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Holomorphic curvature of Kähler Einstein metrics on generalised flag manifolds

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Introduction

One of the most challenging problems in Riemannian geometry is to develop a good understanding of the concept of curvature of a geometric object. That means how to quantify the deviation of a *curved* object from a *flat* Euclidean space. The mathematical answer to this question was given by Bernhard Riemann in his habilitation in 1854 and is now known as the Riemannian curvature tensor

$$R : TM \times TM \times TM \times TM \to \mathbb{R}$$

for the *curved* object of a Riemannian manifold (M, g). The amount of information of M encoded in R is very rich and until today we are far from having a complete understanding of R due to its complexity.

However, the symmetries satisfied by R imply that, at the point $p \in M$, it is completely determined by its values on tuples (X, Y, X, Y) for X, Y being orthonormal in (T_pM, g_p). This results in the definition of the sectional curvature of a tangent plane $\sigma = \langle X, Y \rangle_{\mathbb{R}}$

$$sec(\sigma) = R(X, Y, X, Y)$$

which corresponds to the Gauss curvature of the totally geodesic surface generated by σ.

Since sec still decodes the full complexity of R, mathematicians considered various reductions, such as the Ricci tensor and the scalar curvature, obtaining tractability at the expense of losing information about M. In the analysis of each of these curvature terms, the following fundamental questions arose. What characterises a Riemannian manifold (M, g) that has positive curvature? Is positive curvature a strong restriction? These questions are difficult to answer and we are therefore interested in finding examples in order to develop some intuition.

There are many famous partial results addressing these questions in the literature. We want to point out that these questions are very hard, even in the comparatively easy case of metrics with transitive isometry group. In fact, for example the the classification of homogeneous spaces with $sec > 0$ is the result of a long series of papers by Berger, Wallach, Aloff, Bérard Bergery, Wilking, Xu, Wolf ([Ber61][Wal72][AW75][BB76],[Wil99][XW15]) and has only recently been completed by Wilking and Ziller in [WZ18] closing the last remaining gap. For more details on the contributions of the various authors we refer the interested reader to the exposition on [WZ18].

If the Riemannian manifold had the additional structure of being Kähler, i.e. having a compatible parallel complex structure J, one can define another reduction of the curvature tensor. The so-called holomorphic sectional curvature is given by

$$H : SM \to \mathbb{R} \qquad X \mapsto sec(X \wedge JX)$$

being the sectional curvature of complex planes in the complex vector space (T_pM, J). As with the other curvatures, we ask the following questions. How strong is the condition that H is positive? What examples are known?

Similar to the Bonnet-Myers theorem for Ricci curvature, Tsukamoto proved in [Tsu57] that if the holomorphic curvature is bounded from below by a positive constant then the manifold is compact, simply connected and there is a upper bound for the diameter. Furthermore, as Klingenberg showed in [Kli61] the equality case is only satisfied if M has constant holomorphic curvature and biholomorphically isometric to complex projective space with its standard Kähler structure. Therefore one is left with the impression that the property of having positive holomorphic curvature seems similarly strong as having positive Ricci curvature.

The purpose of this thesis is to present evidence to support the impression that positive holomorphic curvature is not too strong by constructing new examples leading to an interesting conjecture. Following the idea of the Grove symmetry program, we treat the question of positive holomorphic curvature in the setting of a large isometry group, i.e. homogeneous Kähler manifolds. If we have positive holomorphic curvature on a homogeneous Kähler manifold, this immediately implies that H is bounded from below by a positive constant and hence we may restrict our attention to simply connected compact homogeneous Kähler manifolds.

These spaces are called *Kähler C spaces* or *generalised flag manifolds*. Their real representation turns out to be $M = G/K$ where G is a semisimple compact Lie group and the isotropy group K is the centraliser of a torus in G. This implies that K shares a maximal torus with G and, since it is a compact group, it factors up to covering into the product of its center and a semisimple factor K_{ss}. Via the isotropy action, K decomposes $T_K(G/K)$ into irreducible modules and roughly speaking we get the correlation that the larger K_{ss} is, the fewer irreducible modules there are. Resulting in the following rule of thumb:

The smaller the dimension of the center of K the nicer the description of invariant geometric objects on G/K.

Examples of these objects are of course the metric, the Riemannian curvature tensor and the holomorphic sectional curvature.

The only results known to the author pertaining to the positivity of the holomorphic sectional curvature of these spaces were obtained by Itoh in [Ito78]. His examples require that the tangent space decomposes into at most two irreducible modules with respect to the isotropy representation. This forces $b_2(M) = \dim(\mathfrak{z}(K))$ to be 1 and therefore these examples are on the easier side of the spectrum of C spaces. While his theorem covers already all cases where $b_2(M) = 1$ for G being a classical simple group and yields therefore a large family of examples. Nothing seems to be known about the cases of $b_2(M) > 1$.

The first result of the thesis is a generalisation of Itoh's theorem yielding the first C spaces with $b_2(M) > 1$:

Theorem 22. *Let (G, K, J) be a C space, such that $T_K(G/K)$ decomposes into at most three irreducible modules. Then the holomorphic sectional curvature of any Kähler metric is positive.*

This includes infinitely many new examples with $\dim(\mathfrak{z}(K)) = 2$, i.e.

$$SU(n+1)/S(U(k_1)\,U(k_2)\,U(k_3)),\ SO(2n)/SU_i(n-1)\,\mathbb{T}^2,\ E_6/SO(8)\mathbb{T}^2$$

where $k_1 + k_2 + k_3 = n+1$ and $SU_i(n-1) \hookrightarrow SO(2n)$ for $i = 1, 2$ denotes two non equivalent embeddings. Furthermore, we get eight new examples with $\dim(\mathfrak{z}(K)) = 1$, which are all quotients of exceptional groups. However, this result applies also only to *easier* C spaces, i.e. those with small b_2.

At the opposite end of the spectrum, we have the case of K being the maximal torus of G, i.e. $b_2(M) = rk(G)$, whose decomposition into irreducible modules is the most complicated. In particular, the amount of Kähler metrics for a given complex structure is fairly large. However, it is known that there exists, up to scaling, a unique Kähler Einstein metric g^J_{KE} for each C space (G, K, J) to which we will restrict our attention. The first result is that the holomorphic curvature of the Kähler Einstein metric is independent of the complex structure in the following sense:

Corollary 39. *For any two invariant complex structures J, J' on $G/\mathbb{T}$ there exists a biholomorphic isometry*

$$(G, \mathbb{T}, J, g^J_{KE}) \to (G, \mathbb{T}, J', g^{J'}_{KE}).$$

Hence the holomorphic curvature of the corresponding Kähler Einstein metric is independent of the chosen complex structure.

Therefore, we may fix a preferred complex structure J_{std}. In the case of larger isotropy groups the above does not hold any more. However, each classical C space is biholomorphically isometric to one with complex structure induced by J_{std} via the submersion $G/\mathbb{T} \to G/K$, which allows the further restriction to the spaces (G, K, J_{std}, g_{KE}).

In this context, we formulate the following conjecture $H(k)$ depending on $k \in \mathbb{N}$:

Conjecture 51. *Let $(G_k, K, J_{std}, g_{KE})$ be a simple Kähler Einstein C space with G being a classical Lie group of rank k. Then it has positive holomorphic sectional curvature.*

Since $H(k)$ being true implies having examples with $b_2(M) = k$, we see that we are leaving the *"easier"* end of the spectrum of C spaces. The naturally arising questions, we seek to answer are the following:

i) Is there a $k \in \mathbb{N}$ such that $H(k)$ is true?

ii) Is there a relation between $H(k_1)$ and $H(k_2)$ for $k_1 < k_2$?

Assuming that the answer to (i) is *yes* and the independence of the complex structure, the following question arises naturally:

Does every classical Kähler Einstein C space have positive holomorphic sectional curvature, i.e. is $H(k)$ true for all k? If not, what characterises the smallest k for which $H(k)$ is wrong?

The main results of this thesis are the following answers to questions *i)* and *ii)*:

Theorem 52, Theorem 54.

i) *$H(4)$ is true.*

ii) *If $H(n)$ is true so is $H(k)$ for $k \leq n$.*

Itoh's approach, which we also used to prove theorem 22, becomes very complicated and unfeasible in cases of larger $b_2(M)$. Hence, we choose a different strategyby describing the holomorphic sectional curvature H as a restriction of the quadratic form of the so-called holomorphic curvature tensor. This step allows us to use techniques developed in [Tho71] and [GZ81] to prove positivity for the $H(4)$ cases. This results in a variety of new examples with $1 < b_2(M) \leq 4$. As an exemplary application of the techniques we also prove

Theorem 47. *Every Kähler Einstein C space (G_2, K, J, g_{KE}) has positive holomorphic sectional curvature.*

Sadly, even in these cases it is hard to prove positivity and there does not seem to exist an easily generalisable pattern to prove $H(k)$ for an arbitrary k. However, we did not detect any kind of obstruction, which leads us to believe that $H(k)$ might actually be true for all $k \in \mathbb{N}$.

As the last results of this thesis, we analysed the implications of the assumption of $H(k)$ being true for all $k \in \mathbb{N}$. It is clear that $H(k)$ being true implies that $(G_k, \mathbb{T}^k, J_{std}, g_{KE})$ has positive holomorphic curvature. This alone leads to

Theorem 59. *If $(G_k, \mathbb{T}^k, J_{std}, g_{KE})$ has positive holomorphic curvature for all k, then all classical Kähler C spaces have nonnegative holomorphic curvature.*

Hence, finding a negatively curved complex plane in a classical Kähler C space with any complex structure and any compatible Kähler metric would imply that there is also a negatively curved plane for the Kähler Einstein metric of a C space with toric isotropy and hence disproves $H(k)$ for all k larger than a certain k^*. In fact if one relaxes the requirement from positive to nonnegative holomorphic curvature one even obtains an equivalence

Corollary 60. *The spaces $(G_k, \mathbb{T}^k, J_{std}, g_{KE})$ have nonnegative holomorphic curvature for all $k \in \mathbb{N}$ if and only if all classical Kähler C spaces have nonnegative holomorphic curvature.*

The thesis is structured in the following way. In order to be as self-contained as possible we present in chapter 1 all necessary requirements we will use throughout the thesis, including an introduction to the structure of Kähler C spaces, their Riemannian curvature tensor and the holomorphic curvature tensor. Furthermore, we introduce the general techniques of [Tho71] to prove positive curvature. We end the chapter with a description of the algebraic setting for the classical compact simple Lie groups including the definition of their standard complex structures.

Chapter 2 is dedicated to the proof of theorem 22 generalising Itoh's result. This chapter also describes explicitly all spaces to which we can apply the theorem, which are therefore new examples of Kähler C spaces with positive holomorphic curvature.

In chapter 3, we analyse in detail the structure of the holomorphic curvature tensor and how we can apply Thorpe's methods. Furthermore, we present the Kähler Einstein metrics on C spaces in general and in particular for the classical compact simple groups. In the end of the chapter we investigate the question of how strongly the choice of complex structure influences the problems at hand, resulting in the proof of corollary 39.

Chapter 4 serves two purposes. First, it gives a complete answer to the question of positive holomorphic curvature for Kähler Einstein C spaces with G being the exceptional group G_2, i.e. theorem 47. Second, it provides the reader with a small-dimensional example of how we use Thorpe's methods.

In chapter 5, we present a relationship between the holomorphic curvature tensor of the C spaces $(G, \mathbb{T}, J_{std})$ and (G, K, J_{std}). This will turn out useful in calculations in later chapters. However, as we point out, it is not enough to establish some kind of monotonicity of curvature in the sense of O'Neill's formula for Riemannian submersions.

Chapter 6 is dedicated to the formulation of $H(k)$, its monotonicity in k and the discussion of its consequences including proofs of theorem 59 and corollary 60.

Chapters 7 and 8 present the holomorphic curvature tensors in the case of classical compact simple groups and the discussion of the case of rank 4, i.e. the proof of $H(4)$.

Chapter 1

Preliminaries

We begin by giving an accurate description of the spaces considered and present how the symmetry translates all geometric objects involved into purely algebraic ones. To do so throughout this section we follow closely the introductory lines of [Arv92].

1.1 Reduction to the algebraic setting

Definition 1. *A Kähler C space M is a compact simply connected complex homogeneous space that carries a homogeneous Kähler metric.*

Then by [Wan54] and [BFR86], we know that there is a biholomorphism

$$M = (G/K, J)$$

where G is a real semisimple compact Lie group, K is the centralizer of a torus in G and J is a G invariant complex structure. In particular, this implies that G and K share a common maximal torus $\mathbb{T}$ which will allow us to exploit the Lie structure coming from the semisimplicity of G.

Definition 2. *The triple (G, K, J) without the choice of a fixed Kähler metric is called a C space.*

Remark: We call a Kähler C space simple or classical if the group G is simple or classical. If the Kähler metric is Einstein, we call the space a Kähler Einstein C space.
Let $p = eK \in M = G/K$. Then the following is useful to describe the geometric structures at hand using the homogeneity.

Proposition 3. *The restriction to T_pM induces one to one correspondences as follows*

i) Every homogeneous almost complex structure is uniquely determined by an element $J \in Aut(T_pM)^K$ with $J^2 = -id$.

ii) Every homogeneous metric is uniquely determined by a K invariant inner product on T_pM.

where the superscript K means invariance under the isotropy action of K.

We will use the same notation whether referring to the structure on G/K or its restriction to the tangent space.
Now we observe that by homogeneity of g, J and the Nijenhuis tensor

$$N_J(-,-) = [-,-] + J[J-,-] + J[-,J-] - [J-,J-]$$

the properties of J being an isometry with respect to g and integrable, i.e. $N_J = 0$, are true as long as they are true on T_pM. Hence it is natural to describe the tangent space at p in more detail. To that end, let $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$ be the Lie algebras of $\mathbb{T} \subset K \subset G$. Let B denote the Killing form of G and also its complexification to $\mathfrak{g}^{\mathbb{C}}$ and define $\mathfrak{m} = \mathfrak{k}^{\perp}$ with respect to B. Then it is well known , e. g. from [CE75], that the following map

$$\begin{aligned} \pi_K : \mathfrak{m} &\to T_K(G/K) \\ X \mapsto X_K^* &= \frac{d}{dt}\bigg|_{t=0} exp(tX)K \end{aligned}$$

defines a K equivariant linear isomorphism with respect to the Adjoint action on $\mathfrak{m}$ and the isotropy action on $T_K(G/K)$. Therefore, we can describe the metric and complex structure as well as the compatibility and integrability conditions in terms of $\mathfrak{m}$. After complexification, we have that $\mathfrak{t}^{\mathbb{C}} = \mathfrak{h}$ is a Jordan algebra for $\mathfrak{g}^{\mathbb{C}}$ and we obtain a root space decomposition, where $\Delta_{\mathfrak{g}}$ denotes the root system of G:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{g}}} \mathfrak{g}_\alpha$$

and by regularity of K, we get the further decomposition:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{k}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{m}}} \mathfrak{g}_\alpha$$

where $\Delta_{\mathfrak{k}} \subset \Delta_{\mathfrak{g}}$ is the root system of the semisimple part of K and $\Delta_{\mathfrak{m}} = \Delta_{\mathfrak{g}} \setminus \Delta_{\mathfrak{k}}$ is the set of K complementary roots.

1.2 Complex Structure and Metric

By proposition 3 and the isomorphism π_K the almost complex structure corresponds to an Ad_K equivariant map

$$J : \mathfrak{m} \to \mathfrak{m}$$

satisfying $J^2 = -\mathrm{id}$ and the integrability is equivalent to

$$N_J = 0.$$

In our particular case, there are significantly more explicit descriptions for the conditions on J and g. To that end, we consider the complexification of $\mathfrak{g}$ as in the previous section and consider the complexified version of J.

Then we get that J is diagonalizable with eigenvalues $\pm i$ and we define $\mathfrak{m}_\pm = Eig_{\pm i}(J)$. The $Ad(\mathbb{T})$ equivariance implies that the eigenspaces are sums of root spaces, i. e. we obtain a decomposition

$$\Delta_{\mathfrak{m}} = \Delta_{\mathfrak{m}}^+ \cup \Delta_{\mathfrak{m}}^-$$

where $\Delta_{\mathfrak{m}}^{\pm} = \{\alpha \in \Delta_{\mathfrak{m}} |\ \mathfrak{g}_\alpha \subset \mathfrak{m}_\pm\}$. Now we analyse what properties for $\Delta_{\mathfrak{m}}^+$ we can derive from K equivariance and the integrability condition.
First of all, we note that since J is the complexification of a real map it commutes with complex conjugation, which in turn implies

$$\mathfrak{m}_- = \overline{\mathfrak{m}_+}.$$

In addition, the fact that the roots take imaginary values on $\mathfrak{t}$ yields $\overline{\mathfrak{g}_\alpha} = \mathfrak{g}_{-\alpha}$ and therefore

$$\Delta_{\mathfrak{m}}^- = -\Delta_{\mathfrak{m}}^+.$$

The property corresponding to the equivariance of J is obtained from the following. From differentiation, we get that J commutes with ad_k for $k \in \mathfrak{g}_\gamma \subset \mathfrak{k}^{\mathbb{C}}$. Then we have for $X \in \mathfrak{g}_\alpha \subset \mathfrak{m}^+$

$$\begin{aligned} J([k, X]) =& J(ad_k(X)) = ad_k(J(X)) \\ &= ad_k(iX) = i(ad_k(X)) \\ &= i[k, X]. \end{aligned}$$

Since $[k, X]$ is a generator of $\mathfrak{g}_{\alpha+\gamma}$, we obtain the following property for $\Delta^+_{\mathfrak{m}}$. Let $\alpha \in \Delta^+_{\mathfrak{m}}$ and $\gamma \in \Delta_{\mathfrak{k}}$ with $\alpha + \gamma \in \Delta_{\mathfrak{g}}$ that

$$\alpha + \gamma \in \Delta^+_{\mathfrak{m}}.$$

By the Newlander-Nirenberg theorem, $N_J = 0$ is equivalent to $[\mathfrak{m}_+, \mathfrak{m}_+]_{\mathfrak{m}} \subset \mathfrak{m}_+$ and hence we derive the property: Let $\alpha, \beta \in \Delta^+_{\mathfrak{m}}$ with $\alpha + \beta \in \Delta_{\mathfrak{g}}$ then

$$\alpha + \beta \in \Delta^+_{\mathfrak{m}}.$$

In fact we have the following by [Arv92, Proposition 2]:

Proposition 4. *There is a one to one correspondence between complex structures on G/K and decompositions $\Delta_{\mathfrak{m}} = \Delta^+_{\mathfrak{m}} \cup \Delta^-_{\mathfrak{m}}$ satisfying*

i) $\Delta^-_{\mathfrak{m}} = -\Delta^+_{\mathfrak{m}}$

ii) For $\alpha \in \Delta^+_{\mathfrak{m}}$ and $\gamma \in \Delta_{\mathfrak{k}}$ with $\alpha + \gamma \in \Delta_{\mathfrak{g}}$, we have

$$\alpha + \gamma \in \Delta^+_{\mathfrak{m}}$$

iii) For $\alpha, \beta \in \Delta^+_{\mathfrak{m}}$ with $\alpha + \beta \in \Delta_{\mathfrak{g}}$, we have

$$\alpha + \beta \in \Delta^+_{\mathfrak{m}}$$

given by $J|_{\mathfrak{g}_\alpha} = i\ Id|_{\mathfrak{g}_\alpha}$ if and only if $\alpha \in \Delta^+_{\mathfrak{m}}$.

Now we want to determine similar properties for the K invariant inner product

$$g : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$$

defining the metric. Since the killing form B of G is negative definite and biinvariant on $\mathfrak{g}$, we can write

$$g(X, Y) = -B(PX, Y)$$

for all $X, Y \in \mathfrak{m}$ where $P : \mathfrak{m} \to \mathfrak{m}$ is a positive K equivariant isomorphism. Since we know that $\mathfrak{m}$ decomposes into inequivalent irreducible $Ad(\mathbb{T})$ modules $\mathfrak{m}_\alpha = \mathfrak{m} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ for $\alpha \in \Delta^+_{\mathfrak{m}}$, P decomposes into the sum of endomorphisms of the form

$$P_\alpha : \mathfrak{m}_\alpha \to \mathfrak{m}_\alpha$$

with $P_\alpha = g_\alpha Id_{\mathfrak{m}_\alpha}$ by Schur's lemma with $g_\alpha > 0$. Note that this already ensures that J is an isometry of g, i.e. commutes with P. In particular , this implies that J is an isometry for the killing form. Furthermore, the K equivariance of P implies with $k \in \mathfrak{g}_\gamma \subset \mathfrak{k}$ and $X \in \mathfrak{g}_\alpha \subset \mathfrak{m}^+$

$$\begin{aligned} g_{\alpha+\gamma}[k, X] &= P([k, X]) = P(ad_k(X)) \\ &= ad_k(P(X)) = ad_k(g_\alpha X) = g_\alpha(ad_k(X)) \\ &= g_\alpha[k, X]. \end{aligned}$$

So we get $g_{\alpha+\gamma} = g_\alpha$ for $\alpha \in \Delta^+_{\mathfrak{m}}$,$\gamma \in \Delta_{\mathfrak{k}}$ with $\alpha + \gamma \in \Delta^+_{\mathfrak{m}}$. In fact this is also a sufficient condition due to [Bor54]:

Proposition 5. *Every K invariant inner product g on $\mathfrak{m}$ is given by*

$$g = \sum_{\alpha \in \Delta^+_{\mathfrak{m}}} g_\alpha (-B)|_{\mathfrak{m}_\alpha \times \mathfrak{m}_\alpha}$$

for positive constants g_α satisfying

$$g_{\alpha+\gamma} = g_\alpha$$

for $\alpha \in \Delta^+_{\mathfrak{m}}$, $\gamma \in \Delta_{\mathfrak{k}}$ with $\alpha + \gamma \in \Delta^+_{\mathfrak{m}}$.

Now we want to consider the additional conditions g has to satisfy in order to induce a Kähler metric on G/K. By definition a hermitian metric g with a complex structure J is Kähler if and only if the induced 2 form is closed, i.e.

$$d\omega = 0$$

for $\omega(-,-) = g(J-,-)$ being the characteristic two form. In our particular case we see that this implies the following. Let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$ and $Z \in \mathfrak{g}_{-\alpha-\beta}$ with $\alpha, \beta, \alpha+\beta \in \Delta_\mathfrak{m}^+$. Then

$$\begin{aligned} 0 =& d\omega(X,Y,Z) \\ =& \omega([X,Y],Z) + \omega([Z,X],Y) + \omega([Y,Z],X) \\ =& g(J[X,Y],Z) + g(J[Z,X],Y) + g(J[Y,Z],X) \\ =& ig_{\alpha+\beta}(-B)([X,Y],Z) - ig_\beta(-B)([Z,X],Y) - ig_\alpha(-B)([Y,Z],X) \\ =& i(g_{\alpha+\beta} - g_\alpha - g_\beta)(-B)([X,Y],Z) \end{aligned}$$

By semisimplicity of G, that implies

$$g_{\alpha+\beta} = g_\alpha + g_\beta.$$

Hence we get

Proposition 6 ([WG68]). *A inner product on $\mathfrak{m}$ induces a homogeneous Kähler metric on (G,K) with the complex structure J if and only if it is of the form*

$$g = \sum_{\alpha \in \Delta_\mathfrak{m}^+} g_\alpha(-B)|_{\mathfrak{m}_\alpha \times \mathfrak{m}_\alpha}$$

for positive constants g_α satisfying

i) $g_{\alpha+\gamma} = g_\alpha$ *for* $\alpha \in \Delta_\mathfrak{m}^+, \gamma \in \Delta_\mathfrak{k}$, *and* $\alpha + \gamma \in \Delta_m^+$

ii) $g_{\alpha+\beta} = g_\alpha + g_\beta$ *for* $\alpha, \beta, \alpha+\beta \in \Delta_\mathfrak{m}^+$

We see that the second property implies a certain type of additivity. Together with the notion of bases of root systems this can be used to simplify further the construction of Kähler metrics. Since the set $\Delta_\mathfrak{m}$ is not a root system in the classical sense, we need to consider in some detail how $\Delta_\mathfrak{g}, \Delta_\mathfrak{k}$ and $\Delta_\mathfrak{m}$ interact. We do so via the following

Proposition 7. *Let (G,K,J,g) be a Kähler C space. Then for any choice of positive roots of $\Delta_\mathfrak{k}$ there is a unique choice of positive roots $\Delta_\mathfrak{g}^+$ of G such that the following holds*

i) $\Delta_\mathfrak{m}^+$ *determined by* J *is exactly* $\Delta_\mathfrak{m} \cap \Delta_\mathfrak{g}^+$

ii) $\Delta_\mathfrak{k}^+ = \Delta_\mathfrak{g}^+ \cap \Delta_\mathfrak{k}$

In particular, any base of $\Delta_\mathfrak{k}$ can be extended to a base of $\Delta_\mathfrak{g}$.

Proof. From set theoretic considerations it is clear that the two properties determine $\Delta_\mathfrak{g}^+$ uniquely. It remains to show, that it is in fact a set of positive roots. As mentioned before, the existence of a Kähler metric forces K to be the centralizer of a torus. In fact, [BFR86] proved that there has to be an $h_1 \in \mathfrak{z}(\mathfrak{k})$ such that

$$\omega(X,Y) = B(ad_{h_1}(X), Y),$$

i.e. $-P \circ J = ad_{h_1}$. If we apply this to an $X \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta_\mathfrak{m}^+$ we get

$$\begin{aligned} \alpha(ih_1)X &= ad_{ih_1}(X) = iad_{h_1}(X) \\ &= -iP \circ J(X) = P(X) = g_\alpha X. \end{aligned}$$

Hence $\alpha(ih_1) > 0$ for all $\alpha \in \Delta_\mathfrak{m}^+$ and $\gamma(ih_1) = 0$ for all $\gamma \in \Delta_\mathfrak{k}$ since h_1 lies in the center of $\mathfrak{k}$. On the other hand, any choice of positive roots $\Delta_\mathfrak{k}^+$ of $\Delta_\mathfrak{k}$ corresponds to an element

$h_2 \in \mathfrak{h}$ with the property that $\gamma(h_2) > 0$ for all positive roots of $\mathfrak{k}$. This is just the choice of a Weyl Chamber for $\mathfrak{k}$. Then there has to be a $t > 0$ such that with $h_3 = tih_1 + h_2$ we have

$$\alpha(h_3) > 0$$

for all $\alpha \in \Delta_{\mathfrak{m}}^+ \cup \Delta_{\mathfrak{k}}^+$. This proves *i)* and *ii)*. In order to conclude that we can extend bases, we recall that there is a canonical way to obtain a base of a given set of positive roots, i.e. the set of indecomposable roots. The last step is to see that an indecomposable element in $\Delta_{\mathfrak{k}}^+$ stays indecomposable in $\Delta_{\mathfrak{g}}^+$. In fact, let γ be a indecomposable element in $\Delta_{\mathfrak{k}}^+$ and assume $\gamma = \alpha + \beta$ with $\alpha, \beta \in \Delta_{\mathfrak{g}}^+ \setminus \Delta_{\mathfrak{k}} = \Delta_{\mathfrak{m}}^+$ then we have by property *ii)* of proposition 4

$$-\beta = \alpha - \gamma \in \Delta_{\mathfrak{m}}^+$$

which yields a contradiction. □

By the above we can fix a base $\Phi = \Phi_{\mathfrak{k}} \cup \Phi_{\mathfrak{m}}$ of $\Delta_{\mathfrak{g}}^+$ such that $\Phi_{\mathfrak{k}}$ is a base of $\Delta_{\mathfrak{k}}^+$. Let $r = |\Phi_{\mathfrak{m}}|$. Then we define the following projection:

$$\begin{aligned} \rho : \Delta_{\mathfrak{g}} &\to \mathbb{Z}^r \\ \sum_{\alpha \in \Phi} a_\alpha \alpha &\mapsto (a_\alpha)_{\alpha \in \Phi_{\mathfrak{m}}} \end{aligned} \tag{1.1}$$

By the properties of the base of a root system, we know $\rho(\Delta_{\mathfrak{m}}^+) \subset \mathbb{N}^r \setminus \{0\}$. Furthermore, we know that for any positive root α there is a string of simple roots $\alpha_1, ..., \alpha_l$ whose partial sum $\beta_s = \sum_{i=1}^s \alpha_i$ is a positive root and $\beta_l = \alpha$, see [Bou68, p. 159]. These known facts for root systems imply together with proposition 6 the following for the coefficients of a Kähler metric

$$g_\alpha = \sum_{\beta \in \Phi_{\mathfrak{m}}} \rho(\alpha)_\beta g_\beta. \tag{1.2}$$

This implies that a Kähler metric is completely determined by

$$r = |\Phi_{\mathfrak{m}}| = rk(G) - rk(K_{ss}) = \dim(\mathfrak{z}(\mathfrak{k}))$$

positive constants where K_{ss} is the semisimple factor of K. We want to remark here that the image of ρ corresponds to the so called $\mathfrak{t}$ *roots* . For a structured presentation of those we refer the reader to [Arv92, Chapter 2]. We reformulate with our notation the following result concerning the decomposition of $\mathfrak{m}$ into irreducible K modules, which will be significant for our first result:

Proposition 8 ([Arv92, Theorem 2])**.** *The image of ρ from* (1.1) *indexes the decomposition of $\mathfrak{m}$ into irreducible Ad_K modules, i.e.*

$$\mathfrak{m} = \bigoplus_{x \in \rho(\Delta_{\mathfrak{m}}^+)} \mathfrak{m}_x$$

where $\mathfrak{m}_x = \oplus_{\alpha \in \rho^{-1}(x)} \mathfrak{m}_\alpha$ is irreducible.

In the light of proposition 7, we shift the point of view slightly. Instead of starting with a pair (G, K) and constructing J, i.e. extending a $\Phi_{\mathfrak{k}}$ to Φ, we consider a pair (G, J) where G is a compact semisimple Lie group and J is a complex structure on $G/\mathbb{T}$ and we want to determine the groups $K \subset G$ such that J descends to a complex structure on G/K, i.e. given a Φ we consider the groups K with semisimple part induced by a set $\Phi_{\mathfrak{k}} \subset \Phi$.
These observations lead to the following construction of K:

i) Fix a complex structure on $G/\mathbb{T}$, i.e. a choice of positive roots of G.

ii) Fix a base Φ of simple roots for this choice of positive roots.

iii) Choose an arbitrary subset $\Phi_{\mathfrak{k}} \subset \Phi$ and define $\Delta_k = \Delta_{\mathfrak{g}} \cap \langle \Phi_{\mathfrak{k}} \rangle_{\mathbb{R}}$.

iv) Let K be the connected Lie subgroup of G with Lie algebra

$$\mathfrak{k} = \mathfrak{g} \cap (\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{k}}} \mathfrak{g}_\alpha)$$

In fact, the following shows that we obtain all isotropy groups this way.

Proposition 9. *Let K be constructed as above, then $\Delta^+_{\mathfrak{m}} = \Delta^+_{\mathfrak{g}} \backslash \Delta_{\mathfrak{k}}$ satisfies the requirements of proposition 4 and induces a complex structure $\bar{J}$ on G/K. This is the unique complex structure such that the projection*

$$\pi : (G/\mathbb{T}, J) \to (G/K, \bar{J})$$

is holomorphic. Equivalently, J is K equivariant. Furthermore, every K leaving J invariant is obtained this way.

Proof. First, we show that $\Delta^+_{\mathfrak{m}}$ satisfies the requirements of proposition 4, i.e. for $\alpha \in \Delta^+_{\mathfrak{m}}$ and $\beta \in \Delta_{\mathfrak{k}} \cup \Delta^+_{\mathfrak{m}}$ with $\alpha + \beta \in \Delta_{\mathfrak{g}}$ we have $\alpha + \beta \in \Delta^+_{\mathfrak{m}}$. Since every root is a linear combination of the simple roots Φ with either all positive or all negative coefficient it is sufficient to show that there are positive coefficients in the expression of $\alpha + \beta$ in terms of Φ. This follows easily from the simple fact that $\rho(\alpha + \beta) = \rho(\alpha) + \rho(\beta)$ lies in $\mathbb{N}^r \setminus \{0\}$ for $\alpha \in \Delta^+_{\mathfrak{m}}$ and $\beta \in \Delta^+_{\mathfrak{m}} \cup \Delta_{\mathfrak{k}}$. The fact that the projection is holomorphic is immediate since J and $\bar{J}$ coincide when pulled back to $\mathfrak{m}$.

That any K arises this way is just proposition 7 and the trivial observation that an indecomposable element in $\Delta^+_{\mathfrak{g}}$ is also indecomposable in $\Delta^+_{\mathfrak{k}}$. □

1.3 Painting Dynkin diagrams

Uniting propositions 7 and 9 we obtain the following

Theorem 10. *Consider $(G, \mathbb{T}, J)$, where G is a compact semisimple Lie group, $\mathbb{T}$ its maximal torus and J a complex structure on $G/\mathbb{T}$. Let $D = (V, E)$ be the Dynkin diagram corresponding to the base Φ of positive roots determined by J. Then every subgroup $K \subset G$ leaving J invariant corresponds to the choice of a set $V_K \subset V$. In particular, $K = \mathbb{T}K_{ss}$ where the semisimple part K_{ss} has the Dynkin diagram*

$$D_K = (V_K, E_K)$$

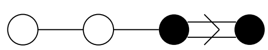
where $E_K = \{(v, w) \in E \mid v, w \in V_K\}$.

Remark:
1) This can be visualized by painting the sub Dynkin diagram $D_K \subset D$ black. This will be used frequently throughout the following chapters.
2) Remember that the vertices of the Dynkin diagram are exactly the simple roots, i.e. V_K corresponds to $\Phi_{\mathfrak{k}}$.

As an example we present the painted Dynkin diagram representing the Kähler C space

$$SO(9)/\mathbb{T}^2 SO(5)$$

for $SO(5)$ being the lower 5×5 block in $SO(9)$ together with the complex structure J_{std} which will be defined later:



1.4 Riemannian Curvature Tensor

1.4.1 Chevalley Basis

We begin this chapter with the choice of a special basis, which will be useful for explicit calculations of the curvature tensor.
By non degeneracy of B on $\mathfrak{h}$ there is a unique element $H_\alpha \in \mathfrak{h}$ satisfying

$$B(H_\alpha, H) = \alpha(H) \tag{1.3}$$

for all $H \in \mathfrak{h}$. It is known, that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional and spanned by H_α. Hence the interesting properties of the following are actually the later two. Then we can get the following basis:

Proposition 11 ([Hel01]). *There exists a basis $\mathcal{B} = \{E_\alpha \in \mathfrak{g}_\alpha |\ \alpha \in \Delta_\mathfrak{g}\}$ of $\bigoplus_{\alpha \in \Delta_\mathfrak{g}} \mathfrak{g}_\alpha$ satisfying the following conditions:*

i) $[E_\alpha, E_{-\alpha}] = z_\alpha H_\alpha$ *for* $\alpha \in \Delta_\mathfrak{g}$ *and* $z_\alpha \in \mathbb{C}$.

ii) $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ *for* $\alpha \neq \beta$

iii) $\overline{E_\alpha} = -E_{-\alpha}$.

The $N_{\alpha,\beta}$ satisfy the following relations:

a) $N_{\alpha,\beta} = 0$ *if* $\alpha + \beta \notin \Delta_\mathfrak{g}$

b) $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$

Remark: Furthermore, it is possible to require the z_α to be 1 for all α. We refrain from requiring it, since the above is strong enough to make actual calculations and it is easier to find a basis with arbitrary z_α. In addition, we observe that it is easy to determine $z_\alpha = B(E_\alpha, E_{-\alpha})$ since

$$\alpha(H) B(E_\alpha, E_{-\alpha}) = B(H, [E_\alpha, E_{-\alpha}]) = z_\alpha \alpha(H)$$

holds for all H. This implies the z_α to be real and $z_{-\alpha} = z_\alpha$.

1.4.2 Curvature formulae

Now we turn over to the Riemannian curvature tensor. To that end, let $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be the symmetric tensor defined by

$$2g(U(X,Y), Z) = g(ad_Z X, Y) + g(X, ad_Z Y).$$

This enables us to define

$$\begin{aligned} \Lambda : \mathfrak{m} \times \mathfrak{m} &\to \mathfrak{m} \\ (X,Y) &\mapsto \frac{1}{2}[X,Y]_\mathfrak{m} + U(X,Y) \end{aligned}$$

where the index $\mathfrak{m}$ denotes the projection. Due to [Nom54] with this notation the curvature tensor is given by

$$\begin{aligned} R(X,Y) : \mathfrak{m} &\to \mathfrak{m} \\ Z &\mapsto -[\Lambda(X), \Lambda(Y)]Z + \Lambda([X,Y]_\mathfrak{m})Z + [[X,Y]_\mathfrak{k}, Z] \end{aligned}$$

As before we will denote the complexifications with the same symbol. Furthermore, we want to remark here that with this sign convention the sectional curvature is given by

$$sec(X \wedge Y) = \frac{g(R(X,Y)X,Y)}{\|X \wedge Y\|^2}$$

In the case of C spaces with the root space decomposition of $\mathfrak{m}$ there are nice descriptions for U and Λ. The following results are due to [Ito78, Chapter 2]. However, he did not seem to have stated all cases we need in proposition 16, which is why we present the results here together with their proofs. Note that in general, for the complexification of a real bilinear form q, we have

$$q(\bar{x}, \bar{y}) = \overline{q(x, y)}$$

and therefore the following determines U and Λ completely:

Proposition 12. *Let $X_\alpha \in \mathfrak{g}_\alpha, X_\beta \in \mathfrak{g}_\beta$. Then we have*

i) $U(X_\alpha, X_\beta) = \frac{g_\beta - g_\alpha}{2g_{\alpha+\beta}}[X_\alpha, X_\beta]_{\mathfrak{m}^\mathbb{C}}$

ii) $U(X_\alpha, \overline{X_\beta}) = \begin{cases} \frac{1}{2}[X_\alpha, \overline{X_\beta}]_{\mathfrak{m}^\mathbb{C}} & \alpha - \beta \in \Delta^-_\mathfrak{m} \\ -\frac{1}{2}[X_\alpha, \overline{X_\beta}]_{\mathfrak{m}^\mathbb{C}} & \alpha - \beta \in \Delta^+_\mathfrak{m} \\ 0 & \text{otherwise} \end{cases}$

This implies for Λ

i) $\Lambda(X_\alpha)(X_\beta) = \frac{g_\beta}{g_{\alpha+\beta}}[X_\alpha, X_\beta]_{\mathfrak{m}^+}$

ii) $\Lambda(X_\alpha)(\overline{X_\beta}) = [X_\alpha, \overline{X_\beta}]_{\mathfrak{m}^-}$

Proof. This follows straightforward from the expression of the metric. For brevity we write only here $X_{-\beta} = \overline{X_\beta}$, note that this is not a Chevalley basis. In fact, we get

$$\begin{aligned} 2g(U(X_\alpha, X_{\pm\beta}), Z) &= g_\beta(-B)([Z, X_\alpha], X_{\pm\beta}) + g_\alpha(-B)(X_\alpha, [Z, X_{\pm\beta}]) \\ &= (g_\beta - g_\alpha)(-B)([X_\alpha, X_{\pm\beta}], Z) \\ &= \frac{(g_\beta - g_\alpha)}{g_\gamma} g([X_\alpha, X_{\pm\beta}], Z) \end{aligned}$$

if $[X_\alpha, X_{\pm\beta}] \in \mathfrak{g}_\gamma$. Going through the different possibilities for γ one obtains the claimed equalities. The equations for Λ are immediate consequences. □

A interesting fact is the following

Proposition 13. *For $X, Y \in \mathfrak{m}^+$ and $Z \in \mathfrak{m}^\mathbb{C}$ we have*

$$R(X, Y)Z = 0 = R(\overline{X}, \overline{Y})Z$$

Proof. Notice, that from proposition 12 we get that $\Lambda(Z) : \mathfrak{m}^\pm \to \mathfrak{m}^\pm$ and therefore also $R(Z, V) : \mathfrak{m}^\pm \to \mathfrak{m}^\pm$ for any $Z, V \in \mathfrak{m}^\mathbb{C}$. If we restrict now to X, Y being both either in $\mathfrak{m}^+$ or $\mathfrak{m}^-$ we obtain for $Z, V \in \mathfrak{m}^\mathbb{C}$

$$\begin{aligned} g(R(X, Y)Z, V) &= g(R(Z, V)X, Y) \\ &= g(J(R(Z, V)X), JY) \\ &= -g(R(Z, V)X, Y) \\ &= -g(R(X, Y)Z, V) \end{aligned}$$

which concludes the proof by non degeneracy of g. □

The above proposition has two crucial consequences, the first being that the only non vanishing curvatures are in

$$\mathfrak{m}^+ \times \mathfrak{m}^- \times \mathfrak{m}^+ \times \mathfrak{m}^-$$

and its complex conjugate. The second consequence being being the following

Lemma 14. *Let $X, Y, V, W \in \mathfrak{m}^+$ then the following symmetry holds*

$$R(X, \overline{Y}, V, \overline{W}) = R(X, \overline{W}, V, \overline{Y}).$$

Proof. By the first Bianchi identity, we have

$$R(X,\overline{Y},V,\overline{W}) + R(\overline{Y},V,X,\overline{W}) + g(R(V,X)\overline{Y},\overline{W}) = 0.$$

The last term is zero by proposition 13 and the second term equals $-R(X,\overline{W},V,\overline{Y})$ by the standard symmetries of the curvature tensor. □

Now, we calculate the curvatures in terms of the Chevalley basis defined in proposition 11. In order to keep the calculations and the notation simple, we assume an ordering on the positive roots with the following properties

i) If $\alpha < \beta$ then $\alpha - \beta \notin \Delta_{\mathfrak{m}}^+$.

ii) If $\alpha < \beta$ and $\gamma \leq \delta$ then $\alpha + \gamma < \beta + \delta$.

It is easy to construct such an ordering in terms of simple roots as we can see in the following.

Proposition 15. *Let $\varepsilon_1,..,\varepsilon_n$ be the set of simple roots of $\Delta_{\mathfrak{g}}^+$. The ordering " $<$ " on $\Delta_{\mathfrak{g}}^+$ given by*

$$\alpha = \sum_{i=1}^{r} n_i\varepsilon_i < \beta = \sum_{i=1}^{r} m_i\varepsilon_i$$

if and only if $n_{i^*} < m_{i^*}$ with $i^* = min\{i \mid m_i \neq n_i\}$ satisfies properties i) and ii) and its restriction to $\Delta_{\mathfrak{m}}^+$ is a ordering as desired.

Proof. For the first part we notice that every positive root is a linear combination of simple roots with nonnegative coefficients. Hence if $\alpha < \beta$ then the first non vanishing coefficient of $\alpha - \beta$ is negative and therefore $\alpha - \beta$ can not be a positive root. The second property is a straightforward calculation. □

Now we turn to the curvature formulas. To that end we fix a Chevalley basis as in proposition 11. Then let

$$R_{\alpha\beta\gamma\delta} = g(R(E_\alpha, E_{-\beta})E_\gamma, E_{-\delta})$$

for $\alpha,\beta,\gamma,\delta \in \Delta_{\mathfrak{m}}^+$.

Proposition 16 ([Ito78]). *Then we have that*

$$R_{\alpha\beta\gamma\delta} = 0 \textit{ unless } \alpha+\gamma = \beta+\delta \tag{1.4}$$

completely determined by $R_{\alpha\beta\gamma\delta}$ with $\alpha \leq \beta, \delta \leq \gamma$ and $\alpha - \beta = \delta - \gamma$. In that case we have

Case	$(\alpha,\beta,\gamma,\delta)$		$R_{\alpha\beta\gamma\delta}$
Ia	$\alpha-\beta=0$	$\gamma-\beta\in\Delta_{\mathfrak{m}}^+$	$-g_\alpha\left(z_\alpha z_\gamma(\alpha,\gamma) + z_{\alpha+\gamma}\frac{g_\alpha}{g_{\alpha+\gamma}}N_{\alpha,\gamma}^2\right)$
Ib		*otherwise*	$-g_\gamma\left(z_\alpha z_\gamma(\alpha,\gamma) + z_{\alpha+\gamma}\frac{g_\gamma}{g_{\alpha+\gamma}}N_{\alpha,\gamma}^2\right)$
IIa	$\alpha-\beta\neq0$	$\gamma-\beta\in\Delta_{\mathfrak{m}}^+$	$-g_\alpha z_{\alpha-\beta}N_{\alpha,-\beta}N_{\gamma,-\delta} - z_{\alpha+\gamma}\frac{g_\alpha g_\beta}{g_{\alpha+\gamma}}N_{\alpha,\gamma}N_{\beta,\delta}$
IIb		*otherwise*	$-g_\delta z_{\alpha-\beta}N_{\alpha,-\beta}N_{\gamma,-\delta} - z_{\alpha+\gamma}\frac{g_\gamma g_\delta}{g_{\alpha+\gamma}}N_{\alpha,\gamma}N_{\beta,\delta}$

Here, we denote $(\alpha,\beta) = B(H_\alpha,H_\beta)$.

Proof. Let us consider $R(E_\alpha,E_{-\beta})E_\gamma$ first.

$$\begin{aligned} R(E_\alpha,E_{-\beta})E_\gamma = &-[\Lambda(E_\alpha),\Lambda(E_{-\beta})]E_\gamma + \Lambda([E_\alpha,E_{-\beta}]_{\mathfrak{m}^{\mathbb{C}}})E_\gamma + [[E_\alpha,E_{-\beta}]_{\mathfrak{k}^{\mathbb{C}}},E_\gamma] \\ = &-\Lambda(E_\alpha)\left(\Lambda(E_{-\beta})(E_\gamma)\right) \\ &+\Lambda(E_{-\beta})\left(\Lambda(E_\alpha)(E_\gamma)\right) \\ &+\Lambda([E_\alpha,E_{-\beta}]_{\mathfrak{m}^{\mathbb{C}}})E_\gamma \\ &+[[E_\alpha,E_{-\beta}]_{\mathfrak{k}^{\mathbb{C}}},E_\gamma] \end{aligned} \tag{1.5}$$

Using the equations from proposition 12, we see that the above equals

$$\begin{aligned}(1.5) = &- \frac{g_{\gamma-\beta}}{g_{\alpha+\gamma-\beta}} [E_\alpha, [E_{-\beta}, E_\gamma]_{\mathfrak{m}^+}]_{\mathfrak{m}^+} && (1.6)\\ &+ \frac{g_\gamma}{g_{\alpha+\gamma}} [E_{-\beta}, [E_\alpha, E_\gamma]_{\mathfrak{m}^+}]_{\mathfrak{m}^+} && (1.7)\\ &+ \frac{g_\gamma}{g_{\gamma+\alpha-\beta}} [[E_\alpha, E_{-\beta}]_{\mathfrak{m}^+}, E_\gamma]_{\mathfrak{m}^+} && (1.8)\\ &+ [[E_\alpha, E_{-\beta}]_{\mathfrak{m}^-}, E_\gamma]_{\mathfrak{m}^+} && (1.9)\\ &+ [[E_\alpha, E_{-\beta}]_{\mathfrak{k}^{\mathbb{C}}}, E_\gamma]_{\mathfrak{m}^+} && (1.10)\end{aligned}$$

Analysing these expressions we get that they either vanish or lie in $\mathfrak{g}_{\alpha+\gamma-\beta}$ if $\alpha+\gamma-\beta \in \Delta^+_{\mathfrak{m}}$. Therefore, we get

$$R_{\alpha\beta\gamma\delta} = 0 \text{ unless } \delta = \alpha + \gamma - \beta \in \Delta^+_{\mathfrak{m}}$$

which proves the first claim. For the second claim, note that the symmetries of the curvature tensor, that it commutes with complex conjugation and the result of lemma 14 imply the following symmetries

$$R_{\alpha\beta\gamma\delta} = \overline{R_{\beta\alpha\delta\gamma}} = R_{\alpha\delta\gamma\beta} = R_{\gamma\delta\alpha\beta}. \tag{1.11}$$

Notice that all of these operations maintain $\alpha - \beta = \delta - \gamma$ and that they allow us to assume that α is a smallest root within $\{\alpha, \beta, \gamma, \delta\}$ with respect to the ordering on $\Delta^+_{\mathfrak{m}}$. That leads to the following case distinction:

I If $\alpha = \beta$, then we have automatically $\gamma = \delta$ and the curvature entry in question is $R_{\alpha\alpha\gamma\gamma}$.

II If $\alpha < \beta$ then $\delta < \gamma$. In fact, if $\gamma \leq \delta$ then $\alpha + \gamma < \delta + \beta$ which is a contradiction to $\alpha - \beta = \delta - \gamma$. Therefore, the entry in question is $R_{\alpha\beta\gamma\delta}$ with $\alpha < \beta$ and $\delta < \gamma$.

Now we continue the calculation of (1.5) with the restriction that $\alpha - \beta = \delta - \gamma$, $\alpha \leq \beta$ and $\delta \leq \gamma$. From the first property of our ordering, we get that $\alpha - \beta \notin \Delta^+_{\mathfrak{m}}$ and therefore $(1.8) = 0$ and $(1.9) + (1.10) = [[E_\alpha, E_{-\beta}], E_\gamma]_{\mathfrak{m}^+}$. Pairing this with $g(-, E_{-\delta}) = g_\delta(-B)((-)_{\mathfrak{g}_\delta}, E_{-\delta})$ yields

$$\begin{aligned}(1.5) = &\begin{cases} -g_{\gamma-\beta} B\left([E_{-\beta}, E_\gamma], [E_\alpha, E_{-\delta}]\right) & \gamma - \beta \in \Delta^+_{\mathfrak{m}} \\ 0 & \text{otherwise}\end{cases}\\ &+ \frac{g_\gamma g_\delta}{g_{\alpha+\gamma}} B\left([E_\alpha, E_\gamma], [E_{-\beta}, E_{-\delta}]\right)\\ &- g_\delta B\left([E_\alpha, E_{-\beta}], [E_\gamma, E_{-\delta}]\right)\end{aligned}$$

Now we consider the two cases of the first term. To that end assume $\gamma - \beta \in \Delta^+_{\mathfrak{m}}$. Using the biinvariance of B and the Jacobi identity, i.e.

$$[E_\alpha, [E_{-\beta}, E_\gamma]] = [[E_\alpha, E_{-\beta}], E_\gamma] + [E_{-\beta}, [E_\alpha, E_\gamma]].$$

We obtain

$$\begin{aligned}-g_{\gamma-\beta} B\left([E_{-\beta}, E_\gamma], [E_\alpha, E_{-\delta}]\right) = &- g_{\gamma-\beta} B\left([E_\alpha, E_\gamma], [E_{-\beta}, E_{-\delta}]\right)\\ &g_{\gamma-\beta} B\left([E_\alpha, E_{-\beta}], [E_\gamma, E_{-\delta}]\right),\end{aligned}$$

which leaves us with

$$\begin{aligned}(1.5) = &\left(\frac{g_\gamma g_\delta}{g_{\alpha+\gamma}} - g_{\gamma-\beta}\right) B\left([E_\alpha, E_\gamma], [E_{-\beta}, E_{-\delta}]\right)\\ &(g_{\gamma-\beta} - g_\delta) B\left([E_\alpha, E_{-\beta}], [E_\gamma, E_{-\delta}]\right)\\ = &\frac{g_\alpha g_\beta}{g_{\alpha+\gamma}} B\left([E_\alpha, E_\gamma], [E_{-\beta}, E_{-\delta}]\right)\\ &- g_\alpha B\left([E_\alpha, E_{-\beta}], [E_\gamma, E_{-\delta}]\right)\end{aligned}$$

On the other hand, if $\gamma - \beta \notin \Delta^+_{\mathfrak{m}}$ then

$$\begin{aligned}(1.5) =& \frac{g_\gamma g_\delta}{g_{\alpha+\gamma}} B\left([E_\alpha, E_\gamma], [E_{-\beta}, E_{-\delta}]\right) \\ &- g_\delta B\left([E_\alpha, E_{-\beta}], [E_\gamma, E_{-\delta}]\right)\end{aligned}$$

Using now the z_α and $N_{\alpha,\beta}$ from proposition 11 the entries of the table follow in a straightforward fashion. Furthermore, we see that $R_{\alpha\beta\gamma\delta}$ is a real number and hence in (1.11) we can actually drop the complex conjugation. □

1.5 The holomorphic curvature tensor

We are interested in the holomorphic sectional curvature of a Kähler C space (G, K, J, g), i.e.

$$H(X) = sec(X \wedge JX)$$

for a non zero vector $X \in \mathfrak{m}$. In particular, we want to show that H is positive. As we will see in theorem 22 the actual function H becomes fairly complicated if $\mathfrak{m}$ decomposes in a large number of K modules, hence we present the alternative approach using the curvature tensor and its modifications. This technique is due to [Tho71] and explicitly for the holomorphic curvature tensor to [GZ81]. The curvature tensor is defined as the symmetric tensor,

$$\begin{aligned}R : \Lambda^2(TM) \times \Lambda^2(TM) &\to \mathbb{R} \\ (X \wedge Y, V \wedge W) &\mapsto R(X, Y, V, W).\end{aligned}$$

Using the identification via π_K, we have

$$\Lambda^2(TM) = \Lambda^2(\mathfrak{m})$$

and the map $J : \mathfrak{m} \to \mathfrak{m}$ induces a map, also denoted as J, on the two forms with the property $J^2 = id_{\Lambda^2(\mathfrak{m})}$. By an easy calculation all complex planes $X \wedge JX$ are in the eigenspace of J of the eigenvalue one. Therefore, we are interested in the restriction of the curvature tensor to this eigenspace.

Definition 17. *We call the restriction of R to $Fix(J) \subset \Lambda^2(\mathfrak{m})$ the holomorphic curvature tensor and denote it by H.*

Remark: If the holomorphic curvature tensor is positive definite, then the holomorphic sectional curvature is positive In fact we have for a unit vector $X \in \mathfrak{m}$

$$0 < H(X \wedge JX, X \wedge JX) = R(X, JX, X, JX) = sec(X \wedge JX) = H(X)$$

Unfortunately, it is possible to have positive holomporphic sectional curvature without the tensor actually being positive, therefore it is of great interest how one may modify the tensor without changing the holomorphic sectional curvature. The first approach is the idea due to Thorpe to add a symmetric tensor ω to H with the property that $\omega(X \wedge JX, X \wedge JX) = 0$. In that case one would have that

$$(H + \omega)(X \wedge JX, X \wedge JX) = H(X) \tag{1.12}$$

but $H + \omega$ is a different tensor, that might be positive definite. Therefore, one way to prove positive holomorphic sectional curvature would be to determine a ω as above such that $H + \omega$ is a positive tensor.

1.6 Modifying the holomorphic curvature tensor

This section is dedicated to the detailed discussion of how one can modify the curvature tensor by adding suitable symmetric tensors on $\Lambda^2(\mathfrak{m})$ introduced by [Tho71]. It turns out the suitable tensors are induced by four forms on $\mathfrak{m}$. We want to determine how adding one of these forms changes the curvature tensor. To that purpose, let us revisit how a four-form may be added to a symmetric form on $\Lambda^2(\mathfrak{m})$. Since we fixed a Kähler C space (G, K, J, g) we simplify notation slightly denoting the metric as

$$g(-,-) = \langle -,- \rangle.$$

This induces an inner product on $\Lambda^l(\mathfrak{m})$ via

$$\langle x_1 \wedge ... \wedge x_l, y_1 \wedge ... \wedge y_l \rangle_{\Lambda^l \mathfrak{m}} = \det(\langle x_i, y_j \rangle).$$

If we extend the action of K as usual via

$$\begin{aligned} Ad_k &: \Lambda^l(\mathfrak{m}) \to \Lambda^l(\mathfrak{m}) \\ x_1 &\wedge ... \wedge x_l \mapsto Ad_k(x_1) \wedge ... \wedge Ad_k(x_l), \end{aligned}$$

K acts via isometries on $\Lambda^l(\mathfrak{m})$. Then we see that a four form ω induces a symmetric bilinear form on $\Lambda^2(\mathfrak{m})$ via

$$\omega \mapsto \omega' = \langle \omega, - \wedge - \rangle_{\Lambda^4 \mathfrak{m}}.$$

As mentioned before, adding such an ω' changes the holomorphic curvature tensor but since

$$\omega'(X \wedge Y, X \wedge Y) = \langle \omega, X \wedge Y \wedge X \wedge Y \rangle_{\Lambda^4(\mathfrak{m})} = 0$$

holds the quadratic form induced by $H + \omega'$ is the same as the holomorphic sectional curvature. Therefore this is a viable technique to prove positivity.

However, it is a hard problem to determine a suitable ω' or proof its non existence. Bearing that in mind there still are some observations simplifying the search for ω'.

We remark that any arbitrary choice of ω' is allowed to be added even though it might not be K invariant and hence $H + \omega'$ would not be well defined on all of G/K. In fact, the positivity of $H + \omega'$ implies the positivity of the holomorphic sectional curvature at this point, which is now invariant under K and therefore positive everywhere. On the other hand, the following lemma is still useful since it allows us to decrease the dimension of the space of allowed forms, which makes the search easier.

Lemma 18. *Let $\omega \in \Lambda^4(\mathfrak{m})$ such that $H + \omega'$ is positive definite then*

$$H + \frac{1}{vol(K)} \int_K Ad_k^* \omega' dK$$

is positive definite where dK is a biinvariant volume form on K.

Proof. The proof is an easy averaging calculation. □

1.7 Relevant invariant forms

Since the holomorphic curvature tensor is the restriction to Fix(J) also for the four forms we are only interested in their inner products with Fix(J) $\wedge$ Fix(J). Hence the four forms in $(\mathrm{Fix}(J) \wedge \mathrm{Fix}(J))^\perp$ do not change the holomorphic curvature tensor at all. Therefore, we may restrict our attention to forms in Fix(J) $\wedge$ Fix(J). Considering that, if at all possible, positivity can be achieved via adding a invariant four form by lemma 18 and the isotropy subgroup of a C space contains always a maximal torus, we can restrict our attention to the $\mathbb{T}$ invariant four forms in Fix(J) $\wedge$ Fix(J). We call those relevant forms. In the following we show what characterizes relevant forms. To that end we define with the basis from proposition 11 the following

$$E_{(\alpha,\beta,\gamma,\delta)} = E_\alpha \wedge E_\beta \wedge E_\gamma \wedge E_\delta$$

Proposition 19. *Let ω be a four form. Then we have that $\mathbb{T}$ invariance is characterized by* *i)* ω *is* $\mathbb{T}$ *invariant iff* $\frac{d}{dt}\big|_{t=0} Ad_{exp(tH)}\omega = 0$ *for all* $H \in \mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$.

ii) $\frac{d}{dt}\big|_{t=0} Ad_{exp(tH)} E_{(\alpha,\beta,\gamma,\delta)} = (\alpha+\beta+\gamma+\delta)(H) E_{(\alpha,\beta,\gamma,\delta)}$

and J *invariance means*

iii) $J(E_{(\alpha,\beta,\gamma,\delta)}) = E_{(\alpha,\beta,\gamma,\delta)}$ *iff* $|\{\alpha,\beta,\gamma,\delta\} \cap \Delta^+_{\mathfrak{m}}| \in \{0,2,4\}$

Proof. Part *i)* follows from the fact that Ad is a group homomorphism and *ii)* is a straight forward calculation using the fact that the E_α are root vectors. Similarly, *iii)* follows by the definition of $\Delta^{\pm}_{\mathfrak{m}}$, i.e. $J(E_{\pm\alpha}) = \pm i E_{\pm\alpha}$ for $\alpha \in \Delta^+_{\mathfrak{m}}$. □

This leads us easily to

Proposition 20. *The real vectorspace of relevant forms is spanned by*

$$E_{\alpha,-\beta,\gamma,-\delta} + \overline{E_{\alpha,-\beta,\gamma,-\delta}} \quad \text{and} \quad i(E_{\alpha,-\beta,\gamma,-\delta} - \overline{E_{\alpha,-\beta,\gamma,-\delta}})$$

with $\alpha+\gamma = \beta+\delta$ *and* $\alpha,\beta,\gamma,\delta \in \Delta^+_{\mathfrak{m}}$.

Remark: Note, that $\overline{E_{\alpha,-\beta,\gamma,-\delta}} = E_{\beta,-\alpha,\delta,-\gamma}$ holds.

Proof. Let $\omega \in \Lambda^4(\mathfrak{m})^{\mathbb{C}}$ be $\mathbb{T}$ and J invariant with $\omega = \overline{\omega}$. Then we can write

$$\omega = \sum_{I=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} \lambda_I E_I$$

with $\lambda_I \in \mathbb{C}$ for linear independent E_I. By Proposition 19 *ii)* and *iii)* we have that $|I \cap \Delta^+_{\mathfrak{m}}| = 2$. After possible reordering inside of I we have $E_I = E_{\alpha,-\beta,\gamma,-\delta}$ and $\alpha+\gamma = \beta+\delta$. Furthermore we obtain from $\omega = \overline{\omega}$ that $\lambda_{-I} = \overline{\lambda_I} = a_I - ib_I$ and therefore

$$\begin{aligned}\omega &= \frac{1}{2}\sum \lambda_I E_I + \overline{\lambda_I E_I}\\ &= \frac{1}{2}\sum a_I(E_I + \overline{E_I}) + b_I i(E_I - \overline{E_I})\end{aligned}$$

□

We observe here that since

$$\langle E_\alpha, E_\beta\rangle = g_\alpha(-B)(E_\alpha,E_\beta) = \begin{cases} -z_\alpha g_\alpha & \beta = -\alpha \\ 0 & \beta \neq -\alpha \end{cases}.$$

the value of $\langle E_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}, E_{(\beta_1,\beta_2,\beta_3,\beta_4)}\rangle$ is a real number and vanishes unless there is a bijection $\sigma \in S_4$ such that $\beta_i = -\alpha_{\sigma(i)}$ and in that case we have

$$\langle E_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}, E_{(-\alpha_{\sigma(1)},-\alpha_{\sigma(2)},-\alpha_{\sigma(3)},-\alpha_{\sigma(4)})}\rangle_{\Lambda^4(\mathfrak{m})} = sign(\sigma)\prod_{i=1}^4 z_{\alpha_i} g_{\alpha_i}. \tag{1.13}$$

1.8 Algebraic structure of the classical groups

Since every Kaehler C space comes from a semisimple compact group G and the centralizer K of a torus in G, we get that the decomposition of $G = G_1 \times ... \times G_l$ into simple factors induces a decomposition of $K = K_1 \times ... \times K_l$ with $K_i \subset G_i$. By representation theory, every K invariant complex structure J and a corresponding Kähler metric g decompose into complex structures and Kähler metrics on G_i/K_i, i.e. we have

$$(G/K, J, g) = \left(\prod_{i=1}^{l} G_i/K_i, \oplus_{i=1}^{l} J_i, \oplus_{i=1}^{l} g_i \right) = \prod_{i=1}^{l} (G_i/K_i, J_i, g_i).$$

A straightforward calculation using the standard properties of the connections of Riemannian product manifolds yields that for two hermitian manifolds (M_i, J_i, g_i) with $i = 1, 2$ we have that the holomorphic sectional curvature of a tangent vector $V = X_1 + X_2$ on the product manifold $(M_1 \times M_2, J_1 \oplus J_2, g_1 \oplus g_2)$ is given by

$$H(X_1 + X_2) = \frac{\|X_1\|^4}{\|X_1 + X_2\|^4} H_1(X_1) + \frac{\|X_2\|^4}{\|X_1 + X_2\|^4} H_2(X_2).$$

Consequently, it is sufficient to show positivity separately for every factor.
Hence we restrict to the Kähler C spaces with simple compact isometry group. These are classified by connected Dynkin diagrams, i.e. the classical groups corresponding to the families A_n, B_n, C_n, D_n for $n \in \mathbb{N}$ and the exceptional ones corresponding to G_2, F_4, E_6, E_7, E_8. In the following, we present the algebraic structure of the simple classical compact groups. The choice of basis used are motivated by [Hel01, Chapter 8]. To that end let $E_{kl} \in \mathfrak{gl}_m(\mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be the matrix with

$$(E_{kl})_{ij} = \delta_{ki}\delta_{lj}$$

where δ denotes the Kronecker delta. Furthermore, let $F_{kl} = E_{lk} - E_{kl}$ be the skew symmetric matrix with -1 at the entry (k, l).

1.8.1 Family $A_n : SU(n+1)$

In this section we describe the algebraic setting of the Lie group $SU(n+1)$, including a Chevalley basis and the structure constants. We choose the maximal torus to be the standard diagonal torus in $SU(n+1)$, i.e.

$$\mathbb{T} = \{Diag(z_1, .., z_{n+1}) \mid \|z_j\| = 1, \prod z_j = 1\}.$$

Then we have

$$\mathfrak{g} = \{A \in \mathfrak{gl}_{n+1}(\mathbb{C}) \mid A^H = -A, tr_{\mathbb{C}}(A) = 0\}$$

and

$$\mathfrak{t} = \{Diag(ia_1, .., ia_{n+1}) \mid a_j \in \mathbb{R}, \sum_{j=1}^{n+1} a_j = 0\}.$$

Complexification yields $\mathfrak{g}^{\mathbb{C}} = \{A \in \mathfrak{gl}_{n+1}(\mathbb{C}) \mid tr_{\mathbb{C}}(A) = 0\}$ and

$$\mathfrak{h} = \mathfrak{t}^{\mathbb{C}} = \{Diag(z_1, .., z_n) \mid z_j \in \mathbb{C}, \sum_{j=1}^{n+1} z_j = 0\}.$$

Obviously, the $H_k = E_{kk}$ form a complex basis of the space of complex diagonal matrices and we denote the dual basis by ε_k. It is easy to see that for $H \in \mathfrak{h}$ we have

$$[H, E_{kl}] = \alpha_{kl}(H) E_{kl}$$

for $\alpha_{kl} = \varepsilon_k - \varepsilon_l \in \mathfrak{h}^*$. In fact it is known that

$$\Delta_{\mathfrak{g}} = \Delta_{\mathfrak{su}(n+1)} = \{\alpha_{kl} \mid 1 \leq k \neq l \leq n+1\}$$

and the root spaces are given by $\mathfrak{g}_{\alpha_{kl}} = \langle E_{kl} \rangle_{\mathbb{C}}$. As we will see in theorem 38 it makes sense to fix a preferred "standard" complex structure J_{std}, i.e a fixed choice of of $\Delta^+_{\mathfrak{m}}$. Furthermore, we fix a normalization of the killing form. In fact, let

a) $\Delta^+_{\mathfrak{m}} = \Delta^+_{\mathfrak{g}} = \{\alpha_{kl} \mid 1 \leq k < l \leq n+1\}$

b) and scale the complexified killing form such that

$$B(X,Y) = tr_{\mathbb{C}}(XY)$$

where $tr_{\mathbb{C}} : \mathfrak{gl}_{n+1}(\mathbb{C}) \to \mathbb{C}$ is the usual trace of complex valued matrices.

Then we get with the notation of the proposition 11 that

$$E_{\alpha_{kl}} = E_{kl} \text{ and } H_{\alpha_{kl}} = H_k - H_l$$

In particular this implies $z_\alpha = 1$ for all $\alpha \in \Delta_{\mathfrak{g}}$ and

$$N_{\alpha_{kl},\alpha_{st}} = \begin{cases} 1 & s = l \ , \ t \neq k \\ -1 & t = k \ , \ s \neq l \\ 0 & \text{otherwise} \end{cases}$$

1.8.2 Family $B_n : SO(2n+1)$

In this section we describe the algebraic setting of the Lie group $SO(2n+1)$, including a Chevalley basis and the structure constants. We choose the maximal torus to be the standard diagonal torus in $SO(2n+1)$, i.e.

$$\mathbb{T} = \{Diag(R(\theta_1), .., R(\theta_n), 1) \mid \theta_i \in \mathbb{R}\}$$

where

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Then we have

$$\mathfrak{g} = \{A \in \mathfrak{gl}_{2n+1}(\mathbb{R}) \mid A^T = -A\}$$

and

$$\mathfrak{t} = \langle H_k \, | k = 1, .., n \rangle_{\mathbb{R}}$$

where $(H_k) = F_{2k-1,2k}$. Complexification yields

$$\mathfrak{g}^{\mathbb{C}} = \{A \in \mathfrak{gl}_{2n+1}(\mathbb{C}) \mid A^T = -A\}$$

and

$$\mathfrak{h} = \mathfrak{t}^{\mathbb{C}} = \langle H_k \mid k = 1, .., n \rangle_{\mathbb{C}}.$$

Obviously, the iH_k form a complex basis of the space of complex diagonal matrices and we denote the dual basis by ε_k. Define for $k < l$ the matrices $E_{\varepsilon_k}, E_{\alpha_{kl}}, E_{\beta_{kl}} \in \mathfrak{g}^{\mathbb{C}}$ as

$$\begin{aligned} E_{\varepsilon_k} &= F_{2k-1,2n+1} + iF_{2k,2n+1} \\ E_{\alpha_{kl}} &= F_{2k-1,2l-1} + F_{2k,2l} - i(F_{2k-1,2l} - F_{2k,2l-1}) \\ E_{\beta_{kl}} &= F_{2k-1,2l-1} - F_{2k,2l} + i(F_{2k-1,2l} + F_{2k,2l-1}). \end{aligned}$$

Notice that the complex conjugation coming from $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is the same as the complex conjugation on $\mathfrak{gl}_{2n+1}(\mathbb{C})$. Hence we define for $\alpha \in \{\varepsilon_k, \alpha_{kl}, \beta_{kl} \mid 1 \leq k < l \leq n\}$

$$E_{-\alpha} = -\overline{E_\alpha}.$$

These will turn out to be root vectors, but before we prove that we consider the following examples in the case $n = 2$.

$$E_{\varepsilon_1} = \left(\begin{array}{cc|cc|c} & & & & -1 \\ & & & & -i \\ \hline & & & & \\ & & & & \\ \hline 1 & i & & & \end{array}\right) \quad E_{\alpha_{12}} = \left(\begin{array}{cc|cc|c} & & -1 & i & \\ & & -i & -1 & \\ \hline 1 & i & & & \\ -i & 1 & & & \\ \hline & & & & \end{array}\right) \quad E_{\beta_{12}} = \left(\begin{array}{cc|cc|c} & & -1 & -i & \\ & & -i & 1 & \\ \hline 1 & i & & & \\ i & -1 & & & \\ \hline & & & & \end{array}\right)$$

Now let $H = \sum_{k=1}^{n} h_k H_k = \sum_{k=1}^{n} -ih_k iH_k \in \mathfrak{h}$ then it is easy to see

$$\begin{aligned} [H, E_{\varepsilon_k}] &= (-ih_k)E_{\varepsilon_k} = \varepsilon_k(H)E_{\varepsilon_k} \\ [H, E_{\alpha_{kl}}] &= (-ih_k + ih_l)E_{\alpha_{kl}} = \alpha_{kl}(H)E_{\alpha_{kl}} \\ [H, E_{\beta_{kl}}] &= (-ih_k - ih_l)E_{\beta_{kl}} = \beta_{kl}(H)E_{\beta_{kl}} \end{aligned}$$

for $\alpha_{kl} = \varepsilon_k - \varepsilon_l$ and $\beta_{kl} = \varepsilon_k + \varepsilon_l$. In fact, we have that

$$\Delta_{\mathfrak{g}} = \Delta_{\mathfrak{so}(2n+1)} = \{\pm\varepsilon_k, \pm\alpha_{kl}, \pm\beta_{kl} \mid k < l\}$$

and the root spaces are given by $\mathfrak{g}_{\varepsilon_k} = \langle E_{\varepsilon_k}\rangle_{\mathbb{C}}$, $\mathfrak{g}_{\alpha_{kl}} = \langle E_{\alpha_{kl}}\rangle_{\mathbb{C}}$ and $\mathfrak{g}_{\beta_{kl}} = \langle E_{\beta_{kl}}\rangle_{\mathbb{C}}$. As we will see in theorem 38 it makes sense to fix a preferred "standard" complex structure J_{std}, i.e a fixed choice of of $\Delta_{\mathfrak{m}}^+$. Furthermore, we fix a normalization of the killing form. In fact, let

a) $\Delta_{\mathfrak{m}}^+ = \Delta_{\mathfrak{g}}^+ = \{\varepsilon_k, \alpha_{kl}, \beta_{kl} \mid k < l\}$

b) and scale the complexified killing form such that

$$B(X, Y) = \frac{1}{8} tr_{\mathbb{C}}(XY)$$

where $tr_{\mathbb{C}} : \mathfrak{gl}_{2n+1}(\mathbb{C}) \to \mathbb{C}$ is the usual trace of complex valued matrices.

Let us determine the structure constants from proposition 11. First of all with this choice of biinvariant form we have

$$z_\alpha = B(E_\alpha, E_{-\alpha}) = \begin{cases} \frac{1}{2} & \alpha = \varepsilon_k \\ 1 & \alpha = \alpha_{kl} \\ 1 & \alpha = \beta_{kl} \end{cases}$$

and the $N_{\alpha\beta}$ are given by the following list together with the properties $N_{-\alpha-\beta} = -N_{\alpha\beta}$ and $N_{\alpha\beta} = -N_{\beta\alpha}$.

(α, β)	$N_{\alpha,\beta}$
$(\varepsilon_k, \varepsilon_l)$	1
$(\varepsilon_k, -\varepsilon_l)$	-1
$(\varepsilon_l, \alpha_{kl})$	2
$(\varepsilon_k, -\alpha_{kl})$	2
$(\varepsilon_k, -\beta_{kl})$	-2
$(\varepsilon_l, -\beta_{kl})$	2
$(\alpha_{kl}, \alpha_{lj})$	-2
$(\alpha_{kl}, -\alpha_{sl})$	-2
$(\alpha_{kl}, -\alpha_{kj})$	2

(α, β)	$N_{\alpha,\beta}$
$(\alpha_{kl}, \beta_{lj})$	-2
$(\alpha_{kl}, \beta_{sl})$	$\begin{cases} 2 & k < s \\ -2 & s < k \end{cases}$
$(\alpha_{kl}, -\beta_{kj})$	$\begin{cases} 2 & l < j \\ -2 & j < l \end{cases}$
$(\alpha_{kl}, -\beta_{sk})$	2
$(\beta_{kl}, -\beta_{lj})$	2
$(\beta_{kl}, -\beta_{kj})$	-2
$(\beta_{kl}, -\beta_{sl})$	-2

where we assume $k < l$.

1.8.3 Family $C_n : Sp(n)$

In this section we describe the algebraic setting of the Lie group $Sp(n)$, including a Chevalley basis and the structure constants. We choose the maximal torus to be the standard diagonal torus in $Sp(n)$, i.e.

$$\mathbb{T} = \{Diag(z_1, .., z_n) \mid \|z_j\| = 1\}.$$

In order to have a nice description of the complexification it is useful to consider it embedded via

$$\begin{aligned} Sp(n) &\hookrightarrow SU(2n) \\ A + Bj &\mapsto \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}. \end{aligned}$$

Then we have

$$\mathfrak{g} = \left\{ \begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathbb{C}) \;\middle|\; X^H = -X, Y^T = Y \right\}$$

and

$$\mathfrak{t} = \{Diag(ia_1, .., ia_n, -ia_1, .., -ia_n) \mid a_i \in \mathbb{R}\}.$$

Complexification yields

$$\mathfrak{g}^{\mathbb{C}} = \left\{ \begin{pmatrix} X & Y_1 \\ Y_2 & -X^T \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathbb{C}) \;\middle|\; Y_l \in Sym(\mathbb{C}) \right\}$$

and

$$\mathfrak{h} = \mathfrak{t}^{\mathbb{C}} = \{Diag(z_1, .., z_n, -z_1, .., -z_n) \mid z_i \in \mathbb{C}\}.$$

Obviously, the elements $H_k = E_{k,k} - E_{n+k,n+k}$ for $k = 1...n$ form a complex basis of $\mathfrak{h}$ and we denote the dual basis by ε_k. Define for $k < l$ the matrices $E_{\gamma_k}, E_{\alpha_{kl}}, E_{\beta_{kl}} \in \mathfrak{g}^{\mathbb{C}}$ as

$$\begin{aligned} E_{\gamma_k} &= E_{k,n+k} \\ E_{\alpha_{kl}} &= E_{k,l} - E_{n+l,n+k} \\ E_{\beta_{kl}} &= E_{k,n+l} + E_{l,n+k}. \end{aligned}$$

Notice that the complex conjugation coming from $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is the same as the restriction of the map $M \mapsto -M^H$ on $\mathfrak{gl}_{2n}(\mathbb{C})$. Hence we define for $\alpha \in \{\varepsilon_k, \alpha_{kl}, \beta_{kl} \mid 1 \le k < l \le n\}$

$$E_{-\alpha} = -\overline{E_\alpha} = E_\alpha^H.$$

These will turn out to be root vectors, but before we prove that we consider the following examples in the case $n = 2$.

$$E_{\gamma_1} = \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 0 & 0 \\ \hline & & & \\ & & & \end{array}\right) E_{\alpha_{12}} = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & -1 & 0 \end{array}\right) E_{\beta_{12}} = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline & & & \\ & & & \end{array}\right)$$

Now let $H = Diag(h_1, .., h_n, -h_1, .., -h_n)$ then it is easy to see that

$$\begin{aligned} [H, E_{\gamma_k}] &= (2h_k)E_{\gamma_k} = \gamma_k(H)E_{\gamma_k} \\ [H, E_{\alpha_{kl}}] &= (h_k - h_l)E_{\alpha_{kl}} = \alpha_{kl}(H)E_{\alpha_{kl}} \\ [H, E_{\beta_{kl}}] &= (h_k + h_l)E_{\beta_{kl}} = \beta_{kl}(H)E_{\beta_{kl}} \end{aligned}$$

for $\gamma_k = 2\varepsilon_k, \alpha_{kl} = \varepsilon_k - \varepsilon_l$ and $\beta_{kl} = \varepsilon_k + \varepsilon_l$. In fact, we have that

$$\Delta_{\mathfrak{g}} = \Delta_{\mathfrak{sp}(n)} = \{\pm\gamma_k, \pm\alpha_{kl}, \pm\beta_{kl} \mid 1 \leq k < l \leq n\}$$

and the root spaces are given by $\mathfrak{g}_{\gamma_k} = \langle E_{\gamma_k}\rangle_{\mathbb{C}}$, $\mathfrak{g}_{\alpha_{kl}} = \langle E_{\alpha_{kl}}\rangle_{\mathbb{C}}$ and $\mathfrak{g}_{\beta_{kl}} = \langle E_{\beta_{kl}}\rangle_{\mathbb{C}}$. As we will see in theorem 38 it makes sense to fix a preferred "standard" complex structure J_{std}, i.e a fixed choice of of $\Delta^+_{\mathfrak{m}}$. Furthermore, we fix a normalization of the killing form. In fact, let

a) $\Delta^+_{\mathfrak{m}} = \Delta^+_{\mathfrak{g}} = \{\gamma_k, \alpha_{kl}, \beta_{kl} \mid 1 \leq k < l \leq n\}$

b) and scale the complexified killing form such that

$$B(X,Y) = \frac{1}{2} tr_{\mathbb{C}}(XY)$$

where $tr_{\mathbb{C}} : \mathfrak{gl}_{2n}(\mathbb{C}) \to \mathbb{C}$ is the usual trace of complex valued matrices.

Let us determine the structure constants from proposition 11. First of all with this choice of biinvariant form we have

$$z_\alpha = B(E_\alpha, E_{-\alpha}) = \begin{cases} \frac{1}{2} & \alpha = \gamma_k \\ 1 & \alpha = \alpha_{kl} \\ 1 & \alpha = \beta_{kl} \end{cases}$$

and the $N_{\alpha\beta}$ are given by the following list together with the properties $N_{-\alpha-\beta} = -N_{\alpha\beta}$ and $N_{\alpha\beta} = -N_{\beta\alpha}$.

(α,β)	$N_{\alpha,\beta}$
$(\alpha_{is}, \alpha_{st})$	1
$(\alpha_{sj}, -\alpha_{st})$	-1
$(\alpha_{it}, -\alpha_{st})$	1
$(\alpha_{is}, \beta_{st})$	1
$(\alpha_{it}, \beta_{st})$	$\begin{cases} 1 & i \neq s \\ 2 & i = s \end{cases}$
$(\alpha_{tj}, -\beta_{st})$	-1

(α,β)	$N_{\alpha,\beta}$
$(\alpha_{sj}, -\beta_{st})$	$\begin{cases} -1 & j \neq t \\ -2 & j = t \end{cases}$
(α_{is}, γ_s)	1
$(\alpha_{sj}, -\gamma_s)$	-1
$(\beta_{ij}, -\beta_{st})$	1 if $\lvert\{i,j\} \cap \{s,t\}\rvert = 1$
$(\beta_{is}, -\gamma_s)$	1
$(\beta_{sj}, -\gamma_s)$	1

1.8.4 Family $D_n : SO(2n)$

In this section we describe the algebraic setting of the Lie group $SO(2n)$, including a Chevalley basis and the structure constants. It is very similar to the B_n series. We choose the maximal torus to be the standard diagonal torus in $SO(2n)$, i.e.

$$\mathbb{T} = \{Diag(R(\theta_1), .., R(\theta_n)) \mid \theta_i \in \mathbb{R}\}$$

where $R(\theta)$ is defined as in the section of $SO(2n+1)$. Then we have

$$\mathfrak{g} = \{A \in \mathfrak{gl}_{2n}(\mathbb{R}) \mid A^T = -A\}$$

and for $(H_k) = F_{2k-1,2k}$ as before

$$\mathfrak{t} = \langle H_k \mid k = 1, .., n\rangle_{\mathbb{R}}.$$

Complexification yields

$$\mathfrak{g}^{\mathbb{C}} = \{A \in \mathfrak{gl}_{2n}(\mathbb{C}) \mid A^T = -A\}$$

and

$$\mathfrak{h} = \mathfrak{t}^{\mathbb{C}} = \langle H_k \mid k = 1, .., n\rangle_{\mathbb{C}}.$$

Obviously, the iH_k form a complex basis of $\mathfrak{h}$ and we denote the dual basis by ε_k. Define for $k < l$ the matrices $E_{\alpha_{kl}}, E_{\beta_{kl}} \in \mathfrak{g}^{\mathbb{C}}$ as

$$E_{\alpha_{kl}} = F_{2k-1,2l-1} + F_{2k,2l} - i(F_{2k-1,2l} - F_{2k,2l-1})$$
$$E_{\beta_{kl}} = F_{2k-1,2l-1} - F_{2k,2l} + i(F_{2k-1,2l} + F_{2k,2l-1}).$$

Notice that the complex conjugation coming from $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$ is the same as the complex conjugation on $\mathfrak{gl}_{2n+1}(\mathbb{C})$. Hence we define for $\alpha \in \{\alpha_{kl}, \beta_{kl} \mid 1 \le k < l \le n\}$

$$E_{-\alpha} = -\overline{E_\alpha}.$$

These will turn out to be root vectors. Now let $H = \sum_{k=1}^n h_k H_k = \sum_{k=1}^n -ih_k iH_k \in \mathfrak{h}$ then it is easy to see

$$[H, E_{\alpha_{kl}}] = (-ih_k + ih_l)E_{\alpha_{kl}} = \alpha_{kl}(H)E_{\alpha_{kl}}$$
$$[H, E_{\beta_{kl}}] = (-ih_k - ih_l)E_{\beta_{kl}} = \beta_{kl}(H)E_{\beta_{kl}}$$

for $\alpha_{kl} = \varepsilon_k - \varepsilon_l$ and $\beta_{kl} = \varepsilon_k + \varepsilon_l$. In fact, we have that

$$\Delta_{\mathfrak{g}} = \Delta_{\mathfrak{so}(2n)} = \{\pm\alpha_{kl}, \pm\beta_{kl} \mid 1 \le k < l \le n\}$$

and the root spaces are given by $\mathfrak{g}_{\alpha_{kl}} = \langle E_{\alpha_{kl}} \rangle_{\mathbb{C}}$ and $\mathfrak{g}_{\beta_{kl}} = \langle E_{\beta_{kl}} \rangle_{\mathbb{C}}$. As we will see in theorem 38 it makes sense to fix a preferred "standard" complex structure J_{std}, i.e a fixed choice of of $\Delta^+_{\mathfrak{m}}$. Furthermore, we fix a normalization of the killing form. In fact, let

a) $\Delta^+_{\mathfrak{m}} = \Delta^+_{\mathfrak{g}} = \{\alpha_{kl}, \beta_{kl} \mid 1 \le k < l \le n\}$

b) and scale the complexified killing form such that

$$B(X, Y) = \frac{1}{8} tr_{\mathbb{C}}(XY)$$

where $tr_{\mathbb{C}} : \mathfrak{gl}_{2n}(\mathbb{C}) \to \mathbb{C}$ is the usual trace of complex valued matrices.

Let us determine the structure constants from proposition 11. First of all with this choice of biinvariant form we have

$$z_\alpha = B(E_\alpha, E_{-\alpha}) = \begin{cases} 1 & \alpha = \alpha_{kl} \\ 1 & \alpha = \beta_{kl} \end{cases}$$

and the $N_{\alpha\beta}$ are given by the following list together with the properties $N_{-\alpha-\beta} = -N_{\alpha\beta}$ and $N_{\alpha\beta} = -N_{\beta\alpha}$.

(α, β)	$N_{\alpha,\beta}$
$(\alpha_{kl}, \alpha_{lj})$	-2
$(\alpha_{kl}, -\alpha_{sl})$	-2
$(\alpha_{kl}, -\alpha_{kj})$	2
$(\alpha_{kl}, \beta_{lj})$	-2
$(\alpha_{kl}, \beta_{sl})$	$\begin{cases} 2 & k < s \\ -2 & s < k \end{cases}$

(α, β)	$N_{\alpha,\beta}$
$(\alpha_{kl}, -\beta_{kj})$	$\begin{cases} 2 & l < j \\ -2 & j < l \end{cases}$
$(\alpha_{kl}, -\beta_{sk})$	2
$(\beta_{kl}, -\beta_{lj})$	2
$(\beta_{kl}, -\beta_{kj})$	-2
$(\beta_{kl}, -\beta_{sl})$	-2

where we assume $k < l$.

Chapter 2

Approach Itoh's and Three Modules

The approach to treat holomorphic curvature on Kähler C spaces so far was to consider the decomposition of $\mathfrak{m}^+$ in irreducible K modules. In this chapter we follow that approach extending a result of Itoh. However, in the proof it will become clear that it is hard and complicated to extend the results even further without changing the method used.

By proposition 8 the irreducible modules are indexed by the positive $\mathfrak{t}$ roots, i.e the image of ρ from (1.1):

$$\mathfrak{m}^+ = \bigoplus_{x \in \rho(\Delta_{\mathfrak{m}}^+)} \mathfrak{m}_x^+$$

where $\mathfrak{m}_x^+ = \oplus_{\alpha \in \rho^{-1}(x)} \mathfrak{g}_\alpha$. Then one notes that for a unit vector $X \in \mathfrak{m}$ and the vector $Z = \frac{1}{\sqrt{2}}(X - iJX) \in \mathfrak{m}^+$, we have

$$H(X) = R(X, JX, X, JX) = -R(Z, \bar{Z}, Z, \bar{Z}).$$

Using the above decomposition, we can write $Z = \sum_{x \in \rho(\Delta_{\mathfrak{m}}^+)} Z_x$ and use the obvious bracket relations

$$[\mathfrak{m}_x^+, \mathfrak{m}_y^+] \subset \mathfrak{m}_{x+y}^+$$

and their counterparts via complex conjugation to get explicit expressions for the curvature. Notice that we have $x \in \mathbb{N}^r$ with

$$r = \dim(\mathfrak{z}(\mathfrak{k})) = b_1(K) = b_2(G/K)$$

where $\mathfrak{z}(\mathfrak{k})$ denotes the center of $\mathfrak{k}$. This is due to Borel and Hirzebruch [BH58].

This was done by Itoh with the following result:

Theorem 21 (Itoh)**.** *Let $M = (G, K, J, g)$ be a simple Kähler C space with $b_2(M) = 1$ such that $\mathfrak{m}^+$ decomposes into two irreducible modules. Then M has positive holomorphic curvature.*

Remark:
1) This covers all classical simple groups G and all with J compatible groups K with $b_1(K) = 1$ and some but not all of the exceptional ones.
2) In the case of $b_2(M) = 1$ and irreducible $\mathfrak{m}^+$, we are in the case of hermitian symmetric spaces which carry a negative multiple of the killing form as Kähler metric and have positive holomorphic curvature as well. These are also the only cases when the killing form induces a Kähler metric.
We extend the result here with similar techniques as Itoh to the following

Theorem 22. *Let $M = (G, K, J, g)$ be a simple Kähler C space such that $\mathfrak{m}^+$ decomposes into three irreducible modules. Then M has positive holomorphic curvature.*

Proof. First of all, we notice that in fact the decomposition into three modules is only possible if $b_1(K) \leq 2$. In fact, assume $b_1(K) > 2$ then, with the decomposition of a base for $\Delta_{\mathfrak{g}}^+$ as in (1.1) $\Phi = \Phi_{\mathfrak{m}} \cup \Phi_{\mathfrak{k}}$, this is equivalent to $|\Phi_{\mathfrak{m}}| > 2$ and hence there are at least three simple roots $\alpha_1, \alpha_2, \alpha_3 \in \Delta_{\mathfrak{g}}^+$ with pairwise different images under ρ. Furthermore, for the highest root $\lambda = \sum_{\alpha \in \Phi} n_\alpha \alpha$, we have $n_\alpha > 0$ for all α. Hence

$$\rho(\lambda) - \rho(\alpha_i) = \left((n_{\alpha_i} - 1)\rho(\alpha_i) + \sum_{\alpha \in \Phi \setminus \{\alpha_i\}} n_\alpha \alpha \right) \neq 0$$

Therefore we have at least four irreducible modules corresponding to $\rho(\alpha_1), \rho(\alpha_2), \rho(\alpha_3)$ and $\rho(\lambda)$ which is a contradiction.
Now we denote the decomposition by $\mathfrak{m}^+ = \mathfrak{m}_{x_1}^+ \oplus \mathfrak{m}_{x_2}^+ \oplus \mathfrak{m}_{x_3}^+$ and we make two case distinctions:

i) If $b_2(K) = 1$ then up to permutation we have $x_2 = 2x_1$ and $x_3 = 3x_1$, where x_1 is the image of ρ of the only simple root in $\Delta_{\mathfrak{m}}^+$. In particular, we have $x_3 = x_1 + x_2$.

ii) If $b_2(K) = 2$ then up to permutation we have $x_3 = x_1 + x_2$ where x_1 and x_2 are the coordinate vectors of the only simple roots in $\Delta_{\mathfrak{m}}^+$.

In both cases we obtain the following table for the brackets where the superscript $*$ means that it is zero in case *ii*). We abbreviate $\mathfrak{m}_{x_i}^+ = \mathfrak{m}_i$.

$[-,-]$	$\mathfrak{m}_1$	$\mathfrak{m}_2$	$\mathfrak{m}_3$	$\overline{\mathfrak{m}_1}$	$\overline{\mathfrak{m}_2}$	$\overline{\mathfrak{m}_3}$
$\mathfrak{m}_1$	$\mathfrak{m}_2^*$	$\mathfrak{m}_3$	0	$\mathfrak{k}$	$\overline{\mathfrak{m}_1}^*$	$\overline{\mathfrak{m}_2}$
$\mathfrak{m}_2$	$\mathfrak{m}_3$	0	0	$\mathfrak{m}_1^*$	$\mathfrak{k}$	$\overline{\mathfrak{m}_1}$
$\mathfrak{m}_3$	0	0	0	$\mathfrak{m}_2$	$\mathfrak{m}_1$	$\mathfrak{k}$

We continue with the proof for this table assuming that $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_2$ may possibly be non zero. The only difference is that we have to consider a few more terms in the expression for the holomorphic curvature. As mentioned above we consider a unit vector X and $Z = \frac{1}{\sqrt{2}}(X - iJX)$. Then

$$H(X) = -R(Z, \bar{Z}, Z, \bar{Z}) = g\left(-R(Z, \bar{Z})Z, \bar{Z}\right).$$

Since $Z \in \mathfrak{m}^+$ we write it as $Z = Z_1 + Z_2 + Z_3$ with $Z_i \in \mathfrak{m}_i$. Let $\lambda_{ij} = \frac{g_j}{g_i + g_j}$ with $i, j = 1, 2$, then we get from the bracket table and section 1.4.2 that the $\mathfrak{m}_i$ components of $-R(Z, \bar{Z})Z$ are given by

$$\begin{aligned}
\left(-R(Z,\bar{Z})Z\right)_1 = &-(\lambda_{12} - \lambda_{21})[\bar{Z}_2, [Z_1, Z_2]] \\
&- [[Z_1, \bar{Z}_2], Z_2] - [[Z_1, \bar{Z}_3], Z_3] - [[Z_2, \bar{Z}_3], Z_2] \\
&- [[Z_1, \bar{Z}_1] + [Z_2, \bar{Z}_2] + [Z_3, \bar{Z}_3], Z_1] \\
\left(-R(Z,\bar{Z})Z\right)_2 = &\ \lambda_{11}[Z_1, [\bar{Z}_1, Z_2]] + \lambda_{11}[Z_1, [\bar{Z}_2, Z_3]] \\
&- \lambda_{11}[[Z_2, \bar{Z}_1], Z_1] - \lambda_{11}[[Z_3, \bar{Z}_2], Z_1] \\
&- (\lambda_{12} - \lambda_{21})[\bar{Z}_1, [Z_1, Z_2]] \\
&- [[Z_1, \bar{Z}_2], Z_3] - [[Z_2, \bar{Z}_3], Z_3] \\
&- [[Z_1, \bar{Z}_1] + [Z_2, \bar{Z}_2] + [Z_3, \bar{Z}_3], Z_2] \\
\left(-R(Z,\bar{Z})Z\right)_3 = &\ \lambda_{12}[Z_1, [\bar{Z}_1, Z_3]] + \lambda_{21}[Z_2, [\bar{Z}_1, Z_2]] + \lambda_{21}[Z_2, [\bar{Z}_2, Z_3]] \\
&- \lambda_{21}[[Z_3, \bar{Z}_1], Z_1] - \lambda_{12}[[Z_2, \bar{Z}_1], Z_2] - \lambda_{12}[[Z_3, \bar{Z}_2], Z_2] \\
&- [[Z_1, \bar{Z}_1] + [Z_2, \bar{Z}_2] + [Z_3, \bar{Z}_3], Z_3]
\end{aligned}$$

Note that $R(Z, \overline{Z})Z \in \mathfrak{m}^+$ by proposition 12.

Using the antisymmetry of the Lie bracket we see that the first two rows in the expression of $(-R(Z,\bar{Z})Z)_2$ cancel and the first two rows of $(-R(Z,\bar{Z})Z)_3$ match except for the coefficient. Furthermore we know that

$$g((-R(Z,\bar{Z})Z)_i,\bar{Z}) = g((-R(Z,\bar{Z})Z)_i,\bar{Z}_i) = g_i(-B)((-R(Z,\bar{Z})Z)_i,\bar{Z}_i)$$

which allows us to use the skew symmetry of ad_X with respect to B. If we denote the hermitian form $\langle X,Y\rangle = -B(X,\bar{Y})$ and $\|X\|^2 = \langle X,X\rangle$ then with $g_3 = g_1 + g_2$ by the Kähler property, we get

$$\begin{aligned}
H(X) = &- g_1(\lambda_{12}-\lambda_{21})\|[Z_1,Z_2]\|^2 \\
&+ g_1\|[Z_1,\bar{Z}_2]\|^2 + g_1\|[Z_1,\bar{Z}_3]\|^2 + g_1\|[Z_1,\bar{Z}_1]\|^2 \\
&+ g_1\langle[Z_2,\bar{Z}_2],[Z_1,\bar{Z}_1]\rangle + g_1\langle[Z_3,\bar{Z}_3],[Z_1,\bar{Z}_1]\rangle + g_1\langle[Z_2,\bar{Z}_3],[Z_1,\bar{Z}_2]\rangle \\
&+ g_2(\lambda_{12}-\lambda_{21})\|[Z_1,Z_2]\|^2 + g_2\|[Z_2,\bar{Z}_3]\|^2 + g_2\|[Z_2,\bar{Z}_2]\|^2 \\
&+ g_2\langle[Z_1,\bar{Z}_1],[Z_2,\bar{Z}_2]\rangle + g_2\langle[Z_3,\bar{Z}_3],[Z_2,\bar{Z}_2]\rangle + g_2\langle[Z_1,\bar{Z}_2],[Z_2,\bar{Z}_3]\rangle \\
&- (g_1+g_2)(\lambda_{12}-\lambda_{21})\|[\bar{Z}_1,Z_3]\|^2 \\
&+ (g_1+g_2)(\lambda_{12}-\lambda_{21})\|[\bar{Z}_2,Z_3]\|^2 \\
&+ (g_1+g_2)\|[Z_3,\bar{Z}_3]\|^2 \\
&+ (g_1+g_2)(\lambda_{12}-\lambda_{21})\langle[\bar{Z}_1,Z_2],[\bar{Z}_2,Z_3]\rangle \\
&+ (g_1+g_2)\langle[Z_1,\bar{Z}_1],[Z_3,\bar{Z}_3]\rangle + (g_1+g_2)\langle[Z_2,\bar{Z}_2],[Z_3,\bar{Z}_3]\rangle
\end{aligned}$$

Before we continue matching and modifying these terms, we observe the following consequence of the skew symmetry of ad with respect to the killing form:

$$\langle[A,\bar{A}],[B,\bar{B}]\rangle = \|[\bar{A},B]\|^2 - \|[A,B]\|^2 \tag{2.1}$$

This implies immediately that

i) The term $\langle[Z_i,\bar{Z}_i],[Z_j,\bar{Z}_j]\rangle$ is a real number and hence is symmetric with respect to i,j.

ii) For i,j with $i+j>3$ we have $\|[Z_i,Z_j]\|^2 = 0$ which implies $\langle[Z_i,\bar{Z}_i],[Z_j,\bar{Z}_j]\rangle = \|[Z_i,\bar{Z}_j]\|^2$.

Using the above and $(\lambda_{12}-\lambda_{21}) = \frac{g_2-g_1}{g_2+g_1}$ we collect terms in the expression of $H(X)$ and obtain:

$$\begin{aligned}
H(X) = &(2g_1+g_2)\|[Z_1,\bar{Z}_2]\|^2 - 4\frac{g_1g_2}{g_1+g_2}\|[Z_1,Z_2]\|^2 \\
&+ 4g_1\|[Z_1,\bar{Z}_3]\|^2 + 4g_2\|[Z_2,\bar{Z}_3]\|^2 \\
&+ g_1\|[Z_1,\bar{Z}_1]\|^2 + g_2\|[Z_2,\bar{Z}_2]\|^2 + (g_2+g_1)\|[Z_3,\bar{Z}_3]\|^2 \\
&+ g_2(\langle[Z_1,\bar{Z}_2],[Z_2,\bar{Z}_3]\rangle + \langle[\bar{Z}_1,Z_2],[\bar{Z}_2,Z_3]\rangle).
\end{aligned}$$

It is left to control the second term in the first row and the two terms of the last row. To that end we use the following equations coming from polarization

$$\begin{aligned}
&\langle[Z_1,\bar{Z}_2],[Z_2,\bar{Z}_3]\rangle + \langle[\bar{Z}_1,Z_2],[\bar{Z}_2,Z_3]\rangle \\
&\qquad = \left\|\frac{1}{2}[Z_1,\bar{Z}_2] + 2[Z_2,\bar{Z}_3]\right\|^2 - \frac{1}{4}\|[Z_1,\bar{Z}_2]\|^2 - 4\|[Z_2,\bar{Z}_3]\|^2
\end{aligned}$$

and this one coming from equation (2.1),

$$-\|[Z_1,Z_2]\|^2 = -\|[Z_1,\bar{Z}_2]\|^2 + \langle[Z_1,\bar{Z}_1],[Z_2,\bar{Z}_2]\rangle$$

Hence

$$\begin{aligned}
H(X) = & \left(2g_1 + \frac{3}{4}g_2 - \frac{4g_1g_2}{g_1+g_2}\right) \|[Z_1, \bar{Z}_2]\|^2 + 4g_1\|[Z_1, \bar{Z}_3]\|^2 \\
& + g_1\|[Z_1, \bar{Z}_1]\|^2 + \frac{4g_1g_2}{g_1+g_2}\langle [Z_1, \bar{Z}_1], [Z_2, \bar{Z}_2]\rangle + g_2\|[Z_2, \bar{Z}_2]\|^2 \\
& + (g_2+g_1)\|[Z_3, \bar{Z}_3]\|^2 + g_2\left\|\frac{1}{2}[Z_1, \bar{Z}_2] + 2[Z_2, \bar{Z}_3]\right\|^2 \\
= & \frac{(16g_1 - 5g_2)^2 + 71g_2^2}{128(g_1+g_2)}\|[Z_1, \bar{Z}_2]\|^2 + 4g_1\|[Z_1, \bar{Z}_3]\|^2 \\
& + \frac{1}{g_1+g_2}\left(\left\|g_1[Z_1, \bar{Z}_1] + g_2[Z_2, \bar{Z}_2]\right\|^2 + g_1g_2\left\|[Z_1, \bar{Z}_1] + [Z_2, \bar{Z}_2]\right\|^2\right) \\
& + (g_2+g_1)\|[Z_3, \bar{Z}_3]\|^2 + g_2\left\|\frac{1}{2}[Z_1, \bar{Z}_2] + 2[Z_2, \bar{Z}_3]\right\|^2
\end{aligned}$$

which is nonnegative. Furthermore, it is immediate that $H(X) = 0$ implies $[Z_3, \bar{Z}_3] = 0$ and $[Z_2, \bar{Z}_2] = -[Z_1, \bar{Z}_1]$. We will show now that this implies $Z_3 = Z_2 = Z_1 = 0$. First of all, we show that the second equation implies $[Z_1, \bar{Z}_1] = [Z_2, \bar{Z}_2] = 0$. The key is to see that for $V \in \mathfrak{m}^+$,

$$[V, \bar{V}]_{\mathfrak{h}} = -\sum_{\gamma \in \Phi} \mu_\gamma H_\gamma$$

with $\mu_\gamma \geq 0$ and H_γ defined as in (1.3). In fact, we have for $V = \sum_{\alpha \in \Delta_{\mathfrak{m}}^+} m_\alpha E_\alpha$ with a Chevalley basis such that $z_\alpha = 1$ and $\bar{E}_\alpha = -E_{-\alpha}$ that

$$\begin{aligned}
[V, \bar{V}]_{\mathfrak{h}} &= \sum_{\alpha \in \Delta_{\mathfrak{m}}^+} \sum_{\beta \in \Delta_{\mathfrak{m}}^+} m_\alpha \bar{m}_\beta [E_\alpha, \bar{E}_\beta]_{\mathfrak{h}} = -\sum_{\alpha \in \Delta_{\mathfrak{m}}^+} \|m_\alpha\|^2 H_\alpha \\
&= -\sum_{\alpha \in \Delta_{\mathfrak{m}}^+} \|m_\alpha\|^2 \sum_{\gamma \in \Phi} n(\alpha)_\gamma H_\gamma = -\sum_{\gamma \in \Phi}\left(\sum_{\alpha \in \Delta_{\mathfrak{m}}^+} n(\alpha)_\gamma \|m_\alpha\|^2\right) H_\gamma \\
&= -\sum_{\gamma \in \Phi} \mu_\gamma H_\gamma
\end{aligned}$$

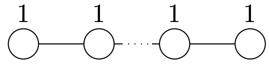
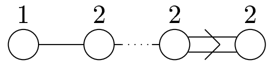
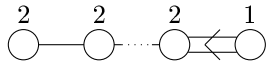
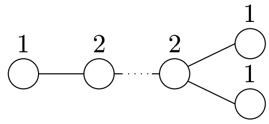
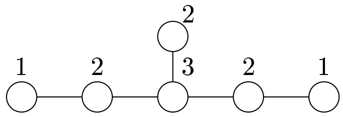
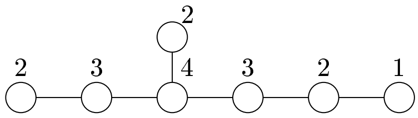
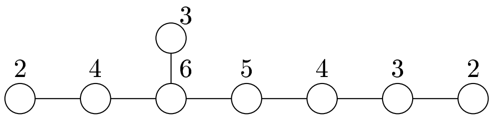
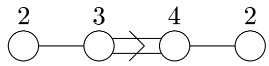
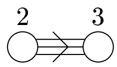
where we used $\alpha = \sum_{\gamma \in \Phi} n(\alpha)_\gamma \gamma$ with $n(\alpha)_\gamma \in \mathbb{N}$. Therefore $[Z_2, \bar{Z}_2] = -[Z_1, \bar{Z}_1]$ is only possible if $[Z_2, \bar{Z}_2]_{\mathfrak{h}} = [Z_1, \bar{Z}_1]_{\mathfrak{h}} = 0$. By the calculation above it is clear that for $V \in \{Z_1, Z_2, Z_3\}$ the equation $[V, \bar{V}]_{\mathfrak{h}} = 0$ implies that $\mu_\gamma = 0$ for all $\gamma \in \Phi$ which in turn implies that $n(\alpha)_\gamma \|m_\alpha\|^2 = 0$ for all $\alpha \in \Delta_{\mathfrak{m}}^+$ and $\gamma \in \Phi$. Since for each α there has to be at least one γ with $n(\alpha)_\gamma > 0$ we get $m_\alpha = 0$ for all $\alpha \in \Delta_{\mathfrak{m}}^+$ and hence $V = 0$. □

The C spaces covered by the above can be read off of the following diagrams, where the number n_α at each simple root represents the coefficient of the highest weight in that basis. Those can be found in [Bou68, p.250f]. Then the result above applies to the spaces obtained by the following two ways of painting the diagrams:

i) Paint all vertices black except for a single one with

$$n_\alpha \begin{cases} \leq 2 & \text{Itoh} \\ = 3 & \text{the above} \end{cases}$$

ii) Paint all vertices black except for two with $n_\alpha = 1$.

Family	G	(V, E)
A_n	$SU(n+1)$	
B_n	$SO(2n+1)$	
C_n	$Sp(n)$	
D_n	$SO(2n)$	
E_6	E_6	
E_7	E_7	
E_8	E_8	
F_4	F_4	
G_2	G_2	

In fact, the new examples of positive holomorphic curvature with $b_2(M) = 2$ are

$$SU(n+1)/S(U(a)\ U(b)\ U(c)),\ SO(2n)/SU_i(n-1)\ \mathbb{T}^2,\ E_6/SO(8)\mathbb{T}^2$$

with $a + b + c = n + 1$ and for two nonequivalent embeddings of $SU_i(n-1) \hookrightarrow SO(2n)$ for $i = 1, 2$. The seven new exceptional $b_2(M) = 1$ examples are

$$E_6/SU(3)\mathbb{S}^1 SU(2)SU(3),\ E_7/SU(2)\mathbb{S}^1 SU(6),\ E_7/SU(5)\mathbb{S}^1 SU(3),$$
$$E_8/SU(8)\mathbb{S}^1,\ E_8/E_6\mathbb{S}^1 SU(2),\ F_4/SU(2)\mathbb{S}^1 SU(3),\ G_2/SU(2)\mathbb{S}^1$$

We remark, that this closes the $b_2(M) = 1$ case for the groups G_2 and E_6 and leaves only 6 open cases for the other exceptional groups.

Chapter 3

Holomorphic sectional curvature

In this chapter we describe the holomorphic curvature tensor from section 1.5 in the special case of a Kähler C space using the fact that the isotropy group contains a maximal torus $\mathbb{T}$ of G. We do so via identifying the tensor with a symmetric endomorphism and use the $\mathbb{T}$ equivariance of said endomorphism to split it into the sum of smaller endomorphisms. Afterwards, we describe the unique Kähler Einstein metric for a C space (G, K, J) and determine it explicitly for the classical groups with the chosen J_{std}. The last section of the chapter is dedicated to the argument, why it is sufficient to consider only our preferred complex structure.

3.1 Structure of a $\mathbb{T}$ invariant symmetric tensor

In this section, we describe the structure of a symmetric $\mathbb{T}$ invariant tensor F on $Fix(J)$, i.e. a symmetric bilinear form

$$F : Fix(J) \times Fix(J) \to \mathbb{R},$$

and will apply this to H and the four forms ω. First of all we identify the tensor with an endomorphism. We do so using the $\mathbb{T}$ invariant inner product $\langle -, - \rangle_{\Lambda^2(\mathfrak{m})}$ from section 1.6 on $\Lambda^2(\mathfrak{m})$. By the invariance of F, we obtain a unique $\mathbb{T}$ equivariant symmetric map

$$\bar{F} : Fix(J) \to Fix(J)$$

such that $F(\omega_1, \omega_2) = \langle \bar{F}(\omega_1), \omega_2 \rangle_{\Lambda^2\mathfrak{m}}$. Complexification gives us the possibility to decompose $Fix(J)$ into weight spaces.

Proposition 23. *The weight space decomposition of $Fix(J)^{\mathbb{C}}$ is given by*

$$Fix(J)^{\mathbb{C}} = \bigoplus_{\alpha \in \Delta_{\mathfrak{m}}^+} \bigoplus_{\beta \in \Delta_{\mathfrak{m}}^+} \mathfrak{g}_\alpha \wedge \mathfrak{g}_{-\beta}.$$

Let $\Delta_H = \{\alpha - \beta \mid \alpha, \beta \in \Delta_{\mathfrak{m}}^+\}/\mathbb{Z}_2$. Then we have the isotypical decomposition

$$Fix(J)^{\mathbb{C}} = \bigoplus_{\eta \in \Delta_H} \mathfrak{m}_\eta$$

where $\mathfrak{m}_\eta = \bigoplus_{\alpha - \beta \in \eta} \mathfrak{g}_\alpha \wedge \mathfrak{g}_{-\beta}$.

Proof. For dimensional reasons it is sufficient to show the inclusion of the right hand side in the left hand side which follows immediately from

$$J(X_\alpha \wedge Y_{-\beta}) = J(X_\alpha) \wedge J(Y_{-\beta}) = i(-i) X_\alpha \wedge Y_{-\beta} = X_\alpha \wedge Y_{-\beta}$$

for $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_{-\beta} \in \mathfrak{g}_{-\beta}$. In fact, it is easy to see that

$$dim_{\mathbb{C}}(\Lambda^2(\mathfrak{m})^{\mathbb{C}}) = \binom{|\Delta_{\mathfrak{m}}|}{2} = (2|\Delta_{\mathfrak{m}}^+| - 1)|\Delta_{\mathfrak{m}}^+|$$

and we see that the following vectors are linearly independent Eigenvectors of J with eigenvalue 1 for $E_\alpha \wedge E_{-\beta}$ for $\alpha, \beta \in \Delta_\mathfrak{m}^+$ and eigenvalue -1 for $E_\alpha \wedge E_\beta$, $E_{-\alpha} \wedge E_{-\beta}$ for $\alpha < \beta \in \Delta_\mathfrak{m}^+$ which give a basis of $\Lambda^2(\mathfrak{m})^\mathbb{C}$ by counting. It remains to show that the summands are weight spaces and that they are equivalent if and only if they induce the same element in Δ_H. Therefore we want to determine the weights of the action of $\mathbb{T} \subset K$. To that end, let $v \in \mathfrak{t}$ and $v(t) = exp(tv)$ then we have,

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} Ad_{v(t)}\left(X_\alpha \wedge Y_{-\beta}\right) &= \left.\frac{d}{dt}\right|_{t=0} Ad_{v(t)}X_\alpha \wedge Ad_{v(t)}Y_{-\beta} \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} Ad_{v(t)}X_\alpha\right) \wedge Y_{-\beta} + X_\alpha \wedge \left(\left.\frac{d}{dt}\right|_{t=0} Ad_{v(t)}Y_{-\beta}\right) \\
&= (ad_v X_\alpha) \wedge Y_{-\beta} + X_\alpha \wedge (ad_v Y_{-\beta}) \\
&= \alpha(v) X_\alpha \wedge Y_{-\beta} - \beta(v) X_\alpha \wedge Y_{-\beta} \\
&= (\alpha - \beta)(v) X_\alpha \wedge Y_{-\beta}
\end{aligned}$$

□

The $\mathfrak{t}$ equivariance of $\bar{F}$ implies that the tensor satisfies $F(\mathfrak{m}_\eta, \mathfrak{m}_{\eta'}) = 0$ otherwise the projection of the restriction of $\bar{F}$, i.e. $pr_{\eta'} \circ F|_{\mathfrak{m}_\eta} : \mathfrak{m}_\eta \to \mathfrak{m}_{\eta'}$, would be a non zero equivariant map between non equivalent modules. Hence $\bar{F}$ decomposes into the sum of maps $\bar{F}_\eta : \mathfrak{m}_\eta \to \mathfrak{m}_\eta$ for $\eta \in \Delta_H$. In order to describe the $\bar{F}_\eta$ more carefully, we will go on to describe them with a useful basis. To that end we identify Δ_H with the set $\{\alpha - \beta \mid \alpha < \beta \in \Delta_\mathfrak{m}^+\} \cup \{0\}$ for a ordering $<$ as in proposition 15 on $\Delta_\mathfrak{m}^+$ and make the following

Definition 24. *We define*

i) for $\alpha < \beta$ the module

$$\mathfrak{m}_{\alpha\beta} = \mathfrak{g}_\alpha \wedge \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_\beta \wedge \mathfrak{g}_{-\alpha},$$

ii) for $\eta \in \Delta_H \setminus \{0\}$ the index set

$$I(\eta) = \{(\alpha, \beta) \mid \alpha < \beta, \alpha - \beta = \eta\}$$

and its cardinality to be $n_\eta = |I(\eta)|$.

Using this notation we have

$$\mathfrak{m}_\eta = \bigoplus_{(\alpha,\beta)\in I(\eta)} \mathfrak{m}_{\alpha\beta} \qquad\qquad \mathfrak{m}_0 = \bigoplus_{\alpha\in\Delta_\mathfrak{m}^+} \mathfrak{g}_\alpha \wedge \mathfrak{g}_{-\alpha}$$

and $\dim_\mathbb{C}(\mathfrak{m}_\eta) = 2n_\eta$ and $\dim_\mathbb{C}(\mathfrak{m}_0) = |\Delta_\mathfrak{m}^+|$.

For a fixed pair $\alpha < \beta$, we choose the ordered basis $A_{\alpha\beta} = E_\alpha \wedge E_{-\beta}$ and $B_{\alpha\beta} = E_\beta \wedge E_{-\alpha}$ of $\mathfrak{m}_{\alpha\beta}$, where the E_α are the Chevalley basis from proposition 11. Obviously, this induces a basis of the real module

$$\mathfrak{m}_{\alpha\beta}^\mathbb{R} = \mathfrak{m} \cap \mathfrak{m}_{\alpha\beta} = \langle \Phi_{\alpha\beta}, \Psi_{\alpha\beta} \rangle_\mathbb{R}$$

where $\Phi_{\alpha\beta} = \frac{1}{\sqrt{2}}(A_{\alpha\beta} - B_{\alpha\beta})$ and $\Psi_{\alpha\beta} = \frac{i}{\sqrt{2}}(A_{\alpha\beta} + B_{\alpha\beta})$. In general, the projection of the restriction of $\bar{F}$ to the submodules of $\mathfrak{m}_\eta$, i.e.

$$\bar{F}_{\gamma\delta}^{\alpha\beta} = pr_{\gamma\delta} \circ \bar{F}|_{\mathfrak{m}_{\alpha\beta}} : \mathfrak{m}_{\alpha\beta} \to \mathfrak{m}_{\gamma\delta}$$

is a intertwining map of equivalent $\mathbb{T}$ modules. However, since these are complex representations the space of intertwining maps is two dimensional and in fact it is fairly easy to see that in the basis A, B of the corresponding modules, we have the matrix representation

$$\bar{F}_{\gamma\delta}^{\alpha\beta} = \frac{1}{\langle A_{\gamma\delta}, B_{\gamma\delta}\rangle_{\Lambda^2(\mathfrak{m})}} \begin{pmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{pmatrix}$$

where $\lambda = F(A_{\gamma\delta}, B_{\gamma\delta})$ and $\mu = F(B_{\gamma\delta}, B_{\gamma\delta})$. However, it will be useful to consider a more restrictive type of tensor F. In fact, we present the following

Proposition 25. *Let $\eta \in \Delta_H$ be non zero. Let F be a symmetric tensor as before, that satisfies for all $(\alpha, \beta), (\gamma, \delta) \in I(\eta)$:*

1) *$F(A_{\alpha\beta}, A_{\gamma\delta})$ is zero and*
2) *$F(A_{\alpha\beta}, B_{\gamma\delta})$ is a real number.*

Then we have that F is positive definite on $\mathfrak{m}_\eta$ if and only if the matrix $M(F_\eta) \in Sym_{n_\eta}(\mathbb{R})$ defined as

$$\big(M(F_\eta)\big)_{(\alpha,\beta)(\gamma,\delta)} = -F(A_{\alpha\beta}, B_{\gamma\delta})$$

with $(\alpha, \beta), (\gamma, \delta) \in I(\eta)$ is positive definite.

Proof. Consider the matrix X representing the bilinear form F on $\mathfrak{m}^{\mathbb{R}}_{\alpha\beta} \times \mathfrak{m}^{\mathbb{R}}_{\gamma\delta}$, i.e.

$$X = \begin{pmatrix} F(\Phi_{\alpha\beta}, \Phi_{\gamma\delta}) & F(\Phi_{\alpha\beta}, \Psi_{\gamma\delta}) \\ F(\Psi_{\alpha\beta}, \Phi_{\gamma\delta}) & F(\Psi_{\alpha\beta}, \Psi_{\gamma\delta}) \end{pmatrix}$$

Then by expanding the expressions of Φ, Ψ in terms of A and B, and using the fact that from $\overline{A_{\alpha\beta}} = B_{\alpha\beta}$ follows $\overline{F(A_{\alpha\beta}, B_{\gamma\delta})} = F(\overline{A_{\alpha\beta}}, \overline{B_{\gamma\delta}}) = F(B_{\alpha\beta}, A_{\gamma\delta})$ and similarly $\overline{F(A_{\alpha\beta}, A_{\gamma\delta})} = F(B_{\alpha\beta}, B_{\gamma\delta})$ we get

$$X = \begin{pmatrix} Re\big(F(A_{\alpha\beta}, A_{\gamma\delta}) - F(A_{\alpha\beta}, B_{\gamma\delta})\big) & -Im\big(F(A_{\alpha\beta}, A_{\gamma\delta}) + F(A_{\alpha\beta}, B_{\gamma\delta})\big) \\ -Im\big(F(A_{\alpha\beta}, A_{\gamma\delta}) - F(A_{\alpha\beta}, B_{\gamma\delta})\big) & -Re\big(F(A_{\alpha\beta}, A_{\gamma\delta}) + F(A_{\alpha\beta}, B_{\gamma\delta})\big) \end{pmatrix}.$$

Now using properties 1) and 2) we obtain

$$X = -F(A_{\alpha\beta}, B_{\gamma\delta}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since this is true for all pairs of pairs $(\alpha, \beta), (\gamma, \delta) \in I(\eta)$ we get that the matrix representing F on $\mathfrak{m}_\eta \times \mathfrak{m}_\eta$ is

$$M(F_\eta) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $\otimes$ denotes the Kronecker product of matrices. Now it is known that the eigenvalues of the Kronecker product of two diagonalizable matrices are exactly the products of their eigenvalues. Hence it follows that F is positive definite on $\mathfrak{m}_\eta$ if and only if $M(F_\eta)$ is positive definite. □

For the trivial module we choose the basis $C_\alpha = E_\alpha \wedge E_{-\alpha}$ and the real version is obviously $\Omega_\alpha = iC_\alpha$. Then the version of the above for the trivial module is given by

Proposition 26. *The tensor F on $\mathfrak{m}_0$ is positive definite if and only if the matrix $M(F_0) \in Sym_{|\Delta^+_{\mathfrak{m}}|}(\mathbb{R})$ defined as*

$$\big(M(F_0)\big)_{(\alpha,\alpha),(\beta,\beta)} = -F(C_\alpha, C_\beta)$$

is positive definite.

Proof. This is even simpler, since the matrix representing F on $\mathfrak{m}^{\mathbb{R}}_0 \times \mathfrak{m}^{\mathbb{R}}_0$ is given by $X_{\alpha\beta} = F(\Omega_\alpha, \Omega_\beta) = -F(C_\alpha, C_\beta)$.

□

Now we want to apply proposition 25 to the holomorphic curvature tensor and to the elementary relevant four forms from proposition 20. To that end, denote the elementary forms by

$$\omega_1 = E_{x,-y,v,-w} + \overline{E_{x,-y,v,-w}} \quad \text{and} \quad \omega_2 = i(E_{x,-y,v,-w} - \overline{E_{x,-y,v,-w}})$$

with $x - y = w - v$ and $x < y, w < v \in \Delta^+_{\mathfrak{m}}$.

Proposition 27. *The holomorphic curvature tensor H and the elementary four forms ω_1 satisfy the requirements of proposition 25.*

Remark: We want to remark that with the same proof also ω_2 satisfies property 1) but not property 2) and hence the changes induced by ω_2 on the curvature tensor can not be represented in the matrices $M(F_\eta)$. We will restrict ourself to the modifications realizable in the $M(F_\eta)$ since they will be sufficient for our proofs. However, it might be worthwhile to investigate further how ω_2 modifies the curvature tensor.

Proof. First we consider the holomorphic curvature tensor H. The first property is true because $H(A_{\alpha\beta}, A_{\gamma\delta}) = R_{\alpha\beta\gamma\delta} = 0$ by equation (1.4) since $\alpha - \beta = \eta = \gamma - \delta \neq \delta - \gamma$ and $H(A_{\alpha\beta}, B_{\gamma\delta}) = R_{\alpha\beta\delta\gamma}$ is a real number by the table of proposition 16. Secondly, we consider ω_1. Then we have by $\mathbb{T}$ invariance as in proposition 19 that

$$\begin{aligned}\omega_1'(A_{\alpha\beta}, A_{\gamma\delta}) &= \langle \omega_1, E_{\alpha,-\beta,\gamma,-\delta} \rangle_{\Lambda^4(\mathfrak{m})}\\ &= \langle Ad_{exp(tv)}(\omega_1), E_{\alpha,-\beta,\gamma,-\delta} \rangle_{\Lambda^4(\mathfrak{m})}\\ &= \langle \omega_1, Ad_{exp(-tv)}(E_{\alpha,-\beta,\gamma,-\delta}) \rangle_{\Lambda^4(\mathfrak{m})}\end{aligned}$$

and differentiation in t yields $0 = (\alpha - \beta + \gamma - \delta)(v) \langle \omega_1, E_{\alpha,-\beta,\gamma,-\delta} \rangle_{\Lambda^4(\mathfrak{m})}$ for $v \in \mathfrak{t}$. As above we have $\alpha - \beta + \gamma - \delta \neq 0$ and hence $\omega_1'(A_{\alpha\beta}, A_{\gamma\delta}) = 0$ as claimed. The second property follows from the fact that the expression (1.13) only yields real values. $\square$

All of the above allows us to reduce the transformation of the holomorphic curvature tensor into a positive definite tensor via forms of the type ω_1 to the matrices $M(F_\eta)$ with $F \in \{H, \omega_1\}$. Before we determine those matrices in detail including the different possibilities of roots x, y, v, w in the choice of ω_1, we describe the strategy we will follow: Notice, that the requirements of proposition 25 are additive in F and hence we have for two symmetric $\mathbb{T}$ invariant tensors F, P satisfying the requirements, that $F + P$ satisfies them as well and we have

$$M\left((F+P)_\eta\right) = M(F_\eta) + M(P_\eta),$$

in particular $M((H+\omega)_\eta) = M(H_\eta) + M(\omega_\eta)$ for ω being a four form in the span of all elementary relevant forms of type ω_1. Hence, once determined $M(H_\eta)$ the proof of positive holomorphic sectional curvature reduces to adding matrices coming from four forms until the resulting matrix is positive definite. By proposition 25 the symmetric tensor corresponding to that matrix is positive definite and by (1.12) the holomorphic sectional curvature is positive.

Now we determine the matrices for the mentioned tensors.

Proposition 28. *The matrices representing the holomorphic curvature tensor are given by the following: For $\eta \in \Delta_H$ being a non zero weight*

$$M(H_\eta)_{(\alpha,\beta)(\gamma,\delta)} = -R_{\alpha\beta\delta\gamma}$$

with $(\alpha, \beta), (\gamma, \delta) \in I(\eta)$ and for the trivial module

$$M(H_0)_{(\alpha,\alpha)(\beta,\beta)} = -R_{\alpha\alpha\beta\beta}$$

for $\alpha, \beta \in \Delta_{\mathfrak{m}}^+$.

In the case of ω_1 we distinguish two cases:

Proposition 29 (Case I)**.** *Assume $|\{x, y, v, w\}| = 4$. Then we may assume $x < y, v, w$.In this case the only non vanishing entries of the matrices $M((\omega_1)_\eta)$ are the following:*

$$M((\omega_1)_{(x-y)})_{(x,y)(w,v)} = -c \text{ and } M((\omega_1)_{(x-w)})_{(x,w)(y,v)} = c$$

and their symmetric counterparts, where $c = z_\alpha z_\beta z_\gamma z_\delta g_\alpha g_\beta g_\gamma g_\delta$.

Proof. First of all notice that $\omega_1(x,y,w,v) = E_{x,-y,v,-w} + E_{y,-x,w,-v} = \omega_1(w,v,x,y)$ which allows us to assume $x < y, v, w$. Then we have to check when

$$M((\omega_1)_\eta)_{(\alpha,\beta)(\gamma,\delta)} = -\langle \omega_1, E_{(\alpha,-\beta,\delta,-\gamma)}\rangle_{\Lambda^4(\mathfrak{m})}$$

with $\alpha \le \beta, \gamma \le \delta$ and $\alpha - \beta = \gamma - \delta = \eta$ is non zero. Note that the equality cases $\alpha = \beta$ and $\gamma = \delta$ are orthogonal to ω_1 since all x, y, v, w are different. It is easy to see that the only non zero cases are

$$\langle \omega_1, E_{(\alpha,-\beta,\delta,-\gamma)}\rangle_{\Lambda^4(\mathfrak{m})} = \begin{cases} c & (\alpha,\beta,\gamma,\delta) = (x,y,w,v) \\ c & (\alpha,\beta,\gamma,\delta) = (w,v,x,y) \\ -c & (\alpha,\beta,\gamma,\delta) = (x,w,y,v) \\ -c & (\alpha,\beta,\gamma,\delta) = (y,v,x,w) \end{cases}$$

Here we see that the first two are the symmetric entries in $M((\omega_1)_{(x-y)})$ and the later two are the symmetric entries in $M((\omega_1)_{(x-w)})$ as claimed. □

Proposition 30 (Case II). *Assume $|\{x,y,v,w\}| = 2$. Then $x = w$ and $y = v$ and the only non vanishing entries of the matrices $M((\omega_1)_\eta)$ are the following*

$$M((\omega_1)_{(x-y)})_{(x,y)(x,y)} = 2c \text{ and } M((\omega_1)_0)_{(x,x)(y,y)} = -2c$$

and its symmetric counterpart. Here we have $c = z_x^2 z_y^2 g_x^2 g_y^2$.

Proof. It is clear that $x = w$ and $y = v$. In that case we have $\omega_1(x,y,x,y) = 2E_{x,-y,y,-x}$ which is orthogonal to $E_{(\alpha,-\beta,\delta,-\beta)}$ unless

$$(\alpha,\beta,\gamma,\delta) \in \{(x,x,y,y),(y,y,x,x),(x,y,x,y)\}$$

and we have

$$\langle 2E_{x,-y,y,-x}, E_{(\alpha,-\beta,\delta,-\gamma)}\rangle_{\Lambda^4(\mathfrak{m})} = \begin{cases} -2c & (\alpha,\beta,\gamma,\delta) = (y,y,x,x) \\ -2c & (\alpha,\beta,\gamma,\delta) = (x,x,y,y) \\ 2c & (\alpha,\beta,\gamma,\delta) = (x,y,x,y) \end{cases}$$

□

Remark: The value of c is actually not important since we can scale ω_1 arbitrarily. The important results of the above was to determine the non vanishing entries and the signs of the modification.

The summary of the above is the following recipe on how to prove positive holomorphic sectional curvature:

i) Determine the set Δ_H of weights.

ii) Determine the matrices $M(H_\eta)$ representing the holomorphic curvature tensor.

iii) Modify the matrices to turn them positive definite using an arbitrary amount of the following modifications:
I : Adding the value $s \in \mathbb{R}$ on the diagonal of an $M(H_\eta)$ with $\eta \in \Delta_H \setminus \{0\}$ and subtracting s symmetrically off the entry of $M(H_0)$ as given in proposition 30.

II : Adding the value s symmetrically to an off diagonal entry of $M(H_\eta)$ with $\eta \in \Delta_H \setminus \{0\}$ and subtracting s symmetrically from the matrix $M(H_{\eta'})$ as given in proposition 29.

A useful observation is that it is sufficient to have all $M(H_\eta)$ positive semidefinite and only $M(H_0)$ positive definite.

Proposition 31. *If there are four forms such that the modified matrices $M(H_\eta)$ are positive semidefinite and $M(H_0)$ is positive definite, then there are also four forms such that all of them are positive definite.*

Proof. If we take for $\varepsilon > 0$ the four form ω that has the matrix expression

$$M(\omega_0)_{(\alpha,\beta)} = \begin{cases} 0 & \alpha = \beta \\ -\varepsilon & \alpha \neq \beta \end{cases}$$

and

$$M(\omega_\eta)_{(\alpha,\beta)(\gamma,\delta)} = \begin{cases} 0 & (\alpha,\beta) \neq (\gamma,\delta) \\ \varepsilon & (\alpha,\beta) = (\gamma,\delta) \end{cases}$$

and denote by H' the modified tensor as stated in the claim, then we have that for ε small enough but not zero $M(H'_0) + M(\omega_0)$ is still positive definite since it is an open condition and at the same time $M(H'_\eta) + M(\omega_\eta)$ is positive definite since it is the sum of a positive definite and a positive semidefinite matrix. □

Notice that the symmetry coming from the Bianchi identity in lemma 14 and the symmetry coming from the four form, i.e. $\omega(A \wedge B, C \wedge D) = -\omega(A \wedge C, B \wedge D)$, imply

$$M(H_\eta)_{(\alpha,\beta)(\gamma,\delta)} = M(H_{\eta+\beta-\gamma})_{(\alpha,\gamma)(\beta,\delta)} \tag{3.1}$$

and

$$M(\omega_\eta)_{(\alpha,\beta)(\gamma,\delta)} = -M(\omega_{\eta+\beta-\gamma})_{(\alpha,\gamma)(\beta,\delta)}. \tag{3.2}$$

Hence, we can see here as stated by [Tho71] that the four forms are exactly the tensors breaking the Bianchi identity.

3.2 Kähler Einstein metric

As mentioned in the introduction, we will restrict ourself to a particular kind of Kähler metric. It is well known due to [Mat72], that for a fixed complex structure on a C space there is up to scaling exactly one homogeneous Kähler Einstein metric. For a detailed exposition we suggest [Arv92, p.36f]. In fact there is a simple and explicit formula for the coefficients g_α in the Kähler Einstein case considering that the complex structure is determined by a particular set $\Delta^+_{\mathfrak{m}}$ of roots as given in proposition 4:

$$g_\alpha = B(\alpha, \delta^*_K),$$

where $\delta^*_K = \sum_{\beta \in \Delta^+_{\mathfrak{m}}} \beta$. Now assume that we have a complex structure on $(G, \mathbb{T})$ compatible with the one on (G, K), i.e a decomposition $\Delta^+_{\mathfrak{g}} = \Delta^+_{\mathfrak{m}} \cup \Delta^+_{\mathfrak{k}}$ as in proposition 7. Then it is useful to consider $\gamma^*_K = \sum_{\alpha \in \Delta^+_{\mathfrak{k}}} \alpha$ to determine the Kähler Einstein metric of (G, K) from the one of $(G, \mathbb{T})$ in the following way:

$$\begin{aligned} (g^K_{KE})_\alpha &= B\big(\delta^*_K, \alpha\big) = B\Bigg(\sum_{\beta \in \Delta^+_{\mathfrak{m}}} \beta, \alpha\Bigg) \\ &= B\Bigg(\sum_{\beta \in \Delta^+_{\mathfrak{g}}} \beta, \alpha\Bigg) - B\Bigg(\sum_{\beta \in \Delta^+_{\mathfrak{k}}} \beta, \alpha\Bigg) \\ &= B\left(\delta^*_{\mathbb{T}}, \alpha\right) - B\left(\gamma^*_K, \alpha\right) \\ &= (g^{\mathbb{T}}_{KE})_\alpha - B\left(\gamma^*_K, \alpha\right). \end{aligned} \tag{3.3}$$

In the following, we determine the Kähler Einstein metrics of $(G, \mathbb{T}, J_{std})$ for the classical groups, their maximal tori and fixed complex structures whilst using the scaled killing form from section 1.8

Proposition 32 (Kähler Einstein metric of $SU(n+1)$)**.** *Up to scaling of the full metric the coefficients of the Kähler Einstein metric are given by*

$$g^{KE}_{\alpha_{ij}} = j - i \tag{3.4}$$

Proof. Notice that

$$\delta^* = \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \alpha_{ij} = \sum_{i=1}^{n+1} (n+2-2i)\varepsilon_i$$

Hence since $B(\varepsilon_i, \varepsilon_i) = \frac{1}{2}$ we have

$$B(\delta^*, \alpha_{kl}) = \frac{1}{2}\left((n+2-2l) - (n+2-2k)\right) = (l-k).$$

□

Proposition 33 (Kähler Einstein metric of $SO(2n+1)$)**.** *Up to scaling of the full metric the coefficients of the Kähler Einstein metric are given by*

$$g_\alpha^{KE} = \begin{cases} 2(k-l) & \alpha = \alpha_{kl} \\ 2(2n+1-(l+k)) & \alpha = \beta_{kl} \\ (2n+1-2k) & \alpha = \varepsilon_k \end{cases} \tag{3.5}$$

Proof. Notice that

$$\delta^* = \left(\sum_{i=1}^{n} \varepsilon_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\alpha_{ij} + \beta_{ij}) \right) = \sum_{i=1}^{n} (2n-2i+1)\varepsilon_i$$

Hence since $B(\varepsilon_i, \varepsilon_i) = 4$ we have

$$\begin{aligned} B(\delta^*, \alpha_{kl}) &= 4\left((2n-2l+1) - (2n-2k+1)\right) = 8(l-k) \\ B(\delta^*, \beta_{kl}) &= 4\left((2n-2l+1) + (2n-2k+1)\right) = 8(2n+1-(l+k)) \\ B(\delta^*, \varepsilon_k) &= 4(2n-2k+1) \end{aligned}$$

□

Proposition 34 (Kähler Einstein metric of $Sp(n)$)**.** *Up to scaling of the full metric the coefficients of the Kähler Einstein metric are given by*

$$g_\alpha^{KE} = \begin{cases} l-k & \alpha = \alpha_{kl} \\ 2n+2-(l+k) & \alpha = \beta_{kl} \\ 2n+2-2k & \alpha = \gamma_k \end{cases} \tag{3.6}$$

Proof. Notice that

$$\delta^* = \left(\sum_{i=1}^{n} 2\varepsilon_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\alpha_{ij} + \beta_{ij}) \right) = 2\sum_{i=1}^{n} (n-i+1)\varepsilon_i$$

Hence since $B(\varepsilon_i, \varepsilon_i) = 1$ we have

$$\begin{aligned} B(\delta^*, \alpha_{kl}) &= 2\left((n-l+1) - (n-k+1)\right) = 2(l-k) \\ B(\delta^*, \beta_{kl}) &= 2\left((n-l+1) + (n-k+1)\right) = 2(2n+2-(l+k)) \\ B(\delta^*, \gamma_k) &= B(\delta^*, 2\varepsilon_k) = 4(n-k+1) \end{aligned}$$

□

Proposition 35 (Kähler Einstein metric of $SO(2n)$)**.** *Up to scaling of the full metric the coefficients of the Kähler Einstein metric are given by*

$$g_\alpha^{KE} = \begin{cases} (k-l) & \alpha = \alpha_{kl} \\ (2n-(l+k)) & \alpha = \beta_{kl} \end{cases} \tag{3.7}$$

Proof. Notice that

$$\delta^* = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\alpha_{ij} + \beta_{ij}) = 2\sum_{i=1}^{n} (n-i)\varepsilon_i$$

Hence since $B(\varepsilon_i, \varepsilon_i) = 4$ we have

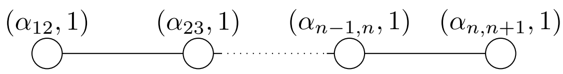
$$B(\delta^*, \alpha_{kl}) = 8\left((n-l) - (n-k)\right) = 8(l-k)$$
$$B(\delta^*, \beta_{kl}) = 8\left((n-l) + (n-k)\right) = 8(2n-(l+k))$$

□

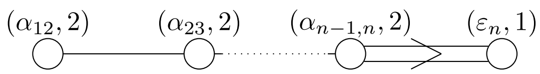
It is well known, e. g. from [Bou68] , that bases for the positive roots of the classical groups are given by

$$\Phi_{SU(n+1)} = \{\alpha_{i,i+1} \mid i = 1...n\}$$
$$\Phi_{SO(2n+1)} = \{\alpha_{i,i+1} \mid i = 1...n-1\} \cup \{\varepsilon_n\}$$
$$\Phi_{Sp(n)} = \{\alpha_{i,i+1} \mid i = 1...n-1\} \cup \{\gamma_n\}$$
$$\Phi_{SO(2n)} = \{\alpha_{i,i+1} \mid i = 1...n-1\} \cup \{\beta_{n-1,n}\}$$

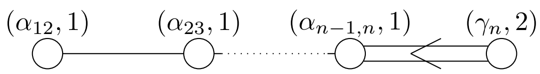
This gives rise to the following Dynkin diagrams, where we write the pair $(\alpha, (g_{KE})_\alpha)$ above each vertex, since the metric is determined by its values on the simple roots by equation (1.2).
$SU(n+1)$:



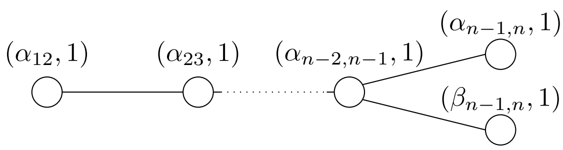
$SO(2n+1)$:



$Sp(n)$:



$SO(2n)$:



We observe that the coefficients of the simple roots of the Kähler Einstein metric do not depend on the rank.

3.3 Independence of complex structure

Before we apply the discussion of the holomorphic curvature tensor to an example, we want to point out the following observations concerning the holomorphic curvature of the Kähler Einstein metric with respect to different complex structures.

By proposition 4 we know that a complex structure on $G/\mathbb{T}$ corresponds to a choice of positive roots which in turn corresponds to a Weyl chamber and we know that the Weyl group acts transitively on those, hence it is a natural question how the Weyl group interacts with complex structures. In fact it is known, e.g. by [Arv92, p.21 f], that thanks to the Weyl group all complex structures on $G/\mathbb{T}$ are equivalent:

Lemma 36. *Let J, J' be two complex structures on $G/\mathbb{T}$. Then there is a biholomorphic equivariant diffeomorphism*

$$\varphi : (G/\mathbb{T}, J) \to (G/\mathbb{T}, J')$$

induced by an automorphism ψ of G,i.e.

$$\varphi(g.g'\mathbb{T}) = \psi(g).\varphi(g'\mathbb{T})$$

Proof. The two complex structures induce sets of positive roots in $\Delta_{\mathfrak{g}}$. By classical results on semisimple Lie algebras two of these orderings are conjugate via the Weyl group and hence there exists a element $g \in N_G(\mathbb{T})$ in the normalizer of the maximal torus such that $\psi(x) = gxg^{-1}$ induces an automorphism of $G/\mathbb{T}$ moving one complex structure into the other. □

This is not true any more if we consider the case G/K. However, in the light of proposition 9 and lemma 36, it makes sense to restrict to a fixed "standard" J_{std} on $G/\mathbb{T}$ and only consider the complex structures induced by this complex structure.

Lemma 37. *Let (G, K, J) a C space with an arbitrary complex structure J. Then there is a C space (G, K', J') and a equivariant biholomorphism*

$$\varphi' : (G/K, J) \to (G/K', J')$$

with the property that the submersion

$$\pi : (G/\mathbb{T}, J_{std}) \to (G/K', J')$$

is holomorphic.

Remark: In other words, every C space (G, K, J) is equivariantly biholomorphic to a C space with complex structure induced by J_{std}.

Proof. By proposition 9 there is a complex structure $\tilde{J}$ such that $(G/\mathbb{T}, \tilde{J}) \to (G/K, J)$ is a holomorphic submersion. By lemma 36, we get a automorphism $\psi : G \to G$ with $\psi(\mathbb{T}) = \mathbb{T}$ that induces an biholomorphic diffeomorphism $\varphi : (G/\mathbb{T}, \tilde{J}) \to (G/\mathbb{T}, J_{std})$. Now with $K' = \varphi(K)$ it is easy to see that J_{std} is K' invariant and therefore letJ' be the complex structure on G/K' induced by J_{std}. This implies already that the submersion is holomorphic. Furthermore, we have a biholomorphic diffeomorphism

$$\varphi' : (G/K, J) \to (G/K', J')$$

by construction. □

This uniqueness property of the Kähler Einstein metric can be used nicely to proof the following theorem.

Theorem 38. *Let (G, K, J, g_{KE}) be a Kähler Einstein C space. Then, up to scaling, the biholomorphism from lemma 37 is an isometry between the Kähler Einstein metrics.*

Proof. Let $\varphi' : (G/K, J) \to (G/K', J')$ be the equivariant biholomorphism from lemma 37. Now, it is only left to show that this is actually an isometry between the Kähler Einstein metrics. Let us denote $g' = \varphi'^{-1*} g_{KE}$. It certainly is an isometry between

$$\varphi' : (G/K, J, g_{KE}) \to (G/K', J', g').$$

So it is only left to show that g' is a homogeneous Kähler Einstein metric. Since the Kähler and Einstein properties are preserved under pullback it is sufficient to prove that g' is G invariant, because of the uniqueness (up to scalar) of homogeneous Kähler Einstein metrics on C spaces. Therefore, we consider for $x \in G$

$$\begin{aligned} x^* g' &= x^*(\varphi'^{-1*} g_{KE}) \\ &= (\varphi'^{-1} \circ x)^* g_{KE} \\ &= (\psi^{-1}(x) \circ \varphi'^{-1})^* g_{KE} \\ &= \varphi'^{-1*}(\psi^{-1}(x)^* g_{KE}) \\ &= \varphi'^{-1*} g_{KE} \\ &= g'. \end{aligned}$$

Here we used the equivariance of φ with respect to ψ. ☐

The proof allows us to deduce the following.

Corollary 39. *For any two invariant complex structures J, J' on $G/\mathbb{T}$ there exists a biholomorphic isometry*

$$(G, \mathbb{T}, J, g^J_{KE}) \to (G, \mathbb{T}, J', g^{J'}_{KE}).$$

Hence the holomorphic curvature of the corresponding Kähler Einstein metric is independent of the chosen complex structure.

Corollary 40. *If (G, K, J_{std}, g_{KE}) has positive holomorphic curvature for all K leaving J_{std} invariant, then the same is true for any complex structure.*

The previous two corollaries allow us to restrict all our efforts to a particular fixed complex structure J_{std}. We will do so for the most part of the rest of the thesis. However, we will recall from time to time in relevant situations the independence of the complex structure.

Chapter 4

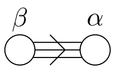
Example of $(G_2, \mathbb{T}^2)$

Since there is only one Kähler Einstein C space with simple group G_2 that is not covered by Itoh and theorem 22, i.e. the case of $K = \mathbb{T}^2$, we will use this case as an detailed example of how the above description of the holomorphic curvature tensor can be used to proof positive holomorphic sectional curvature. First of all we summarize the well known algebraic setting of G_2, which is necessary to determine the curvatures, i.e. z_α and $N_{\alpha,\beta}$ from proposition 11. The following can be found in [FH91, p.346f].

Proposition 41. *The root system of G_2 and its positive roots determining J_{std} are generated by the two simple roots α, β in the following way*

$$\begin{aligned}\Delta_{\mathfrak{g}}^+ =&\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}\\ =&\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}\end{aligned}$$

with the Dynkin diagram



.

We know that the cartan matrix defining the killing form on $\mathfrak{h}$ up to scalar is of the form

$$2\begin{pmatrix} \frac{(\alpha,\alpha)}{(\alpha,\alpha)} & \frac{(\alpha,\beta)}{(\alpha,\alpha)} \\ \frac{(\alpha,\beta)}{(\beta,\beta)} & \frac{(\beta,\beta)}{(\beta,\beta)} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

Furthermore, there is a basis H_1, H_2, E_α with $\alpha \in \Delta$ of $\mathfrak{g}_2^{\mathbb{C}}$ such that H_1, H_2 span $\mathfrak{h}$ and $E_\alpha \in \mathfrak{g}_\alpha$ and the Lie bracket is given by the following table.

$[X,Y]$	H_1	H_2	E_{α_1}	$E_{-\alpha_1}$	E_{α_2}	$E_{-\alpha_2}$	E_{α_3}	$E_{-\alpha_3}$	E_{α_4}	$E_{-\alpha_4}$	E_{α_5}	$E_{-\alpha_5}$	E_{α_6}	$E_{-\alpha_6}$
H_1	0	0	$2E_{\alpha_1}$	$-2E_{-\alpha_1}$	$-3E_{\alpha_2}$	$3E_{-\alpha_2}$	$-E_{\alpha_3}$	$E_{-\alpha_3}$	E_{α_4}	$-E_{-\alpha_4}$	$3E_{\alpha_5}$	$-3E_{-\alpha_5}$	0	0
H_2		0	$-E_{\alpha_1}$	$E_{-\alpha_1}$	$2E_{\alpha_2}$	$-2E_{-\alpha_2}$	E_{α_3}	$-E_{-\alpha_3}$	0	0	$-E_{\alpha_5}$	$E_{-\alpha_5}$	E_{α_6}	$-E_{-\alpha_6}$
E_{α_1}			0	H_1	E_{α_3}	0	$2E_{\alpha_4}$	$-3E_{-\alpha_2}$	$-3E_{\alpha_5}$	$-2E_{-\alpha_3}$	0	$E_{-\alpha_4}$	0	0
$E_{-\alpha_1}$				0	0	$-E_{-\alpha_3}$	$3E_{\alpha_2}$	$-2E_{-\alpha_4}$	$2E_{\alpha_3}$	$3E_{\alpha_5}$	$-E_{\alpha_4}$	0	0	0
E_{α_2}					0	H_2	0	$E_{-\alpha_1}$	0	0	$-E_{\alpha_6}$	0	0	$E_{-\alpha_5}$
$E_{-\alpha_2}$						0	$-E_{\alpha_1}$	0	0	0	0	$-E_{\alpha_6}$	$-E_{\alpha_5}$	0
E_{α_3}							0	H_1+3H_2	$-3E_{\alpha_6}$	$2E_{-\alpha_1}$	0	0	0	$E_{-\alpha_4}$
$E_{-\alpha_3}$								0	$-2E_{\alpha_1}$	$3E_{-\alpha_6}$	0	0	$-E_{\alpha_4}$	0
E_{α_4}									0	$2H_1+3H_2$	0	$-E_{-\alpha_1}$	0	$E_{-\alpha_3}$
$E_{-\alpha_4}$										0	E_{α_1}	0	$-E_{\alpha_3}$	0
E_{α_5}											0	H_1+H_2	0	$-E_{-\alpha_2}$
$E_{-\alpha_5}$												0	E_{α_2}	0
E_{α_6}													0	H_1+2H_2
$E_{-\alpha_6}$														0

We will now extract all necessary information to calculate the holomorphic curvature tensor, i.e. the killing form, g_γ and the z_γ.

Lemma 42. *Up to scaling, the Kähler Einstein metric and the z_γ are given by*

γ	α_1	α_2	α_3	α_4	α_5	α_6
g_γ	1	3	4	5	6	9
z_γ	1	$\frac{1}{3}$	1	1	$\frac{1}{3}$	$\frac{1}{3}$

and the angles between the roots are given by

$$(\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -3 & -1 & 1 & 3 & 0 \\ & 6 & 3 & 0 & -3 & 3 \\ & & 2 & 1 & 0 & 3 \\ & & & 2 & 3 & 3 \\ & & & & 6 & 3 \\ & & & & & 6 \end{pmatrix}$$

Proof. By the cartan matrix, we know $\frac{1}{3}(\beta, \beta) = (\alpha, \alpha) = -\frac{2}{3}(\alpha, \beta)$. If we normalize the killing form such that $(\alpha, \alpha) = 2$ we get

$$B = \begin{pmatrix} (\alpha, \alpha) & (\alpha, \beta) \\ (\beta, \alpha) & (\beta, \beta) \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}.$$

Using the expressions of the α_i in terms of $\alpha = \alpha_1$ and $\beta = \alpha_2$ we obtain the matrix representing the angles between the different roots. Since for the Kähler Einstein metric we have

$$g_\gamma = (\delta^*, \gamma)$$

with $\delta^* = \sum_{i=1}^{6} \alpha_i = 10\alpha + 6\beta$ the g_γ are easily obtained.

To determine the z_γ, we consider the following. Let $H_i = [E_{\alpha_i}, E_{-\alpha_i}]$. Then we see $\alpha_i(H_i) = 2$ for all i and $H_i = cH_{\alpha_i}$. This implies in fact

$$c(\alpha_i, \alpha_i) = B(H_\alpha, H_i) = \alpha(H_i) = 2.$$

Now we have $z_{\alpha_i} = \frac{1}{c} = \frac{2}{(\alpha_i, \alpha_i)}$ which is determined by the previous calculations of the killing form. □

Now we have all the data we need to move towards calculating the curvature tensor considering, that the $N_{\gamma,\delta}$ can be read from the bracket table.

To determine the holomorphic curvature tensor, the first step is to determine the set $\Delta_H = \{\gamma - \delta \mid \gamma < \delta \in \Delta_{\mathfrak{g}}^+\}$. To that end, we use the ordering induced by proposition 15 with $\varepsilon_1 = \beta$ and $\varepsilon_2 = \alpha$, i.e

$$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_6.$$

Proposition 43. *For the C space $(G_2, \mathbb{T}^2)$ let η be a non zero weight of $Fix(J_{std})$, i.e an element in Δ_H. Then η is in the following table together with the pairs $\gamma < \delta$ with $\gamma - \delta = \eta$:*

$\sharp$	η	$I(\eta)$	n_η
1	$-\beta+\alpha$	(α_1, α_2)	1
2	$-\alpha$	$(\alpha_2, \alpha_3), (\alpha_3, \alpha_4), (\alpha_4, \alpha_5)$	3
3	-2α	$(\alpha_2, \alpha_4), (\alpha_3, \alpha_5)$	2
4	-3α	(α_2, α_5)	1
5	$-\beta$	$(\alpha_1, \alpha_3), (\alpha_5, \alpha_6)$	2
6	$-\beta-\alpha$	$(\alpha_1, \alpha_4), (\alpha_4, \alpha_6)$	2
7	$-\beta-2\alpha$	$(\alpha_1, \alpha_5), (\alpha_3, \alpha_6)$	2
8	$-\beta-3\alpha$	(α_2, α_6)	1
9	$-2\beta-2\alpha$	(α_1, α_6)	1

Proof. The elements in the list are obviously weights and these are all of them for dimensional reasons. In fact, we know $dim_{\mathbb{C}}(Fix(J)^{\mathbb{C}}) = |\Delta_{\mathfrak{g}}^+|^2$ from proposition 23 and on the other hand

$$\begin{aligned} dim_{\mathbb{C}}(Fix(J)^{\mathbb{C}}) =& dim_{\mathbb{C}}(\mathfrak{m}_0) + \sum_{\eta \in \Delta_H} dim_{\mathbb{C}}(\mathfrak{m}_\eta) \\ =& |\Delta_{\mathfrak{g}}^+| + \sum_{\eta \in \Delta_H} \sum_{(\alpha,\beta) \in I(\eta)} dim_{\mathbb{C}}(\mathfrak{m}_{\alpha\beta}) \\ =& |\Delta_{\mathfrak{g}}^+| + 2 \sum_{\eta \in \Delta_H} n_\eta. \end{aligned}$$

Hence we only have to verify that

$$\frac{|\Delta_{\mathfrak{g}}^+| \left(|\Delta_{\mathfrak{g}}^+| - 1\right)}{2} = \sum_{\eta \in \Delta_H} n_\eta \tag{4.1}$$

holds. Note that this equation holds for all semisimple groups. Since $|\Delta_{\mathfrak{g}}^+| = 6$ and the right hand side is given by the sum of the entries in of the third column we see easily that both left and right hand side equal 15 and hence we found all weights. □

Having determined Δ_H and for each η also the pairs whose difference equals η, the next step is to determine the matrices $M(H_\eta)$ representing the curvature tensor on $\mathfrak{m}_\eta$. We recall from proposition 25 that each entry of $M(H_\eta)$ is indexed by two pairs of positive roots with difference η, i.e. two pairs from the second column of the table above and we have

$$M(H_\eta)_{(x,y)(v,w)} = -R_{xywv}.$$

To obtain an actual matrix, we need to order these pairs. We choose to order them as given in the second column of the table read from left to right, but any order does the trick. The proofs are straightforward calculations plugging in the values of $N_{\alpha_i \alpha_j}, z_{\alpha_i}$ and (α_i, α_j) into the formulas from proposition 16. For notational simplicity, we write all one dimensional matrices together in one larger diagonal matrix

Proposition 44. *The holomorphic curvature tensor in these cases is represented by the matrix indexed via*

i) the pairs $(\alpha_1, \alpha_2), (\alpha_2, \alpha_5), (\alpha_2, \alpha_6), (\alpha_1, \alpha_6)$ *in the cases* $\sharp 1, \sharp 4, \sharp 8, \sharp 9$*:*

$$\begin{pmatrix} -\frac{3}{4} & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

ii) the pairs $(\alpha_2, \alpha_3), (\alpha_3, \alpha_4), (\alpha_4, \alpha_5)$ *in case* $\sharp 2$*:*

$$\begin{pmatrix} 3 & -6 & -\frac{5}{3} \\ -6 & \frac{28}{3} & -8 \\ -\frac{5}{3} & -8 & 5 \end{pmatrix}$$

iii) the pairs $(\alpha_2, \alpha_4), (\alpha_3, \alpha_5)$ *in case* $\sharp 3$*:*

$$\begin{pmatrix} 0 & -\frac{5}{3} \\ -\frac{5}{3} & 0 \end{pmatrix}$$

iv) the pairs $(\alpha_1, \alpha_3), (\alpha_5, \alpha_6)$ *in case* $\sharp 5$*:*

$$\begin{pmatrix} -\frac{1}{5} & 1 \\ 1 & 2 \end{pmatrix}$$

v) the pairs $(\alpha_1, \alpha_4), (\alpha_4, \alpha_6)$ *in case* $\sharp 6$*:*

$$\begin{pmatrix} \frac{3}{2} & 2 \\ 2 & 5 \end{pmatrix}$$

vi) the pairs $(\alpha_1, \alpha_5), (\alpha_3, \alpha_6)$ *in case* $\sharp 7$*:*

$$\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

Proof. By the explanations above this reduces to calculating $R_\alpha \beta\gamma\delta$ and hence corresponds to evaluating the equations from proposition 16. We will do so exemplary for one case and will omit it throughout the rest of the thesis. We consider the entry $M(H_{-\alpha})_{(\alpha_2,\alpha_3)(\alpha_4,\alpha_5)}$. Since $\alpha_2 - \alpha_3 = -\alpha_1 \neq 0$ and $\alpha_5 - \alpha_3 = 2\alpha_1 \notin \Delta^+_{\mathfrak{g}}$ we are in the case IIb of proposition 16. Note that $\alpha_2 + \alpha_5 = \alpha_6$ and hence we have

$$\begin{aligned} M(H_{-\alpha_1})_{(\alpha_2,\alpha_3)(\alpha_4,\alpha_5)} =& - R_{\alpha_2\alpha_3\alpha_5\alpha_4} \\ =& g_{\alpha_4} z_{\alpha_1} N_{\alpha_2,-\alpha_3} N_{\alpha_5,-\alpha_4} + z_{\alpha_6} \frac{g_{\alpha_5} g_{\alpha_4}}{g_{\alpha_6}} N_{\alpha_2,\alpha_5} N_{\alpha_3,\alpha_4} \\ =& 5 \cdot 1 \cdot 1 \cdot (-1) + \frac{1}{3} \cdot \frac{6 \cdot 5}{9} \cdot (-1) \cdot (-3) \\ =& -\frac{5}{3} \end{aligned}$$

□

We recall from proposition 26 that for the matrix of the trivial module, we have

$$M(H_0)_{ij} = -R_{\alpha_i\alpha_i\alpha_j\alpha_j}$$

and this results in

Proposition 45. *The holomorphic curvature tensor on the trivial module is represented by*

$$M(H_0) = \begin{pmatrix} 2 & -\frac{3}{4} & -\frac{1}{5} & \frac{3}{2} & 1 & 0 \\ -\frac{3}{4} & 2 & 3 & 0 & -\frac{2}{3} & 1 \\ -\frac{1}{5} & 3 & 8 & \frac{28}{3} & 0 & 4 \\ \frac{3}{2} & 0 & \frac{28}{3} & 10 & 5 & 5 \\ 1 & -\frac{2}{3} & 0 & 5 & 4 & 2 \\ 0 & 1 & 4 & 5 & 2 & 6 \end{pmatrix}$$

It is easy to see that these matrices are not positive definite and hence we do not have a positive holomorphic curvature tensor. However, using four forms we may be able to produce a positive modified holomorphic curvature tensor, which is still sufficient for positive holomorphic curvature. We present our general strategy to improve the tensor but want to remark that there are different ways to reach a positive modified tensor and it is not clear to us if there is a preferred one and which one that would be.

Modifications

In the light of proposition 31, we start with the matrices corresponding to $\eta \neq 0$ and want to turn them positive semidefinite. We start with the matrix of the cases $\sharp 1, \sharp 4, \sharp 8, \sharp 9$. Here there is no choice but to add something to the negative values on the diagonal. In detail that means we want to add the four form ω to our tensor such that the matrix N representing ω has the property

$$M(\omega_{-\beta+\alpha})_{(\alpha_1,\alpha_2)(\alpha_1,\alpha_2)} = \frac{3}{4} \qquad M(\omega_{-3\alpha})_{(\alpha_2,\alpha_5)(\alpha_2,\alpha_5)} = \frac{2}{3}$$

by proposition 30 these changes can be realized by a four form if we also have

$$\begin{aligned} M(\omega_0)_{(\alpha_1,\alpha_1)(\alpha_2,\alpha_2)} &= -\frac{3}{4} = M(\omega_0)_{(\alpha_2,\alpha_2)(\alpha_1,\alpha_1)} \\ M(\omega_0)_{(\alpha_2,\alpha_2)(\alpha_5,\alpha_5)} &= -\frac{2}{3} = M(\omega_0)_{(\alpha_5,\alpha_5)(\alpha_2,\alpha_2)} \end{aligned}$$

Hence we can erase the negative diagonals if we subtract $\frac{3}{4}$ from $M(H_0)_{12}$ and $M(H_0)_{21}$ and $\frac{2}{3}$ from $M(H_0)_{25}$ and $M(H_0)_{52}$. We write this compactly as follows

Value	Intended	Forced
$\frac{3}{4}$	$M(H_{-\beta+\alpha})_{(1,1)}$	$M(H_0)_{(1,2)}$
$\frac{2}{3}$	$M(H_{-3\alpha})_{(1,1)}$	$M(H_0)_{(2,5)}$

Note, that we only mention one off diagonal entry in the second column even though the changes must of course be applied symmetrically. Now we have that the matrix for $\sharp 1, \sharp 4, \sharp 8, \sharp 9$ is positive semidefinite, but we have to keep track of the changes to M_0.
Here we did not have much of a choice in other cases there are different ways to achieve positive definiteness. To see this consider the matrix of case $\sharp 3$, here we have the choice to either add something positive (e.g. $\frac{5}{3}$) to the diagonals such that the matrix is positive semidefinite. As above that would have impact to the matrix of the trivial module. The other

option is adding $\frac{5}{3}$ symmetrical to the off diagonal entries resulting in the positive semidefinite 0 matrix. The four form to do that needs to have the property that its representing matrix N satisfies symmetrically

$$M(\omega_{-2\alpha})_{(\alpha_2,\alpha_4)(\alpha_3,\alpha_5)} = \frac{5}{3}$$

in order to realize that by a four form we need by proposition 29 (or more explicitly by equation (3.2)) that

$$M(\omega_{-\alpha})_{(\alpha_2,\alpha_3)(\alpha_4,\alpha_5)} = -\frac{5}{3}$$

holds also symmetrically. Again we denote this change compactly by

Value	Intended	Forced
$\frac{5}{3}$	$M(H_{-2\alpha})_{(1,2)}$	$M(H_{-\alpha})_{(1,3)}$

Following this fashion, we do all the following changes, notice the first three are just the ones already discussed:

Value	Intended	Forced
$\frac{3}{4}$	$M(H_{-\beta+\alpha})_{(1,1)}$	$M(H_0)_{(1,2)}$
$\frac{2}{3}$	$M(H_{-3\alpha})_{(1,1)}$	$M(H_0)_{(2,5)}$
$\frac{5}{3}$	$M(H_{-2\alpha})_{(1,2)}$	$M(H_{-\alpha})_{(1,3)}$
3	$M(H_{-\alpha})_{(1,1)}$	$M(H_0)_{(2,3)}$
10	$M(H_{-\alpha})_{(2,2)}$	$M(H_0)_{(3,4)}$
7	$M(H_{-\alpha})_{(3,3)}$	$M(H_0)_{(4,5)}$
$\frac{1}{5}$	$M(H_{-\beta})_{(1,1)}$	$M(H_0)_{(2,4)}$
-1	$M(H_{-\beta})_{(1,2)}$	$M(H_{-\beta-2\alpha})_{(1,3)}$

Realizing all these changes, we obtain the following modified matrices:

$$Diag(0,0,1,0) \qquad M(H_{-\alpha}) = \begin{pmatrix} 6 & -6 & -\frac{10}{3} \\ -6 & \frac{58}{3} & -8 \\ -\frac{10}{3} & -8 & 12 \end{pmatrix}$$

$$M(H_{-2\alpha}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad M(H_{-\beta}) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$M(H_{-\beta-\alpha}) = \begin{pmatrix} \frac{3}{2} & 2 \\ 2 & 5 \end{pmatrix} \qquad M(H_{-\beta-2\alpha}) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

which are all positive semidefinite. The modified matrix representing the tensor on the trivial module is given by

$$M(H_0) = \begin{pmatrix} 2 & -\frac{3}{2} & -\frac{1}{5} & \frac{3}{2} & 1 & 0 \\ -\frac{3}{2} & 2 & 0 & -\frac{1}{5} & -\frac{4}{3} & 1 \\ -\frac{1}{5} & 0 & 8 & -\frac{2}{3} & 0 & 4 \\ \frac{3}{2} & -\frac{1}{5} & -\frac{2}{3} & 10 & -2 & 5 \\ 1 & -\frac{4}{3} & 0 & -2 & 4 & 2 \\ 0 & 1 & 4 & 5 & 2 & 6 \end{pmatrix}$$

which is sadly not positive definite. In fact for example for $v = (1, -1, -1, -1 - 1, 1)$, we have $vM(H_0)v^t = -2$. Hence further changes to $M(H_0)$ are necessary. At this stage we want to point out that by the arguments as in the proof of proposition 31 and the fact that we are not interested in the actual matrices but that they are positive semidefinite, we may now add symmetrical arbitrary negative values to $M(H_0)$ since the forced changes to the other matrices correspond to increasing the values on the diagonals which maintains semipositivity. This allows us to "erase" rows and columns that have exclusively nonnegative entries without keeping track of the explicit changes on the $M(H_\eta)$. In this case, we see that the sixth row and column of $M(H_0)$ have no negative entries and hence can be erased using four forms. Resulting in the positive definite matrix

$$M(H_0) = \begin{pmatrix} 2 & -\frac{3}{2} & -\frac{1}{5} & \frac{3}{2} & 1 & 0 \\ -\frac{3}{2} & 2 & 0 & -\frac{1}{5} & -\frac{4}{3} & 0 \\ -\frac{1}{5} & 0 & 8 & -\frac{2}{3} & 0 & 0 \\ \frac{3}{2} & -\frac{1}{5} & -\frac{2}{3} & 10 & -2 & 0 \\ 1 & -\frac{4}{3} & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

By proposition 31 we proved that there is a family of four forms that modify the holomorphic curvature tensor into a positive tensor resulting in

Corollary 46. *The Kähler Einstein C space $(G_2, \mathbb{T}^2, J_{std}, g_{KE})$ has positive holomorphic curvature.*

Together with Itoh, theorem 22 and corollary 40, we obtain

Theorem 47. *Every Kähler Einstein C space (G_2, K, J, g_{KE}) has positive holomorphic sectional curvature.*

Chapter 5

From $\mathbb{T}$ to K

In the example of $(G_2, \mathbb{T}^2)$, we have seen how to determine and represent the holomorphic curvature tensor in a simple fashion and how to modify it using four forms. In later sections we are also interested in larger isotropy groups than the maximal torus, which makes it desirable to have a relation between the curvature tensors. Sadly, the description of Kähler metrics in proposition 6 makes it obvious that there is no equivariant Riemannian submersion between Kähler metrics

$$(G/\mathbb{T}, J_{\mathbb{T}}, g_{\mathbb{T}}) \to (G/K, J_K, g_K)$$

because $g_{\mathbb{T}}$ is not K invariant. Therefore an application of O'Neill's curvature formula for submersions is not possible (cf. [O'N66]). Unfortunately, we were not able to find a relation between the holomorphic sectional curvature functions but there is still a connection between the holomorphic curvature tensors simplifying the calculations in later sections.

By lemmata 36 and 37, we can assume to have the complex structure J_{std} of our choice on $G/\mathbb{T}$ such that the submersion to G/K is holomorphic. Then J_{std} and its image J^K_{std} on G/K are determined by a decomposition of a base of the positive roots of G by proposition 7

$$\Phi = \Phi_{\mathfrak{k}} \cup \Phi_{\mathfrak{m}}.$$

For notational reasons, assume that $\Phi = \{\alpha_1, .., \alpha_r\}$ and $\Phi_{\mathfrak{k}} = \{\alpha_1, .., \alpha_s\}$ with $s < r$. By equation (1.2) any Kähler metric on $(G/\mathbb{T}, J_{std})$ is determined by its values $c_{\alpha_i} > 0$ on the root spaces corresponding to simple roots, hence is given as an inner product

$$g_{\mathbb{T}}(c_{\alpha_1}, .., c_{\alpha_r}) : \mathfrak{n} \times \mathfrak{n} \to \mathbb{R}$$

and similarly any Kähler metric on $(G/K, J^K_{std})$ is given by

$$g_K(c_{\alpha_{s+1}}, .., c_{\alpha_r}) : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}.$$

Here we used the B orthogonal decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \oplus \mathfrak{m}$$

with $\mathfrak{n} = \mathfrak{p} \oplus \mathfrak{m}$ and $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{p}$.

Moving $(c_{\alpha_1}, ..., c_{\alpha_s})$ to zero corresponds to shrinking the metric on $\mathfrak{p}$ and therefore collapsing K whilst maintaining the Kähler property and ending up with the desired Kähler metric on G/K. So the natural question is if the holomorphic curvature tensors are related as well. In fact, they are. Before we present this in detail, we want to give a short comment on the fact that we collapse K, because apart from O'Neils curvature formula for Riemannian submersions there is another famous tool to improve curvature which is called Cheeger deformation. For details we suggest the exposition in [Zil07] and [Müt87] .

In words the underlying idea of this technique is that scaling the metric down in the direction of the orbits of a Lie group action of isometries does not decrease and often increases curvature.

However, our comment is that we can not apply Cheeger deformations in our case for two reasons: Firstly, K does not act via isometries in the Kähler metric of $(G, \mathbb{T})$ and secondly, even though we are scaling the metric on $\mathfrak{p}$ to zero, the fact that we maintain the Kähler property implies that we change the metric also on $\mathfrak{m}$.

Now we want to formalize the actual relationship. To that end we fix a Kähler metric on G/K as $g^K = g_K(c_{\alpha_{s+1}}, .., c_{\alpha_r})$ and define the Kähler metric $g^t = g_{\mathbb{T}}(t, .., t, c_{\alpha_{s+1}}, .., c_{\alpha_r})$ on $G/\mathbb{T}$ for $t > 0$. This induces a holomorphic curvature tensor

$$H^t : Fix(J_{std}) \times Fix(J_{std}) \to \mathbb{R}$$

whose restriction H_K^t to $Fix(J_{std}^K) = Fix(J_{std}) \cap \Lambda^2(\mathfrak{m})$ satisfies

Proposition 48. *As elements in $Sym_2(Fix(J_{std}^K))$, we have that*

$$\lim_{t\to 0} H_K^t = H_K$$

where H_K is the curvature tensor of g^K.

Before we proof this we need a short fairly obvious

Lemma 49. *Let $\alpha \in \Delta_{\mathfrak{m}}^+$ and $\beta \in \Delta_{\mathfrak{k}}^+$ then we have*

$$\lim_{t\to 0} g_\alpha^t = g_\alpha^K \text{ and } \lim_{t\to 0} g_\beta^t = 0$$

Proof. By (1.2), we can write

$$\begin{aligned} g_\alpha^t &= \sum_{\gamma\in\Phi} \rho(\alpha)_\gamma g_\gamma^t = \sum_{\gamma\in\Phi_{\mathfrak{k}}} \rho(\alpha)_\gamma g_\gamma^t + \sum_{\gamma\in\Phi_{\mathfrak{m}}} \rho(\alpha)_\gamma g_\gamma^t \\ &= \sum_{\gamma\in\Phi_{\mathfrak{k}}} \rho(\alpha)_\gamma t + \sum_{\gamma\in\Phi_{\mathfrak{m}}} \rho(\alpha)_\gamma c_\gamma \\ &= t\left(\sum_{\gamma\in\Phi_{\mathfrak{k}}} \rho(\alpha)_\gamma\right) + \sum_{\gamma\in\Phi_{\mathfrak{m}}} \rho(\alpha)_\gamma c_\gamma \end{aligned}$$

Now it is obvious that for $\alpha \in \Delta_{\mathfrak{k}}$ the second term is zero and hence the claim follows. For $\alpha \in \Delta_{\mathfrak{m}}$ again by (1.2) the second term equals g_α^K, which concludes the proof. □

Proof of proposition 48. The convergence is equivalent to the convergence of the entries of the matrix representing the tensors in a fixed basis. In our case that means the matrices $M(H_\eta)$. The entries of these matrices are $-R_{\alpha\beta\gamma\delta}^t$ where generally $\alpha, \beta, \gamma, \delta \in \Delta_{\mathfrak{g}}^+$. Since we consider the restriction to $\Lambda^2(\mathfrak{m})$, we are only interested in the case where $\alpha, \beta, \gamma, \delta \in \Delta_{\mathfrak{m}}^+$. Hence, the proof boils down to

$$\lim_{t\to 0} R_{\alpha\beta\gamma\delta}^t = R_K(E_\alpha, E_{-\beta}, E_\gamma, E_{-\delta})$$

for those roots. As before, we know that that these terms are zero in both cases unless $\alpha + \gamma = \beta + \delta$ which we assume to be true from now on. Now note that from (1.5), we have

$$\begin{aligned} R_{\alpha\beta\gamma\delta}^t = &- \frac{g_{\gamma-\beta}^t}{g_{\alpha+\gamma-\beta}^t} g^t\left([E_\alpha, [E_{-\beta}, E_\gamma]_{\mathfrak{n}^+}]_{\mathfrak{n}^+}, E_{-\delta}\right) \\ &+ \frac{g_\gamma^t}{g_{\alpha+\gamma}^K} g^t\left([E_{-\beta}, [E_\alpha, E_\gamma]_{\mathfrak{n}^+}]_{\mathfrak{n}^+}, E_{-\delta}\right) \\ &+ \frac{g_\gamma^t}{g_{\gamma+\alpha-\beta}^t} g^t\left([[E_\alpha, E_{-\beta}]_{\mathfrak{n}^+}, E_\gamma]_{\mathfrak{n}^+}, E_{-\delta}\right) \\ &+ g^t\left([[E_\alpha, E_{-\beta}]_{\mathfrak{n}^-}, E_\gamma]_{\mathfrak{n}^+}, E_{-\delta}\right) \\ &+ g^t\left([[E_\alpha, E_{-\beta}]_{\mathfrak{t}^{\mathbb{C}}}, E_\gamma]_{\mathfrak{n}^+}, E_{-\delta}\right) \end{aligned}$$

Since the projections satisfy $(-)_{\mathfrak{n}^+} = (-)_{\mathfrak{p}^+} + (-)_{\mathfrak{m}^+}$ and $g^t(-, E_{-\delta}) = g^t_\delta B(-, E_{-\delta})$ is zero on $\mathfrak{p}$ because $\delta \in \Delta^+_{\mathfrak{m}}$, we obtain

$$\begin{aligned}
R^t_{\alpha\beta\gamma\delta} = &-\frac{g^t_{\gamma-\beta} g^t_\delta}{g^t_{\alpha+\gamma-\beta}} B\left([E_\alpha, [E_{-\beta}, E_\gamma]_{\mathfrak{n}^+}]_{\mathfrak{m}^+}, E_{-\delta}\right) && (5.1)\\
&+ \frac{g^t_\gamma g^t_\delta}{g^t_{\alpha+\gamma}} B\left([E_{-\beta}, [E_\alpha, E_\gamma]_{\mathfrak{n}^+}]_{\mathfrak{m}^+}, E_{-\delta}\right) && (5.2)\\
&+ \frac{g^t_\gamma g^t_\delta}{g^t_{\gamma+\alpha-\beta}} B\left([[E_\alpha, E_{-\beta}]_{\mathfrak{n}^+}, E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right) && (5.3)\\
&+ g^t_\delta B\left([[E_\alpha, E_{-\beta}]_{\mathfrak{n}^-}, E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right) && (5.4)\\
&+ g^t_\delta B\left([[E_\alpha, E_{-\beta}]_{\mathfrak{t}^{\mathbb{C}}}, E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right) && (5.5)
\end{aligned}$$

Now, we consider these terms separately

$$\begin{aligned}
(5.1) &= -g^t_{\gamma-\beta}\left(B\left([E_\alpha, [E_{-\beta}, E_\gamma]_{\mathfrak{m}^+}]_{\mathfrak{m}^+}, E_{-\delta}\right) + B\left(\left[E_\alpha, [E_{-\beta}, E_\gamma]_{\mathfrak{p}^+}\right]_{\mathfrak{m}^+}, E_{-\delta}\right)\right)\\
(5.2) &= \frac{g^t_\gamma g^t_\delta}{g^t_{\alpha+\gamma}}\left(B\left([E_{-\beta}, [E_\alpha, E_\gamma]_{\mathfrak{m}^+}]_{\mathfrak{m}^+}, E_{-\delta}\right) + B\left(\left[E_{-\beta}, [E_\alpha, E_\gamma]_{\mathfrak{p}^+}\right]_{\mathfrak{m}^+}, E_{-\delta}\right)\right)\\
(5.3) &= g^t_\gamma\left(B\left([[E_\alpha, E_{-\beta}]_{\mathfrak{m}^+}, E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right) + B\left(\left[[E_\alpha, E_{-\beta}]_{\mathfrak{p}^+}, E_\gamma\right]_{\mathfrak{m}^+}, E_{-\delta}\right)\right)\\
(5.4) &= g^t_\delta\left(B\left([[E_\alpha, E_{-\beta}]_{\mathfrak{m}^-}, E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right) + B\left(\left[[E_\alpha, E_{-\beta}]_{\mathfrak{p}^-}, E_\gamma\right]_{\mathfrak{m}^+}, E_{-\delta}\right)\right)\\
(5.5) &= g^t_\delta B\left([[E_\alpha, E_{-\beta}]_{\mathfrak{t}^{\mathbb{C}}}, E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right)
\end{aligned}$$

and observe:

- The second term of (5.1) is only non zero if $\gamma - \beta \in \Delta^+_{\mathfrak{k}}$, but in that case we have that

  $$\lim_{t\to 0} g^t_{\gamma-\beta} = 0$$

  and therefore the second term vanishes for $t \to 0$.

- Since $\alpha, \gamma \in \Delta^+_{\mathfrak{m}}$ we have that either $\alpha + \gamma$ is no root or $\alpha + \gamma \in \Delta^+_{\mathfrak{m}}$ as well. Therefore the second term of (5.2) is constant 0.

- The second terms of (5.3) and (5.4) together with (5.5) yield

  $$(g^t_\gamma - g^t_\delta) B\left(\left[[E_\alpha, E_{-\beta}]_{\mathfrak{p}^+}, E_\gamma\right]_{\mathfrak{m}^+}, E_{-\delta}\right) + g^t_\delta B\left([[E_\alpha, E_{-\beta}]_{\mathfrak{t}^{\mathbb{C}}}, E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right)$$

  where we notice that the first term is either zero or $\alpha - \beta = \delta - \gamma \in \Delta^+_{\mathfrak{k}}$. In the later case we have from the Kähler property that

  $$(g^t_\gamma - g^t_\delta) = -g^t_{\delta-\gamma}$$

  which converges to 0 for $t \to 0$.

- Last but not least, we see that the remaining terms, i.e. the first terms of (5.1), (5.2), (5.3), (5.4) are either 0 because the bracket vanishes or the corresponding root x, i.e $\gamma - \beta, \alpha + \gamma, \gamma, \delta$, are in $\Delta^+_{\mathfrak{m}}$ and for these we know

  $$\lim_{t\to 0} g^t_x = g^K_x$$

Using all these observations, we see

$$\begin{aligned}\lim_{t\to 0} R^t_{\alpha\beta\gamma\delta} = &-\frac{g^K_{\gamma-\beta} g^K_\delta}{g^K_{\alpha+\gamma-\beta}} B\left([E_\alpha,[E_{-\beta},E_\gamma]_{\mathfrak{m}^+}]_{\mathfrak{m}^+}, E_{-\delta}\right)\\ &+\frac{g^K_\gamma g^K_\delta}{g^K_{\alpha+\gamma}} B\left([E_{-\beta},[E_\alpha,E_\gamma]_{\mathfrak{m}^+}]_{\mathfrak{m}^+}, E_{-\delta}\right)\\ &+\frac{g^K_\gamma g^K_\delta}{g^K_{\gamma+\alpha-\beta}} B\left([[E_\alpha,E_{-\beta}]_{\mathfrak{m}^+},E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right)\\ &+g^K_\delta B\left([[E_\alpha,E_{-\beta}]_{\mathfrak{m}^-},E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right)\\ &+g^K_\delta B\left([[E_\alpha,E_{-\beta}]_{\mathfrak{k}^{\mathbb{C}}},E_\gamma]_{\mathfrak{m}^+}, E_{-\delta}\right)\\ &= R^K_{\alpha\beta\gamma\delta}\end{aligned}$$

□

Remark:
We want to remark that the way how we approach 0 on the coefficients g^t of the simple roots in $\mathfrak{k}$ is not relevant. With slight changes the proof would also work for

$$g^{(t_1,..,t_s)} = g_{\mathbb{T}}(t_1,..,t_s,c_{s+1},..,c_r)$$

with $\|(t_1,..,t_s)\| \to 0$ instead of g^t, only the notation would get worse.

The proof of the above allows us to derive the following technique to obtain the curvature matrices of G/K from those of $G/\mathbb{T}$.

Corollary 50. *Let $\eta \in \Delta_H$ be a weight of (G, K, J^K_{std}, g^K) and let $M(H_\eta)^K$ be the curvature matrix corresponding to η. Then for all t, we denote by $M(H_\eta)^t$ the curvature matrix of $(G, \mathbb{T}, J^K_{std}, g^t)$. Then we have*

$$M(H_\eta)^K = \left.\widetilde{M(H_\eta)}^t\right|_{t=0}$$

where $\widetilde{M(H_\eta)}^t$ is the submatrix of $M(H_\eta)^t$ obtained via erasing all rows and columns whose index contains a root in $\Delta^+_{\mathfrak{k}}$.

Chapter 6

The conjecture $H(k)$

This chapter is dedicated to the formulation of our main theorem and its place in the larger context of positive holomorphic curvature for classical Kähler C spaces. Furthermore, we will point out the extraordinary position of the classical Kähler Einstein C space $(G, \mathbb{T}, J_{std}, g_{KE})$ and the consequences of positivity of its holomorphic sectional curvature for arbitrary classical Kähler C spaces even though the previous section showed the difficulties connected to this.

We begin with the definition of the conjecture $H(k)$ for $k \geq 1$ to be that the following statement is true:

Conjecture 51. *Let $(G_k, K, J_{std}, g_{KE})$ be a simple Kähler Einstein C space with G being a classical Lie group of rank k. Then it has positive holomorphic sectional curvature.*

A priori, it might not seem reasonable to define the conjecture in dependency the rank of the group, since there might be a classical group G of rank k with isotropy groups K_1 and K_2 such that G/K_1 has positive holomorphic curvature but G/K_2 does not. However, there is some monotonicity to it if one formulates it this way. In fact, the statement of the following theorem is basically the assertion:

If $H(n)$ is true, then $H(k)$ is true for all $k < n$.

The actual formulation is the following:

Theorem 52. *Let $(G_k, K, J_{std}, g_{KE})$ be a classical Kähler Einstein C space with $rk(G_k) = k$ then there is a holomorphic totally geodesic isometric embedding of*

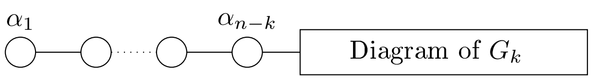
$$(G_k/K, J_{std}, g_{KE}) \hookrightarrow (G_n/\tilde{K}, J^n_{std}, g^n_{KE})$$

where $(G_n, \tilde{K})$ is a C space with G_n being the classical Lie group of rank n of the same family as G_k.

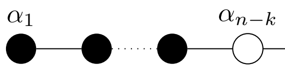
Proof. We define the embedding of G_k into G_n as

$$\begin{aligned} G_k &\hookrightarrow G_n \\ A &\mapsto \begin{pmatrix} \mathrm{id}_{l(n-k)} & 0 \\ 0 & A \end{pmatrix} \end{aligned}$$

where l is two in the case of B_k, D_k and one otherwise. On the level of Dynkin diagrams this corresponds to



Now we define $\tilde{K} = U(n-k) \times K \subset G_n$ by the painted diagram:



Painted Diagram of (G_k, K)

In order to prove that $G_k/K \hookrightarrow G_n/\tilde{K}$ is totally geodesic, we write it as the fixed point set of a subtorus $T \subset \mathbb{T}^n$ of isometries using a technique from [WZ18] which gives us

$$Fix(T) = C_{G_n}(T)/(C_{G_n}(T) \cap \tilde{K})$$

where $C_{G_n}(T)$ denotes the centralizer of T in G_n. Let $\mathfrak{t} = \mathfrak{t}_k^{\perp} \subset \mathfrak{t}_n$ be the Lie algebra of the torus T orthogonal to the maximal torus of G_k, i.e the maximal torus of $U(n-k)$. Since T, as the subset of the maximal torus of G_n, acts via isometries in g^n_{KE} and commutes with J^n_{std}, $Fix(T)$ is a complex totally geodesic submanifold. In fact, it is easy to see from the definition that $C_{G_n}(T) = T \times G_k$ and $(C_{G_n}(T) \cap \tilde{K}) = T \times K$ and therefore we have

$$Fix(T) = G_k/K.$$

It is left to show that the induced structures on G_k/K coming from $G_n/\tilde{K}$ actually coincide with the intrinsic structures of G_k/K. By the choice of $\tilde{K}$ and the embedding in terms of Dynkin diagrams, we see that all positive roots of G_k/K are also positive roots of $G_n/\tilde{K}$. Hence the complex structures coincide. By the Kähler property it is sufficient to check the metric on the simple roots in G_k/K. For such a root α we have that

$$\begin{aligned}(g^n_{KE})_\alpha =&(g^{\mathbb{T}^n}_{KE})_\alpha - (\gamma^*_{\tilde{K}}, \alpha) = g^{\mathbb{T}^n}_{KE} - (\gamma^*_{U(n-k)}, \alpha) - (\gamma^*_K, \alpha)\\ =&(g^{\mathbb{T}^k}_{KE})_\alpha - (\gamma^*_K, \alpha) - (\gamma^*_{U(n-k)}, \alpha)\\ =&(g_{KE})_\alpha - (\gamma^*_{U(n-k)}, \alpha)\\ =&(g_{KE})_\alpha\end{aligned}$$

where we used equation (3.3) and the following facts:

i) $\gamma^*_{\tilde{K}} = \gamma^*_K + \gamma^*_{U(n-k)}$ since $\tilde{K} = U(n-k) \times K$.

ii) $(\gamma^*_{U(n-k)}, \alpha) = 0$ since $U(n-k)$ and G_k commute.

iii) $(g^{\mathbb{T}^n}_{KE})_\alpha = (g^{\mathbb{T}^k}_{KE})_\alpha$ on the simple roots α in G_k by our discussion of the classical groups in the end of section 3.2.

□

Remark: Actually the important property in the definition of $\tilde{K}$ is that the vertex α_{n-k} is not painted. An immediate consequence of the above and corollary 40 is the following:

Theorem 53. *Let G be a simple compact classical Lie group with $rk(G) \leq n$, then, if $H(n)$ is true, any Kähler Einstein C space (G, K, J, g_{KE}) has positive holomorphic sectional curvature curvature.*

Proof. By lemma 38, we know there is a biholomorphic G equivariant isometry

$$(G, K, J, g_{KE}) \to (G, K', J_{std}, g^{std}_{KE}).$$

By theorem 52 there is a totally geodesic holomorphic embedding

$$(G, K', J_{std}, g^{std}_{KE}) \hookrightarrow (G_n, \tilde{K}, J_{std}, g^{std}_{KE})$$

where G_n is the simple compact classical group of rank n of the same family as G and by $H(n)$ we know that $(G_n, \tilde{K}, J_{std}, g^{std}_{KE})$ has positive holomorphic curvature and therefore so does (G, K, J, g_{KE}). □

This leads to the following interesting
Questions:

1) *Is there a k for which $H(k)$ holds?*
2) *Do all classical Kähler Einstein C spaces have positive holomorphic sectional curvature, i.e. is $H(k)$ true for all k?*
3) *If not, what is the maximal k^* for which $H(k^*)$ is true and in that case what is the difference between k^* and k^*+1?*

The next few sections are dedicated to proving our main theorem, that is a answer to 1):

Theorem 54. *The conjecture $H(k)$ is true for $k \leq 4$.*

Even though the technique to prove the statement is finding the right four forms in most cases separately and therefore might not be suitable to proof $H(k)$ for general k, it certainly gives the impression that there does not seem to be an obstruction identifiable in the holomorphic curvature tensor, which leads us to believe that the answer to 2) might actually be yes.

Before we turn to the proof of $H(4)$, we want to dedicate the rest of the section to the discussion of some consequences of $H(k)$ being true for all k, i.e. implications of 2). First of all we want to point out a nice property of the Kähler Einstein metric for the toric isotropy.

Theorem 55. *Let $(G_k, \mathbb{T}^k, J_{std}, g(c_1,..,c_k))$ be a classical Kähler C space of rank k with the metric notation from section 5. Then there is a holomorphic totally geodesic embedding*

$$(G_k, \mathbb{T}^k, J_{std}, g(c_1,..,c_k)) \to (G_n, \mathbb{T}^n, J_{std}, g_{KE})$$

for $n >> k$ if the following condition, depending on the type of G_k, is true:

$A_k :$ $c_i \in \mathbb{N}$ *for* $i = 1...k$

$B_k :$ $c_i \in 2\mathbb{N}$ *for* $i = 1...k-1$ *and* $c_k \in 2\mathbb{N}+1$

$C_k :$ $c_i \in \mathbb{N}$ *for* $i = 1...k-1$ *and* $c_k \in 2\mathbb{N}$

$D_k :$ $c_i \in \mathbb{N}$ *for* $i = 1...k-1$ *and* $c_k + c_{k-1} \in 2\mathbb{N}$

Proof. The idea of the proof is similar to the one of theorem 52 in that we want to find the embedding as the fixed point set of isometries. We begin by identifying the tangent space of our totally geodesic subset having the right metric coefficients. Hence let the c_i satisfy the conditions of the claim. Then the following is a natural number

$$n = \begin{cases} \sum_{i=1}^{k} c_i & A_k \\ \frac{1}{2}\left(\sum_{i=1}^{k} c_i + 1\right) & B_k \\ \frac{1}{2}c_k + \sum_{i=1}^{k-1} c_i & C_k \\ \frac{1}{2}(c_k + c_{k-1}) + \sum_{i=1}^{k-2} c_i + 1 & D_k \end{cases}$$

and we define with $\lambda = \begin{cases} \frac{1}{2} & B_k \\ 1 & otherwise \end{cases}$ the function

$$f : \{1,..,k+1\} \to \mathbb{N} \qquad (6.1)$$
$$s \mapsto 1 + \sum_{i=1}^{s-1} \lambda c_i$$

Now we define the subset R_k of roots of $\Delta_{\mathfrak{g}_n}$ indexed by f, i.e

$$R_k = \begin{cases} \{\alpha_{ij} \mid i,j \in f(\{1,...,k+1\})\} & A_k \\ \{\alpha_{ij}, \beta_{ij}, \varepsilon_i, \mid i,j \in f(\{1,...,k\})\} & B_k \\ \{\alpha_{ij}, \beta_{ij}, \gamma_i, \mid i,j \in f(\{1,...,k\})\} & C_k \\ \{\alpha_{ij}, \beta_{ij} \mid i,j \in f(\{1,...,k\})\} & D_k \end{cases}.$$

It is fairly easy to see that R_k is isomorphic to $\Delta_{\mathfrak{g}_k}$ as root systems via $\eta_I \mapsto \eta_{f(I)}$ for a root $\eta \in \Delta_{\mathfrak{g}_k}$. We will see that later in more detail. For now we want to point out that the space $\bigoplus_{\alpha\in R_k} \mathfrak{g}_\alpha$ would carry the right metric coefficients if it were the tangent space of a homogeneous fixed point set of isometries. Therefore we calculate that on the images of the simple roots we have by the equations for the Kähler Einstein metric, i.e (3.4) (3.5) (3.6) (3.7), in all cases that

$$(g_{KE})_{\alpha_{f(i)f(i+1)}} = \frac{1}{\lambda}(f(i+1) - f(i)) = \frac{1}{\lambda}\lambda c_i = c_i$$

and additionally in the case of B_n

$$(g_{KE})_{\varepsilon_{f(k)}} = 2n + 1 - 2f(k) = \sum_{i=1}^{k} c_i + 2 - 2f(k) = c_k,$$

in the case of C_n

$$(g_{KE})_{\gamma_{f(k)}} = 2n + 2 - 2f(k) = c_k + \sum_{i=1}^{k-1} 2c_i + 2 - 2f(k) = c_k$$

and the case of D_n

$$\begin{aligned}(g_{KE})_{\beta_{f(k-1)f(k)}} &= 2n - (f(k) + f(k-1)) \\ &= c_k + c_{k-1} + \sum_{i=1}^{k-2} 2c_i + 2 - (f(k) + f(k-1)) \\ &= c_k\end{aligned}$$

With that at hand, it is sufficient to show that there is a set $F \subset \mathbb{T}^n$ of isometries of $G_n/\mathbb{T}^n$ with the property that $Fix(F) = G_k/\mathbb{T}^k$ and $T_{\mathbb{T}^n}(Fix(F))^{\mathbb{C}} = \bigoplus_{\alpha\in R_k} \mathfrak{g}_\alpha$. This would also prove the above mentioned isomorphism $R_k = \Delta_{\mathfrak{g}_k}$. The existence of such an F follows from the next propositions. □

Proposition 56. *Let $A \in GL_l(\mathbb{K})$ with $(l, \mathbb{K}) \in \{(n+1, \mathbb{C}), (2n+1, \mathbb{R}), (2n, \mathbb{C}), (2n, \mathbb{R})\}$ depending on whether we consider A_n, B_n, C_n or D_n with the following properties*

i) The conjugation map $C_A : GL_l(\mathbb{K}) \to GL_l(\mathbb{K})$ leaves the pair $(G_n, \mathbb{T}^n)$ invariant.

ii) For $G_k \subset G_n$ being the lower $k \times k$ block as in theorem 52 and its root system $\Delta_{\mathfrak{g}_k} \subset \Delta_{\mathfrak{g}_n}$, we have

$$R_k = \{\alpha \circ Ad_A^{-1} \mid \alpha \in \Delta_{\mathfrak{g}_k}\}.$$

Then there is a set of biholomorphic isometries F of $(G_n, \mathbb{T}^n, J_{std}, g_{KE})$ such that $Fix(F) = G_k/\mathbb{T}^k$ and

$$T_{\mathbb{T}^n}(Fix(F))^{\mathbb{C}} = \bigoplus_{\alpha\in R_k} \mathfrak{g}_\alpha.$$

Proof. As we have seen in theorem 52, there is a subset $\tilde{F}$ of $\mathbb{T}^n \subset Isom(G_n, \mathbb{T}^n, g_{KE})$ with

$$Fix(\tilde{F}) = C_{G_n}(\tilde{F})/C_{G_n}(\tilde{F}) \cap \mathbb{T}^n = G_k\mathbb{T}^{n-k}/\mathbb{T}^n = G_k/\mathbb{T}^k$$

where G_k is embedded as the lower block. We notice that by property *i)* C_A descends to a map which we denote also by $C_A : G_n/\mathbb{T}^n \to G_n/\mathbb{T}^n$. Now we define $F = C_A(\tilde{F}) \subset \mathbb{T}^n$ and see that $Fix(F) = C_A(Fix(\tilde{F})) = C_A(G_k\mathbb{T}^{n-k}/\mathbb{T}^n)$. Therefore $Fix(F) \cong G_k/\mathbb{T}^k$ and by the standard identifications with subspaces of $\mathfrak{g}_n$ we have $T_{\mathbb{T}^n}Fix(\tilde{F}) = \bigoplus_{\alpha\in\Delta_{\mathfrak{g}_k}} \mathfrak{g}_\alpha$ which leads us to

$$\begin{aligned}T_{\mathbb{T}^n}(Fix(F)) &= T_{\mathbb{T}^n}(C_A(Fix(\tilde{F}))) = C_{A*}(T_{\mathbb{T}^n}Fix(\tilde{F})) \\ &= Ad_A\left(\bigoplus_{\alpha\in\Delta_{\mathfrak{g}_k}} \mathfrak{g}_\alpha\right)\end{aligned}$$

Since we know that Ad_A leaves $\mathbb{T}^n$ invariant, we have that Ad_A respects the decomposition in root spaces. In fact we have for $h \in \mathfrak{h} = (\mathfrak{t}^n)^{\mathbb{C}}$ and $X \in \mathfrak{g}_\alpha$ that

$$\begin{aligned} ad_h(Ad_A(X)) &= [h, Ad_A(X)] = Ad_A[Ad_A^{-1}h, X] = Ad_A(\alpha(Ad_A^{-1}(h))X) \\ &= \alpha(Ad_A^{-1}(h))Ad_A(X) = (\alpha \circ Ad_A^{-1})(h)Ad_A(X) \end{aligned}$$

Hence $Ad_A(\mathfrak{g}_\alpha) = \mathfrak{g}_{\alpha \circ Ad_A^{-1}}$, and therefore we have from property *ii)*

$$T_{\mathbb{T}^n}(Fix(F)) = \bigoplus_{\alpha \in \Delta_{\mathfrak{g}_k}} Ad_A \mathfrak{g}_\alpha = \bigoplus_{\alpha \in \Delta_{\mathfrak{g}_k}} \mathfrak{g}_{\alpha \circ Ad_A^{-1}} = \bigoplus_{\alpha \in R_k} \mathfrak{g}_\alpha$$

□

In order to finish the proof of theorem 55, it remains to show that an A as in proposition 56 exists.

Proposition 57. *There exists a permutation matrix $A \in GL_l(\mathbb{K})$ satisfying the requirements of theorem 56.*

Proof. First of all we want to define a suitable permutation. Let κ be 1 in the case of A_n and 0 otherwise. Then we consider the function

$$\begin{aligned} g : \{n-k+1, ..., n+\kappa\} &\to \{1, ..., n+\kappa\} \\ n-k+i &\mapsto f(i) \end{aligned}$$

where f is the function defined in (6.1). Now let g' be the monotonous bijection of $\{1, ..., n-k\} \to \{1, ..., n+\kappa\} \setminus Im(g)$. Then we define the permutation $\sigma \in S_{n+\kappa}$ partially by g and g'. Now we define the matrix $A \in Gl_l(\mathbb{K})$ as follows

$$A = \begin{cases} P_\sigma^{-1} & SU(n+1) \\ Diag\left(P_\sigma^{-1} \otimes id_2, 1\right) & SO(2n+1) \\ id_2 \otimes P_\sigma^{-1} & Sp(n) \subset SU(2n) \\ P_\sigma^{-1} \otimes id_2 & SO(2n) \end{cases}.$$

Here $P_\sigma \in O(n+\kappa)$ is the permutation matrix with respect to σ, i.e.

$$(P_\sigma)_{ij} = \delta_{\sigma(i)j}$$

with the Kronecker delta notation. Since A is lies in $U(l), O(l), Sp(l)$ it is clear that the conjugation map preserves G_n. The maximal torus is also preserved since conjugation with A permutes the diagonal entries of diagonal matrices in the cases of $SU(l)$ and $Sp(l)$. In the case of $SO(l)$, we permute the upper n 2×2-blocks which correspond to the maximal torus. Hence it remains to show that, we have

$$R_k = \{\alpha \circ Ad_A^{-1} \mid \alpha \in \Delta_{\mathfrak{g}_k}\}. \tag{6.2}$$

Before we do this, we observe that A can be seen as the matrix representing the following permutations:

A_n)

$$\begin{aligned} \sigma_1 : \{1, ..., n+1\} &\to \{1, ..., n+1\} \\ i &\mapsto \sigma^{-1}(i) \end{aligned}$$

B_n)

$$\begin{aligned} \sigma_2 : \{1, ..., 2n+1\} &\to \{1, ..., 2n+1\} \\ 2i-1 &\mapsto 2\sigma^{-1}(i) - 1 \\ 2i &\mapsto 2\sigma^{-1}(i) \\ 2n+1 &\mapsto 2n+1 \end{aligned}$$

C_n)

$$\begin{aligned}\sigma_3 : \{1, ..., 2n\} &\to \{1, ..., 2n\}\\ i &\mapsto \sigma^{-1}(i)\\ n+i &\mapsto n+\sigma^{-1}(i)\end{aligned}$$

D_n)

$$\begin{aligned}\sigma_4 : \{1, ..., 2n\} &\to \{1, ..., 2n\}\\ 2i-1 &\mapsto 2\sigma^{-1}(i)-1\\ 2i &\mapsto 2\sigma^{-1}(i)\end{aligned}$$

where i ranges from 1 to n in the last three cases. It is easy to see that for the matrix E_{kl} used to describe all root vectors for the different groups and a permutation matrix P_τ the following holds

$$Ad_{P_\tau}(E_{kl}) = P_\tau E_{kl} P_\tau^{-1} = E_{\tau^{-1}(k)\tau^{-1}(l)}.$$

Now we will show (6.2) separately for the families A_n, B_n, C_n, D_n.

A_n) It is sufficient to show that $Ad_A E_{\alpha_{n-k+i,n-k+j}} = E_{\alpha_{f(i)f(j)}}$ and since $E_{\alpha_{kl}} = E_{kl}$ we have immediately

$$\begin{aligned}(Ad_A(E_{\alpha_{n-k+i,n-k+j}})) &= (P_\sigma^{-1} E_{\alpha_{n-k+i,n-k+j}} P_\sigma)\\ &= (E_{\alpha_{\sigma(n-k+i)\sigma(n-k+j)}})\\ &= E_{\alpha_{f(i)f(j)}}\end{aligned}$$

which finishes the case of A_n.

B_n) Similarly, we want to show the above for $E_{\alpha_{n-k+i,n-k+j}}, E_{\beta_{n-k+i,n-k+j}}$ and $E_{\varepsilon_{n-k+i}}$, we only do the calculation for $E_{\alpha_{n-k+i,n-k+j}}$ since the others go analogously. By the description in section 1.8.2, we have the E_α given in terms of $F_{kl} = E_{lk} - E_{kl}$. It is immediate that $Ad_{P_\tau}(F_{kl}) = F_{\tau^{-1}(k)\tau^{-1}(l)}$ holds. Hence with $s = n-k+i$ and $t = n-k+j$ we have

$$\begin{aligned}Ad_A(E_{\alpha_{n-k+i,n-k+j}}) =& P_{\sigma_2} E_{\alpha_{n-k+i,n-k+j}} P_{\sigma_2}^{-1}\\ =& P_{\sigma_2}(F_{2s-1,2t-1} + F_{2s,2t}\\ & - i(F_{2s-1,2t} - F_{2s,2t-1}))P_{\sigma_2}^{-1}\\ =& F_{2\sigma(s)-1,2\sigma(t)-1} + F_{2\sigma(s),2\sigma(t)}\\ & - i(F_{2\sigma(s)-1,2\sigma(t)} - F_{2\sigma(s),2\sigma(t)-1})\\ =& E_{\alpha_{\sigma(s),\sigma(t)}}\\ =& E_{\alpha_{f(i)f(j)}}\end{aligned}$$

C_n) Similarly, we want to show the above for $E_{\alpha_{n-k+i,n-k+j}} E_{\beta_{n-k+i,n-k+j}}$ and $E_{\gamma_{n-k+i}}$, we only do the calculation for $E_{\alpha_{n-k+i,n-k+j}}$ since the others go analogously. By the description in section 1.8.3, we have $E_{\alpha_{st}} = E_{st} - E_{n+t,n+s}$. Hence with $s = n-k+i$ and $t = n-k+j$ we have

$$\begin{aligned}Ad_A(E_{\alpha_{n-k+i,n-k+j}}) =& P_{\sigma_3} E_{\alpha_{n-k+i,n-k+j}} P_{\sigma_3}^{-1}\\ =& P_{\sigma_3}(E_{s,t} + E_{n+t,n+s})P_{\sigma_3}^{-1}\\ =& E_{\sigma(s),\sigma(t)} + E_{n+\sigma(t),n+\sigma(s)}\\ =& E_{\alpha_{\sigma(s),\sigma(t)}}\\ =& E_{\alpha_{f(i)f(j)}}\end{aligned}$$

D_n) This works exactly as for B_n.

□

The consequences of the existence of these embeddings even though just for particular coefficients are fairly strong. In fact, let $I_{G_k} \subset \mathbb{N}^k$ be the set of coefficients satisfying the conditions of theorem 55 depending on the type of G_k. Then it is easy to see that by

$$\mathbb{Q}_+^k = \mathbb{Q}_+ \cdot I_{G_k}$$

every Kähler metric with rational coefficients c_α has positive holomorphic curvature if this is true for $(G_n, \mathbb{T}^n, J_{std}, g_{KE})$ and all $n \in \mathbb{N}$. In detail that results in the following

Corollary 58. *If $(G_n, \mathbb{T}^n, J_{std}, g_{KE})$ has positive holomorphic curvature for all n, then we have for any $k \in \mathbb{N}$*

i) $(G_k, \mathbb{T}^k, J_{std}, g(c_1, .., c_k))$ has positive holomorphic curvature if $c_i \in \mathbb{Q}$

ii) $(G_k, \mathbb{T}^k, J_{std}, g)$ has nonnegative holomorphic curvature for any Kähler metric g

Proof. The first statement is clear. The second uses that the holomorphic curvature tensor depends continuously on the coefficients c_i and density of $\mathbb{Q}^k \subset \mathbb{R}^k$. □

Combining the second statement with the description of the holomorphic curvature tensor of a Kähler C space with arbitrary isotropy group from section 5 as the limit of holomorphic curvature tensors on a Kähler C space with toric isotropy yields the following result

Theorem 59. *If $(G_n, \mathbb{T}^n, J_{std}, g_{KE})$ has positive holomorphic curvature for all n then all classical Kähler C spaces have nonnegative holomorphic sectional curvature.*

Proof. Let (G, K, J, g) be a classical Kähler C space. Then it is biholomorphically isometric to $(G, K', J_{std}, \tilde{g})$ by lemma 37. Here $\tilde{g}$ is the pullback of g via the biholomorphism

$$(G, K', J_{std}) \to (G, K, J).$$

Now let $k = rk(G)$ and $s = \dim(\mathfrak{z}(\mathfrak{k}')$. Then as in chapter 5 we have positive constants $c_{s+1}, .., c_k$ such that $\tilde{g} = g(c_{s+1}, ..., c_t)$ and we define the Kähler metric on $(G, \mathbb{T}^k, J_{std})$ by $\tilde{g}^t = g^{\mathbb{T}}(t, ..t, c_{s+1}, ..., c_r)$. Then by corollary 58 *ii*), we have that for all $t > 0$ the Kähler C space $(G, \mathbb{T}^k, J_{std}, g^t)$ has nonnegative holomorphic sectional curvature. With the decomposition from section 5

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \oplus \mathfrak{m}$$

where $\mathfrak{m}$ is the tangent space of G/K' and $\mathfrak{n} = \mathfrak{p} \oplus \mathfrak{m}$ is the tangent space of $G/\mathbb{T}^k$. Then for $X \in \mathfrak{m}$ we represent the plane $X \wedge J_{std}(X)$ as the coefficient vector v in terms of the basis $\Omega_\alpha, \Phi_{\alpha\beta}, Psi_{\alpha\beta}$ with $\alpha < \beta \in \Delta_{\mathfrak{m}}^+$ from section 3.1. Then nonnegativity of the holomorphic sectional curvature implies $v^T H^t v \geq 0$ for the holomorphic curvature tensor H^t of $\tilde{g}^t$ and all $t > 0$. Then we have from proposition 48

$$0 \leq \lim_{t \to 0} v^T H^t v = \lim_{t \to 0} \left(v^T H_{K'}^t v \right) = v^T \left(\lim_{t \to 0} H_{K'}^t \right) v = v^t H_{K'} v = H_{K'}(X)$$

where $H_{K'}^t$ is the restriction of H^t from $\mathfrak{n}$ to $\mathfrak{m}$ and $H_{K'}$ denotes the holomorphic curvature tensor and then the holomorphic sectional curvature of $\tilde{g}$. □

Remark:
1.) Note that there is no version of this theorem with bounded rank. As we see in the proof of theorem 55 the choice of the large n enabling the embedding and hence the curvature deduction does depend on the metric and not only on the rank k. Hence, also for fixed rank k the n can grow arbitrarily large depending on the choice of metric coefficients on $(G_k, \mathbb{T}^k)$.
2.) We want to point out that even though it is obvious that every C space carries a homogeneous hermitian metric with non negative holomorphic curvature (i.e. the submersion

metric of the killing form), this metric is only Kähler in the cases of Hermitian symmetric spaces, i.e. when the isotropy representation is irreducible.

A strong consequence of theorem 59 is that to disprove the conjecture $H(k)$ it is sufficient to find any classical Kähler C space admitting a negatively curved complex line. On the other hand, to prove the slightly weaker version of $H(k)$ with nonnegative instead of positive holomorphic curvature it is sufficient to restrict oneself to the toric isotropy and the Kähler Einstein metric, i.e

Corollary 60. *The spaces $(G_n, \mathbb{T}^n, J_{std}, g_{KE})$ have nonnegative holomorphic curvature for all $n \in \mathbb{N}$ if and only if all classical Kähler C spaces have nonnegative holomorphic curvature.*

Chapter 7

Curvature Matrices

Before we restrict ourselves to the classical groups of rank 4 we use this chapter to determine the weights Δ_H and the corresponding matrices representing the holomorphic curvature tensor for the classical C spaces of arbitrary rank $(G_n, \mathbb{T}^n, J_{std})$ with an arbitrary Kähler metric. As we will see the amount of different kind of modules seems overwhelming at first, but careful observations allow us to reduce the problem of positive holomorphic curvature in general to just two to four different kinds of modules depending on G instead of the in the following presented four to eighteen kinds. The remaining modules will be the trivial module and the modules with $\eta \in \Delta_{\mathfrak{m}}^-$. In the following we will not write the representing matrices of the four forms explicitly but rather their effect on the matrices representing the curvature tensor. Therefore, we simplify notation slightly writing

$$M_\eta = M(H_\eta)$$

for the matrices of the holomorphic curvature tensor.
The first observation will be that if $\eta \notin \Delta_{\mathfrak{m}}^-$ then the matrices representing the curvature tensor on these modules cannot be positive definite because they either have negative values on the diagonal or zeros on the diagonal and non zero offdiagonal entries. Hence it is necessary to modify these matrices via four forms to keep the hope alive to be able to prove positive holomorphic curvature via the tensor. As indicated by the recipe in the end of section 1.7, we will do the following:

I) If there is a negative value $-c < 0$ on the diagonal of the matrix M_η at $(\alpha,\beta)(\alpha,\beta)$, we erase it, i.e add its absolute value c to that entry. To realize that via four forms we have to subtract the same absolute value from the corresponding entry of the matrix of the trivial module. That means in the notation of chapter 4 to

Value	Intended	Forced
c	$(M_\eta)_{(\alpha,\beta)(\alpha,\beta)}$	$(M_0)_{(\alpha,\alpha)(\beta,\beta)}$

By the Bianchi identity, the entry of the matrix of the trivial module has the same value as the diagonal entry of M_η, see (3.1). Therefore erasing all negative diagonals results in doubling all negative entries of the matrix of the trivial module.

II) After erasing all negative diagonal entries, we are consider all resulting matrices that have zeros on the diagonal and non zero off diagonal entries. We want to erase those off diagonal entries as well. Assume that we want to erase the off diagonal entry $(M_\eta)_{(\alpha,\beta)(\gamma,\delta)} = c$ via four forms. By (3.2) we are forced to modify the entry $(M_{\eta+\beta-\gamma})_{(\alpha,\gamma)(\beta,\delta)}$ in the opposite direction. In the notation of chapter 4 this means

Value	Intended	Forced
$-c$	$(M_\eta)_{(\alpha,\beta)(\gamma,\delta)}$	$(M_{\eta+\beta-\gamma})_{(\alpha,\gamma)(\beta,\delta)}$

We will see that for every entry of a matrix M_η with zero diagonal the corresponding matrix $M_{\eta+\beta-\gamma}$ has positive diagonal entries. As in case I) the values of these two entries coincide and hence erasing the entry in M_η results in multiplying the entry in $M_{\eta+\beta-\gamma}$ by two.

We specify for each classical group separately in the section *"General modifications"* how these modifications affect the matrices. Exemplary, we will present the full details in the case of $SU(n+1)$.

7.1 $SU(n+1)$

We define the ordering on the positive roots as in proposition 15 with the slight modification that we use the ordered basis ε_j with $j = 1,..,n+1$ instead of the simple roots to induce the ordering. It is easy to see that the desired properties are still true.

Proposition 61. *For the C space $(SU(n+1), \mathbb{T}^n, J_{std})$ let η be a non zero weight of $Fix(J_{std})$, i.e an element in*

$$\Delta_H = \{\alpha - \beta \mid \alpha < \beta \in \Delta_{\mathfrak{g}}^+\}.$$

Then η is in the following table together with the pairs $\alpha < \beta$ with $\alpha - \beta = \eta$:

$\sharp$	η	$I(\eta)$	n_η
1	$-\varepsilon_a + \varepsilon_b$	$(\alpha_{ia}, \alpha_{ib})$, $i = 1..a-1$ $(\alpha_{bj}, \alpha_{aj})$, $j = b+1..n+1$	$n - b + a$
2	$-\varepsilon_a + \varepsilon_b + \varepsilon_c - \varepsilon_d$	$(\alpha_{cd}, \alpha_{ab}), (\alpha_{bd}, \alpha_{ac})$	2
3	$-\varepsilon_a + \varepsilon_b - \varepsilon_c + \varepsilon_d$	$(\alpha_{bc}, \alpha_{ad})$	1
4	$-\varepsilon_a + 2\varepsilon_b - \varepsilon_c$	$(\alpha_{bc}, \alpha_{ab})$	1

Proof. The elements in the list are obviously weights and these are all of them for dimensional reasons. In fact, from (4.1) we only have to verify that

$$\frac{|\Delta_{\mathfrak{g}}^+| \left(|\Delta_{\mathfrak{g}}^+| - 1\right)}{2} = \sum_{\eta \in \Delta_H} n_\eta$$

holds. In fact, we know $|\Delta_{\mathfrak{g}}^+| = \frac{n(n+1)}{2}$ and the right hand side is given by

$$\sum_{a=1}^{n} \sum_{b=a+1}^{n+1} (n - b + a) + \binom{n+1}{3} + 3\binom{n+1}{4}$$

Easy calculations yield that left and right hand side coincide and therefore we found all weights. □

In the following, we determine the curvature matrices M_η. Since the indices of M_η are pairs of roots (α, β) with $\alpha < \beta$ and $\alpha - \beta = \eta$ it makes sense to choose an ordering of these pairs. We will do so in each M_η separately. The proofs reduce to plugging in the values of g_α, $N_{\alpha,\beta}$ and z_α as determined for the classical groups in section 1.8 into the equations of proposition 16 and will therefore be omitted. An exemplary calculation was made in the proof of proposition 44.

Proposition 62 (Case $\sharp$1)**.** *Let $\eta = -\varepsilon_a + \varepsilon_b$ for $1 \le a < b \le n+1$. We order the pairs as follows $x_i = (\alpha_{ia}, \alpha_{ib})$ with $i = 1,..,a-1$ and $y_j = (\alpha_{b,b+j}, \alpha_{a,b+j})$ with $j = 1,..,n+1-b$. Then we have*

$$M_\eta = \begin{pmatrix} X & \langle XY \rangle \\ \langle XY \rangle^T & Y \end{pmatrix}$$

with diagonal blocks

$$X \in Sym_{a-1}(\mathbb{R}) \qquad X_{is} = g_{\alpha_{sa}} \text{ for } i \leq s$$
$$Y \in Sym_{n+1-b}(\mathbb{R}) \qquad Y_{is} = g_{\alpha_{b,b+i}} \text{ for } i \leq s$$

and off diagonal block entries

$$\langle XY \rangle_{is} = -\frac{g_{\alpha_{ia}} g_{\alpha_{b,b+s}}}{g_{\alpha_{i,b+s}}}$$

Proposition 63 (Case ♯2)**.** *Let* $\eta = -\varepsilon_a + \varepsilon_b + \varepsilon_c - \varepsilon_d$ *for* $1 \leq a < b < c < d \leq n+1$. *We order the pairs as follows* $(\alpha_{cd}, \alpha_{ab}), (\alpha_{bd}, \alpha_{ac})$. *Then we have*

$$M_\eta = \begin{pmatrix} 0 & -\frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\alpha_{ad}}} \\ -\frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\alpha_{ad}}} & 0 \end{pmatrix}$$

Proposition 64 (Case ♯3)**.** *Let* $\eta = -\varepsilon_a + \varepsilon_b - \varepsilon_c + \varepsilon_d$ *for* $1 \leq a < b < c < d \leq n+1$. *We only have one entry corresponding to* $(\alpha_{bc}, \alpha_{ad}), (\alpha_{bd}, \alpha_{ac})$. *Then we have*

$$M_\eta = 0$$

Proposition 65 (Case ♯4)**.** *Let* $\eta = -\varepsilon_a + 2\varepsilon_b - \varepsilon_c$ *for* $1 \leq a < b < c \leq n+1$. *We only have one entry corresponding to* $(\alpha_{bc}, \alpha_{ab})$. *Then we have*

$$M_\eta = -\frac{g_{\alpha_{ab}} g_{\alpha_{bc}}}{g_{\alpha_{ac}}}$$

Proposition 66. *The entries of the matrix of the trivial module are determined by the diagonals of the matrices above :*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = (M_{\alpha-\beta})_{(\alpha\beta)(\alpha\beta)}$$

except for the diagonals. For $\alpha = \alpha_{ij}$ *the following holds*

$$(M_0)_{(\alpha\alpha)(\alpha\alpha)} = 2g_\alpha$$

General modifications of $SU(n+1)$

We present the modifications following step I) and II) described in the beginning of the section for $SU(n+1)$ in detail. Roughly speaking we make the following modifications:

1.) We move the diagonals of ♯4 into the matrix of the trivial module.

2.) The off diagonals of ♯2 into the $\langle XY \rangle$ block of ♯1.

In detail, the modifications of 1.) with the notation from chapter 4 correspond to the following changes for all $1 \leq a < b < c \leq n+1$

Value	Intended	Forced
$\frac{g_{\alpha_{ab}} g_{\alpha_{bc}}}{g_{\alpha_{ac}}}$	$(M_{-\varepsilon_a+2\varepsilon_b-\varepsilon_c})_{(\alpha_{bc},\alpha_{ab})(\alpha_{bc},\alpha_{ab})}$	$(M_0)_{(\alpha_{bc},\alpha_{bc})(\alpha_{ab},\alpha_{ab})}$

and 2.) for $1 \leq a < b < c < d \leq n+1$

Value	Intended	Forced
$\frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\alpha_{ad}}}$	$(M_{-\varepsilon_a+\varepsilon_b+\varepsilon_c-\varepsilon_d})_{(\alpha_{cd},\alpha_{ab})(\alpha_{bd},\alpha_{ac})}$	$(M_{-\varepsilon_b+\varepsilon_c})_{(\alpha_{ab},\alpha_{ac})(\alpha_{cd},\alpha_{bd})}$

We want to remark here that even though one needs multiple small matrices of type ♯2 to cover the $\langle X, Y \rangle$ block of one matrix of type ♯1 in the end we cover all of them. In fact, it is easy to see that the map "Intended" to "Forced" is a bijection between off diagonal entries of all matrices of type ♯2 to entries of $\langle X, Y \rangle$ blocks of all matrices of type ♯1.

After these changes the now modified curvature tensor decomposes into the following matrices

Proposition 67 (Case $\sharp 1$). *Let $\eta = -\varepsilon_a + \varepsilon_b$ for $1 \le a < b \le n+1$. We order the pairs as follows $x_i = (\alpha_{ia}, \alpha_{ib})$ and $y_j = (\alpha_{b,b+j}, \alpha_{a,b+j})$ with $i = 1, .., a-1$ and $j = 1, .., n+1-b$. Then we have the modified matrices*

$$M_\eta = \begin{pmatrix} X & \langle XY \rangle \\ \langle XY \rangle^T & Y \end{pmatrix}$$

with diagonal blocks

$$\begin{aligned} X &\in Sym_{a-1}(\mathbb{R}) & X_{is} &= g_{\alpha_{sa}} \text{ for } i \le s \\ Y &\in Sym_{n+1-b}(\mathbb{R}) & Y_{is} &= g_{\alpha_{b,b+i}} \text{ for } i \le s \end{aligned}$$

and off diagonal block entries

$$\langle XY \rangle_{is} = -2 \frac{g_{\alpha_{ia}} g_{\alpha_{b,b+s}}}{g_{\alpha_{i,b+s}}}$$

Proposition 68. *The entries of the matrix of the trivial module are given by:*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = \begin{cases} 2g_\alpha & \alpha = \beta \\ g_{\alpha_{ia}} & (\alpha, \beta) = (\alpha_{ia}, \alpha_{ib}) \\ g_{\alpha_{bj}} & (\alpha, \beta) = (\alpha_{bj}, \alpha_{aj}) \\ -2 \frac{g_{\alpha_{ai}} g_{\alpha_{ib}}}{g_{\alpha_{ab}}} & (\alpha, \beta) = (\alpha_{ai}, \alpha_{ib}) \end{cases}$$

for $\alpha \le \beta$.

7.2 $SO(2n+1)$

As for $SU(n+1)$ we define the ordering on the positive roots as in proposition 15 with the slight modification that we use the ordered basis ε_j with $j = 1, .., n$ instead of the simple roots to induce the ordering. It is easy to see that the desired properties are still true.

Proposition 69. *For the C space $(SO(2n+1), \mathbb{T}^n, J_{std})$ let η be a non zero weight of $Fix(J_{std})$, i.e an element in*

$$\Delta_H = \{\alpha - \beta \mid \alpha < \beta \in \Delta_{\mathfrak{g}}^+\}.$$

Then η is in the following table together with the pairs $\alpha < \beta$ with $\alpha - \beta = \eta$:

$\sharp$	η	$I(\eta)$	n_η
1	$-\varepsilon_a + \varepsilon_b$	$(\alpha_{ia}, \alpha_{ib})$, $i = 1, .., a-1$ $(\alpha_{bj}, \alpha_{aj})$, $j = b+1, .., n$ (β_{tb}, β_{at}) , $t = a+1, .., b-1$ (β_{sb}, β_{sa}) , $s = 1, .., a-1$ (β_{br}, β_{ar}) , $r = b+1, .., n$ $(\varepsilon_b, \varepsilon_a)$	$2n - b + a - 2$
2	$-\varepsilon_a - \varepsilon_b$	$(\alpha_{sa}, \beta_{sb})$, $s = 1, .., a-1$ $(\alpha_{sb}, \beta_{sa})$, $s = 1, .., a-1$ $(\alpha_{tb}, \beta_{at})$, $t = a+1, .., b-1$	$b + a - 3$
3	$-\varepsilon_a$	$(\alpha_{sa}, \varepsilon_s)$, $s = 1, .., a-1$ $(\varepsilon_s, \beta_{sa})$, $s = 1, .., a-1$ $(\varepsilon_r, \beta_{ar})$, $r = a+1, .., n$	$n + a - 2$

$\sharp$	η	$I(\eta)$	n_η
4	$-2\varepsilon_a$	$(\alpha_{sa}, \beta_{sa})$, $s = 1, .., a-1$	$a-1$
5	$-\varepsilon_a + \varepsilon_b + \varepsilon_c - \varepsilon_d$	$(\alpha_{cd}, \alpha_{ab}), (\alpha_{bd}, \alpha_{ac}), (\beta_{bc}, \beta_{ad})$	3
6	$-\varepsilon_a + \varepsilon_b - \varepsilon_c + \varepsilon_d$	$(\alpha_{bc}, \alpha_{ad}), (\beta_{bd}, \beta_{ac})$	2
7	$-\varepsilon_a - \varepsilon_b + \varepsilon_c + \varepsilon_d$	(β_{cd}, β_{ab})	1
8	$-\varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d$	$(\beta_{cd}, \alpha_{ab}), (\beta_{bd}, \alpha_{ac}), (\beta_{bc}, \alpha_{ad})$	3
9	$-\varepsilon_a + \varepsilon_b - \varepsilon_c - \varepsilon_d$	$(\alpha_{bc}, \beta_{ad}), (\alpha_{bd}, \beta_{ac})$	2
10	$-\varepsilon_a - \varepsilon_b + \varepsilon_c - \varepsilon_d$	$(\alpha_{cd}, \beta_{ab})$	1
11	$-\varepsilon_a + 2\varepsilon_b - \varepsilon_c$	$(\alpha_{bc}, \alpha_{ab})$	1
12	$-\varepsilon_a + 2\varepsilon_b + \varepsilon_c$	$(\beta_{bc}, \alpha_{ab})$	1
13	$-\varepsilon_a + \varepsilon_b + 2\varepsilon_c$	$(\beta_{bc}, \alpha_{ac})$	1
14	$-\varepsilon_a + \varepsilon_b - 2\varepsilon_c$	$(\alpha_{bc}, \beta_{ac})$	1
15	$-\varepsilon_a + \varepsilon_b + \varepsilon_c$	$(\varepsilon_c, \alpha_{ab}), (\varepsilon_b, \alpha_{ac}), (\beta_{bc}, \varepsilon_a)$	3
16	$-\varepsilon_a + \varepsilon_b - \varepsilon_c$	$(\alpha_{bc}, \varepsilon_a), (\varepsilon_b, \beta_{ac})$	2
17	$-\varepsilon_a - \varepsilon_b + \varepsilon_c$	$(\varepsilon_c, \beta_{ab})$	1
18	$-\varepsilon_a + 2\varepsilon_b$	$(\varepsilon_b, \alpha_{ab})$	1

Proof. The elements in the list are obviously weights and these are all of them for dimensional reasons. In fact, from (4.1) we only have to verify that

$$\frac{|\Delta_{\mathfrak{g}}^+| \left(|\Delta_{\mathfrak{g}}^+| - 1\right)}{2} = \sum_{\eta \in \Delta_H} n_\eta$$

holds. In fact, we know $|\Delta_{\mathfrak{g}}^+| = n^2$ and the right hand side is given by

$$\sum_{a=1}^{n-1} \sum_{b=a+1}^{n} (2n + 2a - 4) + \sum_{a=1}^{n} (n + 2a - 3)$$
$$+10\binom{n}{3} + 12\binom{n}{4}$$

Easy calculations yield that left and right hand side coincide and therefore we found all weights. □

In the following, we determine the curvature matrices M_η. Since the indices of M_η are pairs of roots (α, β) with $\alpha < \beta$ and $\alpha - \beta = \eta$ it makes sense to choose an ordering of these pairs. We will do so in each M_η separately. The proofs reduce to plugging in the values of g_α, $N_{\alpha,\beta}$ and z_α as determined for the classical groups in section 1.8 into the equations of proposition 16 and will therefore be omitted. An exemplary calculation was made in the proof of proposition 44.

Proposition 70 (Case $\sharp$1)**.** *Let $\eta = -\varepsilon_a + \varepsilon_b$ for $1 \le a < b \le n$. We order the pairs as follows $v_i = (\alpha_{ia}, \alpha_{ib}), w_j = (\alpha_{b,b+j}, \alpha_{a,b+j}), x_t = (\beta_{a+t,b}, \beta_{a,a+t}), y_i = (\beta_{ib}, \beta_{ia}), z_j = (\beta_{b,b+j}, \beta_{a,b+j})$ and $D_1 = (\varepsilon_b, \varepsilon_a)$ with $i = 1, .., a-1, t = 1, .., b-a-1$ and $j = 1, .., n-b$.*

Then we have

$$M_\eta = \begin{pmatrix} V & \langle VW\rangle & \langle VX\rangle & \langle VY\rangle & \langle VZ\rangle & \langle VD\rangle \\ \langle VW\rangle^T & W & \langle WX\rangle & \langle WY\rangle & \langle WZ\rangle & \langle WD\rangle \\ \langle VX\rangle^T & \langle WX\rangle^T & X & \langle XY\rangle & \langle XZ\rangle & \langle XD\rangle \\ \langle VY\rangle^T & \langle WY\rangle^T & \langle XY\rangle^T & Y & \langle YZ\rangle & \langle YD\rangle \\ \langle VZ\rangle^T & \langle WZ\rangle^T & \langle XZ\rangle^T & \langle YZ\rangle^T & Z & \langle ZD\rangle \\ \langle VD\rangle^T & \langle WD\rangle^T & \langle XD\rangle^T & \langle YD\rangle^T & \langle ZD\rangle^T & D \end{pmatrix}$$

with the diagonal matrices being

$$\begin{array}{ll} V \in Sym_{a-1}(\mathbb{R}) & V_{is} = 4g_{\alpha_{sa}} \text{ for } i \le s \\ W \in Sym_{n-b}(\mathbb{R}) & W_{is} = 4g_{\alpha_{b,b+i}} \text{ for } i \le s \\ X \in Sym_{b-a-1}(\mathbb{R}) & X_{is} = 4g_{\beta_{a+s,b}} \text{ for } i \le s \\ Y \in Sym_{a-1}(\mathbb{R}) & Y_{is} = 4g_{\beta_{sb}} \text{ for } i \le s \\ Z \in Sym_{n-b}(\mathbb{R}) & Z_{is} = 4g_{\beta_{b,b+s}} \text{ for } i \le s \\ D \in Sym_1(\mathbb{R}) & D_{11} = \dfrac{(g_{\varepsilon_b})^2}{g_{\beta_{ab}}} \end{array}$$

and the off diagonals being given by the following where i and s vary in the ranges determined by the diagonals

$$\langle VW\rangle_{is} = -4\frac{g_{\alpha_{ia}} g_{\alpha_{b,b+s}}}{g_{\alpha_{i,b+s}}} \qquad \langle VX\rangle_{is} = 4\frac{g_{\alpha_{ia}} g_{\beta_{a+s,b}}}{g_{\beta_{i,a+s}}} \qquad \langle VY\rangle_{is} = \begin{cases} -4g_{\alpha_{ia}} \\ -4\frac{g_{\alpha_{ia}} g_{\beta_{sb}}}{g_{\beta_{is}}} \end{cases}$$

$$\langle VZ\rangle_{is} = -4\frac{g_{\alpha_{ia}} g_{\beta_{b,b+s}}}{g_{\beta_{i,b+s}}} \qquad \langle VD\rangle_{i1} = -2\frac{g_{\alpha_{ia}} g_{\varepsilon_b}}{g_{\varepsilon_i}} \qquad \langle WX\rangle_{is} = -4g_{\alpha_{b,b+i}}$$

$$\langle WY\rangle_{is} = 4g_{\alpha_{b,b+i}} \qquad \langle WZ\rangle_{is} = \begin{cases} 4\frac{g_{\alpha_{b,b+i}} g_{\beta_{b,b+i}}}{g_{\beta_{ab}}} \\ 4g_{\alpha_{b,b+i}} \end{cases} \qquad \langle WD\rangle_{i1} = 2g_{\alpha_{b,b+i}}$$

$$\langle XY\rangle_{is} = -4g_{\beta_{a+i,b}} \qquad \langle XZ\rangle_{is} = -4g_{\beta_{b,b+s}} \qquad \langle XD\rangle_{i1} = -2g_{\varepsilon_b}$$

$$\langle YZ\rangle_{is} = 4g_{\beta_{b,b+s}} \qquad \langle YD\rangle_{i1} = 2g_{\varepsilon_b} \qquad \langle ZD\rangle_{is} = 2g_{\varepsilon_b}$$

In the two case distinctions the upper case corresponds to $i = s$ and the lower case to $i \neq s$.

Proposition 71 (Case $\sharp 2$). *Let $\eta = -\varepsilon_a - \varepsilon_b$ for $1 \le a < b \le n$. We order the pairs as follows $x_i = (\alpha_{ia}, \beta_{ib}), y_i = (\alpha_{ib}, \beta_{ia}), z_t = (\alpha_{a+t,b}, \beta_{a,a+t})$ with $i = 1, .., a-1$ and $t = 1, .., b-a-1$. Then we have*

$$M_\eta = \begin{pmatrix} X & \langle XY\rangle & \langle XZ\rangle \\ \langle XY\rangle^T & Y & \langle YZ\rangle \\ \langle XZ\rangle^T & \langle YZ\rangle^T & Z \end{pmatrix}$$

with the diagonal matrices being

$$\begin{array}{lll} X \in Sym_{a-1}(\mathbb{R}) & X_{is} = 4g_{\alpha_{sa}} & \text{for } i \le s \\ Y \in Sym_{a-1}(\mathbb{R}) & Y_{is} = 4g_{\alpha_{sb}} & \text{for } i \le s \\ Z \in Sym_{b-a-1}(\mathbb{R}) & Z_{is} = 4g_{\alpha_{a+s,b}} & \text{for } i \le s \end{array}$$

and the off diagonals being given by

$$\langle XY\rangle_{is} = \begin{cases} -4g_{\alpha_{sa}} & i = s \\ -4\frac{g_{\alpha_{sb}} g_{\alpha_{ia}}}{g_{\beta_{si}}} & i \neq s \end{cases} \qquad \langle XZ\rangle_{is} = 4\frac{g_{\alpha_{ia}} g_{\alpha_{a+s,b}}}{g_{\beta_{i,a+s}}} \qquad \langle YZ\rangle_{is} = -4g_{\alpha_{a+s,b}}$$

Proposition 72 (Case ♯3). *Let $\eta = -\varepsilon_a$ for $1 \le a \le n$. We order the pairs as follows $x_i = (\alpha_{ia}, \varepsilon_i), y_i = (\varepsilon_i, \beta_{ia}), z_j = (\varepsilon_{a+j}, \beta_{a,a+j})$ with $i = 1,..,a-1$ and $j = 1,..,n-a$. Then we have*

$$M_\eta = \begin{pmatrix} X & \langle XY\rangle & \langle XZ\rangle \\ \langle XY\rangle^T & Y & \langle YZ\rangle \\ \langle XZ\rangle^T & \langle YZ\rangle^T & Z \end{pmatrix}$$

with the diagonal matrices being

$$\begin{array}{lll} X \in Sym_{a-1}(\mathbb{R}) & X_{is} = 2g_{\alpha_{sa}} & \text{for } i \le s \\ Y \in Sym_{a-1}(\mathbb{R}) & Y_{is} = 2g_{\varepsilon_s} & \text{for } i \le s \\ Z \in Sym_{n-a}(\mathbb{R}) & Z_{is} = 2g_{\varepsilon_{a+s}} & \text{for } i \le s \end{array}$$

and the off diagonals being given by

$$\langle XY\rangle_{is} = \begin{cases} -2g_{\alpha_{ia}} & i = s \\ -2\frac{g_{\alpha_{ia}} g_{\varepsilon_s}}{g_{\beta_{si}}} & i \ne s \end{cases} \qquad \langle XZ\rangle_{is} = 2\frac{g_{\alpha_{ia}} g_{\varepsilon_{a+s}}}{g_{\beta_{i,a+s}}} \qquad \langle YZ\rangle_{is} = -2g_{\varepsilon_{a+s}}$$

Proposition 73 (Case ♯4). *Let $\eta = -2\varepsilon_a$ for $1 \le a \le n$. We order the pairs as follows $x_i = (\alpha_{ia}, \beta_{ia})$ for $i = 1...a-1$. Then we have*

$$(M_\eta)_{is} = \begin{cases} 0 & i = s \\ 4\frac{g_{\alpha_{sa}} g_{\beta_{sa}}}{g_{\beta_{is}}} & i < s \end{cases}$$

Proposition 74 (Cases ♯5 − ♯10). *Let $1 \le a < b < c < d \le n$. We order the pairs as given in the table from left to right. Then we have for $\eta = -\varepsilon_a + \varepsilon_b + \varepsilon_c - \varepsilon_d$:*

$$M_\eta = \begin{pmatrix} 0 & -4\frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\alpha_{ad}}} & 4\frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} \\ -4\frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\alpha_{ad}}} & 0 & -4\frac{g_{\alpha_{bd}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} \\ 4\frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} & -4\frac{g_{\alpha_{bd}} g_{\alpha_{ac}}}{g_{\alpha_{ab}}} & 0 \end{pmatrix}.$$

For $\eta = -\varepsilon_a + \varepsilon_b - \varepsilon_c + \varepsilon_d$:

$$M_\eta = \begin{pmatrix} 0 & -4\frac{g_{\alpha_{bc}} g_{\alpha_{ad}}}{g_{\beta_{ab}}} \\ -4\frac{g_{\alpha_{bc}} g_{\alpha_{ad}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a - \varepsilon_b + \varepsilon_c + \varepsilon_d$:

$$M_\eta = 0$$

For $\eta = -\varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d$:

$$M_\eta = \begin{pmatrix} 0 & -4\frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ad}}} & 4\frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} \\ -4\frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ad}}} & 0 & -4\frac{g_{\beta_{bd}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} \\ 4\frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} & -4\frac{g_{\beta_{bd}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a + \varepsilon_b - \varepsilon_c - \varepsilon_d$:

$$M_\eta = \begin{pmatrix} 0 & -4\frac{g_{\alpha_{bc}} g_{\beta_{ad}}}{g_{\beta_{ab}}} \\ -4\frac{g_{\alpha_{bc}} g_{\beta_{ad}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a - \varepsilon_b + \varepsilon_c - \varepsilon_d$:

$$M_\eta = 0$$

Since they are all one dimensional, we write the cases $\sharp 11 - \sharp 14$ in one matrix

Proposition 75 (Cases $\sharp 11 - \sharp 14$)**.** *We consider for $1 \leq a < b < c \leq n$ the ordering $(\alpha_{bc}, \alpha_{ab}), (\beta_{bc}, \alpha_{ab}), (\beta_{bc}, \alpha_{ac}), (\alpha_{bc}, \beta_{ac})$. The direct sum of the corresponding $M'_\eta s$ is given by*

$$\begin{pmatrix} -4\frac{g_{\alpha_{bc}}g_{\alpha_{ab}}}{g_{\alpha_{ac}}} & 0 & 0 & 0 \\ 0 & -4\frac{g_{\beta_{bc}}g_{\alpha_{ab}}}{g_{\beta_{ac}}} & 0 & 0 \\ 0 & 0 & -4\frac{g_{\beta_{bc}}g_{\alpha_{ac}}}{g_{\beta_{ab}}} & 0 \\ 0 & 0 & 0 & -4\frac{g_{\alpha_{bc}}g_{\beta_{ac}}}{g_{\beta_{ab}}} \end{pmatrix}$$

Proposition 76 (Cases $\sharp 15 - \sharp 17$)**.** *Let $1 \leq a < b < c \leq n$. We order the pairs as given in the table from left to right. Then we have for $\eta = -\varepsilon_a + \varepsilon_b + \varepsilon_c$:*

$$M_\eta = \begin{pmatrix} 0 & -2\frac{g_{\varepsilon_c}g_{\alpha_{ab}}}{g_{\varepsilon_a}} & 2\frac{g_{\varepsilon_c}g_{\alpha_{ab}}}{g_{\beta_{ac}}} \\ -2\frac{g_{\varepsilon_c}g_{\alpha_{ab}}}{g_{\varepsilon_a}} & 0 & -2\frac{g_{\varepsilon_b}g_{\alpha_{ac}}}{g_{\beta_{ab}}} \\ 2\frac{g_{\varepsilon_c}g_{\alpha_{ab}}}{g_{\beta_{ac}}} & -2\frac{g_{\varepsilon_b}g_{\alpha_{ac}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}.$$

For $\eta = -\varepsilon_a + \varepsilon_b - \varepsilon_c$:

$$M_\eta = \begin{pmatrix} 0 & -2\frac{g_{\varepsilon_a}g_{\alpha_{bc}}}{g_{\beta_{ab}}} \\ -2\frac{g_{\varepsilon_a}g_{\alpha_{bc}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a - \varepsilon_b + \varepsilon_c$:

$$M_\eta = 0$$

Proposition 77 (Cases $\sharp 18$)**.** *Let $1 \leq a < b \leq n$. We order the pairs as given in the table from left to right. Then we have for $\eta = -\varepsilon_a + 2\varepsilon_b$:*

$$M_\eta = -2\frac{g_{\varepsilon_b}g_{\alpha_{ab}}}{g_{\varepsilon_a}}$$

Proposition 78. *The entries of the matrix of the trivial module are determined by the diagonals of the matrices above :*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = (M_{\alpha-\beta})_{(\alpha\beta)(\alpha\beta)}$$

except for the diagonals. For $\alpha \in \Delta^+_{\mathfrak{m}}$ the following holds

$$(M_0)_{(\alpha\alpha)(\alpha\alpha)} = \begin{cases} 8g_\alpha & \alpha \in \{\alpha_{ij}, \beta_{ij}\} \\ g_\alpha & \alpha = \varepsilon_i \end{cases}$$

General modifications of $SO(2n+1)$

The modifications described in the beginning of the section in the case of $SO(2n+1)$ are the following:

1.) We move the diagonals of $\sharp 11 - 14$ and 18 into the matrix of the trivial module.

2.) The off diagonals of $\sharp 4$ to the diagonal of the $\langle WZ \rangle$ block of $\sharp 1$.

3.) The off diagonals of $\sharp 5$ to the $\langle VW \rangle$ block of $\sharp 1$, the $\langle XZ \rangle$ block and the upper triangular part of the $\langle XY \rangle$ block of $\sharp 2$.

4.) The off diagonal of $\sharp 6$ the lower triangular part of the $\langle XY \rangle$ block of $\sharp 2$.

5.) The off diagonals of $\sharp 8$ to the $\langle VZ \rangle$ block, the $\langle VX \rangle$ block and the upper triangular part of the $\langle VY \rangle$ block of $\sharp 1$.

6.) The off diagonal of $\sharp 9$ the lower triangular part of the $\langle VY\rangle$ block of $\sharp 1$.

7.) The off diagonals of $\sharp 15$ to the $\langle VD\rangle$ block of $\sharp 1$, the $\langle XZ\rangle$ block and the upper triangular part of the $\langle XY\rangle$ block of $\sharp 3$.

8.) The off diagonal of $\sharp 9$ the lower triangular part of the $\langle XY\rangle$ block of $\sharp 3$.

We remark that similarly to the modifications of $SU(n+1)$, in each step have a bijection between the referenced entries of all matrices of the mentioned case $\sharp i$.

Then the modified holomorphic curvature tensor is given by the following matrices

Proposition 79 (Case $\sharp 1$). *Let $\eta=-\varepsilon_a+\varepsilon_b$ for $1\le a<b\le n$. We order the pairs as follows $v_i=(\alpha_{ia},\alpha_{ib}), w_j=(\alpha_{b,b+j},\alpha_{a,b+j}), x_t=(\beta_{a+t,b},\beta_{a,a+t}), y_i=(\beta_{ib},\beta_{ia}), z_j=(\beta_{b,b+j},\beta_{a,b+j})$ and $D_1=(\varepsilon_b,\varepsilon_a)$ with $i=1,..,a-1, t=1,..,b-a-1$ and $j=1,..,n-b$. Then we have*

$$M_\eta=\begin{pmatrix} V & \langle VW\rangle & \langle VX\rangle & \langle VY\rangle & \langle VZ\rangle & \langle VD\rangle \\ \langle VW\rangle^T & W & \langle WX\rangle & \langle WY\rangle & \langle WZ\rangle & \langle WD\rangle \\ \langle VX\rangle^T & \langle WX\rangle^T & X & \langle XY\rangle & \langle XZ\rangle & \langle XD\rangle \\ \langle VY\rangle^T & \langle WY\rangle^T & \langle XY\rangle^T & Y & \langle YZ\rangle & \langle YD\rangle \\ \langle VZ\rangle^T & \langle WZ\rangle^T & \langle XZ\rangle^T & \langle YZ\rangle^T & Z & \langle ZD\rangle \\ \langle VD\rangle^T & \langle WD\rangle^T & \langle XD\rangle^T & \langle YD\rangle^T & \langle ZD\rangle^T & D \end{pmatrix}$$

with the diagonal matrices being

$$\begin{aligned} V &\in Sym_{a-1}(\mathbb{R}) & V_{is} &= 4g_{\alpha_{sa}} \text{ for } i\le s\\ W &\in Sym_{n-b}(\mathbb{R}) & W_{is} &= 4g_{\alpha_{b,b+i}} \text{ for } i\le s\\ X &\in Sym_{b-a-1}(\mathbb{R}) & X_{is} &= 4g_{\beta_{a+s,b}} \text{ for } i\le s\\ Y &\in Sym_{a-1}(\mathbb{R}) & Y_{is} &= 4g_{\beta_{sb}} \text{ for } i\le s\\ Z &\in Sym_{n-b}(\mathbb{R}) & Z_{is} &= 4g_{\beta_{b,b+s}} \text{ for } i\le s\\ D &\in Sym_1(\mathbb{R}) & D_{11} &= \frac{(g_{\varepsilon_b})^2}{g_{\beta_{ab}}} \end{aligned}$$

and the off diagonals being given by the following where i and s vary in the ranges determined by the diagonals

$$\begin{aligned} &\langle VW\rangle_{is}=-8\frac{g_{\alpha_{ia}}g_{\alpha_{b,b+s}}}{g_{\alpha_{i,b+s}}} && \langle VX\rangle_{is}=8\frac{g_{\alpha_{ia}}g_{\beta_{a+s,b}}}{g_{\beta_{i,a+s}}} && \langle VY\rangle_{is}=\begin{cases}-4g_{\alpha_{ia}}\\ -8\frac{g_{\alpha_{ia}}g_{\beta_{sb}}}{g_{\beta_{is}}}\end{cases}\\ &\langle VZ\rangle_{is}=-8\frac{g_{\alpha_{ia}}g_{\beta_{b,b+s}}}{g_{\beta_{i,b+s}}} && \langle VD\rangle_{i1}=-4\frac{g_{\alpha_{ia}}g_{\varepsilon_b}}{g_{\varepsilon_i}} && \langle WX\rangle_{is}=-4g_{\alpha_{b,b+i}}\\ &\langle WY\rangle_{is}=4g_{\alpha_{b,b+i}} && \langle WZ\rangle_{is}=\begin{cases}8\frac{g_{\alpha_{b,b+i}}g_{\beta_{b,b+i}}}{g_{\beta_{ab}}}\\ 4g_{\alpha_{b,b+i}}\end{cases} && \langle WD\rangle_{i1}=2g_{\alpha_{b,b+i}}\\ &\langle XY\rangle_{is}=-4g_{\beta_{a+i,b}} && \langle XZ\rangle_{is}=-4g_{\beta_{b,b+s}} && \langle XD\rangle_{i1}=-2g_{\varepsilon_b}\\ &\langle YZ\rangle_{is}=4g_{\beta_{b,b+s}} && \langle YD\rangle_{i1}=2g_{\varepsilon_b} && \langle ZD\rangle_{i1}=2g_{\varepsilon_b} \end{aligned}$$

In the two case distinctions the upper case corresponds to $i=s$ and the lower case to $i\neq s$.

Proposition 80 (Case $\sharp 2$). *Let $\eta=-\varepsilon_a-\varepsilon_b$ for $1\le a<b\le n$. We order the pairs as follows $x_i=(\alpha_{ia},\beta_{ib}), y_i=(\alpha_{ib},\beta_{ia}), z_t=(\alpha_{a+t,b},\beta_{a,a+t})$ with $i=1,..,a-1$ and $t=1,..,b-a-1$. Then we have*

$$M_\eta=\begin{pmatrix} X & \langle XY\rangle & \langle XZ\rangle \\ \langle XY\rangle^T & Y & \langle YZ\rangle \\ \langle XZ\rangle^T & \langle YZ\rangle^T & Z \end{pmatrix}$$

with the diagonal matrices being

$$\begin{array}{lll} X \in Sym_{a-1}(\mathbb{R}) & X_{is} = 4g_{\alpha_{sa}} & \text{for } i \leq s \\ Y \in Sym_{a-1}(\mathbb{R}) & Y_{is} = 4g_{\alpha_{sb}} & \text{for } i \leq s \\ Z \in Sym_{b-a-1}(\mathbb{R}) & Z_{is} = 4g_{\alpha_{a+s,b}} & \text{for } i \leq s \end{array}$$

and the off diagonals being given by

$$\langle XY\rangle_{is} = \begin{cases} -4g_{\alpha_{sa}} & i = s \\ -8\frac{g_{\alpha_{sb}}g_{\alpha_{ia}}}{g_{\beta_{si}}} & i \neq s \end{cases} \qquad \langle XZ\rangle_{is} = 8\frac{g_{\alpha_{ia}}g_{\alpha_{a+s,b}}}{g_{\beta_{i,a+s}}} \qquad \langle YZ\rangle_{is} = -4g_{\alpha_{a+s,b}}$$

Proposition 81 (Case ♯3)**.** *Let* $\eta = -\varepsilon_a$ *for* $1 \leq a \leq n$. *We order the pairs as follows* $x_i = (\alpha_{ia}, \varepsilon_i), y_i = (\varepsilon_i, \beta_{ia}), z_j = (\varepsilon_{a+j}, \beta_{a,a+j})$ *with* $i = 1, .., a-1$ *and* $j = 1, .., n-a$ *. Then we have*

$$M_\eta = \begin{pmatrix} X & \langle XY\rangle & \langle XZ\rangle \\ \langle XY\rangle^T & Y & \langle YZ\rangle \\ \langle XZ\rangle^T & \langle YZ\rangle^T & Z \end{pmatrix}$$

with the diagonal matrices being

$$\begin{array}{lll} X \in Sym_{a-1}(\mathbb{R}) & X_{is} = 2g_{\alpha_{sa}} & \text{for } i \leq s \\ Y \in Sym_{a-1}(\mathbb{R}) & Y_{is} = 2g_{\varepsilon_s} & \text{for } i \leq s \\ Z \in Sym_{n-a}(\mathbb{R}) & Z_{is} = 2g_{\varepsilon_{a+s}} & \text{for } i \leq s \end{array}$$

and the off diagonals being given by

$$\langle XY\rangle_{is} = \begin{cases} -2g_{\alpha_{ia}} & i = s \\ -4\frac{g_{\alpha_{ia}}g_{\varepsilon_s}}{g_{\beta_{si}}} & i \neq s \end{cases} \qquad \langle XZ\rangle_{is} = 4\frac{g_{\alpha_{ia}}g_{\varepsilon_{a+s}}}{g_{\beta_{i,a+s}}} \qquad \langle YZ\rangle_{is} = -2g_{\varepsilon_{a+s}}$$

Proposition 82. *We give the entries of the matrix of the trivial module just for the upper triangular part by symmetry, i.e. for the pairs* (α, β) *with* $\alpha < \beta$.

i) Along the diagonal we have

$$(M_0)_{(\alpha\alpha)(\alpha\alpha)} = \begin{cases} 8g_\alpha & \alpha \in \{\alpha_{ij}, \beta_{ij}\} \\ g_\alpha & \alpha = \varepsilon_i \end{cases}.$$

ii) For the following pairs $(\alpha, \beta) \in \{(\alpha_{ia}, \alpha_{ib}), (\alpha_{bj}, \alpha_{aj}), (\beta_{kb}, \beta_{ak}), (\beta_{ib}, \beta_{ia}), (\beta_{bj}, \beta_{aj}), (\alpha_{ia}, \beta_{ib}), (\alpha_{ib}, \beta_{ia}), (\alpha_{kb}, \beta_{ak})\}$ *we have*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = 4g_\alpha$$

iii) For the following pairs $(\alpha, \beta) \in \{(\alpha_{ia}, \varepsilon_i), (\varepsilon_i, \beta_{ia}), (\varepsilon_j, \beta_{aj})\}$ *we have*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = 2g_\alpha$$

iv) For the following pairs $(\alpha, \beta) \in \{(\alpha_{ib}, \alpha_{ai}), (\beta_{ib}, \alpha_{ai}), (\beta_{bj}, \alpha_{aj})(\alpha_{bj}, \beta_{aj})\}$ *we have*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = -8\frac{g_\alpha g_\beta}{g_{\alpha+\beta}}$$

v) For the following pairs $(\alpha, \beta) = (\varepsilon_j, \alpha_{aj})$ *we have*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = -4\frac{g_\alpha g_\beta}{g_{\alpha+\beta}}$$

vi) For the following pairs $(\alpha, \beta) = (\varepsilon_b, \varepsilon_a)$ *we have*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = \frac{(g_\alpha)^2}{g_{\alpha+\beta}}$$

Notice that the entries in the trivial module that have been doubled are *iv*) and *v*).

7.3 $Sp(n)$

As for $SU(n+1)$ we define the ordering on the positive roots as in proposition 15 with the slight modification that we use the ordered basis ε_j with $j = 1,..,n$ instead of the simple roots to induce the ordering. It is easy to see that the desired properties are still true.

Proposition 83. *For the C space $(Sp(n), \mathbb{T}^n, J_{std})$ let η be a non zero weight of $Fix(J_{std})$, i.e an element in*

$$\Delta_H = \{\alpha - \beta \mid \alpha < \beta \in \Delta_{\mathfrak{g}}^+\}.$$

Then η is in the following table together with the pairs $\alpha < \beta$ with $\alpha - \beta = \eta$:

$\sharp$	η	$I(\eta)$	n_η
1	$-\varepsilon_a + \varepsilon_b$	$(\alpha_{ia}, \alpha_{ib})$, $i = 1,..,a-1$ $(\alpha_{bj}, \alpha_{aj})$, $j = b+1,..,n$ (β_{tb}, β_{at}) , $t = a+1,..,b-1$ (β_{sb}, β_{sa}) , $s = 1,..,a-1$ (β_{br}, β_{ar}) , $r = b+1,..,n$ (β_{ab}, γ_a) (γ_b, β_{ab})	$2n - b + a - 1$
2	$-\varepsilon_a - \varepsilon_b$	$(\alpha_{sa}, \beta_{sb})$, $s = 1,..,a-1$ $(\alpha_{sb}, \beta_{sa})$, $s = 1,..,a-1$ $(\alpha_{tb}, \beta_{at})$, $t = a+1,..,b-1$ (α_{ab}, γ_a)	$b + a - 2$
3	$-2\varepsilon_a$	$(\alpha_{sa}, \beta_{sa})$, $s = 1,..,a-1$	$a - 1$
4	$-\varepsilon_a + \varepsilon_b + \varepsilon_c - \varepsilon_d$	$(\alpha_{cd}, \alpha_{ab}), (\alpha_{bd}, \alpha_{ac}), (\beta_{bc}, \beta_{ad})$	3
5	$-\varepsilon_a + \varepsilon_b - \varepsilon_c + \varepsilon_d$	$(\alpha_{bc}, \alpha_{ad}), (\beta_{bd}, \beta_{ac})$	2
6	$-\varepsilon_a - \varepsilon_b + \varepsilon_c + \varepsilon_d$	(β_{cd}, β_{ab})	1
7	$-\varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d$	$(\beta_{cd}, \alpha_{ab}), (\beta_{bd}, \alpha_{ac}), (\beta_{bc}, \alpha_{ad})$	3
8	$-\varepsilon_a + \varepsilon_b - \varepsilon_c - \varepsilon_d$	$(\alpha_{bc}, \beta_{ad}), (\alpha_{bd}, \beta_{ac})$	2
9	$-\varepsilon_a - \varepsilon_b + \varepsilon_c - \varepsilon_d$	$(\alpha_{cd}, \beta_{ab})$	1
10	$-\varepsilon_a + 2\varepsilon_b - \varepsilon_c$	$(\alpha_{bc}, \alpha_{ab})(\gamma_b, \beta_{ac})$	2
11	$-\varepsilon_a + 2\varepsilon_b + \varepsilon_c$	$(\beta_{bc}, \alpha_{ab})(\gamma_b, \alpha_{ac})$	2
12	$-\varepsilon_a + \varepsilon_b + 2\varepsilon_c$	$(\beta_{bc}, \alpha_{ac})(\gamma_c, \alpha_{ab})$	2
13	$-\varepsilon_a + \varepsilon_b - 2\varepsilon_c$	$(\alpha_{bc}, \beta_{ac})$	1
14	$-\varepsilon_a - \varepsilon_b + 2\varepsilon_c$	(γ_c, β_{ab})	1
15	$-2\varepsilon_a + \varepsilon_b + \varepsilon_c$	(β_{bc}, γ_a)	1
16	$-2\varepsilon_a + \varepsilon_b - \varepsilon_c$	(α_{bc}, γ_a)	1
17	$-2\varepsilon_a + 2\varepsilon_b$	(γ_b, γ_a)	1
18	$-\varepsilon_a + 3\varepsilon_b$	(γ_b, α_{ab})	1

Proof. The elements in the list are obviously weights and these are all of them for dimensional reasons. The elements in the list are obviously weights and these are all of them for dimensional reasons. In fact, from (4.1) we only have to verify that

$$\frac{|\Delta_{\mathfrak{g}}^+|\left(|\Delta_{\mathfrak{g}}^+|-1\right)}{2} = \sum_{\eta\in\Delta_H} n_\eta$$

holds. In fact, we know $|\Delta_{\mathfrak{g}}^+| = n^2$ and the right hand side is this time given by

$$\sum_{a=1}^{n-1}\sum_{b=a+1}^{n}(2n+2a-1) + \sum_{a=1}^{n}(a-1)$$
$$+10\binom{n}{3} + 12\binom{n}{4}$$

Easy calculations yield that left and right hand side coincide and therefore we found all weights. □

In the following, we determine the curvature matrices M_η. Since the indices of M_η are pairs of roots (α,β) with $\alpha<\beta$ and $\alpha-\beta=\eta$ it makes sense to choose an ordering of these pairs. We will do so in each M_η separately. The proofs reduce to plugging in the values of g_α, $N_{\alpha,\beta}$ and z_α as determined for the classical groups in section 1.8 into the equations of proposition 16 and will therefore be omitted. An exemplary calculation was made in the proof of proposition 44.

Proposition 84 (Case ♯1)**.** *Let $\eta = -\varepsilon_a+\varepsilon_b$ for $1\le a<b\le n$. We order the pairs as follows $v_i = (\alpha_{ia},\alpha_{ib}), w_j = (\alpha_{b,b+j},\alpha_{a,b+j}), x_t = (\beta_{a+t,b},\beta_{a,a+t}), y_i = (\beta_{ib},\beta_{ia}), z_j = (\beta_{b,b+j},\beta_{a,b+j})$, $D_1 = (\beta_{ab},\gamma_a)$ and $E_1 = (\gamma_b,\beta_{ab})$ with $i = 1...(a-1), t = 1...(b-a-1)$ and $j = 1...n-b$. Then we have*

$$M_\eta = \begin{pmatrix} V & \langle VW\rangle & \langle VX\rangle & \langle VY\rangle & \langle VZ\rangle & \langle VD\rangle & \langle VE\rangle \\ \langle VW\rangle^T & W & \langle WX\rangle & \langle WY\rangle & \langle WZ\rangle & \langle WD\rangle & \langle WE\rangle \\ \langle VX\rangle^T & \langle WX\rangle^T & X & \langle XY\rangle & \langle XZ\rangle & \langle XD\rangle & \langle XE\rangle \\ \langle VY\rangle^T & \langle WY\rangle^T & \langle XY\rangle^T & Y & \langle YZ\rangle & \langle YD\rangle & \langle YE\rangle \\ \langle VZ\rangle^T & \langle WZ\rangle^T & \langle XZ\rangle^T & \langle YZ\rangle^T & Z & \langle ZD\rangle & \langle ZE\rangle \\ \langle VD\rangle^T & \langle WD\rangle^T & \langle XD\rangle^T & \langle YD\rangle^T & \langle ZD\rangle^T & D & \langle DE\rangle \\ \langle VE\rangle^T & \langle WE\rangle^T & \langle XE\rangle^T & \langle YE\rangle^T & \langle ZE\rangle^T & \langle DE\rangle & E \end{pmatrix}$$

with the diagonal matrices being

$V \in Sym_{a-1}(\mathbb{R})$	$V_{is} = g_{\alpha_{sa}}$	*for* $i\le s$
$W \in Sym_{n-b}(\mathbb{R})$	$W_{is} = g_{\alpha_{b,b+i}}$	*for* $i\le s$
$X \in Sym_{b-a-1}(\mathbb{R})$	$X_{is} = g_{\beta_{a+s,b}}$	*for* $i\le s$
$Y \in Sym_{a-1}(\mathbb{R})$	$Y_{is} = g_{\beta_{sb}}$	*for* $i\le s$
$Z \in Sym_{n-b}(\mathbb{R})$	$Z_{is} = g_{\beta_{b,b+s}}$	*for* $i\le s$
$D \in Sym_1(\mathbb{R})$	$D_{11} = g_{\beta_{ab}}$	
$E \in Sym_1(\mathbb{R})$	$E_{11} = g_{\gamma_b}$	

and the off diagonals being given by the following where i and s vary in the ranges determined *by the diagonals*

$$\langle VW\rangle_{is} = -\frac{g_{\alpha_{ia}}g_{\alpha_{b,b+s}}}{g_{\alpha_{i,b+s}}} \quad \langle VX\rangle_{is} = -\frac{g_{\alpha_{ia}}g_{\beta_{a+s,b}}}{g_{\beta_{i,a+s}}} \quad \langle VY\rangle_{is} = \begin{cases} -\frac{g_{\alpha_{ia}}g_{\gamma_b}}{g_{\gamma_i}} \\ -\frac{g_{\alpha_{ia}}g_{\beta_{sb}}}{g_{\beta_{is}}} \end{cases}$$

$$\langle VZ\rangle_{is} = -\frac{g_{\alpha_{ia}}g_{\beta_{b,b+s}}}{g_{\beta_{i,b+s}}} \quad \langle VD\rangle_{i1} = -\frac{g_{\alpha_{ia}}g_{\beta_{ab}}}{g_{\beta_{ia}}} \quad \langle VE\rangle_{i1} = -\frac{g_{\alpha_{ia}}g_{\gamma_b}}{g_{\beta_{ib}}}$$

$$\langle WX\rangle_{is} = g_{\alpha_{b,b+i}} \quad \langle WY\rangle_{is} = g_{\alpha_{b,b+i}} \quad \langle WZ\rangle_{is} = \begin{cases} \frac{g_{\alpha_{b,b+i}}(g_{\alpha_{a,b+i}}+g_{\beta_{ab}})}{g_{\beta_{ab}}} \\ g_{\alpha_{b,b+i}} \end{cases}$$

$$\langle WD\rangle_{i1} = g_{\alpha_{b,b+i}} \quad \langle WE\rangle_{i1} = g_{\alpha_{b,b+i}} \quad \langle XY\rangle_{is} = g_{\beta_{a+i,b}}$$
$$\langle XZ\rangle_{is} = g_{\beta_{b,b+s}} \quad \langle XD\rangle_{i1} = g_{\beta_{a+i,b}} \quad \langle XE\rangle_{i1} = g_{\gamma_b}$$
$$\langle YZ\rangle_{is} = g_{\beta_{b,b+s}} \quad \langle YD\rangle_{i1} = g_{\beta_{ab}} \quad \langle YE\rangle_{i1} = g_{\gamma_b}$$
$$\langle ZD\rangle_{i1} = g_{\beta_{b,b+i}} \quad \langle ZE\rangle_{i1} = g_{\beta_{b,b+i}} \quad \langle DE\rangle_{i1} = g_{\gamma_b}$$

In the two case distinctions the upper case corresponds to $i = s$ and the lower case to $i \neq s$.

Proposition 85 (Case ♯2)**.** *Let $\eta = -\varepsilon_a - \varepsilon_b$ for $1 \leq a < b \leq n$. We order the pairs as follows $x_i = (\alpha_{ia}, \beta_{ib}), y_i = (\alpha_{ib}, \beta_{ia}), z_t = (\alpha_{a+t,b}, \beta_{a,a+t})$ and $D_1 = (\alpha_{ab}, \gamma_a)$ with $i = 1, .., a-1$ and $t = 1, .., b-a-1$. Then we have*

$$M_\eta = \begin{pmatrix} X & \langle XY\rangle & \langle XZ\rangle & \langle XD\rangle \\ \langle XY\rangle^T & Y & \langle YZ\rangle & \langle YD\rangle \\ \langle XZ\rangle^T & \langle YZ\rangle^T & Z & \langle ZD\rangle \\ \langle XD\rangle^T & \langle YD\rangle^T & \langle ZD\rangle^T & D \end{pmatrix}$$

with the diagonal matrices being

$$\begin{array}{lll} X \in Sym_{a-1}(\mathbb{R}) & X_{is} = g_{\alpha_{sa}} & \text{for } i \leq s \\ Y \in Sym_{a-1}(\mathbb{R}) & Y_{is} = g_{\alpha_{sb}} & \text{for } i \leq s \\ Z \in Sym_{b-a-1}(\mathbb{R}) & Z_{is} = g_{\alpha_{a+s,b}} & \text{for } i \leq s \\ D \in Sym_1(\mathbb{R}) & D_{11} = g_{\alpha_{ab}} & \end{array}$$

and the off diagonals being given by

$$\langle XY\rangle_{is} = \begin{cases} -\frac{g_{\alpha_{ia}}g_{\gamma_b}}{g_{\gamma_i}} & i = s \\ \frac{g_{\alpha_{sb}}g_{\alpha_{ia}}}{g_{\beta_{si}}} & i \neq s \end{cases} \quad \langle XZ\rangle_{is} = \frac{g_{\alpha_{ia}}g_{\alpha_{a+s,b}}}{g_{\beta_{i,a+s}}} \quad \langle XD\rangle_{i1} = \frac{g_{\alpha_{ia}}g_{\alpha_{ab}}}{g_{\beta_{ia}}}$$

$$\langle YZ\rangle_{is} = g_{\alpha_{a+s,b}} \quad \langle YD\rangle_{i1} = g_{\alpha_{ab}} \quad \langle ZD\rangle_{i1} = g_{\alpha_{a+s,b}}$$

Proposition 86 (Case ♯3)**.** *Let $\eta = -2\varepsilon_a$ for $1 \leq a \leq n$. We order the pairs as follows $x_i = (\alpha_{ia}, \beta_{ia})$ for $i = 1...a-1$. Then we have*

$$(M_\eta)_{is} = \begin{cases} 2\frac{(g_{\alpha_{sa}})^2}{g_{\gamma_s}} & i = s \\ \frac{g_{\alpha_{sa}}(2g_{\beta_{is}} - g_{\beta_{sa}})}{g_{\beta_{is}}} & i < s \end{cases}$$

Proposition 87 (Cases ♯4 − ♯9)**.** *Let $1 \leq a < b < c < d \leq n$. We order the pairs as given in the table from left to right. Then we have for $\eta = -\varepsilon_a + \varepsilon_b + \varepsilon_c - \varepsilon_d$:*

$$M_\eta = \begin{pmatrix} 0 & -\frac{g_{\alpha_{cd}}g_{\alpha_{ab}}}{g_{\alpha_{ad}}} & \frac{g_{\alpha_{cd}}g_{\alpha_{ab}}}{g_{\beta_{ac}}} \\ -\frac{g_{\alpha_{cd}}g_{\alpha_{ab}}}{g_{\alpha_{ad}}} & 0 & \frac{g_{\alpha_{bd}}g_{\alpha_{ac}}}{g_{\beta_{ab}}} \\ \frac{g_{\alpha_{cd}}g_{\alpha_{ab}}}{g_{\beta_{ac}}} & \frac{g_{\alpha_{bd}}g_{\alpha_{ac}}}{g_{\alpha_{ab}}} & 0 \end{pmatrix}.$$

For $\eta = -\varepsilon_a + \varepsilon_b - \varepsilon_c + \varepsilon_d$:

$$M_\eta = \begin{pmatrix} 0 & \frac{g_{\alpha_{bc}} g_{\alpha_{ad}}}{g_{\beta_{ab}}} \\ \frac{g_{\alpha_{bc}} g_{\alpha_{ad}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a - \varepsilon_b + \varepsilon_c + \varepsilon_d$:

$$M_\eta = 0$$

For $\eta = -\varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d$:

$$M_\eta = \begin{pmatrix} 0 & -\frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ad}}} & -\frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} \\ -\frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ad}}} & 0 & -\frac{g_{\beta_{bd}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} \\ -\frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} & -\frac{g_{\beta_{bd}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a + \varepsilon_b - \varepsilon_c - \varepsilon_d$:

$$M_\eta = \begin{pmatrix} 0 & -\frac{g_{\alpha_{bc}} g_{\beta_{ad}}}{g_{\beta_{ab}}} \\ -\frac{g_{\alpha_{bc}} g_{\beta_{ad}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a - \varepsilon_b + \varepsilon_c - \varepsilon_d$:

$$M_\eta = 0$$

Proposition 88 (Cases $\sharp 10 - \sharp 12$)**.** *Let* $1 \leq a < b < c \leq n$. *We order the pairs as given in the table from left to right. Then we have for* $\eta = -\varepsilon_a + 2\varepsilon_b - \varepsilon_c$:

$$M_\eta = \begin{pmatrix} -\frac{g_{\alpha_{bc}} g_{\alpha_{ab}}}{g_{\alpha_{ac}}} & \frac{g_{\alpha_{bc}} g_{\alpha_{ab}}}{g_{\beta_{ab}}} \\ \frac{g_{\alpha_{bc}} g_{\alpha_{ab}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a + 2\varepsilon_b + \varepsilon_c$:

$$M_\eta = \begin{pmatrix} -\frac{g_{\beta_{bc}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} & -\frac{g_{\beta_{bc}} g_{\alpha_{ab}}}{g_{\beta_{ab}}} \\ -\frac{g_{\beta_{bc}} g_{\alpha_{ab}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a + \varepsilon_b + 2\varepsilon_c$:

$$M_\eta = \begin{pmatrix} -\frac{g_{\beta_{bc}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} & -\frac{g_{\gamma_c} g_{\alpha_{ab}}}{g_{\beta_{ac}}} \\ -\frac{g_{\gamma_c} g_{\alpha_{ab}}}{g_{\beta_{ac}}} & 0 \end{pmatrix}$$

Since they are all one dimensional, we write the cases $\sharp 13 - \sharp 16$ in one matrix

Proposition 89 (Cases $\sharp 13 - \sharp 16$)**.** *We consider for* $1 \leq a < b < c \leq n$ *the ordering* $(\alpha_{bc}, \beta_{ac}), (\gamma_c, \beta_{ab}), (\beta_{bc}, \gamma_a), (\alpha_{bc}, \gamma_a)$. *The direct sum of the corresponding* M_η'*s is given by*

$$\begin{pmatrix} -\frac{g_{\alpha_{bc}} g_{\beta_{ac}}}{g_{\beta_{ab}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition 90 (Cases $\sharp 17 - \sharp 18$)**.** *We consider for* $1 \leq a < b \leq n$ *the ordering* (γ_b, γ_a), (γ_b, α_{ab}). *The direct sum of the corresponding* M_η'*s is given by*

$$\begin{pmatrix} 0 & 0 \\ 0 & -\frac{g_{\gamma_b} g_{\alpha_{ab}}}{g_{\beta_{ab}}} \end{pmatrix}$$

Proposition 91. *The entries of the matrix of the trivial module are determined by the diagonals of the matrices above :*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = (M_{\alpha-\beta})_{(\alpha\beta)(\alpha\beta)}$$

except for the diagonals. For $\alpha \in \Delta^+_{\mathfrak{m}}$ the following holds

$$(M_0)_{(\alpha\alpha)(\alpha\alpha)} = \begin{cases} 2g_\alpha & \alpha \in \{\alpha_{ij}, \beta_{ij}\} \\ g_\alpha & \alpha = \gamma_i \end{cases}$$

General modifications of $Sp(n)$

After completing the steps described in the beginning of the section in the case of $Sp(n)$, that is

1.) We move the diagonals of $\sharp 10 - 18$ into the matrix of the trivial module.

2.) The off diagonals of $\sharp 4$ to the $\langle VW \rangle$ block of $\sharp 1$,the $\langle XZ \rangle$ block and the upper triangular part of the $\langle XY \rangle$ block of $\sharp 2$.

3.) The off diagonal of $\sharp 5$ the lower triangular part of the $\langle XY \rangle$ block of $\sharp 2$.

4.) The off diagonals of $\sharp 7$ to the $\langle VZ \rangle$ block, the $\langle VX \rangle$ block and the upper triangular part of the $\langle VY \rangle$ block of $\sharp 1$.

5.) The off diagonal of $\sharp 8$ the lower triangular part of the $\langle VY \rangle$ block of $\sharp 1$.

6.) The off diagonals of $\sharp 10$ to the $\langle XD \rangle$ block of $\sharp 2$.

7.) The off diagonals of $\sharp 11$ to the $\langle VD \rangle$ block of $\sharp 1$.

8.) The off diagonals of $\sharp 12$ to the $\langle VE \rangle$ block of $\sharp 1$.

We remark that, similarly to the modifications of $SU(n+1)$, in each step we have a bijection between the referenced entries of all matrices of the mentioned cases $\sharp i$.

Then the modified holomorphic curvature tensor is given by the following matrices

Proposition 92 (Case $\sharp 1$)**.** *Let $\eta = -\varepsilon_a + \varepsilon_b$ for $1 \le a < b \le n$. We order the pairs as follows $v_i = (\alpha_{ia}, \alpha_{ib}), w_j = (\alpha_{b,b+j}, \alpha_{a,b+j}), x_t = (\beta_{a+t,b}, \beta_{a,a+t}), y_i = (\beta_{ib}, \beta_{ia}), z_j = (\beta_{b,b+j}, \beta_{a,b+j})$, $D_1 = (\beta_{ab}, \gamma_a)$ and $E_1 = (\gamma_b, \beta_{ab})$ with $i = 1...(a-1), t = 1...(b-a-1)$ and $j = 1...n-b$. Then we have*

$$M_\eta = \begin{pmatrix} V & \langle VW \rangle & \langle VX \rangle & \langle VY \rangle & \langle VZ \rangle & \langle VD \rangle & \langle VE \rangle \\ \langle VW \rangle^T & W & \langle WX \rangle & \langle WY \rangle & \langle WZ \rangle & \langle WD \rangle & \langle WE \rangle \\ \langle VX \rangle^T & \langle WX \rangle^T & X & \langle XY \rangle & \langle XZ \rangle & \langle XD \rangle & \langle XE \rangle \\ \langle VY \rangle^T & \langle WY \rangle^T & \langle XY \rangle^T & Y & \langle YZ \rangle & \langle YD \rangle & \langle YE \rangle \\ \langle VZ \rangle^T & \langle WZ \rangle^T & \langle XZ \rangle^T & \langle YZ \rangle^T & Z & \langle ZD \rangle & \langle ZE \rangle \\ \langle VD \rangle^T & \langle WD \rangle^T & \langle XD \rangle^T & \langle YD \rangle^T & \langle ZD \rangle^T & D & \langle DE \rangle \\ \langle VE \rangle^T & \langle WE \rangle^T & \langle XE \rangle^T & \langle YE \rangle^T & \langle ZE \rangle^T & \langle DE \rangle & E \end{pmatrix}$$

with the diagonal matrices being

$V \in Sym_{a-1}(\mathbb{R})$	$V_{is} = g_{\alpha_{sa}}$	*for* $i \le s$
$W \in Sym_{n-b}(\mathbb{R})$	$W_{is} = g_{\alpha_{b,b+i}}$	*for* $i \le s$
$X \in Sym_{b-a-1}(\mathbb{R})$	$X_{is} = g_{\beta_{a+s,b}}$	*for* $i \le s$
$Y \in Sym_{a-1}(\mathbb{R})$	$Y_{is} = g_{\beta_{sb}}$	*for* $i \le s$
$Z \in Sym_{n-b}(\mathbb{R})$	$Z_{is} = g_{\beta_{b,b+s}}$	*for* $i \le s$
$D \in Sym_1(\mathbb{R})$	$D_{11} = g_{\beta_{ab}}$	
$E \in Sym_1(\mathbb{R})$	$E_{11} = g_{\gamma_b}$	

and the off diagonals being given by the following where i and s vary in the ranges determined by the diagonals

$$\langle VW\rangle_{is} = -2\frac{g_{\alpha_{ia}}g_{\alpha_{b,b+s}}}{g_{\alpha_{i,b+s}}} \quad \langle VX\rangle_{is} = -2\frac{g_{\alpha_{ia}}g_{\beta_{a+s,b}}}{g_{\beta_{i,a+s}}} \quad \langle VY\rangle_{is} = \begin{cases} -\frac{g_{\alpha_{ia}}g_{\gamma_b}}{g_{\gamma_i}} \\ -2\frac{g_{\alpha_{ia}}g_{\beta_{sb}}}{g_{\beta_{is}}} \end{cases}$$

$$\langle VZ\rangle_{is} = -2\frac{g_{\alpha_{ia}}g_{\beta_{b,b+s}}}{g_{\beta_{i,b+s}}} \quad \langle VD\rangle_{i1} = -2\frac{g_{\alpha_{ia}}g_{\beta_{ab}}}{g_{\beta_{ia}}} \quad \langle VE\rangle_{i1} = -2\frac{g_{\alpha_{ia}}g_{\gamma_b}}{g_{\beta_{ib}}}$$

$$\langle WX\rangle_{is} = g_{\alpha_{b,b+i}} \quad \langle WY\rangle_{is} = g_{\alpha_{b,b+i}} \quad \langle WZ\rangle_{is} = \begin{cases} \frac{g_{\alpha_{b,b+i}}(g_{\alpha_{a,b+i}}+g_{\beta_{ab}})}{g_{\beta_{ab}}} \\ g_{\alpha_{b,b+i}} \end{cases}$$

$$\langle WD\rangle_{i1} = g_{\alpha_{b,b+i}} \quad \langle WE\rangle_{i1} = g_{\alpha_{b,b+i}} \quad \langle XY\rangle_{is} = g_{\beta_{a+i,b}}$$

$$\langle XZ\rangle_{is} = g_{\beta_{b,b+s}} \quad \langle XD\rangle_{i1} = g_{\beta_{a+i,b}} \quad \langle XE\rangle_{i1} = g_{\gamma_b}$$

$$\langle YZ\rangle_{is} = g_{\beta_{b,b+s}} \quad \langle YD\rangle_{i1} = g_{\beta_{ab}} \quad \langle YE\rangle_{i1} = g_{\gamma_b}$$

$$\langle ZD\rangle_{i1} = g_{\beta_{b,b+i}} \quad \langle ZE\rangle_{i1} = g_{\beta_{b,b+i}} \quad \langle DE\rangle_{11} = g_{\gamma_b}$$

In the two case distinctions the upper case corresponds to $i = s$ and the lower case to $i \neq s$.

Proposition 93 (Case ♯2)**.** *Let $\eta = -\varepsilon_a - \varepsilon_b$ for $1 \leq a < b \leq n$. We order the pairs as follows $x_i = (\alpha_{ia}, \beta_{ib}), y_i = (\alpha_{ib}, \beta_{ia}), z_t = (\alpha_{a+t,b}, \beta_{a,a+t})$ and $D_1 = (\alpha_{ab}, \gamma_a)$ with $i = 1, .., a-1$ and $t = 1, .., b-a-1$. Then we have*

$$M_\eta = \begin{pmatrix} X & \langle XY\rangle & \langle XZ\rangle & \langle XD\rangle \\ \langle XY\rangle^T & Y & \langle YZ\rangle & \langle YD\rangle \\ \langle XZ\rangle^T & \langle YZ\rangle^T & Z & \langle ZD\rangle \\ \langle XD\rangle^T & \langle YD\rangle^T & \langle ZD\rangle^T & D \end{pmatrix}$$

with the diagonal matrices being

$$\begin{aligned} X &\in Sym_{a-1}(\mathbb{R}) & X_{is} &= g_{\alpha_{sa}} & &\text{for } i \leq s \\ Y &\in Sym_{a-1}(\mathbb{R}) & Y_{is} &= g_{\alpha_{sb}} & &\text{for } i \leq s \\ Z &\in Sym_{b-a-1}(\mathbb{R}) & Z_{is} &= g_{\alpha_{a+s,b}} & &\text{for } i \leq s \\ D &\in Sym_1(\mathbb{R}) & D_{11} &= g_{\alpha_{ab}} \end{aligned}$$

and the off diagonals being given by

$$\langle XY\rangle_{is} = \begin{cases} -\frac{g_{\alpha_{ia}}g_{\gamma_b}}{g_{\gamma_i}} & i = s \\ 2\frac{g_{\alpha_{sb}}g_{\alpha_{ia}}}{g_{\beta_{si}}} & i \neq s \end{cases} \quad \langle XZ\rangle_{is} = 2\frac{g_{\alpha_{ia}}g_{\alpha_{a+s,b}}}{g_{\beta_{i,a+s}}} \quad \langle XD\rangle_{i1} = 2\frac{g_{\alpha_{ia}}g_{\alpha_{ab}}}{g_{\beta_{ia}}}$$

$$\langle YZ\rangle_{is} = g_{\alpha_{a+s,b}} \quad \langle YD\rangle_{i1} = g_{\alpha_{ab}} \quad \langle ZD\rangle_{i1} = g_{\alpha_{a+i,b}}$$

Proposition 94 (Case ♯3)**.** *Let $\eta = -2\varepsilon_a$ for $1 \leq a \leq n$. We order the pairs as follows $x_i = (\alpha_{ia}, \beta_{ia})$ for $i = 1...a-1$. Then we have*

$$(M_\eta)_{is} = \begin{cases} 2\frac{(g_{\alpha_{ia}})^2}{g_{\gamma_i}} & i = s \\ \frac{g_{\alpha_{sa}}(2g_{\beta_{is}} - g_{\beta_{sa}})}{g_{\beta_{is}}} & i < s \end{cases}$$

Proposition 95. *We give the entries of the matrix of the trivial module just for the upper triangular part by symmetry, i.e. for the pairs (α, β) with $\alpha < \beta$.*

i) Along the diagonal we have

$$(M_0)_{(\alpha\alpha)(\alpha\alpha)} = \begin{cases} 2g_\alpha & \alpha \in \{\alpha_{ij}, \beta_{ij}\} \\ g_\alpha & \alpha = \gamma_i \end{cases}.$$

ii) For the following $(\alpha,\beta) \in \{(\alpha_{ia},\alpha_{ib}),(\alpha_{bj},\alpha_{aj}),(\beta_{kb},\beta_{ak}),(\beta_{ib},\beta_{ia}),(\beta_{bj},\beta_{aj}),(\beta_{ab},\gamma_a),(\gamma_b,\beta_{ab}),(\alpha_{ia},\beta_{ib}),(\alpha_{ib},\beta_{ia}),(\alpha_{kb},\beta_{ak}),(\alpha_{ab},\gamma_a)\}$ *we have*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = g_\alpha$$

iii) For the following pairs $(\alpha,\beta) \in \{(\alpha_{ib},\alpha_{ai}),(\beta_{ib},\alpha_{ai}),(\beta_{bj},\alpha_{aj}),(\alpha_{bj},\beta_{aj}),\ (\gamma_b,\alpha_{ab})\}$ *we have*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = -2\frac{g_\alpha g_\beta}{g_{\alpha+\beta}}$$

iv) For the following pairs $(\alpha,\beta) = (\alpha_{sa},\beta_{sa})$ *we have*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = 2\frac{(g_\alpha)^2}{g_{\alpha+\beta}}$$

Notice that the entries in the trivial module that have been doubled are *iii*).

7.4 $SO(2n)$

As for $SU(n+1)$ we define the ordering on the positive roots as in proposition 15 with the slight modification that we use the ordered basis ε_j with $j = 1,..,n$ instead of the simple roots to induce the ordering. It is easy to see that the desired properties are still true.

Proposition 96. *For the C space* $(SO(2n),\mathbb{T}^n,J_{std})$ *let* η *be a non zero weight of* $Fix(J_{std})$ *, i.e an element in*

$$\Delta_H = \{\alpha-\beta \mid \alpha<\beta\in\Delta_{\mathfrak{g}}^+\}.$$

Then η *is in the following table together with the pairs* $\alpha<\beta$ *with* $\alpha-\beta=\eta$*:*

♯	η	$I(\eta)$	n_η
1	$-\varepsilon_a+\varepsilon_b$	$(\alpha_{ia},\alpha_{ib})$, $i=1,..,a-1$ $(\alpha_{bj},\alpha_{aj})$, $j=b+1,..,n$ (β_{tb},β_{at}) , $t=a+1,..,b-1$ (β_{sb},β_{sa}) , $s=1,..,a-1$ (β_{br},β_{ar}) , $r=b+1,..,n$	$2n-b+a-3$
2	$-\varepsilon_a-\varepsilon_b$	(α_{sa},β_{sb}) , $s=1,..,a-1$ (α_{sb},β_{sa}) , $s=1,..,a-1$ (α_{tb},β_{at}) , $t=a+1,..,b-1$	$b+a-3$
3	$-2\varepsilon_a$	(α_{sa},β_{sa}) , $s=1,..,a-1$	$a-1$
4	$-\varepsilon_a+\varepsilon_b+\varepsilon_c-\varepsilon_d$	$(\alpha_{cd},\alpha_{ab}),(\alpha_{bd},\alpha_{ac}),(\beta_{bc},\beta_{ad})$	3
5	$-\varepsilon_a+\varepsilon_b-\varepsilon_c+\varepsilon_d$	$(\alpha_{bc},\alpha_{ad}),(\beta_{bd},\beta_{ac})$	2
6	$-\varepsilon_a-\varepsilon_b+\varepsilon_c+\varepsilon_d$	(β_{cd},β_{ab})	1
7	$-\varepsilon_a+\varepsilon_b+\varepsilon_c+\varepsilon_d$	$(\beta_{cd},\alpha_{ab}),(\beta_{bd},\alpha_{ac}),(\beta_{bc},\alpha_{ad})$	3
8	$-\varepsilon_a+\varepsilon_b-\varepsilon_c-\varepsilon_d$	$(\alpha_{bc},\beta_{ad}),(\alpha_{bd},\beta_{ac})$	2
9	$-\varepsilon_a-\varepsilon_b+\varepsilon_c-\varepsilon_d$	(α_{cd},β_{ab})	1
10	$-\varepsilon_a+2\varepsilon_b-\varepsilon_c$	$(\alpha_{bc},\alpha_{ab})$	1
11	$-\varepsilon_a+2\varepsilon_b+\varepsilon_c$	(β_{bc},α_{ab})	1
12	$-\varepsilon_a+\varepsilon_b+2\varepsilon_c$	(β_{bc},α_{ac})	1
13	$-\varepsilon_a+\varepsilon_b-2\varepsilon_c$	(α_{bc},β_{ac})	1

Proof. The elements in the list are obviously weights and these are all of them for dimensional reasons. In fact, from (4.1) we only have to verify that

$$\frac{|\Delta_{\mathfrak{g}}^{+}|\left(|\Delta_{\mathfrak{g}}^{+}|-1\right)}{2}=\sum_{\eta\in\Delta_H}n_\eta$$

holds. In fact, we know $|\Delta_{\mathfrak{g}}^{+}|=n(n-1)$ and the right hand side is given by

$$\sum_{a=1}^{n-1}\sum_{b=a+1}^{n}(2n+2a-6)+\sum_{a=1}^{n}(a-1)$$
$$+4\binom{n}{3}+12\binom{n}{4}$$

Easy calculations yield that left and right hand side coincide and therefore we found all weights. □

In the following, we determine the curvature matrices M_η. Since the indices of M_η are pairs of roots (α,β) with $\alpha<\beta$ and $\alpha-\beta=\eta$ it makes sense to choose an ordering of these pairs. We will do so in each M_η separately. The proofs reduce to plugging in the values of g_α, $N_{\alpha,\beta}$ and z_α as determined for the classical groups in section 1.8 into the equations of proposition 16 and will therefore be omitted. An exemplary calculation was made in the proof of proposition 44.

Proposition 97 (Case ♯1). *Let* $\eta=-\varepsilon_a+\varepsilon_b$ *for* $1\le a<b\le n$. *We order the pairs as follows* $v_i=(\alpha_{ia},\alpha_{ib}),w_j=(\alpha_{b,b+j},\alpha_{a,b+j}),x_t=(\beta_{a+t,b},\beta_{a,a+t}),y_i=(\beta_{ib},\beta_{ia}),z_j=(\beta_{b,b+j},\beta_{a,b+j})$ *with* $i=1,..,a-1,t=1,..,b-a-1$ *and* $j=1,..,n-b$. *Then we have*

$$M_\eta=\begin{pmatrix} V & \langle VW\rangle & \langle VX\rangle & \langle VY\rangle & \langle VZ\rangle\\ \langle VW\rangle^T & W & \langle WX\rangle & \langle WY\rangle & \langle WZ\rangle\\ \langle VX\rangle^T & \langle WX\rangle^T & X & \langle XY\rangle & \langle XZ\rangle\\ \langle VY\rangle^T & \langle WY\rangle^T & \langle XY\rangle^T & Y & \langle YZ\rangle\\ \langle VZ\rangle^T & \langle WZ\rangle^T & \langle XZ\rangle^T & \langle YZ\rangle^T & Z \end{pmatrix}$$

with the diagonal matrices being

$$\begin{aligned} V&\in Sym_{a-1}(\mathbb{R}) & V_{is}&=4g_{\alpha_{sa}} \text{ for } i\le s\\ W&\in Sym_{n-b}(\mathbb{R}) & W_{is}&=4g_{\alpha_{b,b+i}} \text{ for } i\le s\\ X&\in Sym_{b-a-1}(\mathbb{R}) & X_{is}&=4g_{\beta_{a+s,b}} \text{ for } i\le s\\ Y&\in Sym_{a-1}(\mathbb{R}) & Y_{is}&=4g_{\beta_{sb}} \text{ for } i\le s\\ Z&\in Sym_{n-b}(\mathbb{R}) & Z_{is}&=4g_{\beta_{b,b+s}} \text{ for } i\le s \end{aligned}$$

and the off diagonals being given by the following where i *and* s *vary in the ranges determined by the diagonals*

$$\langle VW\rangle_{is}=-4\frac{g_{\alpha_{ia}}g_{\alpha_{b,b+s}}}{g_{\alpha_{i,b+s}}}\qquad \langle VX\rangle_{is}=4\frac{g_{\alpha_{ia}}g_{\beta_{a+s,b}}}{g_{\beta_{i,a+s}}}\qquad \langle VY\rangle_{is}=\begin{cases}-4g_{\alpha_{ia}}\\ -4\frac{g_{\alpha_{ia}}g_{\beta_{sb}}}{g_{\beta_{is}}}\end{cases}$$

$$\langle VZ\rangle_{is}=-4\frac{g_{\alpha_{ia}}g_{\beta_{b,b+s}}}{g_{\beta_{i,b+s}}}\qquad \langle WX\rangle_{is}=-4g_{\alpha_{b,b+i}}\qquad \langle WY\rangle_{is}=4g_{\alpha_{b,b+i}}$$

$$\langle WZ\rangle_{is}=\begin{cases}4\frac{g_{\alpha_{b,b+i}}g_{\beta_{b,b+i}}}{g_{\beta_{ab}}}\\ 4g_{\alpha_{b,b+i}}\end{cases}\qquad \langle XY\rangle_{is}=-4g_{\beta_{a+i,b}}\qquad \langle XZ\rangle_{is}=-4g_{\beta_{b,b+s}}$$

$$\langle YZ\rangle_{is}=4g_{\beta_{b,b+s}}$$

In the two case distinctions the upper case corresponds to $i=s$ *and the lower case to* $i\neq s$.

Proposition 98 (Case $\sharp 2$)**.** *Let $\eta = -\varepsilon_a - \varepsilon_b$ for $1 \le a < b \le n$. We order the pairs as follows $x_i = (\alpha_{ia}, \beta_{ib}), y_i = (\alpha_{ib}, \beta_{ia}), z_t = (\alpha_{a+t,b}, \beta_{a,a+t})$ with $i = 1, .., a-1$ and $t = 1, .., b-a-1$. Then we have*

$$M_\eta = \begin{pmatrix} X & \langle XY \rangle & \langle XZ \rangle \\ \langle XY \rangle^T & Y & \langle YZ \rangle \\ \langle XZ \rangle^T & \langle YZ \rangle^T & Z \end{pmatrix}$$

with the diagonal matrices being

$$\begin{aligned} X \in & Sym_{a-1}(\mathbb{R}) & X_{is} &= 4 g_{\alpha_{sa}} & & \text{for } i \le s \\ Y \in & Sym_{a-1}(\mathbb{R}) & Y_{is} &= 4 g_{\alpha_{sb}} & & \text{for } i \le s \\ Z \in & Sym_{n-b}(\mathbb{R}) & Z_{is} &= 4 g_{\alpha_{a+s,b}} & & \text{for } i \le s \end{aligned}$$

and the off diagonals being given by

$$\langle XY \rangle_{is} = \begin{cases} -4 g_{\alpha_{sa}} & i = s \\ -4 \frac{g_{\alpha_{sb}} g_{\alpha_{ia}}}{g_{\beta_{si}}} & i \neq s \end{cases} \qquad \langle XZ \rangle_{is} = 4 \frac{g_{\alpha_{ia}} g_{\alpha_{a+s,b}}}{g_{\beta_{i,a+s}}} \qquad \langle YZ \rangle_{is} = -4 g_{\alpha_{a+s,b}}$$

Proposition 99 (Case $\sharp 3$)**.** *Let $\eta = -2\varepsilon_a$ for $1 \le a \le n$. We order the pairs as follows $x_i = (\alpha_{ia}, \beta_{ia})$ for $i = 1...a-1$. Then we have*

$$(M_\eta)_{is} = \begin{cases} 0 & i = s \\ 4 \frac{g_{\alpha_{sa}} g_{\beta_{sa}}}{g_{\beta_{is}}} & i < s \end{cases}$$

Proposition 100 (Cases $\sharp 4 - \sharp 9$)**.** *Let $1 \le a < b < c < d \le n$. We order the pairs as given in the table from left to right. Then we have for $\eta = -\varepsilon_a + \varepsilon_b + \varepsilon_c - \varepsilon_d$:*

$$M_\eta = \begin{pmatrix} 0 & -4 \frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\alpha_{ad}}} & 4 \frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} \\ -4 \frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\alpha_{ad}}} & 0 & -4 \frac{g_{\alpha_{bd}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} \\ 4 \frac{g_{\alpha_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} & -4 \frac{g_{\alpha_{bd}} g_{\alpha_{ac}}}{g_{\alpha_{ab}}} & 0 \end{pmatrix}.$$

For $\eta = -\varepsilon_a + \varepsilon_b - \varepsilon_c + \varepsilon_d$:

$$M_\eta = \begin{pmatrix} 0 & -4 \frac{g_{\alpha_{bc}} g_{\alpha_{ad}}}{g_{\beta_{ab}}} \\ -4 \frac{g_{\alpha_{bc}} g_{\alpha_{ad}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a - \varepsilon_b + \varepsilon_c + \varepsilon_d$:

$$M_\eta = 0$$

For $\eta = -\varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d$:

$$M_\eta = \begin{pmatrix} 0 & -4 \frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ad}}} & 4 \frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} \\ -4 \frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ad}}} & 0 & -4 \frac{g_{\beta_{bd}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} \\ 4 \frac{g_{\beta_{cd}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} & -4 \frac{g_{\beta_{bd}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a + \varepsilon_b - \varepsilon_c - \varepsilon_d$:

$$M_\eta = \begin{pmatrix} 0 & -4 \frac{g_{\alpha_{bc}} g_{\beta_{ad}}}{g_{\beta_{ab}}} \\ -4 \frac{g_{\alpha_{bc}} g_{\beta_{ad}}}{g_{\beta_{ab}}} & 0 \end{pmatrix}$$

For $\eta = -\varepsilon_a - \varepsilon_b + \varepsilon_c - \varepsilon_d$:

$$M_\eta = 0$$

Since they are all one dimensional, we write the cases $\sharp 10 - \sharp 13$ in one matrix

Proposition 101 (Cases $\sharp 10 - \sharp 13$). *We consider for $1 \leq a < b < c \leq n$ the ordering $(\alpha_{bc}, \alpha_{ab}), (\beta_{bc}, \alpha_{ab}), (\beta_{bc}, \alpha_{ac}), (\alpha_{bc}, \beta_{ac})$. The direct sum of the corresponding $M'_\eta s$ is given by*

$$\begin{pmatrix} -4\frac{g_{\alpha_{bc}} g_{\alpha_{ab}}}{g_{\alpha_{ac}}} & 0 & 0 & 0 \\ 0 & -4\frac{g_{\beta_{bc}} g_{\alpha_{ab}}}{g_{\beta_{ac}}} & 0 & 0 \\ 0 & 0 & -4\frac{g_{\beta_{bc}} g_{\alpha_{ac}}}{g_{\beta_{ab}}} & 0 \\ 0 & 0 & 0 & -4\frac{g_{\alpha_{bc}} g_{\beta_{ac}}}{g_{\beta_{ab}}} \end{pmatrix}$$

Proposition 102. *The entries of the matrix of the trivial module are determined by the diagonals of the matrices above :*

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = (M_{\alpha-\beta})_{(\alpha\beta)(\alpha\beta)}$$

except for the diagonals. For $\alpha \in \Delta^+_{\mathfrak{m}}$ the following holds

$$(M_0)_{(\alpha\alpha)(\alpha\alpha)} = 8g_\alpha$$

General modifications of $SO(2n)$

After completing the steps described in the beginning of the section in the case of $SO(2n)$, that is

1.) We move the diagonals of $\sharp 10 - 13$ into the matrix of the trivial module.

2.) The off diagonals of $\sharp 3$ to the diagonal of the $\langle WZ \rangle$ block of $\sharp 1$.

3.) The off diagonals of $\sharp 4$ to the $\langle VW \rangle$ block of $\sharp 1$,the $\langle XZ \rangle$ block and the upper triangular part of the $\langle XY \rangle$ block of $\sharp 2$.

4.) The off diagonal of $\sharp 5$ to the lower triangular part of the $\langle XY \rangle$ block of $\sharp 2$.

5.) The off diagonals of $\sharp 7$ to the $\langle VZ \rangle$ block, the $\langle VX \rangle$ block and the upper triangular part of the $\langle VY \rangle$ block of $\sharp 1$.

6.) The off diagonal of $\sharp 8$ the lower triangular part of the $\langle VY \rangle$ block of $\sharp 1$.

We remark that similarly to the modifications of $SU(n+1)$, in each step have a bijection between the referenced entries of all matrices of the mentioned case $\sharp i$.

Then the modified holomorphic curvature tensor is given by the following matrices

Proposition 103 (Case $\sharp 1$). *Let $\eta = -\varepsilon_a + \varepsilon_b$ for $1 \leq a < b \leq n$. We order the pairs as follows $v_i = (\alpha_{ia}, \alpha_{ib}), w_j = (\alpha_{b,b+j}, \alpha_{a,b+j}), x_t = (\beta_{a+t,b}, \beta_{a,a+t}), y_i = (\beta_{ib}, \beta_{ia}), z_j = (\beta_{b,b+j}, \beta_{a,b+j})$ with $i = 1, .., a-1, t = 1, .., b-a-1$ and $j = 1, .., n-b$. Then we have*

$$M_\eta = \begin{pmatrix} V & \langle VW \rangle & \langle VX \rangle & \langle VY \rangle & \langle VZ \rangle \\ \langle VW \rangle^T & W & \langle WX \rangle & \langle WY \rangle & \langle WZ \rangle \\ \langle VX \rangle^T & \langle WX \rangle^T & X & \langle XY \rangle & \langle XZ \rangle \\ \langle VY \rangle^T & \langle WY \rangle^T & \langle XY \rangle^T & Y & \langle YZ \rangle \\ \langle VZ \rangle^T & \langle WZ \rangle^T & \langle XZ \rangle^T & \langle YZ \rangle^T & Z \end{pmatrix}$$

with the diagonal matrices being

$$\begin{array}{ll} V \in Sym_{a-1}(\mathbb{R}) & V_{is} = 4g_{\alpha_{sa}} \text{ for } i \leq s \\ W \in Sym_{n-b}(\mathbb{R}) & W_{is} = 4g_{\alpha_{b,b+i}} \text{ for } i \leq s \\ X \in Sym_{b-a-1}(\mathbb{R}) & X_{is} = 4g_{\beta_{a+s,b}} \text{ for } i \leq s \\ Y \in Sym_{a-1}(\mathbb{R}) & Y_{is} = 4g_{\beta_{sb}} \text{ for } i \leq s \\ Z \in Sym_{n-b}(\mathbb{R}) & Z_{is} = 4g_{\beta_{b,b+s}} \text{ for } i \leq s \end{array}$$

and the off diagonals being given by the following where i and s vary in the ranges determined by the diagonals

$$\langle VW\rangle_{is} = -8\frac{g_{\alpha_{ia}}g_{\alpha_{b,b+s}}}{g_{\alpha_{i,b+s}}} \qquad \langle VX\rangle_{is} = 8\frac{g_{\alpha_{ia}}g_{\beta_{a+s,b}}}{g_{\beta_{i,a+s}}} \qquad \langle VY\rangle_{is} = \begin{cases} -4g_{\alpha_{ia}} \\ -8\frac{g_{\alpha_{ia}}g_{\beta_{sb}}}{g_{\beta_{is}}} \end{cases}$$

$$\langle VZ\rangle_{is} = -8\frac{g_{\alpha_{ia}}g_{\beta_{b,b+s}}}{g_{\beta_{i,b+s}}} \qquad \langle WX\rangle_{is} = -4g_{\alpha_{b,b+i}} \qquad \langle WY\rangle_{is} = 4g_{\alpha_{b,b+i}}$$

$$\langle WZ\rangle_{is} = \begin{cases} 8\frac{g_{\alpha_{b,b+i}}g_{\beta_{b,b+i}}}{g_{\beta_{ab}}} \\ 4g_{\alpha_{b,b+i}} \end{cases} \qquad \langle XY\rangle_{is} = -4g_{\beta_{a+i,b}} \qquad \langle XZ\rangle_{is} = -4g_{\beta_{b,b+s}}$$

$$\langle YZ\rangle_{is} = 4g_{\beta_{b,b+s}}$$

In the two case distinctions the upper case corresponds to $i = s$ and the lower case to $i \neq s$.

Proposition 104 (Case ♯2)**.** *Let $\eta = -\varepsilon_a - \varepsilon_b$ for $1 \leq a < b \leq n$. We order the pairs as follows $x_i = (\alpha_{ia}, \beta_{ib}), y_i = (\alpha_{ib}, \beta_{ia}), z_t = (\alpha_{a+t,b}, \beta_{a,a+t})$ with $i = 1, .., a-1$ and $t = 1, .., b-a-1$. Then we have*

$$M_\eta = \begin{pmatrix} X & \langle XY\rangle & \langle XZ\rangle \\ \langle XY\rangle^T & Y & \langle YZ\rangle \\ \langle XZ\rangle^T & \langle YZ\rangle^T & Z \end{pmatrix}$$

with the diagonal matrices being

$$\begin{array}{lll} X \in Sym_{a-1}(\mathbb{R}) & X_{is} = 4g_{\alpha_{sa}} & \text{for } i \leq s \\ Y \in Sym_{a-1}(\mathbb{R}) & Y_{is} = 4g_{\alpha_{sb}} & \text{for } i \leq s \\ Z \in Sym_{b-a-1}(\mathbb{R}) & Z_{is} = 4g_{\alpha_{a+s,b}} & \text{for } i \leq s \end{array}$$

and the off diagonals being given by

$$\langle XY\rangle_{is} = \begin{cases} -4g_{\alpha_{sa}} & i = s \\ -8\frac{g_{\alpha_{sb}}g_{\alpha_{ia}}}{g_{\beta_{si}}} & i \neq s \end{cases} \qquad \langle XZ\rangle_{is} = 8\frac{g_{\alpha_{ia}}g_{\alpha_{a+s,b}}}{g_{\beta_{i,a+s}}} \qquad \langle YZ\rangle_{is} = -4g_{\alpha_{a+s,b}}$$

Proposition 105. *We give the entries of the matrix of the trivial module just for the upper triangular part by symmetry, i.e. for the pairs (α, β) with $\alpha < \beta$.*

i) Along the diagonal we have

$$(M_0)_{(\alpha\alpha)(\alpha\alpha)} = 8g_\alpha$$

ii) For the following pairs $(\alpha, \beta) \in \{(\alpha_{ia}, \alpha_{ib}), (\alpha_{bj}, \alpha_{aj}), (\beta_{kb}, \beta_{ak}), (\beta_{ib}, \beta_{ia}), (\beta_{bj}, \beta_{aj}), (\alpha_{ia}, \beta_{ib}), (\alpha_{ib}, \beta_{ia}), (\alpha_{kb}, \alpha_{ak})\}$ we have

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = 4g_\alpha$$

iii) For the following pairs $(\alpha, \beta) \in \{(\alpha_{ib}, \alpha_{ai}), (\beta_{ib}, \alpha_{ai}), (\beta_{bj}, \alpha_{aj}), (\alpha_{bj}, \beta_{aj})\}$ we have

$$(M_0)_{(\alpha\alpha)(\beta\beta)} = -8\frac{g_\alpha g_\beta}{g_{\alpha+\beta}}$$

Notice that the entries in the trivial module that have been doubled are *iii*).

Chapter 8

The case of $H(4)$

In the following we consider the holomorphic curvature tensors for $(G, \mathbb{T}, J_{std}, g_{KE})$ for G being a classical Lie group of rank 4 and show that there are four forms turning those tensors positive definite. Afterwards, we go on to larger isotropy groups with their corresponding Kähler Einstein metrics. By the result of section 5, the corresponding curvature matrices are obtained from those of $(G, \mathbb{T})$ via going to submatrices and plugging in different coefficients for the metrics.

Before we continue to the calculations, we want to state the following useful

Lemma 106 (Auxiliary Lemma)**.** *Let $M \in Mat_n(\mathbb{R})$ be given as the symmetric matrix $M_{ij} = a_i$ for $i \leq j$. Then we have*

$$det(M) = a_1 \prod_{i=1}^{n} (a_{i+1} - a_i).$$

In particular, if $0 \leq a_i \leq a_{i+1}$ holds, then M is positive semidefinite. Additionally, if strict inequalities hold then M is positive definite.

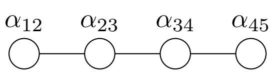
Proof. The expression of the determinant is an easy exercise. The positive (semi-) definiteness, follows from Sylvester's criterion for symmetric matrices once one notices that every minor of M is again of the same type as M and hence the above determinant formula applies also for the minors. □

8.1 The case of $SU(5)$

The positive roots in increasing order are

$$\Delta_{\mathfrak{g}}^{+} = \{\alpha_{45}, \alpha_{34}, \alpha_{35}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}\}. \tag{8.1}$$

and we have the simple roots as follows on the Dynkin diagram:



After applying the general modifications of section 7.1 the matrices M_η representing the modified holomorphic curvature tensor are indexed with $\eta \in \Delta_{\mathfrak{g}}^{-} \cup \{0\}$. In the case of $SU(5)$ this corresponds to the following:

First of all the roots $-\alpha_{1i}$ with $i = 2..5$:

$$M_{-\alpha_{12}} = \begin{pmatrix} g_{\alpha_{23}} & g_{\alpha_{23}} & g_{\alpha_{23}} \\ g_{\alpha_{23}} & g_{\alpha_{24}} & g_{\alpha_{24}} \\ g_{\alpha_{23}} & g_{\alpha_{24}} & g_{\alpha_{25}} \end{pmatrix} \qquad M_{-\alpha_{13}} = \begin{pmatrix} g_{\alpha_{34}} & g_{\alpha_{34}} \\ g_{\alpha_{34}} & g_{\alpha_{35}} \end{pmatrix}$$

$$M_{-\alpha_{14}} = \begin{pmatrix} g_{\alpha_{45}} \end{pmatrix} \qquad M_{-\alpha_{15}} = (\emptyset)$$

Then the roots $-\alpha_{2i}$ with $i = 3..5$:

$$M_{-\alpha_{23}} = \begin{pmatrix} g_{\alpha_{12}} & -2\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\alpha_{14}}} & -2\frac{g_{\alpha_{12}}g_{\alpha_{35}}}{g_{\alpha_{15}}} \\ -2\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\alpha_{14}}} & g_{\alpha_{34}} & g_{\alpha_{34}} \\ -2\frac{g_{\alpha_{12}}g_{\alpha_{35}}}{g_{\alpha_{15}}} & g_{\alpha_{34}} & g_{\alpha_{35}} \end{pmatrix} \quad M_{-\alpha_{24}} = \begin{pmatrix} g_{\alpha_{12}} & -2\frac{g_{\alpha_{12}}g_{\alpha_{45}}}{g_{\alpha_{15}}} \\ -2\frac{g_{\alpha_{12}}g_{\alpha_{45}}}{g_{\alpha_{15}}} & g_{\alpha_{45}} \end{pmatrix}$$

$$M_{-\alpha_{25}} = \begin{pmatrix} g_{\alpha_{12}} \end{pmatrix}$$

and the roots $-\alpha_{3i}$ with $i = 4, 5$:

$$M_{-\alpha_{34}} = \begin{pmatrix} g_{\alpha_{13}} & g_{\alpha_{23}} & -2\frac{g_{\alpha_{13}}g_{\alpha_{45}}}{g_{\alpha_{15}}} \\ g_{\alpha_{23}} & g_{\alpha_{23}} & -2\frac{g_{\alpha_{23}}g_{\alpha_{45}}}{g_{\alpha_{25}}} \\ -2\frac{g_{\alpha_{13}}g_{\alpha_{45}}}{g_{\alpha_{15}}} & -2\frac{g_{\alpha_{23}}g_{\alpha_{45}}}{g_{\alpha_{25}}} & g_{\alpha_{45}} \end{pmatrix} \qquad M_{-\alpha_{35}} = \begin{pmatrix} g_{\alpha_{13}} & g_{\alpha_{23}} \\ g_{\alpha_{23}} & g_{\alpha_{23}} \end{pmatrix}$$

Last but not least $-\alpha_{45}$:

$$M_{-\alpha_{45}} = \begin{pmatrix} g_{\alpha_{14}} & g_{\alpha_{24}} & g_{\alpha_{34}} \\ g_{\alpha_{24}} & g_{\alpha_{24}} & g_{\alpha_{34}} \\ g_{\alpha_{24}} & g_{\alpha_{34}} & g_{\alpha_{34}} \end{pmatrix}.$$

In the increasing ordering of the basis from equation (8.1) we have that the matrix M_0 on the trivial module is given by:

$$\left(\begin{array}{c|cc|ccc|cccc} 2g_{\alpha_{45}} & * & g_{\alpha_{45}} & 0 & * & g_{\alpha_{45}} & 0 & 0 & * & g_{\alpha_{45}} \\ \hline * & 2g_{\alpha_{34}} & g_{\alpha_{34}} & * & g_{\alpha_{34}} & 0 & 0 & * & g_{\alpha_{34}} & 0 \\ g_{\alpha_{45}} & g_{\alpha_{34}} & 2g_{\alpha_{35}} & * & 0 & g_{\alpha_{35}} & 0 & * & 0 & g_{\alpha_{35}} \\ \hline 0 & * & * & 2g_{\alpha_{23}} & g_{\alpha_{23}} & g_{\alpha_{23}} & * & g_{\alpha_{23}} & 0 & 0 \\ * & g_{\alpha_{34}} & 0 & g_{\alpha_{23}} & 2g_{\alpha_{24}} & g_{\alpha_{24}} & * & 0 & g_{\alpha_{24}} & 0 \\ g_{\alpha_{45}} & 0 & g_{\alpha_{35}} & g_{\alpha_{23}} & g_{\alpha_{24}} & 2g_{\alpha_{25}} & * & 0 & 0 & g_{\alpha_{25}} \\ \hline 0 & 0 & 0 & * & * & * & 2g_{\alpha_{12}} & g_{\alpha_{12}} & g_{\alpha_{12}} & g_{\alpha_{12}} \\ 0 & * & * & g_{\alpha_{23}} & 0 & 0 & g_{\alpha_{12}} & 2g_{\alpha_{13}} & g_{\alpha_{13}} & g_{\alpha_{13}} \\ * & g_{\alpha_{34}} & 0 & 0 & g_{\alpha_{24}} & 0 & g_{\alpha_{12}} & g_{\alpha_{13}} & 2g_{\alpha_{14}} & g_{\alpha_{14}} \\ g_{\alpha_{45}} & 0 & g_{\alpha_{35}} & 0 & 0 & g_{\alpha_{25}} & g_{\alpha_{12}} & g_{\alpha_{13}} & g_{\alpha_{14}} & 2g_{\alpha_{15}} \end{array}\right)$$

where $*$ at the entry (α, β) is a place holder for $-2\frac{g_\alpha g_\beta}{g_{\alpha+\beta}}$ for formatting reasons.

Even though the above corresponds to $(SU(5), \mathbb{T}, J_{std}, g)$ with an arbitrary Kähler metric, by corollary 50 the curvature tensor with a different isotropy group K is given by minors of the above together with the property that the metric on the simple roots of K is zero. Hence we see that for any isotropy group the matrices (the corresponding submatrices of them resp.) $M_{-\alpha_{12}}, M_{-\alpha_{13}}, M_{-\alpha_{14}}, M_{-\alpha_{25}}, M_{-\alpha_{35}}, M_{-\alpha_{45}}$ are positive semidefinite, by the auxiliary lemma, which is sufficient proposition 31. Furthermore, we have

$$det(M_{-\alpha_{24}}) = g_{\alpha_{12}}g_{\alpha_{45}}\left(1 - 4\frac{g_{\alpha_{12}}g_{\alpha_{45}}}{g_{\alpha_{15}}^2}\right)$$

which is nonnegative because $g_{\alpha_{15}} = g_{\alpha_{12}} + g_{\alpha_{24}} + g_{\alpha_{45}}$ implies

$$g_{\alpha_{15}}^2 - 4g_{\alpha_{12}}g_{\alpha_{45}} \geq (g_{\alpha_{12}} + g_{\alpha_{45}})^2 - 4g_{\alpha_{12}}g_{\alpha_{45}} = (g_{\alpha_{12}} - g_{\alpha_{45}})^2.$$

Therefore, we only have to consider M_0,$M_{-\alpha_{23}}$ and $M_{-\alpha_{34}}$. Since they are more complicated, we consider them separately in all cases.

Toric Isotropy

Now we consider the Kähler Einstein metric of $(SU(5), \mathbb{T}, J_{std})$, which is given on the simple roots by

α	α_{12}	α_{23}	α_{34}	α_{45}
g_α	1	1	1	1

Then we have

$$M_{-\alpha_{23}} = \begin{pmatrix} 1 & -\frac{2}{3} & -1 \\ -\frac{2}{3} & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \qquad M_{-\alpha_{34}} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -\frac{2}{3} \\ -1 & -\frac{2}{3} & 1 \end{pmatrix}$$

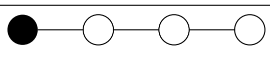
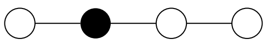
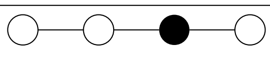
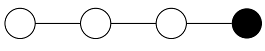
which are obviously positive definite by calculating the minors.

$$M_0 = \left(\begin{array}{c|cc|ccc|cccc} 2 & -1 & 1 & 0 & -\frac{4}{3} & 1 & 0 & 0 & -\frac{3}{2} & 1 \\ \hline -1 & 2 & 1 & -1 & 1 & 0 & 0 & -\frac{4}{3} & 1 & 0 \\ 1 & 1 & 4 & -\frac{4}{3} & 0 & 2 & 0 & -2 & 0 & 2 \\ \hline 0 & -1 & -\frac{4}{3} & 2 & 1 & 1 & -1 & 1 & 0 & 0 \\ -\frac{4}{3} & 1 & 0 & 1 & 4 & 2 & -\frac{4}{3} & 0 & 2 & 0 \\ 1 & 0 & 2 & 1 & 2 & 6 & -\frac{3}{2} & 0 & 0 & 3 \\ \hline 0 & 0 & 0 & -1 & -\frac{4}{3} & -\frac{3}{2} & 2 & 1 & 1 & 1 \\ 0 & -\frac{4}{3} & -2 & 1 & 0 & 0 & 1 & 4 & 2 & 2 \\ -\frac{3}{2} & 1 & 0 & 0 & 2 & 0 & 1 & 2 & 6 & 3 \\ 1 & 0 & 2 & 0 & 0 & 3 & 1 & 2 & 3 & 8 \end{array}\right)$$

By calculating the determinants of the principal minors, we note that all of them are positive and hence no further 4 forms are needed and we get that $(SU(5), \mathbb{T}, J_{std}, g_{KE})$ has positive holomorphic curvature.

Larger Isotropy

By the previous discussion, we know that the isotropy groups we have to consider correspond to painted Dynkin diagrams, which leaves us with the following groups K and the corresponding painted Dynkin diagram. We also calculate γ_K^* from (3.3) which gives us the Kähler Einstein metric. In addition, we only consider the cases, where $dim(\mathfrak{z}(K)) > 2$ since the other cases are covered by the result of Itoh and theorem 22.

K	$D = (V, E)$	γ_K^*	$g_{\alpha_{12}}$	$g_{\alpha_{23}}$	$g_{\alpha_{34}}$	$g_{\alpha_{45}}$
$SU(2)\ \mathbb{T}^3$		$\varepsilon_1 - \varepsilon_2$	0	$\frac{3}{2}$	1	1
$\mathbb{S}^1\ SU(2)\ \mathbb{T}^2$		$\varepsilon_2 - \varepsilon_3$	$\frac{3}{2}$	0	$\frac{3}{2}$	1
$\mathbb{T}^2\ SU(2)\ \mathbb{S}^1$		$\varepsilon_3 - \varepsilon_4$	1	$\frac{3}{2}$	0	$\frac{3}{2}$
$\mathbb{T}^3\ SU(2)$		$\varepsilon_4 - \varepsilon_5$	1	1	$\frac{3}{2}$	0

By the symmetry of the Dynkin diagram of $SU(5)$, it is sufficient to consider the first two cases.

Isotropy group $K = SU(2)\ \mathbb{T}^3$
By proposition 48 we consider the restriction of the holomorphic curvature tensor to $\mathfrak{m}$ and then plug in the coefficients of the metric given above. The act of restricting corresponds by corollary 50 to erasing all rows and columns whose index contains α_{12}. Yielding

$$M_{-\alpha_{23}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad M_{-\alpha_{34}} = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} & -\frac{6}{7} \\ \frac{3}{2} & \frac{3}{2} & -\frac{6}{7} \\ -\frac{6}{7} & -\frac{6}{7} & 1 \end{pmatrix}$$

which are obviously positive semi-definite by calculating the minors. The matrix of the trivial module is given by

$$M_0 = \left(\begin{array}{c|cc|ccc|ccc} 2 & -1 & 1 & 0 & -\frac{10}{7} & 1 & 0 & -\frac{10}{7} & 1 \\ \hline -1 & 2 & 1 & -\frac{6}{5} & 1 & 0 & -\frac{6}{5} & 1 & 0 \\ 1 & 1 & 4 & -\frac{12}{7} & 0 & 2 & -\frac{12}{7} & 0 & 2 \\ \hline 0 & -\frac{6}{5} & -\frac{12}{7} & 3 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & 0 & 0 \\ -\frac{10}{7} & 1 & 0 & \frac{3}{2} & 5 & \frac{5}{2} & 0 & \frac{5}{2} & 0 \\ 1 & 0 & 2 & \frac{3}{2} & \frac{5}{2} & 7 & 0 & 0 & \frac{7}{2} \\ \hline 0 & -\frac{6}{5} & -\frac{12}{7} & \frac{3}{2} & 0 & 0 & 3 & \frac{3}{2} & \frac{3}{2} \\ -\frac{10}{7} & 1 & 0 & 0 & \frac{5}{2} & 0 & \frac{3}{2} & 5 & \frac{5}{2} \\ 1 & 0 & 2 & 0 & 0 & \frac{5}{2} & \frac{3}{2} & \frac{5}{2} & 7 \end{array}\right)$$

which is positive definite.

Isotropy group $K = \mathbb{S}^1\ SU(2)\ \mathbb{T}^2$
In this case we have

$$M_{-\alpha_{23}} = \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & -\frac{15}{8} \\ -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ -\frac{15}{8} & \frac{3}{2} & \frac{5}{2} \end{pmatrix} \qquad M_{-\alpha_{34}} = \begin{pmatrix} 1 & -\frac{3}{4} \\ -\frac{3}{4} & \frac{3}{2} \end{pmatrix}$$

which are obviously positive semi-definite by calculating the minors. The matrix of the trivial module is given by

$$M_0 = \left(\begin{array}{c|cc|cc|cccc} 2 & -\frac{6}{5} & 1 & -\frac{6}{5} & 1 & 0 & 0 & -\frac{3}{2} & 1 \\ \hline -\frac{6}{5} & 3 & \frac{3}{2} & \frac{3}{2} & 0 & 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 1 & \frac{3}{2} & 5 & 0 & \frac{5}{2} & 0 & -\frac{15}{8} & 0 & \frac{5}{2} \\ \hline -\frac{6}{5} & \frac{3}{2} & 0 & 3 & \frac{3}{2} & -\frac{3}{2} & 0 & \frac{3}{2} & 0 \\ 1 & 0 & \frac{5}{2} & \frac{3}{2} & 5 & -\frac{15}{8} & 0 & 0 & \frac{5}{2} \\ \hline 0 & 0 & 0 & -\frac{3}{2} & -\frac{15}{8} & 3 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & -\frac{3}{2} & -\frac{15}{8} & 0 & 0 & \frac{3}{2} & 3 & \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} & 0 & \frac{3}{2} & 0 & \frac{3}{2} & \frac{3}{2} & 6 & 3 \\ 1 & 0 & \frac{5}{2} & 0 & \frac{5}{2} & \frac{3}{2} & \frac{3}{2} & 3 & 8 \end{array}\right).$$

This is not positive definite. In fact, for $v = \left(0, \frac{1}{2}, \frac{2}{5}, -\frac{1}{2}, -\frac{2}{5}, 1, -1, 0, 0\right)$ we have

$$vM_0v^T = -\frac{1}{4}.$$

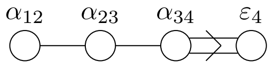
Hence further four forms are necessary. As we saw in proposition 31, we may add any negative value symmetrically to the off diagonal entries of M_0, then the four form realizing this change only improves positivity of the other matrices M_η. In the case of this K, it is enough to add -1 to the entries $(M_0)_{6,7}$ and $(M_0)_{7,6}$. The resulting matrix is now positive definite as one verifies by calculating the principal minors.

8.2 The case of $SO(9)$

The positive roots are

$$\begin{aligned}\Delta_{\mathfrak{g}}^{+} = \{&\alpha_{34}, \alpha_{23}, \alpha_{24}, \alpha_{12}, \alpha_{13}, \alpha_{14},\\ &\beta_{34}, \beta_{23}, \beta_{24}, \beta_{12}, \beta_{13}, \beta_{14},\\ &\varepsilon_4, \varepsilon_3, \varepsilon_2, \varepsilon_1\}\end{aligned} \tag{8.2}$$

and the simple roots are placed on the Dynkin diagram as follows



After applying the general modifications of section 7.2 the matrices M_η representing the modified holomorphic curvature tensor are indexed with $\eta \in \Delta_{\mathfrak{g}}^{-} \cup \{0\}$. In the case of $SO(9)$ this corresponds to the following:

We begin with the ones corresponding to $-\alpha_{ij}$.

$$M_{-\alpha_{12}} = \left(\begin{array}{cc|cc|c} 4g_{\alpha_{23}} & 4g_{\alpha_{23}} & 8\frac{g_{\alpha_{23}}g_{\beta_{23}}}{g_{\beta_{12}}} & 4g_{\alpha_{23}} & 2g_{\alpha_{23}} \\ 4g_{\alpha_{23}} & 4g_{\alpha_{24}} & 4g_{\alpha_{24}} & 8\frac{g_{\alpha_{24}}g_{\beta_{24}}}{g_{\beta_{12}}} & 2g_{\alpha_{24}} \\ \hline 8\frac{g_{\alpha_{23}}g_{\beta_{23}}}{g_{\beta_{12}}} & 4g_{\alpha_{24}} & 4g_{\beta_{23}} & 4g_{\beta_{24}} & 2g_{\varepsilon_2} \\ 4g_{\alpha_{23}} & 8\frac{g_{\alpha_{24}}g_{\beta_{24}}}{g_{\beta_{12}}} & 4g_{\beta_{24}} & 4g_{\beta_{24}} & 2g_{\varepsilon_2} \\ \hline 2g_{\alpha_{23}} & 2g_{\alpha_{24}} & 2g_{\varepsilon_2} & 2g_{\varepsilon_2} & \frac{(g_{\varepsilon_2})^2}{g_{\beta_{12}}} \end{array}\right)$$

$$M_{-\alpha_{13}} = \left(\begin{array}{c|c|c|c} 4g_{\alpha_{34}} & -4g_{\alpha_{34}} & 8\frac{g_{\alpha_{34}}g_{\beta_{34}}}{g_{\beta_{13}}} & 2g_{\alpha_{34}} \\ \hline -4g_{\alpha_{34}} & 4g_{\beta_{23}} & -4g_{\beta_{34}} & -2g_{\varepsilon_3} \\ \hline 8\frac{g_{\alpha_{34}}g_{\beta_{34}}}{g_{\beta_{13}}} & -4g_{\beta_{34}} & 4g_{\beta_{34}} & 2g_{\varepsilon_3} \\ \hline 2g_{\alpha_{34}} & -2g_{\varepsilon_3} & 2g_{\varepsilon_3} & \frac{(g_{\varepsilon_3})^2}{g_{\beta_{13}}} \end{array}\right) \quad M_{-\alpha_{14}} = \left(\begin{array}{cc|c} 4g_{\beta_{24}} & 4g_{\beta_{34}} & -2g_{\varepsilon_4} \\ 4g_{\beta_{34}} & 4g_{\beta_{34}} & -2g_{\varepsilon_4} \\ \hline -2g_{\varepsilon_4} & -2g_{\varepsilon_4} & \frac{(g_{\varepsilon_4})^2}{g_{\beta_{14}}} \end{array}\right)$$

$$M_{-\alpha_{23}} = \left(\begin{array}{c|c|c|c|c} 4g_{\alpha_{12}} & -8\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\alpha_{14}}} & -4g_{\alpha_{12}} & -8\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{14}}} & -4\frac{g_{\alpha_{12}}g_{\varepsilon_3}}{g_{\varepsilon_1}} \\ \hline -8\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\alpha_{14}}} & 4g_{\alpha_{34}} & 4g_{\alpha_{34}} & 8\frac{g_{\alpha_{34}}g_{\beta_{34}}}{g_{\beta_{23}}} & 2g_{\alpha_{34}} \\ \hline -4g_{\alpha_{12}} & 4g_{\alpha_{34}} & 4g_{\beta_{13}} & 4g_{\beta_{34}} & 2g_{\varepsilon_3} \\ \hline -8\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{14}}} & 8\frac{g_{\alpha_{34}}g_{\beta_{34}}}{g_{\beta_{23}}} & 4g_{\beta_{34}} & 4g_{\beta_{34}} & 2g_{\varepsilon_3} \\ \hline -4\frac{g_{\alpha_{12}}g_{\varepsilon_3}}{g_{\varepsilon_1}} & 2g_{\alpha_{34}} & 2g_{\varepsilon_3} & 2g_{\varepsilon_3} & \frac{(g_{\varepsilon_3})^2}{g_{\beta_{23}}} \end{array}\right)$$

$$M_{-\alpha_{24}} = \left(\begin{array}{c|c|c|c} 4g_{\alpha_{12}} & 8\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{13}}} & -4g_{\alpha_{12}} & -4\frac{g_{\alpha_{12}}g_{\varepsilon_4}}{g_{\varepsilon_1}} \\ \hline 8\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{13}}} & 4g_{\beta_{34}} & -4g_{\beta_{34}} & -2g_{\varepsilon_4} \\ \hline -4g_{\alpha_{12}} & -4g_{\beta_{34}} & 4g_{\beta_{14}} & 2g_{\varepsilon_4} \\ \hline -4\frac{g_{\alpha_{12}}g_{\varepsilon_4}}{g_{\varepsilon_1}} & -2g_{\varepsilon_4} & 2g_{\varepsilon_4} & \frac{(g_{\varepsilon_4})^2}{g_{\beta_{24}}} \end{array}\right)$$

$$M_{-\alpha_{34}} = \left(\begin{array}{cc|cc|c}
4g_{\alpha_{13}} & 4g_{\alpha_{23}} & -4g_{\alpha_{13}} & -8\frac{g_{\alpha_{13}}g_{\beta_{24}}}{g_{\beta_{12}}} & -4\frac{g_{\alpha_{13}}g_{\varepsilon_4}}{g_{\varepsilon_1}} \\
4g_{\alpha_{23}} & 4g_{\alpha_{23}} & -8\frac{g_{\alpha_{23}}g_{\beta_{14}}}{g_{\beta_{12}}} & -4g_{\alpha_{23}} & -4\frac{g_{\alpha_{23}}g_{\varepsilon_4}}{g_{\varepsilon_2}} \\
\hline
-4g_{\alpha_{13}} & -8\frac{g_{\alpha_{23}}g_{\beta_{14}}}{g_{\beta_{12}}} & 4g_{\beta_{14}} & 4g_{\beta_{24}} & 2g_{\varepsilon_4} \\
-8\frac{g_{\alpha_{13}}g_{\beta_{24}}}{g_{\beta_{12}}} & -4g_{\alpha_{23}} & 4g_{\beta_{24}} & 4g_{\beta_{24}} & 2g_{\varepsilon_4} \\
\hline
-4\frac{g_{\alpha_{13}}g_{\varepsilon_4}}{g_{\varepsilon_1}} & -4\frac{g_{\alpha_{23}}g_{\varepsilon_4}}{g_{\varepsilon_2}} & 2g_{\varepsilon_4} & 2g_{\varepsilon_4} & \frac{(g_{\varepsilon_4})^2}{g_{\beta_{34}}}
\end{array}\right)$$

Those corresponding to β_{ij} are

$$M_{-\beta_{12}} = \emptyset \qquad M_{-\beta_{13}} = (4g_{\alpha_{23}}) \qquad M_{-\beta_{14}} = \begin{pmatrix} 4g_{\alpha_{24}} & 4g_{\alpha_{34}} \\ 4g_{\alpha_{34}} & 4g_{\alpha_{34}} \end{pmatrix}$$

$$M_{-\beta_{23}} = \left(\begin{array}{c|c} 4g_{\alpha_{12}} & -4g_{\alpha_{12}} \\ \hline -4g_{\alpha_{12}} & 4g_{\alpha_{13}} \end{array}\right) \qquad M_{-\beta_{24}} = \left(\begin{array}{c|c|c} 4g_{\alpha_{12}} & -4g_{\alpha_{12}} & 8\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\beta_{13}}} \\ \hline -4g_{\alpha_{12}} & 4g_{\alpha_{14}} & -4g_{\alpha_{34}} \\ \hline 8\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\beta_{13}}} & -4g_{\alpha_{34}} & 4g_{\alpha_{34}} \end{array}\right)$$

$$M_{-\beta_{34}} = \left(\begin{array}{cc|cc}
4g_{\alpha_{13}} & 4g_{\alpha_{23}} & -4g_{\alpha_{13}} & -8\frac{g_{\alpha_{13}}g_{\alpha_{24}}}{g_{\beta_{12}}} \\
4g_{\alpha_{23}} & 4g_{\alpha_{23}} & -8\frac{g_{\alpha_{23}}g_{\alpha_{14}}}{g_{\beta_{12}}} & -4g_{\alpha_{23}} \\
\hline
-4g_{\alpha_{13}} & -8\frac{g_{\alpha_{23}}g_{\alpha_{14}}}{g_{\beta_{12}}} & 4g_{\alpha_{14}} & 4g_{\alpha_{24}} \\
-8\frac{g_{\alpha_{13}}g_{\alpha_{24}}}{g_{\beta_{12}}} & -4g_{\alpha_{23}} & 4g_{\alpha_{24}} & 4g_{\alpha_{24}}
\end{array}\right)$$

and then we have for the ε_i:

$$M_{-\varepsilon_1} = \begin{pmatrix} 2g_{\varepsilon_2} & 2g_{\varepsilon_3} & 2g_{\varepsilon_4} \\ 2g_{\varepsilon_3} & 2g_{\varepsilon_3} & 2g_{\varepsilon_4} \\ 2g_{\varepsilon_4} & 2g_{\varepsilon_4} & 2g_{\varepsilon_4} \end{pmatrix} \qquad M_{-\varepsilon_2} = \left(\begin{array}{c|c|cc}
2g_{\alpha_{12}} & -2g_{\alpha_{12}} & 4\frac{g_{\alpha_{12}}g_{\varepsilon_3}}{g_{\beta_{13}}} & 4\frac{g_{\alpha_{12}}g_{\varepsilon_4}}{g_{\beta_{14}}} \\
\hline
-2g_{\alpha_{12}} & 2g_{\varepsilon_1} & -2g_{\varepsilon_3} & -2g_{\varepsilon_4} \\
\hline
4\frac{g_{\alpha_{12}}g_{\varepsilon_3}}{g_{\beta_{13}}} & -2g_{\varepsilon_3} & 2g_{\varepsilon_3} & 2g_{\varepsilon_4} \\
4\frac{g_{\alpha_{12}}g_{\varepsilon_4}}{g_{\beta_{14}}} & -2g_{\varepsilon_4} & 2g_{\varepsilon_4} & 2g_{\varepsilon_4}
\end{array}\right)$$

$$M_{-\varepsilon_3} = \left(\begin{array}{cc|cc|c}
2g_{\alpha_{13}} & 2g_{\alpha_{23}} & -2g_{\alpha_{13}} & -4\frac{g_{\alpha_{13}}g_{\varepsilon_2}}{g_{\beta_{12}}} & 4\frac{g_{\alpha_{13}}g_{\varepsilon_4}}{g_{\beta_{14}}} \\
2g_{\alpha_{23}} & 2g_{\alpha_{23}} & -4\frac{g_{\alpha_{23}}g_{\varepsilon_1}}{g_{\beta_{12}}} & -2g_{\alpha_{23}} & 4\frac{g_{\alpha_{23}}g_{\varepsilon_4}}{g_{\beta_{24}}} \\
\hline
-2g_{\alpha_{13}} & -4\frac{g_{\alpha_{23}}g_{\varepsilon_1}}{g_{\beta_{12}}} & 2g_{\varepsilon_1} & 2g_{\varepsilon_2} & -2g_{\varepsilon_4} \\
-4\frac{g_{\alpha_{13}}g_{\varepsilon_2}}{g_{\beta_{12}}} & -2g_{\alpha_{23}} & 2g_{\varepsilon_2} & 2g_{\varepsilon_2} & -2g_{\varepsilon_4} \\
\hline
4\frac{g_{\alpha_{13}}g_{\varepsilon_4}}{g_{\beta_{14}}} & 4\frac{g_{\alpha_{23}}g_{\varepsilon_4}}{g_{\beta_{24}}} & -2g_{\varepsilon_4} & -2g_{\varepsilon_4} & 2g_{\varepsilon_4}
\end{array}\right)$$

$$M_{-\varepsilon_4} = \left(\begin{array}{ccc|ccc}
2g_{\alpha_{14}} & 2g_{\alpha_{24}} & 2g_{\alpha_{34}} & -2g_{\alpha_{14}} & -4\frac{g_{\alpha_{14}}g_{\varepsilon_2}}{g_{\beta_{12}}} & -4\frac{g_{\alpha_{14}}g_{\varepsilon_3}}{g_{\beta_{13}}} \\
2g_{\alpha_{24}} & 2g_{\alpha_{24}} & 2g_{\alpha_{34}} & -4\frac{g_{\alpha_{24}}g_{\varepsilon_1}}{g_{\beta_{12}}} & -2g_{\alpha_{24}} & -4\frac{g_{\alpha_{24}}g_{\varepsilon_3}}{g_{\beta_{23}}} \\
2g_{\alpha_{34}} & 2g_{\alpha_{34}} & 2g_{\alpha_{34}} & -4\frac{g_{\alpha_{34}}g_{\varepsilon_1}}{g_{\beta_{13}}} & -4\frac{g_{\alpha_{34}}g_{\varepsilon_2}}{g_{\beta_{23}}} & -2g_{\alpha_{34}} \\
\hline
-2g_{\alpha_{14}} & -4\frac{g_{\alpha_{24}}g_{\varepsilon_1}}{g_{\beta_{12}}} & -4\frac{g_{\alpha_{34}}g_{\varepsilon_1}}{g_{\beta_{13}}} & 2g_{\varepsilon_1} & 2g_{\varepsilon_2} & 2g_{\varepsilon_3} \\
-4\frac{g_{\alpha_{14}}g_{\varepsilon_2}}{g_{\beta_{12}}} & -2g_{\alpha_{24}} & -4\frac{g_{\alpha_{34}}g_{\varepsilon_2}}{g_{\beta_{23}}} & 2g_{\varepsilon_2} & 2g_{\varepsilon_2} & 2g_{\varepsilon_3} \\
-4\frac{g_{\alpha_{14}}g_{\varepsilon_3}}{g_{\beta_{13}}} & -4\frac{g_{\alpha_{24}}g_{\varepsilon_3}}{g_{\beta_{23}}} & -2g_{\alpha_{34}} & 2g_{\varepsilon_3} & 2g_{\varepsilon_3} & 2g_{\varepsilon_3}
\end{array}\right)$$

Ordering the basis as given in (8.2) the matrix of the trivial module is given by

$$M_0 = \begin{pmatrix} A & \langle AB \rangle & \langle AC \rangle \\ \langle AB \rangle^T & B & \langle BC \rangle \\ \langle AC \rangle^T & \langle BC \rangle^T & C \end{pmatrix}$$

with

$$A = \left(\begin{array}{c|cc|ccc} 8g_{\alpha_{34}} & * & 4g_{\alpha_{34}} & 0 & * & 4g_{\alpha_{34}} \\ \hline * & 8g_{\alpha_{23}} & 4g_{\alpha_{23}} & * & 4g_{\alpha_{23}} & 0 \\ 4g_{\alpha_{34}} & 4g_{\alpha_{23}} & 8g_{\alpha_{24}} & * & 0 & 4g_{\alpha_{24}} \\ \hline 0 & * & * & 8g_{\alpha_{12}} & 4g_{\alpha_{12}} & 4g_{\alpha_{12}} \\ * & 4g_{\alpha_{23}} & 0 & 4g_{\alpha_{12}} & 8g_{\alpha_{13}} & 4g_{\alpha_{13}} \\ 4g_{\alpha_{34}} & 0 & 4g_{\alpha_{24}} & 4g_{\alpha_{12}} & 4g_{\alpha_{13}} & 8g_{\alpha_{14}} \end{array}\right) \quad B = \left(\begin{array}{c|cc|ccc} 8g_{\beta_{34}} & 4g_{\beta_{34}} & 4g_{\beta_{34}} & 0 & 4g_{\beta_{34}} & 4g_{\beta_{34}} \\ \hline 4g_{\beta_{34}} & 8g_{\beta_{23}} & 4g_{\beta_{24}} & 4g_{\beta_{23}} & 4g_{\beta_{23}} & 0 \\ 4g_{\beta_{34}} & 4g_{\beta_{24}} & 8g_{\beta_{24}} & 4g_{\beta_{24}} & 0 & 4g_{\beta_{24}} \\ \hline 0 & 4g_{\beta_{23}} & 4g_{\beta_{24}} & 8g_{\beta_{12}} & 4g_{\beta_{13}} & 4g_{\beta_{14}} \\ 4g_{\beta_{34}} & 4g_{\beta_{23}} & 0 & 4g_{\beta_{13}} & 8g_{\beta_{13}} & 4g_{\beta_{14}} \\ 4g_{\beta_{34}} & 0 & 4g_{\beta_{24}} & 4g_{\beta_{14}} & 4g_{\beta_{14}} & 8g_{\beta_{14}} \end{array}\right)$$

$$C = \begin{pmatrix} g_{\varepsilon_4} & \frac{(g_{\varepsilon_4})^2}{g_{\beta_{34}}} & \frac{(g_{\varepsilon_4})^2}{g_{\beta_{24}}} & \frac{(g_{\varepsilon_4})^2}{g_{\beta_{14}}} \\ \frac{(g_{\varepsilon_4})^2}{g_{\beta_{34}}} & g_{\varepsilon_3} & \frac{(g_{\varepsilon_3})^2}{g_{\beta_{23}}} & \frac{(g_{\varepsilon_3})^2}{g_{\beta_{13}}} \\ \frac{(g_{\varepsilon_4})^2}{g_{\beta_{24}}} & \frac{(g_{\varepsilon_3})^2}{g_{\beta_{23}}} & g_{\varepsilon_2} & \frac{(g_{\varepsilon_2})^2}{g_{\beta_{12}}} \\ \frac{(g_{\varepsilon_4})^2}{g_{\beta_{14}}} & \frac{(g_{\varepsilon_3})^2}{g_{\beta_{13}}} & \frac{(g_{\varepsilon_2})^2}{g_{\beta_{12}}} & g_{\varepsilon_1} \end{pmatrix} \quad \langle AB \rangle = \left(\begin{array}{c|cc|ccc} 0 & 4g_{\alpha_{34}} & * & 0 & 4g_{\alpha_{34}} & * \\ \hline * & 0 & 4g_{\alpha_{23}} & 4g_{\alpha_{23}} & * & 0 \\ * & 4g_{\alpha_{24}} & 0 & 4g_{\alpha_{24}} & 0 & * \\ \hline 0 & * & * & 0 & 4g_{\alpha_{12}} & 4g_{\alpha_{12}} \\ * & * & 0 & 4g_{\alpha_{13}} & 0 & 4g_{\alpha_{13}} \\ * & 0 & * & 4g_{\alpha_{14}} & 4g_{\alpha_{14}} & 0 \end{array}\right)$$

$$\langle AC \rangle = \left(\begin{array}{cccc} ** & 2g_{\alpha_{34}} & 0 & 0 \\ \hline 0 & ** & 2g_{\alpha_{23}} & 0 \\ ** & 0 & 2g_{\alpha_{24}} & 0 \\ \hline 0 & 0 & ** & 2g_{\alpha_{12}} \\ 0 & ** & 0 & 2g_{\alpha_{13}} \\ ** & 0 & 0 & 2g_{\alpha_{14}} \end{array}\right) \quad \langle BC \rangle = \left(\begin{array}{cccc} 2g_{\varepsilon_4} & 2g_{\varepsilon_3} & 0 & 0 \\ \hline 0 & 2g_{\varepsilon_3} & 2g_{\varepsilon_2} & 0 \\ 2g_{\varepsilon_4} & 0 & 2g_{\varepsilon_2} & 0 \\ \hline 0 & 0 & 2g_{\varepsilon_2} & 2g_{\varepsilon_1} \\ 0 & 2g_{\varepsilon_3} & 0 & 2g_{\varepsilon_1} \\ 2g_{\varepsilon_4} & 0 & 0 & 2g_{\varepsilon_1} \end{array}\right)$$

where we used $*$(respectively $**$) at the entry (α, β) as place holder for $-8\frac{g_\alpha g_\beta}{g_{\alpha+\beta}}$ $\Big($respectively $-4\frac{g_\alpha g_\beta}{g_{\alpha+\beta}}\Big)$. In the following, we present the full modification for the toric isotropy.

Toric Isotropy

Now we consider the Kähler Einstein metric of $(SO(9), \mathbb{T}, J_{std})$, which is given on the simple roots by

α	α_{12}	α_{23}	α_{34}	ε_4
g_α	2	2	2	1

First of all, we notice that we do not have to consider the matrices $M_{-\eta}$ for η being one of $\{\beta_{12}, \beta_{13}, \beta_{14}, \beta_{23}\}$, since they are positive semi definite by the auxiliary lemma. We remark here that also $M_{-\varepsilon_1}$ is positive definite but we will need it to compensate for the other matrices. The matrix of the trivial module is given by

$$M_0 = \left(\begin{array}{cccccc|cccccc|cccc}
16 & -8 & 8 & 0 & -\frac{32}{3} & 8 & 0 & 8 & -12 & 0 & 8 & -\frac{64}{5} & -\frac{8}{3} & 4 & 0 & 0 \\
-8 & 16 & 8 & -8 & 8 & 0 & -\frac{32}{3} & 0 & 8 & 8 & -\frac{40}{3} & 0 & 0 & -\frac{24}{5} & 4 & 0 \\
8 & 8 & 32 & -\frac{32}{3} & 0 & 16 & -16 & 16 & 0 & 16 & 0 & -\frac{64}{3} & -\frac{16}{5} & 0 & 8 & 0 \\
0 & -8 & -\frac{32}{3} & 16 & 8 & 8 & 0 & -\frac{64}{5} & -12 & 0 & 8 & 8 & 0 & 0 & -\frac{40}{7} & 4 \\
-\frac{32}{3} & 8 & 0 & 8 & 32 & 16 & -16 & -\frac{64}{3} & 0 & 16 & 0 & 16 & 0 & -\frac{48}{7} & 0 & 8 \\
8 & 0 & 16 & 8 & 16 & 48 & -\frac{96}{5} & 0 & -24 & 24 & 24 & 0 & -\frac{24}{7} & 0 & 0 & 12 \\
\hline
0 & -\frac{32}{3} & -16 & 0 & -16 & -\frac{96}{5} & 32 & 16 & 16 & 0 & 16 & 16 & 2 & 6 & 0 & 0 \\
8 & 0 & 16 & -\frac{64}{5} & -\frac{64}{3} & 0 & 16 & 64 & 24 & 32 & 32 & 0 & 0 & 6 & 10 & 0 \\
-12 & 8 & 0 & -12 & 0 & -24 & 16 & 24 & 48 & 24 & 0 & 24 & 2 & 0 & 10 & 0 \\
0 & 8 & 16 & 0 & 16 & 24 & 0 & 32 & 24 & 96 & 40 & 32 & 0 & 0 & 10 & 14 \\
8 & -\frac{40}{3} & 0 & 8 & 0 & 24 & 16 & 32 & 0 & 40 & 80 & 32 & 0 & 6 & 0 & 14 \\
-\frac{64}{5} & 0 & -\frac{64}{3} & 8 & 16 & 0 & 16 & 0 & 24 & 32 & 32 & 64 & 2 & 0 & 0 & 14 \\
\hline
-\frac{8}{3} & 0 & -\frac{16}{5} & 0 & 0 & -\frac{24}{7} & 2 & 0 & 2 & 0 & 0 & 2 & 1 & \frac{1}{4} & \frac{1}{6} & \frac{1}{8} \\
4 & -\frac{24}{5} & 0 & 0 & -\frac{48}{7} & 0 & 6 & 6 & 0 & 0 & 6 & 0 & \frac{1}{4} & 3 & \frac{9}{8} & \frac{9}{10} \\
0 & 4 & 8 & -\frac{40}{7} & 0 & 0 & 0 & 10 & 10 & 10 & 0 & 0 & \frac{1}{6} & \frac{9}{8} & 5 & \frac{25}{12} \\
0 & 0 & 0 & 4 & 8 & 12 & 0 & 0 & 0 & 14 & 14 & 14 & \frac{1}{8} & \frac{9}{10} & \frac{25}{12} & 7
\end{array}\right)$$

It is not positive definite and hence further four forms are necessary. The other matrices are

$$M_{-\alpha_{12}} = \begin{pmatrix} 8 & 8 & \frac{32}{3} & 8 & 4 \\ 8 & 16 & 16 & 16 & 8 \\ \frac{32}{3} & 16 & 32 & 24 & 10 \\ 8 & 16 & 24 & 24 & 10 \\ 4 & 8 & 10 & 10 & \frac{25}{12} \end{pmatrix} \qquad M_{-\alpha_{13}} = \begin{pmatrix} 8 & -8 & \frac{32}{5} & 4 \\ -8 & 32 & -16 & -6 \\ \frac{32}{5} & -16 & 16 & 6 \\ 4 & -6 & 6 & \frac{9}{10} \end{pmatrix}$$

$$M_{-\alpha_{14}} = \begin{pmatrix} 24 & 16 & -2 \\ 16 & 16 & -2 \\ -2 & -2 & \frac{1}{8} \end{pmatrix} \qquad M_{-\alpha_{23}} = \begin{pmatrix} 8 & -\frac{16}{3} & -8 & -8 & -\frac{24}{7} \\ -\frac{16}{3} & 8 & 8 & 8 & 4 \\ -8 & 8 & 40 & 16 & 6 \\ -8 & 8 & 16 & 16 & 6 \\ -\frac{24}{7} & 4 & 6 & 6 & \frac{9}{8} \end{pmatrix}$$

$$M_{-\alpha_{24}} = \begin{pmatrix} 8 & \frac{32}{5} & -8 & -\frac{8}{7} \\ \frac{32}{5} & 16 & -16 & -2 \\ -8 & -16 & 32 & 2 \\ -\frac{8}{7} & -2 & 2 & \frac{1}{6} \end{pmatrix} \qquad M_{-\alpha_{34}} = \begin{pmatrix} 16 & 8 & -16 & -16 & -\frac{16}{7} \\ 8 & 8 & -\frac{32}{3} & -8 & -\frac{8}{5} \\ -16 & -\frac{32}{3} & 32 & 24 & 2 \\ -16 & -8 & 24 & 24 & 2 \\ -\frac{16}{7} & -\frac{8}{5} & 2 & 2 & \frac{1}{4} \end{pmatrix}$$

The remaining matrices are

$$M_{-\beta_{24}} = \begin{pmatrix} 8 & -8 & \frac{16}{5} \\ -8 & 24 & -8 \\ \frac{16}{5} & -8 & 8 \end{pmatrix} \qquad M_{-\beta_{34}} = \begin{pmatrix} 16 & 8 & -16 & -\frac{32}{3} \\ 8 & 8 & -8 & -8 \\ -16 & -8 & 24 & 16 \\ -\frac{32}{3} & -8 & 16 & 16 \end{pmatrix}$$

and

$$M_{-\varepsilon_1} = \begin{pmatrix} 10 & 6 & 2 \\ 6 & 6 & 2 \\ 2 & 2 & 2 \end{pmatrix} \qquad M_{-\varepsilon_2} = \begin{pmatrix} 4 & -4 & \frac{12}{5} & 1 \\ -4 & 14 & -6 & -2 \\ \frac{12}{5} & -6 & 6 & 2 \\ 1 & -2 & 2 & 2 \end{pmatrix}$$

$$M_{-\varepsilon_3} = \begin{pmatrix} 8 & 4 & -8 & -\frac{20}{3} & 2 \\ 4 & 4 & -\frac{14}{3} & -4 & \frac{4}{3} \\ -8 & -\frac{14}{3} & 14 & 10 & -2 \\ -\frac{20}{3} & -4 & 10 & 10 & -2 \\ 2 & \frac{4}{3} & -2 & -2 & 2 \end{pmatrix} \qquad M_{-\varepsilon_4} = \begin{pmatrix} 12 & 8 & 4 & -12 & -10 & -\frac{36}{5} \\ 8 & 8 & 4 & -\frac{28}{3} & -8 & -6 \\ 4 & 4 & 4 & -\frac{28}{5} & -5 & -4 \\ -12 & -\frac{28}{3} & -\frac{28}{5} & 14 & 10 & 6 \\ -10 & -8 & -5 & 10 & 10 & 6 \\ -\frac{36}{5} & -6 & -4 & 6 & 6 & 6 \end{pmatrix}$$

One observes that in the α_{ij} matrices positive definiteness is prevented by the relatively small last entry on the diagonal. Not using the actual criteria of diagonal dominance for positive definiteness but inspired by its idea , we will use four forms to increase that diagonal entry and decrease the absolute values of the entries in the last column and row of these matrices. Increasing the diagonal will modify M_0 and changing off diagonal entries modifies the other M_η. We represent all elementary operations in the following table, where we use the same notation as in the modifications of $(G_2, \mathbb{T}, J_{std})$ in section 4.

Exemplary, we explain how we obtain the first forced modification from the first intended modification:

Example 107. *The entry* $(1,5)$ *of* $M_{-\alpha_{12}}$ *corresponds by proposition 79 to the pair of pairs of roots* $(\alpha_{23}, \alpha_{13}), (\varepsilon_2, \varepsilon_1)$*. By equation* (3.2) *the other entry we are forced to change corresponds to the pair* $(\alpha_{23}, \varepsilon_2), (\alpha_{13}, \varepsilon_1)$ *which is an entry of the matrix* M_η *with* $\eta = -\alpha_{12} + (\alpha_{13} - \varepsilon_2) = -\varepsilon_3$*. By proposition 81 the pair corresponds to the entry* $(2,1)$ *of said matrix. By symmetry that is the same as* $(1,2)$*.*

Value	Intended	Forced	Value	Intended	Forced
$-\frac{3}{5}$	$(M_{-\alpha_{12}})_{(1,5)}$	$(M_{-\varepsilon_3})_{(1,2)}$	$-\frac{3}{10}$	$(M_{-\alpha_{24}})_{(3,4)}$	$(M_{-\varepsilon_1})_{(1,3)}$
$-\frac{6}{5}$	$(M_{-\alpha_{12}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(1,2)}$	$\frac{1}{10}$	$(M_{-\alpha_{24}})_{(4,4)}$	$(M_0)_{(13,15)}$
$-\frac{3}{2}$	$(M_{-\alpha_{12}})_{(3,5)}$	$(M_{-\varepsilon_3})_{(3,4)}$	$-\frac{37}{80}$	$(M_{-\alpha_{34}})_{(3,5)}$	$(M_{-\varepsilon_1})_{(2,3)}$
$-\frac{3}{2}$	$(M_{-\alpha_{12}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(4,5)}$	$-\frac{1}{2}$	$(M_{-\alpha_{34}})_{(4,5)}$	$(M_{-\varepsilon_2})_{(3,4)}$
$\frac{5}{4}$	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$	$\frac{1}{4}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(13,14)}$
-1	$(M_{-\alpha_{13}})_{(1,4)}$	$(M_{-\varepsilon_4})_{(1,3)}$	$-\frac{5}{4}$	$(M_0)_{(3,15)}$	$(M_{-\varepsilon_4})_{(2,2)}$
$\frac{3}{2}$	$(M_{-\alpha_{13}})_{(2,4)}$	$(M_{-\varepsilon_2})_{(2,3)}$	$-\frac{9}{4}$	$(M_0)_{(4,16)}$	$(M_{-\varepsilon_2})_{(1,1)}$
$-\frac{17}{8}$	$(M_{-\alpha_{13}})_{(3,4)}$	$(M_{-\varepsilon_4})_{(4,6)}$	$\frac{107}{80}$	$(M_0)_{(7,13)}$	$(M_{-\varepsilon_3})_{(5,5)}$
$\frac{9}{10}$	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$	$-\frac{3}{10}$	$(M_0)_{(8,14)}$	$(M_{-\varepsilon_2})_{(3,3)}$
$\frac{3}{10}$	$(M_{-\alpha_{14}})_{(1,3)}$	$(M_{-\varepsilon_2})_{(2,4)}$	$\frac{77}{80}$	$(M_0)_{(9,13)}$	$(M_{-\varepsilon_2})_{(4,4)}$
$\frac{3}{10}$	$(M_{-\alpha_{14}})_{(2,3)}$	$(M_{-\varepsilon_3})_{(3,5)}$	$-\frac{5}{4}$	$(M_0)_{(9,15)}$	$(M_{-\varepsilon_4})_{(5,5)}$
$\frac{3}{40}$	$(M_{-\alpha_{14}})_{(3,3)}$	$(M_0)_{(13,16)}$	$-\frac{51}{200}$	$(M_0)_{(10,15)}$	$(M_{-\varepsilon_1})_{(1,1)}$
-1	$(M_{-\alpha_{23}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(2,3)}$	$-\frac{15}{2}$	$(M_0)_{(10,16)}$	$(M_{-\varepsilon_2})_{(2,2)}$
$-\frac{3}{2}$	$(M_{-\alpha_{23}})_{(3,5)}$	$(M_{-\varepsilon_1})_{(1,2)}$	$-\frac{25}{2}$	$(M_0)_{(11,12)}$	$(M_{-\alpha_{34}})_{(3,3)}$
$-\frac{3}{2}$	$(M_{-\alpha_{23}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(5,6)}$	$-\frac{17}{8}$	$(M_0)_{(11,14)}$	$(M_{-\varepsilon_1})_{(2,2)}$
$\frac{9}{8}$	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(14,15)}$	$-\frac{81}{40}$	$(M_0)_{(11,16)}$	$(M_{-\varepsilon_3})_{(3,3)}$
$\frac{3}{10}$	$(M_{-\alpha_{24}})_{(2,4)}$	$(M_{-\varepsilon_3})_{(4,5)}$	$\frac{5}{4}$	$(M_0)_{(12,13)}$	$(M_{-\varepsilon_1})_{(3,3)}$

By calculating the determinants of the minors of the resulting matrices, we see that all of them are positive except for M_0. In fact, the modified matrices are:

$$M_{-\alpha_{12}} = \begin{pmatrix} 8 & 8 & \frac{32}{3} & 8 & \frac{17}{5} \\ 8 & 16 & 16 & 16 & \frac{34}{5} \\ \frac{32}{3} & 16 & 32 & 24 & \frac{17}{2} \\ 8 & 16 & 24 & 24 & \frac{17}{2} \\ \frac{17}{5} & \frac{34}{5} & \frac{17}{2} & \frac{17}{2} & \frac{10}{3} \end{pmatrix} \qquad M_{-\alpha_{13}} = \begin{pmatrix} 8 & -8 & \frac{32}{5} & 3 \\ -8 & 32 & -16 & -\frac{9}{2} \\ \frac{32}{5} & -16 & 16 & \frac{31}{8} \\ 3 & -\frac{9}{2} & \frac{31}{8} & \frac{9}{5} \end{pmatrix}$$

$$M_{-\alpha_{14}} = \begin{pmatrix} 24 & 16 & -\frac{17}{10} \\ 16 & 16 & -\frac{17}{10} \\ -\frac{17}{10} & -\frac{17}{10} & \frac{1}{5} \end{pmatrix} \qquad M_{-\alpha_{23}} = \begin{pmatrix} 8 & -\frac{16}{3} & -8 & -8 & -\frac{24}{7} \\ -\frac{16}{3} & 8 & 8 & 8 & 3 \\ -8 & 8 & 40 & 16 & \frac{9}{2} \\ -8 & 8 & 16 & 16 & \frac{9}{2} \\ -\frac{24}{7} & 3 & \frac{9}{2} & \frac{9}{2} & \frac{9}{4} \end{pmatrix}$$

$$M_{-\alpha_{24}} = \begin{pmatrix} 8 & \frac{32}{5} & -8 & -\frac{8}{7} \\ \frac{32}{5} & 16 & -16 & -\frac{17}{10} \\ -8 & -16 & 32 & \frac{17}{10} \\ -\frac{8}{7} & -\frac{17}{10} & \frac{17}{10} & \frac{4}{15} \end{pmatrix} \qquad M_{-\alpha_{34}} = \begin{pmatrix} 16 & 8 & -16 & -16 & -\frac{16}{7} \\ 8 & 8 & -\frac{32}{3} & -8 & -\frac{8}{5} \\ -16 & -\frac{32}{3} & \frac{89}{2} & 24 & \frac{123}{80} \\ -16 & -8 & 24 & 24 & \frac{3}{2} \\ -\frac{16}{7} & -\frac{8}{5} & \frac{123}{80} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

The remaining matrices are

$$M_{-\varepsilon_1} = \begin{pmatrix} \frac{2051}{200} & \frac{15}{2} & \frac{23}{10} \\ \frac{15}{2} & \frac{65}{8} & \frac{197}{80} \\ \frac{23}{10} & \frac{197}{80} & \frac{3}{4} \end{pmatrix} \qquad M_{-\varepsilon_2} = \begin{pmatrix} \frac{25}{4} & -4 & \frac{12}{5} & 1 \\ -4 & \frac{43}{2} & -\frac{15}{2} & -\frac{23}{10} \\ \frac{12}{5} & -\frac{15}{2} & \frac{63}{10} & \frac{5}{2} \\ 1 & -\frac{23}{10} & \frac{5}{2} & \frac{83}{80} \end{pmatrix}$$

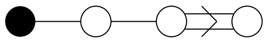
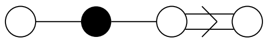
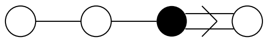
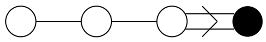
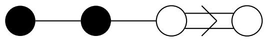
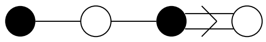
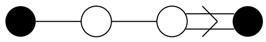
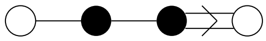
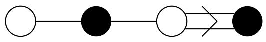
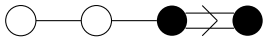
$$M_{-\varepsilon_3} = \begin{pmatrix} 8 & \frac{23}{5} & -8 & -\frac{20}{3} & 2 \\ \frac{23}{5} & 4 & -\frac{14}{3} & -4 & \frac{4}{3} \\ -8 & -\frac{14}{3} & \frac{641}{40} & \frac{23}{2} & -\frac{23}{10} \\ -\frac{20}{3} & -4 & \frac{23}{2} & 10 & -\frac{23}{10} \\ 2 & \frac{4}{3} & -\frac{23}{10} & -\frac{23}{10} & \frac{53}{80} \end{pmatrix} \qquad M_{-\varepsilon_4} = \begin{pmatrix} 12 & \frac{46}{5} & 5 & -12 & -10 & -\frac{36}{5} \\ \frac{46}{5} & 8 & 5 & -\frac{28}{3} & -8 & -6 \\ 5 & 5 & 4 & -\frac{28}{5} & -5 & -4 \\ -12 & -\frac{28}{3} & -\frac{28}{5} & 14 & \frac{23}{2} & \frac{65}{8} \\ -10 & -8 & -5 & \frac{23}{2} & 10 & \frac{15}{2} \\ -\frac{36}{5} & -6 & -4 & \frac{65}{8} & \frac{15}{2} & 6 \end{pmatrix}$$

$$M_0 = \left(\begin{array}{cccccc|cccccc|cccc}
16 & -8 & 8 & 0 & -\frac{32}{3} & 8 & 0 & 8 & -12 & 0 & 8 & -\frac{64}{5} & -\frac{8}{3} & 4 & 0 & 0 \\
-8 & 16 & 8 & -8 & 8 & 0 & -\frac{32}{3} & 0 & 8 & 8 & -\frac{40}{3} & 0 & 0 & -\frac{24}{5} & 4 & 0 \\
8 & 8 & 32 & -\frac{32}{3} & 0 & 16 & -16 & 16 & 0 & 16 & 0 & -\frac{64}{3} & -\frac{16}{5} & 0 & \frac{27}{4} & 0 \\
0 & -8 & -\frac{32}{3} & 16 & 8 & 8 & 0 & -\frac{64}{5} & -12 & 0 & 8 & 8 & 0 & 0 & -\frac{40}{7} & \frac{7}{4} \\
-\frac{32}{3} & 8 & 0 & 8 & 32 & 16 & -16 & -\frac{64}{3} & 0 & 16 & 0 & 16 & 0 & -\frac{48}{7} & 0 & 8 \\
8 & 0 & 16 & 8 & 16 & 48 & -\frac{96}{5} & 0 & -24 & 24 & 24 & 0 & -\frac{24}{7} & 0 & 0 & 12 \\
\hline
0 & -\frac{32}{3} & -16 & 0 & -16 & -\frac{96}{5} & 32 & 16 & 16 & 0 & 16 & 16 & \frac{267}{80} & 6 & 0 & 0 \\
8 & 0 & 16 & -\frac{64}{5} & -\frac{64}{3} & 0 & 16 & 64 & 24 & 32 & 32 & 0 & 0 & \frac{57}{10} & 10 & 0 \\
-12 & 8 & 0 & -12 & 0 & -24 & 16 & 24 & 48 & 24 & 0 & 24 & \frac{237}{80} & 0 & \frac{35}{4} & 0 \\
0 & 8 & 16 & 0 & 16 & 24 & 0 & 32 & 24 & 96 & 40 & 32 & 0 & 0 & \frac{1949}{200} & \frac{13}{2} \\
8 & -\frac{40}{3} & 0 & 8 & 0 & 24 & 16 & 32 & 0 & 40 & 80 & \frac{39}{2} & 0 & \frac{31}{8} & 0 & \frac{479}{40} \\
-\frac{64}{5} & 0 & -\frac{64}{3} & 8 & 16 & 0 & 16 & 0 & 24 & 32 & \frac{39}{2} & 64 & \frac{13}{4} & 0 & 0 & 14 \\
\hline
-\frac{8}{3} & 0 & -\frac{16}{5} & 0 & 0 & -\frac{24}{7} & \frac{267}{80} & 0 & \frac{237}{80} & 0 & 0 & \frac{13}{4} & 1 & 0 & \frac{1}{15} & \frac{1}{20} \\
4 & -\frac{24}{5} & 0 & 0 & -\frac{48}{7} & 0 & 6 & \frac{57}{10} & 0 & 0 & \frac{31}{8} & 0 & 0 & 3 & 0 & 0 \\
0 & 4 & \frac{27}{4} & -\frac{40}{7} & 0 & 0 & 0 & 10 & \frac{35}{4} & \frac{1949}{200} & 0 & 0 & \frac{1}{15} & 0 & 5 & \frac{5}{6} \\
0 & 0 & 0 & \frac{7}{4} & 8 & 12 & 0 & 0 & 0 & \frac{13}{2} & \frac{479}{40} & 14 & \frac{1}{20} & 0 & \frac{5}{6} & 7
\end{array}\right)$$

Now all M_η are positive definite and the following argument turns also M_0 positive definite and therefore finishes the proof for $(SO(9), \mathbb{T}^4, J_{std})$. The rows and columns 10 and 16 of M_0 have nonnegative entries. Hence, by the same argument as in the example of G_2 we erase all off diagonal entries of rows and columns 10 and 16. Then the new M_0 is a block matrix consisting of a positive definite two dimensional diagonal block and a 14×14 block. Calculating the minors of that block proves that it is positive definite and hence we a have positive modified holomorphic curvature tensor.

Larger Isotropy

We will not present the above for all K in the same detail. In fact, we will present here the different possible K and their Kähler Einstein metrics. By corollary 50 the curvature matrices are then obtained by considering the correct submatrices of the matrices of $(SO(9), \mathbb{T}, J_{std})$ with the new metric coefficients. The difficulty lies in determining suitable four forms to turn these matrices positive definite. Even though the four forms are essential to prove positive holomorphic curvature, it does not seem to be very relevant to see the actual values at this point. Therefore we present them in the appendix for the interested reader. For each K, we will give the indices of the rows and columns of the curvature matrices that have to be erased to obtain the restriction and give a detailed table of the used four forms. We only consider the case $dim(\mathfrak{z}(k)) > 1$, since the other cases are covered by Itoh.

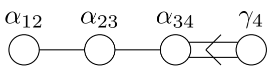
K	$D = (V, E)$	γ_K^*	$g_{\alpha_{12}}$	$g_{\alpha_{23}}$	$g_{\alpha_{34}}$	g_{ε_4}
$SU(2)\ \mathbb{T}^3$		α_{12}	0	3	2	1
$\mathbb{S}^1\ SU(2)\ \mathbb{T}^2$		α_{23}	3	0	3	1
$\mathbb{T}^2\ SU(2)\ \mathbb{S}^1$		α_{34}	2	3	0	2
$\mathbb{T}^3\ SU(2)$		ε_4	2	2	3	0
$SU(3)\ \mathbb{T}^2$		$2\alpha_{13}$	0	0	4	1
$SU(2)\ \mathbb{S}^1\ SU(2)\ \mathbb{S}^1$		$\alpha_{12} - \alpha_{34}$	0	4	0	2
$SU(2)\ \mathbb{T}^2\ SU(2)$		$\alpha_{12} + \varepsilon_4$	0	3	3	0
$\mathbb{S}^1\ SU(3)\ \mathbb{S}^1$		$2\alpha_{24}$	4	0	0	3
$\mathbb{S}^1\ SU(2)\ \mathbb{S}^1\ SU(2)$		$\alpha_{23} + \varepsilon_4$	3	0	4	0
$\mathbb{T}^2\ SO(5)$		$3\varepsilon_3 + \varepsilon_4$	2	5	0	0

8.3 The case of $Sp(4)$

The positive roots are

$$\Delta_{\mathfrak{g}}^+ = \{\alpha_{34}, \alpha_{23}, \alpha_{24}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \beta_{34}, \beta_{23}, \beta_{24}, \beta_{12}, \beta_{13}, \beta_{14}, \gamma_4, \gamma_3, \gamma_2, \gamma_1\} \tag{8.3}$$

and the simple roots are placed on the Dynkin diagram as follows



After applying the general modifications of section 7.3 the matrices M_η representing the modified holomorphic curvature tensor are indexed with $\eta \in \Delta_{\mathfrak{g}}^- \cup \{0\}$. In the case of $Sp(4)$ this corresponds to the following:

We begin with the ones corresponding to $-\alpha_{ij}$.

$$M_{-\alpha_{12}} = \left(\begin{array}{cc|cc|c|c}
g_{\alpha_{23}} & g_{\alpha_{23}} & \frac{g_{\alpha_{23}}(g_{\alpha_{13}}+g_{\beta_{12}})}{g_{\beta_{12}}} & g_{\alpha_{23}} & g_{\alpha_{23}} & g_{\alpha_{23}} \\
g_{\alpha_{23}} & g_{\alpha_{24}} & g_{\alpha_{24}} & \frac{g_{\alpha_{24}}(g_{\alpha_{14}}+g_{\beta_{12}})}{g_{\beta_{12}}} & g_{\alpha_{24}} & g_{\alpha_{24}} \\ \hline
\frac{g_{\alpha_{23}}(g_{\alpha_{13}}+g_{\beta_{12}})}{g_{\beta_{12}}} & g_{\alpha_{24}} & g_{\beta_{23}} & g_{\beta_{24}} & g_{\beta_{23}} & g_{\beta_{23}} \\
g_{\alpha_{23}} & \frac{g_{\alpha_{24}}(g_{\alpha_{14}}+g_{\beta_{12}})}{g_{\beta_{12}}} & g_{\beta_{24}} & g_{\beta_{24}} & g_{\beta_{24}} & g_{\beta_{24}} \\ \hline
g_{\alpha_{23}} & g_{\alpha_{24}} & g_{\beta_{23}} & g_{\beta_{24}} & g_{\beta_{12}} & g_{\gamma_2} \\ \hline
g_{\alpha_{23}} & g_{\alpha_{24}} & g_{\beta_{23}} & g_{\beta_{24}} & g_{\gamma_2} & g_{\gamma_2}
\end{array}\right)$$

$$M_{-\alpha_{13}} = \left(\begin{array}{c|c|c|c|c}
g_{\alpha_{34}} & g_{\alpha_{34}} & \frac{g_{\alpha_{34}}(g_{\alpha_{14}}+g_{\beta_{13}})}{g_{\beta_{13}}} & g_{\alpha_{34}} & g_{\alpha_{34}} \\ \hline
g_{\alpha_{34}} & g_{\beta_{23}} & g_{\beta_{34}} & g_{\beta_{23}} & g_{\gamma_3} \\ \hline
\frac{g_{\alpha_{34}}(g_{\alpha_{14}}+g_{\beta_{13}})}{g_{\beta_{13}}} & g_{\beta_{34}} & g_{\beta_{34}} & g_{\beta_{34}} & g_{\beta_{34}} \\ \hline
g_{\alpha_{34}} & g_{\beta_{23}} & g_{\beta_{34}} & g_{\beta_{13}} & g_{\gamma_3} \\ \hline
g_{\alpha_{34}} & g_{\gamma_3} & g_{\beta_{34}} & g_{\gamma_3} & g_{\gamma_3}
\end{array}\right)$$

$$M_{-\alpha_{14}} = \left(\begin{array}{cc|c|c}
g_{\beta_{24}} & g_{\beta_{34}} & g_{\beta_{24}} & g_{\gamma_4} \\
g_{\beta_{34}} & g_{\beta_{34}} & g_{\beta_{34}} & g_{\gamma_4} \\ \hline
g_{\beta_{24}} & g_{\beta_{34}} & g_{\beta_{14}} & g_{\gamma_4} \\ \hline
g_{\gamma_4} & g_{\gamma_4} & g_{\gamma_4} & g_{\gamma_4}
\end{array}\right)$$

$$M_{-\alpha_{23}} = \left(\begin{array}{c|c|c|c|c|c}
g_{\alpha_{12}} & -2\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\alpha_{14}}} & -\frac{g_{\alpha_{12}}g_{\gamma_3}}{g_{\gamma_1}} & -2\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{14}}} & -2\frac{g_{\alpha_{12}}g_{\beta_{23}}}{g_{\beta_{12}}} & -2\frac{g_{\alpha_{12}}g_{\gamma_3}}{g_{\beta_{13}}} \\ \hline
-2\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\alpha_{14}}} & g_{\alpha_{34}} & g_{\alpha_{34}} & \frac{g_{\alpha_{34}}(g_{\alpha_{24}}+g_{\beta_{23}})}{g_{\beta_{23}}} & g_{\alpha_{34}} & g_{\alpha_{34}} \\ \hline
-\frac{g_{\alpha_{12}}g_{\gamma_3}}{g_{\gamma_1}} & g_{\alpha_{34}} & g_{\beta_{13}} & g_{\beta_{34}} & g_{\beta_{23}} & g_{\gamma_3} \\ \hline
-2\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{14}}} & \frac{g_{\alpha_{34}}(g_{\alpha_{24}}+g_{\beta_{23}})}{g_{\beta_{23}}} & g_{\beta_{34}} & g_{\beta_{34}} & g_{\beta_{34}} & g_{\beta_{34}} \\ \hline
-2\frac{g_{\alpha_{12}}g_{\beta_{23}}}{g_{\beta_{12}}} & g_{\alpha_{34}} & g_{\beta_{23}} & g_{\beta_{34}} & g_{\beta_{23}} & g_{\gamma_3} \\ \hline
-2\frac{g_{\alpha_{12}}g_{\gamma_3}}{g_{\beta_{13}}} & g_{\alpha_{34}} & g_{\gamma_3} & g_{\beta_{34}} & g_{\gamma_3} & g_{\gamma_3}
\end{array}\right)$$

$$M_{-\alpha_{24}} = \left(\begin{array}{c|c|c|c|c}
g_{\alpha_{12}} & -2\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{13}}} & -\frac{g_{\alpha_{12}}g_{\gamma_4}}{g_{\gamma_1}} & -2\frac{g_{\alpha_{12}}g_{\beta_{24}}}{g_{\beta_{12}}} & -2\frac{g_{\alpha_{12}}g_{\gamma_4}}{g_{\beta_{14}}} \\ \hline
-2\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{13}}} & g_{\beta_{34}} & g_{\beta_{34}} & g_{\beta_{34}} & g_{\gamma_4} \\ \hline
-\frac{g_{\alpha_{12}}g_{\gamma_4}}{g_{\gamma_1}} & g_{\beta_{34}} & g_{\beta_{14}} & g_{\beta_{24}} & g_{\gamma_4} \\ \hline
-2\frac{g_{\alpha_{12}}g_{\beta_{24}}}{g_{\beta_{12}}} & g_{\beta_{34}} & g_{\beta_{24}} & g_{\beta_{24}} & g_{\gamma_4} \\ \hline
-2\frac{g_{\alpha_{12}}g_{\gamma_4}}{g_{\beta_{14}}} & g_{\gamma_4} & g_{\gamma_4} & g_{\gamma_4} & g_{\gamma_4}
\end{array}\right)$$

$$M_{-\alpha_{34}} = \left(\begin{array}{cc|cc|c|c}
g_{\alpha_{13}} & g_{\alpha_{23}} & -\frac{g_{\alpha_{13}}g_{\gamma_4}}{g_{\gamma_1}} & -2\frac{g_{\alpha_{13}}g_{\beta_{24}}}{g_{\beta_{12}}} & -2\frac{g_{\alpha_{13}}g_{\beta_{34}}}{g_{\beta_{13}}} & -2\frac{g_{\alpha_{13}}g_{\gamma_4}}{g_{\beta_{14}}} \\
g_{\alpha_{23}} & g_{\alpha_{23}} & -2\frac{g_{\alpha_{23}}g_{\beta_{14}}}{g_{\beta_{12}}} & -\frac{g_{\alpha_{23}}g_{\gamma_4}}{g_{\gamma_2}} & -2\frac{g_{\alpha_{23}}g_{\beta_{34}}}{g_{\beta_{23}}} & -2\frac{g_{\alpha_{23}}g_{\gamma_4}}{g_{\beta_{24}}} \\ \hline
-\frac{g_{\alpha_{13}}g_{\gamma_4}}{g_{\gamma_1}} & -2\frac{g_{\alpha_{23}}g_{\beta_{14}}}{g_{\beta_{12}}} & g_{\beta_{14}} & g_{\beta_{24}} & g_{\beta_{34}} & g_{\gamma_4} \\
-2\frac{g_{\alpha_{13}}g_{\beta_{24}}}{g_{\beta_{12}}} & -\frac{g_{\alpha_{23}}g_{\gamma_4}}{g_{\gamma_2}} & g_{\beta_{24}} & g_{\beta_{24}} & g_{\beta_{34}} & g_{\gamma_4} \\ \hline
-2\frac{g_{\alpha_{13}}g_{\beta_{34}}}{g_{\beta_{13}}} & -2\frac{g_{\alpha_{23}}g_{\beta_{34}}}{g_{\beta_{23}}} & g_{\beta_{34}} & g_{\beta_{34}} & g_{\beta_{34}} & g_{\gamma_4} \\ \hline
-2\frac{g_{\alpha_{13}}g_{\gamma_4}}{g_{\beta_{14}}} & -2\frac{g_{\alpha_{23}}g_{\gamma_4}}{g_{\beta_{24}}} & g_{\gamma_4} & g_{\gamma_4} & g_{\gamma_4} & g_{\gamma_4}
\end{array}\right)$$

Those corresponding to β_{ij} are

$$M_{-\beta_{12}} = (g_{\alpha_{12}}) \qquad M_{-\beta_{13}} = \left(\begin{array}{c|c} g_{\alpha_{23}} & g_{\alpha_{23}} \\ \hline g_{\alpha_{23}} & g_{\alpha_{13}} \end{array}\right)$$

$$M_{-\beta_{14}} = \left(\begin{array}{cc|c} g_{\alpha_{24}} & g_{\alpha_{34}} & g_{\alpha_{24}} \\ g_{\alpha_{34}} & g_{\alpha_{34}} & g_{\alpha_{34}} \\ \hline g_{\alpha_{24}} & g_{\alpha_{34}} & g_{\alpha_{14}} \end{array}\right) \qquad M_{-\beta_{23}} = \left(\begin{array}{c|c|c} g_{\alpha_{12}} & -\frac{g_{\alpha_{12}}g_{\gamma_3}}{g_{\gamma_1}} & 2\frac{g_{\alpha_{12}}g_{\alpha_{23}}}{g_{\beta_{12}}} \\ \hline -\frac{g_{\alpha_{12}}g_{\gamma_3}}{g_{\gamma_1}} & g_{\alpha_{13}} & g_{\alpha_{23}} \\ \hline 2\frac{g_{\alpha_{12}}g_{\alpha_{23}}}{g_{\beta_{12}}} & g_{\alpha_{23}} & g_{\alpha_{23}} \end{array}\right)$$

$$M_{-\beta_{24}} = \left(\begin{array}{c|c|c|c} g_{\alpha_{12}} & -\frac{g_{\alpha_{12}}g_{\gamma_4}}{g_{\gamma_1}} & 2\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\beta_{13}}} & 2\frac{g_{\alpha_{12}}g_{\alpha_{24}}}{g_{\beta_{12}}} \\ \hline -\frac{g_{\alpha_{12}}g_{\gamma_4}}{g_{\gamma_1}} & g_{\alpha_{14}} & g_{\alpha_{34}} & g_{\alpha_{24}} \\ \hline 2\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\beta_{13}}} & g_{\alpha_{34}} & g_{\alpha_{34}} & g_{\alpha_{34}} \\ \hline 2\frac{g_{\alpha_{12}}g_{\alpha_{24}}}{g_{\beta_{12}}} & g_{\alpha_{24}} & g_{\alpha_{34}} & g_{\alpha_{24}} \end{array}\right)$$

$$M_{-\beta_{34}} = \left(\begin{array}{cc|cc|c} g_{\alpha_{13}} & g_{\alpha_{23}} & -\frac{g_{\alpha_{13}}g_{\gamma_4}}{g_{\gamma_1}} & 2\frac{g_{\alpha_{13}}g_{\alpha_{24}}}{g_{\beta_{12}}} & 2\frac{g_{\alpha_{13}}g_{\alpha_{34}}}{g_{\beta_{13}}} \\ g_{\alpha_{23}} & g_{\alpha_{23}} & 2\frac{g_{\alpha_{23}}g_{\alpha_{14}}}{g_{\beta_{12}}} & -\frac{g_{\alpha_{23}}g_{\gamma_4}}{g_{\gamma_2}} & 2\frac{g_{\alpha_{23}}g_{\alpha_{34}}}{g_{\beta_{23}}} \\ \hline -\frac{g_{\alpha_{13}}g_{\gamma_4}}{g_{\gamma_1}} & 2\frac{g_{\alpha_{23}}g_{\alpha_{14}}}{g_{\beta_{12}}} & g_{\alpha_{14}} & g_{\alpha_{24}} & g_{\alpha_{34}} \\ 2\frac{g_{\alpha_{13}}g_{\alpha_{24}}}{g_{\beta_{12}}} & -\frac{g_{\alpha_{23}}g_{\gamma_4}}{g_{\gamma_2}} & g_{\alpha_{24}} & g_{\alpha_{24}} & g_{\alpha_{34}} \\ \hline 2\frac{g_{\alpha_{13}}g_{\alpha_{34}}}{g_{\beta_{13}}} & 2\frac{g_{\alpha_{23}}g_{\alpha_{34}}}{g_{\beta_{23}}} & g_{\alpha_{34}} & g_{\alpha_{34}} & g_{\alpha_{34}} \end{array}\right)$$

and then we have for the γ_i:

$$M_{-\gamma_1} = \emptyset \qquad M_{-\gamma_2} = \left(2\frac{(g_{\alpha_{12}})^2}{g_{\gamma_1}}\right) \qquad M_{-\gamma_3} = \begin{pmatrix} 2\frac{(g_{\alpha_{13}})^2}{g_{\gamma_1}} & \frac{g_{\alpha_{23}}(2g_{\beta_{12}}-g_{\beta_{23}})}{g_{\beta_{12}}} \\ \frac{g_{\alpha_{23}}(2g_{\beta_{12}}-g_{\beta_{23}})}{g_{\beta_{12}}} & 2\frac{(g_{\alpha_{23}})^2}{g_{\gamma_2}} \end{pmatrix}$$

$$M_{-\gamma_4} = \begin{pmatrix} 2\frac{(g_{\alpha_{14}})^2}{g_{\gamma_1}} & \frac{g_{\alpha_{24}}(2g_{\beta_{12}}-g_{\beta_{24}})}{g_{\beta_{12}}} & \frac{g_{\alpha_{34}}(2g_{\beta_{13}}-g_{\beta_{34}})}{g_{\beta_{13}}} \\ \frac{g_{\alpha_{24}}(2g_{\beta_{12}}-g_{\beta_{24}})}{g_{\beta_{12}}} & 2\frac{(g_{\alpha_{24}})^2}{g_{\gamma_2}} & \frac{g_{\alpha_{34}}(2g_{\beta_{23}}-g_{\beta_{34}})}{g_{\beta_{23}}} \\ \frac{g_{\alpha_{34}}(2g_{\beta_{13}}-g_{\beta_{34}})}{g_{\beta_{13}}} & \frac{g_{\alpha_{34}}(2g_{\beta_{23}}-g_{\beta_{34}})}{g_{\beta_{23}}} & 2\frac{(g_{\alpha_{34}})^2}{g_{\gamma_4}} \end{pmatrix}$$

Ordering the basis as given in (8.2) the matrix of the trivial module is given by

$$M_0 = \begin{pmatrix} A & \langle AB\rangle & \langle AC\rangle \\ \langle AB\rangle^T & B & \langle BC\rangle \\ \langle AC\rangle^T & \langle BC\rangle^T & C \end{pmatrix}$$

with

$$A = \left(\begin{array}{c|cc|ccc} 2g_{\alpha_{34}} & * & g_{\alpha_{34}} & 0 & * & g_{\alpha_{34}} \\ \hline * & 2g_{\alpha_{23}} & g_{\alpha_{23}} & * & g_{\alpha_{23}} & 0 \\ g_{\alpha_{34}} & g_{\alpha_{23}} & 2g_{\alpha_{24}} & * & 0 & g_{\alpha_{24}} \\ \hline 0 & * & * & 2g_{\alpha_{12}} & g_{\alpha_{12}} & g_{\alpha_{12}} \\ * & g_{\alpha_{23}} & 0 & g_{\alpha_{12}} & 2g_{\alpha_{13}} & g_{\alpha_{13}} \\ g_{\alpha_{34}} & 0 & g_{\alpha_{24}} & g_{\alpha_{12}} & g_{\alpha_{13}} & 2g_{\alpha_{14}} \end{array}\right) \qquad B = \left(\begin{array}{c|cc|ccc} 2g_{\beta_{34}} & g_{\beta_{34}} & g_{\beta_{34}} & 0 & g_{\beta_{34}} & g_{\beta_{34}} \\ \hline g_{\beta_{34}} & 2g_{\beta_{23}} & g_{\beta_{24}} & g_{\beta_{23}} & g_{\beta_{23}} & 0 \\ g_{\beta_{34}} & g_{\beta_{24}} & 2g_{\beta_{24}} & g_{\beta_{24}} & 0 & g_{\beta_{24}} \\ \hline 0 & g_{\beta_{23}} & g_{\beta_{24}} & 2g_{\beta_{12}} & g_{\beta_{13}} & g_{\beta_{14}} \\ g_{\beta_{34}} & g_{\beta_{23}} & 0 & g_{\beta_{13}} & 2g_{\beta_{13}} & g_{\beta_{14}} \\ g_{\beta_{34}} & 0 & g_{\beta_{24}} & g_{\beta_{14}} & g_{\beta_{14}} & 2g_{\beta_{14}} \end{array}\right)$$

$$C = \begin{pmatrix} g_{\gamma_4} & 0 & 0 & 0 \\ 0 & g_{\gamma_3} & 0 & 0 \\ 0 & 0 & g_{\gamma_2} & 0 \\ 0 & 0 & 0 & g_{\gamma_1} \end{pmatrix} \quad \langle AB \rangle = \left(\begin{array}{c|cc|ccc} 2\frac{(g_{\alpha_{34}})^2}{g_{\gamma_3}} & g_{\alpha_{34}} & * & 0 & g_{\alpha_{34}} & * \\ \hline * & 2\frac{(g_{\alpha_{23}})^2}{g_{\gamma_2}} & g_{\alpha_{23}} & g_{\alpha_{23}} & * & 0 \\ * & g_{\alpha_{24}} & 2\frac{(g_{\alpha_{24}})^2}{g_{\gamma_2}} & g_{\alpha_{24}} & 0 & * \\ \hline 0 & * & * & 2\frac{(g_{\alpha_{12}})^2}{g_{\gamma_1}} & g_{\alpha_{12}} & g_{\alpha_{13}} \\ * & * & 0 & g_{\alpha_{13}} & 2\frac{(g_{\alpha_{13}})^2}{g_{\gamma_1}} & g_{\alpha_{13}} \\ * & 0 & * & g_{\alpha_{14}} & g_{\alpha_{14}} & 2\frac{(g_{\alpha_{14}})^2}{g_{\gamma_1}} \end{array}\right)$$

$$\langle AC \rangle = \left(\begin{array}{cccc} * & g_{\alpha_{34}} & 0 & 0 \\ \hline 0 & * & g_{\alpha_{23}} & 0 \\ * & 0 & g_{\alpha_{24}} & 0 \\ \hline 0 & 0 & * & g_{\alpha_{12}} \\ 0 & * & 0 & g_{\alpha_{13}} \\ * & 0 & 0 & g_{\alpha_{14}} \end{array}\right) \qquad \langle BC \rangle = \left(\begin{array}{cccc} g_{\gamma_4} & g_{\beta_{34}} & 0 & 0 \\ \hline 0 & g_{\gamma_3} & g_{\beta_{23}} & 0 \\ g_{\gamma_4} & 0 & g_{\beta_{24}} & 0 \\ \hline 0 & 0 & g_{\gamma_2} & g_{\beta_{12}} \\ 0 & g_{\gamma_3} & 0 & g_{\beta_{13}} \\ g_{\gamma_4} & 0 & 0 & g_{\beta_{14}} \end{array}\right)$$

where we used $*$ at the entry (α, β) as place holder for $-2\frac{g_\alpha g_\beta}{g_{\alpha+\beta}}$.

Similar to the case of $SO(9)$, we notice that the matrices $M_{-\beta_{12}}, M_{-\beta_{13}}$ and $M_{-\beta_{14}}$ are positive semidefinite by the auxiliary lemma for any kind of metric with $g_{\alpha+\beta} \geq g_\alpha$, which is true for the Kähler Einstein metrics for all isotropy groups K. However, in some cases we might need the matrices for $-\eta = \beta_{12}, \beta_{13}, \beta_{14}$ to compensate for other matrices. In the following, we do the full modification for the toric isotropy.

Toric Isotropy

Now we consider the Kähler Einstein metric of $(Sp(4), \mathbb{T}, J_{std})$, which is given on the simple roots by

α	α_{12}	α_{23}	α_{34}	γ_4
g_α	1	1	1	2

The matrix of the trivial module is given by

$$M_0 = \left(\begin{array}{cccccc|cccccc|cccc}
2 & -1 & 1 & 0 & -\frac{4}{3} & 1 & \frac{1}{2} & 1 & -\frac{8}{5} & 0 & 1 & -\frac{5}{3} & -\frac{4}{3} & 1 & 0 & 0 \\
-1 & 2 & 1 & -1 & 1 & 0 & -\frac{3}{2} & \frac{1}{3} & 1 & 1 & -\frac{12}{7} & 0 & 0 & -\frac{8}{5} & 1 & 0 \\
1 & 1 & 4 & -\frac{4}{3} & 0 & 2 & -\frac{12}{5} & 2 & \frac{4}{3} & 2 & 0 & -\frac{20}{7} & -2 & 0 & 2 & 0 \\
0 & -1 & -\frac{4}{3} & 2 & 1 & 1 & 0 & -\frac{5}{3} & -\frac{8}{5} & \frac{1}{4} & 1 & 1 & 0 & 0 & -\frac{12}{7} & 1 \\
-\frac{4}{3} & 1 & 0 & 1 & 4 & 2 & -\frac{12}{5} & -\frac{20}{7} & 0 & 2 & 1 & 2 & 0 & -\frac{8}{3} & 0 & 2 \\
1 & 0 & 2 & 1 & 2 & 6 & -3 & 0 & -\frac{24}{7} & 3 & 3 & \frac{9}{4} & -\frac{12}{5} & 0 & 0 & 3 \\
\hline
\frac{1}{2} & -\frac{3}{2} & -\frac{12}{5} & 0 & -\frac{12}{5} & -3 & 6 & 3 & 3 & 0 & 3 & 3 & 2 & 3 & 0 & 0 \\
1 & \frac{1}{3} & 2 & -\frac{5}{3} & -\frac{20}{7} & 0 & 3 & 10 & 4 & 5 & 5 & 0 & 0 & 4 & 5 & 0 \\
-\frac{8}{5} & 1 & \frac{4}{3} & -\frac{8}{5} & 0 & -\frac{24}{7} & 3 & 4 & 8 & 4 & 0 & 4 & 2 & 0 & 4 & 0 \\
0 & 1 & 2 & \frac{1}{4} & 2 & 3 & 0 & 2 & 4 & 14 & 6 & 5 & 0 & 0 & 6 & 7 \\
1 & -\frac{12}{7} & 0 & 1 & 1 & 3 & 3 & 2 & 0 & 6 & 12 & 5 & 0 & 4 & 0 & 6 \\
-\frac{5}{3} & 0 & -\frac{20}{7} & 1 & 2 & \frac{9}{4} & 3 & 0 & 4 & 5 & 5 & 10 & 2 & 0 & 0 & 5 \\
\hline
-\frac{4}{3} & 0 & -2 & 0 & 0 & -\frac{12}{5} & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\
1 & -\frac{8}{5} & 0 & 0 & -\frac{8}{3} & 0 & 3 & 4 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 \\
0 & 1 & 2 & -\frac{12}{7} & 0 & 0 & 0 & 5 & 4 & 6 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 7 & 6 & 5 & 0 & 0 & 0 & 8
\end{array}\right)$$

It is not positive definite and hence further four forms are necessary. The other matrices are

$$M_{-\alpha_{12}} = \begin{pmatrix} 1 & 1 & \frac{9}{7} & 1 & 1 & 1 \\ 1 & 2 & 2 & \frac{20}{7} & 2 & 2 \\ \frac{9}{7} & 2 & 5 & 4 & 5 & 5 \\ 1 & \frac{20}{7} & 4 & 4 & 4 & 4 \\ 1 & 2 & 5 & 4 & 7 & 6 \\ 1 & 2 & 5 & 4 & 6 & 6 \end{pmatrix} \qquad M_{-\alpha_{13}} = \begin{pmatrix} 1 & 1 & \frac{3}{2} & 1 & 1 \\ 1 & 5 & 3 & 5 & 4 \\ \frac{3}{2} & 3 & 3 & 3 & 3 \\ 1 & 5 & 3 & 6 & 4 \\ 1 & 4 & 3 & 4 & 4 \end{pmatrix}$$

$$M_{-\alpha_{14}} = \begin{pmatrix} 4 & 3 & 4 & 2 \\ 3 & 3 & 3 & 2 \\ 4 & 3 & 5 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \qquad M_{-\alpha_{23}} = \begin{pmatrix} 1 & -\frac{2}{3} & -\frac{1}{2} & -\frac{6}{5} & -\frac{10}{7} & -\frac{4}{3} \\ -\frac{2}{3} & 1 & 1 & \frac{7}{5} & 1 & 1 \\ -\frac{1}{2} & 1 & 6 & 3 & 5 & 4 \\ -\frac{6}{5} & \frac{7}{5} & 3 & 3 & 3 & 3 \\ -\frac{10}{7} & 1 & 5 & 3 & 5 & 4 \\ -\frac{4}{3} & 1 & 4 & 3 & 4 & 4 \end{pmatrix}$$

$$M_{-\alpha_{24}} = \begin{pmatrix} 1 & -1 & -\frac{1}{4} & -\frac{8}{7} & -\frac{4}{5} \\ -1 & 3 & 3 & 3 & 2 \\ -\frac{1}{4} & 3 & 5 & 4 & 2 \\ -\frac{8}{7} & 3 & 4 & 4 & 2 \\ -\frac{4}{5} & 2 & 2 & 2 & 2 \end{pmatrix} \qquad M_{-\alpha_{34}} = \begin{pmatrix} 2 & 1 & -\frac{1}{2} & -\frac{16}{7} & -2 & -\frac{8}{5} \\ 1 & 1 & -\frac{10}{7} & -\frac{1}{3} & -\frac{6}{5} & -1 \\ -\frac{1}{2} & -\frac{10}{7} & 5 & 4 & 3 & 2 \\ -\frac{16}{7} & -\frac{1}{3} & 4 & 4 & 3 & 2 \\ -2 & -\frac{6}{5} & 3 & 3 & 3 & 2 \\ -\frac{8}{5} & -1 & 2 & 2 & 2 & 2 \end{pmatrix}$$

The remaining matrices are

$$M_{-\beta_{12}} = (1) \qquad M_{-\beta_{13}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$M_{-\beta_{14}} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \qquad M_{-\beta_{23}} = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{2}{7} \\ -\frac{1}{2} & 2 & 1 \\ \frac{2}{7} & 1 & 1 \end{pmatrix}$$

$$M_{-\beta_{24}} = \begin{pmatrix} 1 & -\frac{1}{4} & \frac{1}{3} & \frac{4}{7} \\ -\frac{1}{4} & 3 & 1 & 2 \\ \frac{1}{3} & 1 & 1 & 1 \\ \frac{4}{7} & 2 & 1 & 2 \end{pmatrix} \qquad M_{-\beta_{34}} = \begin{pmatrix} 2 & 1 & -\frac{1}{2} & \frac{8}{7} & \frac{2}{3} \\ 1 & 1 & \frac{6}{7} & -\frac{1}{3} & \frac{2}{5} \\ -\frac{1}{2} & \frac{6}{7} & 3 & 2 & 1 \\ \frac{8}{7} & -\frac{1}{3} & 2 & 2 & 1 \\ \frac{2}{3} & \frac{2}{5} & 1 & 1 & 1 \end{pmatrix}$$

and

$$M_{-\gamma_1} = () \qquad M_{-\gamma_2} = \begin{pmatrix} \frac{1}{4} \end{pmatrix}$$

$$M_{-\gamma_3} = \begin{pmatrix} 1 & \frac{9}{7} \\ \frac{9}{7} & \frac{1}{3} \end{pmatrix} \qquad M_{-\gamma_4} = \begin{pmatrix} \frac{9}{4} & \frac{20}{7} & \frac{3}{2} \\ \frac{20}{7} & \frac{4}{3} & \frac{7}{5} \\ \frac{3}{2} & \frac{7}{5} & \frac{1}{2} \end{pmatrix}$$

Now we have to modify these as follows:

Value	Intended	Forced
$\frac{3}{4}$	$(M_{-\alpha_{12}})_{(1,2)}$	$(M_{-\alpha_{34}})_{(1,2)}$
$\frac{9}{56}$	$(M_{-\alpha_{12}})_{(1,3)}$	$(M_{-\gamma_3})_{(1,2)}$
1	$(M_{-\alpha_{12}})_{(1,4)}$	$(M_{-\beta_{34}})_{(1,2)}$
$\frac{3}{4}$	$(M_{-\alpha_{12}})_{(1,5)}$	$(M_{-\beta_{13}})_{(1,2)}$
$\frac{3}{4}$	$(M_{-\alpha_{12}})_{(1,6)}$	$(M_{-\beta_{23}})_{(2,3)}$
$-\frac{37}{112}$	$(M_{-\alpha_{12}})_{(2,4)}$	$(M_{-\gamma_4})_{(1,2)}$
$\frac{1}{4}$	$(M_{-\alpha_{12}})_{(2,6)}$	$(M_{-\beta_{24}})_{(2,4)}$
$\frac{3}{8}$	$(M_{-\alpha_{12}})_{(1,1)}$	$(M_0)_{(2,5)}$
$\frac{3}{8}$	$(M_{-\alpha_{12}})_{(2,2)}$	$(M_0)_{(3,6)}$
$\frac{5}{8}$	$(M_{-\alpha_{12}})_{(3,3)}$	$(M_0)_{(8,11)}$
$\frac{1}{4}$	$(M_{-\alpha_{12}})_{(4,4)}$	$(M_0)_{(9,12)}$
$-\frac{1}{8}$	$(M_{-\alpha_{13}})_{(1,3)}$	$(M_{-\gamma_4})_{(1,3)}$
$\frac{1}{16}$	$(M_{-\alpha_{14}})_{(2,4)}$	$(M_{-\alpha_{34}})_{(3,6)}$
$-\frac{1}{10}$	$(M_{-\alpha_{23}})_{(2,4)}$	$(M_{-\gamma_4})_{(2,3)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(3,3)}$	$(M_0)_{(10,11)}$
$\frac{1}{2}$	$(M_{-\alpha_{23}})_{(6,6)}$	$(M_0)_{(8,14)}$

Value	Intended	Forced
5	$(M_{-\alpha_{24}})_{(3,3)}$	$(M_0)_{(10,12)}$
$-\frac{9}{8}$	$(M_{-\alpha_{34}})_{(1,3)}$	$(M_{-\beta_{34}})_{(1,3)}$
$-\frac{9}{8}$	$(M_{-\alpha_{34}})_{(2,4)}$	$(M_{-\beta_{34}})_{(2,4)}$
$\frac{1}{4}$	$(M_{-\alpha_{34}})_{(1,1)}$	$(M_0)_{(5,6)}$
$\frac{1}{8}$	$(M_{-\alpha_{34}})_{(2,2)}$	$(M_0)_{(2,3)}$
$\frac{11}{8}$	$(M_{-\alpha_{34}})_{(3,3)}$	$(M_0)_{(11,12)}$
$\frac{5}{4}$	$(M_{-\alpha_{34}})_{(4,4)}$	$(M_0)_{(8,9)}$
$\frac{3}{8}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(7,14)}$
$\frac{1}{6}$	$(M_{-\alpha_{34}})_{(6,6)}$	$(M_0)_{(7,13)}$
$\frac{1}{4}$	$(M_{-\gamma_2})_{(1,1)}$	$(M_0)_{(4,10)}$
1	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
$\frac{1}{3}$	$(M_{-\gamma_3})_{(2,2)}$	$(M_0)_{(2,8)}$
$\frac{9}{4}$	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
$\frac{4}{3}$	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$
$\frac{1}{2}$	$(M_{-\gamma_4})_{(3,3)}$	$(M_0)_{(1,7)}$
$-\frac{3}{4}$	$(M_0)_{(1,11)}$	$(M_{-\beta_{14}})_{(2,2)}$

The modified matrix of the trivial module is given by

$$
M_0 = \left(\begin{array}{cccccc|cccccc|cccc}
2 & -1 & 1 & 0 & -\frac{4}{3} & 1 & 0 & 1 & -\frac{8}{5} & 0 & \frac{1}{4} & -\frac{5}{3} & -\frac{4}{3} & 1 & 0 & 0 \\
-1 & 2 & \frac{7}{8} & -1 & \frac{5}{8} & 0 & -\frac{3}{2} & 0 & 1 & 1 & -\frac{12}{7} & 0 & 0 & -\frac{8}{5} & 1 & 0 \\
1 & \frac{7}{8} & 4 & -\frac{4}{3} & 0 & \frac{13}{8} & -\frac{12}{5} & 2 & 0 & 2 & 0 & -\frac{20}{7} & -2 & 0 & 2 & 0 \\
0 & -1 & -\frac{4}{3} & 2 & 1 & 1 & 0 & -\frac{5}{3} & -\frac{8}{5} & 0 & 1 & 1 & 0 & 0 & -\frac{12}{7} & 1 \\
-\frac{4}{3} & \frac{5}{8} & 0 & 1 & 4 & \frac{7}{4} & -\frac{12}{5} & -\frac{20}{7} & 0 & 2 & 0 & 2 & 0 & -\frac{8}{3} & 0 & 2 \\
1 & 0 & \frac{13}{8} & 1 & \frac{7}{4} & 6 & -3 & 0 & -\frac{24}{7} & 3 & 3 & 0 & -\frac{12}{5} & 0 & 0 & 3 \\
\hline
0 & -\frac{3}{2} & -\frac{12}{5} & 0 & -\frac{12}{5} & -3 & 6 & 3 & 3 & 0 & 3 & 3 & \frac{11}{6} & \frac{21}{8} & 0 & 0 \\
1 & 0 & 2 & -\frac{5}{3} & -\frac{20}{7} & 0 & 3 & 10 & \frac{11}{4} & 5 & \frac{35}{8} & 0 & 0 & \frac{7}{2} & 5 & 0 \\
-\frac{8}{5} & 1 & 0 & -\frac{8}{5} & 0 & -\frac{24}{7} & 3 & \frac{11}{4} & 8 & 4 & 0 & \frac{15}{4} & 2 & 0 & 4 & 0 \\
0 & 1 & 2 & 0 & 2 & 3 & 0 & 5 & 4 & 14 & \frac{19}{4} & 0 & 0 & 0 & 6 & 7 \\
\frac{1}{4} & -\frac{12}{7} & 0 & 1 & 0 & 3 & 3 & \frac{35}{8} & 0 & \frac{19}{4} & 12 & \frac{29}{8} & 0 & 4 & 0 & 6 \\
-\frac{5}{3} & 0 & -\frac{20}{7} & 1 & 2 & 0 & 3 & 0 & \frac{15}{4} & 0 & \frac{29}{8} & 10 & 2 & 0 & 0 & 5 \\
\hline
-\frac{4}{3} & 0 & -2 & 0 & 0 & -\frac{12}{5} & \frac{11}{6} & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\
1 & -\frac{8}{5} & 0 & 0 & -\frac{8}{3} & 0 & \frac{21}{8} & \frac{7}{2} & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 \\
0 & 1 & 2 & -\frac{12}{7} & 0 & 0 & 0 & 5 & 4 & 6 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 7 & 6 & 5 & 0 & 0 & 0 & 8
\end{array}\right)
$$

It is not positive definite and hence further four forms are necessary. The other matrices are

$$M_{-\alpha_{12}} = \begin{pmatrix} \frac{11}{8} & \frac{7}{4} & \frac{81}{56} & 2 & \frac{7}{4} & \frac{7}{4} \\ \frac{7}{4} & \frac{19}{8} & 2 & \frac{283}{112} & 2 & \frac{9}{4} \\ \frac{81}{56} & 2 & \frac{45}{8} & 4 & 5 & 5 \\ 2 & \frac{283}{112} & 4 & \frac{17}{4} & 4 & 4 \\ \frac{7}{4} & 2 & 5 & 4 & 7 & 6 \\ \frac{7}{4} & \frac{9}{4} & 5 & 4 & 6 & 6 \end{pmatrix} \qquad M_{-\alpha_{13}} = \begin{pmatrix} 1 & 1 & \frac{11}{8} & 1 & 1 \\ 1 & 5 & 3 & 5 & 4 \\ \frac{11}{8} & 3 & 3 & 3 & 3 \\ 1 & 5 & 3 & 6 & 4 \\ 1 & 4 & 3 & 4 & 4 \end{pmatrix}$$

$$M_{-\alpha_{14}} = \begin{pmatrix} 4 & 3 & 4 & 2 \\ 3 & 3 & 3 & \frac{33}{16} \\ 4 & 3 & 5 & 2 \\ 2 & \frac{33}{16} & 2 & 2 \end{pmatrix} \qquad M_{-\alpha_{23}} = \begin{pmatrix} 1 & -\frac{2}{3} & -\frac{1}{2} & -\frac{6}{5} & -\frac{10}{7} & -\frac{4}{3} \\ -\frac{2}{3} & 1 & 1 & \frac{13}{10} & 1 & 1 \\ -\frac{1}{2} & 1 & \frac{29}{4} & 3 & 5 & 4 \\ -\frac{6}{5} & \frac{13}{10} & 3 & 3 & 3 & 3 \\ -\frac{10}{7} & 1 & 5 & 3 & 5 & 4 \\ -\frac{4}{3} & 1 & 4 & 3 & 4 & \frac{9}{2} \end{pmatrix}$$

$$M_{-\alpha_{24}} = \begin{pmatrix} 1 & -1 & -\frac{1}{4} & -\frac{8}{7} & -\frac{4}{5} \\ -1 & 3 & 3 & 3 & 2 \\ -\frac{1}{4} & 3 & 10 & 4 & 2 \\ -\frac{8}{7} & 3 & 4 & 4 & 2 \\ -\frac{4}{5} & 2 & 2 & 2 & 2 \end{pmatrix} \qquad M_{-\alpha_{34}} = \begin{pmatrix} \frac{9}{4} & \frac{1}{4} & -\frac{13}{8} & -\frac{16}{7} & -2 & -\frac{8}{5} \\ \frac{1}{4} & \frac{9}{8} & -\frac{10}{7} & -\frac{35}{24} & -\frac{6}{5} & -1 \\ -\frac{13}{8} & -\frac{10}{7} & \frac{51}{8} & 4 & 3 & \frac{31}{16} \\ -\frac{16}{7} & -\frac{35}{24} & 4 & \frac{21}{4} & 3 & 2 \\ -2 & -\frac{6}{5} & 3 & 3 & \frac{27}{8} & 2 \\ -\frac{8}{5} & -1 & \frac{31}{16} & 2 & 2 & \frac{13}{6} \end{pmatrix}$$

The remaining matrices are

$$M_{-\beta_{12}} = (1) \qquad M_{-\beta_{13}} = \begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 2 \end{pmatrix}$$

$$M_{-\beta_{14}} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & \frac{7}{4} & 1 \\ 2 & 1 & 3 \end{pmatrix} \qquad M_{-\beta_{23}} = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{2}{7} \\ -\frac{1}{2} & 2 & \frac{1}{4} \\ \frac{2}{7} & \frac{1}{4} & 1 \end{pmatrix}$$

$$M_{-\beta_{24}} = \begin{pmatrix} 1 & -\frac{1}{4} & \frac{1}{3} & \frac{4}{7} \\ -\frac{1}{4} & 3 & 1 & \frac{7}{4} \\ \frac{1}{3} & 1 & 1 & 1 \\ \frac{4}{7} & \frac{7}{4} & 1 & 2 \end{pmatrix} \qquad M_{-\beta_{34}} = \begin{pmatrix} 2 & 0 & \frac{5}{8} & \frac{8}{7} & \frac{2}{3} \\ 0 & 1 & \frac{6}{7} & \frac{19}{24} & \frac{2}{5} \\ \frac{5}{8} & \frac{6}{7} & 3 & 2 & 1 \\ \frac{8}{7} & \frac{19}{24} & 2 & 2 & 1 \\ \frac{2}{3} & \frac{2}{5} & 1 & 1 & 1 \end{pmatrix}$$

and

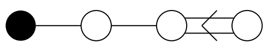
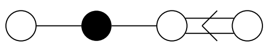
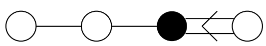
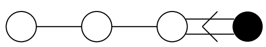
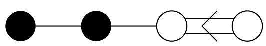
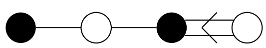
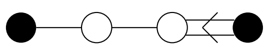
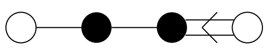
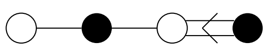
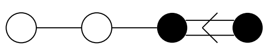
$$M_{-\gamma_1} = () \qquad M_{-\gamma_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$M_{-\gamma_3} = \begin{pmatrix} 2 & \frac{9}{8} \\ \frac{9}{8} & \frac{2}{3} \end{pmatrix} \qquad M_{-\gamma_4} = \begin{pmatrix} \frac{9}{2} & \frac{51}{16} & \frac{13}{8} \\ \frac{51}{16} & \frac{8}{3} & \frac{3}{2} \\ \frac{13}{8} & \frac{3}{2} & 1 \end{pmatrix}$$

These are all positive definite, except for M_0 but as in the previous chapters we can discard rows and columns 10 and 16 since all entries are nonnegative. The resulting 14×14 matrix is positive definite, which finishes the proof for $K = \mathbb{T}$.

Larger Isotropy

As in the case of $SO(9)$, we will not present the above for all K in the same detail. In fact, we will present here the different possible K and their Kähler Einstein metrics. By corollary 50 the curvature matrices are then obtained by considering the correct submatrices of the matrices of $(Sp(4), \mathbb{T}, J_{std})$ with the new metric coefficients. As before, we present the details in the appendix for the interested reader. For each K, we will give the indices of the rows and columns of the curvature matrices that have to be erased to obtain the restriction and give a detailed table of the used four forms. We only consider the case $dim(\mathfrak{z}(k)) > 1$, since the other cases are covered by Itoh. Note, that we scaled the metric by 2 in order to avoid unnecessary fractions.

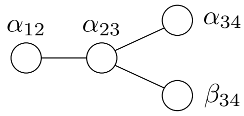
K	$D = (V, E)$	γ_K^*	$g_{\alpha_{12}}$	$g_{\alpha_{23}}$	$g_{\alpha_{34}}$	g_{ε_4}
$SU(2)\ \mathbb{T}^3$		α_{12}	0	3	2	4
$\mathbb{S}^1\ SU(2)\ \mathbb{T}^2$		α_{23}	3	0	3	4
$\mathbb{T}^2\ SU(2)\ \mathbb{S}^1$		α_{34}	2	3	0	6
$\mathbb{T}^3\ SU(2)$		γ_4	2	2	4	0
$SU(3)\ \mathbb{T}^2$		$2\alpha_{13}$	0	0	4	4
$SU(2)\ \mathbb{S}^1\ SU(2)\ \mathbb{S}^1$		$\alpha_{12} + \alpha_{34}$	0	4	0	6
$SU(2)\ \mathbb{T}^2\ SU(2)$		$\alpha_{12} + \gamma_4$	0	3	4	0
$\mathbb{S}^1\ SU(3)\ \mathbb{S}^1$		$2\alpha_{24}$	4	0	0	8
$\mathbb{S}^1\ SU(2)\ \mathbb{S}^1\ SU(2)$		$\alpha_{23} + \gamma_4$	3	0	5	0
$\mathbb{T}^2\ Sp(2)$		$4\varepsilon_3 + 2\varepsilon_4$	2	6	0	0

8.4 The case of $SO(8)$

The positive roots are

$$\Delta_{\mathfrak{g}}^+ = \{\alpha_{34}, \alpha_{23}, \alpha_{24}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \\ \beta_{34}, \beta_{23}, \beta_{24}, \beta_{12}, \beta_{13}, \beta_{14}\} \tag{8.4}$$

and the simple roots are placed on the Dynkin diagram as follows



After applying the general modifications of section 7.4 the matrices M_η representing the modified holomorphic curvature tensor are indexed with $\eta \in \Delta_{\mathfrak{g}}^- \cup \{0\}$. In the case of $SO(8)$ this corresponds to the following:

We begin with the ones corresponding to $-\alpha_{ij}$.

$$M_{-\alpha_{12}} = \left(\begin{array}{cc|cc} 4g_{\alpha_{23}} & 4g_{\alpha_{23}} & 8\frac{g_{\alpha_{23}}g_{\beta_{23}}}{g_{\beta_{12}}} & 4g_{\alpha_{23}} \\ 4g_{\alpha_{23}} & 4g_{\alpha_{24}} & 4g_{\alpha_{24}} & 8\frac{g_{\alpha_{24}}g_{\beta_{24}}}{g_{\beta_{12}}} \\ \hline 8\frac{g_{\alpha_{23}}g_{\beta_{23}}}{g_{\beta_{12}}} & 4g_{\alpha_{24}} & 4g_{\beta_{23}} & 4g_{\beta_{24}} \\ 4g_{\alpha_{23}} & 8\frac{g_{\alpha_{24}}g_{\beta_{24}}}{g_{\beta_{12}}} & 4g_{\beta_{24}} & 4g_{\beta_{24}} \end{array}\right)$$

$$M_{-\alpha_{13}} = \left(\begin{array}{c|c|c} 4g_{\alpha_{34}} & -4g_{\alpha_{34}} & 8\frac{4g_{\alpha_{34}}4g_{\beta_{34}}}{g_{\beta_{13}}} \\ \hline -4g_{\alpha_{34}} & 4g_{\beta_{23}} & -4g_{\beta_{34}} \\ \hline 8\frac{4g_{\alpha_{34}}4g_{\beta_{34}}}{g_{\beta_{13}}} & -4g_{\beta_{34}} & 4g_{\beta_{34}} \end{array}\right) \qquad M_{-\alpha_{14}} = \begin{pmatrix} 4g_{\beta_{24}} & 4g_{\beta_{34}} \\ 4g_{\beta_{34}} & 4g_{\beta_{34}} \end{pmatrix}$$

$$M_{-\alpha_{23}} = \left(\begin{array}{c|c|c|c} 4g_{\alpha_{12}} & -8\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\alpha_{14}}} & -4g_{\alpha_{12}} & -8\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{14}}} \\ \hline -8\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\alpha_{14}}} & 4g_{\alpha_{34}} & 4g_{\alpha_{34}} & 8\frac{g_{\alpha_{34}}g_{\beta_{34}}}{g_{\beta_{23}}} \\ \hline -4g_{\alpha_{12}} & 4g_{\alpha_{34}} & 4g_{\beta_{13}} & 4g_{\beta_{34}} \\ \hline -8\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{14}}} & 8\frac{g_{\alpha_{34}}g_{\beta_{34}}}{g_{\beta_{23}}} & 4g_{\beta_{34}} & 4g_{\beta_{34}} \end{array}\right)$$

$$M_{-\alpha_{24}} = \left(\begin{array}{c|c|c} 4g_{\alpha_{12}} & 8\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{13}}} & -4g_{\alpha_{12}} \\ \hline 8\frac{g_{\alpha_{12}}g_{\beta_{34}}}{g_{\beta_{13}}} & 4g_{\beta_{34}} & -4g_{\beta_{34}} \\ \hline -4g_{\alpha_{12}} & -4g_{\beta_{34}} & 4g_{\beta_{14}} \end{array}\right)$$

$$M_{-\alpha_{34}} = \left(\begin{array}{cc|cc} 4g_{\alpha_{13}} & 4g_{\alpha_{23}} & -4g_{\alpha_{13}} & -8\frac{g_{\alpha_{13}}g_{\beta_{24}}}{g_{\beta_{12}}} \\ 4g_{\alpha_{23}} & 4g_{\alpha_{23}} & -8\frac{g_{\alpha_{23}}g_{\beta_{14}}}{g_{\beta_{12}}} & -4g_{\alpha_{23}} \\ \hline -4g_{\alpha_{13}} & -8\frac{g_{\alpha_{23}}g_{\beta_{14}}}{g_{\beta_{12}}} & 4g_{\beta_{14}} & 4g_{\beta_{24}} \\ -8\frac{g_{\alpha_{13}}g_{\beta_{24}}}{g_{\beta_{12}}} & -4g_{\alpha_{23}} & 4g_{\beta_{24}} & 4g_{\beta_{24}} \end{array}\right)$$

Those corresponding to β_{ij} are

$$M_{-\beta_{12}} = \emptyset \qquad M_{-\beta_{13}} = (4g_{\alpha_{23}}) \qquad M_{-\beta_{14}} = \begin{pmatrix} 4g_{\alpha_{24}} & 4g_{\alpha_{34}} \\ 4g_{\alpha_{34}} & 4g_{\alpha_{34}} \end{pmatrix}$$

$$M_{-\beta_{23}} = \left(\begin{array}{c|c} 4g_{\alpha_{12}} & -4g_{\alpha_{12}} \\ \hline -4g_{\alpha_{12}} & 4g_{\alpha_{13}} \end{array}\right) \qquad M_{-\beta_{24}} = \left(\begin{array}{c|c|c} 4g_{\alpha_{12}} & -4g_{\alpha_{12}} & 8\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\beta_{13}}} \\ \hline -4g_{\alpha_{12}} & 4g_{\alpha_{14}} & -4g_{\alpha_{34}} \\ \hline 8\frac{g_{\alpha_{12}}g_{\alpha_{34}}}{g_{\beta_{13}}} & -4g_{\alpha_{34}} & 4g_{\alpha_{34}} \end{array}\right)$$

$$M_{-\beta_{34}} = \left(\begin{array}{cc|cc} 4g_{\alpha_{13}} & 4g_{\alpha_{23}} & -4g_{\alpha_{13}} & -8\frac{g_{\alpha_{13}}g_{\alpha_{24}}}{g_{\beta_{12}}} \\ 4g_{\alpha_{23}} & 4g_{\alpha_{23}} & -8\frac{g_{\alpha_{23}}g_{\alpha_{14}}}{g_{\beta_{12}}} & -4g_{\alpha_{23}} \\ \hline -4g_{\alpha_{13}} & -8\frac{g_{\alpha_{23}}g_{\alpha_{14}}}{g_{\beta_{12}}} & 4g_{\alpha_{14}} & 4g_{\alpha_{24}} \\ -8\frac{g_{\alpha_{13}}g_{\alpha_{24}}}{g_{\beta_{12}}} & -4g_{\alpha_{23}} & 4g_{\alpha_{24}} & 4g_{\alpha_{24}} \end{array}\right)$$

Ordering the basis as given in (8.4) the matrix of the trivial module is given by

$$M_0 = \begin{pmatrix} A & \langle AB \rangle \\ \langle AB \rangle^T & B \end{pmatrix}$$

with

$$A = \left(\begin{array}{c|cc|ccc} 8g_{\alpha_{34}} & * & 4g_{\alpha_{34}} & 0 & * & 4g_{\alpha_{34}} \\ \hline * & 8g_{\alpha_{23}} & 4g_{\alpha_{23}} & * & 4g_{\alpha_{23}} & 0 \\ 4g_{\alpha_{34}} & 4g_{\alpha_{23}} & 8g_{\alpha_{24}} & * & 0 & 4g_{\alpha_{24}} \\ \hline 0 & * & * & 8g_{\alpha_{12}} & 4g_{\alpha_{12}} & 4g_{\alpha_{12}} \\ * & 4g_{\alpha_{23}} & 0 & 4g_{\alpha_{12}} & 8g_{\alpha_{13}} & 4g_{\alpha_{13}} \\ 4g_{\alpha_{34}} & 0 & 4g_{\alpha_{24}} & 4g_{\alpha_{12}} & 4g_{\alpha_{13}} & 8g_{\alpha_{14}} \end{array}\right) \quad B = \left(\begin{array}{c|cc|ccc} 8g_{\beta_{34}} & 4g_{\beta_{34}} & 4g_{\beta_{34}} & 0 & 4g_{\beta_{34}} & 4g_{\beta_{34}} \\ \hline 4g_{\beta_{34}} & 8g_{\beta_{23}} & 4g_{\beta_{24}} & 4g_{\beta_{23}} & 4g_{\beta_{23}} & 0 \\ 4g_{\beta_{34}} & 4g_{\beta_{24}} & 8g_{\beta_{24}} & 4g_{\beta_{24}} & 0 & 4g_{\beta_{24}} \\ \hline 0 & 4g_{\beta_{23}} & 4g_{\beta_{24}} & 8g_{\beta_{12}} & 4g_{\beta_{13}} & 4g_{\beta_{14}} \\ 4g_{\beta_{34}} & 4g_{\beta_{23}} & 0 & 4g_{\beta_{13}} & 8g_{\beta_{13}} & 4g_{\beta_{14}} \\ 4g_{\beta_{34}} & 0 & 4g_{\beta_{24}} & 4g_{\beta_{14}} & 4g_{\beta_{14}} & 8g_{\beta_{14}} \end{array}\right)$$

$$\langle AB \rangle = \left(\begin{array}{c|cc|ccc} 0 & 4g_{\alpha_{34}} & * & 0 & 4g_{\alpha_{34}} & * \\ \hline * & 0 & 4g_{\alpha_{23}} & 4g_{\alpha_{23}} & * & 0 \\ * & 4g_{\alpha_{24}} & 0 & 4g_{\alpha_{24}} & 0 & * \\ \hline 0 & * & * & 0 & 4g_{\alpha_{12}} & 4g_{\alpha_{12}} \\ * & * & 0 & 4g_{\alpha_{13}} & 0 & 4g_{\alpha_{13}} \\ * & 0 & * & 4g_{\alpha_{14}} & 4g_{\alpha_{14}} & 0 \end{array}\right)$$

where we used $*$ at the entry (α, β) as place holder for $-8\frac{g_\alpha g_\beta}{g_{\alpha+\beta}}$. In the following, we do the full modification for the toric isotropy.

Toric Isotropy

Now we consider the Kähler Einstein metric of $(SO(8), \mathbb{T}, J_{std})$, which is given on the simple roots by

α	α_{12}	α_{23}	α_{34}	β_{34}
g_α	1	1	1	1

First of all, we notice that we do not have to consider the matrices $M_{-\eta}$ for η being one of $\{\alpha_{14}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{23}\}$, since they are positive definite by the auxiliary lemma. The remaining matrices are

$$M_{-\alpha_{12}} = \begin{pmatrix} 4 & 4 & \frac{24}{5} & 4 \\ 4 & 8 & 8 & \frac{32}{5} \\ \frac{24}{5} & 8 & 12 & 8 \\ 4 & \frac{32}{5} & 8 & 8 \end{pmatrix} \qquad M_{-\alpha_{13}} = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 12 & -4 \\ 2 & -4 & 4 \end{pmatrix}$$

$$M_{-\alpha_{23}} = \begin{pmatrix} 4 & -\frac{8}{3} & -4 & -\frac{8}{3} \\ -\frac{8}{3} & 4 & 4 & \frac{8}{3} \\ -4 & 4 & 16 & 4 \\ -\frac{8}{3} & \frac{8}{3} & 4 & 4 \end{pmatrix} \qquad M_{-\alpha_{24}} = \begin{pmatrix} 4 & 2 & -4 \\ 2 & 4 & -4 \\ -4 & -4 & 12 \end{pmatrix}$$

$$M_{-\alpha_{34}} = \begin{pmatrix} 8 & 4 & -8 & -\frac{32}{5} \\ 4 & 4 & -\frac{24}{5} & -4 \\ -8 & -\frac{24}{5} & 12 & 8 \\ -\frac{32}{5} & -4 & 8 & 8 \end{pmatrix}$$

Those corresponding to β_{ij} are

$$M_{-\beta_{24}} = \begin{pmatrix} 4 & -4 & 2 \\ -4 & 12 & -4 \\ 2 & -4 & 4 \end{pmatrix} \qquad M_{-\beta_{34}} = \begin{pmatrix} 8 & 4 & -8 & -\frac{32}{5} \\ 4 & 4 & -\frac{24}{5} & -4 \\ -8 & -\frac{24}{5} & 12 & 8 \\ -\frac{32}{5} & -4 & 8 & 8 \end{pmatrix}$$

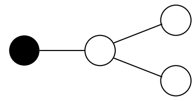
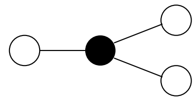
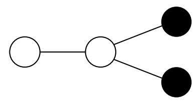
and the matrix of the trivial module is given by

$$M_0 = \left(\begin{array}{cccccc|cccccc} 8 & -4 & 4 & 0 & -\frac{16}{3} & 4 & 0 & 4 & -\frac{16}{3} & 0 & 4 & -6 \\ -4 & 8 & 4 & -4 & 4 & 0 & -4 & 0 & 4 & 4 & -\frac{32}{5} & 0 \\ 4 & 4 & 16 & -\frac{16}{3} & 0 & 8 & -\frac{16}{3} & 8 & 0 & 8 & 0 & -\frac{48}{5} \\ 0 & -4 & -\frac{16}{3} & 8 & 4 & 4 & 0 & -6 & -\frac{16}{3} & 0 & 4 & 4 \\ -\frac{16}{3} & 4 & 0 & 4 & 16 & 8 & -\frac{16}{3} & -\frac{48}{5} & 0 & 8 & 0 & 8 \\ 4 & 0 & 8 & 4 & 8 & 24 & -6 & 0 & -\frac{48}{5} & 12 & 12 & 0 \\ \hline 0 & -4 & -\frac{16}{3} & 0 & -\frac{16}{3} & -6 & 8 & 4 & 4 & 0 & 4 & 4 \\ 4 & 0 & 8 & -6 & -\frac{48}{5} & 0 & 4 & 24 & 8 & 12 & 12 & 0 \\ -\frac{16}{3} & 4 & 0 & -\frac{16}{3} & 0 & -\frac{48}{5} & 4 & 8 & 16 & 8 & 0 & 8 \\ 0 & 4 & 8 & 0 & 8 & 12 & 0 & 12 & 8 & 40 & 16 & 12 \\ 4 & -\frac{32}{5} & 0 & 4 & 0 & 12 & 4 & 12 & 0 & 16 & 32 & 12 \\ -6 & 0 & -\frac{48}{5} & 4 & 8 & 0 & 4 & 0 & 8 & 12 & 12 & 24 \end{array}\right)$$

It turns out that no further changes are necessary, since all of these matrices are positive definite by Sylvester's criterion.

Larger Isotropy

We will not present the above for all K in the same detail. In fact, we will present here the different possible K and their Kähler Einstein metrics. By corollary 50 the curvature matrices are then obtained by considering the correct submatrices of the matrices of $(SO(8), \mathbb{T}, J_{std})$ with the new metric coefficients. As before, we present the details in the appendix for the interested reader. For each K, we will give the indices of the rows and columns of the curvature matrices that have to be erased to obtain the restriction and give a detailed table of the used four forms. We can reduce the different cases for K to the following three using the trialtrian symmetry of D_4 and theorem 22 together with the result of Itoh.

K	$D=(V,E)$	γ_K^*	$g_{\alpha_{12}}$	$g_{\alpha_{23}}$	$g_{\alpha_{34}}$	$g_{\beta_{34}}$
$SU(2)\ \mathbb{T}^3$		α_{12}	0	$\frac{3}{2}$	1	1
$\mathbb{S}^1\ SU(2)\ \mathbb{T}^2$		α_{23}	$\frac{3}{2}$	0	$\frac{3}{2}$	$\frac{3}{2}$
$\mathbb{T}^2\ SU(2)\ SU(2)$		$2\varepsilon_3$	1	2	0	0

Chapter 9

Appendix

We use the appendix to present the remaining part of the proof of conjecture $H(4)$, proving that the holomorphic curvature tensor can be modified into a positive tensor via four forms in all C spaces (G, K, J_{std}, g_{KE}), where G is a simple classical group of rank 4 and K is the centralizer of a torus leaving J_{std} invariant. That means in detail, that we describe the needed $\mathbb{T}$ invariant four forms and the submatrices of the curvature matrices of $(G, \mathbb{T}, J_{std})$ corresponding to the curvature tensor with isotropy K as presented in corollary 50. Notice, that we are allowed to use $\mathbb{T}$ invariant four forms by lemma 18.

The larger isotropy groups of $SU(5)$ are already completely covered in section 8.1 which is why we begin with B_4.

Larger isotropy of $SO(9)$

We treat each case in the table of possible isotropy groups K in section 8.2 separately. In order to keep the indices understandable, we add first the four forms to the matrices from the beginning of section 8.2 and cancel rows and columns afterwards.

Isotropy group $K = SU(2)\ \mathbb{T}^3$

In this case the desired modifications are:

Value	Intended	Forced
-4	$(M_{-\alpha_{12}})_{(1,5)}$	$(M_{-\varepsilon_3})_{(1,2)}$
$-\frac{20}{3}$	$(M_{-\alpha_{12}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(1,2)}$
-8	$(M_{-\alpha_{12}})_{(3,5)}$	$(M_{-\varepsilon_3})_{(3,4)}$
-8	$(M_{-\alpha_{12}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(4,5)}$
4	$(M_{-\alpha_{12}})_{(2,2)}$	$(M_0)_{(3,6)}$
8	$(M_{-\alpha_{12}})_{(3,3)}$	$(M_0)_{(8,11)}$
3	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
$-\frac{4}{5}$	$(M_{-\alpha_{13}})_{(1,4)}$	$(M_{-\varepsilon_4})_{(1,3)}$
$\frac{6}{5}$	$(M_{-\alpha_{13}})_{(2,4)}$	$(M_{-\varepsilon_2})_{(2,3)}$
$-\frac{6}{5}$	$(M_{-\alpha_{13}})_{(3,4)}$	$(M_{-\varepsilon_4})_{(4,6)}$
1	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$
$\frac{1}{7}$	$(M_{-\alpha_{14}})_{(3,3)}$	$(M_0)_{(13,16)}$
$-\frac{8}{5}$	$(M_{-\alpha_{23}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(2,3)}$
$-\frac{12}{5}$	$(M_{-\alpha_{23}})_{(3,5)}$	$(M_{-\varepsilon_1})_{(1,2)}$

Value	Intended	Forced
$-\frac{12}{5}$	$(M_{-\alpha_{23}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(5,6)}$
$\frac{1}{2}$	$(M_{-\alpha_{24}})_{(2,4)}$	$(M_{-\varepsilon_3})_{(4,5)}$
$-\frac{1}{2}$	$(M_{-\alpha_{24}})_{(3,4)}$	$(M_{-\varepsilon_1})_{(1,3)}$
-2	$(M_{-\alpha_{34}})_{(1,3)}$	$(M_{-\beta_{34}})_{(1,3)}$
-2	$(M_{-\alpha_{34}})_{(2,4)}$	$(M_{-\beta_{34}})_{(2,4)}$
$\frac{1}{4}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(13,14)}$
6	$(M_{-\varepsilon_3})_{(1,1)}$	$(M_0)_{(5,16)}$
6	$(M_{-\varepsilon_3})_{(2,2)}$	$(M_0)_{(2,15)}$
12	$(M_{-\varepsilon_3})_{(3,3)}$	$(M_0)_{(11,16)}$
12	$(M_{-\varepsilon_3})_{(4,4)}$	$(M_0)_{(8,15)}$
10	$(M_{-\varepsilon_4})_{(1,1)}$	$(M_0)_{(6,16)}$
10	$(M_{-\varepsilon_4})_{(2,2)}$	$(M_0)_{(3,15)}$
12	$(M_{-\varepsilon_4})_{(4,4)}$	$(M_0)_{(12,16)}$
12	$(M_{-\varepsilon_4})_{(5,5)}$	$(M_0)_{(9,15)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	$\times$	$\times$	1	1	$\times$	$\times$	$\times$	$\times$	1	1	$\times$

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	$\times$	1	$\times$	$\times$	4

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10, 15 and 16 (enumerated prior to erasing the fourth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 12×12 block is now positive definite and hence we have a modified positive holomorphic curvature tensor.

Isotropy group $K = \mathbb{S}^1\ SU(2)\ \mathbb{T}^2$

In this case the desired modifications are:

Value	Intended	Forced
$-\frac{9}{10}$	$(M_{-\alpha_{12}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(1,2)}$
$-\frac{6}{5}$	$(M_{-\alpha_{12}})_{(3,5)}$	$(M_{-\varepsilon_3})_{(3,4)}$
$-\frac{6}{5}$	$(M_{-\alpha_{12}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(4,5)}$
$\frac{16}{11}$	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
$-\frac{9}{10}$	$(M_{-\alpha_{13}})_{(1,4)}$	$(M_{-\varepsilon_4})_{(1,3)}$
$\frac{6}{5}$	$(M_{-\alpha_{13}})_{(2,4)}$	$(M_{-\varepsilon_2})_{(2,3)}$
$-\frac{6}{5}$	$(M_{-\alpha_{13}})_{(3,4)}$	$(M_{-\varepsilon_4})_{(4,6)}$
$\frac{16}{11}$	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$
$\frac{1}{8}$	$(M_{-\alpha_{14}})_{(3,3)}$	$(M_0)_{(13,16)}$

Value	Intended	Forced
1	$(M_{-\alpha_{23}})_{(1,1)}$	$(M_0)_{(4,5)}$
2	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(14,15)}$
$\frac{1}{5}$	$(M_{-\alpha_{24}})_{(4,4)}$	$(M_0)_{(13,15)}$
$\frac{1}{5}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(13,14)}$
1	$(M_{-\varepsilon_4})_{(1,1)}$	$(M_0)_{(6,16)}$
$\frac{27}{800}$	$(M_{-\varepsilon_4})_{(2,2)}$	$(M_0)_{(3,15)}$
$\frac{27}{800}$	$(M_{-\varepsilon_4})_{(3,3)}$	$(M_0)_{(1,14)}$
1	$(M_{-\varepsilon_4})_{(4,4)}$	$(M_0)_{(12,16)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	1	$\times$	$\times$	$\times$	$\times$	2	$\times$	1	$\times$	$\times$	$\times$	2

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	$\times$	$\times$	2	$\times$	2

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10, 11 and 16 (enumerated prior to erasing the second row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 12×12 block is now positive definite and hence we have a modified positive holomorphic curvature tensor.

Isotropy group $K = \mathbb{T}^2\ SU(2)\ \mathbb{S}^1$

In this case the desired modifications are:

Value	Intended	Forced
2	$(M_{-\alpha_{12}})_{(1,4)}$	$(M_{-\beta_{34}})_{(1,2)}$
2	$(M_{-\alpha_{12}})_{(2,3)}$	$(M_{-\beta_{34}})_{(3,4)}$
$\frac{25}{12}$	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
$\frac{3}{8}$	$(M_{-\alpha_{13}})_{(2,4)}$	$(M_{-\varepsilon_2})_{(2,3)}$
$-\frac{3}{8}$	$(M_{-\alpha_{13}})_{(3,4)}$	$(M_{-\varepsilon_4})_{(4,6)}$
$\frac{4}{9}$	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$
$\frac{3}{8}$	$(M_{-\alpha_{14}})_{(1,3)}$	$(M_{-\varepsilon_2})_{(2,4)}$
$\frac{3}{8}$	$(M_{-\alpha_{14}})_{(2,3)}$	$(M_{-\varepsilon_3})_{(3,5)}$
$\frac{4}{9}$	$(M_{-\alpha_{14}})_{(3,3)}$	$(M_0)_{(13,16)}$

Value	Intended	Forced
$\frac{4}{7}$	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(14,15)}$
$\frac{4}{7}$	$(M_{-\alpha_{24}})_{(4,4)}$	$(M_0)_{(13,15)}$
$\frac{253}{400}$	$(M_{-\alpha_{34}})_{(3,5)}$	$(M_{-\varepsilon_1})_{(2,3)}$
$\frac{253}{400}$	$(M_{-\alpha_{34}})_{(4,5)}$	$(M_{-\varepsilon_2})_{(3,4)}$
$\frac{9}{8}$	$(M_{-\alpha_{34}})_{(2,2)}$	$(M_0)_{(2,3)}$
$\frac{15}{2}$	$(M_{-\alpha_{34}})_{(4,4)}$	$(M_0)_{(8,9)}$
$\frac{8}{5}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(13,14)}$
$-\frac{5}{8}$	$(M_{-\varepsilon_1})_{(2,2)}$	$(M_0)_{(11,14)}$
$-\frac{5}{8}$	$(M_{-\varepsilon_1})_{(3,3)}$	$(M_0)_{(12,13)}$

Now all matrices are positive semidefinite except for M_0. We need further changes, but since they all correspond to adding negative off diagonal entries to M_0 by the arguments in the proof of proposition 31 this does not worsen the M_η. Therefore, we do not have to keep track of the forced modifications of the following:

Entry of M_0	$(4,11)$	$(6,11)$	$(2,15)$	$(3,15)$	$(8,15)$	$(9,15)$
Value	-2	-2	$-\frac{87}{40}$	$-\frac{87}{40}$	$-\frac{11}{20}$	$-\frac{11}{20}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	1	$\times$	2	$\times$	$\times$	$\times$	$\times$	2	$\times$	3	$\times$

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	$\times$	$\times$	$\times$	3	1

As in the case of the toric isotropy, we see that column and row 10 and 16 (enumerated prior to erasing the first row and column) have only positive entries and can discarded as in the example of G_2. The remaining 13×13 block is now positive definite and hence we have a modified positive holomorphic curvature tensor.

Isotropy group $K = \mathbb{T}^3\ SU(2)$

In this case the desired modifications are:

Value	Intended	Forced
$-\frac{4}{5}$	$(M_{-\alpha_{12}})_{(1,5)}$	$(M_{-\varepsilon_3})_{(1,2)}$
-2	$(M_{-\alpha_{12}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(1,2)}$
-2	$(M_{-\alpha_{12}})_{(3,5)}$	$(M_{-\varepsilon_3})_{(3,4)}$
-2	$(M_{-\alpha_{12}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(4,5)}$
$\frac{25}{12}$	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
$-\frac{12}{5}$	$(M_{-\alpha_{13}})_{(1,4)}$	$(M_{-\varepsilon_4})_{(1,3)}$
$\frac{12}{5}$	$(M_{-\alpha_{13}})_{(2,4)}$	$(M_{-\varepsilon_2})_{(2,3)}$
$-\frac{12}{5}$	$(M_{-\alpha_{13}})_{(3,4)}$	$(M_{-\varepsilon_4})_{(4,6)}$
$\frac{9}{10}$	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$

Value	Intended	Forced
$-\frac{5}{4}$	$(M_{-\alpha_{23}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(2,3)}$
$-\frac{5}{4}$	$(M_{-\alpha_{23}})_{(3,5)}$	$(M_{-\varepsilon_1})_{(1,2)}$
$-\frac{5}{4}$	$(M_{-\alpha_{23}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(5,6)}$
$\frac{9}{8}$	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(14,15)}$
$\frac{5}{8}$	$(M_{-\varepsilon_3})_{(3,3)}$	$(M_0)_{(11,16)}$
$\frac{3}{8}$	$(M_{-\varepsilon_4})_{(2,2)}$	$(M_0)_{(3,15)}$
$\frac{5}{8}$	$(M_{-\varepsilon_4})_{(3,3)}$	$(M_0)_{(1,14)}$
14	$(M_{-\varepsilon_4})_{(4,4)}$	$(M_0)_{(12,16)}$
$\frac{5}{8}$	$(M_{-\varepsilon_4})_{(6,6)}$	$(M_0)_{(7,14)}$

We need further changes, but since they all correspond to adding negative off diagonal entries to M_0 by the arguments in the proof of proposition 31 this does not worsen the M_η. Therefore, we do not have to keep track of the forced modifications of the following:

Entry of M_0	$(8,15)$	$(9,15)$
Value	$-\frac{5}{4}$	$-\frac{5}{4}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	$\times$	3	$\times$	4	5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	3	4	5	$\times$	13

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10 and 16 (enumerated prior to erasing the thirteenth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 13×13 block is now positive definite and hence we have a modified positive holomorphic curvature tensor.

Isotropy group $K = SU(3)\ \mathbb{T}^2$

In this case the desired modifications are:

Value	Intended	Forced
$\frac{5}{2}$	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
$\frac{5}{2}$	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$
$\frac{1}{6}$	$(M_{-\alpha_{14}})_{(3,3)}$	$(M_0)_{(13,16)}$
$\frac{5}{2}$	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(14,15)}$
$\frac{1}{6}$	$(M_{-\alpha_{24}})_{(4,4)}$	$(M_0)_{(13,15)}$
$\frac{1}{6}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(13,14)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	1	$\times$	$\times$	1	1	$1,2$	$\times$	1	$\times$	$1,2$	1	$1,2$

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	$\times$	1	$1,2$	$\times$	$2,4,5$

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10, 11,14,15 and 16 (enumerated prior to erasing the second fourth and fifth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 8×8 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $K = SU(2)\ \mathbb{S}^1\ SU(2)\ \mathbb{S}^1$

In this case the desired modifications are:

Value	Intended	Forced
8	$(M_{-\alpha_{12}})_{(1,4)}$	$(M_{-\beta_{34}})_{(1,2)}$
8	$(M_{-\alpha_{12}})_{(2,3)}$	$(M_{-\beta_{34}})_{(3,4)}$
$\frac{11}{4}$	$(M_{-\alpha_{12}})_{(1,1)}$	$(M_0)_{(2,5)}$
$\frac{11}{4}$	$(M_{-\alpha_{12}})_{(2,2)}$	$(M_0)_{(3,6)}$
$\frac{11}{4}$	$(M_{-\alpha_{12}})_{(3,3)}$	$(M_0)_{(8,11)}$
$\frac{11}{4}$	$(M_{-\alpha_{12}})_{(4,4)}$	$(M_0)_{(9,12)}$
3	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
$\frac{1}{2}$	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$
$\frac{1}{2}$	$(M_{-\alpha_{14}})_{(3,3)}$	$(M_0)_{(13,16)}$

Value	Intended	Forced
$\frac{1}{2}$	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(14,15)}$
$\frac{1}{2}$	$(M_{-\alpha_{24}})_{(4,4)}$	$(M_0)_{(13,15)}$
-8	$(M_{-\alpha_{34}})_{(1,3)}$	$(M_{-\beta_{34}})_{(1,3)}$
-8	$(M_{-\alpha_{34}})_{(2,4)}$	$(M_{-\beta_{34}})_{(2,4)}$
8	$(M_{-\alpha_{34}})_{(1,1)}$	$(M_0)_{(5,6)}$
8	$(M_{-\alpha_{34}})_{(2,2)}$	$(M_0)_{(2,3)}$
$\frac{11}{4}$	$(M_{-\alpha_{34}})_{(3,3)}$	$(M_0)_{(11,12)}$
$\frac{11}{4}$	$(M_{-\alpha_{34}})_{(4,4)}$	$(M_0)_{(8,9)}$
1	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(13,14)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	1	$\times$	$1,2$	1	$\times$	$\times$	$\times$	2	1	$1,3$	$\times$

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	$\times$	1	$\times$	3	$1,4$

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10, 15 and 16 (enumerated prior to erasing the first and fourth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 11×11 block is now positive definite and hence we have a modified positive holomorphic curvature tensor.

Isotropy group $K = SU(2)\ \mathbb{T}^2\ SU(2)$

In this case the desired modifications are:

Value	Intended	Forced
3	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
$-\frac{9}{5}$	$(M_{-\alpha_{13}})_{(1,4)}$	$(M_{-\varepsilon_4})_{(1,3)}$
$\frac{9}{5}$	$(M_{-\alpha_{13}})_{(2,4)}$	$(M_{-\varepsilon_2})_{(2,3)}$
$-\frac{9}{5}$	$(M_{-\alpha_{13}})_{(3,4)}$	$(M_{-\varepsilon_4})_{(4,6)}$
1	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$
$-\frac{9}{5}$	$(M_{-\alpha_{23}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(2,3)}$
$-\frac{9}{5}$	$(M_{-\alpha_{23}})_{(3,5)}$	$(M_{-\varepsilon_1})_{(1,2)}$

Value	Intended	Forced
$-\frac{9}{5}$	$(M_{-\alpha_{23}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(5,6)}$
1	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(14,15)}$
$\frac{23}{80}$	$(M_{-\varepsilon_4})_{(1,1)}$	$(M_0)_{(6,16)}$
$\frac{23}{80}$	$(M_{-\varepsilon_4})_{(2,2)}$	$(M_0)_{(3,15)}$
$\frac{23}{80}$	$(M_{-\varepsilon_4})_{(3,3)}$	$(M_0)_{(1,14)}$
$\frac{23}{80}$	$(M_{-\varepsilon_4})_{(4,4)}$	$(M_0)_{(12,16)}$
$\frac{23}{80}$	$(M_{-\varepsilon_4})_{(5,5)}$	$(M_0)_{(9,15)}$
$\frac{23}{80}$	$(M_{-\varepsilon_4})_{(6,6)}$	$(M_0)_{(7,14)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	$\times$	3	1	$1,4$	5	$\times$	$\times$	$\times$	1	1	$\times$

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	3	$1,4$	5	$\times$	$4,13$

Now all matrices are positive semidefinite except for M_0. We add $-\frac{1}{4}$ to the symmetrically to the entry $(9,12)$ of M_0. For the resulting M_0 we see as in the case of the toric isotropy that column and row 10, 15 and 16 (enumerated prior to erasing the fourth and thirteenth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 11×11 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $K = \mathbb{S}^1\ SU(3)\ \mathbb{S}^1$

In this case the desired modifications are:

Value	Intended	Forced
$\frac{9}{10}$	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
$\frac{9}{10}$	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$
$\frac{9}{10}$	$(M_{-\alpha_{14}})_{(3,3)}$	$(M_0)_{(13,16)}$
$\frac{23}{8}$	$(M_{-\alpha_{23}})_{(1,1)}$	$(M_0)_{(4,5)}$
$\frac{3}{2}$	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(14,15)}$

Value	Intended	Forced
$\frac{23}{8}$	$(M_{-\alpha_{24}})_{(1,1)}$	$(M_0)_{(4,6)}$
$\frac{3}{2}$	$(M_{-\alpha_{24}})_{(4,4)}$	$(M_0)_{(13,15)}$
$\frac{23}{8}$	$(M_{-\alpha_{34}})_{(1,1)}$	$(M_0)_{(5,6)}$
$\frac{3}{2}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(13,14)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$1,2$	1	$\times$	2	$\times$	2	$\times$	1	$1,2$	$\times$	3	$2,4$

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	$\times$	$\times$	2	$2,3$	$1,2,3$

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10,11,12 and 16 (enumerated prior to erasing the first , second and third row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 9×9 block is now positive definite and hence we have a modified positive holomorphic curvature tensor.

Isotropy group $K = \mathbb{S}^1\ SU(2)\ \mathbb{S}^1\ SU(2)$

In this case the desired modifications are:

Value	Intended	Forced
-2	$(M_{-\alpha_{12}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(1,2)}$
-2	$(M_{-\alpha_{12}})_{(3,5)}$	$(M_{-\varepsilon_3})_{(3,4)}$
-2	$(M_{-\alpha_{12}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(4,5)}$
$\frac{16}{11}$	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
-2	$(M_{-\alpha_{13}})_{(1,4)}$	$(M_{-\varepsilon_4})_{(1,3)}$
2	$(M_{-\alpha_{13}})_{(2,4)}$	$(M_{-\varepsilon_2})_{(2,3)}$
-2	$(M_{-\alpha_{13}})_{(3,4)}$	$(M_{-\varepsilon_4})_{(4,6)}$
$\frac{16}{11}$	$(M_{-\alpha_{13}})_{(4,4)}$	$(M_0)_{(14,16)}$

Value	Intended	Forced
2	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(14,15)}$
$\frac{21}{80}$	$(M_{-\varepsilon_4})_{(1,1)}$	$(M_0)_{(6,16)}$
$\frac{21}{80}$	$(M_{-\varepsilon_4})_{(2,2)}$	$(M_0)_{(3,15)}$
$\frac{21}{80}$	$(M_{-\varepsilon_4})_{(3,3)}$	$(M_0)_{(1,14)}$
$\frac{21}{80}$	$(M_{-\varepsilon_4})_{(4,4)}$	$(M_0)_{(12,16)}$
$\frac{21}{80}$	$(M_{-\varepsilon_4})_{(5,5)}$	$(M_0)_{(9,15)}$
$\frac{21}{80}$	$(M_{-\varepsilon_4})_{(6,6)}$	$(M_0)_{(7,14)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	1	$\times$	3	$\times$	4	2, 5	$\times$	1	$\times$	$\times$	$\times$	2

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	3	4	2, 5	$\times$	2, 13

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10,11 and 16 (enumerated prior to erasing the second and thirteenth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 11×11 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $K = \mathbb{T}^2\ SO(5)$

In this case the desired modifications are:

Value	Intended	Forced
-1	$(M_{-\alpha_{12}})_{(1,5)}$	$(M_{-\varepsilon_3})_{(1,2)}$
-1	$(M_{-\alpha_{12}})_{(2,5)}$	$(M_{-\varepsilon_4})_{(1,2)}$
-1	$(M_{-\alpha_{12}})_{(3,5)}$	$(M_{-\varepsilon_3})_{(3,4)}$
-1	$(M_{-\alpha_{12}})_{(4,5)}$	$(M_{-\varepsilon_4})_{(4,5)}$
$\frac{27}{8}$	$(M_{-\alpha_{12}})_{(1,1)}$	$(M_0)_{(2,5)}$
$\frac{27}{8}$	$(M_{-\alpha_{12}})_{(2,2)}$	$(M_0)_{(3,6)}$
$\frac{27}{8}$	$(M_{-\alpha_{12}})_{(3,3)}$	$(M_0)_{(8,11)}$
$\frac{27}{8}$	$(M_{-\alpha_{12}})_{(4,4)}$	$(M_0)_{(9,12)}$
$\frac{25}{12}$	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(15,16)}$
$\frac{27}{8}$	$(M_{-\alpha_{34}})_{(1,1)}$	$(M_0)_{(5,6)}$
$\frac{27}{8}$	$(M_{-\alpha_{34}})_{(2,2)}$	$(M_0)_{(2,3)}$
$\frac{27}{8}$	$(M_{-\alpha_{34}})_{(3,3)}$	$(M_0)_{(11,12)}$
$\frac{27}{8}$	$(M_{-\alpha_{34}})_{(4,4)}$	$(M_0)_{(8,9)}$

Value	Intended	Forced
$\frac{27}{8}$	$(M_{-\beta_{34}})_{(1,1)}$	$(M_0)_{(5,12)}$
$\frac{27}{8}$	$(M_{-\beta_{34}})_{(2,2)}$	$(M_0)_{(2,9)}$
$\frac{27}{8}$	$(M_{-\beta_{34}})_{(3,3)}$	$(M_0)_{(6,11)}$
$\frac{27}{8}$	$(M_{-\beta_{34}})_{(4,4)}$	$(M_0)_{(3,8)}$
$\frac{3}{4}$	$(M_{-\varepsilon_3})_{(1,1)}$	$(M_0)_{(5,16)}$
$\frac{3}{4}$	$(M_{-\varepsilon_3})_{(2,2)}$	$(M_0)_{(2,15)}$
$\frac{3}{4}$	$(M_{-\varepsilon_3})_{(3,3)}$	$(M_0)_{(11,16)}$
$\frac{3}{4}$	$(M_{-\varepsilon_3})_{(4,4)}$	$(M_0)_{(8,15)}$
$\frac{3}{4}$	$(M_{-\varepsilon_4})_{(1,1)}$	$(M_0)_{(6,16)}$
$\frac{3}{4}$	$(M_{-\varepsilon_4})_{(2,2)}$	$(M_0)_{(3,15)}$
$\frac{3}{4}$	$(M_{-\varepsilon_4})_{(4,4)}$	$(M_0)_{(12,16)}$
$\frac{3}{4}$	$(M_{-\varepsilon_4})_{(5,5)}$	$(M_0)_{(9,15)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	$\times$	3	$\times$	4	5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

and

$-\eta$	ε_1	ε_2	ε_3	ε_4	0
$\sharp$ of M_η	3	4	5	$\times$	1, 7, 13, 14

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10 and 16 (enumerated prior to erasing the first, seventh,thirteenth and fourteenth row and column) have only positive entries and can discarded as in the example of G_2. The remaining 10×10 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Larger isotropy of $Sp(4)$

We treat each case in the table of possible isotropy groups K in section 8.3 separately. In order to keep the indices understandable, we add first the four forms to the matrices from the beginning of section 8.3 and cancel rows and columns afterwards.

Isotropy group $SU(2)$ $\mathbb{T}^3$

In this case the desired modifications are:

Value	Intended	Forced
$\frac{15}{14}$	$(M_{-\alpha_{12}})_{(1,3)}$	$(M_{-\gamma_3})_{(1,2)}$
$\frac{9}{8}$	$(M_{-\alpha_{12}})_{(1,1)}$	$(M_0)_{(2,5)}$
$\frac{9}{8}$	$(M_{-\alpha_{12}})_{(2,2)}$	$(M_0)_{(3,6)}$
$\frac{9}{8}$	$(M_{-\alpha_{12}})_{(3,3)}$	$(M_0)_{(8,11)}$
$\frac{9}{8}$	$(M_{-\alpha_{12}})_{(4,4)}$	$(M_0)_{(9,12)}$
14	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(10,16)}$
14	$(M_{-\alpha_{12}})_{(6,6)}$	$(M_0)_{(10,15)}$

Value	Intended	Forced
-3	$(M_{-\alpha_{34}})_{(1,3)}$	$(M_{-\beta_{34}})_{(1,3)}$
-3	$(M_{-\alpha_{34}})_{(2,4)}$	$(M_{-\beta_{34}})_{(2,4)}$
$\frac{9}{7}$	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
$\frac{9}{7}$	$(M_{-\gamma_3})_{(2,2)}$	$(M_0)_{(2,8)}$
$\frac{25}{7}$	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
$\frac{25}{7}$	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$
1	$(M_{-\gamma_4})_{(3,3)}$	$(M_0)_{(1,7)}$
$-\frac{1}{4}$	$(M_0)_{(7,14)}$	$(M_{-\alpha_{34}})_{(5,5)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	$\times$	$\times$	1	1	$\times$	$\times$	$\times$	$\times$	1	1	$\times$

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	1	$\times$	$\times$	4

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10,15 and 16 (enumerated prior to erasing the fourth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 12×12 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $\mathbb{S}^1$ $SU(2)$ $\mathbb{T}^2$

In this case the desired modifications are:

Value	Intended	Forced
$-\frac{1}{4}$	$(M_{-\alpha_{12}})_{(2,4)}$	$(M_{-\gamma_4})_{(1,2)}$
$-\frac{1}{4}$	$(M_{-\alpha_{13}})_{(1,3)}$	$(M_{-\gamma_4})_{(1,3)}$
$\frac{5}{8}$	$(M_{-\alpha_{23}})_{(2,4)}$	$(M_{-\gamma_4})_{(2,3)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(1,1)}$	$(M_0)_{(4,5)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(2,2)}$	$(M_0)_{(1,3)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(3,3)}$	$(M_0)_{(10,11)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(4,4)}$	$(M_0)_{(7,9)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(8,15)}$

Value	Intended	Forced
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(6,6)}$	$(M_0)_{(8,14)}$
10	$(M_{-\alpha_{24}})_{(3,3)}$	$(M_0)_{(10,12)}$
10	$(M_{-\alpha_{34}})_{(3,3)}$	$(M_0)_{(11,12)}$
$\frac{9}{8}$	$(M_{-\gamma_2})_{(1,1)}$	$(M_0)_{(4,10)}$
$\frac{9}{8}$	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
$\frac{9}{2}$	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
$\frac{9}{5}$	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$
$\frac{9}{5}$	$(M_{-\gamma_4})_{(3,3)}$	$(M_0)_{(1,7)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	1	$\times$	$\times$	$\times$	$\times$	2	$\times$	1	$\times$	3	$\times$	2

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	$\times$	2	$\times$	2

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10,11 and 16 (enumerated prior to erasing the second row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 12×12 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $\mathbb{T}^2$ $SU(2)$ $\mathbb{S}^1$

In this case the desired modifications are:

Value	Intended	Forced
$-\frac{1}{4}$	$(M_{-\alpha_{12}})_{(1,3)}$	$(M_{-\gamma_3})_{(1,2)}$
$-\frac{1}{4}$	$(M_{-\alpha_{12}})_{(2,4)}$	$(M_{-\gamma_4})_{(1,2)}$
$\frac{1}{2}$	$(M_{-\alpha_{12}})_{(1,1)}$	$(M_0)_{(2,5)}$
$\frac{1}{2}$	$(M_{-\alpha_{12}})_{(2,2)}$	$(M_0)_{(3,6)}$
$\frac{3}{2}$	$(M_{-\alpha_{12}})_{(3,3)}$	$(M_0)_{(8,11)}$
$\frac{3}{2}$	$(M_{-\alpha_{12}})_{(4,4)}$	$(M_0)_{(9,12)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(3,3)}$	$(M_0)_{(10,11)}$
$\frac{5}{4}$	$(M_{-\alpha_{24}})_{(3,3)}$	$(M_0)_{(10,12)}$
$-\frac{15}{4}$	$(M_{-\alpha_{34}})_{(1,3)}$	$(M_{-\beta_{34}})_{(1,3)}$
$-\frac{15}{4}$	$(M_{-\alpha_{34}})_{(2,4)}$	$(M_{-\beta_{34}})_{(2,4)}$

Value	Intended	Forced
$\frac{5}{4}$	$(M_{-\alpha_{34}})_{(1,1)}$	$(M_0)_{(5,6)}$
$\frac{7}{8}$	$(M_{-\alpha_{34}})_{(2,2)}$	$(M_0)_{(2,3)}$
$\frac{15}{8}$	$(M_{-\alpha_{34}})_{(3,3)}$	$(M_0)_{(11,12)}$
$\frac{15}{8}$	$(M_{-\alpha_{34}})_{(4,4)}$	$(M_0)_{(8,9)}$
$\frac{5}{8}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(7,14)}$
$\frac{5}{8}$	$(M_{-\alpha_{34}})_{(6,6)}$	$(M_0)_{(7,13)}$
$\frac{1}{2}$	$(M_{-\gamma_2})_{(1,1)}$	$(M_0)_{(4,10)}$
$\frac{25}{8}$	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
$\frac{3}{2}$	$(M_{-\gamma_3})_{(2,2)}$	$(M_0)_{(2,8)}$
$\frac{25}{8}$	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
$\frac{3}{2}$	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	1	$\times$	2	$\times$	$\times$	$\times$	$\times$	2	$\times$	3	5

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	$\times$	$\times$	3	1

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10 and 16 (enumerated prior to erasing the first row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 13×13 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $\mathbb{T}^3$ $SU(2)$

In this case the desired modifications are:

Value	Intended	Forced
$-\frac{4}{7}$	$(M_{-\alpha_{12}})_{(1,3)}$	$(M_{-\gamma_3})_{(1,2)}$
$-\frac{24}{7}$	$(M_{-\alpha_{12}})_{(2,4)}$	$(M_{-\gamma_4})_{(1,2)}$
$-\frac{11}{4}$	$(M_{-\alpha_{13}})_{(1,3)}$	$(M_{-\gamma_4})_{(1,3)}$
$-\frac{25}{8}$	$(M_{-\alpha_{23}})_{(2,4)}$	$(M_{-\gamma_4})_{(2,3)}$
$\frac{5}{2}$	$(M_{-\alpha_{23}})_{(3,3)}$	$(M_0)_{(10,11)}$
$-\frac{5}{2}$	$(M_{-\alpha_{34}})_{(1,3)}$	$(M_{-\beta_{34}})_{(1,3)}$
$-\frac{5}{2}$	$(M_{-\alpha_{34}})_{(2,4)}$	$(M_{-\beta_{34}})_{(2,4)}$

Value	Intended	Forced
$\frac{1}{2}$	$(M_{-\gamma_2})_{(1,1)}$	$(M_0)_{(4,10)}$
2	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
$\frac{2}{3}$	$(M_{-\gamma_3})_{(2,2)}$	$(M_0)_{(2,8)}$
8	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
6	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$
$\frac{21}{4}$	$(M_{-\gamma_4})_{(3,3)}$	$(M_0)_{(1,7)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	$\times$	4	$\times$	5	6	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	$\times$	$\times$	$\times$	13

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10 and 16 (enumerated prior to erasing the thirteenth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 13×13 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $SU(3)$ $\mathbb{T}^2$

In this case the desired modifications are:

Value	Intended	Forced
$\frac{8}{3}$	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
$\frac{8}{3}$	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$
$\frac{8}{3}$	$(M_{-\gamma_4})_{(3,3)}$	$(M_0)_{(1,7)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	1	$\times$	$\times$	1	1	$1,2$	$\times$	$\times$	$\times$	$\times$	1	$1,2$

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	1	2	$\times$	$2,4,5$

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10,11,14,15 and 16 (enumerated prior to erasing the second, fourth and fifth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 8×8 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $SU(2)\ \mathbb{S}^1\ SU(2)\ \mathbb{S}^1$

In this case the desired modifications are:

Value	Intended	Forced
$\frac{4}{7}$	$(M_{-\alpha_{12}})_{(1,3)}$	$(M_{-\gamma_3})_{(1,2)}$
$\frac{4}{7}$	$(M_{-\alpha_{12}})_{(2,4)}$	$(M_{-\gamma_4})_{(1,2)}$
$\frac{7}{4}$	$(M_{-\alpha_{12}})_{(1,1)}$	$(M_0)_{(2,5)}$
$\frac{7}{4}$	$(M_{-\alpha_{12}})_{(2,2)}$	$(M_0)_{(3,6)}$
$\frac{7}{4}$	$(M_{-\alpha_{12}})_{(3,3)}$	$(M_0)_{(8,11)}$
$\frac{7}{4}$	$(M_{-\alpha_{12}})_{(4,4)}$	$(M_0)_{(9,12)}$
$\frac{7}{4}$	$(M_{-\alpha_{12}})_{(5,5)}$	$(M_0)_{(10,16)}$
$\frac{7}{4}$	$(M_{-\alpha_{12}})_{(6,6)}$	$(M_0)_{(10,15)}$
-4	$(M_{-\alpha_{34}})_{(1,3)}$	$(M_{-\beta_{34}})_{(1,3)}$

Value	Intended	Forced
-4	$(M_{-\alpha_{34}})_{(2,4)}$	$(M_{-\beta_{34}})_{(2,4)}$
$\frac{3}{8}$	$(M_{-\alpha_{34}})_{(1,1)}$	$(M_0)_{(5,6)}$
$\frac{3}{8}$	$(M_{-\alpha_{34}})_{(4,4)}$	$(M_0)_{(8,9)}$
$\frac{3}{8}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(7,14)}$
$\frac{3}{8}$	$(M_{-\alpha_{34}})_{(6,6)}$	$(M_0)_{(7,13)}$
$\frac{16}{7}$	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
$\frac{16}{7}$	$(M_{-\gamma_3})_{(2,2)}$	$(M_0)_{(2,8)}$
$\frac{16}{7}$	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
$\frac{16}{7}$	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	1	$\times$	$1,2$	1	$\times$	$\times$	$\times$	2	1	$1,3$	5

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	1	$\times$	3	$1,4$

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10,15 and 16 (enumerated prior to erasing the first and fourth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 11×11 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $SU(2)$ $\mathbb{T}^2$ $SU(2)$

In this case the desired modifications are:

Value	Intended	Forced
$\frac{15}{14}$	$(M_{-\alpha_{12}})_{(1,3)}$	$(M_{-\gamma_3})_{(1,2)}$
$-\frac{7}{2}$	$(M_{-\alpha_{12}})_{(2,4)}$	$(M_{-\gamma_4})_{(1,2)}$
$-\frac{15}{4}$	$(M_{-\alpha_{13}})_{(1,3)}$	$(M_{-\gamma_4})_{(1,3)}$
$-\frac{15}{4}$	$(M_{-\alpha_{23}})_{(2,4)}$	$(M_{-\gamma_4})_{(2,3)}$
-3	$(M_{-\alpha_{34}})_{(1,3)}$	$(M_{-\beta_{34}})_{(1,3)}$
-3	$(M_{-\alpha_{34}})_{(2,4)}$	$(M_{-\beta_{34}})_{(2,4)}$

Value	Intended	Forced
$\frac{9}{7}$	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
$\frac{9}{7}$	$(M_{-\gamma_3})_{(2,2)}$	$(M_0)_{(2,8)}$
7	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
$\frac{57}{8}$	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$
4	$(M_{-\gamma_4})_{(3,3)}$	$(M_0)_{(1,7)}$
$-\frac{1}{4}$	$(M_0)_{(9,12)}$	$(M_{-\alpha_{12}})_{(4,4)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	$\times$	4	1	$1,5$	6	$\times$	$\times$	$\times$	1	1	$\times$

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	1	$\times$	$\times$	$4,13$

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10,15 and 16 (enumerated prior to erasing the fourth and thirteenth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 11×11 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $\mathbb{S}^1$ $SU(3)$ $\mathbb{S}^1$

In this case the desired modifications are:

Value	Intended	Forced
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(1,1)}$	$(M_0)_{(4,5)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(3,3)}$	$(M_0)_{(10,11)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(4,4)}$	$(M_0)_{(7,9)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(8,15)}$
$\frac{5}{4}$	$(M_{-\alpha_{23}})_{(6,6)}$	$(M_0)_{(8,14)}$
$\frac{5}{4}$	$(M_{-\alpha_{24}})_{(1,1)}$	$(M_0)_{(4,6)}$
$\frac{5}{4}$	$(M_{-\alpha_{24}})_{(2,2)}$	$(M_0)_{(7,8)}$
$\frac{5}{4}$	$(M_{-\alpha_{24}})_{(3,3)}$	$(M_0)_{(10,12)}$
$\frac{5}{4}$	$(M_{-\alpha_{24}})_{(4,4)}$	$(M_0)_{(9,15)}$

Value	Intended	Forced
$\frac{5}{4}$	$(M_{-\alpha_{24}})_{(5,5)}$	$(M_0)_{(9,13)}$
$\frac{5}{4}$	$(M_{-\alpha_{34}})_{(1,1)}$	$(M_0)_{(5,6)}$
$\frac{5}{4}$	$(M_{-\alpha_{34}})_{(3,3)}$	$(M_0)_{(11,12)}$
$\frac{5}{4}$	$(M_{-\alpha_{34}})_{(4,4)}$	$(M_0)_{(8,9)}$
$\frac{5}{4}$	$(M_{-\alpha_{34}})_{(5,5)}$	$(M_0)_{(7,14)}$
$\frac{5}{4}$	$(M_{-\alpha_{34}})_{(6,6)}$	$(M_0)_{(7,13)}$
2	$(M_{-\gamma_2})_{(1,1)}$	$(M_0)_{(4,10)}$
2	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
2	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$1,2$	1	$\times$	2	$\times$	2	$\times$	1	$1,2$	3	$3,4$	$2,4,5$

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	$\times$	2	$2,3$	$1,2,3$

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10,11,12 and 16 (enumerated prior to erasing the first three rows and columns) have only positive entries and can be discarded as in the example of G_2. The remaining 9×9 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy groups $\mathbb{S}^1$ $SU(2)$ $\mathbb{S}^1$ $SU(2)$

In this case the desired modifications are:

Value	Intended	Forced
$-\frac{25}{8}$	$(M_{-\alpha_{12}})_{(2,4)}$	$(M_{-\gamma_4})_{(1,2)}$
$-\frac{25}{8}$	$(M_{-\alpha_{13}})_{(1,3)}$	$(M_{-\gamma_4})_{(1,3)}$
$\frac{5}{2}$	$(M_{-\alpha_{23}})_{(1,1)}$	$(M_0)_{(4,5)}$
$\frac{5}{2}$	$(M_{-\alpha_{23}})_{(2,2)}$	$(M_0)_{(1,3)}$
13	$(M_{-\alpha_{23}})_{(3,3)}$	$(M_0)_{(10,11)}$
$\frac{5}{2}$	$(M_{-\alpha_{23}})_{(4,4)}$	$(M_0)_{(7,9)}$
$\frac{5}{2}$	$(M_{-\alpha_{23}})_{(5,5)}$	$(M_0)_{(8,15)}$

Value	Intended	Forced
$\frac{5}{2}$	$(M_{-\alpha_{23}})_{(6,6)}$	$(M_0)_{(8,14)}$
$\frac{9}{8}$	$(M_{-\gamma_2})_{(1,1)}$	$(M_0)_{(4,10)}$
$\frac{9}{8}$	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
8	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
5	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$
5	$(M_{-\gamma_4})_{(3,3)}$	$(M_0)_{(1,7)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	1	$\times$	4	$\times$	5	$2,6$	$\times$	1	$\times$	3	$\times$	2

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	$\times$	2	$\times$	$2,13$

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10,11 and 16 (enumerated prior to erasing the second and thirteenth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 11×11 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Isotropy group $\mathbb{T}^2$ $Sp(2)$

In this case the desired modifications are:

Value	Intended	Forced
$-\frac{24}{7}$	$(M_{-\alpha_{12}})_{(1,3)}$	$(M_{-\gamma_3})_{(1,2)}$
$-\frac{24}{7}$	$(M_{-\alpha_{12}})_{(2,4)}$	$(M_{-\gamma_4})_{(1,2)}$
$-\frac{55}{8}$	$(M_{-\alpha_{34}})_{(1,3)}$	$(M_{-\beta_{34}})_{(1,3)}$
$-\frac{55}{8}$	$(M_{-\alpha_{34}})_{(2,4)}$	$(M_{-\beta_{34}})_{(2,4)}$
2	$(M_{-\alpha_{34}})_{(2,2)}$	$(M_0)_{(2,3)}$

Value	Intended	Forced
2	$(M_{-\alpha_{34}})_{(4,4)}$	$(M_0)_{(8,9)}$
8	$(M_{-\gamma_3})_{(1,1)}$	$(M_0)_{(5,11)}$
6	$(M_{-\gamma_3})_{(2,2)}$	$(M_0)_{(2,8)}$
8	$(M_{-\gamma_4})_{(1,1)}$	$(M_0)_{(6,12)}$
6	$(M_{-\gamma_4})_{(2,2)}$	$(M_0)_{(3,9)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}
$\sharp$ of M_η	$\times$	$1,3,5$	$2,4$	$2,4,6$	$2,5$	$5,6$	$\times$	$\times$	2	$\times$	3	5

and

$-\eta$	γ_1	γ_2	γ_3	γ_4	0
$\sharp$ of M_η	$\times$	1	$\times$	3	$1, 7, 13, 14$

Now all matrices are positive semidefinite except for M_0. As in the case of the toric isotropy, we see that column and row 10 and 16 (enumerated prior to erasing the first, seventh, thirteenth and fourteenth row and column) have only positive entries and can be discarded as in the example of G_2. The remaining 10×10 block is now positive definite and hence we have a positive modified holomorphic curvature tensor.

Larger isotropy $SO(8)$

We treat each case in the table of possible isotropy groups K in section 8.4 separately. In order to keep the indices understandable, we add first the four forms to the matrices from the beginning of section 8.4 and cancel rows and columns afterwards.

Isotropy group $SU(2)$ $\mathbb{T}^3$

In this case the desired modifications are:

Value	Intended	Forced
-1	$(M_0)_{(9,12)}$	$(M_{-\alpha_{12}})_{(4,4)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}	0
$\sharp$ of M_η	$\times$	$\times$	$\times$	1	1	$\times$	$\times$	$\times$	$\times$	1	1	$\times$	4

Now all matrices are positive semidefinite and M_0 is positive definite. Hence we can modify the holomorphic curvature tensor into a positive tensor by proposition 31 and hence we have positive holomorphic curvature.

Isotropy group $\mathbb{S}^1$ $SU(2)$ $\mathbb{T}^2$

In this case the desired modifications are:

Value	Intended	Forced
-1	$(M_0)_{(4,5)}$	$(M_{-\alpha_{23}})_{(1,1)}$
-1	$(M_0)_{(7,9)}$	$(M_{-\alpha_{23}})_{(4,4)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}	0
$\sharp$ of M_η	1	$\times$	$\times$	$\times$	$\times$	2	$\times$	1	$\times$	$\times$	$\times$	2	2

Now all matrices are positive semidefinite and M_0 is positive definite. Hence we can modify the holomorphic curvature tensor into a positive tensor by proposition 31 and hence we have positive holomorphic curvature.

Isotropy group $\mathbb{T}^2$ $SU(2)$ $SU(2)$

In this case the desired modifications are:

Value	Intended	Forced
2	$(M_{-\alpha_{12}})_{(1,1)}$	$(M_0)_{(2,5)}$
2	$(M_{-\alpha_{12}})_{(2,2)}$	$(M_0)_{(3,6)}$
2	$(M_{-\alpha_{12}})_{(3,3)}$	$(M_0)_{(8,11)}$
2	$(M_{-\alpha_{12}})_{(4,4)}$	$(M_0)_{(9,12)}$
2	$(M_{-\alpha_{34}})_{(1,1)}$	$(M_0)_{(5,6)}$
2	$(M_{-\alpha_{34}})_{(2,2)}$	$(M_0)_{(2,3)}$

Value	Intended	Forced
2	$(M_{-\alpha_{34}})_{(3,3)}$	$(M_0)_{(11,12)}$
2	$(M_{-\alpha_{34}})_{(4,4)}$	$(M_0)_{(8,9)}$
2	$(M_{-\beta_{34}})_{(1,1)}$	$(M_0)_{(5,12)}$
2	$(M_{-\beta_{34}})_{(2,2)}$	$(M_0)_{(2,9)}$
2	$(M_{-\beta_{34}})_{(3,3)}$	$(M_0)_{(6,11)}$
2	$(M_{-\beta_{34}})_{(4,4)}$	$(M_0)_{(3,8)}$

Now we restrict the tensor, i.e. we consider the submatrices obtained by erasing the following rows and columns:

$-\eta$	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}	β_{12}	β_{13}	β_{14}	β_{23}	β_{24}	β_{34}	0
$\sharp$ of M_η	$\times$	$1,3$	2	$2,4$	2	$\times$	$\times$	$\times$	2	$\times$	3	$\times$	$1,7$

Now all matrices are positive semidefinite and M_0 is positive definite. Hence we can modify the holomorphic curvature tensor into a positive tensor by proposition 31 and hence we have positive holomorphic curvature.

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