M. Girlando et al. "Conditional beliefs: from neighbourhood semantics to sequent calculus", The Review of Symbolic Logic, 11, pp. 736-779.

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Reviewer Name: Bruni, Riccardo

Mathematical Reviews/MathSciNet Reviewer Number: 138582

Address:
Dipartimento di Lettere e Filosofia
Università di Firenze
via della Pergola 58-60
50121 Florence
ITALY
riccardo.bruni@unifi.it

Author: Girlando, Marianna; Negri, Sara; Olivetti, Nicola; Risch, Vincent

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This is a thorough investigation on the logic CDL of conditional belief considered by O. Board [Games Econom. Behav. 49 (2004), no. 1, 49-80; MR2089181] and A. Baltag and S. Smets [Proceedings of the 13th Workshop on Logic, Language, Information and Computation (WoLLIC 2006), 5-21, Electron. Notes Theor. Comput. Sci., 165, Elsevier Sci. B. V., Amsterdam, 2006; MR2321761], [Logic and the foundations of game and decision theory (LOFT 7), 11-58, Texts Log. Games, 3, Amsterdam Univ. Press, Amsterdam, 2008; MR2985071], [New perspectives on games and interaction, 9-31, Texts Log. Games, 4, Amsterdam Univ. Press, Amsterdam, 2008; MR2985106]. The idea is to model belief and knowledge in a multi-agent setting by taking as primitive a notion of "conditional belief" ($Bel_i(A|B)$ in symbols), the meaning of which is that agent i would believe A in case B was added to her set of beliefs. The idea then is not to model an agent's actual beliefs directly, but rather to capture a hypothetical description of how the agent's set of beliefs would be like provided some further information was acquired by her. Actual, unconditional belief ($Bel_i(A)$ for: "agent i believes A") and even knowledge ($K_i(A)$ for: "agent i knows A"), are then defined out of conditional belief by putting

$$\begin{aligned} Bel_i(A) &:= Bel_i(A|\top) \\ K_i(A) &:= Bel_i(\bot|\neg A) \end{aligned}$$

where $\top$ and $\bot$ represent truth and falsity respectively as usual. The aim of the paper is, granted the Hilbert-style axiomatization of CDL, to present an alternative semantics for this system based on neighbourhood models, which are essentially a multi-agent version of D.K. Lewis' sphere models for the logic of counterfactuals [Harvard University Press, Cambridge, Mass., 1973; MR0421986]. This novel approach serves the purpose of introducing a labelled sequent calculus in the style of S. Negri [J. Philos. Logic 34 (2005), no. 5-6, 507-544; MR2189371], that is, by incorporating elements of the semantics into the syntax.

The new semantics the calculus CDL is provided with, comes in form of a triple $\mathcal{M} = \langle W, \{I_i\}_{i \in \mathcal{A}}, \nu \rangle$ for any given set $\mathcal{A}$ of agents, where: W is a non-empty set of worlds, I_i is, for every $i \in \mathcal{A}$, the neighbourhood function associating with every world $w \in W$ its neighbourhood, i.e. a set $I_i(w)$ of nested sets of worlds (that is, such that for every $\alpha, \beta \in I_i(w)$, it is either $\alpha \subseteq \beta$, or $\beta \subseteq \alpha$), and ν is the propositional evaluation function associating with each atomic formula a set of worlds. Neighbourhoods are then used to provide formulas of the form $Bel_i(A|B)$ with an interpretation that reflects the conditional nature of the situation regarding the agent's beliefs they are supposed to express. Hence, it is said that $Bel_i(A|B)$ is true at a world x (in symbols: $x \Vdash Bel_i(A|B)$) if and only if either $\neg B$ is everywhere true in every element of $I_i(x)$ (i.e., if, for every $\alpha \in I_i(x)$, it is $y \Vdash \neg B$ for every $y \in \alpha$), or B holds true somewhere in i-th neighbourhood of x, and $B \to A$, where $\to$ is the usual material conditional connective, is everywhere true in that element (that is, if there exists $\beta \in I_i(x)$ such that $y \Vdash B$ for some $y \in \beta$, and $z \Vdash B \to A$ for every $z \in \beta$).

The axiom system CDL is then proved to be sound with respect to the neighbourhood semantics and a completeness proof via construction of a canonical model of maximal consistent sets of formulas as worlds is also given. Then, as it was said, a labelled sequent calculs **G3CDL** is introduced, where two sorts of labels, one for worlds and one for neighbourhoods, are present. Each semantic condition on neighbourhood models is reflected in the sequent system by means of a rule. Rules, however, are carefully devised in order to achieve the desired structural properties, hence a closure condition to avoid redundant duplication of formulas in the premise(s) that may clash with admissibility of contraction is applied. Rules for knowledge and unconditional belief are shown to be admissible. The calculus has several, desirable structural properties: height-preserving intersubstitutivity of labels is proved to be admissible, as well as height-preserving invertibility of the rules of the calculus; contraction is shown to be height-preserving admissible, and a double-induction argument proves that the cut rule is also admissible. For this last result to yield a subformula property for the calculus as in the usual cases, it is noticed that a new definition of the notion of "subformula" the may circumvent the peculiar features of the formalism of CDL is required.

Equally important are the properties that **G3CDL** has with respect to the neighbourhood semantics. As a matter of fact, a soundness proof is given. Also, it is defined a strategy for proof-search that yields both a decision procedure and completeness for **G3CDL**. The strategy is required since root-first proof-search can be non-terminating owing to redundant backwards applications of rules. The proof-search strategy that gets defined has also exact bounds. As mentioned, completeness under this strategy is also achieved. In particular, completeness is shown to follow from the fact that a finite countermodel can be constructed out of a branch of the derivation tree of a sequent that is "saturated" in the sense of the article. Therefore, the finite model property also holds for **G3CDL**.

Those interested in complexity issues will also find some useful remarks, as validity of a formula in CDL is shown to be decidable in NEXPTIME and a conjecture is advanced that PSPACE might be the exact bound for it.

The last two sections of the paper are devoted to two more topics: on the one hand, to relate the new neighbourhood models with the semantics for CDL given in terms of what are called here "plausibility models", and have different names in the literature depending on the context of use (which can be many, as surveyed by E. Pacuit [Philosophy Compass 8 (2017), no. 9, 798-814]); on the other hand, to consider extensions of the basic formalism with other epistemic operators. In the first direction, it is shown that neighbourhood models are equivalent to plausibility ones, which are essentially Kripke structures featuring an equivalence relation over worlds for defining knowledge, and a plausibility relation for beliefs. In the second direction, the operator for "safe belief", that captures the attitude of agents which remains stable under the acquisition of further information of R. Stalnaker [The logic of strategy, 3-38, Oxford Univ. Press, New York, 1999; MR1715037], N. Malcolm [Mind 61 (1952), no. 242, 178-189], and J. Hintikka [Cornell University Press, 1962], and the operator of "strong belief" from P. Battigalli and M. Siniscalchi [J. Econom. Theory 106 (2002), no. 2, 356-391; MR1946502], A. Baltag and S. Smets [Logic and the foundations of game and decision theory (LOFT 7), 11-58, Texts Log. Games, 3, Amsterdam Univ. Press, Amsterdam, 2008; MR2985071], and E. Pacuit [Philosophy Compass 8 (2017), no. 9, 798-814], are added to the formalism by giving both their semantic conditions and the corresponding sequent rules. In addition, the operator of "weakly safe belief" as well as the unary operator for belief revision are shown to be treatable by the article novel means. Some final comments suggesting that some further operators might be considered for extensions are also given.

Comments to the MR Editors (not part of the Review Text):
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