Deformations of coisotropic submanifolds in abstract Jacobi manifolds

DEFORMATIONS OF COISOTROPIC SUBMANIFOLDS IN ABSTRACT JACOBI MANIFOLDS

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Abstract. In this paper, using the Atiyah algebroid and first order multi-differential calculus on non-trivial line bundles, we attach an L_∞-algebra to any coisotropic submanifold S in an abstract (or Kirillov's) Jacobi manifold. Our construction generalizes and unifies analogous constructions in [32] (symplectic case), [4] (Poisson case), [25] (locally conformal symplectic case). As a new special case, we attach an L_∞-algebra to any coisotropic submanifold in a contact manifold, including Legendrian submanifolds. The L_∞-algebra of a coisotropic submanifold S governs the (formal) deformation problem of S.

Contents

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1. Introduction

Jacobi structures were independently introduced by Lichnerowicz [27] and Kirillov [22], and they are a combined generalization of symplectic or Poisson structures and contact structures. Note that Kirillov *local Lie algebras with one dimensional fiber* [22] are slightly more general than Lichnerowicz *Jacobi manifolds*. In particular, the former encompass non-coorientable contact manifolds, while the latter does not. In this note we adopt Kirillov's approach. To make it clear the distinction with the more popular Lichnerowicz's approach we speak about *abstract Jacobi manifolds* (see also [7]). On the other hand we sometimes call "standard" Jacobi manifolds in the sense of Lichenowicz. Many constructions in standard Jacobi geometry have an "abstract version" and (generically, non-trivial) line-bundles play a distinguished role in the "abstract setting".

Coisotropic submanifolds in (standard) Jacobi manifolds have been first studied by Ibáñez-de León-Marrero-Martín de Diego [16]. They showed that these submanifolds play a similar role as coisotropic submanifolds in Poisson manifolds. For instance, the graph of a conformal Jacobi morphism $f : M_1 \to M_2$ between Jacobi manifolds is a coisotropic submanifold in $M_1 \times M_2 \times \mathbb{R}$ equipped with an appropriate Jacobi structure (note that this remark has an "abstract version"). Other important examples of coisotropic submanifolds in a Jacobi manifold M are closed leaves of the characteristic distribution of M, which have been intensively studied by many authors. Since the property of being coisotropic does not change in the same conformal class of a standard Jacobi manifold (see Remark 2.22 and Lemma 3.1), it seems to us that we should not restrict the study of coisotropic submanifolds to those inside Poisson manifolds, and, even more, we should in fact consider the general case of coisotropic submanifolds in abstract Jacobi manifolds.

One purpose of the present article is to extend the construction of an L_∞-algebra attached to a coisotropic submanifold S in a Jacobi manifold, generalizing analogous constructions in [32] (symplectic case), [4] (Poisson case), [25] (locally conformal symplectic case). Our construction encompasses all the known cases as special cases and reveals the prominent role of the Atiyah algebroid $\operatorname{der} L$ of a line bundle L. In all previous cases L is a trivial line bundle while it is not necessarily so for general Jacobi manifolds. As a new special case, our construction canonically applies to coisotropic submanifolds in any (not necessarily co-orientable) contact manifold. We also provide a global tensorial description of our L_∞-algebra, in the spirit of [4], originally given in the language of (formal) Q-manifolds [1] for the symplectic case (see [32, Appendix]).

The L_∞-algebra of a coisotropic submanifold S governs the formal deformation problem of S. In this respect, another purpose of the present article is to present necessary and sufficient conditions under which the L_∞-algebra of S governs the *non-formal* deformation problem as well. Our Proposition 4.15 extends - even in the Poisson setting - the sufficient condition given by Schätz and Zambon in [36] to a necessary and sufficient condition. We also discuss the relation between Hamiltonian equivalence of coisotropic sections and gauge equivalence of Maurer-Cartan elements. We obtain a satisfactory description of this relations (Proposition 4.21) and discuss its consequences (Theorem 4.23 and Corollary 4.22)

Note that Jacobi manifolds can be understood as Poisson manifolds (of a special kind) via the "Poissonization trick" (see Appendix C). However, not all coisotropic submanifolds in the Poissonization come from coisotropic submanifolds in the original Jacobi manifold. On the other hand, if we regard a Poisson manifold as a Jacobi manifold, all its coisotropic submanifolds are coisotropic in the Jacobi sense as well. In particular, the deformation problem of a coisotropic submanifold in a Jacobi manifold is genuinely more general than its analogue in the Poisson setting.

Our paper is organised as follows. In Section 2 we attach important algebraic and geometric structures to an *abstract* Jacobi manifold (see therein for a definition of abstract Jacobi manifold). Our approach, via Atiyah algebroids and first order multi-differential calculus on non-trivial line bundles, unifies and simplifies previous, analogous constructions for Poisson manifolds and locally conformal symplectic manifolds. In Section 3, using results in Section 2, we attach an L_∞-algebra to any closed coisotropic submanifold in an abstract Jacobi manifold. In Section 4 we study the deformation problem of coisotropic submanifolds. In particular we discuss the relation between smooth coisotropic deformations and formal coisotropic deformations as well as the moduli problem under Hamiltonian equivalence. In Sections 5 and 6 we apply the theory to the contact case, which is, in a sense, analogous to the symplectic case analysed by Oh-Park [32], and interpret the results obtained (Remark 6.3, Corollary 6.5).

Finally, the paper contains four appendices. The first two collect some facts about Atiyah algebroids and *abstract* Gersternhaber-Jacobi algebras that are needed in the main body of the paper. The third one discusses the "Poissonization" of the L_∞-algebra of a coisotropic submanifold in an abstract Jacobi manifold. The fourth one provides a complete proof of the expected result that Catteneo-Felder L_∞-algebra [4] reduces to Oh-Park one [32] in the symplectic case.

2. Abstract Jacobi manifolds and associated algebraic and geometric structures

In this section we recall the definition of Jacobi manifolds and present important examples (Definition 2.1, Examples 2.3) of them. Our primary sources are [22], [27], [30], [13], and the recent paper by Crainic and Salazar [7] whose philosophy/approach *á la* Kirillov we adopt. Accordingly, we will speak about *abstract Jacobi manifolds*, retaining the term (*standard*) *Jacobi manifolds* for Jacobi manifolds in the sense of Lichnerowicz. Generically non-trivial line bundles and first order multi-differential calculus on them (see Appendix B) play a prominent role in abstract Jacobi geometry. We also associate important algebraic and geometric structures with abstract Jacobi manifolds. Namely, we recall the notion of *Jacobi algebroid* (see [13] and [17] for the equivalent notion of *Lie algebroid with a* 1-*cocycle*), but we adopt a slightly more general approach in the same spirit as that of abstract Jacobi manifolds and abstract Gerstenhaber-Jacobi algebras (see Appendix B). Accordingly, we will speak about *abstract Jacobi algebroids*. In particular, we discuss the existence of an abstract Jacobi algebroid structure on the first jet bundle J^1L of the Jacobi bundle of an abstract Jacobi manifold $(M, L, \{-,-\})$ (Example 2.10), first discovered by Kerbat and Souci-Behammadi in the special case $L = M \times \mathbb{R}$ [20] (see [7] for the general case). We also present an abstract version of the Iglesias-Marrero [17] construction of a fiber-wise linear Jacobi structure on the dual bundle E^* of a Jacobi algebroid E (Proposition 2.15). This construction provides a natural lifting of an abstract Jacobi structure $(L, \{-,-\})$ on M to an abstract Jacobi structure on the total space of the Atiyah algebroid $\operatorname{der} L$ of L (Example 2.17, see Appendix A for a definition of the Atiyah algebroid of a vector bundle and its relation with differential operators). Finally, we discuss the notion of morphisms of Jacobi manifolds. We explain the relation between certain geometric properties of an abstract Jacobi manifold $(M, L, \{-,-\})$ involving morphisms and cohomological invariants of the associated Jacobi algebroid (Proposition 2.24).

2.1. **Abstract Jacobi manifolds and their canonical bi-linear forms.** Let M be a smooth manifold.

Definition 2.1. An *abstract Jacobi structure* on M is a pair $(L, \{-,-\})$ where $L \to M$ is a (generically non-trivial) line bundle, and $\{-,-\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ is a Lie bracket which, moreover, is a first order differential operator on both entries. An *abstract Jacobi manifold* is a manifold equipped with an abstract Jacobi structure. The bundle L and the bracket $\{-,-\}$ will be referred to as the *Jacobi bundle* and the *Jacobi bracket* respectively.

Remark 2.2. One can recover the standard definition [27] by letting L be the trivial line bundle $\mathbb{R}_M := M \times \mathbb{R}$. Indeed, in this case a Jacobi bracket $\{-,-\}$ on $\Gamma(L) = C^\infty(M)$ is the same as a pair (Λ, Γ), where Λ is a bi-vector field and Γ is a vector field satisfying the following equations [13] (see Remark 2.12):

$$[\Gamma, \Lambda]^{SN} = \mathcal{L}_\Gamma \Lambda = 0 \quad \text{and} \quad [\Lambda, \Lambda]^{SN} = -2\Gamma \wedge \Lambda, \tag{2.1}$$

where $[-,-]^{SN}$ is the Schouten-Nijenhuis bracket on multi-vector fields. Accordingly, a *standard Jacobi manifold* is a smooth manifold M equipped with a pair (Λ, Γ) as above. The corresponding Jacobi bracket $\{-,-\}$ identifies with the bi-differential operator $\Lambda + \Gamma \wedge \mathrm{id}$, where id denotes the identity map. Namely, for $f, g \in C^\infty(M)$

$$\{f, g\} := (\Lambda - \Gamma \wedge \mathrm{id})(f, g) = \Lambda(df, dg) + f \cdot \Gamma(g) - g \cdot \Gamma(f). \tag{2.2}$$

On the other hand, since every line bundle is locally trivial, abstract Jacobi manifolds look locally like standard Jacobi manifolds.

Let $\{-,-\}$ be a Jacobi bracket. When we want to stress that $\{-,-\}$ is a bi-differential operator, we also denote it by J. We collect basic facts, including our notations and conventions, about (multi-)differential operators in Appendix A and Appendix B. In the following, we will often refer to them for details.

Example 2.3.

(1) Any (possibly non-coorientable) contact manifold (M, C) is naturally equipped with an abstract Jacobi structure, with Jacobi bundle given by the (possibly non-trivial) line bundle TM/C (see Section 5).
(2) Recall that a locally conformal symplectic (l.c.s.) manifold is naturally equipped with a standard Jacobi structure sometimes called the *associated locally conformal Poisson structure*. There is a slight generalization of a l.c.s. manifold in the same spirit as abstract Jacobi manifolds (see Appendix A of [38]). Call it an *abstract l.c.s. manifold*. Then, any abstract l.c.s. manifold is naturally equipped with an abstract Jacobi structure [38].
(3) Let $\{\omega_t\}_{t\in I}$ be a smooth l.c.s. deformation of a l.c.s. form ω_0 on a manifold M, where I is an open interval in $\mathbb{R}$ containing 0. Denote by J_t the standard Jacobi structure on M associated with ω_t, and let $\tilde{J} : C^\infty(M \times I) \times C^\infty(M \times I) \to C^\infty(M \times I)$ be defined by $\tilde{J}(\tilde{g}, \tilde{f})(x, t) := J_t(\tilde{f}(-, t), \tilde{g}(-, t))(x)$. Then it is not hard to verify that $(M \times I, \tilde{J})$ is a standard Jacobi manifold.

Let $(M, L, J = \{-,-\})$ be an abstract Jacobi manifold and $\lambda \in \Gamma(L)$. Then $\Delta_\lambda := \{\lambda, -\}$ is a first order differential operator, hence a derivation of L. The (scalar) symbol of Δ_λ (see Appendix A) will be denoted by X_λ.

Remark 2.4. By definition, a Jacobi bracket $\{-,-\}$ on sections of a line bundle $L \to M$ satisfies the following generalized Leibniz rule

$$\{\lambda, f\mu\} = f\{\lambda,\mu\} + X_\lambda(f)\mu, \tag{2.3}$$

$\lambda,\mu \in \Gamma(L)$, $f \in C^\infty(M)$. In the standard case, $\Gamma(L) = C^\infty(M)$ is what is called a *Jacobi algebra* [12].

Denote by J^1L the bundle of 1-jets of section of L and let $j^1 : \Gamma(L) \to \Gamma(J^1L)$ be the first jet prolongation (see Appendix A for more details). The bi-differential operator J can be also interpreted as an L-valued, skew-symmetric, bi-linear form $\widehat{\Lambda}_J : \wedge^2 J^1L \to L$. Namely, $\widehat{\Lambda}_J$ is uniquely determined by

$$\widehat{\Lambda}_J(j^1\lambda, j^1\mu) = \{\lambda,\mu\},$$

for all $\lambda,\mu \in \Gamma(L)$. Denote by $\operatorname{der} L = \operatorname{Hom}(J^1L, L)$ the Atiyah algebroid of the line bundle L (see Appendix A for details). Then, the bi-linear form $\widehat{\Lambda}_J$ determines an obvious morphism of vector bundles $\widehat{\Lambda}_J^\# : J^1L \to \operatorname{der} L$, defined by $\widehat{\Lambda}_J^\#(\alpha)\lambda := \widehat{\Lambda}_J(\alpha, j^1\lambda)$, where $\alpha \in \Gamma(J^1L)$ and $\lambda \in \Gamma(L)$. The *bi-symbol* Λ_J of $\widehat{\Lambda}_J$ will be also useful. It is defined as follows: $\Lambda_J : \wedge^2(T^*M \otimes L) \to L$ is the bi-linear form obtained by restriction of $\widehat{\Lambda}_J$ to the module $\Omega^1(M,L)$ of L-valued one forms on M, regarded as a submodule in $\Gamma(J^1L)$ via the *co-symbol* $\gamma : \Omega^1(M,L) \to \Gamma(J^1L)$ (see Appendix A). Namely,

$$\Lambda_J(\eta,\theta) := \widehat{\Lambda}_J(\gamma(\eta),\gamma(\theta)),$$

for all $\eta,\theta \in \Omega^1(M,L)$. It immediately follows from the definition that

$$\Lambda_J(df\otimes\lambda, dg\otimes\mu) = \{f\lambda, g\mu\} - fg\{\lambda,\mu\} - fX_\lambda(g)\mu + gX_\mu(f)\lambda = (X_{f\lambda}(g) - fX_\lambda(g))\,\mu, \tag{2.4}$$

where $f,g \in C^\infty(M)$, and $\lambda,\mu \in \Gamma(L)$.

Remark 2.5. When $L = \mathbb{R}_M$, then J is the same as a standard Jacobi structure, i.e. a pair (Λ,Γ) as in Remark 2.2, and Λ_J is just a bi-vector fields. Actually, we have $\Lambda_J = \Lambda$.

The skew-symmetric form Λ_J determines an obvious morphism of vector bundles $\Lambda_J^\# : T^*M \otimes L \to TM$, implicitly defined by $\langle \Lambda_J^\#(\eta\otimes\lambda), \theta\rangle\mu := \Lambda_J(\eta\otimes\lambda, \theta\otimes\mu)$, where $\eta,\theta \in \Omega^1(M)$, $\lambda,\mu \in \Gamma(L)$, and $\langle -,-\rangle$ is the duality pairing. In other words,

$$\Lambda_J^\#(df\otimes\lambda) = X_{f\lambda} - fX_\lambda, \tag{2.5}$$

$f \in C^\infty(M)$, $\lambda \in \Gamma(L)$. Morphism $\Lambda_J^\#$ can be alternatively defined as follows. Recall that $\operatorname{der} L$ projects onto TM via the symbol σ. It is easy to see that diagram

$$\begin{tikzcd} T^*M \otimes L \arrow[r, "\Lambda_J^\#"] \arrow[d, "\gamma"'] & TM \\ J^1L \arrow[r, "\widehat{\Lambda}_J^\#"] & \operatorname{der} L \arrow[u, "\sigma"'] \end{tikzcd}$$

commutes, i.e. $\Lambda_J^\# = \sigma \circ \widehat{\Lambda}_J^\# \circ \gamma$, which can be used as an alternative definition of $\Lambda_J^\#$. Finally, note that

$$(\widehat{\Lambda}_J^\# \circ \gamma)(df\otimes\lambda) = \Delta_{f\lambda} - f\Delta_\lambda.$$

2.2. **Abstract Jacobi algebroid associated with an abstract Jacobi manifold.**

Definition 2.6. An *abstract Jacobi algebroid* is a pair (A, L) where $A \to M$ is a Lie algebroid, and $L \to M$ is a line bundle equipped with a representation of A.

Let A be a Lie algebroid with anchor ρ and Lie bracket $[-,-]_A$. Recall that a representation of A in a vector bundle $E \to M$ is a *flat A-connection in E*, i.e. a Lie algebroid morphism $\nabla : A \to \operatorname{der} E$, written $\alpha \mapsto \nabla_\alpha$, with values in the Atiyah algebroid $\operatorname{der} E$ of L. In other words ∇ is an $\mathbb{R}$-linear map $\Gamma(A) \to \operatorname{Diff}_1(E,E)$, where $\operatorname{Diff}_1(E,E)$ is the module of first order differential operators $\Gamma(E) \to \Gamma(E)$, such that

$$
\begin{aligned}
\nabla_{f\alpha} e &= f \nabla_\alpha e,\\
\nabla_\alpha (fe) &= \rho(\alpha)(f)e + f\nabla_\alpha e,\\
[\nabla_\alpha, \nabla_\beta] e &= \nabla_{[\alpha,\beta]_A} e,
\end{aligned}
$$

where $\alpha, \beta \in \Gamma(A)$, $f \in C^\infty(M)$, and $e \in \Gamma(E)$.

Let $(\Gamma(\wedge^\bullet A^*), d_A)$ be the de Rham complex of the Lie algebroid A. A representation ∇ of A in E defines a degree one differential on sections of $\wedge^\bullet A^* \otimes E$, denoted by $d_{A,E}$, and uniquely determined by

$$
\begin{aligned}
d_{A,E}\lambda &= \nabla\lambda,\\
d_{A,E}(\omega \wedge \Omega) &= d_A\omega \wedge \Omega + (-)^{|\omega|}\omega \wedge d_{A,E}\Omega,
\end{aligned}
\tag{2.6}
$$

where $\lambda \in \Gamma(L)$, $\omega \in \Gamma(\wedge^* A^*)$, $\Omega \in \Gamma(\wedge^\bullet A^* \otimes E)$ and we used the obvious $\Gamma(\wedge^\bullet A^*)$-module structure on $\Gamma(\wedge^\bullet A^* \otimes E)$. Complex $(\Gamma(\wedge^\bullet A^* \otimes E), d_{A,E})$ is called *de Rham complex of A with values in E*. Its cohomology is denoted by $H(A,E)$ and called *de Rham cohomology of A with values in E*.

Remark 2.7. In the case $L = \mathbb{R}_M$, a representation of A in L is the same as a 1-cocycle in the de Rham complex $(\Gamma(\wedge^\bullet A^*), d_A)$ of A. Namely, in this case $\Gamma(\wedge^\bullet A^* \otimes L) = \Gamma(\wedge^\bullet A^*)$ and, in view of (2.6) ∇ is completely determined by $\omega_\nabla := d_{A,L}1 \in \Gamma(A^*)$. It is easy to see that ω_∇ is a d_A-cocycle, i.e. $d_A\omega_\nabla = 0$. Conversely, a d_A-cocycle $\omega \in \Gamma(A^*)$ determines a unique representation ∇ in $L = \mathbb{R}_M$ such that $d_{A,L}1 = \omega$. This shows that, in the case $L = \mathbb{R}_M$, Definition 2.6 recovers the definition of Lie algebroid with a 1-cocycle proposed in [17], which is in turn equivalent to the definition of Jacobi algebroid proposed in [13].

Let $A \to M$ be a vector bundle, and let $L \to M$ be a line bundle. Consider vector bundle $A_L := A \otimes L^*$. The parallel between Lie algebroid structures on A and Gerstenhaber brackets on the associated Grassmann algebra $\Gamma(\wedge^\bullet A)$ is well-known (see e.g. [23], [13, Theorem 3]). There is an analogous parallel between abstract Jacobi algebroid structures on (A, L) and abstract Gerstenhaber-Jacobi algebra structures on $(\Gamma(\wedge^\bullet A_L), \Gamma(\wedge^\bullet A_L \otimes L)[1])$ (see Appendix B about Gerstenhaber-Jacobi algebras, moreover, see Example 2.10 about the relevance of Gerstenhaber-Jacobi structures for Jacobi geometry). Proposition 2.8 below clarifies this parallel.

Proposition 2.8. *(see also* [13, Theorem 5]*) There is a one-to-one correspondence between abstract Jacobi algebroid structures on (A, L) and abstract Gerstenhaber-Jacobi algebra structures on $(\Gamma(\wedge^\bullet A_L), \Gamma(\wedge^\bullet A_L \otimes L)[1])$.*

Proof. Note preliminarily that an abstract Gerstenhaber-Jacobi structure on $(\Gamma(\wedge^\bullet A_L), \Gamma(\wedge^\bullet A_L \otimes L)[1])$ is completely determined by

1. the action of degree zero elements of $\Gamma(\wedge^\bullet A_L \otimes L)[1]$ on degree zero elements of $\Gamma(\wedge^\bullet A_L)$,
2. the Lie bracket between degree zero elements of $\Gamma(\wedge^\bullet A_L \otimes L)[1]$, and
3. the Lie bracket between degree zero elements and degree -1 elements of $\Gamma(\wedge^\bullet A_L \otimes L)[1]$.

Now, let $(\Gamma(\wedge^\bullet A_L), \Gamma(\wedge^\bullet A_L \otimes L)[1])$ possess the structure of an abstract Gerstenhaber-Jacobi structure, with graded Lie bracket $[-,-]$ and action of $\Gamma(\wedge^\bullet A_L \otimes L)[1]$ on $\Gamma(\wedge^\bullet A_L)$ written $\alpha \mapsto X_\alpha$. Define an anchor, a Lie bracket, and a flat connection by putting

$$
\begin{aligned}
\rho(\alpha)(f) &= X_\alpha(f),\\
[\alpha,\beta]_A &= [\alpha,\beta],\\
\nabla_\alpha\lambda &= [\alpha,\lambda],
\end{aligned}
\tag{2.7}
$$

where $\alpha,\beta \in \Gamma(A) = \Gamma(\wedge^1 A_L \otimes L)$, $f \in C^\infty(M) = \Gamma(\wedge^0 A_L)$, and $\lambda \in \Gamma(L) = \Gamma(\wedge^0 A_L \otimes L)$. It is easy to see that the above operations form a well-defined abstract Jacobi algebroid structure on (A, L). Conversely, let (A, L) be an abstract Jacobi algebroid with anchor ρ, Lie bracket $[-,-]_A$, and representation ∇. Read Equations (2.7) from the right to the left to define an abstract Gerstenhaber-Jacobi structure on $(\Gamma(\wedge^\bullet A_L), \Gamma(\wedge^\bullet A_L \otimes L)[1])$. □

Now, let M be a manifold and let $L \to M$ be a line bundle. Denote by $J_1 L$ the dual bundle of $J^1 L$. Sections of $J_1 L$ are first order differential operators $\Gamma(L) \to C^\infty(M)$, i.e. $J_1 L := \operatorname{diff}_1(L, \mathbb{R}_M)$. Moreover, denote by $\operatorname{Der}^\bullet L = \Gamma(\wedge^\bullet J_1 L \otimes L)$ the space of multi-differential operators $\Gamma(L)\times\cdots\times\Gamma(L) \to \Gamma(L)$ (see Appendix B for more details).

Example 2.9. (cf. [13, (26)]) The Atiyah algebroid $\operatorname{der} L$ of L is equipped with the tautological representation $\mathrm{id}_{\operatorname{der} L}$ in L. Accordingly, $(\operatorname{der} L, L)$ is an abstract Jacobi algebroid. It follows from Proposition 2.8 that there is an abstract Gerstenhaber-Jacobi algebra structure on $(\Gamma(\wedge^\bullet J_1 L), (\operatorname{Der}^\bullet L)[1])$. The Lie bracket on $(\operatorname{Der}^\bullet L)[1]$ will be also called the *Schouten-Jacobi bracket* and denoted by $[-,-]^{SJ}$. See Appendix B for explicit formulas.

Example 2.10. (cf. [20, Theorem 1], [18, (2.7)], [13, Theorem 13]) Let $(M, L, J = \{-,-\})$ be an abstract Jacobi manifold. It is not hard to see (see, e.g., [7]) that there is a unique Jacobi algebroid structure on $(J^1 L, L)$ with anchor ρ_J, Lie bracket $[-,-]_J$, and representation ∇^J such that

$$
\begin{aligned}
\rho_J(j^1\lambda) &= X_\lambda, \\
[j^1\lambda, j^1\mu]_J &= j^1\{\lambda,\mu\}, \\
\nabla^J_{j^1\lambda}\mu &= \{\lambda,\mu\},
\end{aligned}
\tag{2.8}
$$

for all $\lambda,\mu \in \Gamma(L)$. Using the fact that every section α of $J^1 L$ can be uniquely written as

$$
\alpha = j^1\lambda + \gamma(\eta), \quad \lambda \in \Gamma(L), \quad \eta \in \Gamma(T^*M \otimes L) \tag{2.9}
$$

where, we recall, γ is the co-symbol (see Appendix A), explicit formulas for the structure maps ρ_J, $[-,-]_J$ and ∇^J can be found. Namely, use decomposition $\Gamma(J^1 L) = \Gamma(L) \oplus \Omega^1(M, L)$ to identify $\alpha,\beta \in \Gamma(J^1 L)$ with pairs $(\lambda,\eta), (\mu,\theta) \in \Gamma(L) \oplus \Omega^1(M,L)$, and let $\nu \in \Gamma(L)$. A straightforward computation shows that

$$
\rho_J(\alpha) = X_\lambda + \Lambda_J^\#(\eta), \tag{2.10}
$$

$$
[\alpha,\beta]_J = j^1\left(\{\lambda,\mu\} + i_{\Lambda_J^\#(\eta)}\theta\right) + \gamma\left(\mathcal{L}_{\Delta_\lambda + \widehat{\Lambda}_J^\#(\eta)}\theta - \mathcal{L}_{\Delta_\mu + \widehat{\Lambda}_J^\#(\theta)}\eta\right), \tag{2.11}
$$

$$
\nabla^J_\alpha = \Delta_\lambda + \widehat{\Lambda}_J^\#(\eta).
$$

In Formula (2.11) it appears the *Lie derivative of an L-valued form on M along a derivation of L*. See Appendix A for details about the main definitions and formulas in vector valued Cartan calculus.

Lemma 2.11. *Let $J \in \operatorname{Der}^2 L$ be an alternating, first order bi-differential operator $J : \Gamma(L)\times\Gamma(L) \to \Gamma(L)$. Then*

(1) *for all $\lambda,\mu \in \Gamma(L)$,*

$$
J(\lambda,\mu) = -[[J,\lambda]^{SJ},\mu]^{SJ}. \tag{2.12}
$$

(2) *(cf.* [13, Theorem 1.b, (28), (29)]*) J is a Jacobi bracket, i.e. it defines a Lie algebra structure on $\Gamma(L)$ iff*

$$
[J,J]^{SJ} = 0, \tag{2.13}
$$

Proof. The first assertion is a consequence of the explicit form of the Schouten-Jacobi bracket (see Appendix B). The second assertion is a particular case of Theorem 3.3 in [26]. □

Remark 2.12. Denote by $\mathfrak{X}^\bullet(M) = \bigoplus_k \mathfrak{X}^k(M)$ the space of (skew-symmetric) multi-vector fields on M. When $L = \mathbb{R}_M$, the space $\mathrm{Der}^{k+1}L$ of alternating first order multi-differential operators on $\Gamma(L)$ with $k+1$ entries, identifies with $\mathfrak{X}^{k+1}(M) \oplus \mathfrak{X}^k(M)$ (see Appendix B). In particular, an alternating, first order bi-differential operator J identifies with a pair (Λ, Γ) where Λ is a bi-vector field and Γ is a vector field on M. In this case, Equation (2.13) is equivalent to (2.1).

Remark 2.13. Let (M, Λ) be a Poisson manifold, with Poisson bi-vector Λ, and Poisson bracket $\{-,-\}_\Lambda$. Differential $d_* := [\Lambda, -]^{SN} : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(M)$, where $[-,-]^{SN}$ is the Schouten-Nijenhuis bracket on multi-vectors, has been introduced by Lichnerowicz to define what is known as the Lichnerowicz-Poisson cohomology of (M, Λ). Note that complex $(\mathfrak{X}^\bullet(M), d_*)$ can be seen as a sub-complex of the Chevalley-Eilenberg complex associated with the Lie algebra $(C^\infty(M), \{-,-\}_\Lambda)$ and its adjoint representation. For more general abstract Jacobi manifolds $(M, L, J = \{-,-\})$ it is natural to replace multi-vector fields, with a suitable subcomplex of the Chevalley-Eilenberg complex associated with the Lie algebra $(\Gamma(L), \{-,-\})$ and its adjoint representation, specifically, the subcomplex of first order, multi-differential operators, i.e. elements of $\mathrm{Der}^\bullet L$. In particular, the Lichnerowicz-Poisson differential is replaced with the differential $d_*^J := [J, -]^{SJ}$. The resultant cohomology is called the *Chevalley-Eilenberg cohomology* of $(M, L, \{-,-\})$, and we denote it by $H_{CE}(M, L, J)$ [14, 27]. Furthermore, for a general abstract Jacobi manifold $(M, L, J = \{-,-\})$, the action of $(\mathrm{Der}^\bullet L)[1]$ on $\Gamma(\wedge^\bullet J_1 L)$ gives rise to another cohomology, namely cohomology of the complex $(\Gamma(\wedge^\bullet J_1 L), X_J)$ (see Appendix B for a definition of X_J), also called the *Lichnerowicz-Jacobi (LJ-)cohomology of* $(M, L, \{-,-\})$ (see, e.g., [8]). It is easy to see that the complex $(\Gamma(\wedge^\bullet J_1 L), X_J)$ is nothing but the de Rham complex of the Lie algebroid $(J^1 L, \rho_J, [-,-]_J)$. Similarly, complex $(\mathrm{Der}^\bullet L, d_*^J)$ is the de Rham complex of $(J^1 L, \rho_J, [-,-]_J)$ with values in L.

Remark 2.14. Many properties of Poisson manifolds have analogues for Jacobi manifolds, and abstract Jacobi manifolds. Sometimes these analogues can be found using the "*Poissonization trick*" which consists in the remark that Jacobi brackets on a line bundle $L \to M$ are in one-to-one correspondence with homogeneous Poisson brackets on the principal $R^\times$-bundle $L^* \smallsetminus \mathbf{0}$, where $\mathbf{0}$ is the (image of the) zero section of L^* (see Appendix C). For instance, using the Poissonization trick, Iglesias and Marrero established a one-to-one correspondence between Jacobi structures and Jacobi bialgebroids [18, Theorem 3.9]. In our paper we prefer to adopt an intrinsic approach to abstract Jacobi structures in the spirit of [7] (see Remarks 2.13, 2.25, and Proposition 3.6 in this paper). We only use the Poissonization trick in one case (Theorem 5.27) as a technical tool to avoid lengthy computations and get a quick proof.

2.3. **The fiber-wise linear abstract Jacobi structure on the adjoint bundle of an abstract Jacobi algebroid.** Let M be a smooth manifold. It is well known that, if A is a Lie algebroid over M, then the total space of the dual bundle A^* is equipped with a fiber-wise linear Poisson structure Λ_A, see e.g. [3, §16.5] or [9, §8.2]. Namely, fiber-wise constant functions on A^* are the same as functions on M. Moreover, fiber-wise linear functions on A^* are the same as sections of A. Then Λ_A is uniquely determined by

$$\Lambda_A(d\alpha, df) = \rho(\alpha)(f),$$
$$\Lambda_A(d\alpha, d\beta) = [\alpha, \beta]_A,$$

where $\alpha, \beta \in \Gamma(A)$ are also interpreted as fiber-wise linear functions and $f \in C^\infty(M)$ is also interpreted as a fiber-wise constant function on A^*.

A similar result holds for Jacobi algebroids [17, Theorems 1,2]. Here, we provide its abstract version. Thus, let $A \to M$ be a vector bundle, and let $L \to M$ be a line bundle. Vector bundle $A^* \otimes L = A_L{}^*$ will be called the *L-adjoint bundle of* A (or simply the *adjoint bundle*). Now, assume (A, L) is a Jacobi algebroid, with anchor ρ, Lie bracket $[-,-]_A$ and representation ∇ in L as usual. Let $\pi : A_L{}^* \to M$ be the projection and consider the line bundle $\pi^* L \to A_L{}^*$.

Proposition 2.15. *(*[17, Theorems 1,2]*) The adjoint bundle ${A_L}^*$ is equipped with a fiber-wise linear abstract Jacobi structure $\{-,-\}_{(A,L)}$.*

Before sketching the proof of Proposition 2.15, let us specify better what we mean by fiber-wise linear abstract Jacobi structure on a vector bundle. Thus, let $\pi : E \to M$ be a vector bundle, $L \to M$ a line bundle and $\pi^*L \to E$ the induced line bundle. There is a natural isomorphism $\Gamma(\pi^*L) \simeq C^\infty(E) \otimes \Gamma(L)$ (where the tensor product is over $C^\infty(M)$). Accordingly,

1. Sections λ of $L \to M$ identify with pull-back sections $\pi^*\lambda$ of π^*L, and we call them *fiber-wise constant*.
2. sections of the L-adjoint bundle ${E_L}^* = E^* \otimes L$ identify with certain sections of π^*L and we call them *fiber-wise linear*.

Now, an abstract Jacobi structure $(\pi^*L, \{-,-\})$ is *fiber-wise linear* if it preserves fiber-wise linear sections, i.e. the Jacobi bracket between two fiber-wise linear sections of π^*L is fiber-wise linear as well.

Proof of Proposition 2.15: a sketch. Let $\pi : {A_L}^* \to M$ be the projection and let $\pi^*L \to {A_L}^*$ be the induced line bundle. Fiber-wise constant sections of π^*L are the same as sections of L. Moreover, fiber-wise linear sections of π^* are the same as sections of $({A_L}^*)_L{}^* = A$. It is easy to see that $\{-,-\}_{(A,L)}$ is uniquely determined by

$$\{\alpha, \lambda\}_{(A,L)} := \nabla_\alpha \lambda,$$
$$\{\alpha, \beta\}_{(A,L)} := [\alpha, \beta]_A,$$

where $\alpha, \beta \in \Gamma(A)$ are also interpreted as fiber-wise linear sections, and $\lambda \in \Gamma(L)$ is also interpreted as a fiber-wise constant section of $\pi^*L \to {A_L}^*$. □

Example 2.16. Let $L \to M$ be a line bundle and let $\operatorname{der} L$ be its Atiyah algebroid. Since $(\operatorname{der} L, L)$ is a Jacobi algebroid, Proposition 2.15 provides a fiber-wise linear abstract Jacobi structure on the L-adjoint bundle of $\operatorname{der} L$, which is J^1L. This is nothing but the abstract Jacobi structure determined by the canonical contact structure on J^1L (see Example 5.5).

Example 2.17. (cf. [17, Theorem 1, §3 Example 5]) Let $(M, L, \{-,-\})$ be an abstract Jacobi manifold. Since (J^1L, L) is a Jacobi algebroid, there is a fiber-wise linear abstract Jacobi structure on the L-adjoint bundle of J^1L which is $\operatorname{der} L$.

2.4. **Morphisms of abstract Jacobi manifolds.**

Definition 2.18. A *morphism of Jacobi manifolds*, or a *Jacobi map*,

$$(M, L, \{-,-\}) \to (M', L', \{-,-\}')$$

is a vector bundle morphism $\phi : L \to L'$, covering a smooth map $\underline{\phi} : M \to M'$, such that ϕ is an isomorphism on fibers, and $\phi^*\{\lambda, \mu\}' = \{\phi^*\lambda, \phi^*\mu\}$ for all $\lambda, \mu \in \Gamma(\overline{L'})$.

Definition 2.19. An *infinitesimal automorphism of a Jacobi manifold* $(M, L, \{-,-\})$, or a *Jacobi derivation*, is a derivation Δ of the line bundle L, equivalently, a section of the Atiyah algebroid $\operatorname{der} L$ of L, such that Δ generates a flow of automorphisms of $(M, L, \{-,-\})$ (see Appendix A). A *Jacobi vector field* is the (scalar-type) symbol of a Jacobi derivation.

Remark 2.20. Let Δ be a derivation of L, $\{\varphi_t\}$ be its flow, and let $\square$ be a first order multi-differential operator on L with k entries, i.e. $\square \in \operatorname{Der}^k L$. Since L is a line bundle, a derivation Δ of L is the same as a first order differential operator on $\Gamma(L)$, i.e. an element of $\operatorname{Der} L = \operatorname{Der}^1 L$. It is easy to see that (similarly as for vector fields)

$$\left.\frac{d}{dt}\right|_{t=0} (\varphi_t)_*\square = [\square, \Delta]^{SJ} \tag{2.14}$$

where we denoted by $\varphi_*\Box$ the *push forward* of $\Box$ along a line bundle isomorphism $\varphi : L \to L'$, defined by $(\varphi_*\Box)(\lambda'_1, \ldots, \lambda'_k) := (\varphi^{-1})^*(\Box(\varphi^*\lambda_1, \ldots, \varphi^*\lambda_k))$, for all $\lambda'_1, \ldots, \lambda'_k \in \Gamma(L')$ (see also Appendix A about pushing forward derivations along vector bundle morphisms). In particular, Δ is an infinitesimal automorphism of $(M, L, \{-,-\})$ iff $[J, \Delta]^{SJ} = 0$. Since

$$[J, \Delta]^{SJ}(\lambda, \mu) = \{\Delta\lambda, \mu\} + \{\lambda, \Delta\mu\} - \Delta\{\lambda, \mu\}, \tag{2.15}$$

we conclude that Δ is an infinitesimal automorphism of $(M, L, \{-,-\})$ iff

$$\Delta\{\lambda, \mu\} = \{\Delta\lambda, \mu\} + \{\lambda, \Delta\mu\} \tag{2.16}$$

for all $\lambda, \mu \in \Gamma(L)$. In other words Δ is a *derivation* of the Jacobi bracket.

Remark 2.21. More generally, let $\{\Delta_t\}$ be a one parameter family of derivations of L, generating the one parameter family of automorphisms $\{\varphi_t\}$, and let $\Box \in \mathrm{Der}^\bullet L$. Then

$$\frac{d}{dt}(\varphi_t)_*\Box = [(\varphi_t)_*\Box, \Delta_t]^{SJ}. \tag{2.17}$$

Remark 2.22. Definitions 2.18 and 2.19 encompass the notions of *conformal morphisms* and *infinitesimal conformal automorphisms* of standard Jacobi manifolds, respectively. In particular two standard Jacobi structures are *conformally equivalent* iff they are isomorphic as abstract Jacobi structures.

Example 2.23. Let $(M, L, \{-,-\})$ be an abstract Jacobi manifold. The values of all Hamiltonian vector fields generate a distribution $\mathcal{K} \subset TM$ which is, generically, non-constant-dimensional. Distribution $\mathcal{K}$ is called *the characteristic distribution* of $(M, L, \{-,-\})$. The Jacobi manifold $(M, L, \{-,-\})$ is said to be *transitive* if its characteristic distribution $\mathcal{K}$ is the whole tangent bundle TM. Identity (2.19) implies that $\mathcal{K}$ is involutive. Moreover, it is easy to see that $\mathcal{K}$ is constant-dimensional along the flow lines of a Hamiltonian vector field. Hence, it is completely integrable in the sense of Stefan and Sussmann. In particular, it defines a (singular) foliation $\mathcal{K}$ on M. Each leaf $\mathcal{C}$ of $\mathcal{K}$, is called a *characteristic leaf* and possesses a unique transitive abstract Jacobi structure defined by the restriction of the Jacobi bracket to $L|_{\mathcal{C}}$, see Corollary 3.3.2 for a precise expression. In other words, the inclusion $L|_{\mathcal{C}} \hookrightarrow L$ is a Jacobi map. Moreover, a transitive Jacobi manifold $(M, L, \{-,-\})$ is either an abstract l.c.s. manifold (if $\dim M$ is even) or a contact manifold (if $\dim M$ is odd) [22].

Let $(M, L, J = \{-,-\})$ be an abstract Jacobi manifold and $\lambda \in \Gamma(L)$. Note that

$$\Delta_\lambda = \{\lambda, -\} = -[J, \lambda]^{SJ}. \tag{2.18}$$

The Jacobi identity for the Jacobi bracket immediately implies that not only Δ_λ is a derivation of L, but even more, it is an infinitesimal automorphism of $(M, L, \{-,-\})$, called the *Hamiltonian derivation associated with the section* λ. Similarly, the scalar symbol X_λ of Δ_λ will be called the *Hamiltonian vector field associated with* λ. Clearly we have

$$[\Delta_\lambda, \Delta_\mu] = \Delta_{\{\lambda,\mu\}}, \quad \text{and} \quad [X_\lambda, X_\mu] = X_{\{\lambda,\mu\}}, \tag{2.19}$$

for all $\lambda, \mu \in \Gamma(L)$. Jacobi automorphisms $L \to L$ generated by Hamiltonian derivations will be called *Hamiltonian automorphisms*. Similarly, diffeomorphisms $M \to M$ generated by Hamiltonian vector fields will be called *Hamiltonian diffeomorphisms*.

Recall that an infinitesimal automorphism of $(M, L, \{-,-\})$ is, in particular, an element of $\mathrm{Der}\, L = \mathrm{Der}^1 L$. Hamiltonian derivations are interpreted as *inner infinitesimal automorphisms*. The following proposition provides a geometric interpretation of the first and the second Chevalley-Einlenberg cohomologies of $(M, L, \{-,-\})$.

Proposition 2.24.

(1) *A derivation $\Delta : \Gamma(L) \to \Gamma(L)$ of L is an infinitesimal automorphism of $(M, L, \{-,-\})$ iff $d_*^J \Delta = 0$, hence the set of outer infinitesimal automorphisms of $(M, L, \{-,-\})$ is $H^1_{CE}(M, L, J)$.*
(2) *An infinitesimal deformation $\bar{J}$ of J is a Jacobi deformation, if and only if $d_*^J \bar{J} = 0$, hence the set of infinitesimal Jacobi deformations of J modulo infinitesimal automorphisms of the bundle L is $H^2_{CE}(M, L, J)$.*

Proof.

(1) The first part of the assertion follows from Remark 2.20. Using this and taking into account (2.18), which interprets inner infinitesimal automorphisms as degree one co-boundaries, we immediately obtain the second part.
(2) The first part of the assertion follows from Lemma 2.11.(2). To prove the second part it suffices to show that the trivial infinitesimal deformation of J induced by an infinitesimal automorphism $Y \in \operatorname{Der} L$ is of the form $[J, Y]^{SN}$. Clearly (2.14) proves what we need and this completes the proof.

□

Remark 2.25. Proposition 2.24 generalizes a known interpretation of Lichnerowicz-Poisson cohomology, see e.g. [9, §2.1.2], and fits into deformation theory of Lie algebras, since any infinitesimal Jacobi deformation $\bar{J}$ of a Jacobi bracket J is also an infinitesimal deformation of the Lie algebra $(\Gamma(L), \{-,-\})$ and, therefore, $\bar{J}$ is closed in the Chevalley-Eilenberg complex (see also [31]).

3. Coisotropic submanifolds in abstract Jacobi manifolds and their invariants

In this section we propose some equivalent characterizations of coisotropic submanifolds S in an abstract Jacobi manifold $(M, L, \{-,-\})$ (Lemma 3.1, Corollary 3.3.(3)). Then we establish a one-to-one correspondence between coisotropic submanifolds and certain Jacobi subalgebroids of the Jacobi algebroid $(J^1 L, L)$ (Proposition 3.6). In particular, this yields a natural L_∞-isomorphism class of L_∞-algebras associated with each coisotropic submanifold (Proposition 3.12 and Proposition 3.18).

3.1. Differential geometry of a coisotropic submanifold.

Let $(M, L, J = \{-,-\})$ be an abstract Jacobi manifold, and let $x \in M$. A subspace $T \subset T_x M$ is said to be *coisotropic* (wrt the abstract Jacobi structure $(L, J = \{-,-\})$, if $\Lambda_J^\#(T^0 \otimes L) \subset T$, where $T^0 \subset T_x^* M$ denotes the annihilator of T (cf. [16, Definition 4.1]). Equivalently, $T^0 \otimes L$ is isotropic wrt the L-valued bi-linear form Λ_J.

A submanifold $S \subset M$ is called *coisotropic* (wrt the abstract Jacobi structure $(L, J = \{-,-\})$, if its tangent space $T_x S$ is coisotropic for all $x \in S$.

Lemma 3.1. *Let $S \subset M$ be a submanifold, and let Γ_S denote the set of sections λ of the Jacobi bundle such that $\lambda|_S = 0$. The following three conditions are equivalent:*

(1) *S is a coisotropic submanifold,*
(2) *Γ_S is a Lie subalgebra in $\Gamma(L)$,*
(3) *X_λ is tangent to S, for all $\lambda \in \Gamma_S$.*

Proof. Let $S \subset M$ be a submanifold. We may assume, without loss of generality, that L is trivial. Then $\Gamma_S = I(S) \cdot \Gamma(L)$, where $I(S)$ denotes the ideal in $C^\infty(M)$ consisting of functions that vanish on S. In particular, if λ is a generator of $\Gamma(L)$, then every section in Γ_S is of the form $f\lambda$ for some $f \in I(S)$. Now, let $f, g \in I(S)$. Putting $\mu = \lambda$ in (2.4) restricting to S, we find

$$\{f\lambda, g\lambda\}|_S = \langle \Lambda_J^\#(df \otimes \lambda), dg \rangle \lambda|_S.$$

This shows that (1) $\iff$ (2). The equivalence (2) $\iff$ (3) follows from the identity $X_\lambda(f)\mu|_S = \{\lambda, f\mu\}|_S$, for all $\lambda \in \Gamma_S$, $\mu \in \Gamma(L)$, and $f \in I(S)$. □

A version of Lemma 3.1 for Poisson manifold is well known [4, §2]. (See also [32, Lemma 13.3] for the symplectic case.)

Now, let $S \subset M$ be a coisotropic submanifold and let $T^0S \subset T^*M|_S$ be the annihilator of TS. The (generically non constant-dimensional) distribution $\mathcal{K}_S := \Lambda_J^\#(T^0S \otimes L) \subset TS$ on S is called the *characteristic distribution of* S.

Remark 3.2. In view of (2.5), $\mathcal{K}_S$ is generated by the (restrictions to S of) the Hamiltonian vector fields of the kind X_λ, with $\lambda \in \Gamma_S$. In particular, if $S = M$, then $\mathcal{K}_S = \mathcal{K}$ is the characteristic distribution of $(M, L, \{-,-\})$ as defined in Section 2.1.

From Lemma 3.1 we derive the following

Corollary 3.3.

(1) *(cf.* [4, §2]*) The characteristic distribution $\mathcal{K}_S$ of any coisotropic submanifold S is integrable (hence, it determines a foliation on S, called the* characteristic foliation *of S).*
(2) *(cf.* [22]*) Every characteristic leaf $\mathcal{C}$, i.e. any leaf of the characteristic distribution $\mathcal{K} = \mathcal{K}_M$ has an induced Jacobi structure $(L|_\mathcal{C}, \{-,-\}_\mathcal{C})$ well-determined by $\{\lambda|_\mathcal{C}, \mu|_\mathcal{C}\}_\mathcal{C} = \{\lambda, \mu\}|_C$, for all $\lambda, \mu \in \Gamma(L)$. The induced Jacobi structure is transitive.*
(3) *A submanifold $S \subset M$ is coisotropic, iff $TS \cap T\mathcal{C}$ is coisotropic in the tangent bundle $T\mathcal{C}$, for all characteristic leaves $\mathcal{C}$ intersecting S, where $\mathcal{C}$ is equipped with the induced Jacobi structure.*

Proof.
(1) Recall that, for $\lambda, \mu \in \Gamma_S$, X_λ, X_μ are in $\mathcal{K}_S$ and their commutator

$$[X_\lambda, X_\mu] = X_{\{\lambda,\mu\}} \tag{3.1}$$

is in $\mathcal{K}_S$ as well in view of Lemma 3.1.2. Since the X_λ's are the symbols of infinitesimal automorphisms $\{\lambda, -\}$ of J, the Stefan-Sussmann theorem applies and we obtain the required assertion.

(2) To prove the first assertion it suffices to show that if $\lambda|_\mathcal{C} = 0$, then $\{\lambda, \mu\}|_\mathcal{C} = 0$ for all $\mu \in \Gamma(L)$, i.e. the subspace $\Gamma_\mathcal{C}$ of sections of L vanishing on $\mathcal{C}$ is also an ideal of the Lie algebra $(\Gamma(L), \{-,-\})$. Similarly as in the proof of Lemma 3.1, we may assume, without loss of generality, that L is trivial. Thus, let λ be a generator of $\Gamma(L)$ and $f \in I(\mathcal{C})$, the ideal of functions vanishing on $\mathcal{C}$. Then, putting $\lambda = \mu$ in (2.4), and restricting to $\mathcal{C}$, one gets

$$\{f\lambda, g\lambda\}|_\mathcal{C} = (X_{f\lambda}(g) - gX_\lambda(f))\,\lambda|_C.$$

The claim follows taking into account skew-symmetry in f, g, and noting that $X_{g\lambda}$ is tangent to $\mathcal{C}$ for all g. Now, let $J_\mathcal{C} = \{-,-\}_\mathcal{C}$ be the induced Jacobi bracket on $\Gamma(L|_\mathcal{C})$. The transitivity of $(\mathcal{C}, L|_\mathcal{C}, \{-,-\}_\mathcal{C})$ is equivalent to the surjectivity of the map $\Lambda^\#_{J_\mathcal{C}}$, which follows from the identities

$$\langle \Lambda^\#_{J_\mathcal{C}}(d\,f|_\mathcal{C} \otimes \mu|_\mathcal{C}), d\,g|_\mathcal{C}\rangle \nu|_\mathcal{C} = \Lambda_{J_\mathcal{C}}(d\,f|_\mathcal{C} \otimes \mu|_\mathcal{C}, d\,g|_\mathcal{C} \otimes \nu|_\mathcal{C}) = \Lambda_J(df \otimes \mu, dg \otimes \nu)|_\mathcal{C},$$

for $f, g \in C^\infty(M)$, and $\mu, \nu \in \Gamma(L)$.

(3) For $V \subset T\mathcal{C}$ let V^{0c} denote the annihilator of V in $T^*\mathcal{C}$. Noting that $\mathcal{C}$ is coisotropic, the transitivity of $(\mathcal{C}, L|_\mathcal{C}, \{-,-\}_\mathcal{C})$ and (3.1) imply that the restriction to $\mathcal{C}$ of an Hamiltonian vector field on M is an Hamiltonian vector field. Denote by $i : \mathcal{C} \hookrightarrow M$ the inclusion. Then for any $\xi \in T^*M$, $\lambda \in L$ and for any submanifold S in M we have

$$\Lambda^\#_{J_\mathcal{C}}(i^*\xi \otimes \lambda) = \Lambda^\#_J(\xi \otimes \lambda) \quad \text{and} \quad (TS \cap T\mathcal{C})^{0c} \otimes L|_\mathcal{C} = i^*(T^0S) \otimes L|_\mathcal{C}. \tag{3.2}$$

Hence, if S is coisotropic, we have

$$\Lambda^\#_{J_\mathcal{C}}((TS \cap T\mathcal{C})^{0c} \otimes L|_\mathcal{C}) \subset TS \cap T\mathcal{C},$$

i.e. $TS \cap T\mathcal{C}$ is coisotropic in $T\mathcal{C}$. Conversely, assume that $TS \cap T\mathcal{C}$ is coisotropic in $T\mathcal{C}$, i.e. $\Lambda^{\#}_{J_{\mathcal{C}}}((TS \cap T\mathcal{C})^{0c} \otimes L|_{\mathcal{C}}) \subset TS \cap T\mathcal{C}$. Using (3.2) we obtain immediately

$$\Lambda^{\#}(T^0 S \otimes L\,|_{\mathcal{C}}) = \Lambda^{\#}_{J_{\mathcal{C}}}(i^*(T^0 S) \otimes L|_{\mathcal{C}}) = \Lambda^{\#}_{J_{\mathcal{C}}}((TS \cap T\mathcal{C})^{0c} \otimes L|_{\mathcal{C}}) \subset TS \cap T\mathcal{C}.$$

□

Example 3.4.

(1) Any coisotropic submanifold (in particular a Legendrian submanifold) in a contact manifold is a coisotropic submanifold wrt the associated abstract Jacobi structure (see Section 5.1 for details).

(2) Let S be a coisotropic submanifold of the abstract Jacobi manifold $(M, L, \{-,-\})$, and let $X \in \mathfrak{X}(M)$ be a Jacobi vector field such that $X_x \notin T_x S$, for all $x \in S$. Then T, the flowout of S along X_μ, is a coisotropic submanifold as well. Indeed, let $\{\phi_t\}$ be the flow of X. Clearly, whenever defined, $\phi_t(S)$ is a coisotropic submanifold, and the claim immediately follows from Lemma 3.1.

3.2. **Jacobi subalgebroid associated with a closed coisotropic submanifold.** We are interested in deformations of a *closed* coisotropic submanifold, so, from now on, we assume that S is a closed submanifold in a smooth manifold M. Let $A \to M$ be a Lie algebroid. Recall that a *subalgebroid of A over S* is a vector subbundle $B \to S$, with embeddings $j : B \hookrightarrow A$ and $\underline{j} : S \hookrightarrow M$, such that the anchor $\rho : A \to TM$ descends to a (necessarily unique) vector bundle morphism $\rho_B : B \to TS$, making diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "\rho_B"'] & A \arrow[d, "\rho"] \\ TS \arrow[r, "d\underline{j}"] & TM \end{tikzcd}$$

commutative and, moreover, for all $\beta, \beta' \in \Gamma(B)$ there exists a (necessarily unique) section $[\beta, \beta']_B \in \Gamma(B)$ such that whenever $\alpha, \alpha' \in \Gamma(A)$ are j-related to β, β' (i.e. $j \circ \beta = \alpha \circ \underline{j}$, in other words $\alpha|_S = \beta$, and similarly for β', α') then $[\alpha, \alpha']_A$ is j-related to $[\beta, \beta']_B$. In this case B, equipped with ρ_B and $[-,-]_B$, is a Lie algebroid itself. One can also give a notion of *Jacobi subalgebroid* as follows.

Let (A, L) be a Jacobi algebroid with representation ∇.

Definition 3.5. A *Jacobi subalgebroid of* (A, L) *over* S is a pair (B, ℓ), where $B \to S$ is a Lie subalgebroid of A over $S \subset M$, and $\ell := L|_S \to S$ is the pull-back line subbundle of L, such that ∇ descends to a (necessarily unique) vector bundle morphism $\nabla|_\ell$ making diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "\nabla|_\ell"'] & A \arrow[d, "\nabla"] \\ \operatorname{der} \ell \arrow[r, "\operatorname{der} j_\ell"] & \operatorname{der} L \end{tikzcd}$$

commutative (see Appendix A for the definition of morphism $\operatorname{der} j_\ell$).

If (B, ℓ) is a Jacobi subalgebroid, then the restriction $\nabla|_\ell$ is a representation so that (B, ℓ), equipped with $\nabla|_\ell$, is a Jacobi algebroid itself.

Now, let $(M, L, J = \{-,-\})$ be a Jacobi manifold, and let S be a submanifold. In what follows, we denote by

- $\ell := L|_S$ the restricted line bundle,
- $NS := TM|_S / TS$ the normal bundle of S in M,
- $N^*S := (NS)^* \simeq T^0 S \subset T^*M$ the conormal bundle of S in M,

- $N_\ell S := NS \otimes \ell^*$, and by
- $N_\ell{}^* S := (N_\ell S)^* = N^* S \otimes \ell$ the ℓ-adjoint bundle of NS.

Vector bundle $N_\ell{}^* S$ will be also regarded as a vector subbundle of $(J^1 L)|_S$ via the vector bundle embedding

$$N_\ell{}^* S \longrightarrow (T^* M \otimes L)|_S \xrightarrow{\gamma} J^1 L|_S,$$

where γ is the co-symbol. If $\lambda \in \Gamma(L)$, we have that $(j^1\lambda)|_S \in \Gamma(N_\ell{}^* S)$ if and only if $\lambda|_S = 0$, i.e. $\lambda \in \Gamma_S$.

The following Proposition establishes a one-to-one correspondence between coisotropic submanifolds and certain Lie subalgebroids of $J^1 L$.

Proposition 3.6. *(cf.* [19, Proposition 5.2]*) The submanifold $S \subset M$ is coisotropic iff $(N_\ell{}^* S, \ell)$ is a Jacobi subalgebroid of $(J^1 L, L)$.*

Proof. Let $S \subset M$ be a coisotropic submanifold. We want to show that $N_\ell{}^* S$ is a Jacobi subalgebroid of $J^1 L$. We propose a proof which is shorter than the one in [19]. Since S is coisotropic, we have

$$\rho_J(N_\ell{}^* S) \subset TS, \tag{3.3}$$

and similarly

$$\nabla^J(N_\ell{}^* S) \subset \operatorname{der} \ell. \tag{3.4}$$

Next we shall show that for any $\alpha, \beta \in \Gamma(J^1 L)$ such that $\alpha|_S, \beta|_S \in \Gamma(N_\ell{}^* S)$ we have

$$[\alpha, \beta]_J|_S \in \Gamma(N_\ell{}^* S). \tag{3.5}$$

First we note that if $\alpha|_S \in \Gamma(N_\ell{}^* S)$ then $\alpha = \sum f j^1\lambda$ for some $\lambda \in \Gamma_S$. Using the Leibniz properties of the Jacobi bracket we can restrict to the case $\alpha, \beta \in j^1\Gamma_S$. The latter case can be handled taking into account (2.8) and Lemma 3.1. Moreover, using (2.11), we easily check that

$$[\alpha, \beta]_J|_S = 0 \text{ if } \alpha|_S = 0 \text{ and } \beta|_S \in \Gamma(N_\ell{}^* S).$$

This completes the "only if part" of the proof.

To prove the "if part" it suffices to note that condition (3.3), regarded as a condition on the image of the anchor map of the Lie subalgebroid $N_\ell{}^* S$, implies, in view of (2.10), that S is a coisotropic submanifold. □

Remark 3.7. Different versions of Proposition 3.6 were proved for the Poisson case [40, Proposition 3.1.3], [5, Proposition 5.1], [29, Theorem 10.4.2].

3.3. L_∞-algebra associated with a coisotropic submanifold.

Let M be as above, and let $S \subset M$ be a closed submanifold. Let

$$P_0 : \Gamma(J_1 L) \longrightarrow \Gamma(N_\ell S)$$

be the projection adjoint to the embedding $\gamma : N_\ell{}^* S \hookrightarrow J^1 L$, i.e. $\langle P_0(\Delta)_x, \alpha_x \rangle = \langle \Delta_x, \gamma(\alpha_x) \rangle$, where $\Delta \in \Gamma(J_1 L)$, $\alpha \in \Gamma(N_\ell{}^* S)$, and $x \in M$. Tensorizing by $\Gamma(L)$ we also get a projection

$$P : \operatorname{Der} L \longrightarrow \Gamma(NS).$$

It is not hard to see that P coincides with the

$$\operatorname{Der} L \xrightarrow{\sigma} \mathfrak{X}(M) \longrightarrow \Gamma(TM|_S) \longrightarrow \Gamma(NS), \tag{3.6}$$

where the second arrow is the restriction, and the last arrow is the canonical projection. Projection P_0 extends uniquely to a (degree zero) morphism of graded algebras $\Gamma(\wedge^\bullet J_1 L) \to \Gamma(\wedge^\bullet N_\ell S)$ which we denote again by P_0. Similarly, P extends uniquely to a (degree zero) morphism of graded modules $(\operatorname{Der}^\bullet L)[1] \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ which we denote again by P. As in the Poisson case (see, e.g., [6]), projection $P : (\operatorname{Der}^\bullet L)[1] \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ allows to formulate a further characterization of coisotropic submanifolds.

Proposition 3.8. *Submanifold S is coisotropic iff $P(J) = 0$.*

Remark 3.9. Let $S \subset M$ be any submanifold, then $P(J)$ does only depend on the bi-symbol Λ_J of J. To see this, note, first of all, that the symbol $\sigma : \operatorname{Der} L \to \mathfrak{X}(M)$ induces an obvious projection $(\operatorname{Der}^\bullet L) \to \Gamma(\wedge^\bullet(TM \otimes L^*) \otimes L)$. Moreover, in view of its very definition, $P : (\operatorname{Der}^\bullet L)[1] \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ descends to an obvious projection

$$\Gamma(\wedge^\bullet(TM \otimes L^*) \otimes L)[1] \longrightarrow \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1],$$

which, abusing the notation, we denote again by P. Now, recall that $\Lambda_J \in \Gamma(\wedge^2(TM \otimes L^*) \otimes L)$. It immediately follows from the definition of P that, actually,

$$P(J) = P(\Lambda_J).$$

In particular S is coisotropic iff $P(\Lambda_J) = 0$.

From now on we assume that S is coisotropic. In this case, the Jacobi algebroid structure on $(N_\ell{}^* S, \ell)$ (Proposition 3.6) turns the graded space $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)$ into the de Rham complex of $N_\ell{}^* S$, with values in ℓ. To express the differential $d_{N_\ell{}^* S, \ell}$ in terms of the differential $d_*^J = [J, -]^{SJ}$ on $\operatorname{Der}^\bullet P$ it suffices to find a right inverse $I : \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] \to (\operatorname{Der}^\bullet L)[1]$ of P. However, there is no natural way to do this unless further structure is available. In what follows we use a *fat tubular neighborhood* as an additional structure. Before giving a definition, recall that a *tubular neighborhood of S* is an embedding of the normal bundle NS into M which identifies the zero section $\mathbf{0}$ of $NS \to S$ with the inclusion $i : S \hookrightarrow M$. Denote by $\pi : NS \to S$ the projection and consider the pull-back line bundle $L_{NS} := \pi^* \ell = NS \times_S \ell$ over NS. Moreover, let $i_L : \ell \hookrightarrow L$ be the inclusion.

Definition 3.10. A *fat tubular neighborhood of $\ell \to S$ in $L \to M$ over a tubular neighborhood* $\underline{\tau} : NS \hookrightarrow M$ is an embedding $\tau : L_{NS} \hookrightarrow L$ of vector bundles over $\underline{\tau} : NS \hookrightarrow M$ such that diagram

$$\begin{tikzcd}
L_{NS} = \pi^*\ell \arrow[rr, "\tau"] \arrow[dd] & & L \arrow[dd] \\
& \ell \arrow[ul] \arrow[ur, "i_L"'] \arrow[dd] & \\
NS \arrow[rr, "\underline{\tau}"] \arrow[dr, shift left, "\pi"] & & M \\
& S \arrow[ul, shift left, "\mathbf{0}"] \arrow[ur, "i"'] &
\end{tikzcd}$$

commutes.

In particular, it follows from the above definition that τ is an isomorphism when restricted to fibers. A fat tubular neighborhood can be understood as a "tubular neighborhood in the category of line bundles". In the following we regard S as a submanifold of NS identifying it with the image of the zero section $\mathbf{0} : S \to NS$.

Lemma 3.11. *There exist fat tubular neighborhoods of ℓ in L.*

Proof. Since fibers of $NS \to S$ are contractible, for every vector bundle $V \to NS$ over NS there is a, generically non-canonical, isomorphism of line bundle $NS \times_S V|_S \simeq V$ over the identity of NS. Now, let $\underline{\tau} : NS \hookrightarrow M$ be a tubular neighborhood of S. According to the above remark, the pull-back bundle $\underline{\tau}^* L \to NS$ is (non-canonically) isomorphic to L_{NS}. Pick any isomorphism $\phi : L_{NS} \to \underline{\tau}^* L$. Then the composition

$$L_{NS} \xrightarrow{\phi} \underline{\tau}^* L \longrightarrow L,$$

where the second arrow is the canonical map, is a fat tubular neighborhood of ℓ over $\underline{\tau}$. $\square$

Choose once for all a fat tubular neighborhood $\tau : L_{NS} \hookrightarrow L$ of ℓ over a tubular neighborhood $\underline{\tau} : NS \hookrightarrow M$ of S. We identify NS with the open neighborhood $\underline{\tau}(NS)$ of S in M. Similarly, we identify L_{NS} with $L|_{\underline{\tau}(NS)}$. In particular NS inherits from $\underline{\tau}(NS)$ a Jacobi structure with Jacobi bundle given by L_{NS}. Abusing the notation we denote by J again the Jacobi bracket on $\Gamma(L_{NS})$. Moreover, in view of Proposition 3.8, there is a projection $P : (\mathrm{Der}^\bullet L_{NS})[1] \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ such that $P(J) = 0$.

Now, regard the vertical bundle $V(NS) := \ker d\pi$ as a Lie algebroid and note preliminarily that

(1) There is a natural splitting $T(NS)|_S = TS \oplus NS$: projection $T(NS)|_S \to TS$ is $d\pi$, while projection $T(NS)|_S \to NS$ is the natural one. In particular, sections of NS can be understood as vector fields on NS along the submanifold S and vertical wrt π.
(2) Since $\pi : NS \to S$ is a vector bundle, the vertical bundle $V(NS)$ identifies canonically with the induced bundle $\pi^* NS \to NS$. In particular, there is an embedding $\pi^* : \Gamma(NS) \hookrightarrow \mathfrak{X}(NS)$ that takes a section ν of NS to the unique vertical vector field $\pi^*\nu$ on NS, which is constant along the fibers of π, and agrees with ν on S.
(3) Since $L_{NS} = \pi^*\ell = NS \times_S \ell$, there is a natural flat connection $\mathbb{D}$ in L, along the Lie algebroid $V(NS)$, uniquely determined by $\mathbb{D}_X \pi^*\lambda = 0$, for all vertical vector fields X on NS, and all fiber-wise constant sections $\pi^*\lambda$ of L_{NS}, $\lambda \in \Gamma(\ell)$.

With these preliminary remarks we are finally ready to define a right inverse $I : \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] \to (\mathrm{Der}^\bullet L_{NS})[1]$ of $P : (\mathrm{Der}^\bullet L_{NS})[1] \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$. First of all, let

$$I : \Gamma(NS) \hookrightarrow \mathrm{Der}\, L_{NS}$$

be the embedding given by $I(\nu) := \mathbb{D}_{\pi^*\nu}$. Tensorizing it by $\Gamma(L_{NS}^*)$ we also get an embedding

$$I_0 : \Gamma(N_\ell S) \hookrightarrow \Gamma(J_1 L_{NS}).$$

Inclusion I_0 extends uniquely to a (degree zero) morphism of graded algebras $\Gamma(\wedge^\bullet N_\ell S) \to \Gamma(\wedge^\bullet J_1 L_{NS})$ which we denote again by I_0. Similarly, I extends uniquely to a (degree zero) morphism of graded modules $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] \to (\mathrm{Der}^\bullet L_{NS})[1]$ which we denote again by I. It is straightforward to check that

$$P_0 \circ I_0 = \mathrm{id} \quad \text{and} \quad P \circ I = \mathrm{id}\,.$$

Using I and the explicit expression for the Schouten-Jacobi bracket, one can check that

$$d_{N_\ell{}^* S,\ell}\alpha = (P \circ d_*^J \circ I)(\alpha) = P[J, I(\alpha)]^{SJ} \tag{3.7}$$

for all $\alpha \in \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$.

The rightmost hand side of (3.7) reminds us of the Voronov construction of L_∞-algebras via derived brackets. We refer the reader to [39] for details. Our conventions about L_∞-algebras are the same as those in [39]. In particular, multi-brackets in L_∞-algebras in this paper will always be (graded) symmetric. Now, using the derived bracket construction, we are going to define an L_∞-algebra structure $\{\mathfrak{m}_k\}$ on $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ whose first (unary) bracket $\mathfrak{m}_1$ coincides with the differential $d_{N_\ell{}^* S,\ell}$. The following Proposition is an analogue of Lemma 2.2 in [10], see also [4] and [32, Appendix].

Proposition 3.12. *Let $I : \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] \hookrightarrow (\mathrm{Der}^\bullet L_{NS})[1]$ be the embedding defined above. There is an L_∞-algebra structure on $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ given by the following family of graded multi-linear maps $\mathfrak{m}_k : \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]^{\otimes k} \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$*

$$\mathfrak{m}_k(\xi_1, \cdots, \xi_k) := P[\cdots[[J, I(\xi_1)]^{SJ}, I(\xi_2)]^{SJ} \cdots, I(\xi_l)]^{SJ}. \tag{3.8}$$

Proof. First, we observe that the image of I is an abelian subalgebra of the graded Lie algebra $((\mathrm{Der}^\bullet L_{NS})[1], [-,-]^{SJ})$, or equivalently, the Schouten-Jacobi bracket $[I(\alpha), I(\beta)]^{SJ}$ vanishes for any two sections $\alpha, \beta \in \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$. The last assertion is a consequence of the (generalized) Leibniz property (B.1) for the Schouten-Jacobi bracket, and the fact that if α and β are sections of NS then derivations $I(\alpha)$ and $I(\beta)$ commute.

Next, we will show that the kernel of the projection P is a graded Lie subalgebra of $(\mathrm{Der}^\bullet L_{NS})[1]$. Clearly, $\ker P$ is the $\Gamma(\wedge^\bullet J_1 L_{NS})$-submodule generated by those sections of $\operatorname{der} L_{NS}$ whose symbol is tangent to S. Since such sections are preserved by the Schouten-Jacobi bracket, it is easy to check that $\ker P$ is also preserved, using the generalized Leibniz property (B.1) again.

Finally, recall that $J \in \ker P$. It follows that $((\mathrm{Der}^\bullet L_{NS})[1], \operatorname{im} I, P, J)$ are V-data [39, Theorem 1, Corollary 1]. See also [10, §1.2, Lemma 2.2] and [6] where the terminology *V-data* has been introduced for the first time. This completes the proof. □

Remark 3.13.

(1) In view of (3.7), differential $\mathfrak{m}_1$ coincides with the Jacobi algebroid differential $d_{N_\ell{}^*S,\ell}$.
(2) If (M,ω) is a l.c.s. manifold and S is a coisotropic submanifold in M, then $\mathfrak{m}_1$ can be identified, via $\Lambda^\#$, with a deformation of the foliation differential of the characteristic foliation of S [25].

3.4. **Coordinate formulas for the multi-brackets.** In this subsection we propose some more efficient formulas for the multi-brackets in the L_∞-algebra of a coisotropic submanifold. Let $(M, L, J = \{-,-\})$ be a Jacobi manifold and let $S \subset M$ be a coisotropic submanifold. Moreover, as in the previous subsection, we equip S with a fat tubular neighborhood $\tau : L_{NS} \hookrightarrow L$.

Remark 3.14. By their very definition, the $\mathfrak{m}_k$'s satisfy the following properties:

(a) $\mathfrak{m}_k$ is a graded $\mathbb{R}$-linear map of degree one,
(b) $\mathfrak{m}_k$ is a first order differential operator with scalar-type symbol in each entry separately.

Because of (b) the $\mathfrak{m}_k$'s are completely determined by their action on all $\lambda \in \Gamma(\ell) = \Gamma(\wedge^0 N_\ell S \otimes \ell)$, and on all $s \in \Gamma(NS) = \Gamma(\wedge^1 N_\ell S \otimes \ell)$. Moreover (a) implies that, if $\xi_1, \ldots, \xi_k \in \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ have non-positive degrees, then $\mathfrak{m}_k(\xi_1, \ldots, \xi_k) = 0$ whenever more than two arguments have degree -1.

From now on, in this section, we identify

- a section $\lambda \in \Gamma(\ell)$, with its pull-back $\pi^*\lambda \in \Gamma(L_{NS})$,
- a section $s \in \Gamma(NS)$, with the corresponding vertical vector field $\pi^* s \in \Gamma(\pi^* NS) \simeq \Gamma(V(NS))$,
- a section $\varphi \in \Gamma(N_\ell{}^*S)$ of the ℓ-adjoint bundle $N_\ell{}^*S = N^*S \otimes \ell$ with the corresponding fiber-wise linear section of L_{NS}.

Moreover, we denote by $\langle -,- \rangle : NS \otimes N_\ell{}^*S \to \ell$ the obvious (ℓ-twisted) duality pairing.

Proposition 3.15. *The multi-bracket $\mathfrak{m}_{k+1}$ is completely determined by*

$$\mathfrak{m}_{k+1}(s_1, \ldots, s_{k-1}, \lambda, \nu) = (-)^k \mathbb{D}_{s_1} \cdots \mathbb{D}_{s_{k-1}} \{\lambda, \nu\}|_S \tag{3.9}$$

$$\langle \mathfrak{m}_{k+1}(s_1, \ldots, s_k, \lambda), \varphi \rangle = -(-)^k \left(\mathbb{D}_{s_1} \cdots \mathbb{D}_{s_k} \{\lambda, \varphi\} - \sum_i \mathbb{D}_{s_1} \cdots \widehat{\mathbb{D}_{s_i}} \cdots \mathbb{D}_{s_k} \{\lambda, \langle s_i, \varphi \rangle\} \right) \Bigg|_S \tag{3.10}$$

$$\begin{aligned} \langle \mathfrak{m}_{k+1}(s_1, \ldots, s_{k+1}), \varphi \otimes \psi \rangle = -(-)^k \Bigg(& \mathbb{D}_{s_1} \cdots \mathbb{D}_{s_{k+1}} \{\varphi, \psi\} \\ & + \sum_{i<j} \mathbb{D}_{s_1} \cdots \widehat{\mathbb{D}_{s_i}} \cdots \widehat{\mathbb{D}_{s_j}} \cdots \mathbb{D}_{s_{k+1}} \left(\{\langle s_i, \varphi \rangle, \langle s_j, \psi \rangle\} + \{\langle s_j, \varphi \rangle, \langle s_i, \psi \rangle\} \right) \\ & - \sum_i \mathbb{D}_{s_1} \cdots \widehat{\mathbb{D}_{s_i}} \cdots \mathbb{D}_{s_{k+1}} \left(\{\langle s_i, \varphi \rangle, \psi\} + \{\varphi, \langle s_i, \psi \rangle\} \right) \Bigg) \Bigg|_S, \end{aligned} \tag{3.11}$$

where $\lambda, \nu \in \Gamma(\ell)$, $s_1, \ldots, s_{k+1} \in \Gamma(NS)$, $\varphi, \psi \in \Gamma(N_\ell{}^*S)$, and a hat "$\widehat{-}$" denotes omission.

Proof. Equation (3.9) immediately follows from (3.8), (2.12), and the easy remark that $[\Delta, \lambda]^{SJ} = \Delta(\lambda)$ for all $\Delta \in \operatorname{Der} L_{NS} = \operatorname{Der}^1 L_{NS}$, and $\lambda \in \Gamma(L_{NS}) = \operatorname{Der}^0 L_{NS}$. Equation (3.10) follows from (3.8), (2.15), and the obvious remark that $\langle s, \varphi \rangle = \mathbb{D}_s \varphi$, hence $\mathbb{D}_{s_1} \mathbb{D}_{s_2} \varphi = 0$, for all $s, s_1, s_2 \in \Gamma(NS)$, and $\varphi \in \Gamma(N_\ell{}^* S)$. Equation (3.11) can be proved in a similar way. □

Let z^α be local coordinates on M, and let μ be a local generator of $\Gamma(L)$. Define local sections μ^* and ∇_α of $J_1 L$ by putting

$$\mu^*(f\mu) = f, \quad \nabla_\alpha(f\mu) = \partial_\alpha f,$$

where $f \in C^\infty(M)$, and $\partial_\alpha = \partial / \partial z^\alpha$. Then $\Gamma(\wedge^\bullet J_1 L)$ is locally generated, as a $C^\infty(M)$-module, by

$$\nabla_{\alpha_1} \wedge \ldots \wedge \nabla_{\alpha_k}, \qquad \nabla_{\alpha_1} \wedge \ldots \wedge \nabla_{\alpha_{k-1}} \wedge \mu^*, \quad k > 0,$$

with $\alpha_1 < \ldots < \alpha_k$. In particular, any $\Delta \in \Gamma(\wedge^\bullet J_1 L)$ is locally expressed as

$$\Delta = X^{\alpha_1 \ldots \alpha_k} \nabla_{\alpha_1} \wedge \ldots \wedge \nabla_{\alpha_k} + g^{\alpha_1 \ldots \alpha_{k-1}} \nabla_{\alpha_1} \wedge \ldots \wedge \nabla_{\alpha_{k-1}} \wedge \mu^*,$$

where $X^{\alpha_1 \ldots \alpha_k}, g^{\alpha_1 \ldots \alpha_{k-1}} \in C^\infty(M)$. Here and in what follows, we adopt the Einstein summation convention over pair of upper-lower repeated indexes. Hence, $(\operatorname{Der}^\bullet L)[1]$ is locally generated, as a $C^\infty(M)$-module, by

$$\nabla_{\alpha_1} \wedge \ldots \wedge \nabla_{\alpha_k} \otimes \mu, \qquad \nabla_{\alpha_1} \wedge \ldots \wedge \nabla_{\alpha_{k-1}} \wedge \mathrm{id}, \quad k > 0,$$

with $\alpha_1 < \ldots < \alpha_k$, and any $\square \in (\operatorname{Der}^\bullet L)[1]$ is locally expressed as

$$\square = X^{\alpha_1 \ldots \alpha_k} \nabla_{\alpha_1} \wedge \ldots \wedge \nabla_{\alpha_k} \otimes \mu + g^{\alpha_1 \ldots \alpha_{k-1}} \nabla_{\alpha_1} \wedge \ldots \wedge \nabla_{\alpha_{k-1}} \wedge \mathrm{id}\,.$$

Remark 3.16. Let $J \in \operatorname{Der}^2 L$. Locally,

$$J = J^{\alpha\beta} \nabla_\alpha \wedge \nabla_\beta \otimes \mu + J^\alpha \nabla_\alpha \wedge \mathrm{id}, \tag{3.12}$$

for some local functions $J^{\alpha\beta}, J^\alpha$.

Now, identify L_{NS} with its image in L under τ and assume that

- Coordinates z^α are fibered, i.e. $z^\alpha = (x^i, y^a)$, with x^i coordinates on S, and y^a linear coordinates along the fibers of $\pi : NS \to S$,
- local generator μ is fiber-wise constant so that, locally, $\Gamma(\ell) \subset \Gamma(L_{NS})$ consists exactly of sections λ which are vertical, i.e. $\nabla_a \lambda = 0$.

In particular, local expression (3.12) for J expands as

$$J = \left(J^{ab} \nabla_a \wedge \nabla_b + 2 J^{ai} \nabla_a \wedge \nabla_i + J^{ij} \nabla_i \wedge \nabla_j \right) \otimes \mu + \left(J^a \nabla_a + J^i \nabla_i \right) \wedge \mathrm{id}\,. \tag{3.13}$$

We have the following

Corollary 3.17. *Locally, the multi-bracket* $\mathfrak{m}_{k+1}$ *is uniquely determined by*

$$\begin{aligned}
&\mathfrak{m}_{k+1}\left(\partial_{a_1}, \ldots, \partial_{a_{k-1}}, f\mu, g\mu\right) = (-)^k \partial_{a_1} \cdots \partial_{a_{k-1}} \left[2J^{ij} \partial_i f \partial_i g + J^i (f \partial_i g - g \partial_i f) \right]\Big|_S \mu, \\
&\mathfrak{m}_{k+1}\left(\partial_{a_1}, \ldots, \partial_{a_k}, f\mu\right) = (-)^k \partial_{a_1} \cdots \partial_{a_k} \left(2J^{ai} \partial_i f + J^a f \right)\Big|_S \partial_a, \\
&\mathfrak{m}_{k+1}\left(\partial_{a_1}, \ldots, \partial_{a_{k+1}}\right) = -(-)^k\, \partial_{a_1} \cdots \partial_{a_{k+1}} J^{ab}\Big|_S \delta_a \wedge \delta_b \otimes \mu,
\end{aligned}$$

where $f, g \in C^\infty(S)$, *and* $\delta_a := \partial_a \otimes \mu^*$.

3.5. **Independence of the tubular embedding.** Now we show that, as already in the symplectic [32, Appendix], the Poisson [6], and the l.c.s. [25, Theorem 9.5] cases, the L_∞-algebra in Proposition 3.12 does not really depend on the choice of a fat tubular neighborhood, in the sense clarified by Proposition 3.18 below. As a consequence, its L_∞-isomorphism class is an *invariant* of the coisotropic submanifold.

Proposition 3.18. *Let S be a coisotropic submanifold of the Jacobi manifold $(M, L, J = \{-,-\})$. Then the L_∞-algebra structures on $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)$ associated to different choices of the fat tubular neighborhood $L_{NS} \hookrightarrow L$ of ℓ in L are L_∞-isomorphic.*

The proof is just an adaptation of the one given by Cattaneo and Schätz in the Poisson setting (see Subsections 4.1 and 4.2 of [6]) and it is based on Theorem 3.2 of [6] and the fact that *any two fat tubular neighborhoods are isotopic* (in the sense of Lemma 3.20 below). Before proving Proposition 3.18, let us recall Cattaneo-Schätz Theorem. We will present a "minimal version" of it, adapted to our purposes. The main ingredients are the following.

We work in a category of real topological vector spaces. Let $(\mathfrak{h}, \mathfrak{a}, P, \Delta_0)$ and $(\mathfrak{h}, \mathfrak{a}, P, \Delta_1)$ be V-data [10]. We identify $\mathfrak{a}$ with the target space of P. Note that $(\mathfrak{h}, \mathfrak{a}, P, \Delta_0)$ and $(\mathfrak{h}, \mathfrak{a}, P, \Delta_1)$ differ for the last entry only. Voronov construction associates L_∞-algebras to $(\mathfrak{h}, \mathfrak{a}, P, \Delta_0)$ and $(\mathfrak{h}, \mathfrak{a}, P, \Delta_1)$. Denote them $\mathfrak{a}_0$ and $\mathfrak{a}_1$ respectively. Cattaneo and Schätz main idea is proving that when

- Δ_0 and Δ_1 are gauge equivalent elements of the graded Lie algebra $\mathfrak{h}$, and
- they are intertwined by a gauge transformation preserving $\ker P$,

then $\mathfrak{a}_0$ and $\mathfrak{a}_1$ are L_∞-isomorphic. Specifically, Δ_0 and Δ_1 are *gauge equivalent* if they are interpolated by a smooth family $\{\Delta_t\}_{t\in[0,1]}$ of elements $\Delta_t \in \mathfrak{h}$, and there exists a smooth family $\{\xi_t\}_{t\in[0,1]}$ of degree zero elements $\xi_t \in \mathfrak{h}$ such that the following evolutionary differential equation is satisfied:

$$\frac{d}{dt}\Delta_t = [\xi_t, \Delta_t]. \tag{3.14}$$

One usually assumes that the family $\{\xi_t\}_{t\in[0,1]}$ integrates to a family $\{\phi_t\}_{t\in[0,1]}$ of automorphisms $\phi_t : \mathfrak{h} \to \mathfrak{h}$ of the Lie algebra $\mathfrak{h}$, i.e. $\{\phi_t\}_{t\in[0,1]}$ is a solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\phi_t(-) = [\phi_t(-), \xi_t] \\ \phi_0 = \mathrm{id} \end{cases}. \tag{3.15}$$

Finally we say that Δ_0 and Δ_1 are intertwined by a gauge transformation preserving $\ker P$ if family $\{\xi_t\}_{t\in[0,1]}$ above satisfies the following conditions:

(1) the only solution $\{a_t\}_{t\in[0,1]}$, where $a_t \in \mathfrak{a}$, of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}a_t = P[a_t, \xi_t] \\ a_0 = 0 \end{cases} \tag{3.16}$$

is the trivial one: $a_t = 0$ for all $t \in [0,1]$,

(2) $[\xi_t, \ker P] \subset \ker P$ for all $t \in [0,1]$.

Theorem 3.19 (Cattaneo & Schätz, cf. [6, Theorem 3.2])**.** *Let $(\mathfrak{h}, \mathfrak{a}, P, \Delta_0)$ and $(\mathfrak{h}, \mathfrak{a}, P, \Delta_1)$ be V-data, and let $\mathfrak{a}_0$ and $\mathfrak{a}_1$ be the associated L_∞-algebras. If Δ_0 and Δ_1 are gauge equivalent and they are intertwined by a gauge transformation preserving* $\ker P$*, then $\mathfrak{a}_0$ and $\mathfrak{a}_1$ are L_∞-isomorphic.*

The last ingredient needed to prove Proposition 3.18 is provided by the following

Lemma 3.20. *Any two fat tubular neighborhoods τ_0 and τ_1 of S are* isotopic*, i.e. there is a smooth one parameter family of fat tubular neighborhoods $\mathcal{T}_t$ of ℓ in L, and an automorphism $\psi : L_{NS} \to L_{NS}$*

of L_{NS} covering an automorphism $\underline{\psi} : NS \to NS$ of NS over the identity, such that $\mathcal{T}_0 = \tau_0$, and $\mathcal{T}_1 = \tau_1 \circ \psi$.

Proof. In view of the tubular neighborhood Theorem [15, Theorem 5.3], there is a smooth one parameter family of tubular neighborhoods $\underline{\mathcal{T}}_t : NS \hookrightarrow M$ of S in M, and an automorphism $\underline{\psi} : NS \to NS$ over the identity such that $\underline{\mathcal{T}}_0 = \underline{\tau}_0$, and $\underline{\mathcal{T}}_1 = \underline{\tau}_1 \circ \underline{\psi}$. Denote by $\underline{\mathcal{T}} : NS \times [0,1] \to M$ the map defined by $\underline{\mathcal{T}}(\nu, t) = \underline{\mathcal{T}}_t(\nu)$ and consider the line bundle

$$p : L^*_{NS} \otimes_{NS} \underline{\mathcal{T}}^* L \longrightarrow NS \times [0,1].$$

Note that

(1) fibers of $NS \times [0,1]$ over $S \times [0,1]$ are contractible,
(2) $L^*_{NS} \otimes_{NS} \underline{\mathcal{T}}^* L$ reduces to $\operatorname{End} \ell \times [0,1] = \mathbb{R}_{S\times[0,1]}$ over $S \times [0,1]$.

It follows from (1) and (2) that $L^*_{NS} \otimes_{NS} \underline{\mathcal{T}}^* L$ is isomorphic to the pull-back over $NS \times [0,1]$ of the trivial line bundle $\mathbb{R}_{S\times[0,1]}$ over $S \times [0,1]$. In particular, p is a trivial bundle. Moreover, p admits a nowhere zero section υ defined on $(S \times [0,1]) \cup (NS \times \{0,1\})$ and given by id_ℓ on $S \times [0,1]$, by $\mathcal{T}_0$ on $NS \times \{0\}$ and by $\mathcal{T}_1$ on $NS \times \{1\}$. By triviality, υ can be extended to a nowhere zero section Υ on the whole $NS \times [0,1]$. Section Υ is the same as a one parameter family of vector bundle isomorphisms $\Upsilon_t : L_{NS} \to \underline{\mathcal{T}}_t^* L$ over the identity of NS. Denote by $\mathcal{T}_t : L_{NS} \to L$ the composition

$$L_{NS} \xrightarrow{\Upsilon_t} \underline{\mathcal{T}}_t^* L \hookrightarrow L,$$

where the second arrow is the natural inclusion. By construction, the $\mathcal{T}_t$'s are line bundle embedding covering the $\underline{\mathcal{T}}_t$'s. Finally, there exists a unique automorphism $\psi : L_{NS} \to L_{NS}$ over $\underline{\psi}$ such that $\mathcal{T}_1 = \tau_1 \circ \psi$. We conclude that the $\mathcal{T}_t$'s and ψ possess all the required properties. □

Proof of Proposition 3.18. Let $\tau_0, \tau_1 : L_{NS} \hookrightarrow L$ be fat tubular neighborhoods over tubular neighborhoods $\underline{\tau}_0, \underline{\tau}_1 : NS \hookrightarrow M$. Denote by J_0 and J_1 the Jacobi brackets induced on $\Gamma(L_{NS})$ by τ_0 and τ_1 respectively, i.e. $J_0 = (\tau_0^{-1})_* J$, and $J_1 = (\tau_1^{-1})_* J$ (see Remark 2.20 about pushing forward a multi-differential operator along a line bundle isomorphism). In view of Lemma 3.20 it is enough to consider the following two cases:

Case I: *$\tau_1 = \tau_0 \circ \psi$ for some automorphism $\psi : L_{NS} \to L_{NS}$ covering an automorphism $\underline{\psi}$: $NS \to NS$ of NS over the identity.* Obviously, ψ identify the V-data $((\mathrm{Der}^\bullet L_{NS})[1], \operatorname{Im} I, P, J_0)$ and $((\mathrm{Der}^\bullet L_{NS})[1], \operatorname{Im} I, P, J_1)$. As an immediate consequence, the L_∞-algebra structures on $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ determined by τ_0 and τ_1 are (strictly) L_∞-isomorphic.

Case II: *τ_0 and τ_1 are interpolated by a smooth one parameter family of fat tubular neighborhoods τ_t.* Consider $\phi_t := \tau_t^{-1} \circ \tau_0$. It is a local automorphism of L_{NS} covering a local automorphism $\underline{\varphi}_t = \underline{\tau}_t^{-1} \circ \underline{\tau}_0$ over id, well defined in a suitable neighborhood of S in NS, and such that $\varphi_0 = \mathrm{id}$. Let ξ_t be infinitesimal generators of the family $\{\varphi_t\}$. They are derivations of L_{NS} well defined around S. Our strategy is using ξ_t and φ_t to prove that J_0 and J_1 are gauge equivalent Maurer-Cartan elements of $(\mathrm{Der}^\bullet L_{NS})[1]$ intertwined by a gauge transformation preserving $\ker P$, and then applying Theorem 3.19. However, the φ_t's are well-defined only around S in NS. In order to remedy this minor drawback, we slightly change the graded space $\mathrm{Der}^\bullet L_{NS}$ underlying our V-data, passing to the graded space $\mathrm{Der}^\bullet_{for} L_{NS}$ of *alternating, first order, multi-differential operators on L_{NS} in a formal neighborhood of S in NS*. The space $\mathrm{Der}^\bullet_{for} L_{NS}$ is defined as the inverse limit

$$\varprojlim \mathrm{Der}^\bullet L_{NS} / I(S)^n \mathrm{Der}^\bullet L_{NS},$$

where $I(S) \subset C^\infty(S)$ is the ideal of functions vanishing on S, and consists of "Taylor series normal to S" of multi-differential operators. V-data $((\mathrm{Der}^\bullet L_{NS})[1], \operatorname{Im} I, P, J)$ induce obvious V-data $((\mathrm{Der}^\bullet_{for} L_{NS})[1], \operatorname{Im} I_{for}, P_{for}, J_{for})$. In particular, $I_{for} : \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] \hookrightarrow (\mathrm{Der}^\bullet_{for} L_{NS})[1]$ is the natural embedding, and J_{for} is the class of J in $(\mathrm{Der}^\bullet_{for} L_{NS})[1]$. Moreover, in view of Corollary 3.17, the L_∞-algebra determined by $((\mathrm{Der}^\bullet L_{NS})[1], \mathrm{Im}\, I, P, J)$ does only depend on J_{for}. Therefore, V-data $((\mathrm{Der}^\bullet_{\mathrm{for}} L_{NS})[1], \mathrm{Im}\, I_{\mathrm{for}}, P_{\mathrm{for}}, J_{\mathrm{for}})$ determine the same L_∞-algebra as $((\mathrm{Der}^\bullet L_{NS})[1], \mathrm{Im}\, I, P, J)$.

Now, being well defined around S, the φ_t's determine well-defined automorphisms $\phi_t := (\varphi_t)_* : (\mathrm{Der}^\bullet_{\mathrm{for}} L_{NS})[1]$ such that $\phi_0 = \mathrm{id}$. Similarly the ξ_t's descend to zero degree elements of $(\mathrm{Der}^\bullet_{\mathrm{for}} L_{NS})[1]$ which we denote by ξ_t again. Clearly, family $\{\phi_t(J_0)_{\mathrm{for}}\}$ interpolates between $(J_0)_{for}$ and $(J_1)_{for}$ and, in view of Equation (2.17), the ϕ_t's satisfy Cauchy problem (3.15). Finally,

1. from uniqueness of the one parameter family of automorphisms φ_t generated by the one parameter family of derivation ξ_t, it follows that Cauchy problem (3.16) possesses a unique solution,
2. $\varphi_t|_\ell = \mathrm{id}$ so that the ξ_t's vanish on S, hence $[\xi_t, \ker P] \subset \ker P$ for all t.

The above considerations show that $(J_0)_{\mathrm{for}}$ and $(J_1)_{\mathrm{for}}$ are gauge equivalent and they are intertwined by a gauge transformation preserving $\ker P$. Hence, from Theorem 3.19, the L_∞-algebra structures on $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ associated to the two choices τ_0 and τ_1 of the fat tubular neighborhood $L_{NS} \hookrightarrow L$ are actually L_∞-isomorphic. □

Remark 3.21. In the contact case, as already in the l.c.s. one, there exists a tubular neighborhood theorem for coisotropic submanifolds. As a consequence, the proof of Proposition 3.18 simplifies. In particular, it does not require using any *formal neighborhood technique*.

4. Deformations of coisotropic submanifolds in Jacobi manifolds

In this section, using the Baker-Campbell-Hausdorff formula, we introduce the notion of formal coisotropic deformation of a coisotropic submanifold (Definition 4.6). We prove that formal coisotropic deformations are in one-to-one correspondence with (degree 0) Maurer-Cartan elements of the associated L_∞-algebra (Proposition 4.9). We also give a necessary and sufficient condition for the convergence of the Maurer-Cartan series $MC(s)$ for any smooth section s (Proposition 4.15), extending a previous sufficient condition given by Schätz and Zambon in [36]. Analysing the notion of Hamiltonian equivalence of coisotropic deformations (Proposition 4.19) leads to a definition of Hamiltonian equivalence of formal deformations (Definition 4.20). We show that Hamiltonian equivalence of formal coisotropic deformations coincides with gauge equivalence of the corresponding Maurer-Cartan elements (Proposition 4.21) and derive consequences of this fact (Theorem 4.23, Corollary 4.22). Finally we compare our results with related results obtained earlier (Remark 4.25).

4.1. **Smooth coisotropic deformations.** Let $(M, L, J = \{-,-\})$ be an abstract Jacobi manifold and let $S \subset M$ be a closed coisotropic submanifold. We equip S with a fat tubular neighborhood $\tau : L_{NS} \hookrightarrow L$ and use it to identify L_{NS} with its image. Accordingly, and similarly as above, from now on in this section, we abuse the notation and denote by $(L, J = \{-,-\})$ (instead of $(L_{NS}, \tau_*^{-1} J)$) the Jacobi structure on NS (unless otherwise specified). It is well known that a C^1-small deformation of S in NS can be identified with a sections $S \to NS$ of NS. We say that a section $s : S \to NS$ is *coisotropic* if its image $s(S)$ is a coisotropic submanifold in (NS, L, J).

Definition 4.1. A smooth one parameter family of smooth sections of $NS \to S$ starting from the zero section is a *smooth coisotropic deformation of S* if each section in the family is coisotropic. A section s of $NS \to S$ is an *infinitesimal coisotropic deformation of S* if εs is a coisotropic section up to infinitesimals $O(\varepsilon^2)$, where ε is a formal parameter.

Remark 4.2. Let $\{s_t\}$ be a smooth coisotropic deformation of S. Then

$$\left.\frac{d}{dt}\right|_{t=0} s_t$$

is an infinitesimal coisotropic deformation.

Recall that a section $s : N \to NS$ is mapped, via $I : \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] \to (\mathrm{Der}^\bullet L)[1]$, to a derivation $I(s) := \mathbb{D}_{\pi^* s}$ of L, where $\pi : NS \to S$ is the projection. Let $\{\Phi_t\}$ be the one parameter group of automorphisms of L generated by $I(s)$ and denote $\exp I(s) := \Phi_1$. Clearly $\exp I(s)(\nu, \lambda) = (\nu + s(x), \lambda)$, for all $(\nu, \lambda) \in L = NS \times_S \ell$, $x = \pi(\nu)$. Further, let $\mathrm{pr} : J^1 L \to NS$ be the projection, denote by $j^1 \exp I(s) : J^1 L \to J^1 L$ the first jet prolongation of $\exp I(s)$, and consider the following commutative diagram

$$\begin{tikzcd}
N_\ell{}^* S \arrow[r, "\gamma"] \arrow[d] & J^1 L \arrow[r, "j^1 \exp I(s)"] \arrow[d] & J^1 L \arrow[d] \\
S \arrow[r, shift left, "\mathbf{0}"] & NS \arrow[l, shift left, "\pi"] \arrow[r, shift left, "\exp I(s)"] & NS \arrow[l, shift left, "\exp I(-s)"]
\end{tikzcd}$$

where $\mathbf{0}$ is the zero section. Note that $s = \exp I(s) \circ \mathbf{0}$.

Proposition 4.3. *Let $s : S \to NS$ be a section of π. The following three conditions are equivalent*

(1) *s is coisotropic,*
(2) *$P(\exp I(-s)_* J) = 0$ (cf. [36]),*
(3) *vector bundle $\mathrm{pr} \circ j^1 \exp I(s) \circ \gamma : N_\ell{}^* S \to s(N)$ is a Jacobi subalgebroid of $J^1 L$.*

Proof.
(1) $\Longleftrightarrow$ (2). Let $P^s : \mathrm{Der}\, L \to NS$ be composition

$$\mathrm{Der}\, L \xrightarrow{\sigma} \mathfrak{X}(M) \longrightarrow \Gamma(TM|_{s(S)}), \longrightarrow \Gamma(NS),$$

where second arrow is the restriction, and last arrow is the canonical projection (cf. (3.6)). Surjection P^s extends to a surjection of graded modules $(\mathrm{Der}^\bullet L)[1] \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ which we denote again by P^s (and is defined analogously as $P : (\mathrm{Der}^\bullet L)[1] \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$). By Proposition 3.8, s is coisotropic iff $P^s(J) = 0$. Since

$$\mathrm{der}\, \ell = \exp I(-s)_* \mathrm{der}\, L|_{s(S)} \quad \text{and} \quad \exp I(-s)_* NS = NS,$$

we obtain

$$P^s = P \circ \exp I(-s)_*. \tag{4.1}$$

In particular, $P^s(J) = P(\exp I(-s)_* J) = 0$ iff s is coisotropic.
(1) $\Longleftrightarrow$ (3). Note that $\mathrm{pr} \circ j^1 \exp I(s) \circ \gamma : N_\ell{}^* S \to s(N)$ is the ℓ-adjoint bundle of the normal bundle of $s(S)$ in NS. Now the claim follows immediately from Proposition 3.6. □

Remark 4.4. Let s be a section of NS. In view of Remark 3.9, $P^s(J) = P^s(\Lambda_J)$, where, in the rhs, P^s denotes the extension $\Gamma(\wedge^\bullet(T(NS) \otimes L^*) \otimes L) \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)$ of composition $T(NS) \to T(NS)|_{s(S)} \to NS$ defined analogously as $P : (\mathrm{Der}^\bullet L)[1] \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$. Moreover, it is clear that

$$\Lambda_{\exp I(-s)_* J} = \exp I(-s)_* \Lambda_J,$$

where $\Lambda_{\exp I(-s)_* J}$ is the bi-symbol of $\exp I(-s)_* J$, and, in the rhs, $\exp I(-s)_* : \Gamma(\wedge^\bullet(T(NS) \otimes L^*) \otimes L) \to \Gamma(\wedge^\bullet(T(NS) \otimes L^*) \otimes L)$ denotes the isomorphism induced by the line bundle automorphism $\exp I(-s)$. It immediately follows that s is coisotropic iff $P(\exp I(-s)_* \Lambda_J) = 0$.

4.2. **Formal coisotropic deformations.** Let ε be a formal parameter.

Definition 4.5. A formal series $s(\varepsilon) = \sum_{i=0}^\infty \varepsilon^i s_i \in \Gamma(NS)[[\varepsilon]]$, $s_i \in \Gamma(NS)$, such that $s_0 = 0$, is called a *formal deformation of S*.

Formal series $I(s(\varepsilon)) := \sum_{i=0}^{\infty} \varepsilon^i I(s_i) \in (\operatorname{Der} L)[[\varepsilon]]$ is a formal derivation of L. It is easy to see that the space $(\operatorname{Der} L)[[\varepsilon]]$ of formal derivations of L is a Lie algebra, which has a linear representation in the space $(\operatorname{Der}^\bullet L)[[\varepsilon]]$ of formal first order multi-differential operators on L via the following *Lie derivative*:

$$\mathcal{L}_{\xi(\varepsilon)}\Delta(\varepsilon) \equiv [\xi(\varepsilon), \Delta(\varepsilon)]^{SJ} := \sum_{k=0}^{\infty} \varepsilon^k \sum_{i+j=k} [\xi_i, \Delta_j]^{SJ}, \tag{4.2}$$

for $\xi(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \xi_i$, $\xi_i \in \operatorname{Der} L$, and $\Delta(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \Delta_i$, $\Delta_i \in \operatorname{Der}^\bullet L$.

Using the Baker-Campbell-Hausdorff formula we define the exponential of the Lie derivative $\mathcal{L}_{\xi(\varepsilon)}$ as the following formal power series

$$\exp \mathcal{L}_{\xi(\varepsilon)} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{L}^n_{\xi(\varepsilon)}. \tag{4.3}$$

Proposition 4.3 motivates the following

Definition 4.6. A formal deformation $s(\varepsilon)$ of S is said *coisotropic*, if $P(\exp \mathcal{L}_{I(s(\varepsilon))} J) = 0$.

Remark 4.7. Let $\xi(\varepsilon) \in (\operatorname{Der} L)[[\varepsilon]]$. Define a Lie derivative

$$\mathcal{L}_{\xi(\varepsilon)} : \Gamma(\wedge^\bullet(T(NS) \otimes L^*) \otimes L)[[\varepsilon]] \to \Gamma(\wedge^\bullet(T(NS) \otimes L^*) \otimes L)[[\varepsilon]],$$

in the obvious way. It is easy to see that

$$P(\exp \mathcal{L}_{I(s(\varepsilon))} J) = P(\exp \mathcal{L}_{I(s(\varepsilon))} \Lambda_J), \tag{4.4}$$

for all formal deformations $s(\varepsilon)$ of S (cf. Remarks 3.9 and 4.4). In particular, $s(\varepsilon)$ is coisotropic iff $P(\exp \mathcal{L}_{I(s(\varepsilon))} \Lambda_J) = 0$.

Remark 4.8 (Formal deformation problem)**.** The *formal deformation problem for a coisotropic submanifold* S consists in finding formal coisotropic deformations of S. Let $s(\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i s_i$ be a formal coisotropic deformation of S. Then s_1 is an infinitesimal coisotropic deformation. On the other hand, in general, not all infinitesimal coisotropic deformations can be "prolonged" to a formal coisotropic deformation. If this is the case, one says that *the formal deformation problem is unobstructed*. Otherwise, *the formal deformation problem is obstructed*. The formal deformation problem of S is *governed by the* L_∞*-algebra* $(\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1], \{\mathfrak{m}_k\})$ in the sense clarified by the following proposition.

Proposition 4.9. *A formal deformation $s(\varepsilon)$ of S is coisotropic iff $-s(\varepsilon)$ is a solution of the (formal) Maurer-Cartan equation*

$$MC(-s(\varepsilon)) := \sum_{k=1}^{\infty} \frac{1}{k!} \mathfrak{m}_k(-s(\varepsilon), \cdots, -s(\varepsilon)) = 0. \tag{4.5}$$

Proof. The expression $MC(-s(\varepsilon))$ should be interpreted as an element of $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[[\varepsilon]]$. The proposition is then a consequence of (4.3), $P(J) = 0$, and the following identities

$$P(\mathcal{L}^k_{I(\xi)} J) = \mathfrak{m}_k(-\xi, \cdots, -\xi), \quad k \geq 1, \tag{4.6}$$

for $\xi \in \Gamma(NS)$, which immediately follow from the definition of $\mathfrak{m}_k$. □

Let s be a section of NS. The *Maurer-Cartan series of* s is the series

$$MC(-s) := \sum_{k=1}^{\infty} \frac{1}{k!} \mathfrak{m}_k(-s, \ldots, -s).$$

In general, $MC(-s)$ does not converge, not even for a coisotropic s. However, we have the obvious

Corollary 4.10. *Let s be a section of NS such that the* Maurer-Cartan series $MC(-s)$ *converges. Then s is a coisotropic deformation of S iff $MC(-s) = 0$.*

Corollary 4.11. *A section s of NS is an infinitesimal coisotropic deformation of S iff*

$$\mathfrak{m}_1(s) = 0. \tag{4.7}$$

By Remark 3.13.(1), $\mathfrak{m}_1$ coincides with the Jacobi algebroid de Rham differential $d_{N_\ell{}^*S,\ell}$. Hence, a similar argument as in the proof of Theorem 11.2 in [32] yields

Corollary 4.12. *Assume that the second cohomology group $H^2(N_\ell{}^*S, \ell)$ of the Jacobi subalgebroid $N_\ell{}^*S \subset J^1L$ with values in ℓ is zero. Then every infinitesimal coisotropic deformation can be prolonged to a formal coisotropic deformation, i.e. for any given class $\alpha \in H^1(N_\ell{}^*S, \ell)$ Equation (4.5) has a solution $s(\varepsilon) = \sum_{i=i}^{\infty} \varepsilon^i s_i$ such that $\mathfrak{m}_1(s_1) = 0$ and $[s_1] = \alpha$. In other words, the formal deformation problem is unobstructed.*

There is also a simple criterion for non-prolongability of an infinitesimal coisotropic deformation to a formal coisotropic deformation based on the *Kuranishi map*:

$$Kr : H^1(N_\ell{}^*S, \ell) \longrightarrow H^2(N_\ell{}^*S, \ell), \quad [s] \longmapsto [\mathfrak{m}_2(s, s)].$$

Since $\mathfrak{m}_1$ is a derivation of the binary bracket $\mathfrak{m}_2$, the Kuranishi map is well-defined. Moreover, similarly as in [32] (Theorem 11.4) we have the following

Proposition 4.13. *Let $\alpha = [s] \in H^1(N_\ell{}^*S, \ell)$, where $s \in \Gamma(NS)$ is an infinitesimal coisotropic deformation, i.e. $d_{N_\ell{}^*S,\ell}s = \mathfrak{m}_1 s = 0$. If $Kr(\alpha) \neq 0$, then s cannot be prolonged to a formal coisotropic deformation. In particular, the formal deformation problem is obstructed.*

We also have

Corollary 4.14. *Let $\alpha = [s]$ be as in the above proposition. If $\mathfrak{m}_k = 0$ for all $k > 2$, then s can be prolonged to a formal coisotropic deformation iff $Kr(\alpha) = 0$.*

4.3. **Formal deformations and smooth deformations.** In this subsection we establish a connection between formal coisotropic deformations and smooth coisotropic deformations. We do this introducing the notion of fiber-wise entire bi-symbol, which is a slight generalization of the notion of fiber-wise entire Poisson structure introduced by Schätz and Zambon in [36], and is motivated by the Taylor expansion of the bi-linear form $P(\exp I(-s)_*\Lambda_J)$ (Proposition 4.15).

Let $E \to S$ be a vector bundle. Recall that a smooth function on E is called *fiber-wise entire* if its restriction to each fiber of E is *entire*, i.e. it is real analytic on the whole fiber. Now, let $\ell \to S$ be a line bundle, and $L := E \times_S \ell$. A section of L is called *fiber-wise entire* if it is a linear combination of fiber-wise constant sections, with coefficients being fiber-wise entire functions. Let $\Theta \in \Gamma(\wedge^k(TE \otimes L^*) \otimes L)$. We regard Θ as a multi-linear map

$$\Theta : \wedge^k(T^*E \otimes L) \longrightarrow L.$$

The multi-linear map Θ is called *fiber-wise entire* if

$$\Theta(df_1 \otimes \lambda_1, \ldots, df_k \otimes \lambda_k)$$

is fiber-wise entire, whenever $f_1, \ldots, f_k \in C^\infty(M)$ and $\lambda_1, \ldots, \lambda_k$ are fiber-wise linear. Equivalently Θ is fiber-wise entire if its components in some (and therefore any) system of vector bundle coordinates are fiber-wise entire functions (cf. [36, Lemmas 1.4, 1.7]).

Now, let S and $(NS, L, J = \{-,-\})$ be as in Subsection 4.1. The following proposition generalizes the main result of [36] establishing a necessary and sufficient condition for the convergence of the Maurer-Cartan series $MC(-s)$ of a generic section $s \in \Gamma(NS)$.

Proposition 4.15. *The bi-symbol Λ_J of the Jacobi bi-differential operator J is fiber-wise entire iff the Maurer-Cartan series $MC(-s)$ of any smooth section $s \in \Gamma(NS)$ converges to $P(\exp I(s)_* J) = P(\exp I(s)_* \Lambda_J)$.*

Proof. Let $z^\alpha = (x^i, y^a)$ be vector bundle coordinates on NS, with x^i coordinates on S, and y^a linear coordinates along the fibers of NS. Moreover, let μ be a fiber-wise constant local generator of $\Gamma(L)$. The Jacobi bi-differential operator J is locally given by Equation (3.12), or, equivalently, Equation (3.13)

$$\begin{aligned} J &= J^{\alpha\beta}\nabla_\alpha \wedge \nabla_\beta \otimes \mu + J^\alpha \nabla_\alpha \wedge \mathrm{id} \\ &= \left(J^{ab}\nabla_a \wedge \nabla_b + 2J^{ai}\nabla_a \wedge \nabla_i + J^{ij}\nabla_i \wedge \nabla_j\right) \otimes \mu + \left(J^a \nabla_a + J^i \nabla_i\right) \wedge \mathrm{id}\,. \end{aligned}$$

Accordingly, the bi-symbol Λ_J is locally given by

$$\begin{aligned} \Lambda_J &= J^{\alpha\beta}\delta_\alpha \wedge \delta_\beta \otimes \mu \\ &= \left(J^{ab}\delta_a \wedge \delta_b + 2J^{ai}\delta_a \wedge \delta_i + J^{ij}\delta_i \wedge \delta_j\right) \otimes \mu \end{aligned}$$

where $\delta_\alpha := \partial_\alpha \otimes \mu^*$. In particular, Λ_J is fiber-wise entire iff its components J^{ab}, J^{ai}, J^{ij} are fiber-wise entire functions. Now, let $s \in \Gamma(NS)$ and denote by $\{\Phi_t\}$ the one parameter group of automorphisms of L generated by $I(s)$. Then, from $P(J) = P(\Lambda_J) = 0$, Equations (4.6), (4.4), and the very definition of the Lie derivative, we get

$$MC(-s) = P\sum_{k=0}^{\infty} \left.\frac{\partial^k (\Phi_{-t_1-\cdots-t_k})_* \Lambda_J}{\partial t_1 \cdots \partial t_k}\right|_{t_1=\cdots=t_k=0} = P\sum_{k=0}^{\infty} \frac{1}{k!}\left.\frac{d^k}{dt^k}\right|_{t=0} (\Phi_{-t})_* \Lambda_J.$$

Let $(x, y, \lambda) \in L$, $x \in S$, $y \in N_x S$, $\lambda \in L_x$. Then $\Phi_{-t}(x, y, \lambda) = (x, y - ts(x), \lambda)$ and

$$\begin{aligned} &(\Phi_{-t})_* \Lambda_J \\ &= \left[(J^{ab}\circ \Phi_t)\delta_a \wedge \delta_b + 2(J^{ai}\circ\Phi_t)\delta_a \wedge (\delta_i - ts_i^b\delta_b) + (J^{ij}\circ\Phi_t)(\delta_i - ts_i^a\delta_a)\wedge(\delta_j - ts_j^b\delta_b)\right] \otimes \mu, \end{aligned}$$

where s_i^a denotes the partial derivative wrt x^i of the a-th local component of s in the local basis (∂_a) of $\Gamma(NS)$. Hence

$$MC(-s) = \sum_{k=0}^{\infty} \frac{1}{k!}\left.\frac{d^k}{dt^k}\right|_{t=0} \left[J^{ab}\circ ts - 2ts_i^b(J^{ai}\circ ts) + t^2 s_i^a s_j^b (J^{ij}\circ ts)\right]\delta_a \wedge \delta_b \otimes \mu. \tag{4.8}$$

Assume that Λ_J is fiber-wise entire. Then the Taylor expansions in t, around $t = 0$, of $J^{ab}\circ ts$, $J^{ai}\circ ts$, and $J^{ij}\circ ts$ converge for all t's, in particular for $t = 1$. It immediately follows that the series in the rhs of (4.8) converges as well. This proves the "only if" part of the proposition (cf. the proof of the analogous proposition in [36]).

For the "if part" of the proposition assume that the series in the rhs of (4.8) converges for all s. First of all, locally, we can choose s to be "constant" wrt coordinates (x^i, y^a). Then $s_i^a = 0$ and (4.8) reduces to

$$MC(-s) = \sum_{k=0}^{\infty} \frac{1}{k!}\left.\frac{d^k}{dt^k}\right|_{t=0} \left(J^{ab}\circ ts\right)\delta_a \wedge \delta_b \otimes \mu. \tag{4.9}$$

Since s is arbitrary, (4.9) shows that the J^{ab}'s are entire on any straight line through the origin in the fibers of NS. Since the Taylor series of the restriction to such a straight line is the same as the restriction of the Taylor series, we conclude that the J^{ab}'s are fiber-wise entire. Now, fix values i_0, a_0 for the indexes i, a respectively, and choose s so that $s_i^a = \delta_i^{i_0}\delta_{a_0}^a$ to see that the J^{ai_0}'s are fiber-wise entire for all a, i_0. One can prove that the J^{ij}'s are fiber-wise entire in a similar way. This concludes the proof. $\square$

Corollary 4.16. *Let $(M, L, J = \{-,-\})$ be a Jacobi manifold, and let $S \subset M$ be a coisotropic submanifold equipped with a fat tubular neighborhood $\tau : L_{NS} \hookrightarrow L$. If $\tau_*^{-1}\Lambda_J$ is fiber-wise entire, then a section $s : S \to NS$ of NS is coisotropic iff the Maurer-Cartan series $MC(-s)$ converges to zero.*

4.4. **Moduli of coisotropic sections.** Jacobi diffeomorphisms, in particular Hamiltonian diffeomorphisms, preserve coisotropic submanifolds. Two coisotropic submanifolds are *Hamiltonian equivalent* if there is an Hamiltonian isotopy (i.e. a one parameter family of Hamiltonian diffeomorphisms) interpolating them. With this definition at hand one can define a moduli space of coisotropic submanifolds under *Hamiltonian equivalence*. Now, let S be a coisotropic submanifold. In this section we adapt the definition of Hamiltonian equivalence to the case of coisotropic sections of $NS \to S$ [25, Definition 6.3]. In this way we define a local version of the moduli space under Hamiltonian equivalence.

Definition 4.17. (cf. [25, Definition 10.2]).

(1) Two coisotropic sections $s_0, s_1 \in \Gamma(NS)$ are called *Hamiltonian equivalent* if they are interpolated by a smooth family of sections $s_t \in \Gamma(NS)$ and there exists a family of Hamiltonian diffeomorphisms $\psi_t : NS \to NS$ of $(NS, L, J = \{-,-\})$ (i.e. the family $\{\psi_t\}$ is generated by a family $\{X_{\lambda_t}\}$ of Hamiltonian vector fields, where the λ_t's depend smoothly on t) and a family of diffeomorphisms $g_t : S \to S$, $t \in [0,1]$, such that $g_0 = \mathrm{id}_S$, $\psi_0 = \mathrm{id}_{NS}$ and $s_t = \psi_t \circ s_0 \circ g_t^{-1}$. A coisotropic deformation of S is *trivial* if it is Hamiltonian equivalent to the zero section.
(2) Two coisotropic sections $s_0, s_1 \in \Gamma(NS)$ are called *infinitesimally Hamiltonian equivalent* if $s_1 - s_0$ is the vertical component along S of an Hamiltonian vector field. An infinitesimal coisotropic deformation is trivial if it is infinitesimally Hamiltonian equivalent to the zero section.

Note that both Hamiltonian equivalence and infinitesimal Hamiltonian equivalence are equivalence relations. The notion of infinitesimal Hamiltonian equivalence is motivated by the following remark.

Remark 4.18. Let s_0, s_1 be Hamiltonian equivalent coisotropic sections, and let s_t be the family of sections interpolating them as in Definition 4.17.(1). Then s_t is obviously a coisotropic section for all t. Moreover, s_0 and

$$s_0 + \left.\frac{d}{dt}\right|_{t=0} s_t$$

are infinitesimally Hamiltonian equivalent coisotropic sections.

Proposition 4.19. *Two coisotropic sections $s_0, s_1 \in \Gamma(NS)$ are Hamiltonian equivalent iff they are interpolated by a smooth family of sections $s_t \in \Gamma(NS)$ and there exists a smooth family of sections λ_t of the Jacobi bundle L such that s_t is a solution of the following evolutionary equation:*

$$\frac{d}{dt} s_t = P(\exp I(-s_t)_* \Delta_{\lambda_t}). \tag{4.10}$$

Proof. Denote by $\pi : NS \to S$ the projection. First of all, let s_0, s_1 be Hamiltonian equivalent coisotropic sections, and let s_t, ψ_t, g_t be as in Definition 4.17.(1). The g_t's are completely determined by the ψ_t's via $g_t = \pi \circ \psi_t \circ s_0$. In their turn, the ψ_t's are generated by a smooth family $\{X_{\lambda_t}\}$ of Hamiltonian vector fields, $\lambda_t \in \Gamma(L)$. Differentiating the identity $s_t = \psi_t \circ s_0 \circ g_t^{-1}$ with respect to t, one finds

$$\frac{d}{dt} s_t = P^{s_t}(\Delta_{\lambda_t}), \tag{4.11}$$

where, for a generic section $s \in \Gamma(NS)$, the projection $P^s : (\mathrm{Der}^\bullet L)[1] \to \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ is defined as in the proof of Proposition 4.3. To see this, interpret the s_t's as smooth maps, and consider their pull-backs $s_t^* : C^\infty(NS) \to C^\infty(S)$. Then $s_t^* = (g_t^{-1})^* \circ s_0^* \circ \psi_t^*$ and a straightforward computation shows that

$$\frac{d}{dt} s_t^* = s_t^* \circ X_{\lambda_t} \circ (\mathrm{id} - \pi^* \circ s_t^*).$$

which is equivalent to 4.11. Equation (4.10) now follows from (4.1).

Conversely, let s_t be a solution of Equation (4.10) interpolating s_0 and s_1, and let $\{\psi_t\}$ be the one parameter family of Hamiltonian diffeomorphisms $NS \to NS$ generated by $\{X_{\lambda_t}\}$. In view of (4.1) again, s_t is the (unique) solution of (4.11) starting at s_0. In particular, ψ_t maps diffeomorphically the image of s_0 to the image of s_t. Hence, the map $g_t = \pi \circ \psi_t \circ s_0$ is a diffeomorphism and $s_t = \psi_t \circ s_0 \circ g_t^{-1}$. □

Note that if $\{s_t\}$ is a solution of (4.10) interpolating coisotropic sections s_0, s_1, then s_t is a coisotropic section for all t. Proposition 4.19 motivates the following

Definition 4.20. Two formal coisotropic deformations $s_0(\varepsilon), s_1(\varepsilon) \in \Gamma(NS)[[\varepsilon]]$ are called *Hamiltonian equivalent* if they are interpolated by a smooth family of formal coisotropic deformations $s_t(\varepsilon) \in \Gamma(NS)[[\varepsilon]]$ (i.e. $s_t(\varepsilon) = \sum_i s_{t,i}\varepsilon^i$ and the $s_{t,i}$'s depend smoothly on t) and there exists a smooth family of formal sections $\lambda_t(\varepsilon) \in \Gamma(L)[[\varepsilon]]$ of the Jacobi bundle such that

$$\frac{d}{dt}s_t(\varepsilon) = P(\exp \mathcal{L}_{I(s_t(\varepsilon))}\Delta_{\lambda_t(\varepsilon)}).$$

We now show that formal coisotropic deformations $s_0(\varepsilon), s_1(\varepsilon)$ are Hamiltonian equivalent iff $-s_0(\varepsilon), -s_1(\varepsilon)$ are gauge equivalent solutions of the Maurer-Cartan equation $MC(\xi(\varepsilon)) = 0$. Two solutions $\xi_0(\varepsilon), \xi_1(\varepsilon)$ of the Maurer-Cartan equation are *gauge equivalent* if, by definition, they are interpolated by a smooth family of formal sections $\xi_t(\varepsilon) \in \Gamma(NS)[[\varepsilon]] = \Gamma(\wedge^1 N_\ell S \otimes \ell)[[\varepsilon]]$ and there exists a smooth family of formal sections $\lambda_t(\varepsilon) \in \Gamma(\ell)[[\varepsilon]] = \Gamma(\wedge^0 N_\ell S \otimes \ell)[[\varepsilon]]$ such that

$$\frac{d}{dt}\xi_t(\varepsilon) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathfrak{m}_{k+1}(\xi_t(\varepsilon), \ldots, \xi_t(\varepsilon), \lambda_t(\varepsilon)). \tag{4.12}$$

Gauge equivalence is an equivalence relation. Moreover, it follows from Equation (4.12) that the $\xi_t(\varepsilon)$ is a solution of the Maurer-Cartan equation for any t.

Proposition 4.21. *Two formal coisotropic deformations $s_0(\varepsilon), s_1(\varepsilon) \in \Gamma(NS)[[\varepsilon]]$ are Hamiltonian equivalent iff they are gauge equivalent solutions of the Maurer-Cartan equation.*

Proof. Recall that $\ker P \subset (\mathrm{Der}^\bullet L)[1]$ is a Lie subalgebra. As Voronov notes [39], this can be rephrased as:

$$P[\square_1, \square_2]^{SJ} = P[IP\square_1, \square_2]^{SJ} + P[\square_1, IP\square_2]^{SJ}, \tag{4.13}$$

$\square_1, \square_2 \in (\mathrm{Der}^\bullet L)[1]$. Now, let $\{s_t(\varepsilon)\}$ be a family of formal coisotropic deformations, and let $\{\lambda_t(\varepsilon)\}$ be a family of formal sections of L. Put

$$J_k(\varepsilon) := [\cdots[J, \underbrace{I(-s(\varepsilon))]^{SJ} \cdots, I(-s(\varepsilon))]^{SJ}}_{k \text{ times}},$$

In particular, $PJ_k(\varepsilon) = \mathfrak{m}_k(-s(\varepsilon), \dots, -s(\varepsilon))$. Compute

$$\begin{aligned} P(\exp \mathcal{L}_{I(s_t(\varepsilon))} \Delta_{\lambda_t(\varepsilon)}) &= -\sum_{k=0}^{\infty} \frac{1}{k!} P[\cdots [[J, \lambda_t(\varepsilon)]^{SJ}, I(-s(\varepsilon))]^{SJ} \cdots, I(-s(\varepsilon))]^{SJ} \\ &= -\sum_{k=0}^{\infty} \frac{1}{k!} P[J_k(\varepsilon), \lambda_t(\varepsilon)]^{SJ} \\ &= -\sum_{k=0}^{\infty} \frac{1}{k!} P[IPJ_k(\varepsilon), \lambda_t(\varepsilon)]^{SJ} - \sum_{k=0}^{\infty} \frac{1}{k!} P[J_k(\varepsilon), IP\lambda_t(\varepsilon)]^{SJ} \\ &= -P[I(MC(-s_t(\varepsilon))), \lambda_t(\varepsilon)]^{SJ} - \sum_{k=0}^{\infty} \frac{1}{k!} P[J_k(\varepsilon), I(\lambda_t(\varepsilon)|_S)]^{SJ} \\ &= -\sum_{k=0}^{\infty} \frac{1}{k!} \mathfrak{m}_{k+1}(-s(\varepsilon), \cdots, -s(\varepsilon), \lambda_t(\varepsilon)|_S), \end{aligned}$$

where we used (4.13), and the fact that $MC(-s_t(\varepsilon)) = 0$ for all t. This concludes the proof. □

Corollary 4.22. *Two solutions of Equation (4.7) are infinitesimal Hamiltonian equivalent iff they are cohomologous as element in the complex* $(\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1], \mathfrak{m}_1)$. *Hence, the moduli space (i.e. the set of infinitesimal Hamiltonian equivalence classes) of infinitesimal coisotropic deformations of* S *is* $H^0(\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1], \mathfrak{m}_1) = H^1(N_\ell{}^* S, \ell)$.

Now, we establish necessary and sufficient conditions for the convergence of both the Maurer-Cartan series $MC(-s)$ and the series

$$\delta_\lambda MC(-s) := \sum_{k=0}^{\infty} \frac{1}{k!} \mathfrak{m}_{k+1}(-s, \dots, -s, \lambda) \tag{4.14}$$

for generic sections $s \in \Gamma(NS)$ and $\lambda \in \Gamma(\ell)$. In this way, we can describe moduli of coisotropic sections in terms of gauge equivalence classes of non-formal solutions of the Maurer-Cartan equation. First of all, let E and L be as in the beginning of Section 4.3. A multi-differential operator $\Delta \in (\mathrm{Der}^\bullet L)[1]$ is *fiber-wise entire* if it maps linear sections (of L) to fiber-wise entire sections. Equivalently, Δ is fiber-wise entire if its component in vector bundle coordinates are fiber-wise entire.

Theorem 4.23. *The Jacobi bi-differential operator* J *is fiber-wise entire iff the Maurer-Cartan series* $MC(-s)$ *converges to* $P(\exp I(s)_* J)$*, and the series* $\delta_{\lambda|_S} MC(-s)$ *(4.14) converges to* $P(\exp I(s)_* \Delta_\lambda)$*, for all smooth sections* $s \in \Gamma(NS)$*, and* $\lambda \in \Gamma(L)$*.*

Proof. We already know that the bi-linear form Λ_J is fiber-wise entire iff $MC(-s)$ converges for all s. Now, it is easy to see that $P(\exp \mathcal{L}_{I(s)} \Delta_\lambda) = P(\exp \mathcal{L}_{I(s)} X_\lambda)$ for all $s \in \Gamma(NS)$, and $\lambda \in \Gamma(L)$ (cf. (4.4)). Moreover, from the proof of Proposition 4.21, we get

$$\delta_{\lambda|_S} MC(-s) = -P(\exp \mathcal{L}_{I(s)} \Delta_\lambda) = -P(\exp \mathcal{L}_{I(s)} X_\lambda).$$

Therefore, similarly as in the proof of Proposition 4.15, we find

$$\delta_{\lambda|_S} MC(-s) = -P \sum_{k=0}^{\infty} \frac{1}{k!} \left. \frac{d^k}{dt^k} \right|_{t=0} (\Phi_{-t})_* X_\lambda.$$

The bi-differential operator J is locally given by (3.13), hence a straightforward computation shows that

$$\delta_{\lambda|_S} MC(-s) = \sum_{k=0}^{\infty} \frac{1}{k!} \left. \frac{d^k}{dt^k} \right|_{t=0} \left[2\partial_i g(J^{ai} \circ ts) + g(J^a \circ ts) - ts^a_i g(J^i \circ s) \right] \partial_a,$$

where we used the same notations as in the proof of Proposition 4.21, and g is the component of $\lambda|_S$ in the basis μ. The assertion now follows in a very similar way as in the proof of Proposition 4.21. □

Corollary 4.24. *Let $(M, L, J = \{-,-\})$ be a Jacobi manifold, and let $S \subset M$ be a coisotropic submanifold equipped with a fat tubular neighborhood $\tau : \ell \hookrightarrow L$. If $\tau_*^{-1}J$ is fiber-wise entire, then two solutions $s_0, s_1 : S \to NS$ of the (well-defined) Maurer-Cartan equation $MC(-s) = 0$ are Hamiltonian equivalent iff they are interpolated by a smooth family of sections $s_t \in \Gamma(NS)$ and there exists a smooth family of sections λ_t of ℓ such that s_t is a solution of the following well-defined evolutionary equation:*

$$\frac{d}{dt}s_t = \delta_{\lambda_t} MC(-s_t).$$

Remark 4.25. Corollary 4.22 generalizes [25, Lemma 6.6], which has been proved by a different method.

5. The contact case

Contact manifolds form a distinguished class of (transitive) abstract Jacobi manifolds. In this section we consider in some details (regular) coisotropic submanifolds in a contact manifold (M, C). A normal form theorem is available in this case. As a consequence, the L_∞-algebra of a regular coisotropic submanifold S in (M, C) does only depend on the intrinsic pre-contact geometry of S. In particular, we get rather efficient formulas (from a computational point of view) for the multibrackets, analogous to those of Oh and Park in the symplectic case [32, Equation (9.17)].

5.1. **Coisotropic submanifolds in contact manifolds.** Let C be an hyperplane distribution on a smooth manifold M. Denote by L the quotient line bundle TM/C, and by $\theta : TM \to L$, $X \mapsto \theta(X) := X \operatorname{mod} C$ the projection. We will often interpret θ as an L-valued differential 1-form, and call it the *structure form of* C. The *curvature form* of (M, C) is the vector bundle morphism $\omega : \wedge^2 C \to L$ well-defined by $\omega(X, Y) = \theta([X, Y])$, with $X, Y \in \Gamma(C)$. Consider also the vector bundle morphism $\omega^\flat : C \to C^* \otimes L$, $X \mapsto \omega^\flat(X) := \omega(X, -)$. The *characteristic distribution* of (M, C), is the (generically singular) distribution $\ker \omega^\flat = C^{\perp_\omega}$, where we denoted by $V^{\perp_\omega}$ the ω-orthogonal complement of a subbundle $V \subset C$. Note that the definition of curvature form works verbatim for distribution of arbitrary codimension (See also [32, section 4] for a detailed exposition on the curvature form).

Remark 5.1. The characteristic distribution of an hyperplane distribution C is involutive.

Definition 5.2. A *pre-contact structure* on a smooth manifold M is an hyperplane distribution C on M such that its characteristic distribution $\ker \omega^\flat$ has constant dimension. A *pre-contact manifold* (M, C) is a smooth manifold M equipped with a pre-contact structure C. The integral foliation of $\ker \omega^\flat$ is called the *characteristic foliation* of C and will be denoted by $\mathcal{F}$.

See [33, Section 5] where essentially the same definition was given in terms of the one-form generating the hyperplane distribution in relation to the study of normal forms of the contact form of Morse-Bott type.

Remark 5.3. The curvature form ω of (M, C) measures how far is C from being integrable. Indeed, C is integrable iff $\omega = 0$, or, equivalently, $\omega^\flat = 0$. Accordingly, C is said to be *maximally non-integrable* when ω is non degenerate, or, equivalently, $\ker \omega^\flat = 0$. If C is maximally non-integrable, then C is even-dimensional, M is odd-dimensional, and $\omega^\flat$ is a vector bundle isomorphism, whose inverse will be denoted by $\omega^\# : C^* \otimes L \to C$.

Definition 5.4. A *contact structure* on a smooth manifold M is a maximally non-integrable hyperplane distribution C on M. A *contact manifold* is a smooth manifold M equipped with a contact structure C.

Example 5.5. Let $L \to M$ be a line bundle. There is a canonical contact structure C on J^1L, sometimes called the *Cartan distribution* and defined as follows. Let $\pi : J^1L \to M$, and $\mathrm{pr} : J^1L \to L$ be canonical projections. Consider the pull-back line bundle $\pi^*L \to J^1L$. There is a canonical π^*L-valued one form θ on J^1L given by

$$\theta(\xi_\alpha) := (d\mathrm{pr} - d\lambda \circ d\pi)(\xi_\alpha), \quad \xi_\alpha \in T_\alpha J^1L,$$

where $\alpha = (j^1\lambda)(x) \in J^1L$, and $x = \pi(\alpha)$, $\lambda \in \Gamma(L)$. The Cartan distribution is then defined as the kernel of θ. In particular, the line bundle $T(J^1L)/C$ identifies canonically with π^*L.

Remark 5.6. Let (M, C) be a contact manifold. There exists a natural one-to-one correspondence between

(1) local trivializations (or nowhere zero local sections) of the line bundle $L \to M$ and
(2) local contact forms of (M, C), i.e. 1-forms $\alpha \in \Omega^1(U)$, with U open in M, such that $C|_U = \ker \alpha$.

Let (M, C) and (M', C') be contact manifolds. A contactomorphism from (M, C) to (M', C') is a diffeomorphism $\phi : M \to M'$ such that

$$(d\phi)C = C'.$$

An *infinitesimal contactomorphism* (or *contact vector field*) of a contact manifold (M, C) is a vector field $X \in \mathfrak{X}(M)$ whose flow consists of local contactomorphisms. Equivalently, $X \in \mathfrak{X}(M)$ is a contact vector field if $[X, \Gamma(C)] \subset \Gamma(C)$. Contact vector fields of (M, C) form a Lie subalgebra of $\mathfrak{X}(M)$ which will be denoted by $\mathfrak{X}_C$ (see e.g. [33, Proposition 2.3]).

Proposition 5.7 (cf. [7], [33, Proposition 2.3])**.** *Let (M, C) be a contact manifold. There is a natural direct sum decomposition of $\mathbb{R}$-vector spaces: $\mathfrak{X}(M) = \mathfrak{X}_C \oplus \Gamma(C)$.*

Proof. For $X \in \mathfrak{X}(M)$, let $\phi_X \in \Gamma(C^* \otimes L)$ be defined by $\phi_X(Y) = \theta([X, Y])$, $Y \in \Gamma(C)$. The first order differential operator $\phi : \mathfrak{X}(M) \to \Gamma(C^* \otimes L)$, $X \mapsto \phi_X$, sits in a short exact sequence of $\mathbb{R}$-linear maps

$$0 \longrightarrow \mathfrak{X}_C \hookrightarrow \mathfrak{X}(M) \xrightarrow{\phi} \Gamma(C^* \otimes L) \longrightarrow 0, \tag{5.1}$$

where the second arrow is the inclusion. Now the $C^\infty(M)$-linear map $\Gamma(C^* \otimes L) \to \mathfrak{X}(M)$ given by the composition

$$\Gamma(C^* \otimes L) \xrightarrow{\omega^\#} \Gamma(C) \longrightarrow \mathfrak{X}(M)$$

splits sequence (5.1). □

In what follows, for $\lambda \in \Gamma(L)$, we denote by X_λ the unique contact vector field such that $\theta(X_\lambda) = \lambda$.

Proposition 5.8. *A contact structure C induces a canonical Jacobi structure $(L, \{-,-\})$, where the Lie bracket $\{-,-\}$ on $\Gamma(L)$ is uniquely determined by $X_{\{\lambda,\mu\}} = [X_\lambda, X_\mu]$. The symbol of the first order differential operator $\Delta_\lambda := \{\lambda, -\} \in \mathrm{Der}\, L$ is X_λ.*

Now, let (M, C) be a contact manifold, and let $S \subset M$ be a submanifold. The intersection $C_S := C \cap TS$ is a generically singular distribution on S. More precisely S is the union of two disjoint subsets S_0, S_1 defined by

- $p \in S_0$ iff $\dim(C_S)_p = \dim S$,
- $p \in S_1$ iff $\dim(C_S)_p = \dim S - 1$.

If $S = S_0$ then S is said to be an *isotropic submanifold* of (M, C). In other words, an isotropic submanifold of (M, C) is an integral manifold of the contact distribution C. Locally maximal isotropic, or, equivalently, locally maximal integral submanifolds of C are *Legendrian submanifolds*.

Proposition 5.9. *Let $S = S_1$. The following conditions are equivalent:*

(1) *C_S is a pre-contact structure on S, with characteristic distribution given by $(C_S)^{\perp_\omega} \subset C|_S$,*

(2) $(C_S)_p$ *is a coisotropic subspace in the symplectic vector space* (C_p, ω_p)*, i.e.* $(C_S)_p^{\perp_\omega} \subset (C_S)_p$*, for all* $p \in S$*,*
(3) S *is a coisotropic submanifold of the associated Jacobi manifold* $(M, L, J = \{-,-\})$*.*

Proof. The equivalence 1) $\Longleftrightarrow$ 2) amounts to a standard argument in symplectic linear algebra. The equivalence 2) $\Longleftrightarrow$ 3) is based on the following facts. Let $(L, J = \{-,-\})$ be the Jacobi structure associated to (M, C). For $\lambda \in \Gamma(L)$, and $f \in C^\infty(M)$ put $Y_{f,\lambda} := \Lambda_J^\#(df \otimes \lambda) = X_{f\lambda} - f X_\lambda$. We have the following:

- $Y_{f,\lambda} \in \Gamma(C)$.
- Let $I(S) \subset C^\infty(M)$ be the ideal of functions vanishing on S. Then $Y_{f,\lambda}$ is tangent to S iff $X_{f\lambda}$ is tangent to S, for all $f \in I(S)$, and $\lambda \in \Gamma(L)$.
- $\omega(Y_{f,\lambda}, X) = X(f)\lambda$, for all $f \in C^\infty(M)$, $\lambda \in \Gamma(L)$, and $X \in \Gamma(C)$.
- Let $\Gamma_S \subset \Gamma(L)$ be the submodule consisting of sections vanishing on S. Then $\Gamma_S = I(S) \cdot \Gamma(L)$.

Now it is easy to see that $(C_S)^{\perp_\omega} \subset C_S$ if and only if S is coisotropic in $(M, L, \{-,-\})$. □

Definition 5.10. If equivalent conditions (1)-(3) in Proposition 5.9 are satisfied, then S is said to be a *regular coisotropic submanifold* of (M, C).

Remark 5.11. Differently from equivalence (1) $\Longleftrightarrow$ (2), in Proposition 5.9, equivalence (2) $\Longleftrightarrow$ (3) continues to hold also without assuming that $S = S_1$.

Remark 5.12. Let $(M, L, \{-,-\})$ be a Jacobi manifold. Then $(L, \{-,-\})$ is the Jacobi structure induced by a (necessarily unique) contact structure iff the associated bi-linear form $\widehat{\Lambda}_J : \wedge^2 J^1 L \to L$ is non-degenerate. In particular, Hamiltonian derivations of a contact manifold, exhaust all infinitesimal Jacobi automorphisms, and Hamiltonian vector fields exhaust all Jacobi vector fields.

5.2. **Coisotropic embeddings and L_∞-algebras from pre-contact manifolds.** From now till the end of this section we consider only closed regular coisotropic submanifolds. The intrinsic pre-contact geometry of a regular coisotropic submanifold S in a contact manifold M, contains a full information about the coisotropic embedding of S into M, at least locally around S. This is an immediate consequence of the *Tubular Neighborhood Theorem* in contact geometry (see [28], [33, Section 6], see also [11] for the analogous result in symplectic geometry).

Let (S, C_S) be a pre-contact manifold, with characteristic foliation $\mathcal{F}$.

Definition 5.13. A *coisotropic embedding* of (S, C_S) into a contact manifold (M, C) is an embedding $i : S \hookrightarrow M$ such that $(di)C_S = C|_{i(S)}$, and $(di)T\mathcal{F} = C|_{i(S)}^{\perp_\omega}$, where ω is the curvature form of (M, C).

Remark 5.14. Clearly, in view of Proposition 5.9, if $i : S \hookrightarrow M$ is a coisotropic embedding of (S, C_S) into (M, C), then $i(S)$ is a coisotropic submanifold of (M, C).

Let i_1 and i_2 be coisotropic embeddings of (S, C_S) into contact manifolds (M_1, C_1) and (M_2, C_2), respectively.

Definition 5.15. Coisotropic embeddings i_1 and i_2 are said to be *locally equivalent* if there exist open neighborhoods U_j of $\operatorname{Im} i_j$ in M_j, $j = 1, 2$, and a contactomorphism $\phi : (U_1, C_1) \to (U_2, C_2)$ such that $\phi \circ i_1 = i_2$.

Theorem 5.16 (Coisotropic embedding of pre-contact manifolds: existence and uniqueness)**.** *Every pre-contact manifold admits a coisotropic embedding. Additionally, any two coisotropic embeddings of a given pre-contact manifold are locally equivalent.*

Theorem 5.16 is a special case of Theorem 3 in [28]. We do not repeat the "uniqueness part" of the proof here. The "existence part" can be proved constructively via *contact thickening*. This is done for later purposes in the next subsection.

Corollary 5.17 (L_∞-algebra of a pre-contact manifold)**.** *Every pre-contact manifold determines a natural isomorphism class of L_∞-algebras.*

Proof. The "existence part" of Theorem 5.16 and Proposition 3.12 guarantee that a pre-contact manifold (S, C_S) determines a unique L_∞-algebra up to the choice of a coisotropic embedding $(S, C_S) \subset (M, C)$, a fat tubular neighborhood $\tau : NS \times_S \ell \hookrightarrow L$ of ℓ in L, where $\ell = TS/C_S$ and L is the Jacobi bundle of (M, C). Any two such L_∞-algebras are L_∞-isomorphic because of Proposition 3.18 and the "uniqueness part" of Theorem 5.16 □

5.3. **Contact thickening.** We now show that every pre-contact manifold (S, C_S) admits a coisotropic embedding into a suitable contact manifold uniquely determined by (S, C_S) up to the choice of a complementary distribution to the characteristic distribution. Thus, let (S, C_S) be a pre-contact manifold, $\mathcal{F}$ its characteristic foliation, $\ell = TS/C_S$ the quotient line bundle, and let $\theta : TS \to \ell$ be the structure form. Theorem 5.16 is a "contact version" of a theorem by Gotay [11] and can be proved by a similar technique as the *symplectic thickening* of [32]. Accordingly, we will speak about *contact thickening*. See also [33] for a relevant discussion on contact thickening in a different context.

Pick a distribution G on S complementary to $T\mathcal{F}$, and let $p_{T\mathcal{F};G} : TS \to T\mathcal{F}$ be the projection determined by the splitting $TS = G \oplus T\mathcal{F}$. Put $T_\ell{}^*\mathcal{F} := T^*\mathcal{F} \otimes \ell$, and let $\tau : T_\ell{}^*\mathcal{F} \to S$ be the natural projection map. We equip the manifold $E := T_\ell{}^*\mathcal{F}$ with the line bundle $L := \tau^*\ell$. The ℓ-valued 1-form θ can be pulled-back via τ to an L-valued 1-form $\tau^*\theta$ on $T_\ell{}^*\mathcal{F}$. There is also another L-valued 1-form θ_G on $T_\ell{}^*\mathcal{F}$. It is defined as follows: for $\alpha \in T_\ell{}^*\mathcal{F}$, and $\xi \in T_\alpha(T_\ell{}^*\mathcal{F})$

$$(\theta_G)_\alpha(\xi) := (\alpha \circ p_{T\mathcal{F};G} \circ d\tau)(\xi) \in \ell_x = L_\alpha, \quad x := \tau(\alpha),$$

where α is interpreted as a linear map $T_x\mathcal{F} \to L_x$. By definition, θ_G depends on the choice of splitting G.

Proposition 5.18. *Distribution $C := \ker(\theta_G + \tau^*\theta)$ is a contact structure on a neighborhood U of* $\operatorname{im}\mathbf{0}$*, the image of the zero section $\mathbf{0}$ of τ. Additionally $\mathbf{0}$ is a coisotropic embedding of (S, C_S) into the contact manifold $(U, C|_U)$.*

Proof. First of all, there is a local frame $X_1, \ldots, X_d, Y_1, \ldots, Y_{2n}, Z$ on S such that, locally

$$\Gamma(T\mathcal{F}) = \langle X_1, \ldots, X_d \rangle, \quad \Gamma(C_S) = \langle X_1, \ldots, X_d, Y_1, \ldots, Y_{2n} \rangle,$$
$$[X_i, X_j] = [X_i, Y_a] = 0, \quad (1 \le i \le j \le d,\ 1 \le a \le 2n).$$

Let $\alpha^1, \ldots, \alpha^d, \beta^1, \ldots, \beta^{2n}, \gamma$ be the dual co-frame. Then $\lambda := \theta(Z)$ is a local generator of $\Gamma(\ell)$. Moreover θ is locally given by $\theta = \gamma \otimes \lambda$, and the curvature form ω_S of C_S is locally given by

$$\omega_S = \frac{1}{2}\omega_{ab}\beta^a \wedge \beta^b \otimes \lambda,$$

for some local functions ω_{ab}. In particular, the skew-symmetric matrix $\mathbb{W} := (\omega_{ab})$ is non-degenerate. We will use the following local frame on S adapted to both C_S and G:

$$X_1, \ldots, X_d, V_1, \ldots, V_{2n}, W,$$

where $V_a := (\mathrm{id} - p_{T\mathcal{F};G})(Y_a)$, and $W := (\mathrm{id} - p_{T\mathcal{F};G})(Z)$. Denote by

$$\epsilon^1, \ldots, \epsilon^d, \beta^1, \ldots, \beta^{2n}, \gamma.$$

the dual co-frame. Now, let $p = (p_1, \ldots, p_d)$ be linear coordinates along the fibers of $\tau : T_\ell{}^*\mathcal{F} \to S$ associated with the local frame $\epsilon^1 \otimes \lambda, \ldots, \epsilon^d \otimes \lambda$. Then $X_1, \ldots, X_d, V_1, \ldots, V_{2n}, W, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_d}$ is a local frame on $T_\ell{}^*\mathcal{F}$. It is easy to check that locally

$$\Gamma(C) = \left\langle X'_1, \ldots, X'_d, V_1, \ldots, V_{2n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_d} \right\rangle,$$

where $X_i' := X_i - p_i W$, $(1 \leq i \leq d)$. Finally, the representative matrix of the curvature of C wrt the local frames $X_1', \ldots, X_d', V_1, \ldots, V_{2n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_d}$ of C and $W \operatorname{mod} C'$ of $T(T_\ell{}^*\mathcal{F})/C = L$ is

$$\begin{array}{c} \\ X' \\ V \\ \partial/\partial p \end{array} \begin{array}{c} \begin{array}{ccc} X' & V & \partial/\partial p \end{array} \\ \left\| \begin{array}{ccc} \mathbb{O} & \mathbb{O} & \mathbb{I} \\ \mathbb{O} & \mathbb{W} & \mathbb{O} \\ -\mathbb{I} & \mathbb{O} & \mathbb{O} \end{array} \right\| \end{array} \quad \text{up to infinitesimals } O(p) \tag{5.2}$$

This shows that C is maximally non-integrable around the zero section of $T_\ell{}^*\mathcal{F}$. Moreover, it immediately follows from (5.2) that the zero section of $T_\ell{}^*\mathcal{F}$ is a coisotropic embedding (transversal to fibers of τ). This concludes the proof. $\square$

The contact manifold $(U, C|_U)$ is called a *contact thickening* of (S, C_S). Now, let NS be the normal bundle of S in U. Clearly $NS = T_\ell{}^*\mathcal{F}$, hence $N_\ell S = T^*\mathcal{F}$. According to the proof of Corollary 5.17 the choice of the complementary distribution G determines an L_∞-algebra structure on $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] = \Gamma(\wedge^\bullet T^*\mathcal{F} \otimes \ell)[1]$. Moreover, such L_∞-structure is actually independent on the choice of G up to L_∞-isomorphisms. Sections of $\wedge^\bullet T^*\mathcal{F} \otimes \ell$ are ℓ-valued leaf-wise differential forms on S and we also denote them by $\Omega^\bullet(\mathcal{F}, \ell)$ (see below).

5.4. **The transversal geometry of the characteristic foliation.** Similarly as in the symplectic case (cf. [32, Section 9.3]), the multi-brackets in the L_∞-algebra of a pre-contact manifold can be expressed in terms of the "geometry transversal to the characteristic foliation". To write down this expression, the relevant transversal geometry needs to be described. Let (S, C_S) be a pre-contact manifold, with characteristic foliation $\mathcal{F}$. Denote by $N\mathcal{F} := TS/T\mathcal{F}$ the normal bundle to $\mathcal{F}$, and by $N^*\mathcal{F} = (N\mathcal{F})^* = T^0\mathcal{F} \subset T^*S$ the conormal bundle of $\mathcal{F}$.

Recall that $T\mathcal{F}$ is a Lie algebroid. The standard Lie algebroid differential in $\Omega^\bullet(\mathcal{F}) := \Gamma(\wedge^\bullet T^*\mathcal{F})$ will be denoted by $d_\mathcal{F}$ and called the *leaf-wise de Rham differential*. There is a flat $T\mathcal{F}$-connection ∇ in $N^*\mathcal{F}$ well-defined by

$$\nabla_X \eta := L_X \eta, \quad X \in \Gamma(T\mathcal{F}), \quad \eta \in \Gamma(N^*\mathcal{F}).$$

Remark 5.19. Connection ∇ is "dual to the Bott connection" in $N\mathcal{F}$.

As usual, ∇ determines a differential in $\Omega^\bullet(\mathcal{F}, N^*\mathcal{F}) := \Gamma(\wedge^\bullet T^*\mathcal{F} \otimes N^*\mathcal{F})$ denoted again by $d_\mathcal{F}$. There exists also a flat $T\mathcal{F}$-connection in ℓ, denoted again by ∇, and defined by

$$\nabla_X \theta(Y) := \theta([X, Y]), \quad X \in \Gamma(T\mathcal{F}), \quad Y \in \mathfrak{X}(M).$$

The corresponding differential in $\Omega^\bullet(\mathcal{F}, \ell) := \Gamma(\wedge^\bullet T^*\mathcal{F} \otimes \ell)$ will be also denoted by $d_\mathcal{F}$. Now, let $J^1_\perp \ell$ be the vector subbundle of $J^1\ell$ given by the kernel of the vector bundle epimorphism

$$\varphi_\nabla : J^1\ell \longrightarrow T^*\mathcal{F} \otimes \ell, \quad j^1_x \lambda \longmapsto (d_\mathcal{F}\lambda)_x.$$

Sections of $J^1_\perp \ell$ will be interpreted as sections of $J^1\ell$ "transversal to $\mathcal{F}$". Note also that the Spencer sequence $0 \to T^*S \otimes \ell \to J^1\ell \to \ell \to 0$ restricts to a "transversal Spencer sequence" $0 \to N^*\mathcal{F} \otimes \ell \to$

$J^1_\perp \ell \to \ell \to 0$ and the two fit in the following exact commutative diagram of vector bundle morphisms

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N^*\mathcal{F}\otimes \ell \arrow[r] \arrow[d] & J^1_\perp \ell \arrow[r] \arrow[d] & \ell \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T^*S\otimes \ell \arrow[r] \arrow[d] & J^1\ell \arrow[r] \arrow[d, "\varphi_\nabla"] & \ell \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T^*\mathcal{F}\otimes \ell \arrow[r, equal] \arrow[d] & T^*\mathcal{F}\otimes \ell \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
.
$$

In what follows embeddings $\gamma : T^*S \otimes \ell \hookrightarrow J^1\ell$ and $N^*\mathcal{F} \otimes \ell \hookrightarrow J^1_\perp \ell$ will be understood, and we will identify $df \otimes \lambda$ with $j^1(f\lambda) - fj^1\lambda$, for any $f \in C^\infty(S)$, and $\lambda \in \Gamma(\ell)$. Recall that an arbitrary $\alpha \in \Gamma(J^1\ell)$ can be uniquely decomposed as $\alpha = j^1\lambda + \eta$, with $\lambda \in \Gamma(\ell)$, and $\eta \in \Gamma(T^*S \otimes \ell)$. Then, by definition, for $p \in S$, α_p is in $J^1_\perp \ell$ iff $\varphi_\nabla(\eta_p) = -(d_\mathcal{F}\lambda)_p$. Finally, there is a flat $T\mathcal{F}$-connection in $J^1_\perp \ell$, also denoted by ∇, well-defined as follows. For $X \in \Gamma(T\mathcal{F})$ and $\alpha = j^1\lambda + \eta \in \Gamma(J^1_\perp \ell)$, with $\lambda \in \Gamma(\ell)$, $\eta \in \Omega^1(S, \ell)$ such that $\varphi_\nabla(\eta) = -d_\mathcal{F}\lambda$, put

$$\nabla_X(j^1\lambda + \eta) = j^1(\nabla_X\lambda) + \mathcal{L}_{\nabla_X}\eta, \tag{5.3}$$

where $\mathcal{L}_{\nabla_X}$ is the Lie derivative of ℓ-valued forms on S along derivation $\nabla_X \in \operatorname{Der}\ell$. Accordingly, there is a differential in $\Omega^\bullet(\mathcal{F}, J^1_\perp \ell) := \Gamma(\wedge^\bullet T^*\mathcal{F} \otimes J^1_\perp \ell)$ which we also denote by $d_\mathcal{F}$.

Now, note that the curvature form of (S, C_S), $\omega_S : \wedge^2 C_S \to \ell$, descends to a(n ℓ-valued) symplectic form $\omega_\perp : \wedge^2(C_S/T\mathcal{F}) \to \ell$. In particular, it determines a vector bundle isomorphism $\omega^\flat_\perp : C_S/T\mathcal{F} \to (C_S/T\mathcal{F})^* \otimes \ell$ (cf. Section 5.1).

Remark 5.20. Let $p \in S$, $X \in \mathfrak{X}(S)$, and $\lambda = \theta(X)$. Recall that $\phi_X \in \Gamma(C^*_S \otimes \ell)$ is defined by $\phi_X(Y) = \theta([X, Y])$, for all $Y \in \Gamma(C_S)$ (cf. Section 5.1). Then we have that $j^1_p\lambda \in J^1_\perp \ell$ iff $(\phi_X)_p \in (C_S/T\mathcal{F})^* \otimes \ell$. Furthermore it is easy to check that $j^1_p\lambda = 0$ if and only if the following holds:

(1) $X_p \in (C_S)_p$, and
(2) $\omega(X_p, Y_p) = \theta([X, Y]_p)$, for all $Y \in \mathfrak{X}(S)$ with $Y_p \in (C_S)_p$.

Therefore, if $j^1_p\lambda = 0$, then $X_p \bmod T_p\mathcal{F} = (\omega^\flat_\perp)^{-1}(\phi_X)_p$, and the following definition is well-posed.

Definition 5.21. Let $\sigma\widehat{\Lambda}^\#_\perp : J^1_\perp \ell \to N\mathcal{F}$ be the vector bundle morphism uniquely determined by:

$$\sigma\widehat{\Lambda}^\#_\perp(j^1_p\lambda) = X_p \bmod T_p\mathcal{F} - (\omega^\flat_\perp)^{-1}(\phi_X)_p, \tag{5.4}$$

where $p \in M$, $\lambda \in \Gamma(\ell)$, and $X \in \mathfrak{X}(S)$, such that $j^1_p\lambda \in J^1_\perp L$, and $\lambda = \theta(X)$.

Proposition 5.22. *There exists a vector bundle morphism $\widehat{\Lambda}_\perp : \wedge^2 J^1_\perp \ell \to \ell$ uniquely determined by putting*

$$\widehat{\Lambda}_\perp(\alpha, \alpha') = \theta([Y, Y']_p),$$

where $p \in M$, $\alpha, \alpha' \in (J^1_\perp \ell)_p$, and $Y, Y' \in \mathfrak{X}(S)$ are such that $\sigma\widehat{\Lambda}^\#_\perp(\alpha) = Y_p \bmod T_p\mathcal{F}$ and $\sigma\widehat{\Lambda}^\#_\perp(\alpha') = Y'_p \bmod T_p\mathcal{F}$.

Proof. Let $\alpha = j^1_p\lambda \in (J^1_\perp \ell)_p$, $p \in S$, and let Y be as in the statement. Equation (5.4) implies that $\theta(Y_p) = \lambda_p$ and Y can be chosen so that $\theta(Y) = \lambda$ globally. Thus, from (5.4) again, we get $(\phi_Y)_p = 0$. Now, let $\alpha' = 0$. Then $Y'_p \in T_p\mathcal{F} \subset (C_S)_p$ and $\theta([Y, Y']_p) = (\phi_Y)_p(Y'_p) = 0$. This shows that $\widehat{\Lambda}_\perp$ is well-defined. □

Vector bundle morphism $\widehat{\Lambda}_\perp : \wedge^2 J^1_\perp \ell \to \ell$ will be interpreted as the *transversal version* of the bi-linear form $\widehat{\Lambda}_J$ associated to a Jacobi bi-differential operator J.

5.5. **An explicit formula for the multi-brackets.** Retaining notations from previous subsection, choose a distribution G on S which is complementary to $T\mathcal{F}$, i.e. $TS = G \oplus T\mathcal{F}$. There is a dual splitting $T^*S \simeq T^*\mathcal{F} \oplus N^*\mathcal{F}$ and there are identifications $N\mathcal{F} \simeq G$, $T^*\mathcal{F} \simeq G^0$. Furthermore the induced splitting of $0 \to N^*\mathcal{F} \otimes \ell \to T^*S \otimes \ell \to T^*\mathcal{F} \otimes \ell \to 0$ lifts to a splitting of $0 \to J^1_\perp \ell \to J^1\ell \to T^*\mathcal{F} \otimes \ell \to 0$. Hence $J^1\ell \simeq J^1_\perp \ell \oplus (T^*\mathcal{F} \otimes \ell)$. Let $F \in \Gamma(\wedge^2 G^* \otimes TS/G)$ be the curvature form of G. The curvature F will be also understood as an element $F \in \Gamma(\wedge^2 N^*\mathcal{F} \otimes T\mathcal{F}) \subset \Gamma(\wedge^2(J^1_\perp \ell \otimes \ell^*) \otimes T\mathcal{F})$, where we used embedding $N^*\mathcal{F} \otimes \ell \hookrightarrow J^1_\perp \ell$.

Let $d_G : C^\infty(S) \to \Gamma(N^*\mathcal{F})$ be the composition of the de Rham differential $d : C^\infty(S) \to \Omega^1(S)$ followed by the projection $\Omega^1(S) \to \Gamma(N^*\mathcal{F})$ determined by decomposition $T^*S = T^*\mathcal{F} \oplus N^*\mathcal{F}$. Then d_G is a $\Gamma(N^*\mathcal{F})$-valued derivation of $C^\infty(S)$ and will be interpreted as "transversal de Rham differential".

Proposition 5.23. *There exists a unique degree zero, graded $\mathbb{R}$-linear map $\varepsilon : \Omega(\mathcal{F}) \to \Omega(\mathcal{F}, N^*\mathcal{F})$ such that*

1. $\varepsilon|_{C^\infty(S)} = d_G$,
2. $[\varepsilon, d_\mathcal{F}] = 0$, *and*
3. *the following identity holds*

$$\varepsilon(\tau \wedge \tau') = \tau \wedge \varepsilon(\tau') + (-)^{|\tau||\tau'|}\tau' \wedge \varepsilon(\tau),$$

for all homogeneous $\tau, \tau' \in \Omega(\mathcal{F})$.

In order to prove Proposition 5.23, the following Lemma will be useful:

Lemma 5.24. *Let f be a leaf-wise constant local function on S, i.e. $d_\mathcal{F} f = 0$, then $d_\mathcal{F} d_G f = 0$ as well.*

Proof. Let f be as in the statement. First of all, note that df takes values in $N^*\mathcal{F}$, so that $d_G f = df$. Now recall that $d_\mathcal{F} d_G f = 0$ iff $0 = \langle d_\mathcal{F} d_G f, X\rangle = \nabla_X d_G f = \mathcal{L}_X d_G f$ for all $X \in \Gamma(T\mathcal{F})$, where ∇ is the canonical $T\mathcal{F}$-connection in $N^*\mathcal{F}$. But $\mathcal{L}_X d_G f = \mathcal{L}_X df = d(Xf) = 0$. This completes the proof. □

Proof of Proposition 5.23. The graded algebra $\Omega(\mathcal{F})$ is generated in degree 0 and 1. In order to define ε, we first define it on the degree one piece $\Omega^1(\mathcal{F})$ of $\Omega(\mathcal{F})$. Thus, note that $\Omega^1(\mathcal{F})$ is generated, as a $C^\infty(S)$-module, by leaf-wise de Rham differentials $d_\mathcal{F} f \in \Omega^1(\mathcal{F})$ of functions $f \in C^\infty(S)$. The only relations among these generators are the following

$$\begin{aligned} d_\mathcal{F}(f+g) &= d_\mathcal{F} f + d_\mathcal{F} g, \\ d_\mathcal{F}(fg) &= f d_\mathcal{F} g + g d_\mathcal{F} f, \\ d_\mathcal{F} f &= 0 \text{ on every open domain where } f \text{ is leaf-wise constant,} \end{aligned} \tag{5.5}$$

where $f, g \in C^\infty(S)$, and $U \subset S$ is an open subset. Now define $\varepsilon : \Omega^1(\mathcal{F}) \to \Omega^1(\mathcal{F}, N^*\mathcal{F})$ on generators by putting

$$\varepsilon f := d_G f \quad \text{and} \quad \varepsilon d_\mathcal{F} f := d_\mathcal{F} d_G f,$$

and extend it to the whole $\Omega^1(S)$ by prescribing $\mathbb{R}$-linearity and the following Leibniz rule:

$$\varepsilon(f\sigma) = f\varepsilon(\sigma) + \sigma \otimes d_G f, \tag{5.6}$$

for all $f \in C^\infty(S)$, and $\sigma \in \Omega^1(S)$. In order to see that ε is well defined it suffices to check that it preserves relations (5.5). Compatibility with the first two relations can be checked by a straightforward computation that we omit. Compatibility with the third relation immediately follows from Lemma 5.24. Finally, in view of Leibniz rule (5.6), d_G and ε combine and extend to a well-defined derivation $\Omega(\mathcal{F}) \to \Omega(\mathcal{F}, N^*\mathcal{F})$. By construction, the extension satisfies all required properties. Uniqueness is obvious. □

The graded differential operator ε will be also denoted by d_G.

Similarly, there is a "transversal version of the first jet prolongation j^1". Namely, let $j^1_G : \Gamma(\ell) \to \Gamma(J^1_\perp \ell)$ be the composition of the first jet prolongation $j^1 : \Gamma(\ell) \to \Gamma(J^1\ell)$ followed by the projection $\Gamma(J^1\ell) \to \Gamma(J^1_\perp \ell)$ determined by decomposition $J^1\ell = J^1_\perp \ell \oplus (N^*\mathcal{F} \otimes \ell)$. Then j^1_G is a first order differential operator from $\Gamma(\ell)$ to $\Gamma(J^1_\perp \ell)$ such that

$$j^1_G(f\lambda) = f j^1_G \lambda + (d_G f) \otimes \lambda, \tag{5.7}$$

$\lambda \in \Gamma(\ell)$ and $f \in C^\infty(S)$, where, similarly as above, we understood the embedding $N^*\mathcal{F} \otimes \ell \hookrightarrow J^1_\perp \ell$. As announced, the operator j^1_G will be interpreted as "transversal first jet prolongation".

Proposition 5.25. *There exists a unique degree zero graded $\mathbb{R}$-linear map $\delta : \Omega(\mathcal{F}, \ell) \to \Omega(\mathcal{F}, J^1_\perp \ell)$ such that*

1. $\delta|_{\Gamma(\ell)} = j^1_G$,
2. $[\delta, d_\mathcal{F}] = 0$, *and*
3. *the following identity holds*

$$\delta(\tau \wedge \Omega) = \tau \wedge \delta(\Omega) + d_G \tau \otimes \Omega,$$

for all $\tau \in \Omega(\mathcal{F})$, and $\Omega \in \Omega(\mathcal{F}, \ell)$, where the tensor product is over $\Omega(\mathcal{F})$, and we understood both the isomorphism

$$\Omega(\mathcal{F}, N^*\mathcal{F}) \underset{\Omega(\mathcal{F})}{\otimes} \Omega(\mathcal{F}, \ell) \simeq \Omega(\mathcal{F}, N^*\mathcal{F} \otimes \ell) \tag{5.8}$$

and embedding $N^*\mathcal{F} \otimes \ell \hookrightarrow J^1_\perp \ell$.

In order to prove Proposition 5.25, the following Lemma will be useful:

Lemma 5.26. *Let μ be a leaf-wise constant local section of ℓ, i.e. $d_\mathcal{F}\mu = 0$, then $d_\mathcal{F} j^1_G \mu = 0$ as well.*

Proof. Let μ be as in the statement. First of all note that, by the very definition of $J^1_\perp \ell$, $j^1\mu$ takes values in $J^1_\perp \ell$ so that $j^1_G \mu = j^1 \mu$. Now recall that $d_\mathcal{F} j^1_G \mu = 0$ iff $0 = \langle d_\mathcal{F} j^1_G \mu, X\rangle = \nabla_X j^1_G \mu$ for all $X \in \Gamma(T\mathcal{F})$, where ∇ is the canonical $T\mathcal{F}$-connection in $J^1_\perp \ell$. But $\nabla_X j^1_G \mu = \nabla_X j^1 \mu = j^1 \nabla_X \mu = 0$, where we used (5.3). This completes the proof. □

Proof of Proposition 5.25. In this proof a tensor product $\otimes$ will be over $C^\infty(S)$ unless otherwise stated. We can regard $\Omega(\mathcal{F}, \ell) = \Omega(\mathcal{F}) \otimes \Gamma(\ell)$ as a quotient of $\Omega(\mathcal{F}) \otimes_\mathbb{R} \Gamma(\ell)$ in the obvious way. Our strategy is defining an operator $\delta' : \Omega(\mathcal{F}) \otimes_\mathbb{R} \Gamma(\ell) \to \Omega(\mathcal{F}, J^1_\perp \ell)$ and prove that it descends to an operator $\delta : \Omega(\mathcal{F}, \ell) \to \Omega(\mathcal{F}, J^1_\perp \ell)$ with the required properties. Thus, for $\sigma \in \Omega(\mathcal{F})$ and $\lambda \in \Gamma(\ell)$ put

$$\delta'(\sigma \otimes_\mathbb{R} \lambda) := \sigma \otimes j^1_G \lambda + d_G \sigma \otimes_{\Omega(\mathcal{F})} \lambda \in \Omega(\mathcal{F}, J^1_\perp \ell), \tag{5.9}$$

where, in the second summand, we understood both isomorphism (5.8) and embedding $N^*\mathcal{F} \otimes \ell \hookrightarrow J^1_\perp \ell$ (just as in the statement of the proposition). In order to prove that δ' descends to an operator δ on $\Omega(\mathcal{F}, \ell)$ it suffices to check that $\delta'(f\sigma \otimes_\mathbb{R} \lambda) = \delta'(\sigma \otimes_\mathbb{R} f\lambda)$ for all σ, λ as above, and all $f \in C^\infty(S)$. This can be easily obtained using the derivation property of d_G and (5.7). Now, Properties (1) and (3) immediately follows from (5.9). In order to prove Property (2), it suffices to check that $\delta d_\mathcal{F} \lambda = d_\mathcal{F} j^1_G \lambda$

for all $\lambda \in \Gamma(\ell)$ (and then use Property (3)). It is enough to work locally. Thus, let μ be a local generator of $\Gamma(\ell)$ with the further property that $d_{\mathcal{F}}\mu = 0$. Moreover, let $f \in C^\infty(S)$, and compute

$$\begin{aligned}\delta d_{\mathcal{F}}(f\mu) &= \delta(d_{\mathcal{F}} f \otimes \mu) = d_{\mathcal{F}} f \otimes j^1_G\mu + d_G d_{\mathcal{F}} f \otimes \mu = d_{\mathcal{F}} f \otimes j^1_G \mu + d_{\mathcal{F}} d_G f \otimes \mu \\ &= d_{\mathcal{F}}(f j^1_G \mu + d_G f \otimes \mu) = d_{\mathcal{F}}(j^1_G f\mu),\end{aligned}$$

where we used $d_{\mathcal{F}}\mu = 0$, Proposition 5.23, Lemma 5.26, and (5.7). Uniqueness of δ is obvious. □

The graded differential operator δ will be also denoted by j^1_G.

Now, interpret $\widehat{\Lambda}_\perp \in \Gamma(\wedge^2(J^1_\perp \ell)^* \otimes \ell)$ as a section $\# \in \Gamma((J^1_\perp \ell \otimes \ell^*)^* \otimes J^1_\perp \ell)$. The interior product of $\#$ and $F \in \Gamma(\wedge^2(J^1_\perp \ell \otimes \ell^*) \otimes T\mathcal{F})$ is a section $F^\# \in \Gamma(\operatorname{End}(J^1_\perp \ell) \otimes T\mathcal{F} \otimes \ell^*)$. For any $\mu \in \Omega(\mathcal{F}, \ell)$, the interior product of $F^\#$ and μ is a section $i_{F^\#}\mu \in \Omega(\mathcal{F}, \operatorname{End} J^1_\perp \ell)$. Now, we extend

(1) the bi-linear map $\widehat{\Lambda}_\perp : \wedge^2 J^1_\perp \ell \to \ell$ to a degree -1, $\Omega(\mathcal{F})$-bilinear, symmetric form
$$\langle -, - \rangle_C : \Omega(\mathcal{F}, J^1_\perp \ell)[1] \times \Omega(\mathcal{F}, J^1_\perp \ell)[1] \longrightarrow \Omega(\mathcal{F}, \ell)[1]$$
(2) the natural bilinear map $\circ : \operatorname{End} J^1_\perp \ell \otimes \operatorname{End} J^1_\perp \ell \to \operatorname{End} J^1_\perp \ell$ to a degree -1, $\Omega(\mathcal{F})$-bilinear map
$$\Omega(\mathcal{F}, \operatorname{End} J^1_\perp \ell)[1] \times \Omega(\mathcal{F}, \operatorname{End} J^1_\perp \ell)[1] \longrightarrow \Omega(\mathcal{F}, \operatorname{End} J^1_\perp \ell)[1],$$
also denoted by $\circ$, and
(3) the tautological action $\operatorname{End} J^1_\perp \ell \otimes J^1_\perp \ell \to J^1_\perp \ell$ to a degree -1, $\Omega(\mathcal{F})$-linear action
$$\Omega(\mathcal{F}, \operatorname{End} J^1_\perp \ell)[1] \times \Omega(\mathcal{F}, J^1_\perp \ell)[1] \longrightarrow \Omega(\mathcal{F}, J^1_\perp \ell)[1].$$

Theorem 5.27. *The first (unary) bracket in the L_∞-algebra structure on $\Omega(\mathcal{F}, \ell)[1]$ is $-d_{\mathcal{F}}$. Moreover, for $k > 1$, the k-th multi-bracket is given by*

$$\mathfrak{m}_k(\mu_1, \ldots, \mu_k) = \frac{1}{2} \sum_{\sigma \in S_k} \epsilon(\sigma, \boldsymbol{\mu}) \left\langle j^1_G \mu_{\sigma(1)}, (i_{F^\#}\mu_{\sigma(2)} \circ \cdots \circ i_{F^\#}\mu_{\sigma(k-1)}) j^1_G \mu_{\sigma(k)} \right\rangle_C, \tag{5.10}$$

for all $\mu_1 \ldots, \mu_k \in \Omega(\mathcal{F}, \ell)[1]$, where $\epsilon(\sigma, \boldsymbol{\mu})$ is the Koszul sign prescribed by the permutations of the μ's.

Proof. See Appendix C. □

Remark 5.28. The explicit form of the contact thickening (see Subsection 5.3) shows that the Jacobi bracket is actually fiber-wise entire. In particular Corollaries 4.16 and 4.24 always apply to the contact case.

6. Toy examples

In this short section we briefly discuss the formal deformation problem for the "simplest possible" coisotropic submanifolds, namely *Legendrian submanifolds* in a contact manifold, and their *flowout along a Jacobi vector field* (or, which is the same in this case, a contact vector field). Recall that the flowout along a Jacobi vector field of a coisotropic submanifold is again coisotropic (Example 3.4.(2)).

Now, let (M, C) be a contact manifold, and let $(L, J = \{-,-\})$ be the associated Jacobi structure. In particular, $\dim M = 2n + 1$ for some $n > 0$. Recall that a *Legendrian submanifold* of (M, C) is a locally maximal, hence n-dimensional, integral submanifold of the contact distribution. Equivalently, a Legendrian submanifold is as isotropic submanifold, which is additionally coisotropic wrt the Jacobi structure $(L, \{-,-\})$. Let $S \subset M$ be a Legendrian submanifold, $\ell = L|_S$, and let $\mu \in \Gamma(L)$ be such that $\mu_x \neq 0$, hence $(X_\mu)_x \notin T_x S$, for all $x \in S$. In what follows, we denote by $\mathcal{T}$ the flowout of S along the Hamiltonian vector field X_μ.

Remark 6.1. There exists a canonical vector bundle isomorphism $J^1\ell \to NS$ (over the identity) given by $j^1\lambda|_S \mapsto X_\lambda|_S \bmod TS$, for $\lambda \in \Gamma(L)$. Accordingly, there are canonical vector bundle isomorphisms $N^*S \simeq J_1\ell$ and $N_\ell{}^*S \simeq \operatorname{der}\ell$.

Recall that $J^1\ell$ is equipped with a canonical contact structure (see Example 5.5). The Legendrian tubular neighborhood theorem [28] asserts that there is a tubular neighborhood $NS \hookrightarrow M$ of S in M such that composition $J^1\ell \to NS \to M$ is a contactomorphism onto its image. Since we are interested in C^1-small coisotropic deformations of S, we can assume that $M = J^1\ell$ and identify S with the image of the zero section of the natural projection $J^1\ell \to S$.

Proposition 6.2. *Let $\{\mathfrak{m}_k\}$ be the L_∞-algebra structure on $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] = (\mathrm{Der}^\bullet \ell)[1]$ associated to the coisotropic submanifold S in the contact manifold $J^1\ell$. Then $\mathfrak{m}_k = 0$ for $k > 1$, and $\mathfrak{m}_1 = -d_{\operatorname{der}\ell,\ell}$, the opposite of the de Rham differential of the Atiyah algebroid $\operatorname{der}\ell$ with values in its tautological representation on ℓ.*

Proof. Recall that the Jacobi structure on $J^1\ell$ is fiber-wise linear (Example 2.16). Accordingly, the Jacobi bracket between

- fiber-wise constant sections is trivial,
- a fiber-wise constant and a fiber-wise linear section is fiber-wise constant,
- fiber-wise linear sections is fiber-wise linear.

Now, the assertion immediately follows from Equations (3.9), (3.10), (3.11). □

Remark 6.3. As a consequence of the above proposition, the formal deformation problem for Legendrian submanifolds is unobstructed. Even more, one can exhibit a canonical contracting homotopy for the complex $(\mathrm{Der}^\bullet\ell, d_{\operatorname{der}\ell,\ell})$ (see, for instance [34]). Hence, $\mathfrak{m}_1$ is acyclic and, as known to experts, all coisotropic, hence Lengendrian, sections of $J^1\ell \to S$ are actually trivial, i.e. they are Hamiltonian equivalent to S. In other words the moduli space of coisotropic deformations of a Legendrian submanifold is zero dimensional.

Proposition 6.4. *Let $\{\mathfrak{m}_k\}$ be the L_∞-algebra structure on $\Gamma(\wedge^\bullet(N\mathcal{T} \otimes L|^*_{\mathcal{T}}) \otimes L|_{\mathcal{T}})$ associated to the coisotropic submanifold $\mathcal{T}$ in the contact manifold $J^1\ell$. Then $\mathfrak{m}_k = 0$ for $k > 2$.*

Proof. The characteristic foliation $\mathcal{F}$ of $\mathcal{T}$ is one-co-dimensional. Accordingly, any distribution G complementary to $T\mathcal{F}$ is one-dimensional and, therefore, involutive. In particular, its curvature F vanishes. Now the assertion immediately follows from Theorem 5.27.

□

Corollary 6.5 (from Corollary 4.14)**.** *Let $\alpha = [s] \in H^1(N^*\mathcal{T} \otimes L|_{\mathcal{T}}, L|_{\mathcal{T}})$, where $s \in \Gamma(N\mathcal{T})$ is an infinitesimal coisotropic deformation of $\mathcal{T}$, i.e. $\mathfrak{m}_1 s = 0$. Then s can be prolonged to a formal coisotropic deformation iff $Kr(\alpha) = 0$.*

Remark 6.6. Let $\mu \in \Gamma(L)$ be as above. Since $\mu_x \neq 0$ for all $x \in S$, local contactomorphism $J^1\ell \to M$ can be chosen in such a way that μ identifies with a no-where zero, fiber-wise constant section of the Jacobi bundle on $J^1\ell$. In particular, $J^1\ell \simeq J^1(M) := J^1\mathbb{R}_M$ and X_μ identifies with the Reeb vector field on $J^1(M)$. It follows that Propositions 6.2 and 6.4 can be also proved from Proposition 3.17 and the explicit form of the Jacobi structure on $J^1(M)$ in jet coordinates (see, for instance, [2, Exercise 2.7]).

Appendix A. Derivations and infinitesimal automorphisms of vector bundles

Let M be a smooth manifold and let E, F be vector bundles over M. Recall that a (linear) k-th order differential operator from E to F is an $\mathbb{R}$-linear map $\Delta : \Gamma(E) \to \Gamma(F)$ such that

$$[[\cdots[[\Delta, a_0], a_1]\cdots], a_k] = 0$$

for all $a_0, a_1, \cdots, a_k \in C^\infty(M)$, where we interpret the functions a_i as operators (multiplication by a_i). There is a natural isomorphism between the $C^\infty(M)$-module $\mathrm{Diff}_k(E,F)$ of k-th order differential operators from E to F and the $C^\infty(M)$-module of sections of the vector bundle $\mathrm{diff}_k(E,F) := \mathrm{Hom}(J^kE, F)$, where J^kE is the vector bundle of k-jets of sections of E. The isomorphism $\Gamma(\mathrm{Hom}(J^kE,F)) \simeq \mathrm{Diff}_k(E,F)$ is given by $\phi \mapsto \phi \circ j^k$, where $\phi : \Gamma(J^kE) \to \Gamma(F)$ is a $C^\infty(M)$-linear map, and $j^k : \Gamma(E) \to \Gamma(J^kE)$ is the k-th jet prolongation. In particular $\mathrm{diff}_k(E, \mathbb{R}_M)$ is the dual bundle of J^kE. In this paper we often denote $J_1E := \mathrm{diff}_1(E, \mathbb{R}_M) = (J^1E)^*$.

Let $\Delta : \Gamma(E) \to \Gamma(F)$ be a k-th order differential operator. The correspondence

$$(a_1, \ldots, a_k) \longmapsto [[\cdots[\Delta, a_1]\cdots], a_k],$$

$a_1, \ldots, a_k \in C^\infty(M)$, is a well-defined symmetric, k-multi-derivation of the algebra $C^\infty(M)$ with values in $C^\infty(M)$-linear maps $\Gamma(E) \to \Gamma(F)$. In other words, it is a section of the vector bundle $S^kTM \otimes \mathrm{Hom}(E,F)$, called the *symbol of* Δ and denoted by σ_Δ. The symbol map $\sigma : \Delta \mapsto \sigma_\Delta$ sits in a short exact sequence

$$0 \longrightarrow \mathrm{Diff}_{k-1}(E,F) \longrightarrow \mathrm{Diff}_k(E,F) \xrightarrow{\sigma} \Gamma(S^kTM \otimes \mathrm{Hom}(E,F)) \longrightarrow 0, \tag{A.1}$$

of $C^\infty(M)$-modules. Note that Sequence (A.1) can be also obtained applying the contravariant functor $\mathrm{Hom}(-, \Gamma(\mathrm{F}))$ to the *Spencer sequence*

$$0 \longleftarrow \Gamma(J^{k-1}E) \longleftarrow \Gamma(J^kE) \xleftarrow{\gamma} \Gamma(S^kT^*M \otimes E) \longleftarrow 0,$$

where the inclusion γ, sometimes called the *co-symbol*, is given by

$$da_1 \cdot \cdots \cdot da_k \otimes e \longmapsto [[\cdots[j^k, a_1]\cdots], a_k]e,$$

$a_1, \ldots, a_k \in C^\infty(M)$, and $e \in \Gamma(E)$.

Now we focus on first order differential operators. In general, there is no natural $C^\infty(M)$-linear splitting of the Spencer sequence

$$0 \longleftarrow \Gamma(E) \longleftarrow \Gamma(J^1E) \xleftarrow{\gamma} \Gamma(T^*M \otimes E) \longleftarrow 0. \tag{A.2}$$

However, Sequence (A.2) splits via the first order differential operator $j^1 : \Gamma(E) \to \Gamma(J^1E)$. In particular, $\Gamma(J^1E) = \Gamma(E) \oplus \Gamma(T^*M \otimes E)$, and any section α of J^1E can be uniquely written as $\alpha = j^1\lambda + \gamma(\eta)$, for some $\lambda \in \Gamma(E)$, and $\eta \in \Gamma(T^*M \otimes E)$.

Now, let $\Delta : \Gamma(E) \to \Gamma(E)$ be a first order differential operator. The symbol of Δ is *scalar-type* if it is of the kind $X \otimes \mathrm{id}_{\Gamma(E)}$ for some (necessarily unique) vector field X. In other words $\Delta(fe) = X(f)e + f\Delta e$ for all $f \in C^\infty(M)$, and $e \in \Gamma(E)$. In this case we identify $\sigma(\Delta)$ with X, and call Δ a *derivation* of the vector bundle E (over the vector field X). The space of derivations of E will be denoted by $\mathrm{Der}\, E$. It is the space of section of a (transitive) Lie algebroid $\mathrm{der}\, E \to M$ over M, sometimes called the *Atiyah algebroid of* E, whose Lie-bracket is the commutator of derivations, and whose anchor is the symbol $\sigma : \mathrm{der}\, E \to TM$ (see, e.g., [24, Theorem 1.4] for details).

Remark A.1. If E is a line bundle, then every first order differential operator $\Gamma(E) \to \Gamma(E)$ is a derivation of E. Consider the line bundle $\mathbb{R}_M := M \times \mathbb{R}$. Then $\Gamma(\mathbb{R}_M) = C^\infty(M)$. First order differential operators $\Gamma(\mathbb{R}_M) \to \Gamma(\mathbb{R}_M)$ or, equivalently, derivations of $\mathbb{R}_M$, are the operators of the form $X + a : C^\infty(M) \to C^\infty(M)$, where X is a vector field on M and $a \in C^\infty(M)$ is interpreted as an operator (multiplication by a). Accordingly, in this case, there is a natural direct sum decomposition $\mathrm{Der}\, E = \mathfrak{X}(M) \oplus C^\infty(M)$, the projection $\mathrm{Der}\, E \to C^\infty(M)$ being given by $\Delta \mapsto \Delta 1$.

The construction of the Atiyah algebroid of a vector bundle is functorial, in the following sense. Let $\phi : E \to F$ be a morphism of vector bundles $E \to M$, $F \to N$, over a smooth map $\underline{\phi} : M \to N$. We assume that ϕ is *regular*, in the sense that it is an isomorphism when restricted to fibers. In particular, there is a morphism $\phi^\vee : E^* \to F^*$ of dual vector bundles over the same map $\underline{\phi}$. Morphism ϕ gives also rise to a morphism $\operatorname{der}\phi : \operatorname{der} E \to \operatorname{der} F$ of the associated Atiyah algebroids (over the same map $\underline{\phi}$) which is uniquely defined by the property that diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{End} E \arrow[r] \arrow[d, "\phi^\vee \otimes \phi"] & \operatorname{der} E \arrow[r, "\sigma"] \arrow[d, "\operatorname{der}\phi"] & TM \arrow[r] \arrow[d, "d\underline{\phi}"] & 0 \\ 0 \arrow[r] & \operatorname{End} F \arrow[r] & \operatorname{der} F \arrow[r, "\sigma"] & TN \arrow[r] & 0 \end{tikzcd}$$

commutes. We also denote $\phi_* := \operatorname{der}\phi$.

Derivations of a vector bundle E can be also understood as infinitesimal automorphisms of E as follows. First of all, a derivation Δ of E determines a derivation Δ^* of the dual bundle E^*, with the same symbol as Δ. Derivation Δ^* is defined by $\Delta^*\varphi := \sigma(\Delta) \circ \varphi - \varphi \circ \Delta$, where $\varphi : \Gamma(E) \to C^\infty(M)$ is a $C^\infty(M)$-linear form, i.e. a section of E^*. Now, recall that an automorphism of E is a fiber-wise linear, bijective bundle map $\phi : E \to E$. In particular, ϕ covers a (unique) diffeomorphism $\underline{\phi} : M \to M$. One can pull-back sections of E along an automorphism ϕ: the pull-back of section e is $\phi^* e := \phi^{-1} \circ e \circ \underline{\phi}$. An *infinitesimal automorphism* of E is a vector field Y on E whose flow consists of (local) automorphisms. In particular, Y projects onto a (unique) vector field $\underline{Y} \in \mathfrak{X}(M)$. Note that one parameter families of infinitesimal automorphisms generate one parameter families of automorphisms and vice-versa, and one parameter family of automorphisms is generated by a one parameter family of infinitesimal automorphisms. Infinitesimal automorphisms of E are sections of a (transitive) Lie algebroid over M, whose Lie-bracket is the commutator of vector fields on E, and whose anchor is $Y \mapsto \underline{Y}$. It can be proved that a vector field Y on E is an infinitesimal automorphism iff it preserves fiber-wise linear functions on E, i.e. sections of the dual bundle E^*. Finally, note that the restriction of an infinitesimal automorphism to fiber-wise linear functions $Y|_{\Gamma(E^*)} : \Gamma(E^*) \to \Gamma(E^*)$ is a derivation of E^*, and the correspondence $Y \mapsto Y|^*_{\Gamma(E^*)}$ is a well-defined isomorphism between the Lie algebroid of infinitesimal automorphisms and the Atiyah algebroid of E.

If Δ is a derivation of E, Y is the corresponding infinitesimal automorphism, and $\{\phi_t\}$ is its flow, then we will also say that Δ *generates the flow of automorphisms* $\{\phi_t\}$. We have

$$\left.\frac{d}{dt}\right|_{t=0} \phi_t^* e = \Delta e,$$

for all $e \in \Gamma(E)$. Similarly, if $\{\Delta_t\}$ is a smooth one parameter family of derivations of E, $\{Y_t\}$ is the corresponding one parameter family of infinitesimal automorphisms, and $\{\psi_t\}$ is the associated one parameter family of automorphisms, then we will say that $\{\Delta_t\}$ *generates* $\{\psi_t\}$. We have

$$\frac{d}{dt}\psi_t^* e = (\psi_t^* \circ \Delta_t)e.$$

A.1. **Vector valued Cartan calculus.** There is a vector bundle valued version of the standard Cartan calculus which is useful when dealing with abstract Jacobi manifolds. The following material can be presented in terms of Cartan calculus on Atiyah algebroids. Here we propose the simplest presentation for the purposes of this paper.

Let $E \to M$ be a vector bundle. A section of the graded bundle $\wedge^\bullet T^*M \otimes E$ is called an E-valued differential form on M. Differential forms with values in E form a graded $\Omega(M)$-module which we denote by $\Omega(M, E)$. Note that vector fields on M can be *contracted* with E-valued differential forms in an obvious way. For $X \in \mathfrak{X}(M)$ we denote by $i_X : \Omega(M, E) \to \Omega(M, E)$ the contraction operator. It is a degree -1 operator. On the other hand, in general there is no natural way how to define the Lie derivative of an E-valued form along X. Nonetheless, there is a natural notion of Lie derivative of an E-valued form along a derivation of E. Namely, let $\Delta \in \operatorname{Der} E$ be a derivation of E. There is a unique degree zero operator $\mathcal{L}_\Delta : \Omega(M,E) \to \Omega(M,E)$ such that

$$\mathcal{L}_\Delta e = \Delta e$$
$$\mathcal{L}_\Delta(\omega \wedge \Omega) = \mathcal{L}_{\sigma(\Delta)}\omega \wedge \Omega + \omega \wedge \mathcal{L}_\Delta \Omega,$$

for $e \in \Gamma(E)$ a zero degree E-valued form, $\omega \in \Omega(M)$, and $\Omega \in \Omega(M,E)$. Contractions with vector fields and Lie derivatives along derivations form a *vector valued Cartan calculus* in the sense that the following identities hold:

$$[i_X, i_Y] = 0, \qquad [\mathcal{L}_\Delta, i_X] = i_{[\sigma(\Delta),X]}, \qquad [\mathcal{L}_\Delta, \mathcal{L}_\nabla] = \mathcal{L}_{[\Delta,\nabla]},$$

where $X, Y \in \mathfrak{X}(M)$, $\Delta, \nabla \in \operatorname{Der} E$, and the bracket $[-,-]$ denotes the graded commutator. Moreover

$$\mathcal{L}_{f\Delta} = f\mathcal{L}_\Delta + df \wedge i_{\sigma(\Delta)}$$

for all $f \in C^\infty(M)$, and $\Delta \in \operatorname{Der} E$.

Appendix B. Gerstenhaber-Jacobi algebras

In this Appendix we recall the notion of a *Gerstenhaber-Jacobi algebra* [13]. We mainly follow Ref. [13]. However, we adopt a slightly more general approach in the same spirit as that of *abstract Jacobi manifolds* of Section 2. Accordingly, we will speak about *abstract Gerstenhaber-Jacobi algebras*.

Definition B.1. An *abstract Gerstenhaber-Jacobi algebra* is given by a graded commutative, (associative) unital algebra $\mathcal{A}$, a graded $\mathcal{A}$-module $\mathcal{L}$, and, moreover, a graded Lie bracket $[-,-]$ on $\mathcal{L}$ and an action by derivations, $\lambda \mapsto X_\lambda$, of $\mathcal{L}$ on $\mathcal{A}$ such that

$$[\lambda, a\mu] = X_\lambda(a)\mu + (-)^{|\lambda||a|}a[\lambda,\mu], \quad a \in A, \quad \lambda,\mu \in \mathcal{L}. \tag{B.1}$$

In particular $[\lambda,-]$ is a degree $|\lambda|$ graded first order differential operator with scalar-type symbol X_λ.

Remark B.2. In the case $\mathcal{L} = \mathcal{A}[1]$ we recover the notion of Gerstenhaber-Jacobi algebra as defined in [13].

Remark B.3. If $\operatorname{Ann}_{\mathcal{A}} \mathcal{L} = 0$, then condition $X_{[\lambda,\mu]} = [X_\lambda, X_\mu]$, for any $\lambda, \mu \in \mathcal{L}$, in Definition B.1, is redundant.

Remark B.4. Abstract Gerstenhaber-Jacobi algebras encompass several well known notions. Namely

- an abstract Jacobi structure $(L, \{-,-\})$ on a manifold M is the same as an abstract Gerstenhaber-Jacobi algebra with $\mathcal{A} = C^\infty(M)$, and $\mathcal{L} = \Gamma(L)$,
- a (graded) Jacobi algebra is the same as a Gerstenhaber–Jacobi algebra with $\mathcal{L} = \mathcal{A}$,
- a Gerstenhaber algebra is the same as a Gerstenhaber–Jacobi algebra with $\mathcal{L} = \mathcal{A}[1]$ and $X_a = [a,-]$, for all $a \in \mathcal{A}$,
- a graded Lie–Rinehart algebra is the same as a Gerstenhaber–Jacobi algebra such that $\lambda \mapsto X_\lambda$ is $\mathcal{A}$-linear.

We now describe the main Gerstenhaber-Jacobi algebra of interest in this paper. Given a smooth manifold M, a vector bundle $A \to M$, and a line bundle $L \to M$, we consider:

- the vector bundle $A_L := A \otimes L^*$,
- the graded algebra $\Gamma(\wedge^\bullet A_L)$ of sections of the exterior bundle $\wedge^\bullet A_L$,
- the graded $\Gamma(\wedge^\bullet A_L)$-module $\Gamma(\wedge^\bullet A_L \otimes L)[1]$ (note the shift in the degree).

The main reason why considering such objects is that a *Jacobi algebroid structure on* (A, L) *is equivalent to a Gerstenhaber-Jacobi algebra structure on* $(\Gamma(\wedge^\bullet A_L), \Gamma(\wedge^\bullet A_L \otimes L)[1])$ (Proposition 2.8).

In particular, let $A = \operatorname{der} L$ be the Atiyah algebroid of L, and note that $\operatorname{der} L \otimes L^* = \operatorname{diff}_1(L, \mathbb{R}_M)$. In the paper we often adopt the following notation: $J_1 L := \operatorname{diff}_1(L, \mathbb{R}_M)$. In this case, $\Gamma(\wedge^\bullet A_L) = \Gamma(\wedge^\bullet J_1 L)$ and it consists of alternating, first order multi-differential operators from $\Gamma(L)$ to $C^\infty(M)$, i.e. $\mathbb{R}$-multi-linear maps which are first order differential operators on each entry separately. Let $\Delta \in \Gamma(\wedge^k J_1 L)$, and $\Delta' \in \Gamma(\wedge^{k'} J_1 L)$. If we interpret Δ and Δ' as multi-differential operators, then their exterior product is given by

$$(\Delta \wedge \Delta')(\lambda_1, \ldots, \lambda_{k+k'}) = \sum_{\sigma \in S_{k,k'}} (-)^\sigma \Delta(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(k)}) \Delta'(\lambda_{\sigma(k+1)}, \ldots, \lambda_{\sigma(k+k')}) \tag{B.2}$$

where $\lambda_1, \ldots, \lambda_{k+k'} \in \Gamma(L)$, and $S_{k,k'}$ denotes (k,k')-unshuffles. Similarly, $\Gamma(\wedge^\bullet A_L \otimes L) = \Gamma(\wedge^\bullet J_1 L \otimes L)$ and it consists of alternating, first order multi-differential operators from $\Gamma(L)$ to itself. For this reason we often denote $\mathrm{Der}^\bullet L := \Gamma(\wedge^\bullet J_1 L \otimes L)$, where $\mathrm{Der}^0 L := \Gamma(L)$. Beware that an element of $\mathrm{Der}^k L$ is a multi-differential operator with k-entries but its degree in $(\mathrm{Der}^\bullet L)[1]$ is $k-1$. The $\Gamma(\wedge^\bullet J_1 L)$-module structure on $(\mathrm{Der}^\bullet L)[1]$ is given by the same formula (B.2) as above.

Remark B.5. A Jacobi bracket $\{-,-\}$ on L (see Section 2) will be also interpreted as an element J of $\mathrm{Der}^2 L$.

Proposition B.6. *For any line bundle L, there is a natural Gerstenhaber–Jacobi algebra structure on* $(\Gamma(\wedge^\bullet J_1 L), (\mathrm{Der}^\bullet L)[1])$.

Proof. Since the Atiyah algebroid of a line bundle is a Jacobi algebroid (Example 2.9), the proposition is an immediate corollary of Proposition 2.8. □

Finally, we describe explicitly the Gerstenhaber-Jacobi structure on $(\Gamma(\wedge^\bullet J_1 L), (\mathrm{Der}^\bullet L)[1])$. The Lie bracket on $(\mathrm{Der}^\bullet L)[1]$ is a "Jacobi version" of the Schouten bracket between multi-vector fields, therefore we call it the *Schouten-Jacobi bracket* and denote it by $[-,-]^{SJ}$. It is easy to see that

$$[\Box, \Box']^{SJ} := (-)^{kk'} \Box \circ \Box' - \Box' \circ \Box,$$

where $\Box \in \mathrm{Der}^{k+1} L$, $\Box' \in \mathrm{Der}^{k'+1} L$, and $\Box \circ \Box'$ is given by the following "*Gerstenhaber formula*":

$$(\Box \circ \Box')(\lambda_1, \ldots, \lambda_{k+k'+1}) = \sum_{\tau \in S_{k'+1,k}} (-)^\tau \Box(\Box'(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(k'+1)}), \lambda_{\tau(k'+2)}, \ldots, \lambda_{\tau(k+k'+1)}),$$

where $\lambda_1, \ldots, \lambda_{k+k'+1} \in \Gamma(L)$.

A direct computation shows that the action $\Box \mapsto X_\Box$ of $(\mathrm{Der}^\bullet L)[1]$ on $\Gamma(\wedge^\bullet J_1 L)$ is defined as follows. For $\Box \in \mathrm{Der}^{k+1} L$, the *symbol* of $\Box$, denoted by $\sigma_\Box \in \Gamma(TM \otimes \wedge^k J_1 L)$, is, by definition, the $\wedge^k J_1 L$-valued vector field on M implicitly defined by:

$$\sigma_\Box(f)(\lambda_1, \ldots, \lambda_k)\lambda := \Box(f\lambda, \lambda_1, \ldots, \lambda_k) - f\Box(\lambda, \lambda_1, \ldots, \lambda_k),$$

where $f \in C^\infty(M)$. Finally, for any $\Delta \in \Gamma(\wedge^l J_1 L)$, and $\Box \in \mathrm{Der}^{k+1} L$, section $X_\Box(\Delta) \in \Gamma(\wedge^{k+l} J_1 L)$ is given by

$$\begin{aligned} X_\Box(\Delta)(\lambda_1, \ldots, \lambda_{k+l}) := (-)^{k(l-1)} &\sum_{\tau \in S_{l,k}} (-)^\tau \sigma_\Box(\Delta(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(l)}))(\lambda_{\tau(l+1)}, \ldots, \lambda_{\tau(k+l)}) \\ &- \sum_{\tau \in S_{k+1,l-1}} (-)^\tau \Delta(\Box(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(k+1)}), \lambda_{\tau(k+2)}, \ldots, \lambda_{\tau(k+l)}). \end{aligned} \tag{B.3}$$

Remark B.7. Denote by $\mathfrak{X}^\bullet(M) = \Gamma(\wedge^\bullet TM)$ the Gerstehaber algebra of (skew-symmetric) multi-vector fields on M. When $L = \mathbb{R}_M$, then $\mathrm{Der}^k L = \Gamma(\wedge^k J_1 L)$. Moreover, there is a canonical direct sum decomposition $\mathrm{Der}^{k+1} L = \mathfrak{X}^{k+1}(M) \oplus \mathfrak{X}^k(M)$, where projection $\mathrm{Der}^{k+1} L \to \mathfrak{X}^k(M)$ is given by $\Box \mapsto \Box(1, -, \ldots, -)$. In particular, the Schouten-Jacobi bracket on $(\mathrm{Der}^\bullet L)[1]$ can be expressed in terms of the Schouten-Nijenhuis bracket on multi-vector fields (see [13] for more details).

Appendix C. Poissonization and pre-symplectization

C.1. **Poissonization of Jacobi manifolds.** The category of Poisson manifolds can be regarded as a (non-full) subcategory of the category of abstract Jacobi manifolds. Interestingly enough, the converse is also true: the category of abstract Jacobi manifolds can be regarded as a (non-full) subcategory of the category of Poisson manifolds. Specifically, an abstract Jacobi manifold can be regarded as an *homogeneous Poisson manifold*. Recall that an *homogeneous Poisson manifold* is a principal $\mathbb{R}^\times$-bundle $P \to M$ equipped with an *homogeneous Poisson bivector* Π_P, i.e., $[\Pi_P, \mathcal{E}]^{SN} = \Pi_P$, where $[-,-]^{SN}$ is the Schouten-Nijenhuis bracket and $\mathcal{E}$ is the *Euler vector field on* P, that is the fundamental vector field corresponding to the canonical generator 1 in the Lie algebra $\mathbb{R}$ of the structure group $\mathbb{R}^\times$ of P. The correspondence (actually a faithful but not full functor)

$$\{\text{abstract Jacobi manifolds}\} \longrightarrow \{\text{homogeneous Poisson manifolds}\}$$

can be described as follows. Let L be a line bundle on a smooth manifold M and let $L^* \to M$ be the dual line bundle of $L \to M$. Consider $\widetilde{M} := L^* \smallsetminus \mathbf{0}$, where $\mathbf{0}$ is the (image of) the zero section of L^*. For later purposes, denote by $\mathrm{pr} : \widetilde{M} \to M$ the projection. Note that $\widetilde{M} \to M$ is a principal $\mathbb{R}^\times$-bundle and every principal $\mathbb{R}^\times$-bundle is actually of this kind. The Euler vector field on L^* restricts to the Euler vector field $\mathcal{E}$ on $\widetilde{M}$ and a function $f \in C^\infty(\widetilde{M})$ is *homogeneous* if $\mathcal{E}(f) = f$. Clearly, sections of L are in one-to-one correspondence with homogeneous functions on $\widetilde{M}$. Denote by $\widetilde{\lambda} \in C^\infty(\widetilde{M})$ the homogeneous function corresponding to $\lambda \in \Gamma(L)$. Finally, let $\{-,-\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ be a Jacobi bracket. It is easy to see that there exists a unique homogeneous Poisson bracket $\{-,-\}_{\widetilde{M}} : C^\infty(\widetilde{M}) \times C^\infty(\widetilde{M}) \to C^\infty(\widetilde{M})$ such that

$$\{\widetilde{\lambda}, \widetilde{\mu}\}_{\widetilde{M}} = \widetilde{\{\lambda, \mu\}}.$$

The Poisson manifold $(\widetilde{M}, \{-,-\}_{\widetilde{M}})$ is, by definition, the *Poissonization* of the abstract Jacobi manifold $(M, L, \{-,-\})$. Note that if S is a coisotropic submanifold of $(M, L, \{-,-\})$, then $\widetilde{S} := \mathrm{pr}^{-1}(S)$ is a coisotropic submanifold of $(\widetilde{M}, \{-,-\}_{\widetilde{M}})$.

C.2. **Poissonization and L_∞-algebras from coisotropic submanifolds.** Let $(\widetilde{M}, \Pi = \{-,-\}_{\widetilde{M}})$ be a Poisson manifold, where Π is the Poisson bi-vector and $\{-,-\}_{\widetilde{M}}$ is the Poisson bracket. We can regard $\widetilde{M}$ as an abstract Jacobi manifold with Jacobi bundle $\mathbb{R}_{\widetilde{M}} = \widetilde{M} \times \mathbb{R}$ and Jacobi bracket $\{-,-\}_{\widetilde{M}}$. For simplicity we assume that $\widetilde{M} \to \widetilde{S}$ is a vector bundle over a manifold $\widetilde{S}$, and that $\widetilde{S}$ is a coisotropic submanifold. In particular we can construct an L_∞-algebra structure on $\Gamma(\wedge^\bullet N\widetilde{S})[1]$, applying the Voronov derived bracket construction [39] to the V-data $((\mathrm{Der}^\bullet \mathbb{R}_{\widetilde{M}})[1], \mathrm{Im}\, I, P, \{-,-\}_{\widetilde{M}})$. However, in this case, Cattaneo and Felder [4] indicate a slightly simpler way how to get the L_∞-algebra of $\widetilde{S}$. Namely, consider the graded Lie algebra $(\mathfrak{X}^\bullet(\widetilde{M})[1], [-,-]^{S\widetilde{N}})$ of multi-vector fields on $\widetilde{M}$, where $[-,-]^{SN}$ is the Schouten-Nijenhuis bracket. The Poisson bi-vector Π is a (degree one) Maurer-Cartan element in $\mathfrak{X}^\bullet(\widetilde{M})[1]$, i.e., $[\Pi, \Pi]^{SN} = 0$. There are V-data $(\mathfrak{X}^\bullet(\widetilde{M})[1], \mathrm{Im}\, \widetilde{I}, \widetilde{P}, \Pi)$ determining the same L_∞-algebra structure on $\Gamma(\wedge^\bullet N\widetilde{S})[1]$ as above. Namely, $\widetilde{P} : \mathfrak{X}^\bullet(\widetilde{M})[1] \to \Gamma(\wedge^\bullet N\widetilde{S})[1]$ is the composition of "restriction to S" and "projection over the normal part", while $\widetilde{I} : \Gamma(\wedge^\bullet N\widetilde{S})[1] \to \mathfrak{X}^\bullet(\widetilde{M})[1]$ is the "vertical lift" (we refer to [4] for more details). In the case when $(\widetilde{M}, \{-,-\}_{\widetilde{M}})$ is the Poissonization of a Jacobi manifold $(M, L, J = \{-,-\})$ and $\widetilde{S} = \mathrm{pr}^{-1}(S)$ for some coisotropic submanifold S in M, then the L_∞-algebra structure on $\Gamma(\wedge^\bullet N\widetilde{S})[1]$ can be understood as the *Poissonization of the L_∞-algebra structure on* $\Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1]$ in the sense explained below.

Let $(M, L, J = \{-,-\})$ be an abstract Jacobi manifold, and let $(\widetilde{M}, \Pi = \{-,-\}_{\widetilde{M}})$ be its Poissonization. Moreover, let $S \subset M$ be a coisotropic submanifold. For simplicity, we assume, additionally, that there is a vector bundle structure $M \to S$ (such that S identifies with the image of the zero section), and an isomorphism of line bundles $L := M \times_S \ell \to M$, where $\ell = L|_S$. Note, for later purposes, that $\widetilde{S} := \mathrm{pr}^{-1}(S) = \ell^* \smallsetminus \mathbf{0}$.

Theorem C.1 ("Poissonization" of the L_∞-algebra of a coisotropic submanifold)**.**

(1) *There exists a unique degree zero graded Lie algebra embedding* $(\mathrm{Der}^\bullet L)[1] \hookrightarrow \mathfrak{X}^\bullet(\widetilde{M})[1]$, $\square \mapsto \widetilde{\square}$ *such that*
$$\widetilde{\square}(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_k) = \square(\widetilde{\lambda_1, \ldots, \lambda_k}),$$
for all $\square \in \mathrm{Der}^k L$, *and* $\lambda_1, \ldots, \lambda_k \in \Gamma(L)$. *In particular,* $\Pi = \widetilde{J}$.

(2) *There exists a unique degree zero embedding of graded vector spaces* $j : \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] \hookrightarrow \Gamma(\wedge^\bullet N\widetilde{S})[1]$ *such that diagrams*

$$\begin{tikzcd} (\mathrm{Der}^\bullet L)[1] \arrow[r] & \mathfrak{X}^\bullet(\widetilde{U})[1] \\ \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] \arrow[u, "I"] \arrow[r, "j"] & \Gamma(\wedge^\bullet N\widetilde{S})[1] \arrow[u, "\widetilde{I}"'] \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} (\mathrm{Der}^\bullet L)[1] \arrow[r] \arrow[d, "P"'] & \mathfrak{X}^\bullet(\widetilde{U})[1] \arrow[d, "\widetilde{P}"] \\ \Gamma(\wedge^\bullet N_\ell S \otimes \ell)[1] \arrow[r, "j"] & \Gamma(\wedge^\bullet N\widetilde{S})[1] \end{tikzcd}$$

commute. In particular, j is a strict L_∞-algebra monomorphism.

The above theorem can be easily proved, e.g. in local coordinates, and we leave details to the reader.

C.3. **Pre-symplectization.** Not only one can "Poissonize an abstract Jacobi manifold", one can also "pre-symplectize a pre-contact manifold". In the contact case, the two constructions agree. Some details follow.

Let (S, C_S) be a pre-contact manifold, $\mathcal{F}$ its characteristic foliation, and $\ell := TS/C_S$. Recall that there is a canonical flat $T\mathcal{F}$-connection ∇ in ℓ (see subsection 5.4). Accordingly, there is a flat $T\mathcal{F}$-connection in L^*, the dual connection. Geometrically this corresponds to a foliation $\widetilde{\mathcal{F}}$ in L^*, such that $T\widetilde{\mathcal{F}}$ projects fiber-wise isomorphically onto $T\mathcal{F}$ via the bundle map $\ell^* \to S$, and, moreover, $\widetilde{\mathcal{F}}$ is linear in a suitable sense. Restrict $\widetilde{\mathcal{F}}$ to the open submanifold $\widetilde{M} := L^* \smallsetminus 0$, and denote again by $\widetilde{\mathcal{F}}$ the restriction. Since L^* identifies canonically with the annihilator of C_S in T^*S, $\widetilde{M}$ can be regarded as a submanifold in the symplectic manifold T^*S. Denote by $\widetilde{\omega}$ the pull-back to $\widetilde{S}$ of the canonical 2-form on T^*S (which we assume to be minus the tautological, Liouville, one form). The pair $(\widetilde{M}, \widetilde{\omega})$ is a pre-symplectic manifold with characteristic distribution given by $(T\widetilde{M})^{\perp_{\widetilde{\omega}}} = T\widetilde{\mathcal{F}}$ (see, e.g., [37, Theorem 15, case $n = 1$]). The pair $(\widetilde{S}, \widetilde{\omega})$ is the *pre-symplectization of* (S, C_S).

Remark C.2. Since $T\widetilde{\mathcal{F}}$ projects fiber-wise isomorphically onto $T\mathcal{F}$ via the projection $\mathrm{pr} : \widetilde{S} \to S$, then sections of $T\mathcal{F}$ can be canonically lifted to sections of $T\widetilde{\mathcal{F}}$. We denote by $X \mapsto \widehat{X}$ this lifting. Now, the pull-back $\mathrm{pr}^*\eta$ of a 1-form $\eta \in \Gamma(T^0\mathcal{F}) = \Gamma(N^*\mathcal{F})$ clearly belongs to $\Gamma(T^0\widetilde{\mathcal{F}}) = \Gamma(N^*\widetilde{\mathcal{F}})$. As a consequence, $\Omega(\widetilde{\mathcal{F}}) = C^\infty(\widetilde{S}) \otimes \Omega(\mathcal{F})$ (where the tensor product is over $C^\infty(S)$), and there is a unique well-defined morphism of DG algebras $\mathrm{pr}^* : \Omega(\mathcal{F}) \to \Omega(\widetilde{\mathcal{F}})$, $\sigma \mapsto \mathrm{pr}^*\sigma$ such that
$$(\mathrm{pr}^*\sigma)(\widehat{X}_1, \ldots, \widehat{X}_k) = \mathrm{pr}^*(\sigma(X_1, \ldots, X_k))$$
for all $X_1, \ldots, X_k \in \Gamma(T\mathcal{F})$, $\sigma \in \Omega^k(\mathcal{F})$. Equivalently, $\mathrm{pr}^*\sigma = 1 \otimes \sigma$, where 1 is the unit function on $\widetilde{S}$.

Our next aim is threefold:

(1) showing that the "symplectization trick" intertwines the "transversal geometries" of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$,
(2) showing that the symplectization of the contact thickening of (S, C_S) identifies with the symplectic thickening of $(\widetilde{S}, \widetilde{\omega})$ (see below, se also [32] for details about the symplectic thickening),

(3) using the above two points to regard the L_∞-algebra of (S, C_S) as a strict L_∞-subalgebra of the L_∞-algebra of $(\widetilde{S}, \widetilde{\omega})$ (see [32] for details about the L_∞-algebra of a pre-symplectic manifold).

As a corollary we will obtain a simple proof of Theorem 5.27 from the analogous result by Oh & Park in the symplectic case [32, Formula (9.17)]. In what follows, we only sketch the main proofs and leave obvious details and easy computations to the reader.

As in Subsection 5.5, choose a distribution G on S complementary to $T\mathcal{F}$. Obviously, the distribution $\widetilde{G} := (d\mathrm{pr})^{-1}(G)$ on $\widetilde{S}$ is complementary to $T\widetilde{\mathcal{F}}$. The main ingredients in Oh & Park formulas for the multi-brackets in the L_∞-algebra of $(\widetilde{S}, \widetilde{\omega})$ are

- the transversal Poisson bivector $\Pi_\perp \in \Gamma(\wedge^2 N\widetilde{\mathcal{F}})$,
- the associated graded symmetric $\Omega(\widetilde{\mathcal{F}})$-bilinear form
$$\langle -,-\rangle_{\widetilde{\omega}} : \Omega(\widetilde{\mathcal{F}}, N^*\widetilde{\mathcal{F}})[1] \times \Omega(\widetilde{\mathcal{F}}, N^*\widetilde{\mathcal{F}})[1] \longrightarrow \Omega(\widetilde{\mathcal{F}})[1],$$
- the transversal de Rham differential $d_{\widetilde{G}} : \Omega(\widetilde{\mathcal{F}}) \to \Omega(\widetilde{\mathcal{F}}, N^*\widetilde{\mathcal{F}})$,
- the curvature form $\widetilde{F}$ of $\widetilde{G}$.

We now show that the above items are uniquely determined by their contact analogues:

- the transversal bi-linear form $\widehat{\Lambda}_\perp \in \Gamma(\wedge^2 (J^1_\perp \ell)^* \otimes \ell)$,
- the associated graded symmetric $\Omega(\mathcal{F})$-bilinear form
$$\langle -,-\rangle_C : \Omega(\mathcal{F}, J^1_\perp \ell)[1] \times \Omega(\mathcal{F}, J^1_\perp \ell)[1] \longrightarrow \Omega(\mathcal{F}, \ell)[1],$$
- the transversal jet prolongation $j^1_G : \Omega(\widetilde{\mathcal{F}}, \ell) \to \Omega(\widetilde{\mathcal{F}}, J^1_\perp \ell)$,
- the curvature form F of G,

respectively. The first two items are actually independent of G. We start from them. Recall that sections of ℓ identify with homogeneous functions on $\widetilde{M}$ and we denote by $\lambda \mapsto \widetilde{\lambda}$ the identification. Moreover, $(\mathrm{Der}^\bullet \ell)[1]$ embeds into $\mathfrak{X}^\bullet(\widetilde{S})[1]$ (Theorem C.1.(1)), and we denote by $\square \mapsto \widetilde{\square}$ the embedding. We want to show that there is a "transversal version" of the latter embedding. Thus, denote $\mathrm{Der}^\bullet_\perp \ell := \Gamma(\wedge^\bullet (J^1_\perp \ell)^* \otimes \ell)$, and $\mathfrak{X}^\bullet_\perp := \Gamma(\wedge^\bullet N\widetilde{\mathcal{F}})$.

Lemma C.3. *There is a canonical embedding* $(\mathrm{Der}^\bullet_\perp \ell)[1] \hookrightarrow \mathfrak{X}^\bullet_\perp[1]$, $\square \mapsto \widetilde{\square}$.

Proof. Note that $\mathrm{Der}^\bullet_\perp \ell$ (resp. $\mathfrak{X}^\bullet_\perp$) is a quotient of $\mathrm{Der}^\bullet \ell$ (resp. $\mathfrak{X}^\bullet(\widetilde{S})$). Namely $\mathrm{Der}^\bullet_\perp \ell = \mathrm{Der}^\bullet \ell / I$, where I is the $\Gamma(\wedge^\bullet J_1 \ell)$-submodule generated by covariant derivatives along ∇, the $T\mathcal{F}$-connection in ℓ. Similarly $\mathfrak{X}^\bullet_\perp = \mathfrak{X}^\bullet(\widetilde{S})/\widetilde{I}$, where $\widetilde{I}$ is the (associative) ideal generated by $\Gamma(T\widetilde{\mathcal{F}})$. It is easy to see, for instance using local coordinates, that $\widetilde{\nabla_X} \in \widetilde{I}$ for all $X \in \Gamma(T\mathcal{F})$. This shows that embedding $\mathrm{Der}^\bullet \ell[1] \hookrightarrow \mathfrak{X}^\bullet(\widetilde{S})[1]$ descends to a well-defined map $(\mathrm{Der}^\bullet_\perp \ell)[1] \longrightarrow \mathfrak{X}^\bullet_\perp[1]$. Moreover, the latter is injective (again, use, for instance, local coordinates). $\square$

There is a transversal version of the Poisson bi-vector defined as follows. The pre-symplectic form $\widetilde{\omega}$ descends to a non-degenerate two-form $\widetilde{\omega} \in \Gamma(\wedge^2 N^*\widetilde{\mathcal{F}})$, whose inverse we denote by $\Pi_\perp \in \Gamma(\wedge^2 N\widetilde{\mathcal{F}}) = \mathfrak{X}^2_\perp$ and interpret as *transversal Poisson bi-vector*. it is easy to see that, if $\widehat{\Lambda}_\perp \in \mathrm{Der}^2_\perp \ell$ is the bi-linear form associated to (S, C_S). Then
$$\Pi_\perp = \widetilde{\widehat{\Lambda}_\perp}. \tag{C.1}$$

Our next aim is relating $\langle -,-\rangle_{\widetilde{\omega}}$ and $\langle -,-\rangle_C$. Pairing $\langle -,-\rangle_{\widetilde{\omega}} : \Omega(\widetilde{\mathcal{F}}, N^*\widetilde{\mathcal{F}})[1] \times \Omega(\widetilde{\mathcal{F}}, N^*\widetilde{\mathcal{F}})[1] \to \Omega(\widetilde{\mathcal{F}})[1]$ is the unique degree one, $\Omega(\widetilde{\mathcal{F}})$-bilinear, symmetric form extending $\Pi_\perp : \wedge^2 N^*\widetilde{\mathcal{F}} \to \mathbb{R}_{\widetilde{M}}$. In order to relate it to $\langle -,-\rangle_C$ we have to relate $\Omega(\widetilde{\mathcal{F}}, N^*\widetilde{\mathcal{F}})$ and $\Omega(\mathcal{F}, J^1_\perp L)$ first. Thus, note that embedding $\Gamma(\ell) \hookrightarrow C^\infty(\widetilde{S})$ uniquely extends to an $\Omega(\mathcal{F})$-linear embedding $\Omega(\mathcal{F}, \ell) \hookrightarrow \Omega(\widetilde{\mathcal{F}})$, $\sigma \mapsto \widetilde{\sigma}$. The latter does actually coincides with the embedding j of Proposition C.1, up to understanding S as a coisotropic submanifold in its contact thickening (see Proposition C.5.(2)).

Lemma C.4. *There is a canonical embedding* $\Gamma(J^1_\perp L) \hookrightarrow \Gamma(N^*\widetilde{\mathcal{F}})$, $\psi \mapsto \widetilde{\psi}$ *such that*

$$\langle \widetilde{\Box}, \widetilde{\psi} \rangle = \widetilde{\langle \Box, \psi \rangle}, \quad \Box \in \Gamma((J^1_\perp \ell)^*), \quad \psi \in \Gamma(J^1_\perp \ell). \tag{C.2}$$

Proof. First of all note that

$$d_{\widetilde{\mathcal{F}}} \widetilde{\lambda} = \widetilde{d_{\mathcal{F}} \lambda} \tag{C.3}$$

for all $\lambda \in \Gamma(\ell)$. Now, correspondence $\Gamma(\ell) \to \Omega^1(\widetilde{S})$, $\lambda \mapsto d\widetilde{\lambda}$, is a well-defined first order differential operator between $C^\infty(S)$-modules. Therefore, it determines a $C^\infty(S)$-linear map $\Gamma(J^1\ell) \to \Omega^1(\widetilde{S})$, which restricts to a well-defined $C^\infty(S)$-linear map $\Gamma(J^1_\perp \ell) \to \Gamma(N^*\widetilde{\mathcal{F}})$. Indeed, diagram

$$\begin{tikzcd}
\Omega^1(\widetilde{S}) \arrow[r] & \Omega^1(\widetilde{\mathcal{F}}) \\
\Gamma(J^1\ell) \arrow[r] \arrow[u] & \Omega^1(\mathcal{F}, \ell) \arrow[u]
\end{tikzcd}$$

commutes in view of Equation (C.3). Finally, Equation (C.2) can be easily checked, for instance, in local coordinates. $\Box$

Extend the embedding $\Gamma(J^1_\perp \ell) \hookrightarrow \Gamma(N^*\widetilde{\mathcal{F}})$ to an $\Omega(\mathcal{F})$-linear embedding $\Omega(\mathcal{F}, J^1_\perp \ell) \hookrightarrow \Omega(\widetilde{\mathcal{F}}, N^*\widetilde{\mathcal{F}})$, also denoted $\sigma \mapsto \widetilde{\sigma}$. It immediately follows from (C.1) that

$$\langle \widetilde{\sigma}, \widetilde{\tau} \rangle_{\widetilde{\omega}} = \widetilde{\langle \sigma, \tau \rangle}_C \tag{C.4}$$

for all $\sigma, \tau \in \Omega(\mathcal{F}, J^1_\perp \ell)$.

Now, let G and $\widetilde{G}$ be as above. Define $d_{\widetilde{G}}$ in the same way as d_G, and note that

$$d_{\widetilde{G}} \widetilde{\sigma} = \widetilde{j^1_G \sigma}. \tag{C.5}$$

It remains to relate the curvatures F and $\widetilde{F}$ of G and $\widetilde{G}$. Recall that F is a section of $\wedge^2 G^* \otimes G^0$, but $G^* \simeq N^*\mathcal{F}$, and $G^0 \simeq T\mathcal{F}$. Thus, F can be regarded as a section of $\wedge^2 N^*\mathcal{F} \otimes T\mathcal{F}$. Similarly, $\widetilde{F}$ is a section of $\wedge^2 N^*\widetilde{\mathcal{F}} \otimes T\widetilde{\mathcal{F}}$. Obviously, there is a well-defined map $\mathrm{pr}^* : \Gamma(\wedge^\bullet N^*\mathcal{F} \otimes T\mathcal{F}) \to \Gamma(\wedge^\bullet N^*\widetilde{\mathcal{F}} \otimes T\widetilde{\mathcal{F}})$, $\eta \otimes X \longmapsto \mathrm{pr}^*(\eta) \otimes \widehat{X}$, and it is easy to see that curvatures F and $\widetilde{F}$ are related via

$$\widetilde{F} = \mathrm{pr}^* F. \tag{C.6}$$

Now, we have to show that the symplectization of the contact thickening of (S, C_S) coincides with the *symplectic thickening* of $(\widetilde{S}, \widetilde{\omega})$, which is defined as follows (see [32] for more details). Take the cotangent bundle $T^*\widetilde{\mathcal{F}}$ to $\widetilde{\mathcal{F}}$, and let $\widetilde{\tau} : T^*\widetilde{\mathcal{F}} \to \widetilde{S}$ be the projection. The 2-form $\widetilde{\omega}$ can be pulled-back to $T^*\widetilde{\mathcal{F}}$ via $\widetilde{\tau}$. There is also another 2-form $\widetilde{\omega}_G$ on $T^*\widetilde{\mathcal{F}}$, defined as follows. First, define a 1-form $\widetilde{\theta}_G \in \Omega^1(T^*\widetilde{\mathcal{F}})$ by putting, for $\alpha \in T^*\widetilde{\mathcal{F}}$, and $\xi \in T_\alpha(T^*\widetilde{\mathcal{F}})$

$$(\widetilde{\theta}_G)_\alpha(\xi) := (\alpha \circ p_{T\widetilde{\mathcal{F}};\widetilde{G}} \circ d\widetilde{\tau})(\xi), \quad x := \widetilde{\tau}(\alpha),$$

where $p_{T\widetilde{\mathcal{F}};\widetilde{G}} : T\widetilde{S} \to T\widetilde{\mathcal{F}}$ is the projection induced by the splitting $T\widetilde{S} = \widetilde{G} \oplus T\widetilde{\mathcal{F}}$. Finally, put $\widetilde{\omega}_G := -d\theta_G$. The 2-form $\widetilde{\Omega} := \widetilde{\omega}_G + \widetilde{\tau}^*\omega$ is obviously closed. Moreover, it is non-degenerate in a neighborhood of the zero section of $\widetilde{\tau}$ called the *symplectic thickening of* $(\widetilde{S}, \widetilde{\omega})$, and the zero section of $\widetilde{\tau}$ is a coisotropic embedding of the pre-symplectic manifold $(\widetilde{S}, \widetilde{\omega})$.

Proposition C.5.

(1) Symplectization and thickening commute*, i.e., there is a canonical symplectomorphism* ψ *between the symplectization of the contact thickening and the symplectic thickening of the pre-symplectization of* (S, C_S).

(2) *Regard S as a coisotropic submanifold in its contact thickening so that* $\Omega(\mathcal{F},\ell) \simeq \Gamma(\wedge^\bullet N_\ell S \otimes \ell)$. *Moreover, regard $\widetilde{S}$ as a coisotropic submanifold in its symplectic thickening so that* $\Omega(\widetilde{\mathcal{F}}) \simeq \Gamma(\wedge^\bullet N\widetilde{S})$. *Then, symplectomorphism ψ identifies the canonical embedding* $\Omega(\mathcal{F},\ell)[1] \hookrightarrow \Omega(\widetilde{\mathcal{F}})[1]$ *with embedding* $j : \Gamma(\wedge^\bullet N_\ell S \otimes \ell) \hookrightarrow \Gamma(\wedge^\bullet N\widetilde{S})$ *of Theorem C.1.(2).*

Proof. The contact thickening is an open neighborhood of the zero section of $\tau : T^*\mathcal{F} \otimes \ell \to S$. The manifold $T^*\mathcal{F} \otimes \ell$ is equipped with the hyper-plane distribution $\ker(\theta_G + \tau^*\theta)$ (see Subsection 5.3). The Jacobi bundle $T(T^*\mathcal{F}\otimes\ell)/\ker(\theta_G+\tau^*\theta)$ identifies with the pull-back bundle $\tau^*\ell = (T^*\mathcal{F}\otimes\ell)\times_S \ell$. Hence, the symplectization of the contact thickening is an open submanifold in

$$\tau^*\ell^* \smallsetminus \mathbf{0} = (T^*\mathcal{F} \otimes \ell) \times_S (\ell^* \smallsetminus \mathbf{0}) = (T^*\mathcal{F} \otimes \ell) \times_S \widetilde{S}$$

As usual, we understand $\tau^*\ell^*\smallsetminus\mathbf{0}$ as a submanifold in $T^*(T^*\mathcal{F}\otimes\ell)$, identifying $\tau^*\ell^*$ with the annihilator of $\ker(\theta_G + \tau^*\theta)$. In particular, $\tau^*\ell^* \smallsetminus \mathbf{0}$ is equipped with a 2-form Ω' given by the pull-back of the canonical 2-form on $T^*(T^*\mathcal{F} \otimes \ell)$. Notice that Ω' is non-degenerate at a point p iff the curvature of $\ker(\theta_G + \tau^*\theta)$ is non-degenerate at the projection of p down to $T^*\mathcal{F} \otimes \ell$. We want to show that there is a canonical diffeomorphism

$$\psi : \tau^*\ell^* \smallsetminus \mathbf{0} \to T^*\widetilde{\mathcal{F}}$$

making the following diagram commutative

$$\begin{tikzcd} T^*\widetilde{\mathcal{F}} \arrow[d] & \tau^*\ell^* \smallsetminus \mathbf{0} \arrow[l, "\psi"'] \arrow[r] & T^*\mathcal{F} \otimes \ell \arrow[d] \\ \widetilde{S} \arrow[rr] & & S \end{tikzcd} , \tag{C.7}$$

and such that $\psi^*\widetilde{\Omega} = \Omega'$. In order to define ψ recall that $T\widetilde{\mathcal{F}}$ projects fiber-wise isomorphically to $T\mathcal{F}$, hence $T\widetilde{\mathcal{F}} = \widetilde{S} \times_S T\mathcal{F}$ and $T^*\widetilde{\mathcal{F}} = \widetilde{S} \times_S T^*\mathcal{F}$. Now, for $(\alpha,\varphi) \in \tau^*\ell^* \smallsetminus \mathbf{0} = (T^*\mathcal{F} \otimes \ell) \times_S \widetilde{S}$, put

$$\psi(\alpha,\varphi) = (\varphi, \varphi \circ \alpha),$$

where, in the second entry of the rhs, we interpret α as a linear map $T\mathcal{F} \to \ell$. A direct check shows that diagram (C.7) does actually commute. Moreover, ψ is invertible, its inverse $\psi^{-1} : T^*\widetilde{\mathcal{F}} \to \tau^*\ell^*\smallsetminus\mathbf{0}$ being given by

$$\psi^{-1}(\varphi, p) = (p \otimes \varphi^*, \varphi),$$

for all $(\varphi, p) \in T^*\widetilde{\mathcal{F}} = \widetilde{S} \times_S T^*\mathcal{F}$, where we interpret φ as a basis in a fiber of ℓ^*, and $\varphi^* \in \ell$ is its dual basis. One can easily check that $\psi^*\widetilde{\Omega} = \Omega'$ in local coordinates. Finally, an easy check in local coordinates proves (2). Details are left to the reader. □

With this preparation, we are finally ready to prove Theorem 5.27.

Proof of Theorem 5.27. Denote by $\widetilde{\mathfrak{m}}_k$ the multi-brackets in the L_∞-algebra of $(\widetilde{S},\widetilde{\omega})$ (determined by $\widetilde{G}$). According to Propositions C.5 and C.1, the $\mathfrak{m}_k$'s can be obtained by restricting the $\widetilde{\mathfrak{m}}_k$'s to forms in the image of embedding $j : \Omega(\mathcal{F},\ell)[1] \hookrightarrow \Omega(\widetilde{\mathcal{F}})[1]$. Now, the $\widetilde{\mathfrak{m}}_k$'s are given by Oh-Park formula [32, Formula (9.17)] up to a global normalization factor (See Appendix D). Specifically

$$\widetilde{\mathfrak{m}}_k(\varpi_1,\ldots,\varpi_k) = \frac{1}{2}\sum_{\sigma\in S_k} \epsilon(\sigma,\boldsymbol{\varpi}) \left\langle d_G\varpi_{\sigma(1)}, (i_{\widetilde{F}^\sharp}\varpi_{\sigma(2)} \circ \cdots \circ i_{\widetilde{F}^\sharp}\varpi_{\sigma(k-1)})\, d_G\varpi_{\sigma(k)} \right\rangle_C , \tag{C.8}$$

$\varpi_1,\ldots,\varpi_k \in \Omega(\widetilde{\mathcal{F}})$, where we extended the natural bilinear map $\circ : \operatorname{End} N\widetilde{S} \otimes \operatorname{End} N\widetilde{S} \to \operatorname{End} N\widetilde{S}$ to a degree -1, $\Omega(\widetilde{\mathcal{F}})$-bilinear map

$$\Omega(\widetilde{\mathcal{F}}, \operatorname{End} N\widetilde{S})[1] \times \Omega(\widetilde{\mathcal{F}}, \operatorname{End} N\widetilde{S})[1] \longrightarrow \Omega(\widetilde{\mathcal{F}}, \operatorname{End} N\widetilde{S})[1],$$

also denoted by $\circ$, and the tautological action $\operatorname{End} N\widetilde{S} \otimes N\widetilde{S} \to N\widetilde{S}$ to a degree -1, $\Omega(\mathcal{F})$-linear action

$$\Omega(\widetilde{\mathcal{F}}, \operatorname{End} N\widetilde{S})[1] \times \Omega(\widetilde{\mathcal{F}}, N\widetilde{S})[1] \longrightarrow \Omega(\widetilde{\mathcal{F}}, N\widetilde{S})[1].$$

To conclude the proof, it is enough to set $\varpi_i = \widetilde{\mu}_i$ and then using Equations (C.4), (C.5) and (C.6), and Lemma C.6 below. □

Lemma C.6. *For all $\mu \in \Omega(\mathcal{F}, \ell)$ and $\eta \in \Omega(\mathcal{F}, J^1_\perp \ell)$, $\widetilde{(i_{F^\sharp}\mu)\eta} = (i_{\widetilde{F}^\sharp}\widetilde{\mu})\widetilde{\eta}$.*

Proof. Let μ and η be as in the statement. Note that $i_{F^\sharp}\mu$ belongs to $\Omega(\mathcal{F}, \operatorname{End} J^1_\perp \ell)$ and it acts on η giving an element $(i_{F^\sharp}\mu)\eta$ in $\Omega(\mathcal{F}, J^1_\perp \ell)$. In its turn, $(i_{F^\sharp}\mu)\eta$ can be "lifted" to an element $\widetilde{(i_{F^\sharp}\mu)\eta}$ in $\Omega(\widetilde{\mathcal{F}}, N^*\widetilde{\mathcal{F}})$. Similarly, $(i_{\widetilde{F}^\sharp}\widetilde{\mu})\widetilde{\eta}$ belongs to $\Omega(\widetilde{\mathcal{F}}, N^*\widetilde{\mathcal{F}})$. The assertion can now be checked easily in local coordinates. □

Appendix D. The L_∞-algebra of a pre-symplectic manifold

In [32] the second named author and Park attach an L_∞-algebra to any coisotropic submanifold in a symplectic manifold (in fact, to any pre-symplectic manifold). They define the multi-brackets first [32, Formula (9.17)] and then prove the higher Jacobi identities. On another hand, in [4] Cattaneo and Felder use the Voronov construction [39] to attach an L_∞-algebra to any coisotropic submanifold in a Poisson manifold. Despite, in some paper [4, 35, 21, 36], it is implicitly stated that Cattaneo-Felder L_∞-algebra gives back Oh-Park L_∞-algebra in the symplectic case, a proof has not yet been provided. We provide such a proof in this section.

Let $(\widetilde{S}, \widetilde{\omega})$ be a pre-symplectic manifold, with characteristic foliation $\widetilde{\mathcal{F}}$, and let $\widetilde{G}$ be a complementary distribution to $T\widetilde{\mathcal{F}}$, i.e., $T\widetilde{S} = \widetilde{G} \oplus T\widetilde{\mathcal{F}}$. The bundle $T^*\widetilde{\mathcal{F}}$ is then equipped with a 2-form $\widetilde{\Omega}$ which is non-degenerate in a neighborhood of the zero section $\mathbf{0}$, called the *symplectic thickening of* $(\widetilde{S}, \widetilde{\omega})$. Moreover $\mathbf{0}$ is a coisotropic embedding and, therefore, every pre-symplectic manifold is a coisotropic submanifold in its symplectic thickening (see Subsection C.3 of previous Appendix for details). In particular, in view of Proposition 3.12, there is an L_∞-algebra $(\Gamma(\wedge^\bullet N\widetilde{S})[1], \{\widetilde{\mathfrak{m}}_k\})$ attached to $(\widetilde{S}, \widetilde{\omega})$. Notice that, in this case, $N\widetilde{S} = T^*\widetilde{\mathcal{F}}$, so that $\Gamma(\wedge^\bullet N\widetilde{S}) = \Omega(\mathcal{F})$. In what follows we will understand the latter identification. We will show below that the multi-brackets $\widetilde{\mathfrak{m}}_k$ are given precisely by formula (C.8) which is precisely Oh-Park formula (see [32, Formula (9.17)]) up to a global normalization factor 1/2. We will do this in local coordinates, using Corollary 3.17. From now on, we freely use notations and conventions from Subsection C.3 of previous Appendix.

Let (x^i, u^a) be local coordinates on $\widetilde{S}$ adapted to $\widetilde{\mathcal{F}}$, i.e. $T\widetilde{\mathcal{F}}$ is spanned by coordinate vector fields $\partial/\partial x^i$. Distribution $\widetilde{G}$ is then spanned by vector fields of the form

$$\mathbb{G}_a = \frac{\partial}{\partial u^a} + G^i_a \frac{\partial}{\partial x^i}.$$

The pre-symplectic form $\widetilde{\omega}$ is locally given by

$$\widetilde{\omega} = \frac{1}{2}\omega_{ab} du^a \wedge du^b.$$

Let p_i be linear coordinates along the fibers of $T^*\widetilde{\mathcal{F}}$ corresponding to the local frame $(\partial/\partial x^i)$ of $T\mathcal{F}$. It is shown in [32] that the symplectic form on the symplectic thickening is locally given by [32, Formula (6.8)]

$$\widetilde{\Omega} = \frac{1}{2}\left(\omega_{ab} + p_i F^i_{ab}\right) du^a \wedge du^b - \left(dp_i + p_k \frac{\partial G^k_a}{\partial x^i} du^a\right) \wedge \left(dx^i - G^i_b du^b\right)$$

where $F^i_{ab} := \mathbb{G}_a(G^i_b) - \mathbb{G}_b(G^i_a)$ are components of the curvature $\widetilde{F}$ of $\widetilde{G}$. More precisely,

$$\widetilde{F} = \frac{1}{2} F^i_{ab} du^a \wedge du^b \otimes \frac{\partial}{\partial x^i}.$$

Accordingly, the Poisson structure is

$$\Pi = -\frac{1}{2}\widetilde{\omega}^{ab} X_a \wedge X_b - \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial x^i},$$

where $(\widetilde{\omega}^{ab})$ is the inverse matrix of $(\widetilde{\omega}_{ab})$, $\widetilde{\omega}_{ab} := \omega_{ab} + p_i F^i_{ab}$, and

$$X_a := \mathbb{G}_a - p_j \frac{\partial G^j_a}{\partial x^i} \frac{\partial}{\partial p_i}.$$

The $\widetilde{\mathfrak{m}}_k$ are graded first order multi-differential operators. In particular, they are completely determined by their action on (local) generators of $\Omega(\widetilde{\mathcal{F}})$, i.e. on smooth functions $f \in C^\infty(\widetilde{S})$, and leaf-wise differentials $d_{\widetilde{\mathcal{F}}} x^i$. The rhs of Equation (C.8) is also a graded first order multi-differential operator in its arguments. We conclude that Equation (C.8) is satisfied, provided only it is satisfied for $\varpi_1, \ldots, \varpi_k$ being generators of the above mentioned kind.

An easy computation in local coordinates shows that $\widetilde{\mathfrak{m}}_1 = -d_{\mathcal{F}}$. Moreover, from Corollary 3.17 we easily find

$$\begin{aligned}
\widetilde{\mathfrak{m}}_{k+1}\left(d_{\widetilde{\mathcal{F}}} x^{i_1}, \ldots, d_{\widetilde{\mathcal{F}}} x^{i_{k-1}}, f, g\right) &= -(-)^k \left.\frac{\partial^{k-1}\widetilde{\omega}^{ab}}{\partial p_{i_1}\cdots\partial p_{i_{k-1}}}\right|_{\boldsymbol{p}=0} (\mathbb{G}_a f)(\mathbb{G}_b g),\\
\widetilde{\mathfrak{m}}_{k+1}\left(d_{\widetilde{\mathcal{F}}} x^{i_1}, \ldots, d_{\widetilde{\mathcal{F}}} x^{i_k}, f\right) &= (-)^k \sum_r \left.\frac{\partial^{k-1}\widetilde{\omega}^{ab}}{\partial p_{i_1}\cdots\widehat{\partial p_{i_r}}\cdots\partial p_{i_k}}\right|_{\boldsymbol{p}=0} \frac{\partial G^{i_r}_a}{\partial x^i}\, \mathbb{G}_b f\, \frac{\partial}{\partial p_i},\\
\widetilde{\mathfrak{m}}_{k+1}\left(d_{\widetilde{\mathcal{F}}} x^{i_1}, \ldots, d_{\widetilde{\mathcal{F}}} x^{i_{k+1}}\right) &= (-)^k \frac{1}{2}\sum_{r,s} \left.\frac{\partial^{k-1}\widetilde{\omega}^{ab}}{\partial p_{i_1}\cdots\widehat{\partial p_{i_r}}\cdots\widehat{\partial p_{i_s}}\cdots\partial p_{i_{k+1}}}\right|_{\boldsymbol{p}=0} \frac{\partial G^{i_r}_a}{\partial x^i}\frac{\partial G^{i_s}_b}{\partial x^j}\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial p_j},
\end{aligned} \tag{D.1}$$

for all $f, g \in C^\infty(\widetilde{S})$, $k > 0$. Now, we compute partial derivatives of $(\widetilde{\omega}^{ab})$. Denote $\mathbb{W} := (\omega_{ab})$, $\mathbb{F}^i := (F^i_{ab})$, and $\widetilde{\mathbb{W}} = (\widetilde{\omega}_{ab})$, so that $\widetilde{\mathbb{W}} = \mathbb{W} + p_i \mathbb{F}^i$, and $\widetilde{\mathbb{W}}^{-1} = (\widetilde{\omega}^{ab})$. Moreover, $\widetilde{\mathbb{W}}|_{\boldsymbol{p}=0} = \mathbb{W}$, and $\partial\widetilde{\mathbb{W}}/\partial p_i = \mathbb{F}^i$. Hence, it follows by induction on m that

$$\left.\frac{\partial^k \widetilde{\mathbb{W}}^{-1}}{\partial p_{i_1}\ldots\partial p_{i_m}}\right|_{\boldsymbol{p}=0} = (-)^m \sum_{\sigma\in S_k} \mathbb{W}^{-1}\mathbb{F}^{i_{\sigma(1)}}\mathbb{W}^{-1}\cdots\mathbb{F}^{i_{\sigma(m)}}\mathbb{W}^{-1},$$

which, used in (D.1), gives

$$\begin{aligned}
\widetilde{\mathfrak{m}}_{k+1}\left(d_{\widetilde{\mathcal{F}}} x^{i_1}, \ldots, d_{\widetilde{\mathcal{F}}} x^{i_{k-1}}, f, g\right) &= \sum_{\sigma\in S_{k-1}} \left(\mathbb{W}^{-1}\mathbb{F}^{i_{\sigma(1)}}\mathbb{W}^{-1}\cdots\mathbb{F}^{i_{\sigma(k-1)}}\mathbb{W}^{-1}\right)^{ab}(\mathbb{G}_a f)(\mathbb{G}_b g),\\
\widetilde{\mathfrak{m}}_{k+1}\left(d_{\widetilde{\mathcal{F}}} x^{i_1}, \ldots, d_{\widetilde{\mathcal{F}}} x^{i_k}, f\right) &= -\sum_{\sigma\in S_k} \left(\mathbb{W}^{-1}\mathbb{F}^{i_{\sigma(1)}}\mathbb{W}^{-1}\cdots\mathbb{F}^{i_{\sigma(k-1)}}\mathbb{W}^{-1}\right)^{ab}\frac{\partial G^{i_{\sigma(k)}}_a}{\partial x^i}\, \mathbb{G}_b f\, \frac{\partial}{\partial p_i},\\
\widetilde{\mathfrak{m}}_{k+1}\left(d_{\widetilde{\mathcal{F}}} x^{i_1}, \ldots, d_{\widetilde{\mathcal{F}}} x^{i_{k+1}}\right) &= -\frac{1}{2}\sum_{\sigma\in S_{k+1}} \left(\mathbb{W}^{-1}\mathbb{F}^{i_{\sigma(1)}}\mathbb{W}^{-1}\cdots\mathbb{F}^{i_{\sigma(k-1)}}\mathbb{W}^{-1}\right)^{ab}\frac{\partial G^{i_{\sigma(k)}}_a}{\partial x^i}\frac{\partial G^{i_{\sigma(k+1)}}_b}{\partial x^j}\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial p_j}.
\end{aligned} \tag{D.2}$$

Finally, from the easy remark that

$$d_{\widetilde{G}} f = \mathbb{G}_a f du^a, \quad \text{and} \quad d_{\widetilde{G}} d_{\widetilde{\mathcal{F}}} x^i = d_{\widetilde{\mathcal{F}}} d_{\widetilde{G}} x^i = \frac{\partial G^i_a}{\partial x^j} d_{\widetilde{\mathcal{F}}} x^j \otimes du^a,$$

it follows, after a straightforward computation, that the rhs of (D.2) agrees with Equation (C.8) which is, therefore, correct.

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