{R_A(E)} |\nabla\zeta|^2 + \zeta^2\, d\mathcal{H}^{n-1}\,, \tag{A.13}$$

whenever $\zeta \in C^1_c(A)$ with $\mathrm{spt}\zeta \cap \Sigma_A(E) = \emptyset$.

Proof. As proved in [DPM14, Section 2.4] we have

$$\nabla\Phi(\nu_E(x))\cdot\nu_H = 0 \qquad \forall\, x \in R_A(E) \cap \partial H\,.$$

If $\zeta \in C^1_c(A \setminus \Sigma_A(E))$, then there exists $N \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$N = \nu_E\,, \qquad \text{on } R_A(E) \cap \mathrm{spt}\,\zeta\,, \tag{A.14}$$

$$\nabla\Phi(N)\cdot\nu_H = 0\,, \qquad \text{on } R_A(E) \cap \partial H \cap \mathrm{spt}\,\zeta\,. \tag{A.15}$$

We set $T = \zeta\,\nabla\Phi(N) \in C^1_c(A; \mathbb{R}^n)$ and we note that, by (A.15), $f_t(x) = x + tT(x)$ defines a family of admissible variations for $|t| \leq \varepsilon_0$ and ε_0 suitably small. Since f_t is affine in t, by (A.2), one has $Z = 0$. In particular, by Lemma A.1, Lemma A.2, and by minimality of E,

$$0 = \frac{d}{dt}\Big|_{t=0} (\Phi + \textstyle\int g)(f_t(E)) = \displaystyle\int_{A\cap H\cap\partial^* E} g\,(T\cdot\nu_E) + \Phi\,\mathrm{div}\,T - (\nabla T)^*[\nu_E]\cdot\nabla\Phi\, d\mathcal{H}^{n-1}\,, \tag{A.16}$$

$$0 \leq \frac{d^2}{dt^2}\Big|_{t=0} (\Phi + \textstyle\int g)(f_t(E)) = \displaystyle\int_{A\cap H\cap\partial^* E} \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4\, d\mathcal{H}^{n-1}\,,$$

where, setting for simplicity $\Phi = \Phi(\nu_E)$, $\nabla\Phi = \nabla\Phi(\nu_E)$, and $\nabla^2\Phi = \nabla^2\Phi(\nu_E)$, one has

$$\begin{aligned} \Gamma_1 &= \mathrm{div}\,(g\,T)\,(T\cdot\nu_E) - g\,\nabla T[T]\cdot\nu_E\,, \\ \Gamma_2 &= \big((\mathrm{div}\,T)^2 - \mathrm{tr}((\nabla T)^2)\big)\,\Phi\,, \\ \Gamma_3 &= 2\Big((\nabla T^*)^2[\nu_E]\cdot\nabla\Phi - \mathrm{div}\,T\,\nabla T^*[\nu_E]\cdot\nabla\Phi\Big)\,, \\ \Gamma_4 &= \nabla^2\Phi\,\big[\nabla T^*[\nu_E]\big]\cdot\nabla T^*[\nu_E]\,. \end{aligned}$$

We start by noticing that (A.14) gives

$$\nabla N(x) = \mathrm{II}_E(x) + a(x)\otimes\nu_E(x) \qquad \forall\, x \in R_A(E)\cap\mathrm{spt}\zeta\,,$$

where $\mathrm{II}_E(x)$ is extended to be zero on $(T_x R_A(E))^\perp$ and $a : R_A(E) \to \mathbb{R}^n$ is a continuous vector field. Hence

$$\nabla T = \nabla\Phi\otimes\nabla\zeta + \zeta\,\mathrm{II}_E^\Phi + \zeta\,\nabla^2\Phi[a]\otimes\nu_E\,, \qquad \text{on } R_A(E)\,.$$

By $\nabla^2\Phi\,[\nu_E] = 0$ and the symmetry of $\nabla^2\Phi$ one finds $\mathrm{tr}(\nabla^2\Phi[a]\otimes\nu_E) = 0$, so that

$$\mathrm{div}\,T = \nabla\Phi\cdot\nabla\zeta + \zeta\,H_E^\Phi\,, \qquad \text{on } R_A(E)\,, \tag{A.17}$$

where we have set

$$H_E^\Phi = \mathrm{tr}(\mathrm{II}_E^\Phi) = \mathrm{tr}(\nabla^2\Phi\,\mathrm{II}_E)\,.$$

Moreover, by $\nabla\Phi\cdot\nu_E = \Phi$ and again by $\nabla^2\Phi\,[\nu_E] = 0$ we find $(\nabla T)^*[\nu_E] = \Phi\,\nabla\zeta$ and $T\cdot\nu_E = \zeta\Phi$, so that (A.16) gives

$$0 = \int_{A\cap H\cap\partial^* E} (g + H_E^\Phi)\,\Phi\,\zeta\, d\mathcal{H}^{n-1}\,.$$

The validity of this condition for every $\zeta \in C^1_c(A \setminus \Sigma_A(E))$ gives the well-know stationarity condition

$$H^\Phi_E + g = 0\,, \qquad \forall\, x \in R_A(E)\,. \tag{A.18}$$

We now compute Γ_1. By $\nabla\Phi \cdot \nu_E = \Phi$, we find

$$\nabla T[T] = \zeta\,(\nabla\zeta\cdot\nabla\Phi)\,\nabla\Phi + \zeta^2\,\mathrm{II}^\Phi_E[\nabla\Phi] + \zeta^2\,\Phi\,\nabla^2\Phi[a]\,,$$

so that, by $\mathrm{II}^\Phi_E[\nabla\Phi]\cdot\nu_E = 0$ and by $\nabla^2\Phi[a]\cdot\nu_E = 0$ (which follow by the symmetry of $\nabla^2\Phi$ and by $\nabla^2\Phi[\nu] = 0$), we find

$$\nabla T[T]\cdot\nu_E = \zeta\,\Phi\,(\nabla\zeta\cdot\nabla\Phi)\,.$$

By (A.17), (A.18) and a simple computation one gets

$$\Gamma_1 = \Big((\nabla\Phi\cdot\nabla g) + g\,H^\Phi_E\Big)\zeta^2\,\Phi = \Big((\nabla\Phi\cdot\nabla g) - (H^\Phi_E)^2\Big)\zeta^2\,\Phi\,.$$

We now start computing Γ_2. By (A.17) we have

$$(\mathrm{div}\,T)^2 = (\nabla\Phi\cdot\nabla\zeta)^2 + \zeta^2\,(H^\Phi_E)^2 + 2\,\zeta\,H^\Phi_E\,(\nabla\Phi\cdot\nabla\zeta)\,;$$

at the same time, writing $\nabla T = X + Y$ where $X = \nabla\Phi\otimes\nabla\zeta + \zeta\mathrm{II}^\Phi_E$ and $Y = \zeta\nabla^2\Phi[a]\otimes\nu_E$, and noticing that $Y^2 = 0$, while

$$\begin{aligned}
\mathrm{tr}(Y\,X) = \mathrm{tr}(X\,Y) &= \mathrm{tr}\Big(\zeta(\nabla\zeta\cdot\nabla^2\Phi[a])\,\nabla\Phi\otimes\nu_E + \zeta^2\,\mathrm{II}^\Phi_E\,\nabla^2\Phi[a]\otimes\nu_E\Big)\\
&= \zeta(\nabla\zeta\cdot\nabla^2\Phi[a])\,\Phi\,,\\
X^2 &= (\nabla\zeta\cdot\nabla\Phi)\nabla\Phi\otimes\nabla\zeta + \zeta^2(\mathrm{II}^\Phi_E)^2 + \zeta\,\mathrm{II}^\Phi_E[\nabla\Phi]\otimes\nabla\zeta + \zeta\,\nabla\Phi\otimes(\mathrm{II}^\Phi_E)^*[\nabla\zeta]\,,
\end{aligned}$$

we find that,

$$\mathrm{tr}((\nabla T)^2) = (\nabla\zeta\cdot\nabla\Phi)^2 + \zeta^2\mathrm{tr}[(\mathrm{II}^\Phi_E)^2] + 2\,\zeta\,(\nabla\zeta\cdot\mathrm{II}_\Phi[\nabla\Phi]) + 2\,(\nabla\zeta\cdot\nabla^2\Phi[a])\,\Phi\,.$$

Hence,

$$\begin{aligned}
\Gamma_2 &= \zeta^2\,(H^\Phi_E)^2\Phi + 2\zeta(\nabla\zeta\cdot\nabla\Phi)H^\Phi_E\,\Phi - \zeta^2\mathrm{tr}[(\mathrm{II}^\Phi_E)^2]\,\Phi\\
&\quad -2\zeta\,(\nabla\zeta\cdot\mathrm{II}^\Phi_E[\nabla\Phi])\Phi - 2\,(\nabla\zeta\cdot\nabla^2\Phi[a])\,\Phi^2\,.
\end{aligned}$$

We now compute Γ_3. By (A.17) and $(\nabla T)^*[\nu_E] = \Phi\,\nabla\zeta$, we find

$$\mathrm{div}\,T\,\nabla T^*[\nu_E]\cdot\nabla\Phi = (\nabla\zeta\cdot\nabla\Phi)^2\Phi + \zeta H^\Phi_E\,(\nabla\zeta\cdot\nabla\Phi)\,\Phi\,.$$

At the same time, writing $\nabla T = X + Y$ with X and Y as above, we find

$$\begin{aligned}
(X^*)^2 &= (\nabla\zeta\cdot\nabla\Phi)\nabla\zeta\otimes\nabla\Phi + \zeta^2(\mathrm{II}^\Phi_E)^2 + \zeta\,\nabla\zeta\otimes\mathrm{II}^\Phi_E[\nabla\Phi] + \zeta\,(\mathrm{II}^\Phi_E)^*[\nabla\zeta]\otimes\nabla\Phi\,,\\
Y^*X^* &= \zeta(\nabla\zeta\cdot\nabla^2\Phi[a])\,\nu_E\otimes\nabla\Phi + \zeta^2\,(\nu_E\otimes\nabla^2\Phi[a])\,\mathrm{II}^\Phi_E\\
X^*Y^* &= \zeta\,\Phi\,\nabla\zeta\otimes\nabla^2\Phi[a]\,.
\end{aligned}$$

By taking into account that $(Y^*)^2 = 0$ (as $Y^2 = 0$) and by exploiting once more that $\nabla^2\Phi[\nu_E] = 0$ and $\mathrm{II}^\Phi_E[\nu_E] = 0$, we find that

$$[(\nabla T)^*]^2[\nu_E] = (\nabla\zeta\cdot\nabla\Phi)\,\Phi\,\nabla\zeta + \zeta\,\Phi\,(\mathrm{II}^\Phi_E)^*[\nabla\zeta] + \zeta(\nabla\zeta\cdot\nabla^2\Phi[a])\,\Phi\,\nu_E\,,$$

so that

$$[(\nabla T)^*]^2[\nu_E]\cdot\nabla\Phi = (\nabla\zeta\cdot\nabla\Phi)^2\,\Phi + \zeta\,\nabla\zeta\cdot\,\mathrm{II}^\Phi_E[\nabla\Phi]\,\Phi + \zeta(\nabla\zeta\cdot\nabla^2\Phi[a])\,\Phi^2\,.$$

In conclusion,

$$\Gamma_3 = 2\Big(\zeta\,\nabla\zeta\cdot\mathrm{II}^\Phi_E[\nabla\Phi]\,\Phi + \zeta(\nabla\zeta\cdot\nabla^2\Phi[a])\,\Phi^2 - \zeta H^\Phi_E\,(\nabla\zeta\cdot\nabla\Phi)\,\Phi\Big)\,,$$

so that

$$\Gamma_1 + \Gamma_2 + \Gamma_3 = \Big(\nabla\Phi\cdot\nabla g - \mathrm{tr}[(\mathrm{II}^\Phi_E)^2]\Big)\,\zeta^2\,\Phi\,.$$

On noticing that $\Gamma_4 = \Phi^2 \nabla^2\Phi \left[\nabla\zeta\right] \cdot \nabla\zeta$, we conclude the proof of (A.12). By (1.1), one has $\nabla^2\Phi \geq (1/\lambda)\mathrm{Id}_{T_x(R_A(E))}$ for every $x \in R_A(E)$, and thus $\mathrm{tr}[(\mathrm{II}_E^\Phi)^2] \geq \lambda^{-2}\,|\mathrm{II}_E|^2$. Hence, (A.12) implies (A.13). □

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