

SURFACE SHEAR WAVES IN A HALF-PLANE WITH DEPTH-VARIANT STRUCTURE

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ABSTRACT. We consider the propagation of surface shear waves in a half-plane, whose shear modulus $\mu(y)$ and density $\rho(y)$ depend continuously on the depth coordinate y . The problem amounts to studying the parametric Sturm-Liouville equation on a half-line with frequency ω and wave number k as the parameters. The Neumann (traction-free) boundary condition and the requirement of decay at infinity are imposed. The condition of solvability of the boundary value problem determines the dispersion spectrum $\omega(k)$ for the corresponding surface wave. We establish the criteria for non-existence of surface waves and for the existence of $N(k)$ surface wave solutions, with $N(k) \rightarrow \infty$ as $k \rightarrow \infty$. The most intriguing result is a possibility of the existence of infinite number of solutions, $N(k) = \infty$, for any given k . These three options are conditioned by the properties of $\mu(y)$ and $\rho(y)$.

functionally graded medium and surface shear waves and parametric Sturm-Liouville problem

1. INTRODUCTION

We consider the 2D wave equation

$$(1) \quad \rho \hat{u}_{tt} - \nabla (M \nabla \hat{u}) = 0$$

in a half-plane $\{(x, y) : y > 0\}$. One imposes the Neumann boundary condition

$$(2) \quad \hat{u}'_y|_{y=0} = 0.$$

We seek the solutions, which decay at infinity:

$$(3) \quad \lim_{y \rightarrow +\infty} \hat{u} = 0.$$

We make an assumption of M, ρ depending only on y and M being a scalar matrix $M = \mu(y)\text{Id}$.

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In the physical context, this is a problem of the existence of surface shear waves in functionally graded semi-infinite media with a traction-free boundary. Surface acoustic waves find numerous applications in various fields extending from seismology to microelectronics. Their localization near the surface (decay into the depth) makes them extremely advantageous in non-destructive material testing for detection of surface and subsurface defects (surface wave sensors). Small wavelength of surface waves enables their application in filters and transducers used in modern miniature devices [5]. Functionally graded materials may be of natural origin (e.g. bones), they may occur due to material aging, or they may be specially manufactured to realize desired combination of physical properties [8].

Under the adopted assumptions equation (1) reads as

$$(4) \quad \rho(y)\hat{u}_{tt} = \mu(y)\hat{u}_{xx} + \partial_y(\mu(y)\partial_y\hat{u}).$$

We will seek solutions of the form

$$\hat{u}(x, y) = u(y)e^{i(kx - \omega t)}.$$

Substituting $\hat{u}(x, y)$ into (4) and cancelling $e^{i(kx - \omega t)}$, one gets for $u(y)$ the equation

$$\rho(y)u(y)(-\omega^2) = \mu(y)u(y)(-k^2) + \partial_y(\mu(y)\partial_y u(y)).$$

We denote the (total) derivative ∂_y by $'$ arriving at the second-order linear differential equation

$$(5) \quad (\mu(y)u'(y))' + (\omega^2\rho(y) - k^2\mu(y))u = 0.$$

The boundary conditions (2) and (3) formulated for $u(y)$ become

$$(6) \quad u'(0) = 0,$$

$$(7) \quad \lim_{y \rightarrow +\infty} u(y) = 0.$$

We assume both functions $\rho(y)$ and $\mu(y)$ to be continuous and positive on $[0, +\infty)$; further assumptions are introduced in Sections 2,3.

It is known that for generic ω, k there are no solutions of (5), which satisfy both boundary conditions (6) and (7). For many bi-parametric problems the set of admissible ω, k is known to be a union of a number of *eigencurves* ([3, Ch.6]) in ωk -plane, called in the physical context *dispersion curves*. Our goal is to characterize the pairs (ω, k) , for which the solutions of the boundary value problems (5)-(6)-(7) exist.

The situation is elementary, when $\rho(y), \mu(y)$ are constants, and is relatively uncomplicated, when $\rho(y), \mu(y)$ become constants on an interval $[y_s, +\infty)$. In Section 4 we briefly consider the latter *homogeneous substrate case* as a particular case of our general treatment. There has been a number of studies, which either treat the problem asymptotically for high ω, k or assume that $\rho(y), \mu(y)$ are periodic [1], [11], [12], [9], [10]. We address the case, where no bounds for ω, k are imposed and neither periodicity nor (piecewise) constancy for $\rho(y), \mu(y)$ is assumed.

The paper has the following structure. Section 2 contains the auxiliary results. In Section 3 we formulate the corresponding parametric Sturm-Liouville problem on a half-line and introduce the assumptions for the material coefficients $\rho(y), \mu(y)$. Section 4 contains the formulations of the main results, which are the criteria for non-existence of surface waves (Theorem 4.1) and for the existence of $N(k)$ surface wave solutions, with $N(k) \rightarrow \infty$ as $k \rightarrow \infty$ (Theorem 4.2). The most intriguing result is a possibility of the existence of infinite number of solutions, $N(k) = \infty$, for any given k (Theorem 4.3). These three options are conditioned by the properties of $\mu(y)$ and $\rho(y)$. Section 5 contains the proofs of the above Theorems.

2. SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION ON A HALF-LINE: AUXILIARY RESULTS

2.1. Second-order linear equation. Equation (5) is a particular type of the second-order linear differential equation

$$(8) \quad (\mu(y)u'(y))' + \gamma(y)u = 0$$

defined on a half line $[0, +\infty)$.

Assumption 2.1. *We assume from now on that the function $\mu(s) \geq \underline{\mu} > 0$ on $[0, +\infty)$, is continuous on $[0, +\infty)$ and admits a finite limit $\lim_{s \rightarrow \infty} \mu(s) = \mu_\infty > 0$. \square*

The following substitution of the independent variable

$$(9) \quad \tau(y) = \int_0^y (\mu(s))^{-1} ds$$

is invertible ($\tau(y)$ is strictly growing) and satisfies the relation: $\frac{d}{d\tau} = \mu \frac{d}{dy}$.

By Assumption 2.1, the functions $\mu(s), (\mu(s))^{-1}$ are both bounded on $[0, +\infty)$ and therefore the function $\tau(y)$ and its inverse $y(\tau)$ are Lipschitzian. Besides $\int_0^{+\infty} (\mu(s))^{-1} ds = \infty$, i.e. $\tau(y)$ is Lipschitzian homeomorphism of $[0, +\infty)$ onto $[0, +\infty)$.

This substitution transforms (8) into the standard form

$$(10) \quad \frac{d^2 \bar{u}}{d\tau^2} + \bar{\gamma}(\tau) \bar{u}(\tau) = 0,$$

where $\bar{u}(\tau) = u(y(\tau))$ and $\bar{\gamma}(\tau) = \mu(y(\tau))\gamma(y(\tau))$.

Another form of (8) is its representation as a system of first-order differential equations for the variables $u(y), w(y) = \mu(y)u'(y)$:

$$(11) \quad u'(y) = \frac{w(y)}{\mu(y)}, \quad w'(y) = -\gamma(y)u(y),$$

or in the matrix form for $Z = \begin{pmatrix} w \\ u \end{pmatrix}$:

$$(12) \quad Z'(y) = \frac{dZ}{dy} = C(y)Z(y), \quad C(y) = \begin{pmatrix} 0 & -\gamma(y) \\ (\mu(y))^{-1} & 0 \end{pmatrix}.$$

Performing substitution (9), we transform (12) into the system for the function $\bar{Z}(\tau) = Z(y(\tau))$

$$(13) \quad \frac{d\bar{Z}}{d\tau} = \bar{C}(\tau)\bar{Z}(\tau), \quad \bar{C}(\tau) = \begin{pmatrix} 0 & -\bar{\gamma}(\tau) \\ 1 & 0 \end{pmatrix}.$$

We concentrate for a moment on the asymptotic properties of the solutions of (8), (10), (12), (13) at infinity.

2.2. Asymptotic properties of solutions for $y \rightarrow +\infty$. The matrix of the coefficients $C(y)$ of the system (12) for each y is traceless, hence, by the Liouville formula, the Wronskian of a fundamental system of solutions is constant in y . This precludes a possibility of having two independent solutions, which would both tend to zero at infinity.

Important characteristics of the asymptotics of the system at infinity are determined by the limit of the coefficient matrix for $y \rightarrow +\infty$ (whenever it exists):

$$C_\infty = \lim_{y \rightarrow +\infty} C(y) = \begin{pmatrix} 0 & -\gamma_\infty \\ (\mu_\infty)^{-1} & 0 \end{pmatrix},$$

where $\mu_\infty = \lim_{y \rightarrow +\infty} \mu(y)$, $\gamma_\infty = \lim_{y \rightarrow +\infty} \gamma(y)$.

Whenever $\det C_\infty = \gamma_\infty(\mu_\infty)^{-1} > 0$, or, equivalently, $\gamma_\infty > 0$, the eigenvalues of C_∞ are purely imaginary and one can conclude (see Proposition 2.4 below) the non-existence of a solution of system (12) with $\lim_{y \rightarrow +\infty} u(y) = 0$.

If on the contrary $\det C_\infty < 0$, then the eigenvalues of C_∞ are real numbers of opposite signs and the existence of a solution of (12) with $\lim_{y \rightarrow \infty} u(y) = 0$ is guaranteed under some additional conditions on the functions $\mu(y), \gamma(y)$.

Note that $\det \bar{C}_\infty = \mu_\infty^2 \det C_\infty$ and therefore a similar conclusion holds for the solutions of system (13).

Later on we use a number of results which follow the quasi-classical or WKB-approximation paradigm ([6, Ch.2]). We formulate the results for equations (8) or (10).

Let us introduce linear space \mathcal{G} of the coefficients $\gamma(y)$ of equations (8) as a space of functions $\gamma(y) = \gamma_\infty + \beta(y)$, with γ_∞ being a constant and $\beta(y)$ a continuous function on $[0, +\infty)$ such that:

$$(14) \quad \lim_{y \rightarrow +\infty} \beta(y) = 0,$$

$$(15) \quad \int_0^{+\infty} |\beta(y)| dy < \infty.$$

Evidently $\lim_{y \rightarrow \infty} \gamma(y) = \gamma_\infty$.

Introduce in \mathcal{G} the norm

$$(16) \quad \|\gamma(\cdot)\|_{01} = |\gamma_\infty| + \|\beta(\cdot)\|_{C^0} + \|\beta(\cdot)\|_{L^1}.$$

For each $y_0 \in [0, +\infty)$ we define a subset $\mathcal{G}^-(y_0) \subset \mathcal{G}$ (respectively $\mathcal{G}^+(y_0) \subset \mathcal{G}$), consisting of the functions $\gamma(y) = \gamma_\infty + \beta(y)$, for which $\gamma_\infty < 0$ (respectively > 0) and $\gamma_\infty + \beta(y) < 0$ (respectively > 0) on $[y_0, +\infty)$. Both $\mathcal{G}^-(y_0)$ and $\mathcal{G}^+(y_0)$ are open subsets of \mathcal{G} in the above introduced norm. It is easy to verify that substitution (9) transforms the space \mathcal{G} into itself and the sets $\mathcal{G}^-(y_0), \mathcal{G}^+(y_0)$ into $\mathcal{G}^-(\tau(y_0)), \mathcal{G}^+(\tau(y_0))$, correspondingly.

The first classical result regards the so called non-elliptic case for equation (10), where the coefficient $\bar{\gamma}(\cdot) \in \mathcal{G}^-(\tau_0)$.

Proposition 2.1. (see [4, §6.12]). *Consider the equation*

$$(17) \quad u''(\tau) + \bar{\gamma}(\tau)u = u''(\tau) + (-\lambda^2 + \beta(\tau))u = 0, \quad \lambda > 0.$$

Assume $\beta(\tau)$ to be continuous and to satisfy (14). Then for equation (17) there exist $\tau_0 \geq 0$, constants $c_1, c_2, d_1, c'_1, c'_2, d'_1$ and two solutions $u_\lambda(\tau), u_{-\lambda}(\tau)$ such that $\forall \tau \geq \tau_0$:

$$(18) \quad \begin{aligned} c'_2 \exp \left[\lambda\tau - d'_1 \int_{\tau_0}^{\tau} |\beta(\theta)| d\theta \right] &\leq u_\lambda(\tau) \leq \\ &\leq c'_1 \exp \left[\lambda\tau + d'_1 \int_{\tau_0}^{\tau} |\beta(\theta)| d\theta \right], \end{aligned}$$

$$(19) \quad \begin{aligned} c_2 \exp \left[-\lambda\tau - d_1 \int_{\tau_0}^{\tau} |\beta(\theta)| d\theta \right] &\leq u_{-\lambda}(\tau) \leq \\ &\leq c_1 \exp \left[-\lambda\tau + d_1 \int_{\tau_0}^{\tau} |\beta(\theta)| d\theta \right]. \end{aligned}$$

Corollary 2.2. (see [7, §XI.9]). *Assume the assumptions of Proposition 2.1 to hold and $\beta(\cdot)$ to satisfy (15). Then the solutions $u_\lambda, u_{-\lambda}$ satisfy*

$$u_\lambda \sim \frac{u'_\lambda}{\lambda} \sim e^{\lambda\tau}, \quad u_{-\lambda} \sim -\frac{u'_{-\lambda}}{\lambda} \sim e^{-\lambda\tau}$$

as $\tau \rightarrow +\infty$.

Corollary 2.3. *For each $\tilde{\gamma}(\cdot)$ sufficiently close to $\bar{\gamma}(\cdot)$ in the norm (16) the equation*

$$u''(\tau) + \tilde{\gamma}(\tau)u(\tau) = 0$$

has a decaying solution.

Next we pass on to the elliptic case (see [7, §XI.8]; Corollary 8.1), where the coefficient $\bar{\gamma}(\cdot) \in \mathcal{G}^+(y_0)$.

Proposition 2.4. *Consider the equation*

$$(20) \quad u''(\tau) + \bar{\gamma}(\tau)u = u''(\tau) + (\lambda^2 + \beta(\tau))u = 0, \quad \lambda > 0$$

with $\bar{\gamma}(\cdot) \in \mathcal{G}^+(y_0)$. Then for any real a, b there is a unique solution of equation (20) with the asymptotics

$$(21) \quad \begin{aligned} u(\tau) &= (a + o(1)) \cos \lambda\tau + (b + o(1)) \sin \lambda\tau, \\ u'(\tau) &= (-\lambda a + o(1)) \sin \lambda\tau + (\lambda b + o(1)) \cos \lambda\tau, \end{aligned}$$

as $\tau \rightarrow +\infty$.

2.3. Prüfer's coordinates. We consider Prüfer's coordinates (see [7, 3]):

$$(22) \quad r = (u^2 + \mu^2 u'^2)^{\frac{1}{2}} = (u^2 + w^2)^{\frac{1}{2}}, \quad \varphi = \operatorname{Arctg} \frac{u}{w},$$

where again $w = \mu u'$. For the vector function $Z = \begin{pmatrix} w \\ u \end{pmatrix}$ we denote φ by $\operatorname{Arg} Z$ (the choice of a continuous branch is done in a standard way). In coordinates (22) system (8) takes the form:

$$(23) \quad r' = (\mu^{-1}(y) - \gamma(y)) r \sin \varphi \cos \varphi, \quad \varphi' = \gamma(y) \sin^2 \varphi + \mu^{-1}(y) \cos^2 \varphi;$$

note that the second equation is decoupled from the first one.

We list some facts concerning the evolution of $\operatorname{Arg} Z(y)$. Recall that $\mu(y)$ in equation (8) meets Assumption 2.1.

- Proposition 2.5.**
- i) *If $\gamma(y) \geq 0$ (respectively $\gamma(y) > 0$) on an interval, then for a solution $Z(y)$ of (11) Prüfer's angle variable $\varphi = \operatorname{Arg} Z$ is non-decreasing (increasing) on the interval.*
 - ii) *If $\gamma(y) < 0$ on an interval I , then the first and the third quadrants – $\operatorname{Arg} Z \in (0, \pi/2)$ and $\operatorname{Arg} Z \in (\pi, 3\pi/2)$ – are invariant for system (11) on I .*
 - iii) *For any $\gamma(y)$ there is a kind of weakened monotonicity for $\operatorname{Arg} Z$: if $\operatorname{Arg} Z(\tilde{y}) > m\pi$, then $\operatorname{Arg} Z(y) > m\pi$ for any $y > \tilde{y}$.*

Property i) follows from (23). So does property ii), since, according to (23), $\varphi'(\pi m) > 0$ and $\varphi'(\pi/2 + \pi m) < 0$ for negative γ . Property iii) follows from the fact that in (23) $\varphi'(m\pi) = \mu^{-1}(m\pi) > 0$.

2.4. Oscillatory equations. Second-order linear differential equation is *oscillatory* ([7, §XI.5]) on $[0, +\infty)$ when its every solution has infinite number of zeros on $[0, +\infty)$, or equivalently the set of zeros of any solution has no upper limit, or equivalently for every solution its Prüfer's coordinate $\operatorname{Arg} Z$ (see the previous Subsection) satisfies

$$\limsup_{y \rightarrow +\infty} \operatorname{Arg} Z(y) = +\infty.$$

An obvious example of oscillatory equation is (20), when the assumptions of Proposition 2.4 are met.

We are interested in conditions, under which the same equation is oscillatory for vanishing λ . We formulate the result (see [7, §XI.5], [6, Ch.2,§6]) for equation (10).

Proposition 2.6. *Let $\bar{\gamma}(\cdot)$ in (10) be continuous of bounded variation on every interval $[0, T]$, $\bar{\gamma}(\tau) > 0$ on some interval $[\tau_0, +\infty)$, and*

$$(24) \quad \int_{\tau_0}^{+\infty} (\bar{\gamma}(\tau))^{1/2} d\tau = +\infty,$$

$$(25) \quad \int_{\tau_0}^T (\bar{\gamma}(\tau))^{-1} |d\bar{\gamma}(\tau)| = o\left(\int_{\tau_0}^T (\bar{\gamma}(\tau))^{1/2} d\tau\right), \quad \text{as } T \rightarrow +\infty.$$

Then equation (10) is oscillatory.

2.5. Hamiltonian form. One can rewrite the system (12) in the Hamiltonian form

$$(26) \quad u' = \frac{\partial H}{\partial w} = \frac{w}{\mu(y)}, \quad w' = -\frac{\partial H}{\partial u} = -\gamma(y)u$$

with the Hamiltonian

$$H = \frac{1}{2} \left(\frac{w^2}{\mu(y)} + \gamma(y)u^2 \right).$$

We denote by \vec{h} the (Hamiltonian) vector field at the right-hand side of (26).

For Prüfer's angle $\varphi = \text{Arctan} \left(\frac{u}{w} \right)$ there holds

$$\varphi' = \frac{-w'u + wu'}{u^2 + w^2} = \frac{\gamma u^2 + w^2/\mu}{u^2 + w^2} = \frac{2H}{u^2 + w^2}.$$

The last equation is equivalent to the differential equations (23) for Prüfer's coordinate φ .

Remark 2.1. *A simple but relevant (see [2]) computation is provided by derivation of $u(y)w(y)$ along the trajectories of Hamiltonian system (26):*

$$(27) \quad \frac{d}{dy}(uw) = \partial_{\vec{h}}(uw) = \left(\partial_{\vec{h}} u \right) w + u \left(\partial_{\vec{h}} w \right) = -\gamma u^2 + \frac{w^2}{\mu},$$

wherefrom it follows, among other things, that uw is nondecreasing (respectively increasing) on the intervals where $\gamma(y) \leq 0$ (respectively $\gamma(y) < 0$).

Proposition 2.5 and Remark 2.1 allow us to arrive at a conclusion on qualitative behaviour of solutions on an interval, where $\gamma(\tau) < 0$ in (17).

According to Proposition 2.1, there is a decaying solution, along which (according to Remark 2.1) uw grows. Hence the solution approaches the origin either in the second or in the fourth quadrant, where $uw < 0$.

Proposition 2.7. *Let $\bar{\gamma}(\tau)$ meet the assumptions of Proposition 2.1 and $\bar{\gamma}(\tau) < 0$ for $\tau \in [\tau_0, +\infty)$. Then the decaying solutions $\pm u(\tau)$ of (17) correspond to the solutions $\pm Z(\tau)$ of (11) with $\text{Arg } Z(\tau) \in [\pi/2, \pi]$, $\text{Arg } (-Z)(\tau) \in [3\pi/2, 2\pi]$ for $\tau \in [\tau_0, +\infty)$.*

Other solutions, which start in the same quadrants, escape to either the first or the third quadrant, which, according to Proposition 2.5, are invariant for (17) whenever $\gamma(\tau) < 0$. According to Remark 2.1, the product uw (positive in these quadrants) grows along the respective trajectories, which tend to infinity.

2.6. Sturmian properties of trajectories. We provide few results from the Sturm theory. First result is classical ([6],[3], [7, Ch. X,XI]) and follows directly from the second equation (23).

Proposition 2.8 (comparison result). *Consider a pair of second-order equations*

$$(28) \quad (\mu(y)u'(y))' + \gamma(y)u = 0, \quad (\mu(y)u'(y))' + \tilde{\gamma}(y)u(y) = 0,$$

where $\mu(y)$ meets Assumption 2.1 and

$$\tilde{\gamma}(y) \geq \gamma(y), \quad \forall y \in [y_0, +\infty).$$

If for $y_1 \geq y_0$ and a pair of vector solutions $Z = \begin{pmatrix} w \\ u \end{pmatrix}$, $\tilde{Z} = \begin{pmatrix} \tilde{w} \\ \tilde{u} \end{pmatrix}$ of the first and the second equations (28)

$$\text{Arg } \tilde{Z}(y_1) = \text{Arg } Z(y_1),$$

then

$$\forall y \geq y_1 : \text{Arg } \tilde{Z}(y) \geq \text{Arg } Z(y)$$

and

$$(29) \quad \forall y \in [y_0, y_1] : \text{Arg } \tilde{Z}(y) \leq \text{Arg } Z(y).$$

We provide analogue of the comparison result (in particular, of relation (29)) for the decaying solutions of (28), when $y_1 = +\infty$. We were not able to trace it in the literature and provide a (short) proof.

Proposition 2.9. (comparison result for decaying solutions on a half-line) *Consider the pair of second-order equations (28) with the coefficient $\mu(y)$ meeting Assumption 2.1 and with $\gamma(y), \tilde{\gamma}(y)$ belonging to $\mathcal{G}^-(y_0)$. Let*

$$(30) \quad 0 > \tilde{\gamma}(y) \geq \gamma(y), \quad \forall y \in [y_0, +\infty).$$

If Z, \tilde{Z} are the decaying solutions of equations (28), then

$$(31) \quad \text{Arg } \tilde{Z}(y) \leq \text{Arg } Z(y), \quad \forall y \geq y_0.$$

Proof. Without lack of generality we may assume $\mu(y) \equiv 1$; otherwise we perform substitution (9) of the independent variable, which preserves relation (30) for the coefficients.

By (30) and (27), the functions uw and $\tilde{u}\tilde{w}$ are increasing on $[y_0, +\infty)$. As long as the limits of these functions at $+\infty$ are null, we conclude that $(uw)(y) < 0, (\tilde{u}\tilde{w})(y) < 0$ on $[y_0, +\infty)$ and then without lack of generality we may assume that $u(y), \tilde{u}(y)$ are positive, while $w(y), \tilde{w}(y)$ are negative on $[y_0, +\infty)$.

Denote $\tilde{\gamma}(y) - \gamma(y)$ by $\Delta\gamma(y)$ and represent the second one of equations (28) as

$$(32) \quad \tilde{u}'' + \gamma(y)\tilde{u} = -\Delta\gamma(y)\tilde{u};$$

$\Delta\gamma(y) > 0$ by (30).

Applying the integral form of the Lagrange identity (or Green's formula, see [7, §XI.2]) to the respective vector solutions $Z = \begin{pmatrix} w \\ u \end{pmatrix}$, $\tilde{Z} = \begin{pmatrix} \tilde{w} \\ \tilde{u} \end{pmatrix}$ of equations (28), of which the second one is written as (32), we conclude:

$$\forall y \geq y_0 : (u\tilde{w} - w\tilde{u})|_y^{+\infty} = \int_y^{+\infty} -\Delta\gamma(s)\tilde{u}(s)u(s)ds < 0.$$

Given that $(u\tilde{w} - w\tilde{u})$ vanishes at $+\infty$, we obtain:

$$(33) \quad \forall y \geq y_0 : -u(y)\tilde{w}(y) + w(y)\tilde{u}(y) = \int_y^{+\infty} -\Delta\gamma(s)\tilde{u}(s)u(s)ds < 0.$$

Dividing the inequality in (33) by the positive value $w(y)\tilde{w}(y)$, we get

$$\forall y \geq y_0 : \frac{\tilde{u}(y)}{\tilde{w}(y)} \leq \frac{u(y)}{w(y)},$$

wherefrom (31) follows. \square

We establish the continuous dependence of decaying solutions on the coefficient $\gamma(\cdot)$ in $\|\cdot\|_{01}$ -norm.

Proposition 2.10. (continuous dependence of decaying solutions on the right-hand side) *Consider equations (28). Let $\gamma(\cdot) = -\lambda^2 + \beta(\cdot) \in \mathcal{G}_{y_0}^-$ for some $y_0 \in [0, +\infty)$. Then for any $\tilde{\gamma}(\cdot) = -\tilde{\lambda}^2 + \tilde{\beta}(\cdot)$ sufficiently close to $\gamma(\cdot)$ in $\|\cdot\|_{01}$ -norm:*

i) *both equations (28) possess the decaying vector solutions $Z(\cdot), \tilde{Z}(\cdot)$ with $\text{Arg } Z, \text{Arg } \tilde{Z} \in [\pi/2, \pi]$;*

ii) *for each $y \in [y_0, +\infty)$*

$$\left| \text{Arg } \tilde{Z}(y) - \text{Arg } Z(y) \right| \rightarrow 0, \text{ as } \|\tilde{\gamma}(\cdot) - \gamma(\cdot)\|_{01} \rightarrow 0.$$

Proof. Again we may proceed assuming $\mu(y) \equiv 1$.

i) Any $\tilde{\gamma}(\cdot)$ sufficiently close to $\gamma(\cdot)$ in $\|\cdot\|_{01}$ -norm belongs to $\mathcal{G}_{y_0}^-$, which is open with respect to the norm. The existence of the decaying solutions $Z(y), \tilde{Z}(y)$ follows from Corollary 2.3. Since both γ and $\tilde{\gamma}$ are negative on $[y_0, +\infty)$, we conclude by Proposition 2.7 that $\text{Arg } Z(y)$ and $\text{Arg } \tilde{Z}(y)$ lie in $[\pi/2, \pi]$ for $y \in [y_0, +\infty)$.

This implies that for $s \in [y_0, +\infty)$, $w(s), \tilde{w}(s)$ are negative, while $u(s), \tilde{u}(s)$ are positive and by (11) decrease.

ii) Recall that $\Delta\gamma(\cdot) = \tilde{\gamma}(\cdot) - \gamma(\cdot)$. Invoking the equality in (33) and dividing it by $-u(y)\tilde{u}(y)$, we get

$$(34) \quad \frac{\tilde{w}(y)}{\tilde{u}(y)} - \frac{w(y)}{u(y)} = \int_y^{+\infty} \Delta\gamma(s) \frac{\tilde{u}(s)}{\tilde{u}(y)} \frac{u(s)}{u(y)} ds = \int_y^{+\infty} \Delta\gamma(s) \nu(s) \tilde{\nu}(s) d\tau,$$

where $\nu(s) = \frac{\tilde{u}(s)}{\tilde{u}(y)}$, $\tilde{\nu}(s) = \frac{u(s)}{u(y)}$ are the solutions of the first and second equation (28), which are normalized by the condition: $\nu(y) = \tilde{\nu}(y) = 1$.

By the aforesaid $\nu(s)$, $\tilde{\nu}(s)$ decrease; hence

$$(35) \quad \nu(s) \leq 1, \tilde{\nu}(s) \leq 1, \text{ for } s \geq y.$$

According to Proposition 2.1, there exist $c_1, d_1 > 0, s_0 > y$ such that

$$(36) \quad \nu(s) \leq c_1 \exp\left(-\lambda s + d_1 \int_{s_0}^s |\beta(\sigma)| d\sigma\right), \quad \forall s > s_0.$$

From the proof of the Proposition (see [4, §6.12, §2.6]) it follows that one can choose in (36) any $c_1 > 1$, a sufficiently large d_1 and then choose s_0 such that $d_1 \sup_{s \geq s_0} |\beta(s)| < \lambda$. The same holds for the second one of equations (28).

For each $\tilde{\gamma}$ from a small neighborhood of γ in $\|\cdot\|_{01}$ -norm, $\tilde{\lambda}$ and λ as well as $\sup_{\tau \geq \tau_0} |\beta(\tau)|$ and $\sup_{\tau \geq \tau_0} |\tilde{\beta}(\tau)|$ are close. Thus one can choose common c_1, d_1, τ_0 for all the equations with the coefficient $\tilde{\gamma}$ from the neighborhood. Besides there is a common upper bound B for the corresponding norms $\|\tilde{\beta}(\cdot)\|_{L_1}$. Then by (34),(35) and (36)

$$\left| \int_y^{+\infty} \Delta\gamma(s) \nu(s) \tilde{\nu}(s) ds \right| \leq \int_y^{s_0} |\Delta\gamma(s)| ds + c_1^2 e^{2d_1 B} \int_{s_0}^{\infty} e^{-\lambda s} |\Delta\gamma(s)| ds$$

with the right-hand side tending to 0 as $\|\Delta\gamma(s)\|_{01} \rightarrow 0$.

Note that $\text{Arg } Z = \text{Arccot } \frac{w(y)}{u(y)}$, $\text{Arg } \tilde{Z} = \text{Arccot } \frac{\tilde{w}(y)}{\tilde{u}(y)}$ and since the function $z \mapsto \text{Arccot } z$ is Lipschitzian with constant 1:

$$\left| \text{Arg } Z(y) - \text{Arg } \tilde{Z}(y) \right| = \left| \text{Arccot } \frac{w(y)}{u(y)} - \text{Arccot } \frac{\tilde{w}(y)}{\tilde{u}(y)} \right| \leq \left| \frac{w(y)}{u(y)} - \frac{\tilde{w}(y)}{\tilde{u}(y)} \right|$$

and the left-hand side tends to 0 as $\|\Delta\gamma(\tau)\|_{01} \rightarrow 0$. □

3. EXISTENCE OF SURFACE WAVES AND PARAMETRIC STURM-LIOUVILLE PROBLEM

We come back to equation (5) and simplify the notations putting $\Omega = \omega^2$, $K = k^2$, $A = (K, \Omega)$,

$$(37) \quad \gamma_A(y) = \Omega \rho(y) - K \mu(y),$$

thus arriving at the equation

$$(38) \quad (\mu(y)u'(y))' + \underbrace{(\Omega \rho(y) - K \mu(y))}_{\gamma_A(y)} u(y) = 0$$

with the parameter A .

Performing the substitution of the independent variable the way it is done in (9), we get the equation:

$$(39) \quad \frac{d^2 \bar{u}}{d\tau^2} + \underbrace{\bar{\mu}(\tau) (\Omega \bar{\rho}(\tau) - K \bar{\mu}(\tau))}_{\tilde{\gamma}_A(\tau)} \bar{u}(\tau) = 0,$$

where

$$(40) \quad \bar{\rho}(\tau) = \rho(y(\tau)), \quad \bar{\mu}(\tau) = \mu(y(\tau)), \quad \bar{u}(\tau) = u(y(\tau)).$$

In equations (38) and (39) the dependence of the coefficients on the parameters Ω, K is linear; the functions $\rho(y)$, $\mu(y)$, $\bar{\rho}(\tau)$, $\bar{\mu}(\tau)$ are positive. Note that $\bar{\mu}(0) = \mu(0)$, $\bar{\rho}(0) = \rho(0)$ and

$$\bar{\mu}(+\infty) = \mu(+\infty), \quad \bar{\rho}(+\infty) = \rho(+\infty), \quad \bar{\gamma}_A(+\infty) = \Omega\mu_\infty\rho_\infty - K\mu_\infty^2.$$

We know from the previous Section that if equation (39) meets the assumptions of Proposition 2.1, then it has a solution, which satisfies the boundary condition at infinity (7). We are interested, though, in the solutions, which satisfy at the same time the boundary condition (6), and it is not possible for generic combinations of $\bar{\rho}(y)$, $\bar{\mu}(y)$, Ω, K , which enter (39) via the coefficient $\bar{\gamma}_A(\cdot)$. In other words we get parametric Sturm-Liouville problem on a half-line for equation (39) (or (38)) with the boundary conditions (6)-(7).

Let us introduce the vector-function $a(y) = (\rho(y), \mu(y))$, which characterizes our medium, and formulate the assumptions for the medium in terms of $a(y)$.

Assumption 3.1 (Lipschitz continuity). *The function $a(y) = (\rho(y), \mu(y))$ is Lipschitz continuous on $[0, +\infty)$. There exists a finite limit*

$$\lim_{y \rightarrow +\infty} a(y) = a_\infty, \quad a_\infty = (\rho_\infty, \mu_\infty), \quad \rho_\infty > 0, \quad \mu_\infty > 0.$$

Assumption 3.2 (integral boundedness). *The function*

$$\hat{a}(y) = (\hat{\rho}(y), \hat{\mu}(y)) = a(y) - a(+\infty) = (\rho(y) - \rho_\infty, \mu(y) - \mu_\infty)$$

is integrable on $[0, +\infty)$: $\int_0^\infty |\hat{a}(y)| dy < \infty$.

We now introduce monotonicity assumptions formulated in terms of polar coordinates representation for $a(y)$. Let

$$|a(y)| = ((\rho(y))^2 + (\mu(y))^2)^{1/2}, \quad \text{Arg } a(y) = \text{Arctan } \frac{\mu(y)}{\rho(y)},$$

$$a(y) = (\rho(y), \mu(y)) = |a(y)| (\cos \text{Arg } a(y), \sin \text{Arg } a(y)).$$

The values $a(y)$ have both positive coordinates; hence the values of $\text{Arg } a(y)$ lie in $[0, \pi/2]$. As long as $a_\infty \neq 0$, $\text{Arg } a_\infty$ is properly defined.

Assumption 3.3 (monotonicity at infinity). *There exists an interval $\bar{I} = (\bar{y}, +\infty)$ such that either: i) $\text{Arg } a(y) < \text{Arg } a_\infty$ on \bar{I} - positive monotonicity at infinity, or ii) $\text{Arg } a(y) > \text{Arg } a_\infty$ on \bar{I} - negative monotonicity at infinity.*

Examples of the curves $a(y) = (\rho(y), \mu(y))$ are drawn in Figure 1 together with the vector $A = (K, \Omega)$. The curves (1) and (2) are negatively monotonous at infinity, while the curve (3) is positively monotonous at infinity.

Assume the vector of parameters $A = (K, \Omega)$ to belong to (the positive quadrant of) the oriented plane, in which the curve $y \mapsto a(y)$, $y \in [0, +\infty]$

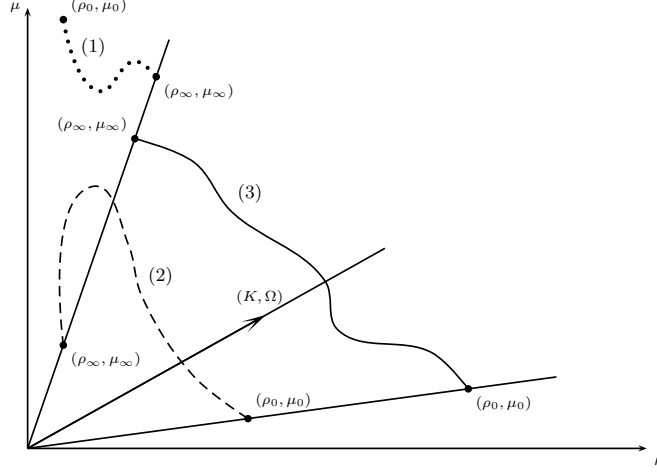


FIGURE 1. Curves parameterized by y , which exhibit different types of monotonicity at infinity

is contained. We define $\text{Arg } A = \text{Arctan } \frac{\Omega}{K}$. The following remarks are important.

Remark 3.1. Fix an admissible A . If $\text{Arg } a(y) < \text{Arg } A$ (respectively $\text{Arg } a(y) > \text{Arg } A$) for some y , then $\gamma_A(y)$ defined by (37) is positive (respectively negative). \square

Remark 3.2. Under Assumptions 3.1 and 3.2, for any admissible A and $\gamma_A(\cdot)$ defined by (37) there holds:

- (1) $\gamma_A(y) - \gamma_A(+\infty) \xrightarrow{y \rightarrow +\infty} 0$;
- (2) $\int_0^\infty |\gamma_A(y) - \gamma_A(+\infty)| dy < \infty$;
- (3) if $\gamma_A(y)$ admits positive values, then so does $\gamma_{A'}(y)$ with any A' such that $\text{Arg } A'$ is sufficiently close to $\text{Arg } A$. \square

We wish to check what occurs with Assumptions 3.1-3.2-3.3 after substitution (9).

Proposition 3.1. Let Assumption 2.1 hold and let equation (38) meet Assumptions 3.1, 3.2, 3.3 for any admissible A . Then equation (39) meets the same Assumptions.

Proof. By Assumption 2.1, $\tau(y)$ defined by (9) is Lipschitzian homeomorphism of $[0, +\infty)$ onto itself. Hence the functions $\bar{\mu}, \bar{\rho}$ defined by (40) are bounded, Lipschitzian, with finite limits at infinity, i.e. Assumption 3.1 is valid for them.

Under substitution (9), the vector-function $a(y) = (\rho(y), \mu(y))$ is transformed into $\bar{a}(\tau) = \bar{\mu}(\tau) (\bar{\rho}(\tau), \bar{\mu}(\tau))$. Hence $\text{Arg } a(y) = \text{Arg } \bar{a}(\tau(y))$ and all the monotonicity properties listed in Assumption 3.3 are maintained.

Regarding Assumption 3.2 we perform substitution (9) and obtain:

$$\begin{aligned} \int_0^{+\infty} |\bar{\gamma}_A(\tau) - \bar{\gamma}_A(+\infty)| d\tau &= \int_0^{+\infty} \frac{|\mu(y)\gamma_A(y) - \mu(+\infty)\gamma_A(+\infty)|}{\mu(y)} dy = \\ &= \int_0^{+\infty} |(\gamma_A(y) - \gamma_A(+\infty)) + \gamma_A(+\infty)(\mu(y) - \mu(+\infty))(\mu(y))^{-1}| dy < \infty, \end{aligned}$$

since $(\mu(y))^{-1}$ is bounded on $[0, +\infty)$. \square

For the *limit case*, in which $A_\infty = (K_\infty, \Omega_\infty) = \beta a_\infty$, $\beta > 0$, or in other words $\text{Arg } A_\infty = \text{Arg } a_\infty$, we get

$$\gamma_{A_\infty}(y) = \Omega_\infty \rho(y) - K_\infty \mu(y) = \beta(\mu_\infty(\rho_\infty + \hat{\rho}(y)) - \rho_\infty(\mu_\infty + \hat{\mu}(y))) = \beta \hat{\gamma}_\infty(y),$$

where

$$(41) \quad \hat{\gamma}_\infty(y) = \mu_\infty \hat{\rho}(y) - \rho_\infty \hat{\mu}(y).$$

Remark 3.3. (1) *Under Assumption 3.1, for each $A = (K, \Omega)$ with $\text{Arg } A < \text{Arg } a_\infty$ there exists an interval, $[y_-, +\infty)$, on which $\gamma_A(y) < 0$.*

(2) *Under Assumption 3.3i) (respectively 3.3ii)), there is an interval $[\bar{y}, +\infty)$, on which $\gamma_\infty(y)$ is positive (respectively negative).*

4. RESULTS

Key information for our treatment is provided by the *limit-case equation*, which corresponds to the vectors of parameters $A_\infty = (K_\infty, \Omega_\infty) = \beta a_\infty$, $\beta > 0$. For such choice of parameters equation (38) takes the form

$$(42) \quad (\mu(y)u')' + \beta \hat{\gamma}_\infty(y)u = 0$$

with $\hat{\gamma}_\infty(y)$ as in (41).

We formulate here main results of the paper; the proofs are provided in the next Section. Our first result establishes non-existence of solutions under a kind of global negative monotonicity of $a(y)$ at infinity.

Theorem 4.1. *Let assumptions 3.1-3.2 hold and*

$$(43) \quad \forall y \in [0, +\infty) : \text{Arg } a(y) \geq \text{Arg } a_\infty.$$

Then there are no admissible values of parameters K, Ω , for which solutions of (38)-(6)-(7) exist.

Remark 4.1. *The assumptions of the theorem are met by curve (1) in Fig. 1. The proof of the result is based on the following fact: for any $A = (K, \Omega)$ such that $\text{Arg } A \leq \min_{y \in [0, +\infty)} \text{Arg } a(y)$, or the same,*

$$\Omega \leq K \min_{y \in [0, +\infty)} \frac{\mu(y)}{\rho(y)} = \frac{\mu(\check{y})}{\rho(\check{y})} = \frac{\check{\mu}}{\check{\rho}},$$

solutions of (38)-(6)-(7) do not exist. \square

If (43) does not hold, then one can guarantee existence of solutions at least for sufficiently large K, Ω .

Theorem 4.2. *Let assumptions 3.1-3.2-3.3 hold and in addition $\text{Arg } a(y) < \text{Arg } a_\infty$ for $y \in I$ - a non-null sub-interval of $[0, +\infty)$. Then for each $N > 0$ $\exists K_N$ such that $\forall K > K_N$ there are at least N values $\Omega_j \in \left(\frac{\underline{\mu}}{\underline{\rho}}K, \frac{\mu_\infty}{\rho_\infty}K\right)$, $j = 1, \dots, N$, such that for each (K, Ω_j) the solution of (38)-(6)-(7) exists.*

Remark 4.2. *The curves (2) and (3) in Fig. 1 meet assumptions of the Theorem. \square*

Finally there is a case, in which for each $K > 0$ one finds a numerable set of $\Omega_j \in \left(\frac{\underline{\mu}}{\underline{\rho}}K, \frac{\mu_\infty}{\rho_\infty}K\right)$ such that the solution exists for (K, Ω_j) . It happens when the limit-case equation (42) is oscillatory (see Subsection 2.4).

Theorem 4.3. *Let assumptions 3.1-3.2-3.3i) hold and the limit-case equation (42) be oscillatory¹.*

Then for each $\bar{K} > 0$ there exists a numerable set of $\Omega_m \in \left(\frac{\underline{\mu}}{\underline{\rho}}\bar{K}, \frac{\mu_\infty}{\rho_\infty}\bar{K}\right)$, $m = 1, \dots$, such that:

- i) for $A_m = (\bar{K}, \Omega_m)$ the solution of (38)-(6)-(7) exists;*
- ii) Ω_m increase with m and accumulate (only) to $\bar{\Omega} = \frac{\mu_\infty}{\rho_\infty}\bar{K}$;*
- iii) for the vector solutions $Z(y; A_m)$ there holds*

$$\text{Arg } Z(y; A_m) \in [(m - 1/2)\pi, m\pi] \text{ for } y \text{ sufficiently large.}$$

Remark 4.3. *Assumptions 3.1-3.2-3.3i) hold for curve (3) in Fig. 1, but the oscillatory property for the limit-case equation can not be concluded from the curve only, since it also depends on its parametrization. \square*

4.1. Homogeneous substrate example. This is a particular case, in which the properties of the medium become depth-independent starting from some depth. For the model under discussion this means existence of y_s such that $\mu(y)$ and $\rho(y)$ are constant on the interval $[y_s, +\infty)$: $\mu(y) \equiv \mu_s$, $\rho(y) \equiv \rho_s$ on $[y_s, +\infty)$ (see Fig. 2).

We denote $a_s = (\rho_s, \mu_s)$. Then $a_\infty = \lim_{y \rightarrow \infty} a(y) = a_s$ and $\hat{a}(y) = a(y) - a_\infty$ vanishes on $[y_s, +\infty)$.

If $\forall y \in [0, +\infty) : \text{Arg } a(y) \geq \text{Arg } a_s$ or, the same

$$\forall y \in [0, +\infty) : \frac{\mu(y)}{\rho(y)} \geq \frac{\mu_s}{\rho_s},$$

then we are under assumptions of Theorem 4.1 and solutions of (38)-(6)-(7) do not exist.

If $\frac{\mu(y)}{\rho(y)} < \frac{\mu_s}{\rho_s}$ on some non-null subinterval of $[0, +\infty)$, then we fall under assumptions of Theorem 4.2 and hence its claim holds.

¹We assume (24) and (25) to hold

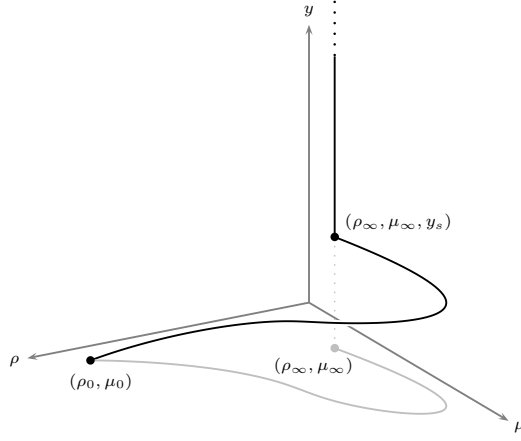


FIGURE 2. The functions $\rho(y)$ and $\mu(y)$ in the homogeneous substrate example become constant when $y \geq y_s$, as illustrated by the curve in black. Note that its projection (in gray) on the (ρ, μ) -plane is a curve, which exhibits negative monotonicity at infinity.

5. PROOFS

Since substitution (9) transforms parametric equation (38) into its standard form (39) and Assumptions 3.1,3.2,3.3 are maintained under (9), we may take, without loss of generality, $\mu(y) \equiv 1$ in (38).

The *proof of Theorem 4.1* is easy. Pick some $A = (K, \Omega)$. There are two options: $\text{Arg } a_\infty \leq \text{Arg } A$ or $\text{Arg } a_\infty > \text{Arg } A$.

In the first case, by monotonicity and continuity assumptions, the coefficient $\gamma_A(y)$ in equation (38) is non-negative on some interval $[y_0, +\infty)$. Then, by Proposition 2.4, there exists a fundamental system of solutions of the form (21) and none of the solutions of (38) tend to the origin as $y \rightarrow +\infty$.

If $\text{Arg } A < \text{Arg } a_\infty \leq \text{Arg } a(y) \forall y \in [0, +\infty)$, then $\gamma_A(y) < 0$ on $[0, +\infty)$. By (27), for a solution $Z(y, A) = \begin{pmatrix} w \\ u \end{pmatrix}$ there holds $\frac{d}{dy}(u(y)w(y)) > 0$. This enters in contradiction with the boundary conditions (6)-(7), according to which $u(0)w(0) = 0$ and $\lim_{y \rightarrow +\infty}(u(y)w(y)) = 0$.

The latter reasoning also validates the claim of Remark 4.1.

Proof of Theorem 4.3. We start with a *sketch of the proof*.

Take a vector of parameters $A_\infty = (\bar{K}, \bar{\Omega})$ collinear to $a_\infty = (\rho_\infty, \mu_\infty)$ and consider its perturbation $A_{\infty,s} = (\bar{K}, \bar{\Omega} - s)$. It is immediate to see that for each $s > 0$ equation (38) with $A = A_{\infty,s}$ and the coefficient

$$(44) \quad \gamma_{A_{\infty,s}}(y) = \gamma_{A_\infty}(y) - s\rho(y) = -s\rho_\infty + (\bar{\Omega} - s)\hat{\rho}(y) - \bar{K}\hat{\mu}(y)$$

meets the assumptions of Proposition 2.1, and hence the equation

$$(45) \quad u'' + \gamma_{A_{\infty, \bar{s}}}(y)u = 0$$

possesses a decaying solution $Z^+(y, A_{\infty, \bar{s}})$.

Simultaneously we consider the solutions $Z^0(y, A_{\infty, s})$ of the same equation with the boundary condition (6). The goal is to detect the values $s > 0$, for which the solutions $Z^0(y, A_{\infty, s})$ and $Z^+(y, A_{\infty, s})$ *meet* at some intermediate point $\bar{y} \in [0, +\infty)$, i.e admit at \bar{y} the same value (mod π). In such a case they (or their opposites) can be concatenated into solutions of (38)-(6)-(7). The possibility of such *meeting* follows from Propositions 2.8 and 2.9, according to which for a sufficiently large intermediate point $\bar{y} \in [0, +\infty)$ the vectors $Z^0(\bar{y}, A_{\infty, s})$ and $Z^+(\bar{y}, A_{\infty, s})$ rotate in opposite directions as s grows from some $\bar{s} > 0$.

One can assume (increasing \bar{y} if necessary) that $\forall s \geq \bar{s}$ one has $\gamma_{A_{\infty, s}}(y) < 0$ on $(\bar{y}, +\infty)$ and $\text{Arg } Z^+(\bar{y}, A_{\infty, s}) \in (\pi/2, \pi)$. On the other hand, for small $s > 0$, $\text{Arg } Z^0(\bar{y}, A_{\infty, s})$ is close to $\text{Arg } Z^0(\bar{y}, A_{\infty})$, which, due to the oscillation property of the limit-case equation, tends to $+\infty$ as $\bar{y} \rightarrow +\infty$. Therefore for each natural m one can find (again increasing \bar{y} when necessary) small $\bar{s} > 0$ such that $\text{Arg } Z^0(\bar{y}, A_{\infty, \bar{s}}) > \pi m$. As s will grow from \bar{s} to $\bar{\Omega}$, $\text{Arg } Z^0(\bar{y}, A_{\infty, s})$ will decrease from the value greater than πm to the value less than π and during this evolution it becomes equal (mod π) to $\text{Arg } Z^+(\bar{y}, A_{\infty, s})$ for m distinct values of s .

Now we provide the detailed proofs of the statements i)-iii) of the Theorem.

i) Pick $\bar{K} > 0$ and take $\bar{\Omega} = \frac{\mu_{\infty}}{\rho_{\infty}}\bar{K}$, so that $A_{\infty} = (\bar{K}, \bar{\Omega})$ is collinear with a_{∞} . Consider the limit-case equation (42) with the parameter A_{∞} and choose the solution $Z^0(\cdot; A_{\infty})$, which satisfies the boundary condition (6). As long as equation (42) is oscillatory, $\text{Arg } Z^0(y; A_{\infty})$ tends to infinity as $y \rightarrow +\infty$. Hence, *for each natural $m \exists y_m \in [0, +\infty)$ such that $\text{Arg } Z^0(y_m; A_{\infty}) > \pi m$.*

By the continuity of the trajectories of (38) with respect to the parameter A , one can find $\bar{s} > 0$ such that for any $s \in (0, \bar{s}]$ and for $A_{\infty, s} = (\bar{K}, \bar{\Omega} - s)$ there holds $\text{Arg } Z^0(y_m; A_{\infty, s}) > \pi m$.

For the function $\gamma_{A_{\infty, \bar{s}}}(y)$ defined by (44) one can find $\bar{y} \geq y_m$ such that $\gamma_{A_{\infty, \bar{s}}}(y) < 0$ on $[\bar{y}, +\infty)$. It follows from Remark 2.5iii) that $\text{Arg } Z^0(\bar{y}; A_{\infty, \bar{s}}) > \pi m$.

The second-order equation (45) for $s = \bar{s}$ meets the assumptions of Proposition 2.1 and hence has the decaying solution $Z^+(y; A_{\infty, \bar{s}})$. By Proposition 2.7, there holds:

$$\forall y \geq \bar{y} : \text{Arg } Z^+(y; A_{\infty, \bar{s}}) \in (\pi/2, \pi) \pmod{\pi}.$$

Letting s grow from \bar{s} towards $\bar{\Omega}$, we note that the values of $\gamma_{A_{\infty, s}}(y) = \gamma_{A_{\infty}}(y) - s\rho(y)$ on $[0, +\infty)$ diminish; in particular, $\gamma_{A_{\infty, s}}(y) < 0$ for $y \in [\bar{y}, +\infty)$ for all $s \geq \bar{s}$. According to Proposition 2.8, the function

$s \rightarrow \text{Arg } Z^0(\bar{y}; A_{\infty, s})$ decreases monotonously from the value $\text{Arg } Z^0(\bar{y}; A_{\infty, \bar{s}}) > \pi m$ to the value $\text{Arg } Z^0(\bar{y}; A_{\infty, \bar{\Omega}}) \in (0, \pi)$.

Consider now the decaying solutions $Z^+(y; A_{\infty, s})$. Proposition 2.9 implies that for chosen \bar{y} $\text{Arg } Z^+(\bar{y}; A_{\infty, s})$ grows with the growth of s , remaining (mod π) in the interval $(\pi/2, \pi)$. During the evolution there occur (at least) m values of s_j , $j = 1, \dots, m$, for which

$$\text{Arg } Z^+(\bar{y}; A_{\infty, s_j}) = \text{Arg } Z^0(\bar{y}; A_{\infty, s_j}) - \pi n \quad (n - \text{integer}).$$

Then the concatenations

$$(46) \quad Z(y; A_{\infty, s_j}) = \begin{cases} Z^0(y; A_{\infty, s_j}), & y \leq \bar{y}, \\ (-1)^n Z^+(y; A_{\infty, s_j}), & y \geq \bar{y}, \end{cases}$$

are the decaying solutions of the corresponding equations

$$u'' + ((\bar{\Omega} - s_j)\rho(y) - \bar{K}\mu(y))u = 0,$$

and (46) satisfies the boundary condition (6)-(7).

ii) Let $\tilde{\Omega} \in (0, \bar{\Omega})$ be a limit point of $\Omega_n = \bar{\Omega} - s_n$, $n = 1, \dots$. Then $\tilde{\Omega} = \bar{\Omega} - \tilde{s} < \bar{\Omega}$.

Consider $\gamma_{A_{\infty, \tilde{s}}}$. There exists \tilde{y} , such that $\gamma_{A_{\infty, \tilde{s}}} < 0$ on $[\tilde{y}, +\infty)$. Pick the decaying solution $Z^+(y; A_{\infty, \tilde{s}})$. According to the aforesaid $\forall y \in [\tilde{y}, +\infty)$: $\text{Arg } Z^+(y; A_{\infty, \tilde{s}}) \in (\pi/2, \pi) \pmod{\pi}$.

Consider the solution $Z^0(y; A_{\infty, \tilde{s}})$, which meets the initial condition (6). If $\text{Arg } Z^+(\tilde{y}; A_{\infty, \tilde{s}}) \neq \text{Arg } Z^0(\tilde{y}; A_{\infty, \tilde{s}}) \pmod{\pi}$, then the inequality holds for values of s close to \tilde{s} , and in particular for all s_n , but finite number of them, and this results in a contradiction.

Let $\text{Arg } Z^0(\tilde{y}; A_{\infty, \tilde{s}}) - \text{Arg } Z^+(\tilde{y}; A_{\infty, \tilde{s}}) = \pi m$. Since $\text{Arg } Z^0(\tilde{y}; A_{\infty, s}) - \text{Arg } Z^+(\tilde{y}; A_{\infty, s})$ decreases with the growth of s , one concludes:

$$\text{Arg } Z^+(\tilde{y}; A_{\infty, s}) \neq \text{Arg } Z^0(\tilde{y}; A_{\infty, s}) \pmod{\pi}$$

for all $s \neq \tilde{s}$ from a sufficiently small neighborhood of \tilde{s} and hence for all s_n but a finite number of them, which leads us to the same contradiction.

iii) By the construction provided in i), for each natural m , there exist $A_m = (\bar{K}, \Omega_m)$ and the decaying solution $Z(y, A_m)$ of (38)-(6)-(7), which converges to the origin in such a way that $\text{Arg } Z(y, A_m) \in [\pi(m - 1/2), \pi m]$ for sufficiently large y .

To prove its uniqueness, we assume on the contrary that there exists another $A' = (\bar{K}, \Omega')$ and a decaying solution of (38)-(6)-(7) such that for $y \in [y_0, +\infty)$ $\gamma_{A_m}(y) < 0$, $\gamma_{A'}(y) < 0$ and both $\text{Arg } Z(y, A')$, $\text{Arg } Z(y, A_m)$ belong to $[\pi(m - 1/2), \pi m]$ for $y \in [y_0, +\infty)$.

Let for example $\Omega' > \Omega_m$. Then $\gamma_{A_m}(y) < \gamma_{A'}(y)$ and hence $\text{Arg } Z(y_0, A_m) < \text{Arg } Z(y_0, A')$. This enters in contradiction with the result of Proposition 2.9.

Proof of Theorem 4.2. Let us pick $\bar{K} > 0$ and take $\bar{\Omega} = \bar{\Omega} = \frac{\mu_\infty}{\rho_\infty} \bar{K}$, so that $A_\infty = (\bar{K}, \bar{\Omega})$ is collinear with a_∞ . By assumptions of the Theorem, the

function $\gamma_{A_\infty}(y)$ admits positive values on some non-null subinterval $(c, c) \subset [0, +\infty)$. The same holds true for $\gamma_{\beta A_\infty}$ with $\beta A_\infty = (\beta \bar{K}, \beta \bar{\Omega})$, $\beta > 0$.

Our proof can be accomplished along the lines of the proof of Theorem 4.3 if one proves that for any N there exists $\beta_N > 0$, for which the solution $Z^0(y, \beta_N A_\infty)$ with initial condition (6) satisfies $\text{Arg } Z^0(c, \beta_N A_\infty) > \pi N$.

Consider the equation

$$u''(y) + \gamma_{\beta A_\infty}(y)u = u''(y) + \beta \gamma_{A_\infty}(y)u = 0$$

on the interval $[0, c]$. It is known ([3, §A.3, §A.5]) that the number of zeros of the solution $u(y, \gamma_{\beta A_\infty}(\cdot))$, or, the same, the increment of Prüfer's angle

$$\text{Arg } Z(y, \gamma_{\beta A_\infty}(\cdot)) - \text{Arg } Z(0, \gamma_{\beta A_\infty}(\cdot))$$

grows as

$$(47) \quad \pi^{-1} \beta^{1/2} \int_0^y (\max(\gamma_{A_\infty}(\eta), 0))^{1/2} d\eta + O(\beta^{1/3})$$

as $\beta \rightarrow +\infty$. Hence choosing sufficiently large $\beta > 0$, we can get a solution $Z^0(y; \beta A_\infty)$ with the boundary condition (6) and such that $\text{Arg } Z^0(c; \beta A_\infty) > N\pi$. It follows from Proposition 2.5 that $Z^0(y; \beta A_\infty) > N\pi$, $\forall y > c$.

The rest of the proof follows the proof of Theorem 4.3. One can also conclude from (47) that $N(k) \sim k$ as $k \rightarrow \infty$, where $N(k)$ is the number of surface wave solutions with a given wave number $k = K^{1/2}$.

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